Codomain rigidity of the Dirichlet to Neumann operator for the Riemannian wave equation

by

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Abstract

We study the Dirichlet to Neumann operator for the Riemannian wave equation on a compact Riemannian manifold. If the Riemannian manifold is modelled as an elastic medium, this operator represents the data available to an observer on the boundary of the manifold when the manifold is set into motion through boundary vibrations. We study the Dirichlet to Neumann operator when vibrations are imposed and data recorded on disjoint sets, a useful setting for applications. We prove that this operator determines the Dirichlet to Neumann operator where sources and observations are on the same set, provided a spectral condition on the Laplace-Beltrami operator for the manifold is satisfied. We prove this by providing an implementable procedure for determining a portion of the Riemannian manifold near the area where sources are applied. Drawing on established results, an immediate corollary is that a compact Riemannian manifold can be reconstructed from the Dirichlet to Neumann operator where sources and observations are on disjoint sets.

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Chapter 1

Introduction

1.1 Problem Setting

Let (M,g) be a connected, compact, smooth Riemannian manifold with boundary ∂M. The Riemannian wave equation with Dirichlet boundary condition $f \in C_0^\infty((0,\infty)\times\partial M)$ — the set of compactly supported smooth functions on $(0,\infty)\times\partial M$ — is given by

$$\begin{aligned} (\partial_t^2 - \Delta_g)u(t,x) = 0 \quad &\text{in } (0,\infty)\times M,\\ u = f \quad &\text{in } (0,\infty)\times \partial M,\\ u|_{t=0} = \partial_t u|_{t=0} = 0 \quad &\text{in } M, \end{aligned} \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator for the manifold (M,g), given by $\Delta_g = \text{div}\circ\nabla_g$, where div : $\chi(M) \to C^\infty(M)$ is the divergence operator on the manifold sending smooth vector fields on M to smooth scalar functions, and $\nabla_g : C^\infty(M) \to \chi(M)$ is the gradient map induced by g on M. We will denote the unique, smooth solution to Equation 1.1 by u^f. For a proof of the existence and uniqueness of solutions to Equation (1.1), as well as regularity estimates, see Chapter 2 of [12]. Equation 1.1 is a physical model for the propagation of vibrations through an elastic medium given by (M,g), where $u(t,x)$ is the one-dimensional displacement of the point $x \in M$ at time t, and f is an imposed vibration on the boundary. Let $\mathcal{S},\mathcal{R} \subset \partial M$ be open sets with $\mathcal{S}\cap\mathcal{R} = \emptyset$. We define the Dirichlet to Neumann operator as follows:

$$\Lambda_{\mathcal{S},\mathcal{R}} : C_0^\infty((0,\infty)\times\mathcal{S}) \to C^\infty((0,\infty)\times\mathcal{R})$$

$$\Lambda_{\mathcal{S},\mathcal{R}}(f) = \partial_\nu u^f|_{(0,\infty)\times\mathcal{R}}, \tag{1.2}$$

where $\partial_\nu u^f$ denotes the inward normal derivative of u^f, and where the inward normal vector is defined by the Riemannian metric. The Dirichlet to Neumann map is often interpreted as the experimental data available to an observer on $\mathcal{R}$ when vibrations are imposed by f on $\mathcal{S}$. A common question in the field of inverse problems is whether or not (M, g) is uniquely determined up to isometry from $\Lambda_{\mathcal{S},\mathcal{R}}$. In physical terms, this amounts to asking if the elastic medium (M, g) is determined by the experimental data obtained from imposing vibrations over $\mathcal{S}$ and observing the response on $\mathcal{R}$. This question is motivated by applications such as mineral prospection, where the determination of the Riemannian metric g can give insight into the material properties, and in turn the mineral composition, of the Earth's crust. In this thesis, we study the dependence of the solvability of this inverse problem on the codomain $C^\infty((0,\infty)\times\mathcal{R})$ of the Dirichlet to Neumann map. Our main result is the following:

Theorem 1.1.1. *Suppose that $\mathcal{S}\cap\mathcal{R} = \emptyset$ and that there exists $F_1, F_2 : \mathbb{R}^+ \to \mathbb{R}^+$ so that $\frac{F_2}{F_1}$ is bounded and*

$$F_1(\lambda_j) \leq \|\partial_\nu \phi_j\|_{L^2(\mathcal{S})} \leq F_2(\lambda_j) \quad j = 1, 2, \ldots \tag{1.3}$$

where $\{\lambda_j\}_{j=1}^\infty$ are the Dirichlet eigenvalues of Δ_g and $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis for the corresponding Dirichlet eigenfunctions. Then $\Lambda_{\mathcal{S},\mathcal{R}}$ determines $\Lambda_{\Gamma,\Gamma}$, where $\Gamma \subset \partial M$ is any open set satisfying $\overline{\Gamma} \subset \mathcal{S}$.

Remark on Theorem 1.1.1 : The phrase "$\Lambda_{\mathcal{S},\mathcal{R}}$ determines $\Lambda_{\Gamma,\Gamma}$" in this theorem means that $\Lambda_{\Gamma,\Gamma}$ can be written explicitly as a function of $\Lambda_{\mathcal{S},\mathcal{R}}$.

Corollary 1.1.2. *Under the conditions of Theorem* 1.1.1*, $\Lambda_{\mathcal{S},\mathcal{R}}$ determines (M, g) up to isometry.*

1.2 Background

The problem of determining (M, g) from the Dirichlet to Neumann operator has a long history. Problems of this type were first posed by Calderon [5], who asked if the Dirichlet to Neumann operator for a second-order elliptic PDE known as the conductivity equation uniquely determines the electrical conductivity of an open region with smooth boundary in $\mathbb{R}^n$, and if so, asked for constructive procedures for determining this conductivity. The PDE involved here is similar to Equation 1.1 except with no time dependence; as such, the conductivity equation can be thought of as a steady-state version of the wave equation. Despite this connection, solution methods for the inverse problem for the conductivity equation bear little resemblance to solution methods for the Riemannian wave equation. For example, dimension appears to play a large role in the conductivity equation inverse problem, as there was an 11 year gap between solving the problem for dimensions $n \geq 3$ [20] and the solution for $n = 2$ [15]. In contrast, the same inverse problem for the Riemannian wave equation has no such dependence, as the technique proven in this thesis will work for all dimensions.

The time dependence of the Riemannian wave equation plays a key role here. In particular, the finite propagation speed of waves allows one to gather local information from the Dirichlet to Neumann map, which is critical in the reconstruction procedure we develop.

In this thesis we study PDEs over a Riemannian manifold rather than an open region in $\mathbb{R}^n$, as was the historical setting for the conductivity equation inverse problem. It may seem unmotivated to examine this problem using the framework of a Riemannian manifold, since most applications (e.g., medical imaging or mineral prospection) fit within the setting of an open region in $\mathbb{R}^n$. There may be, however, different coordinate representations for the same domain, and one would like to obtain a solution represented in a coordinate free manner; Riemannian manifolds are the natural mathematical setting for such an undertaking.

For the Dirichlet to Neumann operator for the Riemannian wave equation, research has been focused in three broad categories. The first is when the vibrational sources and responses are obtained on the entirety of the boundary — $\mathcal{S} = \mathcal{R} = \partial M$ in our notation — which is commonly referred to as the complete data problem. The complete data problem was first solved by Belishev and Kurylev in [2] using the boundary control method. The setting in this paper is slightly different from the one we consider; rather than assuming $\Lambda_{\mathcal{S},\mathcal{R}}$ is known for $\mathcal{S} = \mathcal{R} = \partial M$, the authors assume knowledge of the eigenvalues for Δ_g and traces of Neumann eigenfunctions on ∂M, and proceed to show that this uniquely determines (M, g). The equivalence of this problem to the problem where the Dirichlet to Neumann operator is known is made clear in [12]; in particular, they show that each set of data can be obtained from the other. From the point of view of applications, however, it makes more sense to assume knowledge of the Dirichlet to Neumann operator directly, as this represents the available experimental data. For a review of the boundary control method as applied to a variety of inverse problems, see [1].

The second category of problems is when stimuli and responses are obtained on the same open proper subset of the boundary, $\mathcal{S} = \mathcal{R} \neq \partial M$ in our notation. We shall call this type of problem the partial data problem. This is a natural extension of the complete data problem, since in certain cases it may be difficult to obtain data on the entirety of the boundary. For example, if one wishes to use these techniques for mineral prospection, the boundary of the manifold would be the surface of the Earth. The partial data problem was initially solved in [11], which also assumes the boundary spectral data alluded to above.

Other interesting works on the partial data problem include [3] and [16]. In both of these publications, the Neumann to Dirichlet operator is assumed known, as opposed to the Dirichlet to Neumann operator. The Neumann to Dirichlet operator is similar to the Dirichlet to Neumann operator, except that Neumann boundary conditions are imposed (i.e. conditions on the normal derivative of a function), and the value of the solution u is observed on the boundary. In [3], optimization techniques are used to produce a sequence of waves controlled from the boundary which focus their support to a single point. This phenomenon is then used to find geodesics that intersect the studied portion of the boundary, which in turn determines the boundary distance functions and the manifold. This idea is extended in [16] to computing the volumes of domains of influence, which are regions of the manifold influenced by a boundary control supported over a certain time span. For example, if a boundary control is supported within $(0,T)\times\mathcal{S}$, the domain of influence $M(\mathcal{S},T)$ will be the closure of the set of all points within M which are less than T units of distance away from $\mathcal{S}$; by the finite propagation speed of waves, this is exactly the portion of the manifold that a control supported within $(0,T)\times\mathcal{S}$ can affect. These volumes are then used to compute the boundary distance functions, which determine an isometric copy of the manifold. Both of these techniques are designed for implementation and rely heavily on the fact that the stimuli and responses are obtained on the same sets, precluding them for use in the third class of problem, to be discussed presently.

The third class of problem is called the disjoint partial data problem. In this setting, we assume that $\Lambda_{\mathcal{S},\mathcal{R}}$ is known for $\mathcal{S}\cap\mathcal{R}=\emptyset$. Note that the case $\mathcal{S}\cap\mathcal{R}\neq\emptyset$ can be reduced to the partial data problem once the Dirichlet to Neumann operator is restricted to $\mathcal{S}\cap\mathcal{R}$. The disjoint partial data problem is the most relevant of the three categories of problems discussed here for applications, since it is often difficult to record observations over the same region in which stimuli are imposed. For example, in mineral prospection, vibrations are often imposed using explosives, which makes gathering data in the same region a dangerous activity. Of the three categories of problems discussed here, however, the least amount of work has been done for the disjoint partial data problem. Early results include [18] which solves the problem of recovering the material properties of a finite one dimensional elastic medium where sources are supported on one end and observations are taken on another. In [13], uniqueness results are proved for the disjoint partial data problem under a variety of settings. The authors show first that, if $\overline{\mathcal{S}}\cap\overline{\mathcal{R}}\neq\emptyset$ and there is some portion of the boundary which contains $\overline{\mathcal{S}}$ and $\overline{\mathcal{R}}$ and is known as a smooth manifold, then $\Lambda_{\mathcal{S},\mathcal{R}}$ determines the isometry class of (M,g). Second, they show that, if the pairwise Dirichlet to Neumann operator is known between three pairwise disjoint open sets, then this uniquely determines the isometry class of (M,g).

More recently, in [14], the authors prove that the Riemannian manifold is determined up to isometry by $\Lambda_{\mathcal{S},\mathcal{R}}$ for the disjoint partial data problem under the assumption of exact controllability from $\mathcal{S}$, which is that for some $T > 0$, the map

$$\mathcal{U} : L^2((0,T) \times \mathcal{S}) \to L^2(M) \times H^{-1}(M),$$

$$\mathcal{U}(f) = (u^f(T), \partial_t u^f(T)),$$

is surjective. They also prove that, under some spectral conditions which are stronger than those in Theorem 1.1.1, $\Lambda_{\mathcal{S},\mathcal{R}}$ determines the isometry class of (M, g) when $\mathcal{S} \cap \mathcal{R} = \emptyset$. The spectral conditions they use are strictly weaker than exact controllability, and greatly reduce the constraints on the set $\mathcal{S}$. For example, if (M, g) is the unit disc in $\mathbb{R}^2$ with the standard Euclidean metric, exact controllability only holds for $\mathcal{S}$ containing one hemisphere, whereas the spectral conditions hold for any $\mathcal{S}$ open. The technique these authors describe is to find the Riemannian distance function between points on $\mathcal{R}$ and points near $\mathcal{R}$ using domains of influence. More specifically, they test the inclusion of domains of influence in other domains of influence, as this inclusion relation contains some geometric information on the Riemannian distance function. This inclusion relation is tested by computing the inner products of solutions to the Riemannian wave equation, which are determined through Blagovestchenskii's identity. For $\psi \in C_0^\infty((0,T) \times \mathcal{R})$ and $f \in C_0^\infty((0,T) \times \mathcal{S})$, Blagovestchenskii's identity is

$$(u^\psi(T), u^f(T))_{L^2(M)} = (\psi, (J\Lambda_{\mathcal{S},\mathcal{R}} - R\Lambda_{\mathcal{S},\mathcal{R}}RJ)f)_{L^2((0,T)\times\mathcal{R}},$$

where J is a time filtering operator and R is a time reversal operator. Testing inclusion of these domains of influence is a difficult procedure, however, as it requires the computation of uncountably many inner products through Blagovestchenskii's identity. Once the Riemannian distance function is determined through the testing of inclusion relations, the authors of [14] show that this distance function uniquely determines the Riemannian metric over a coordinate patch. No procedure is given, however, for actually finding the Riemannian metric over this patch. As such, even though the authors of [14] prove that $\Lambda_{\mathcal{S},\mathcal{R}}$ contains enough information to uniquely determine the isometry class of (M, g), the procedure they provide should not be viewed as one that is suitable for implementation.

This thesis is inspired by [14], and our contribution is three-fold:

(i) We prove that the Dirichlet to Neumann operator is codomain rigid; more precisely, we show that the Dirichlet to Neumann operator for the disjoint partial data problem determines the Dirichlet to Neumann operator for the partial data problem. This result is of mathematical interest independent of manifold reconstruction, but, additionally, allows for previous work on the partial data problem to be applied to the disjoint partial data problem. This stands in contrast to [14], in which the technique developed makes little use of previous results for the partial data problem. There are many efficient techniques ([3], [16]) for reconstructing a manifold from the Dirichlet to Neumann map for the partial data problem, and the technique we provide here has the advantage of making use of these previous results, whereas the technique in [14] does not.

(ii) We provide an implementable procedure for reconstructing a portion of the Riemannian manifold from the Dirichlet to Neumann operator. This stands in contrast to [14], where the method employed to reconstruct the manifold is not implementable as it requires uncountably many tests to complete. We also prove some stability estimates which give an estimate of the accuracy of our approach when calculations are halted in finite time, which is useful for implementation. This stands again in contrast to [14], in which the described procedure yields no results if halted in finite time.

(iii) Finally, we broaden the set of manifolds whose isometry class is uniquely determined by $\Lambda_{\mathcal{S},\mathcal{R}}$. We do this by generalizing the spectral condition necessary for unique reconstruction. This is an important extension, since manifolds as simple as the upper hemisphere of $\mathbb{S}^2$ are excluded from the uniqueness result obtained in [14].

We will now sketch the procedure we use for proving our main theorem. In Chapter 2 we use the Laplace transform of the Dirichlet to Neumann map to determine some spectral information about Δ_g. In particular, we will determine the spectrum of Δ_g and the linear span of the normal derivatives of the Dirichlet eigenfunctions of Δ_g restricted to the boundary. Transformation of the problem in this way resembles the techniques used in [15] for the conductivity equation, where a Fourier transform of the problem forms the core of the reconstruction technique. This spectral information allows us to define a functional which, upon minimization, gives information on the volumes of certain sets which are obtained from domains of influence using elementary set operations. These volumes are then related to the Riemannian distance function through Lemma 2.1.1. In this way, our approach resembles the optimization techniques used in [3] and [16], as we are effectively using a variational approach to produce waves which have support focussed in an area of interest. One key difference between our approach and that of [16] is that while they determine the exact volume of a certain set, we are only able to determine whether or not that set has zero measure, which turns out to be enough information to determine the Riemannian distance function. Testing the volumes of these sets also mirrors the technique of [14], as one of the tests we use (see (i) of Lemma 2.2.4) determines whether or not a certain set has non-zero measure, and is equivalent to testing the inclusions of domains of influence used in [14]. We also establish some other tests which are not covered in [14] which improve the implementability of our approach (see (ii) of Lemma 2.1.1 and Lemma 2.1.3).

In Chapter 3 we assume that the Riemannian distance function is known between points on the boundary and points on the interior, and we then compute the gradient of this function with respect to the interior variable in boundary normal coordinates. Elementary Riemannian geometry, or alternatively, considerations from optimal control, then show that this gradient and the Riemannian metric must satisfy a relationship known as the Eikonal equation. We use this equation as a basis for a variational approach to finding the Riemannian metric in a coordinate patch near $\mathcal{S}$ once the gradient of the distance function is known. We also use some elementary convex analysis to prove that if the gradient of the distance function is known up to some small error, then a perturbed functional can be used to find an estimate of the underlying Riemannian metric. This result is essential for implementation, as in practice the Riemannian distance function, and hence the gradient of this function, will only be known up to some precision.

Finally, in Chapter 4, knowledge of the Riemannian metric in a coordinate patch near $\mathcal{S}$ then allows us to determine the Dirichlet to Neumann map $\Lambda_{\Gamma,\Gamma}$, where Γ is as in the statement of Theorem 1.1.1. We prove this first in a non-constructive way under no additional assumptions, and then again in a constructive manner with more hypothesis. This result establishes that under the spectral conditions of Theorem 1.1.1, the disjoint partial data problem is actually equivalent to the partial data problem. At this point in our reconstruction procedure, any of the established results on the partial data problem can be used to finish the reconstruction of the manifold. Again, this stands in contrast to the reconstruction technique used in [14], which is completely isolated from previous results.

In keeping with Calderon's original question about the inverse conductivity problem, we have proved that the Dirichlet to Neumann map for the Riemannian wave equation uniquely determines the underlying Riemannian metric up to isometry, and have provided a constructive method for doing so. This is similar in spirit to Nachman's result in [15], where a uniqueness result is proved for the inverse conductivity problem in two dimensions and a step-by-step procedure is outlined for reconstructing the electric conductivity. Because of his algorithmic approach, Nachman's technique has seen some numerical implementations (see e.g. [19]). For the Riemannian wave equation, very few numerical implementations have been completed even for the complete data problem (see e.g. [17]), and, to our knowledge, no numerical results have been obtained for the disjoint partial data problem. As such, it is our hope that the methods outlined here are of sufficient clarity as to follow [15] in terms of future numerical implementations.

Chapter 2

Reconstructing the distance function from the Dirichlet to Neumann operator

2.1 Relating volumes to the Riemannian distance

We start by defining some geometric parameters, namely the exit time and the distance to the cut locus from $\mathcal{S}$. We define the exit time $\tau_M : \mathcal{S} \to \mathbb{R}$ by

$$\tau_M(y) = \inf\{s \in (0, \infty) \mid \gamma(s; y, \nu) \in \partial M\},$$

where, ν denotes the element of $T_y M$ that is normal to ∂M, inward pointing, and of unit norm, as determined by the Riemannian metric, and $\gamma(s; y, \nu)$ denotes the position at time s of a geodesic with initial conditions $\gamma(0) = y, \dot\gamma(0) = \nu$. The exit time from y, $\tau_M(y)$, is the smallest non-zero time at which the geodesic starting at y in the normal direction hits the boundary. It is a fact (see [10]) that τ_M is lower semi-continuous and $\tau_M(y) > 0$ for all $y \in \partial M$. We define the distance to the cut locus from $\mathcal{S}$, $\sigma_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}$, by

$$\sigma_{\mathcal{S}}(y) = \sup\{s \in (0, \tau_M(y)] \mid d(\gamma(s; y, \nu), \mathcal{S}) = s\}.$$

We remark that this definition is not the traditional distance to the cut locus, which would replace $\mathcal{S}$ in the definition with y (see e.g. [6]). We have $\sigma_{\mathcal{S}}(y) > 0$ for all $y \in \Gamma$, and $\sigma_{\mathcal{S}}$ is a lower semi-continuous function of y [14]. We set

$$\sigma = \inf_{y \in \mathcal{S}} \sigma_{\mathcal{S}}(y).$$

Let $\Gamma \subset \partial M$, and let $h : \Gamma \to \mathbb{R}$ be a piecewise continuous function. Following [14], we define the modified distance to Γ as follows,

$$d_h(x,y) = d(x,y) - h(y) \quad x \in M, y \in \Gamma,$$
$$d_h(x,\Gamma) = \inf_{y\in\Gamma} d_h(x,y) \quad x \in M,$$

where $d : M \times M \to \mathbb{R}$ is the Riemannian distance function. The domain of influence corresponding to h is given by

$$M(\Gamma, h) = \{x \in M \mid d_h(x,\Gamma) \leq 0\}.$$

If h is a constant function, then $M(\Gamma, h)$ is the closure of the set of all points in M that are less than h units of distance away from Γ, as determined by the Riemannian metric. By the finite speed of wave propagation (see e.g. [7], Chapter 7), if $f \in C_0^\infty((0,h) \times \Gamma)$, then $\mathrm{supp}(u^f) \subset M(\Gamma, h)$. Thus, $M(\Gamma, h)$ is the domain that $f \in C_0^\infty((0,h) \times \Gamma)$ can influence.

We now present a series of lemmas that will allow us to calculate the Riemannian distance between points on the interior and points on the boundary, provided the volumes of certain sets formed from domains of influence are known.

Lemma 2.1.1. *Let $t, s, \delta > 0$, $s > \delta$, $y_0, y_1 \in \mathcal{S}$, and let $\Gamma_0, \Gamma_1 \subset \mathcal{S}$, open sets so that $\bar{\Gamma}_i \subset \mathcal{S}$ $y_i \in \Gamma_i$ and $d(y, y_i) \leq \delta$ for all $y \in \Gamma_i$, $i = 0, 1$. Set $x_0 = \gamma(s; y_0, \nu)$ and let $\epsilon(\delta) = \sup\{d(x, x_0) \mid x \in M(\Gamma_0, s) \setminus (M(\mathcal{S}, s - \delta))\}$. Then*

(i) If $\mathrm{Vol}(M(\Gamma_0, s) \setminus (M(\mathcal{S}, s-\delta) \cup M(\Gamma_1, t))) > 0$, then $d(y_1, x_0) \geq t - \epsilon(\delta)$

(ii) If $\mathrm{Vol}(M(\Gamma_0, s) \cap M(\Gamma_1, t) \setminus M(\mathcal{S}, s-\delta)) > 0$, then $d(y_1, x_0) \leq t + \delta + \epsilon(\delta)$

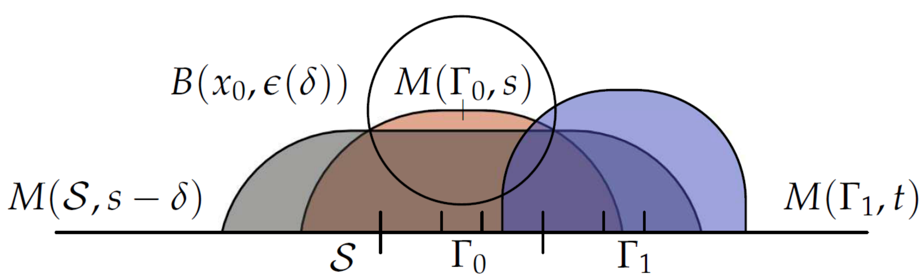

Figure 2.1: The sets of Lemma 2.1.1

Proof. Assuming the hypothesis of (i), there exists $x \in M(\Gamma_0, s) \setminus (M(\mathcal{S}, s-\delta) \cup M(\Gamma_1, t))$. Using the triangle inequality we find

$$\begin{aligned} d(y_1, x_0) &\geq d(y_1, x) - d(x, x_0), \\ &\geq t - \epsilon(\delta). \end{aligned}$$

Here, the second inequality follows from the fact that $x \in M(\Gamma_1, t)^c \cap \overline{B(x_0, \epsilon(\delta))}$, the latter set being the closed geodesic ball of radius $\epsilon(\delta)$ centred at x_0.

We prove (ii) in a similar way. The hypothesis implies that there exists $x \in M(\Gamma_0, s) \cap M(\Gamma_1, t) \setminus M(\mathcal{S}, s-\delta)$. Again using the triangle inequality, we find

$$\begin{aligned} d(y_1, x_0) &\leq d(y_1, x) + d(x, x_0), \\ &\leq t + \delta + \epsilon(\delta). \end{aligned}$$

Here, the second inequality follows from the fact that $x \in M(\Gamma_1, t) \cap \overline{B(x_0, \epsilon(\delta))}$ □

We would like to get an estimate on $\epsilon(\delta)$, which is easiest to do if we are working within a single coordinate chart. We now show that, if $s < \sigma$ and δ is small enough, $M(y_0, s) \setminus M(\Gamma_0, s-\delta)$ is entirely contained in the domain of a boundary normal coordinate chart. Let $s_0 < \sigma$ and let $\exp : (0, s_0) \times \Gamma \to M$ be the exponential map, defined so that $\exp(t, y) = \gamma(t; y, \nu)$. It is well known that $s_0 < \sigma$ implies that exp will be a diffeomorphism onto its image.

Lemma 2.1.2. *Let $\Gamma \subset \partial M$ be an open set such that $\bar{\Gamma} \subset \mathcal{S}$. If $s < \sigma$ there exists $t \in (s, \sigma)$ and $\delta > 0$ small enough so that $M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta) \subset \exp((0,t) \times \Gamma_0')$, where Γ_0' is any open set satisfying $\bar{\Gamma}_0 \subset \Gamma_0'$*

Proof. Suppose that this is not the case. Then, for any $\{\delta_n\}_{n=1}^\infty$ so that $\lim_{n\to\infty} \delta_n = 0$ and $t \in (s, \sigma)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ such that

$$x_n \in (M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta_n)) \cap \exp((0,t) \times \Gamma_0')^c,$$

where A^c denotes the complement of a set $A \subset M$ relative to M. The latter set is closed since exp is a diffeomorphism on the domain it is defined. Since M is compact, $\exp((0,t) \times \Gamma_0')^c$ is compact. Taking a subsequence if necessary, we can assume that $\lim_{n\to\infty} x_n = x \in \exp((0,t) \times \Gamma_0')^c$. Furthermore, $x \in M(\Gamma_0, s) \setminus M(\mathcal{S}, s)^{\text{int}}$, where $M(\mathcal{S}, s)^{\text{int}}$ denotes the interior of $M(\mathcal{S}, s)$. Indeed,

$$\begin{aligned} d(x, \Gamma_0) &= \lim_{n\to\infty} d(x_n, \Gamma_0), \\ &\leq s. \end{aligned}$$

Further,

$$\begin{aligned} d(x,\mathcal{S}) &= \lim_{n\to\infty} d(x_n,\mathcal{S}), \\ &\geq \lim_{n\to\infty} s-\delta_n, \\ &= s. \end{aligned}$$

We claim that $M(\Gamma_0,s)\setminus M(\mathcal{S},s)^{\mathrm{int}} = \{\gamma(s;y,\nu)\mid y\in\bar{\Gamma}_0\} =: A$. Indeed, let $z\in A$. Letting $y\in\bar{\Gamma}_0$ so that $z=\gamma(s;y,\nu)$, we see that $z\in M(\Gamma_0,s)$. Further, $s<\sigma$, and so by definition of σ, $d(z,\mathcal{S})=s$. As a result, $z\in(M(\mathcal{S},s)^{\mathrm{int}})^c$, and therefore $z\in M(\Gamma_0,s)\setminus(M(\mathcal{S},s)^{\mathrm{int}})$.

Now let $z\in M(\Gamma_0,s)\setminus(M(\mathcal{S},s)^{\mathrm{int}})$. Then $d(z,\Gamma_0)\leq s$ and $d(z,\mathcal{S})\geq s$. But $\bar{\Gamma}_0\subset\mathcal{S}$, so we get $s\geq d(z,\Gamma_0)\geq d(z,\mathcal{S})\geq s$. As such, there exists $y\in\bar{\Gamma}_0\subset\mathcal{S}$ so that $d(z,y)=d(z,\mathcal{S})=s$. Let γ be a unit-speed shortest path connecting y and z so that $\gamma(0)=y$. Observe that there exists some $\rho>0$ so that $\gamma|_{(0,\rho)}\subset M^{\mathrm{int}}$. Indeed, if no such ρ existed, then by openness of $\mathcal{S}$, $s>d(z,\mathcal{S})$. But a shortest path in the interior of a manifold must be a geodesic, so $\gamma|_{(0,\rho)}$ coincides with some geodesic that starts at y. Let $r\in(0,\rho)$ be small enough so that $\gamma|_{[0,r]}$ lies within a boundary normal coordinate patch (U,ϕ), where $\phi:U\to\mathbb{R}^n$ is a diffeomorphism onto its image, and set $z':=\gamma(r)$. Then $d(z',y)=d(z',\mathcal{S})=r$, since otherwise we would have $d(z,\mathcal{S})<s$. Sending U to $\mathbb{R}^n$ through the coordinates ϕ, and identifying the points z' and y with their images under ϕ, we get

$$y = \operatorname*{arg\,min}_{w\in\phi(U)\cap(\mathbb{R}^{n-1}\times\{0\})} d(\phi^{-1}(z'),\phi^{-1}(w)).$$

Note that since $z'\notin\mathbb{R}^{n-1}\times\{0\}$, $f:=d(\phi^{-1}(z'),\phi^{-1}(\cdot)):\phi(U)\cap(\mathbb{R}^{n-1}\times(-\epsilon,\epsilon))\to\mathbb{R}$, with $\epsilon>0$, is a smooth function, provided r is small enough. Then, by standard constrained optimization techniques (see e.g. [8]) we must have

$$\nabla f(y) = ce_n,$$

where e_n is the nth element of the canonical basis for $\mathbb{R}^n$, and c is some constant. If ϕ is expressed in components as $\phi=(x^1,\dots,x^n)$, then the differential of $d(z',\cdot)$ is

$$\mathrm{d}(d(z',\cdot))|_y = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(y)\partial x^i = c\partial x^n.$$

Noting the form of the Riemannian metric in boundary normal coordinates, as in the preamble for Lemma 2.1.3, and the relationship between the Riemannian gradient and the differential, we get

$$\nabla_g d(z', \cdot)|_y = c\frac{\partial}{\partial x^n}.$$

Since we are using boundary normal coordinates, this shows us that $\nabla_g d(z', \cdot)|_y$ is normal to ∂M. We also have that $\nabla_g d(z', \cdot)|_y \in T_y M$ is parallel to the geodesic connecting z' and y at y (see e.g. Lemma 11 of [14]). As such, γ is normal to the boundary at y, and so $\gamma(t) = \gamma(t; y, \nu)$ provided $t \leq \tau_M(y)$ by uniqueness of geodesics. Since $s < \sigma \leq \tau_M$, we must have $z = \gamma(s; y, \nu) \in A$. This proves the claim.

As a result of the claim, there exists $y \in \bar{\Gamma}_0 \subset \Gamma_0'$ so that $x = \gamma(s; y, \nu)$. But then $x \in \exp((0,t) \times \Gamma_0')$, a contradiction to what was asserted above.

□

Remark on Lemma 2.1.2: By choosing δ sufficiently small, $\bar{\Gamma}_0$ will be in the domain of some coordinate chart sending Γ_0 to $\mathbb{R}^{n-1} \times \{0\}$. This guarantees that $M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)$ will be in the domain of a boundary normal coordinate chart. Note also that under the conditions of the lemma, $M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta) \subset \exp(\Gamma_0' \times (s-\delta, s))$.

We now prove an estimate on $\epsilon(\delta)$ that provides an upper bound and shows that $\lim_{\delta \to 0} \epsilon(\delta) = 0$. To start with, let s and δ be as in Lemma 2.1.2 so that $M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)$ is contained in the domain of a boundary normal coordinate chart. Let $U \subset \mathbb{R}^n$ be the image of $M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)$ under these coordinates, and set U_0 as the image of $(0,t) \times \Gamma_0'$ under the same coordinates, where $\bar{\Gamma}_0 \subset \Gamma_0'$. Finally, let $\mathbf{G} : U \to M^n(\mathbb{R})$ be a matrix representation of the Riemannian metric in these coordinates so that $\mathbf{G}_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Assuming that x_n corresponds to the distance of a point in M from $\mathcal{S}$, we recall that $\mathbf{G}$ has the following form:

$$\mathbf{G}(x) = \begin{bmatrix} \tilde{\mathbf{G}}(x) & 0 \\ 0 & 1 \end{bmatrix}.$$

Define the following quantities

$$\beta = \inf_{x \in U_0} \{\lambda \mid \lambda = \min\{\text{spec}(\mathbf{G}(x))\}\},$$
$$\alpha = \sup_{x \in U_0} \{\lambda \mid \lambda = \max\{\text{spec}(\mathbf{G}(x))\}\},$$

Since $\mathbf{G}(x)$ is positive-definite for all $x \in U_0$ and the metric is smooth over M, which is compact, we have that $\beta > 0$ and $\alpha < \infty$.

Lemma 2.1.3. *With s, δ and $\epsilon(\delta)$ as in Lemmas 2.1.1 and 2.1.2, and δ small enough, there exists some positive constant C so that*

$$\epsilon(\delta) \leq C(\frac{\alpha}{\beta}\delta)^{1/2}. \tag{2.1}$$

Proof. We consider two additional Riemannian metrics over U_0.

$$\mathbf{G}^*(x) = \alpha \mathbf{I}_n \qquad x \in U_0,$$
$$\mathbf{G}_*(x) = \begin{bmatrix} \beta \mathbf{I}_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \qquad x \in U_0.$$

We also let $d^* : U_0 \times U_0 \to \mathbb{R}$ and $d_* : U_0 \times U_0 \to \mathbb{R}$ be the distance functions induced by the Riemannian metrics $\mathbf{G}^*$ and $\mathbf{G}_*$. Let $x_0, x_1 \in U$ and let $\gamma : [0,1] \to U_0$ be any C^1 curve so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Let also $l_g(\gamma)$ denote the length of the curve γ under g. We have:

$$\begin{aligned} l_g(\gamma) &= \int_0^1 \sqrt{\dot{\gamma}(t)^T \mathbf{G}(\gamma(t))\dot{\gamma}(t)}dt, \\ &= \int_0^1 \sqrt{\dot{\gamma}(t)^T \tilde{\mathbf{G}}(\gamma(t))\dot{\gamma}(t) + \dot{\gamma_n}(t)^2}dt, \\ &\geq \int_0^1 \sqrt{\beta(\dot{\gamma_1}(t)^2 + \dots \dot{\gamma_{n-1}}(t)^2) + \dot{\gamma_n}(t)^2}dt, \\ &= l_{g_*}(\gamma). \end{aligned}$$

Here the third inequality follows from the fact that every eigenvalue of $\tilde{\mathbf{G}}(x)$ is also an eigenvalue of $\mathbf{G}(x)$. One can similarly show that $l_g(\gamma) \leq l_{g^*}(\gamma)$. Minimizing over all C^1 curves γ connecting x_0 and x_1 in U_0, we have that

$$d_*(x_0, x_1) \leq d(x_0, x_1) \leq d^*(x_0, x_1).$$

Note that the minimizing geodesics between the points x_0 and x_1 will lie in U_0 for δ small enough, since U will be contained in some Euclidean ball of small radius, and every geodesic ball of small enough radius is geodesically convex. Letting $M_*(y_0, s) = \{x \in U_0 \mid d_*(x, y_0) \leq s\}$, we observe that

$$M(y_0, s) \setminus M(\mathcal{S}, s-\delta) \subset M_*(y_0, s) \setminus M(\mathcal{S}, s-\delta).$$

Note that here we have identified y_0 and its coordinate representation in U_0. Without loss of generality, $y_0 = 0$, and

$$\Gamma_0 \subset \{x \in \mathbb{R}^n \mid x_n = 0, x_1^2 + \ldots x_{n-1}^2 \leq \frac{\delta^2}{\beta}\} =: B.$$

Furthermore, since x_n determines the distance of a point in U_0 from $\mathcal{S}$, we get that $M(\Gamma_0, s-\delta) = \{x_n \leq s-\delta\}$. Now,

$$\begin{aligned}
\epsilon(\delta) &= \sup\{d(x, x_0) \mid x \in M(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)\}, \\
&\leq \sup\{d(x, x_0) \mid x \in M_*(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)\}, \\
&\leq \sup\{d^*(x, x_0) \mid x \in M_*(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)\}, \\
&= \sqrt{\alpha} \sup\{\|x_0 - x\| \mid x \in M_*(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta)\}.
\end{aligned}$$

Here, $\|\cdot\|$ denotes the usual euclidean norm. Now,

$$M_*(\Gamma_0, s) \setminus M(\mathcal{S}, s-\delta) = \overline{(\cup_{y \in \Gamma_0} M_*(y, s))} \setminus \{x_n \leq s-\delta\}.$$

Let $x \in M_*(y, s) \setminus \{x_n \leq s-\delta\}$ for some $y \in \Gamma_0$. Let $z = \gamma(s; y, \nu)$. Then

$$\begin{aligned}
\|x_0 - x\| &\leq \|x_0 - z\| + \|z - x\|, \\
&\leq \frac{\delta}{\sqrt{\beta}} + \|z - x\|.
\end{aligned}$$

The second inequality holds because $y \in \Gamma_0 \subset B$, and the last coordinate of x_0 and z are the same. Hence,

$$\sup_{x \in M_*(y,s) \setminus \{x_n \leq s-\delta\}} \|x_0 - x\| \leq \frac{\delta}{\sqrt{\beta}} + \sup_{x \in M_*(y,s) \setminus \{x_n \leq s-\delta\}} \|z - x\|. \tag{2.2}$$

One can show that the domain of influence $M_*(y, s)$ is given by

$$M_*(y, s) = \{x \in U \mid \beta((x_1 - y_1)^2 + \ldots (x_{n-1} - y_{n-1})^2) + x_n^2 \leq s^2\}.$$

Using standard constrained optimization techniques [8], one can determine that the solution to the optimization problem on the right hand side of Equation (2.2) occurs when both constraints are active. In other words, when

$$\beta((x_1 - y_1)^2 + \ldots (x_{n-1} - y_{n-1})^2) + x_n^2 = s^2,$$

and $x_n = s - \delta$. Solving this maximization problem, we get

$$\sup_{x\in M_*(y,s)\setminus\{x_n\leq s-\delta\}}\|x_0-x\|\leq\frac{\delta}{\sqrt{\beta}}+(\frac{2s\delta}{\beta})^{1/2}.$$

This bound holds independently of $y\in\Gamma_0$, and so for all $\delta>0$ small enough,

$$\epsilon(\delta)\leq(\frac{\alpha}{\beta})^{1/2}(\delta+(2s\delta)^{1/2})\leq C(\frac{\alpha}{\beta}\delta)^{1/2}.$$

□

Remark on Lemma 2.1.3: Lemma 2.1.3 provides estimates on the accuracy of the distance function calculated using 2.1.1, which is useful for implementation.

We now show that the Dirichlet to Neumann operator provides enough information to test the conditions of Lemma 2.1.1.

2.2 Testing volumes of domains of influence

We consider the case of disjoint source and receiver domains. In this setting we assume knowledge of the Dirichlet to Neumann operator $\Lambda_{\mathcal{S},\mathcal{R}}:C_0^\infty((0,\infty)\times\mathcal{S})\to C^\infty((0,\infty)\times\mathcal{R})$ with

$$\Lambda_{\mathcal{S},\mathcal{R}}(f)=\partial_\nu u^f|_{(0,\infty)\times\mathcal{R}},\quad f\in C_0^\infty((0,\infty)\times\mathcal{S}),\tag{2.3}$$

where $\mathcal{S},\mathcal{R}\subset\partial M$, and $\mathcal{S}\cap\mathcal{R}=\emptyset$. Select $d\tilde{S}$ as an arbitrary positive smooth measure on $\mathcal{S}$. Then there exists a smooth, positive function $\mu:\mathcal{S}\to\mathbb{R}$ so that

$$d\tilde{S}=\mu dS,$$

where dS is the measure on ∂M induced by the Riemannian metric. We begin with the following lemma, for which we sketch the proof.

Lemma 2.2.1 (From [12]). *Let the Dirichlet eigenfunctions for the Laplace–Beltrami operator Δ_g be indexed so that, for $j=1,2\ldots$, the functions*

$$\{\phi_{jk}\mid k=1\ldots K_j\}$$

form a basis for the jth eigenspace of Δ_g, with eigenvalue λ_j so that $\lambda_j<\lambda_{j+1}$ for all j. Then $\Lambda_{\mathcal{S},\mathcal{R}}$ determines the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and the spaces E_j where

$$E_j=span\{\mu^{-1}\partial_\nu\phi_{jk}|_{\mathcal{S}}\mid k=1\ldots K_j\}$$

Proof. $\Lambda_{\mathcal{S},\mathcal{R}}$ determines $\Lambda_{\mathcal{R},\mathcal{S}}$ by Lemma A.1.4, and so we assume knowledge of $\Lambda_{\mathcal{R},\mathcal{S}}$ here. By the estimate (4.31) from [12], for $\psi \in C_0^\infty((0,\infty)\times\mathcal{R})$, there exists $C_1, C_2 > 0$ so that for all $t \geq 0$,

$$\left\|\partial_\nu u^\psi\right\|_{H^1((0,t)\times\partial M)} \leq C_1 e^{C_2 t}\|\psi\|_{H^2((0,t)\times\partial M)},$$

where $H^i((0,t)\times\partial M)$ is the Sobolev space of i times weakly differentiable functions on $(0,t)\times\partial M$ for which all derivatives are square integrable. This estimate guarantees that, for all $k \in \mathbb{C}$ such that $\mathrm{Re}(k) < -C_2$, the Laplace transform of $\Lambda_{\mathcal{R},\mathcal{S}}(f)$ in the t variable exists. Let $\hat{u}^f : \{k \in \mathbb{C} \mid \mathrm{Re}(k) < -C_2\} \times M \to \mathbb{C}$ be the Laplace transform of u^f, which is an analytic function of k in its domain [12]. Taking the Laplace transform of equation 1.1 in the t variable, we get the following PDE

$$\begin{aligned} (k^2 - \Delta_g)\hat{u}(k,x) = 0 \quad &\text{in } \{k \in \mathbb{C} \mid \mathrm{Re}(k) < -C_2\} \times M,, \\ \hat{u} = \hat{f} \quad &\text{in } \{k \in \mathbb{C} \mid \mathrm{Re}(k) < -C_2\} \times \partial M. \end{aligned} \tag{2.4}$$

Let $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}} : C_0^\infty(\mathcal{R}) \to C^\infty(\mathcal{S})$ be the map

$$\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\psi) = \partial_\nu u^\psi_{k^2}|_{\mathcal{S}}.$$

where $u^\psi_{k^2}$ is the unique solution to equation 2.4 with boundary condition ψ for fixed $k \in \mathbb{C} \setminus \mathrm{spec}(\Delta_g)$ (see Chapter 2, [12]). By our transformation of the wave equation, we see that, for all $k \in \mathbb{C} \setminus \mathrm{spec}(\Delta_g)$ such that $\mathrm{Re}(k) < -C_2$,

$$\widehat{\Lambda_{\mathcal{R},\mathcal{S}}(\psi)}(k) = \Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi}). \tag{2.5}$$

As a function of k, $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi})$ is a meromorphic function taking values in the Sobolev space $H^2(M)$ [12]. Equation 2.5 proves that $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi})$ is the unique meromorphic continuation of $\widehat{\Lambda_{\mathcal{R},\mathcal{S}}(\psi)}(k)$ beyond the half plane $\{k \in \mathbb{C} \mid \mathrm{Re}(k) < -C_2\}$. The poles of $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi})$ are all simple, and are contained in the spectrum of Δ_g [12]. By varying ψ and determining the poles of $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi})$, we are able to determine the eigenvalues of Δ_g. The residue of $\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi})$ at $\lambda_j \in \mathrm{spec}(\Delta_g)$ is given by

$$\mathrm{res}_{k^2=\lambda_j}\Lambda^{k^2}_{\mathcal{R},\mathcal{S}}(\hat{\psi}) = \sum_{k=1}^{K_j}\left(\int_{\partial M}\hat{\psi}\partial_\nu\phi_{jk}dS(x)\right)\partial_\nu\phi_{jk}|_{\mathcal{S}}.$$

These residues determine an integral operator, that in turn determines the spaces E_j. See [12], pages 205-216 for a more a complete treatment. □

We will make use of the following spectral condition on the Laplace-Beltrami operator Δ_g:

Condition 1: The Riemannian manifold (M, g) satisfies Condition 1 if there exists $F_1, F_2 : \mathbb{R}^+ \to \mathbb{R}^+$ so that for all $j = 1, 2, \dots$ and $k \in 1, \dots K_j$,

$$F_1(\lambda_j) \leq \|\partial_\nu \phi_{jk}\|_{L^2(\mathcal{S})} \leq F_2(\lambda_j),$$
$$\frac{F_2(\lambda)}{F_1(\lambda)} \leq K \quad \text{for all} \quad \lambda > 0,$$

for some $K > 0$. Here, $\|\cdot\|_{L^2(\mathcal{S})}$ refers to the L^2 norm induced by the Riemannian metric on $\mathcal{S}$; we shall use the notation $\|\cdot\|_{L^2(\mathcal{S};d\tilde{S})}$ to refer to the L^2 norm in the measure we have selected.

Remark: Condition 1 is a generalization of the Hassell–Tao condition employed in [14]. Indeed, from [9], for a smooth compact Riemannian manifold with boundary the following equation holds

$$\|\partial_\nu \phi_j\|_{L^2(\mathcal{S})} \leq C_1 \sqrt{\lambda_j},$$

where C_1 is some positive constant and ϕ_j is any normal Dirichlet eigenfunction with eigenvalue λ_j. The Hassell–Tao condition employed in [14] holds for $\mathcal{S} = \partial M$ if M can be embedded in the interior of a compact manifold with boundary, N, of the same dimension, such that every geodesic in M intersects the boundary of N. The Hassell–Tao condition used in [14] is given by the equation

$$\sqrt{\lambda_j} \leq C_0 \|\partial_\nu \phi_j\|_{L^2(\mathcal{S})}.$$

It is clear that, if the Hassell–Tao condition holds for $\mathcal{S}$, then the manifold satisfies Condition 1. The examples in [9] point out that there are manifolds for which the Hassell–Tao condition does not hold, but for which Condition 1 holds if $\mathcal{S} = \partial M$.

We now define a functional that will allow us to recast reconstruction of the Riemannian metric as a variational problem. Let $\Gamma_0, \Gamma_1 \subset \mathcal{S}$ be open sets, and $T > \sigma$. Then, for $f \in L^2((0,T) \times \Gamma_0)$, we define $E_f : H \to \mathbb{R}^+$, where H is some closed subspace of $L^2((0,T) \times \Gamma_1)$,

$$E_f[g] = \sum_{j=1}^{\infty} \sup_{v \in D_j} \Big(\int_0^T \int_{\partial M} s_j(t)(g(t,x) - f(t,x))v(x) d\tilde{S}(x) dt \Big)^2,$$

where

$$D_j = \{v \in E_j \mid \|v\|_{L^2(\mathcal{S},d\tilde{S})} \leq F_2(\lambda_j)\},$$

and $s_j(t) = \sin(\sqrt{\lambda_j}(T-t))/\sqrt{\lambda_j}$. We must first show that this sum converges.

Since $v \in E_j$, there exists some eigenfunction $\phi_j(v)$ and some constant $c(v)$ so that $v = \mu^{-1}c(v)\partial_\nu\phi_j(v)$. We have the following identity, which we prove in the appendix:

$$(u^f(T) - u^g(T), c(v)\phi_j(v))_{L^2(M)} = \int_0^T \int_{\partial M} s_j(t)(g(t,x) - f(t,x))v(x)d\tilde{S}(x)dt. \quad (2.6)$$

We observe that for $v \in E_j$ such that $\|v\|_{L^2(\mathcal{S};d\tilde{S})} \leq F_2(\lambda)$, the coefficients $c(v)$ are bounded in j. Indeed,

$$\begin{aligned} \|v\|_{L^2(\mathcal{S};d\tilde{S})} &\leq F_2(\lambda_j), \\ |c(v)|\|\partial_\nu\phi_j(v)\|_{L^2(\mathcal{S})} &\leq F_2(\lambda_j), \\ |c(v)| &\leq \frac{F_2(\lambda_j)}{F_1(\lambda_j)}, \\ |c(v)| &\leq K. \end{aligned}$$

As such,

$$\begin{aligned} E_f[g] &= \sum_{j=1}^{\infty} \sup_{v \in D_j} (u^f(T) - u^g(T), c(v)\phi_j(v))^2_{L^2(M)}, \\ &\leq \sum_{j=1}^{\infty} \sup_{v \in D_j} K^2(u^f(T) - u^g(T), \phi_j(v))^2_{L^2(M)}, \\ &\leq K^2\left\|u^f(T) - u^g(T)\right\|^2_{L^2(M)}, \end{aligned} \quad (2.7)$$

the last inequality holding because $\phi_j(v)$ is of unit norm and $(\phi_j(v), \phi_i(v))_{L^2(M)} = 0$ for $i \neq j$. So the functional E_f is indeed well defined. We also have a lower bound. Indeed, suppose ϕ_j is the projection of $u^f(T) - u^g(T)$ on the jth eigenspace of Δ_g. Then, by assumption, $\mu^{-1}\partial_\nu\phi_j \in E_j$, and $\|\mu^{-1}\partial_\nu\phi_j\|_{L^2(\mathcal{S};d\tilde{S})} \leq F_2(\lambda_j)$. As such,

$$\begin{aligned} \left\|u^f(T) - u^g(T)\right\|^2_{L^2(M)} &= \sum_{j=1}^{\infty}(u^f(T) - u^g(T), \phi_j)^2_{L^2(M)}, \\ &\leq \sum_{j=1}^{\infty} \sup_{v \in D_j} (u^f(T) - u^g(T), c(v)\phi_j(v))^2_{L^2(M)}, \\ &= E_f[g]. \end{aligned} \quad (2.8)$$

We now regularize E_f to ensure coercivity and the existence of a minimizer. For $\alpha > 0$, set $E_f^\alpha : H \to \mathbb{R}^+$ where

$$E_f^\alpha[g] = E_f[g] + \alpha \|g\|^2_{L^2(H)}.$$

Note that since dS is not assumed known, $\|g\|^2_{L^2(H)}$ is not known explicitly. We can calculate $\|g\|^2_{L^2(H;d\tilde{S})}$, however, and since none of the following lemmas require anything specific of the measure we use to calculate $\|g\|$, they will still hold for E_f^α defined using our selected measure.

Lemma 2.2.2. *For all $\alpha > 0$, E_f^α has a unique minimizer in H, where H is any closed subset of $L^2((0,T) \times \mathcal{S})$.*

Proof. Let $\{g_n\}_{n=1}^\infty$ be a sequence in H so that $\lim_{n\to\infty} E_f^\alpha[g_n] = \inf_{g\in H} E_f^\alpha[g]$. Then $\{g_n\}_{n=1}^\infty$ is bounded. Indeed,

$$\limsup_{n\to\infty} \alpha \|g_n\|^2_{L^2(H)} \leq \lim_{n\to\infty} E_f^\alpha[g_n] < \infty.$$

Since H is a closed subspace of a Hilbert space, it is itself a Hilbert space and therefore any bounded set is weakly relatively compact. So, taking a subsequence if necessary, we may assume that there exists $g^\alpha \in H$ so that g_n converges weakly to g^α in H. We also observe that E_f^α is continuous and convex. Convexity is immediate, so we only prove continuity. Clearly, continuity of E_f^α holds if E_f is continuous. Let $g_1, g_2 \in H$, and let $v_j \in E_j$ so that $\|v_j\|_{L^2(\mathcal{S};d\tilde{S})} \leq F_2(\lambda_j)$ and

$$\sup_{v\in D_j} (u^{f-g_1}(T), c(v)\phi_j(v))^2_{L^2(M)} \leq (u^{f-g_1}(T), c(v_j)\phi_j(v_j))^2_{L^2(M)} + \frac{\epsilon}{j^2},$$

for all $j = 1, 2 \ldots$. Then,

$$
\begin{aligned}
E_f[g_1] - E_f[g_2] &\leq \sum_{j=1}^{\infty} (u^{f-g_1}(T), c(v_j)\phi_j(v_j))^2 + \frac{\epsilon}{j^2} - (u^{f-g_2}(T), c(v_j)\phi_j(v_j))^2, \\
&\leq \sum_{j=1}^{\infty} ((u^{f-g_1}(T), c(v_j)\phi_j(v_j))^2 - (u^{f-g_2}(T), c(v_j)\phi_j(v_j))^2 + C\epsilon, \\
&\leq \sum_{j=1}^{\infty} \underbrace{(u^{2f-g_1-g_2}(T), c(v_j)\phi_j(v_j))}_{A_i} \underbrace{(u^{g_2-g_1}(T), c(v_j)\phi_j(v_j))}_{B_i} + C\epsilon, \\
&\leq C\epsilon \sum_{j=1}^{\infty} A_i^2 + \frac{C}{\epsilon} \sum_{j=1}^{\infty} B_i^2 + C\epsilon, \\
&\leq C\epsilon + \frac{C}{\epsilon} \| u^{g_2}(T) - u^{g_1}(T) \|_{L^2(M)}^2 + C\epsilon, \\
&\leq C\epsilon + \frac{C}{\epsilon} \| g_2 - g_1 \|_{L^2(M)}^2 + C\epsilon.
\end{aligned}
$$

taking ϵ small, and then taking g_2 so that $\|g_2 - g_1\| \leq \epsilon^2$, this difference can be made arbitrarily small. We have continuity once we reproduce this same set of inequalities with g_1 and g_2 interchanged. In the fourth inequality above, we employed Cauchy's inequality with ϵ, and in the final inequality we used the continuity of the map $g \mapsto u^g(T)$ from H to $L^2(M)$ [16]. This establishes continuity of the functional E_f^α. It is a classical fact that a continuous convex functional on a Hilbert space possesses a non-empty subdifferential at all points [4]. Letting the subdifferential of E_f^α at the point $g \in H$ be denoted by $\partial E_f^\alpha[g]$, and letting $w \in \partial E_f^\alpha[g^\alpha]$, we have

$$E_f^\alpha[g_n] \geq E_f^\alpha[g^\alpha] + (w, g_n - g^\alpha)_H.$$

Letting $n \to \infty$, we obtain, by the weak convergence of g_n to g^α,

$$\lim_{n\to\infty} E_f^\alpha[g_n] \geq E_f^\alpha[g^\alpha].$$

Hence,

$$\inf_{g\in H} E_f^\alpha[g] = E_f^\alpha[g^\alpha].$$

This proves existence of a minimizer of E_f^α for every $\alpha > 0$. To prove uniqueness, let $g_1, g_2 \in H$ be two minimizers of E_f^α. Take $g_3 = (g_1 + g_2)/2$. Let $w \in \partial E_f[g_3]$. We have

$$E_f[g_1] \geq E_f[g_3] + (w, g_1 - g_3)_H,$$
$$E_f[g_2] \geq E_f[g_3] + (w, g_2 - g_3)_H.$$

Adding and subtracting the appropriate terms, we get

$$E_f^\alpha[g_1] \geq E_f^\alpha[g_3] + (w, g_1 - g_3)_H + \alpha(\|g_1\|^2 - \|g_3\|^2),$$
$$E_f^\alpha[g_2] \geq E_f^\alpha[g_3] + (w, g_2 - g_3)_H + \alpha(\|g_2\|^2 - \|g_3\|^2).$$

Adding these inequalities and dividing by two, we get,

$$E_f^\alpha[g_1] = \frac{E_f^\alpha[g_1] + E_f^\alpha[g_2]}{2} \geq E_f^\alpha[g_3] + \frac{\alpha}{4}(\|g_1 - g_2\|^2).$$

This proves uniqueness, since if $g_1 \neq g_2$ then g_3 reduces the value of E_f^α, contradicting the assumption that g_1 is a minimizer. □

The value of this functional at a minimizer turns out to be key in determining the Riemannian distance function. Let $M_0, M_1 \subset M$ be domains of influence whose definition we will make precise when necessary. Let $f \in H_0 := \{f \in L^2([0,T] \times \mathcal{S}) \mid \operatorname{supp}(u^f(T)) \subset M_0\}$, and let $g \in H_1 := \{g \in L^2([0,T] \times \mathcal{S}) \mid \operatorname{supp}(u^g(T)) \subset M_1\}$.

Lemma 2.2.3. *Let* $g^\alpha = \arg\min_{g \in H_1} E_f^\alpha[g]$ *for* $\alpha > 0$. *Then* $\lim_{\alpha \to 0} E_f^\alpha[g^\alpha]$ *exists, and*

$$\left\|u^f(T)\right\|^2_{L^2(M_0 \setminus M_1)} \leq \lim_{\alpha \to 0} E_f^\alpha[g^\alpha] \leq K^2 \left\|u^f(T)\right\|^2_{L^2(M_0 \setminus M_1)}.$$

Proof. To prove that the limit exists, we observe that, as $\alpha \to 0$, $E_f^\alpha[g^\alpha]$ is a monotonically decreasing sequence. Indeed, let $\alpha_1 < \alpha_2$. Since g^{α_1} is the minimizer of $E_f^{\alpha_1}$, we get

$$\begin{aligned} E_f^{\alpha_1}[g^{\alpha_1}] &\leq E_f^{\alpha_1}[g^{\alpha_2}], \\ &= E_f[g^{\alpha_2}] + \alpha_1 \|g^{\alpha_2}\|, \\ &< E_f[g^{\alpha_2}] + \alpha_2 \|g^{\alpha_2}\|, \\ &= E_f^{\alpha_2}[g^{\alpha_2}]. \end{aligned}$$

Since this sequence is bounded below by zero, it must converge. The first inequality of the lemma is the easier of the two. Indeed

$$\begin{aligned}\left\|u^f(T)\right\|^2_{L^2(M_0\setminus M_1)} &\leq \left\|u^f(T) - u^{g^\alpha}(T)\right\|^2_{L^2(M)},\\ &\leq E_f[g^\alpha],\\ &\leq E_f^\alpha[g^\alpha].\end{aligned}$$

which holds for all $\alpha > 0$, where we used equation (2.8) in the second inequality. To prove the second inequality of the lemma, we use the fact the g^α is the minimizer of E_f^α in H_1, and employ equation (2.7). Let $g \in H_1$. Then,

$$\begin{aligned}E_f^\alpha[g^\alpha] &\leq E_f^\alpha[g],\\ &\leq K^2\left\|u^f(T) - u^g(T)\right\|^2_{L^2(M)} + \alpha\|g\|^2,\\ &= K^2\left\|u^f(T)\right\|^2_{L^2(M_0\setminus M_1)} + K^2\left\|u^f(T) - u^g(T)\right\|^2_{L^2(M_1)} + \alpha\|g\|^2.\end{aligned}$$

By approximate controllability, which is proven in the appendix for a simple case, the set $\{u^g(T) \mid g \in H_1\}$ is dense in $L^2(M_1)$ provided M_1 is of the form $M(\Gamma, h)$, for $\Gamma \subset \partial M$ some open set and h a piecewise continuous function [14]. Hence, we can select $g \in H_1$ so that the middle term of the above becomes smaller than ϵ for some $\epsilon > 0$. Selecting such a g, we have

$$E_f^\alpha[g^\alpha] \leq K^2\left\|u^f(T)\right\|^2_{L^2(M_0\setminus M_1)} + K^2\epsilon + \alpha\|g\|^2.$$

Taking $\alpha \to 0$, and then $\epsilon \to 0$, we get

$$\lim_{\alpha\to 0} E_f^\alpha[g^\alpha] \leq K^2\left\|u^f(T)\right\|^2_{L^2(M_0\setminus M_1)}$$

□

We now connect the value of this functional at a minimizer with the volumes of domains of influence.

Lemma 2.2.4. *The following are equivalent:*

(i) $\mathrm{Vol}(M_0 \setminus M_1) > 0$*;*

(ii) $\sup_{f\in H_0} \lim_{\alpha\to 0} E_f^\alpha[g^\alpha] > 0$.

Proof. Assuming the hypothesis of (i), by approximate controllability there exists $f \in H_0$ so that $\left\|u^f(T)\right\|_{L^2(M_0\setminus M_1)} > 0$. By Lemma 2.2.3,

$$E_f^\alpha[g^\alpha] \geq \left\|u^f(T)\right\|^2_{L^2(M_0 \setminus M_1)} > 0,$$

for all $\alpha > 0$, providing the result.

Assuming the hypothesis of (ii), let $f \in H_0$ so that $\lim_{\alpha\to 0} E_f^\alpha[g^\alpha] > 0$. By Lemma 2.2.3, we obtain

$$K^2 \left\|u^f(T)\right\|^2_{L^2(M_0 \setminus M_1)} > 0.$$

This implies $\mathrm{Vol}(M_0 \setminus M_1) > 0$. □

Let now $M_0 = M(\Gamma_0, s)$ and $M_1(t) = M(\mathcal{S}, s-\delta) \cup M(\Gamma_1, t)$ where Γ_0, Γ_1 are as in Lemma 2.1.1. It is not hard to see that the union of two domains of influence can be written as a single modified domain of influence with a piecewise constant function h; approximate controllability will hold for this type of domain. We will also make the t dependence of H_1 more explicit by denoting it $H_1(t)$. The preceding lemma tells us something about the Riemannian distance function; specifically, if $f \in H_0$ is such that $\lim_{\alpha\to 0} E_f^\alpha[g^\alpha] > 0$, then $\mathrm{Vol}(M(\Gamma_0, s) \setminus (M(\mathcal{S}, s-\delta) \cup M(\Gamma_1, t))) > 0$. Using Lemma 2.1.1, we are then able to get a lower bound on $d(x_0, y_1)$. To get an upper bound on this quantity, we must prove the following lemma. First, let $H_2 := L^2((T-(s-\delta), T) \times \mathcal{S})$, so that $M_2 = M(\mathcal{S}, s-\delta)$. Now set

$$E^\alpha_{f,1,t} = E^\alpha_f|_{H_1(t)}, \qquad E^\alpha_{f,2} = E^\alpha_f|_{H_2},$$
$$g^\alpha_{1,t} = \operatorname*{arg\,min}_{g \in H_1(t)} E^\alpha_{f,1,t}[g], \qquad g^\alpha_2 = \operatorname*{arg\,min}_{g \in H_2} E^\alpha_{f,2}[g].$$

Lemma 2.2.5. *Suppose that*

$$\lim_{\alpha\to 0} E^\alpha_{f,1,t}[g^\alpha_{1,t}] < \frac{1}{K^2} \lim_{\alpha \to 0} E^\alpha_{f,2}[g^\alpha_2], \tag{2.9}$$

then

$$\mathrm{Vol}(M(\Gamma_0, s) \cap M(\Gamma_1, t) \setminus M(\mathcal{S}, s-\delta)) > 0.$$

Proof. We have the following calculation

$$\begin{aligned}\|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t)))} &\leq \lim_{\alpha\to 0} E^\alpha_{f,1,t}[g^\alpha_{1,t}],\\ &< \frac{1}{K^2}\lim_{\alpha\to 0} E^\alpha_{f,2}[g^\alpha_2],\\ &\leq \frac{K^2}{K^2}\|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta))}.\end{aligned}$$

In other words,

$$\|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t)))} < \|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta))}.$$

This is only possible if

$$\mathrm{Vol}(M(\Gamma_0,s)\cap M(\Gamma_1,t)\setminus M(\mathcal{S},s-\delta)) > 0.$$

□

We are almost ready to describe an algorithm that allows for the calculation of the Riemannian distance function to a fixed precision. We require two more lemmas.

Lemma 2.2.6. *Suppose that $f \in H_0$ so that*

$$\lim_{\alpha\to 0} E^\alpha_{f,2}[g^\alpha_2] > 0.$$

Then there exists an open, non-empty interval $I \subset \mathbb{R}$ so that, if $t \in I$,

$$0 < \lim_{\alpha\to 0} E^\alpha_{f,1,t}[g^\alpha_{1,t}] < \frac{1}{K^2}\lim_{\alpha\to 0} E^\alpha_{f,2}[g^\alpha_2].$$

Proof. Letting f be as in the lemma, define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\phi(t) = \|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t)))}.$$

Then ϕ is a continuous, non-increasing function. Further,

$$\phi(0) = \|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus M(\mathcal{S},s-\delta))} > 0,$$

$$\phi(t) = 0 \quad t > C,$$

where C is some large enough constant. Set

$$I := \phi^{-1}((0, \frac{1}{K^4}\|u^f(T)\|^2_{L^2(M(\Gamma_0,s)\setminus M(\mathcal{S},s-\delta))})).$$

Clearly, I is non-empty by the continuity of ϕ and its behaviour between 0 and C, and I is open by continuity of ϕ. Further, letting $t \in I$, we get

$$\begin{aligned} 0 &< \phi(t), \\ &= \left\|u^f(T)\right\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t)))}, \\ &\le \lim_{\alpha\to 0} E^\alpha_{f,1,t}[g^\alpha_{1,t}]. \end{aligned}$$

This shows that the first inequality of the lemma holds. Continuing,

$$\begin{aligned} \lim_{\alpha\to 0} E^\alpha_{f,1,t}[g^\alpha_{1,t}] &\le K^2\left\|u^f(T)\right\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t)))}, \\ &= K^2\phi(t), \\ &< \frac{1}{K^2}\left\|u^f(T)\right\|^2_{L^2(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta))}, \\ &\le \frac{1}{K^2}\lim_{\alpha\to 0} E^\alpha_{f,2}[g^\alpha_2]. \end{aligned}$$

This proves that equation (2.9) holds for $t \in I$. □

The existence of $f \in H_0$ such that $\lim_{\alpha\to 0} E^\alpha_{f,2}[g^\alpha_2] > 0$ is proven using the following lemma.

Lemma 2.2.7. *Let* $y_0 \in \mathcal{S}$, $s \le \sigma_{\mathcal{S}}(y_0)$. *Then* $\mathrm{Vol}(M(y_0,s)\setminus M(\mathcal{S},s-\delta)) > 0$ *for all* $\delta \in (0,s)$

Proof. We show that the assumption $s \le \sigma_{\mathcal{S}}(y_0)$ implies that $B(x_0,\frac{\delta}{2}) \subset M(y_0,s)\setminus M(\mathcal{S},s-\delta)$, where $x_0 = \gamma(s-\frac{\delta}{2};y_0,\nu)$. Indeed, let $x \in B(x_0,\frac{\delta}{2})$. Then $d(x,y_0) \le s$ by the triangle inequality. Further, letting $y \in \mathcal{S}$ and applying the triangle inequality, we see that

$$\begin{aligned} d(x,y) &\ge d(x_0,y) - d(x_0,x), \\ &\ge d(x_0,y_0) - d(x_0,x), \\ &> s-\delta, \end{aligned}$$

the second inequality following from the fact that $d(x_0,y) \ge d(x_0,y_0) = s-\frac{\delta}{2}$, which holds since $s \le \sigma_{\mathcal{S}}(y_0)$, and because $d(x_0,x) < \delta$. Since $d(x,\mathcal{S}) = \inf_{y\in\mathcal{S}} d(x,y)$, we have that $d(x,\mathcal{S}) > s-\delta$. As such, $B(x_0,\frac{\delta}{2}) \subset M(y_0,s)\setminus M(\mathcal{S},s-\delta)$. This implies the latter set has non-zero measure. □

Remark on Lemma 2.2.7: By Lemma 2.2.4, we have that $\mathrm{Vol}(M(y_0,s)\setminus M(\mathcal{S},s-\delta))>0$ implies $\lim_{\alpha\to 0}E^{\alpha}_{f,2}[g^{\alpha}_2]>0$.

We are now ready to state the process by which we obtain the Riemannian distance function between points $x_0=\gamma(s;y_0,\nu)$ and $y_1\in\mathcal{S}$, where $s<\sigma$.

Step 1: Find $f\in H_0$ so that $C_2:=\lim_{\alpha\to 0}E^{\alpha}_{f,2}[g^{\alpha}_2]>0$.

Step 2: Compute $C_1:=\lim_{\alpha\to 0}E^{\alpha}_{f,1,t}[g^{\alpha}_{1,t}]$. If

(i) $C_1>0$, then using Lemma 2.2.4, $\mathrm{Vol}(M(\Gamma_0,s)\setminus(M(\mathcal{S},s-\delta)\cup M(\Gamma_1,t))>0$. Using Lemma 2.1.1, we obtain a lower bound on $d(x_0,y_1)$ in terms of t. Increase t unless,

(ii) $C_1<\frac{1}{K^2}C_2$. Then by Lemma 2.2.5, $\mathrm{Vol}(M(\Gamma_0,s)\cap M(\Gamma_1,t)\setminus M(\mathcal{S},s-\delta))>0$. Using Lemma 2.1.1, we obtain an upper bound on $d(x_0,y_1)$ in terms of t. Decrease t.

Step 3: Iterate Step 2 until both conditions (i) and (ii) hold. By Lemma 2.2.6, there is always a non-empty open interval I where this is true.

Step 4: If conditions (i) and (ii) hold, then $d(x_0,y_1)$ is determined up to a known tolerance. (Lemmas 2.1.1 and 2.1.3). If more precision is required, decrease δ and iterate Steps 2 and 3.

We remark also that one of conditions (i) and (ii) must hold, and so some new information on the distance function is produced each time C_1 is calculated. Additionally, we note that precise knowledge of K is not required; substituting any $\tilde{K}>K$ in the algorithm will work. As such, this algorithm constructively determines $d(x_0,y_1)$ for $x_0\in M^{\mathrm{int}}$ and $y\in\mathcal{S}$, provided $d(x_0,\mathcal{S})<\sigma$.

2.3 Determining the distance to the cut locus

Finding a lower bound on $\sigma=\inf_{y\in\mathcal{S}}\sigma_{\mathcal{S}}(y)$ is an important part of our algorithm for reconstructing the Riemannian distance function. Indeed, Step 1 of the algorithm relies on $s\le\sigma_{\mathcal{S}}(y_0)$, and the estimate in Lemma 2.1.3 requires that $s<\sigma$. We now discuss how to get an estimate on $\sigma_{\mathcal{S}}$. We will mostly use a technique published in [14], with a few minor modifications. Let $h\in C^1(\overline{\mathcal{S}})$, and define $\tilde{\sigma}_{\mathcal{S}}:\mathcal{S}\times C^1(\overline{\mathcal{S}})\to\mathbb{R}$ by

$$\tilde{\sigma}_{\mathcal{S}}(y,h)=\sup\{t>0\mid \text{for all } s<t, M(y,t)\not\subset M(\mathcal{S},s+h)\}.$$

Paraphrasing the results of [14], we show how to calculate $\sigma_{\mathcal{S}}(y,h)$ assuming $\tilde{\sigma}_{\mathcal{S}}$ is known.

Lemma 2.3.1 (Lemma 14 from [14])**.** *Let $y_0 \in \mathcal{S}$. Then*

$$\liminf_{(y,h)\to(y_0,0)} \tilde{\sigma}_{\mathcal{S}}(y,h) = \sigma_{\mathcal{S}}(y_0),$$

where the lim inf *is taken over all sequences $\{(y_n, h_n)\}_{n=1}^{\infty} \subset \mathcal{S} \times C^1(\overline{\mathcal{S}})$ such that $h_n(y_n) = 0$ for all n.*

We now show how to calculate $\tilde{\sigma}_{\mathcal{S}}$

Lemma 2.3.2. *Let $y \in \mathcal{S}$ and $h \in C^1(\overline{\mathcal{S}})$. Then the following are equivalent;*

(i) $M(y,t) \not\subset M(\mathcal{S}, s+h))$;

(ii) $\mathrm{Vol}(M(y,t) \setminus M(\mathcal{S}, s+h)) > 0$.

Proof. It is clear that (ii) implies (i). We now show the converse. Let $x \in M(y,t) \setminus M(\mathcal{S}, s+h)$, and set $\gamma : I \to M$ a shortest path so that $\gamma(0) = y$ and $\gamma(l) = x$ for some $l > 0$. Then, since $M(\mathcal{S}, s+h)^c$ is open, there exists $\epsilon_0 > 0$ so that for all $\epsilon \in (0, \epsilon_0), \gamma(l-\epsilon) \in M(\mathcal{S}, s+h)^c$ and $d(\gamma(l-\epsilon), y) < t$. As such, $\gamma(l-\epsilon) \in M(y,t)^{\mathrm{int}} \setminus M(\mathcal{S}, s+h)$, which is an open set. A non-empty open set must have non-zero measure, and this completes the proof. □

Set $M_0 = M(\Gamma_0, t)$, and $M_1 = M(\mathcal{S}, s+h)$, where $\overline{\Gamma}_0 \subset \mathcal{S}$ is so that $d(y,z) \leq \delta$ for all $z \in \Gamma_0$. Let H_0, H_1 be defined as above. Then Lemma 2.2.4 provides the means to determine whether or not $\mathrm{Vol}(M(\Gamma_0, t) \setminus M(\mathcal{S}, s+h)) > 0$ using the functional we have developed. We note that, by [14], approximate controllability holds over the domain of influence $M(\mathcal{S}, s+h)$ for all $h \in C^1(\overline{\mathcal{S}})$. Taking the limit as $\delta \to 0$, we can calculate $\mathrm{Vol}(M(y,t) \setminus M(\mathcal{S}, s+h))$, which in turn allows us to calculate $\tilde{\sigma}_{\mathcal{S}}(y,h)$, which upon taking a limit infimum, determines $\sigma_{\mathcal{S}}(y_0)$.

It follows that $\sigma_{\mathcal{S}}$ can be determined, that allows for the implementation the algorithm described above. The calculation of $\sigma_{\mathcal{S}}$ is quite onerous however, as it is necessary to test if $\mathrm{Vol}(M(y,t) \setminus M(\mathcal{S}, s+h)) > 0$ for an infinite number of t and s values, and all $y \to y_0$ and $h \to 0$. Calculation of this volume itself involves solving infinitely many infinite dimensional optimization problems. As such, this method for finding $\sigma_{\mathcal{S}}$ should not be viewed as an implementable procedure. It turns out, however, that if a lower bound on the exit time $\tau_M(y_0)$ is known, $\sigma_{\mathcal{S}}(y_0)$ can be calculated in a much simpler way. This is the content of the following lemma.

Lemma 2.3.3. *Let $y_0 \in \mathcal{S}$, $\Gamma_0(\delta) \subset \mathcal{S}$ open with $\delta > 0$ so that $y_0 \in \Gamma_0(\delta)$ and $d(y, y_0) \leq \delta$ for all $y \in \Gamma_0(\delta)$, and suppose that $s \leq \tau_M(y_0)$. The following are equivalent:*

(i) $s \leq \sigma_{\mathcal{S}}(y_0)$;

(ii) $\mathrm{Vol}(M(\Gamma_0(\delta), s) \setminus M(\mathcal{S}, s-\delta)) > 0$ *for all* $\delta > 0$.

Proof. The proof that (i) implies (ii) follows from the proof of Lemma 2.2.7, since for all $\delta > 0, M(y_0, s) \subset M(\Gamma_0(\delta), s)$. We now prove that (ii) implies (i). What follows is a modification of the proofs of Lemmas 8 and 10 from [14]. Assuming (ii), for all $\delta > 0$ there exists $x_\delta \in M(\Gamma_0(\delta), s) \setminus M(\mathcal{S}, s-\delta)$. By compactness of the manifold, we can assume that $\lim_{\delta\to 0} x_\delta = x$ for some $x \in M$, re-labelling a subsequence if necessary. Let $\epsilon > 0$, and for each x_δ, let $y_\delta \in \Gamma_0(\delta)$ so that $d(x_\delta, y_\delta) \le s + \epsilon$. Then

$$\begin{aligned} d(x, y_0) &= \lim_{\delta\to 0} d(x_\delta, y_\delta), \\ &\le s + \epsilon. \end{aligned}$$

As such, $x \in M(y_0, s)$. Further,

$$\begin{aligned} d(x, \mathcal{S}) &= \lim_{\delta\to 0} d(x_\delta, \mathcal{S}), \\ &\ge \lim_{\delta\to 0}(s - \delta), \\ &= s. \end{aligned}$$

So $s = d(x, y_0) = d(x, \mathcal{S})$. Let $\gamma : [0, s] \to M$ be a unit speed shortest path between y_0 and x so that $\gamma(0) = y_0$ and $\gamma(s) = x$. By an identical argument to that used in the proof of Lemma 2.1.2, $\gamma(r)$ must coincide with $\gamma(r; y_0, \nu)$ as long as $r \le \tau_M(y_0)$, as this is the first point at which uniqueness may fail. Fortunately, $s \le \tau_M(y_0)$, and therefore $x = \gamma(s) = \gamma(s; y_0, \nu)$. As such,

$$d(\gamma(s; y_0, \nu), \mathcal{S}) = d(\gamma(s; y_0, \nu), y_0) = s,$$

and, therefore, $s \le \sigma_{\mathcal{S}}(y_0)$. □

Remark: Lemma 2.3.3 shows that having a lower bound on $\tau_M(y_0)$ allows one to calculate $\sigma_{\mathcal{S}}(y_0)$ without having to do the intermediate step of calculating $\tilde{\sigma}(y, h)$ for some $y \in \mathcal{S}$ and $h \in C^1(\overline{\mathcal{S}})$, and then taking a limit infimum over all such pairs converging to $(y_0, 0)$. This greatly economizes the calculation of $\sigma_{\mathcal{S}}(y_0)$, and in practice estimates on the exit time may be available from other sources. We also remark that, for our algorithm to work, we need only a lower bound on σ, rather than knowledge of σ itself, and Lemma 2.3.3 gives a compact way of finding such a lower bound.

Chapter 3

Determining the Riemannian metric from the distance function

3.1 Relating the Riemannian metric and distance

Using the techniques in the last section, we can calculate the distance function restricted to $U_0 \times \mathcal{S}$ to an arbitary precision in the $C^0(U_0 \times \mathcal{S})$ norm, where $\mathcal{S}$ is identified with its image under our coordinate chart, restricting $\mathcal{S}$ if necessary. Given $\epsilon > 0$, let $d_\epsilon : U_0 \times \mathcal{S} \to \mathbb{R}$ be an estimated distance function so that

$$|d(x,y) - d_\epsilon(x,y)| \leq \epsilon \quad \text{for all} \;\; x \in U_0, y \in \mathcal{S},$$

It turns out that the gradient of the distance function with respect to the x variable is more useful in determining the Riemannian metric over U_0 than the distance function itself. Fortunately, we are able to get an estimate of the gradient through a finite difference quotient, and it turns out that by controlling the step carefully, we are able to approximate a finite difference quotient of the true distance function with our estimated distance function. Indeed, for e_i being the ith element of a standard basis in $\mathbb{R}^n$, we have the following

$$\begin{aligned}\left|\frac{d(x+\epsilon e_i,y)-d(x,y)}{\epsilon} - \frac{d_{\epsilon^2}(x+\epsilon e_i,y)-d_{\epsilon^2}(x,y)}{\epsilon}\right| & \\ \leq \left|\frac{d(x+\epsilon e_i,y)-d_{\epsilon^2}(x+\epsilon e_i,y)}{\epsilon}\right| &+ \left|\frac{d(x,y)-d_{\epsilon^2}(x,y)}{\epsilon}\right|, \\ \leq 2\epsilon. &\end{aligned}$$

Let us now restrict U_0 away from the boundary so that $d(U_0, \mathcal{S}) > 0$. Having done this, recall that the Riemannian distance function $d(\cdot, y)$ is smooth away from y and the cut locus [6]. Restricting U_0 if necessary, the latter will hold for all $y \in \mathcal{S}$, and the former holds by our restriction of U_0 away from $\mathcal{S}$. As a result, $d(\cdot, y) : U_0 \to \mathbb{R}$ is smooth for all $y \in \mathcal{S}$. This allows us to assume that $\nabla d(\cdot, y)$ is Lipschitz. Let $C > 0$ so that, for all $i = 1 \ldots n$,

$$\left| \frac{\partial}{\partial x_i} d(z, y) - \frac{\partial}{\partial x_i} d(x, y) \right| \leq C\|z - x\|,$$

This allows us to estimate the difference between a finite difference quotient and the derivative as follows

$$\begin{aligned} \left| \frac{d(x + \epsilon e_i, y) - d(x, y)}{\epsilon} - \frac{\partial}{\partial x_i} d(x, y) \right| &= \left| \frac{\partial}{\partial x_i} d(z, y) - \frac{\partial}{\partial x_i} d(x, y) \right|, \\ &\leq C\|z - x\| \leq C\epsilon, \end{aligned}$$

for some z with $z - x = ce_i$ with $|c| \leq \epsilon$. This calculation shows that our estimated distance function allows us to estimate the gradient of the true distance function, with accuracy depending on the Lipschitz constant of $\nabla d(\cdot, y)$ for $x \in U_0$.

We now recall the following result, which we prove for completeness.

Lemma 3.1.1. *Let $\mathbf{G}(x)$ be the matrix representation of the Riemannian metric g at the point x in the coordinate patch U_0. Then, for every $x \in U_0$, $\mathbf{G}(x)$ is the unique symmetric matrix satisfying*

$$\nabla d(x, y)^T \mathbf{G}(x)^{-1} \nabla d(x, y) = 1, \tag{3.1}$$

for all $y \in \mathcal{S}$ (Eikonal equation)

Proof. We start by showing that $\mathbf{G}(x)$ is a solution to (3.1). Let $\gamma : I \to \mathbb{R}^n$ with $0 \in I$ be any unit speed geodesic, as determined by the metric $\mathbf{G}$, such that $\gamma(0) = x$. We observe

$$\begin{aligned} \frac{d(\gamma(t), y) - d(\gamma(0), y)}{t} &\geq -\frac{d(\gamma(t), \gamma(0)}{t}, \\ &= -1. \end{aligned}$$

Taking the limit as $t \to 0$, we discover that

$$\mathrm{d}(d(\cdot, y))|_x \cdot \dot{\gamma}(0) \geq -1.$$

Here d denotes the differential of a smooth function on a Riemannian manifold. Recalling the relationship between the differential of a function and the gradient induced by the Riemannian metric g, which we will denote using the notation ∇_g, we get

$$\nabla_g d(x,y)^T \mathbf{G}(x)\dot{\gamma}(0) \geq -1.$$

This holds for any unit speed γ. Select $\dot{\gamma}(0) = -\frac{\nabla_g d(x,y)}{\|\nabla_g d(x,y)\|_G}$, the denominator denoting the norm of this vector as defined by the Riemannian metric. Plugging this into the above equation, we get

$$\nabla_g d(x,y)^T \mathbf{G}(x)\nabla_g d(x,y) \leq 1.$$

Now let $\gamma : I \to \mathbb{R}^n$ be a unit speed shortest path connecting x and y so that $\gamma(0) = x$. Then

$$\begin{aligned} d(x,y) &= d(\gamma(0),y), \\ &= d(\gamma(0),\gamma(t)) + d(\gamma(t),y), \\ &= t + d(\gamma(t),y). \end{aligned}$$

In turn, this gives

$$\frac{d(\gamma(t),y) - d(\gamma(0),y)}{t} = -1.$$

Taking a limit as $t \to 0$, we get

$$\nabla_g d(x,y)^T \mathbf{G}(x)\dot{\gamma}(0) = -1.$$

This implies

$$\min_{\|\dot{\gamma}(0)\|_G = 1} \nabla_g d(x,y)^T \mathbf{G}(x)\dot{\gamma}(0) \leq -1.$$

This minimum is reached when $\dot{\gamma}(0)$ is anti-parallel with $\nabla_g d(x,y)$. Selecting the right $\dot{\gamma}(0)$, we get

$$\nabla_g d(x,y)^T \mathbf{G}(x)\nabla_g d(x,y) \geq 1.$$

Combining this with the above inequality, we have equality. We now state the relationship between ∇, the standard Euclidean gradient on the x variables, and ∇_g, which can be easily proven:

$$\mathbf{G}(x)\nabla_g d(x,y) = \nabla d(x,y).$$

Using this, we get

$$\nabla d(x,y)^T \mathbf{G}^{-1}(x)\nabla d(x,y) = 1,$$

as claimed. We now prove uniqueness. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices satisfying equation (3.1) at a point $x \in U_0$. Then

$$\nabla d(x,y)^T(\mathbf{A}-\mathbf{B})\nabla d(x,y) = 0,$$

for all $y \in \mathcal{S}$, which is an open set in ∂M. Fixing x, and letting $S_x M_{\mathbf{G}^{-1}(x)} = \{v \in T_x M \mid v^T G^{-1}(x) v = 1\}$ we see that $\nabla d(x,\cdot) : \mathcal{S} \to S_x M_{\mathbf{G}^{-1}(x)}$ is an open map from [14]; denote the open image of this function by V. Let $f : T_x M \setminus \{0\} \to S_x M_{\mathbf{G}^{-1}(x)}$ be defined by

$$f(v) = \frac{v}{(v^T \mathbf{G}^{-1}(x) v)^{1/2}}.$$

Then f is continuous, and so $f^{-1}(V) = \{cv \mid c \in \mathbb{R} \setminus \{0\}, v \in V\}$ is an open set in $T_x M$. Further, for all $w \in f^{-1}(V)$, we have

$$w^T(\mathbf{A}-\mathbf{B})w = 0.$$

Letting $z \in T_x M$, for all $\epsilon > 0$ small enough, $w + \epsilon z \in f^{-1}(V)$ by openness. As such,

$$\begin{aligned}(w+\epsilon z)^T(\mathbf{A}-\mathbf{B})(w+\epsilon z) &= 0,\\ 2\epsilon z^T(\mathbf{A}-\mathbf{B})w + \epsilon^2 z^T(\mathbf{A}-\mathbf{B})z &= 0.\end{aligned}$$

Dividing by ϵ and sending ϵ to zero, we get

$$z^T(\mathbf{A}-\mathbf{B})w = 0,$$

and therefore

$$z^T(\mathbf{A}-\mathbf{B})z = 0,$$

for all $z \in T_x M$. This forces all eigenvalues of $\mathbf{A}-\mathbf{B}$ to be zero, and since this matrix is symmetric, this in turn implies $\mathbf{A} = \mathbf{B}$. This proves uniqueness. □

Remark: Here we have proved Lemma 3.1.1 using some elementary Riemannian geometry. Incidentally, if one views $d(x,y)$ as a value function for the optimal control problem of moving from y to x while minimizing path length, one can re-derive Equation 3.1 as the Hamilton-Jacobi-Bellman equation for this optimal control problem.

Lemma 3.1.1 allows us to use equation (3.1) as a characterization of the underlying Riemannian metric. We now use this to show how the Riemannian metric itself can be determined through a variational approach.

3.2 Determining the Riemannian metric through minimization of a functional

Let $Sym(n)$ denote the vector space of real $n \times n$ symmetric matrices. To simplify notation, let $v(x, y) = \nabla d(x, y)$ where $x \in U_0$ and $y \in \mathcal{S}$. We then define $L : Sym(n) \times U_0 \to \mathbb{R}$ by

$$L(A, x) = \int_{\mathcal{S}} (v(x, y)^T A^2 v(x, y) - 1)^2 dy.$$

Let $L^4(U_0; Sym(n)) = \{\mathbf{A} : U_0 \to Sym(n) \mid \int_{U_0} tr(\mathbf{A}(x)^4)dx < \infty\}$. Note that we have some information on the Riemannian metric by our choice of coordinates; namely, $\mathbf{G}(x)_{in} = \delta^i_n$, which is to say that the nth column of our Riemannian metric is equal to the nth element of the canonical basis of $\mathbb{R}^n$. Thus, we lose nothing by restricting our search to the Hilbert space $H = \{\mathbf{A} \in L^4(U_0; Sym(n)) \mid \mathbf{A}(x)_{in} = \delta^i_n \quad x \in U_0\}$. We define a functional $I : H \to \mathbb{R}$ by

$$I[\mathbf{A}] = \int_{U_0} L(\mathbf{A}(x), x)dx.$$

Lemma 3.2.1. *The functional I has a minimizer in H that is unique up to the sign of its eigenvalues.*

Proof. Clearly, $I[\mathbf{A}] \geq 0$ for all $\mathbf{A} \in H$, and by equation (3.1), if $\mathbf{A} = (\mathbf{G}^{-1})^{1/2}$ we see that $I[\mathbf{A}] = 0$. So there is a minimizer of this functional given by the square root of $\mathbf{G}^{-1}$. In addition, up to the sign of its eigenvalues, $(\mathbf{G}^{-1})^{1/2}$ is the unique minimizer in H. Indeed, if $\mathbf{A}_1, \mathbf{A}_2 \in H$ are two minimizers of I, then

$$v(x, y)^T (\mathbf{A}_1^2(x) - \mathbf{A}_2^2(x))v(x, y) = 0,$$

for almost all $(x, y) \in U_0 \times \mathcal{S}$. By a similar argument to the above, this implies

$$\mathbf{A}_1^2(x) = \mathbf{A}_2^2(x),$$

for almost all $x \in U_0$. Letting $\{\lambda^i_j(x)\}_{i=1}^n$ be the eigenvalues of $\mathbf{A}_j(x)$ for $j = 1, 2$, this implies that $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same eigenspaces almost everywhere and,

$$\lambda^i_1(x)^2 = \lambda^i_2(x)^2,$$

almost everywhere. Hence, the minimizer of I is unique up to the sign of its eigenvalues. □

Remark: Once $v(x, y) = \nabla d(x, y)$ is computed using the results of previous sections, one can find $\mathbf{G}|_{U_0}$ by minimizing I. Note that since $d(U_0, \mathcal{S}) > 0$, if one wants to know the Riemannian metric up to $\mathcal{S}$ it would be necessary to iterate this process with a sequence U_0^i so that $\lim_{i\to\infty} d(U_0^i, \mathcal{S}) = 0$. In practice, however, we do not think that this will be an issue, as after finding $\mathbf{G}|_{U_0}$, the unknown part of $\mathbf{G}$ that we would determine by iterating is near the boundary and thus is easily accessed or estimated.

We now address the problem of finding an approximation of $\mathbf{G}|_{U_0}$ when we do not know $\nabla d(x,y)$ precisely, but rather up to some error term. This is a natural problem if one wishes to implement this technique, as the algorithm outlined in Chapter 2 will only produce results up to a non-zero precision in finite time.

3.3 Determining the Riemannian metric with uncertainty

We now assume that, instead of knowing $\nabla d(x,y)$ precisely for all $x \in U_0$, $y \in \mathcal{S}$, we know $\nabla d(x,y) + \epsilon(x,y)$, where $\epsilon : U_0 \times \mathcal{S} \to \mathbb{R}^n$ is some essentially bounded error term. We define the approximate Lagrangian $L_\epsilon : Sym(n) \times U_0 \to \mathbb{R}$ by

$$L_\epsilon(A,x) = \int_{\mathcal{S}} ((v(x,y) + \epsilon(x,y))^T A^2 (v(x,y) + \epsilon(x,y)) - 1)^2 dy,$$

and the approximate functional $I_\epsilon : H \to \mathbb{R}$ by

$$I_\epsilon[\mathbf{A}] = \int_{U_0} L_\epsilon(\mathbf{A}(x), x) dx.$$

We now prove a series of lemmata which will establish the existence of a minimizer of I_ϵ. In addition, we will prove that a minimizer of I_ϵ is close to $\mathbf{G}|_{U_0}$ in the appropriate norm.

Lemma 3.3.1. *If $\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}$ is small enough, then there exists $C_1, C_2 > 0$ and $q \in \mathbb{Z}^+$ so that*

$$L_\epsilon(A,x) \geq C_1\|A\|^q - C_2. \tag{3.2}$$

Here, $\|A\| = (tr(A^2))^{(1/2)}$ (Frobenius norm).

Proof. We have:

$$L_\epsilon^{1/2}(A,x) \geq \left\|(v(x,\cdot) + \epsilon(x,\cdot))^T A^2 (v(x,\cdot) + \epsilon(x,\cdot))\right\|_{L^2(\mathcal{S})} - \|1\|_{L^2(\mathcal{S})}.$$

As such, there exist constants $C, C_2 > 0$ so that

$$L_\epsilon(A,x) \geq C\left\|(v(x,\cdot) + \epsilon(x,\cdot))^T A^2 (v(x,\cdot) + \epsilon(x,\cdot))\right\|^2_{L^2(\mathcal{S})} - C_2.$$

Set $\phi(x) = \left\|(v(x,\cdot)+\epsilon(x,\cdot))^T A^2 (v(x,\cdot)+\epsilon(x,\cdot))\right\|_{L^2(\mathcal{S})}$. If we can show that

$$\phi(x) \geq C_1 \|A\|^2,$$

then we are done. Let $\{v_i\}_{i=1}^n$ be a set of orthonormal eigenvectors for A, and let $\{\lambda_i\}_{i=1}^n$ be the corresponding eigenvalues. Then for all $x \in U_0$ and $y \in \mathcal{S}$, there exists $f_1(x,y),\ldots,f_n(x,y) \in \mathbb{R}$ and $g_1(x,y),\ldots,g_n(x,y) \in \mathbb{R}$ so that

$$\begin{aligned} v(x,y) &= \sum_{i=1}^n f_i(x,y) v_i \\ \epsilon(x,y) &= \sum_{i=1}^n g_i(x,y) v_i. \end{aligned}$$

Then, we have

$$\begin{aligned} \phi(x)^2 &= \int_{\mathcal{S}} (v(x,y)-\epsilon(x,y))^T A^2 (v(x,y)-\epsilon(x,y))^2 dy, \\ &= \int_{\mathcal{S}} (\sum_{i=1}^n \lambda_i^2 (f_i(x,y)-g_i(x,y))^2)^2 dy, \\ &\geq \lambda_i^4 \int_{\mathcal{S}} ((f_i(x,y)-g_i(x,y))^4 dy, \end{aligned}$$

where this last inequality holds for all $i = 1 \ldots n$. We now seek to prove that the integral in the last inequality is strictly positive if $\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}$ is small enough. Recall that $V = \{\nabla d(x,y) \mid y \in \mathcal{S}\}$ is an open subset of $S_x M_{\mathbf{G}^{-1}(x)}$. As such,

$$\sup_{y\in\mathcal{S}} |f_i(x,y)| > 0,$$

for all i and $x \in U_0$, since otherwise $f_i(x,y) = 0$ for all $y \in \mathcal{S}$ and some i, which would contradict openness of V. Note also that, since $\nabla d(x,y)$ is smooth over $U_0 \times \mathcal{S}$, each f_i is continuous. Now, suppose that

$$\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} < \inf_{i,x\in U_0} \frac{1}{2} \sup_{y\in\mathcal{S}} |f_i(x,y)|.$$

Then, by continuity of f_i,

$$\int_{\mathcal{S}} ((f_i(x,y)-g_i(x,y))^4 dy > 0,$$

for all $i = 1, \dots n$ and all $x \in U_0$. As such, there exists $C_1 > 0$ so that

$$\phi(x)^2 \geq C_1 \lambda_i^4$$

for all i. As such,

$$\phi(x)^2 \geq C_1 \sum_{i=1}^{n} \lambda_i^4 \geq C_1 (\sum_{i=1}^{n} \lambda_i^2)^2 = C_1 \|A\|^4.$$

This last line holds because $\|A\|^2 = tr(A^2)$, which is a sum of the eigenvalues of A^2. Since A is symmetric, every eigenvalue of A^2 is the square of an eigenvalue of A. The middle line holds by the equivalence of p norms on finite dimensional vector spaces. As such,

$$L_\epsilon(A, x) \geq C_1 \|A\|^4 - C_2.$$

□

Lemma 3.3.2. *A minimizer of I_ϵ exists in H.*

Proof. Let $\{\mathbf{A}_n\}_{n=1}^{\infty}$ be a minimizing sequence of I_ϵ in H. Then, by the coercivity of L_ϵ proved in the previous lemma, $\{\mathbf{A}_n\}_{n=1}^{\infty}$ is a bounded sequence. Indeed,

$$\begin{aligned} I_\epsilon[\mathbf{A}] &= \int_{U_0} L_\epsilon(\mathbf{A}(x), x) dx, \\ &\geq C_1 \int_{U_0} \|\mathbf{A}(x)\|^4 dx - C_2, \\ &= C_1 \|\mathbf{A}\|_H^4 - C_2. \end{aligned}$$

The subspace H is a closed subspace of $L^2(U_0; Sym(n))$ and therefore is itself a Hilbert space. As such, any bounded set is weakly relatively compact, and so, upon taking a subsequence if necessary, we can assume that $\mathbf{A_n} \rightharpoonup \mathbf{A}$ for some $\mathbf{A} \in H$. In addition, we have that I_ϵ is convex. This will follow if we prove convexity of the map $A \to v^T A^2 v$, as all other components of I_ϵ are either linear or quadratic and thus obviously convex. Let $A, B \in Sym(n)$, and $\lambda \in [0, 1]$

$$v^T(\lambda A + (1-\lambda)B)^2 v = \lambda^2 v^T A^2 V + \lambda(1-\lambda)(AB + BA) + (1-\lambda)^2 v^T B^2 V.$$

A quick calculation reveals that

$$\lambda^2 v^T A^2 V + \lambda(1-\lambda)(AB + BA) + (1-\lambda)^2 v^T B^2 V \leq \lambda v^T A^2 V + (1-\lambda) v^T B^2 v,$$

if and only if $0 \leq v^T(A-B)^2 v$, which obviously holds. This proves convexity of I_ϵ. Continuity of I_ϵ will follow using standard techniques. As a result, $\partial I_\epsilon[\mathbf{A}]$ is non-empty. Let $\mathbf{B} \in \partial I_\epsilon[\mathbf{A}]$. We get

$$I_\epsilon[\mathbf{A}_n] \geq I_\epsilon[\mathbf{A}] + (\mathbf{B}, \mathbf{A}_n - \mathbf{A})_H.$$

Taking $n \to \infty$, and using $\mathbf{A_n} \rightharpoonup \mathbf{A}$, we get that $\mathbf{A}$ is a minimizer of I_ϵ in H. □

We now turn our attention to showing the dependence of a minimizer of I_ϵ on ϵ. In particular, we will show that as $\epsilon \to 0$, the square of the minimizer of I_ϵ converges to $\mathbf{G}^{-1}$ in L^∞ norm.

Lemma 3.3.3. *Let $\mathbf{A}_\epsilon$ be a minimizer of I_ϵ in H. For $\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}$ small enough, there exists $C > 0$ so that*

$$\left\|\mathbf{A}_\epsilon^2 - \mathbf{G}^{-1}\right\|_{L^\infty(U_0\times\mathcal{S})} \leq C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}.$$

Proof. To start with, we note that if ϵ is small, then the value of L_ϵ is small at a minimizer. Indeed, letting $\mathbf{A}_\epsilon$ be a minimizer of I_ϵ, we get, for almost all $x \in U_0$,

$$\begin{aligned} L_\epsilon(\mathbf{A}_\epsilon(x), x) &\leq L_\epsilon((\mathbf{G}^{-1})^{1/2}(x), x), \\ &= \int_{\mathcal{S}} (2v^T(x,y)\mathbf{G}^{-1}(x)\epsilon(x,y) + \epsilon^T(x,y)\mathbf{G}^{-1}(x)\epsilon^T(x,y))^2 dy, \\ &\leq C\|\epsilon\|^2_{L^\infty(U_0\times\mathcal{S})}. \end{aligned} \tag{3.3}$$

for some constant C depending on the problem data. The first inequality holds almost everywhere because if this inequality did not hold over some set of positive measure, we could replace $\mathbf{A}$ over that set by $(\mathbf{G}^{-1})^{1/2}$, and this new matrix valued function would reduce the value of I_ϵ, contradicting the assumption that $\mathbf{A}_\epsilon$ is a minimizer. Set $U_1 = \{x \in U_0 \mid (3.3) \text{ holds at } x\}$. We now observe,

$$
\begin{aligned}
L_\epsilon(\mathbf{A}_\epsilon(x), x) &= \int_{\mathcal{S}} ((v(x,y)+\epsilon(x,y))^T \mathbf{A}_\epsilon^2(x)(v(x,y)+\epsilon(x,y)) - 1)^2 dy, \\
&= \int_{\mathcal{S}} ((v(x,y)+\epsilon(x,y))^T \mathbf{A}_\epsilon^2(x)(v(x,y)+\epsilon(x,y)) \\
&\qquad\qquad - v(x,y)^T \mathbf{G}^{-1}(x) v(x,y))^2 dy, \\
&\geq C \int_{\mathcal{S}} (v(x,y)^T(\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x))v(x,y))^2 dy, \\
&\qquad - C \int_{\mathcal{S}} (2v^T(x,y)\mathbf{A}_\epsilon^2(x)\epsilon(x,y) + \epsilon^T(x,y)\mathbf{A}_\epsilon^2(x)\epsilon^T(x,y))^2 dy,
\end{aligned} \tag{3.4}
$$

for C some positive constant, possibly varying between these two terms. Now, using inequalities (3.2) and (3.3), we see that for all $x \in U_1$

$$C_1\|\mathbf{A}_\epsilon(x)\|^4 - C_2 \leq L_\epsilon(\mathbf{A}(x), x) \leq C\|\epsilon\|^2_{L^\infty(U_0\times\mathcal{S})}.$$

So the matrix norm of $\mathbf{A}_\epsilon(x)$ is bounded for all $x \in U_1$, uniformly in ϵ provided ϵ is small enough. Returning to inequality (3.4), we see

$$
\begin{aligned}
C \int_{\mathcal{S}} (v(x,y)^T(\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x))v(x,y))^2 dy &\leq L_\epsilon(\mathbf{A}_\epsilon(x), x) + C\|\epsilon\|^2_{L^\infty(U_0\times\mathcal{S})}, \\
&\leq C\|\epsilon\|^2_{L^\infty(U_0\times\mathcal{S})}.
\end{aligned}
$$

As such, there exists some constant C so that

$$\int_{\mathcal{S}} (v(x,y)^T(\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x))v(x,y))^2 dy \leq C\|\epsilon\|^2_{L^\infty(U_0\times\mathcal{S})}.$$

As a result, for all $x \in U_1$ there must exist some maximal open set $\mathcal{S}'(x) \subset \mathcal{S}$ so that for all $y \in \mathcal{S}'(x)$,

$$\left|v(x,y)^T(\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x))v(x,y)\right| \leq \sqrt{\frac{2C}{\mathrm{Vol}(\mathcal{S})}}\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}.$$

Setting $V(x) = \{\nabla d(x,y) \mid y \in \mathcal{S}'(x)\}$, we observe that $V(x) \subset S_x M_{\mathbf{G}^{-1}(x)}$ is open since it is the image of an open set under a local diffeomorphism composed with an open map [14]. Note that by definition of H, $e_n \in V(x)$ for all $x \in U_1$. In addition, there exists some open neighbourhood of e_n, call it V', so that $V' \subset V(x)$ for all $x \in U_1$. This is due to the fact that the matrix norm of $\mathbf{A}_\epsilon$ is bounded over U_1 and uniformly in ϵ provided this is small enough.

Let now $\{\lambda_i(x)\}_{i=1}^n$ be the eigenvalues of $\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}$, with associated basis of orthonormal eigenvectors $\{v_i(x)\}_{i=1}^n$. Let $\mathbf{P}(x)$ be an orthogonal matrix with the eigenvectors $\{v_i(x)\}_{i=1}^n$ as its columns. Let $W(x) \subset T_xM$ be defined as $W(x) := P^{-1}(x)(V(x))$. Then $W(x)$ is also a non-empty open set in $S_xM_{\mathbf{P}(x)\mathbf{G}^{-1}(x)\mathbf{P}(x)}$. Further, for all $w \in W(x)$,

$$\left|\sum_{i=1}^n \lambda_i(x) w_i(x,y)^2\right| \le C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

Restrict $W(x)$ so that each component of its vectors is strictly negative or strictly positive; this is done to ensure the upcoming map Φ is invertible. This leads to corresponding changes in $V(x)$ and V', but none of the useful properties of these sets will change. We define $\Phi : \mathbb{R}^n \to \mathbb{S}^{n-1}$ by

$$\Phi(w) = P_{\mathbb{S}^{n-1}}\left(\begin{bmatrix} w_1^2 \\ \vdots \\ w_n^2 \end{bmatrix}\right),$$

where $P_{\mathbb{S}^{n-1}}$ is the projection map onto the unit sphere. Then since $P_{\mathbb{S}^{n-1}} : W(x) \to \mathbb{S}^{n-1}$ is a diffeomorphism onto its image with inverse given by projection onto $S_xM_{\mathbf{P}(x)^T\mathbf{G}^{-1}(x)\mathbf{P}(x)}$, and the "componentwise square" map on the domain $W(x)$ is a diffeomorphism onto its image, $\Phi(W(x))$ is a non-empty open set in $\mathbb{S}^{n-1}$. Set also

$$\Lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \vdots \\ \lambda_n(x) \end{bmatrix}.$$

Then the above inequality becomes

$$|\langle \Lambda(x), z\rangle| \le C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})},$$

for C some positive constant, and all $z \in \Phi(W(x))$. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. This inequality implies that $\Lambda(x)$ lives in an intersection of half spaces, each defined by $z \in \Phi(W(x))$. Let $\{z_1, \dots, z_n\} \subset \Phi(W(x))$ be any linearly independent set; such a set exists due to the fact that $\Phi(W(x))$ is open in $\mathbb{S}^{n-1}$. Then,

$$-C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \le \langle \Lambda(x), z_i\rangle \le C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})},$$

for all $i = 1, \dots, n$. We can place an upper bound on the norm of $\Lambda(x)$ by bounding the maximum norm of any vector $\Lambda \in \mathbb{R}^n$ satisfying the above inequality. It is clear that the maximum norm will occur at one of the vertices of the convex polyhedron $K(x) = \{\Lambda \in \mathbb{R}^n \mid -C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \leq \langle \Lambda, z_i \rangle \leq C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \quad i = 1, \ldots, n\}$. Setting

$$\mathbf{B}(x) = \begin{bmatrix} z_1 \ldots z_n \end{bmatrix}^T,$$

a vector Λ is at a vertex of the convex polyhedron $K(x)$ if any only if

$$\mathbf{B}(x)\Lambda = \begin{bmatrix} \pm C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \\ \vdots \\ \pm C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \end{bmatrix},$$

for some arrangement of positives and negatives. But since the vectors $z_1 \ldots z_n$ are linearly independent, $\mathbf{B}$ is invertible, and so

$$\Lambda = \mathbf{B}^{-1}(x) \begin{bmatrix} \pm C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \\ \vdots \\ \pm C\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})} \end{bmatrix}.$$

As such,

$$\|\Lambda(x)\| \leq C\left|\mathbf{B}^{-1}(x)\right|\|\epsilon\|_{L^\infty(U_0\times\mathcal{S})},$$

where $|\mathbf{B}^{-1}(x)|$ denotes the operator norm of the linear map defined by $\mathbf{B}^{-1}(x)$. Let $s_1, \ldots, s_n$ be the singular values of $\mathbf{B}(x)$, in increasing order. We have the following facts:

$$\frac{1}{|\mathbf{B}(x)|} = s_1 \leq \ldots \leq s_n = \left|\mathbf{B}^{-1}(x)\right|, \quad \det\left((\mathbf{B}^{-1}(x))\right) = s_1 \cdot \ldots \cdot s_n.$$

Then

$$\begin{aligned} \left|\mathbf{B}^{-1}(x)\right| &= s_n, \\ &= \frac{\det(\mathbf{B}^{-1}(x))}{s_1 \cdot \ldots \cdot s_{n-1}}, \\ &\leq |\mathbf{B}(x)|^{n-1} \det\left((\mathbf{B}^{-1}(x))\right), \\ &\leq \|\mathbf{B}(x)\|^{n-1} \det\left((\mathbf{B}^{-1}(x))\right), \end{aligned}$$

the last inequality holding because the Frobenius norm of a matrix always dominates the operator norm. The rows of $\mathbf{B}(x)$ are of unit norm, however, so $\|\mathbf{B}(x)\| = \sqrt{n}$. This gives us

$$\left|\mathbf{B}^{-1}(x)\right| \leq C \det\left(\mathbf{B}^{-1}(x)\right) = \frac{C}{\det\left(\mathbf{B}(x)\right)}.$$

In turn,

$$\|\Lambda(x)\| \leq \frac{C}{\det\left(\mathbf{B}(x)\right)} \|\epsilon\|_{L^\infty(U_0 \times \mathcal{S})}.$$

But, by definition of the Frobenius norm, $\|\Lambda(x)\| = \|\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x)\|$. So,

$$\left\|\mathbf{A}_\epsilon^2(x) - \mathbf{G}^{-1}(x)\right\| \leq \frac{C}{\det\left(\mathbf{B}(x)\right)} \|\epsilon\|_{L^\infty(U_0 \times \mathcal{S})}. \tag{3.5}$$

Recall that $\mathbf{B}(x)$ was formed by a selection of linearly independent vectors $z_1, \ldots, z_n$. Let $C(x)$ be the supremum of $\det\left(\mathbf{B}(x)\right)$ over all such selections. If $\inf_{x \in U_1} C(x) > 0$ we will be done. Proceeding by contradiction, assume that there exists $\{x_n\}_{n=1}^\infty \subset U_1$ so that $\lim_{n\to\infty} C(x_n) = 0$. But this implies that $\lim_{n\to\infty} \mathrm{Vol}(\Phi(W(x_n))) = 0$, which in turn implies that $\lim_{n\to\infty} \mathrm{Vol}(V(x_n)) = 0$ since $\Phi \circ P^{-1}(x_n)$ is a diffeomorphism on $V(x_n)$. This is impossible, however, since $\mathrm{Vol}(V(x_n)) \geq \mathrm{Vol}(V') > 0$. As a result, there exists some constant $C > 0$ so that

$$\left\|\mathbf{A}_\epsilon^2 - \mathbf{G}^{-1}\right\|_{L^\infty(U_0 \times \mathcal{S})} \leq C \|\epsilon\|_{L^\infty(U_0 \times \mathcal{S})}.$$

□

Remark on inequality 3.5: This inequality has a nice heuristic argument associated with it. It makes sense that the more of $S_x M_{\mathbf{G}^{-1}(x)}$ that $v(x,y)$ covers as y varies over $\mathcal{S}$, the more constraints are put on the matrix $\mathbf{A}^2 - \mathbf{G}^{-1}$ from the inequality $\left|v(x,y)^T(\mathbf{A}^2 - \mathbf{G}^{-1})v(x,y)\right| \leq \epsilon$. As such, a "wider" set $W(x)$, which gives rise to a "wider" set $\Phi(W(x))$, should produce a more stable estimate on $\|\mathbf{A}_\epsilon^2 - \mathbf{G}^{-1}\|_{L^\infty(U_0 \times \mathcal{S})}$. $\det\left(\mathbf{B}(x)\right)$ reflects the width of this set, and so it makes sense that the reciprocal of this value shows up in the stability estimate.

Chapter 4

Proof of the Main Theorem

4.1 A first proof

We are now ready to prove our main result. Using the techniques of the previous sections, $\Lambda_{\mathcal{S},\mathcal{R}}$ determines the Riemannian metric and manifold in a boundary normal coordinate patch around $\mathcal{S}$, call it U_0. Let Γ be some open set so that $\bar{\Gamma} \subset \mathcal{S}$. By the finite propagation speed of waves, it is clear that there exists $T_0 > 0$ determined by U_0 and Γ such that if $f \in C_0^\infty((0, T_0) \times \Gamma)$ then $\text{supp}(u^f(t)) \subset U_0$ for all $t \in [0, T_0]$. For such an f, we have full knowledge of $u^f|_{(0,T_0)\times U_0}$, since it solves the following PDE

$$\begin{aligned}
(\partial_t^2 - \Delta_g)u^f(t,x) = 0 \quad &\text{in } (0,T_0) \times U_0, \\
u = f \quad &\text{in } (0,T_0) \times \Gamma, \\
u = 0 \quad &\text{in } (0,T_0) \times (\partial U_0 \setminus \Gamma), \\
u|_{t=0} = \partial_t u|_{t=0} = 0 \quad &\text{in } U_0,
\end{aligned}$$

and the manifold $(U_0, g|_{U_0})$ is known. Knowledge of the Riemannian manifold and metric over U_0 also gives us the map

$$L_{\mathcal{R},M(\Gamma,T_0)} : C_0^\infty((0,\infty) \times \mathcal{R}) \to C^\infty((0,\infty) \times M(\Gamma, T_0)),$$

where

$$L_{\mathcal{R},M(\Gamma,T_0)}(\psi) = u^\psi|_{(0,\infty)\times M(\Gamma,T_0)},$$

where u^ψ is the solution to the wave equation (1.1) with boundary condition ψ. To see this, fix $t \in (0,\infty)$; we aim to determine $u^\psi(t)$ as an element of $L^2(M(\Gamma, T_0))$, which will determine it as an element of $C^\infty(M(\Gamma, T_0))$. Using Blagovestchenskii's identity (see the appendix for a proof and definitions), we have, for any function $f \in C_0^\infty((t - T_0, t) \times \Gamma)$,

$$(u^f(t), u^\psi(t))_{L^2(M)} = (f, (J\Lambda_{\mathcal{R},\mathcal{S}} - R\Lambda_{\mathcal{R},\mathcal{S}}RJ)(\psi))_{L^2((0,t)\times\mathcal{S})}, \tag{4.1}$$

where J is a time-filtering operator and R is a time reversal operator, defined in the appendix. We note that due to the support of f, the integral on the left hand side occurs only over $M(\Gamma, T_0)$. The right hand side of the above equality is determined by $\Lambda_{\mathcal{S},\mathcal{R}}$ due to the fact that

$$\Lambda^*_{\mathcal{S},\mathcal{R}} = R\Lambda_{\mathcal{R},\mathcal{S}}R,$$

which is also proved in the appendix (Lemma A.1.4). Since we have determined the metric over U_0, which by continuous extension determines the metric over $\mathcal{S}$, we can compute the inner product on the right hand side of equation (4.1). Furthermore, by approximate controllability, the set $\{u^f(t) \mid f \in C_0^\infty((t-T_0,t)\times\Gamma)\}$ is dense in $L^2(M(\Gamma,T_0))$. As such, varying f over $C_0^\infty((t-T_0,t)\times\Gamma)$, equality (4.1) and the fact that $u^f(t)|_{M(\Gamma,T_0)}$ is known can be used to uniquely determine $u^\psi(t)|_{M(\Gamma,T_0)}$ as an element of $L^2(M(\Gamma,T_0))$. By the invariance of equation (1.1) in time, we can simply vary t to uniquely determine u^ψ as an element of $C^\infty((0,\infty)\times M(\Gamma,T_0))$. So $L_{\mathcal{R},M(\Gamma,T_0)}$ is uniquely determined by $\Lambda_{\mathcal{S},\mathcal{R}}$.

We claim that, for $f \in C_0^\infty((0,\infty)\times\Gamma)$, $u^f|_{(0,\infty)\times M(\Gamma,T_0)}$ is uniquely determined by $\Lambda_{\mathcal{S},\mathcal{R}}$. Fix $t > 0$. According to [12], for T large enough there exists a set $\{\psi_i\}_{i=1}^\infty \subset C_0^\infty((-T,\infty)\times\mathcal{R})$ so that $\{u^{\psi_i}(t)\}_{i=1}^\infty$ is an orthonormal basis for $L^2(M)$. In this context we understand that the initial conditions for u^ψ occur at time $-T$. Using an orthonormal decomposition, we have

$$u^f|_{M(\Gamma,T_0)}(t) = \sum_{i=1}^\infty (u^f(t), u^{\psi_i}(t))_{L^2(M)} u^{\psi_i}|_{M(\Gamma,T_0)}(t),$$
$$u^f|_{M(\Gamma,T_0)}(t) = \sum_{i=1}^\infty (u^f(t), u^{\psi_i}(t))_{L^2(M)} L_{\mathcal{R},M(\Gamma,T_0)}(\psi_i)(t).$$

Everything on the right hand side of the above equation is known, as the inner products are computable through Blagovestchenskii's identity, and the map $L_{\mathcal{R},M(\Gamma,T_0)}$ is known. So, $u^f(t)|_{M(\Gamma,T_0)}$ is determined, and varying t, $u^f|_{(0,\infty)\times M(\Gamma,T_0)}$ is determined. This determines $\partial_\nu u^f|_\Gamma$ for all $t > 0$, which is precisely the image of f under $\Lambda_{\Gamma,\Gamma}$. □

Proof of Corollary 1.1.2. Determining the Riemannian manifold (M, g) from the operator $\Lambda_{\Gamma,\Gamma}$ for $\Gamma \subset \partial M$ open is exactly the partial data problem, and has been solved in the literature [11], [12]. □

Remark on this section: We recognize that our proof that $\Lambda_{\mathcal{S},\mathcal{R}}$ determines $\Lambda_{\Gamma,\Gamma}$ is not constructive because the boundary controls giving rise to an orthonormal basis of $L^2(M)$ are not known explicitly. To remedy this, we give in the next section a constructive proof of the same fact under some additional assumptions.

4.2 A constructive proof

Theorem 4.2.1. *In addition to the hypothesis of Theorem 1.1.1, suppose that*

(i) Condition 1 holds for $\mathcal{R}$ replacing $\mathcal{S}$.

(ii) The Dirichlet to Neumann operator $\Lambda_{\mathcal{S}\cup\mathcal{R},\mathcal{P}}$ is known for $\mathcal{P} \subset \partial M$ some open set disjoint from $\mathcal{S} \cup \mathcal{R}$

Then there exists a constructive procedure for determining $\Lambda_{\Gamma,\Gamma}$ from $\Lambda_{\mathcal{S},\mathcal{R}}$ for any $\Gamma \subset \partial M$ open so that $\bar{\Gamma} \subset \mathcal{S}$.

Remark on condition (i): It is not too much to ask that Condition 1 holds for $\mathcal{S}$ and $\mathcal{R}$ simultaneously. Indeed, the example in [14] shows that for M as the unit disc in $\mathbb{R}^2$, condition 1 holds for any open set in ∂M.

Proof. We start by claiming that $\Lambda_{\mathcal{S},\mathcal{R}}$ constructively determines $L_{\Gamma,M(\Gamma,T_0)}$, $T_0 > 0$ such that $M(\Gamma, T_0) \subset U_0$. To see this, suppose that we are given a set $\{\psi_i\}_{i=1}^{\infty} \subset C_0^\infty((-T,\infty) \times \mathcal{R})$ satisfying the following properties for some fixed $t > 0$:

(i) For some fixed $\epsilon > 0$, $\left\|u^{\psi_i}(t)\right\|_{L^2(M\setminus M(\Gamma,T_0))} \leq \frac{\epsilon}{i^2}$ for all i.

(ii) $\{u^{\psi}(t)|_{M(\Gamma,T_0)}\}_{i=1}^{\infty}$ forms an orthonormal basis for $L^2(M(\Gamma, T_0))$

Then we get, for $f \in C_0^\infty((0,\infty) \times \Gamma)$,

$$\begin{aligned} u^f(t)|_{M(\Gamma,T_0)} &= \sum_{i=1}^{\infty} (u^f(t), u^{\psi_i}(t)|_{M(\Gamma,T_0)}) u^{\psi_i}(t)|_{M(\Gamma,T_0)}, \\ &= \sum_{i=1}^{\infty} (u^f(t), u^{\psi_i}(t)) u^{\psi_i}(t)|_{M(\Gamma,T_0)} \\ &\quad - \sum_{i=1}^{\infty} (u^f(t), u^{\psi_i}(t)|_{M\setminus M(\Gamma,T_0)}) u^{\psi_i}(t)|_{M(\Gamma,T_0)}. \end{aligned} \tag{4.2}$$

Every term of the first sum is known by Blagovestchenskii's identity and our knowledge of the operator $L_{\mathcal{R},M(\Gamma,T_0)}$. By the first condition on $\{\psi_i\}_{i=1}^{\infty}$, the second sum is bounded above by $C\epsilon$ for some constant $C > 0$ that depends on f. As such, we can determine $u^f(t)|_{M(\Gamma,T_0)}$ for $f \in C_0^\infty((0,T_0)\times\Gamma)$ as an element of $L^2(M(\Gamma,T_0))$ up to an arbitrarily small precision controlled by ϵ. In other words, $L_{\Gamma,M(\Gamma,T_0)}$ is known provided that for any $\epsilon > 0$ we can find a set $\{\psi_i\}_{i=1}^\infty$ satisfying conditions (i) and (ii). Such a sequence must exist by approximate controllability, and we now show how to find one.

Finding $\{\psi_i\}_{i=1}^\infty$ satisfying condition (ii) is possible by approximate controllability if $T > 0$ is large enough and since both $M(\Gamma,T_0)$ and $L_{\mathcal{R},M(\Gamma,T_0)}$ are known. So we focus on testing condition (i). Suppose that the map $\Phi_{\mathcal{R}} : \cup_{j=1}^\infty E_j|_{\mathcal{R}} \to L^2(M(\Gamma,T_0))$ is known, where

$$\Phi_{\mathcal{R}}(v) = c(v)\phi_j(v)|_{M(\Gamma,T_0)},$$

where $E_j|_{\mathcal{R}} = \text{span}\{\mu^{-1}\partial_\nu\phi_{jk}|_{\mathcal{R}} \mid k = 1\ldots K_j\}$; we note that, as above, the sets $E_j|_{\mathcal{R}}$ are determined by $\Lambda_{\mathcal{S},\mathcal{R}}$. Selecting a positive measure $d\tilde{S}$ on $\mathcal{R}$ as above, we define the following functional $E : C_0^\infty((-T,\infty)\times\mathcal{R}) \to \mathbb{R}$, where

$$E[\psi] = \sum_{i=1}^\infty \sup_{v\in D_j|_{\mathcal{R}}} ((\psi, s_j v)_{L^2((-T,t)\times\partial M;d\tilde{S})} - (L_{\mathcal{R},M(\Gamma,T_0)}(\psi)(t), \Phi_{\mathcal{R}}(v))_{L^2(M(\Gamma,T_0))})^2.$$

where

$$D_j|_{\mathcal{R}} = \{v \in E_j|_{\mathcal{R}} \mid \|v\|_{L^2(\mathcal{R};d\tilde{S})} \le F_2(\lambda_j)\}.$$

This functional is of interest because it provides an estimate on $\left\|u^\psi(t)\right\|_{L^2(M\setminus M(\Gamma,T_0))}$. Indeed, using (2.6), we get:

$$\begin{aligned}
E[\psi] &= \sum_{i=1}^\infty \sup_{v\in D_j|_{\mathcal{R}}} ((u^\psi(t), c(v)\phi_j(v))_{L^2(M)} - (u^\psi(t), c(v)\phi_j(v))_{L^2(M(\Gamma,T_0))})^2,\\
&= \sum_{i=1}^\infty \sup_{v\in D_j|_{\mathcal{R}}} (u^\psi(t), c(v)\phi_j(v))^2_{L^2(M\setminus M(\Gamma,T_0))},\\
&\ge \left\|u^\psi(t)\right\|^2_{L^2(M\setminus M(\Gamma,T_0))}.
\end{aligned}$$

The final inequality following from the discussion of the functional in Section 3.1. Note that since Condition 1 holds for $\mathcal{R}$, the coefficients $c(v)$ are bounded, which guarantees that $E[\psi] < \infty$ for all ψ in its domain. We are also able to show similarly that,

$$E[\psi] \le K^2\left\|u^\psi(t)\right\|^2_{L^2(M\setminus M(\Gamma,T_0))}.$$

So, evaluating $E[\psi]$, we are able to test condition (i) for $\psi \in C_0^\infty((-T,\infty)\times\mathcal{R})$, under the assumption that $\Phi_{\mathcal{R}}$ is known. We now show that this map is determined by $\Lambda_{\mathcal{S}\cup\mathcal{R},\mathcal{P}}$. To see this, observe first that $\Lambda_{\mathcal{S}\cup\mathcal{R},\mathcal{P}}$ determines the sets $E_j|_{\mathcal{S}\cup\mathcal{R}}$, which in turn determines the map $\Psi : E_j|_{\mathcal{R}} \to E_j|_\Gamma$ defined by

$$\Psi(v|_{\mathcal{R}}) = v|_\Gamma.$$

We also have the map $L^{T_0}_{\Gamma,M(\Gamma,T_0)} : C_0^\infty((0,T_0),\Gamma) \to C^\infty((0,T_0)\times M(\Gamma,T_0))$ defined by

$$L^{T_0}_{\Gamma,M(\Gamma,T_0)}(f) = u^f|_{(0,T_0)\times M(\Gamma,T_0)}.$$

This follows from the fact that $M(\Gamma,T_0)\subset U_0$, the known portion of the manifold. For $v\in E_j|_\Gamma$ and $t\in(0,T_0)$,

$$\begin{aligned}\int_0^t\int_{\partial M} s_j(r)f(r,x)v(x)d\tilde{S}(x)dr &= (u^f(t),c(v)\phi_j(v))_{L^2(M)},\\ &= (L^{T_0}_{\Gamma,M(\Gamma,T_0)}(f)(t),c(v)\phi_j(v))_{M(\Gamma,T_0)}.\end{aligned}$$

By approximate controllability, this inner product determines $c(v)\phi_j(v)|_{M(\Gamma,T_0)}$. As such, the operator $\Phi_\Gamma : \cup_{j=1}^\infty E_j|_\Gamma \to L^2(M(\Gamma,T_0))$ is known, where

$$\Phi_\Gamma(v) = c(v)\phi_j(v)|_{M(\Gamma,T_0)}.$$

Observing that $\Phi_{\mathcal{R}} = \Phi_\Gamma\circ\Psi$, we see that $\Phi_{\mathcal{R}}$ is known. This concludes the proof that $L_{\Gamma,M(\Gamma,T_0)}$ is known. To see that this determines $\Lambda_{\Gamma,\Gamma}(f)$ for $f\in C_0^\infty((0,\infty)\times\Gamma))$, we start with Green's formula with an inward facing normal ν. For any $t>0$, let $h\in C_0^\infty((t-T_0,t)\times\Gamma)$

$$\begin{aligned}(\partial_\nu u^h, f)_{L^2((t-T_0,t)\times\Gamma)} &- (h,\partial_\nu u^f)_{L^2((t-T_0,t)\times\Gamma)}\\ &= (u^h,\Delta_g u^f)_{L^2((t-T_0,t)\times M)} - (\Delta_g u^h,u^f)_{L^2((t-T_0,t)\times M)},\\ &= (u^h,\partial_t^2 u^f)_{L^2((t-T_0,t)\times M)} - (\partial_t^2 u^h,u^f)_{L^2((t-T_0,t)\times M)},\\ &= (u^h(t),\partial_t u^f(t))_{L^2(M)} - (\partial_t u^h(t),u^f(t))_{L^2(M)},\\ &= (u^h(t),u^{\partial_t f}(t))_{L^2(M)} - (u^{\partial_t h}(t),u^f(t))_{L^2(M)}.\end{aligned} \quad (4.3)$$

The second equality follows from the fact that u^f and u^h satisfy the wave equation, the third inequality follows from Green's theorem applied in the time domain, and the final equality follows from the invariance of g in time. Recalling the definition of $L_{\Gamma,M(\Gamma,T_0)}$, we have

$$\begin{aligned}(u^h(t), u^{\partial_t f}(t))_{L^2(M)} - (u^{\partial_t h}(t), u^f(t))_{L^2(M)} \\ = (L_{\Gamma,M(\Gamma,T_0)}(h)(t), L_{\Gamma,M(\Gamma,T_0)}(\partial_t f)(t))_{L^2(M)} \\ - (L_{\Gamma,M(\Gamma,T_0)}(\partial_t h)(t), L_{\Gamma,M(\Gamma,T_0)}(f)(t))_{L^2(M)}.\end{aligned}$$

Hence, the right hand side of equation (4.3) is determined. The first term on the left hand side is also known since $\partial_\nu u^h|_{(t-T_0,t)} = \partial_\nu L^{T_0}_{\Gamma,M(\Gamma,T_0)}(h)$. As such, the inner product $(h, \partial_\nu u^f)_{L^2((t-T_0,t)\times\Gamma)}$ is known for all h, which allows for the computation of $\partial_\nu u^f_{(t-T_0,t)\times\Gamma}$. Varying t, we determine $\Lambda_{\Gamma,\Gamma}(f)$ constructively. □

Chapter 5

Conclusion

We examined the problem of reconstructing a Riemannian manifold from the Dirichlet to Neumann map $\Lambda_{\mathcal{S},\mathcal{R}}$ when $\mathcal{S}$ and $\mathcal{R}$ are disjoint. Through a variational approach we were able to constructively determine a portion of the manifold near $\mathcal{S}$, and provide stability estimates. We then showed how this portion of the manifold, along with $\Lambda_{\mathcal{S},\mathcal{R}}$, uniquely determines $\Lambda_{\Gamma,\Gamma}$, where $\Gamma \subset \partial M$ is any open set such that $\bar{\Gamma} \subset \mathcal{S}$. We also showed that, under some additional but reasonable assumptions, $\Lambda_{\Gamma,\Gamma}$ can be determined constructively. We then argued, by drawing on previous results for the partial data problem, that $\Lambda_{\Gamma,\Gamma}$ determines the Riemannian manifold up to isometry. The contributions of our approach are that it allows for the use of previous results on the partial data problem to be used on the disjoint partial data problem, is implementable, and applies to a potentially larger class of manifolds than previous results.

For future research directions, a numerical implementation of our reconstruction technique would provide some intriguing results. Furthermore, it would be interesting to see how our approach can be extended to Riemannian manifolds which are not smooth, as very rarely in our approach did we require infinite differentiability of any functions. This type of problem is certainly well motivated by practical concerns, since in an application such as mineral prospection it is likely that the most regularity one could expect from the underlying metric is that it is essentially bounded. Finally, it would be enlightening to see if the spectral condition we assume is also a necessary condition for unique reconstruction, in addition to being sufficient.

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Appendix A

Some useful facts

A.1 Identities and Approximate Controllability

In this appendix we prove some of the identities used throughout the main body of the thesis. We start by proving equation (2.6), which was critical in our approach for estimating the Riemannian distance function. This identity was originally proven in [14], and we reproduce it here for completeness.

Lemma A.1.1 (From [14]). *Let $f \in C_0^\infty((0,\infty) \times \partial M)$. Let $\lambda > 0$ be an eigenvalue of the Laplace-Beltrami operator Δ_g, and let $\phi \in C^\infty(M)$ be an associated Dirichlet eigenfunction. For all $T > 0$, we have*

$$(u^f(T), \phi)_{L^2(M)} = \int_0^T \int_{\partial M} s(t) f(t,x) \partial_\nu \phi(x) dS(x) dt,$$

where $s(t) = \sin(\sqrt{\lambda}(T-t))/\sqrt{\lambda}$.

Proof. We consider the map $t \mapsto (u^f(t), \phi)_{L^2(M)}$, which will be a smooth function of t. Differentiating this twice with respect to t, we get:

$$\begin{aligned}
\partial_t^2 (u^f(t), \phi)_{L^2(M)} &= (\partial_t^2 u^f(t), \phi)_{L^2(M)}, \\
&= (\Delta_g u^f(t), \phi)_{L^2(M)}, \\
&= (\Delta_g u^f(t), \phi)_{L^2(M)} - (u^f(t), \Delta_g \phi)_{L^2(M)} - \lambda (u^f(t), \phi)_{L^2(M)}, \\
&= (f(t), \partial_\nu \phi)_{L^2(\partial M)} - \lambda (u^f(t), \phi)_{L^2(M)}.
\end{aligned}$$

In the final line we have used Green's formula with an inward pointing normal, and the fact that $\phi|_{\partial M} = 0$. Setting $g(t) := (u^f(t), \phi)_{L^2(M)}$, we find that g satisfies the following second order ordinary differential equation:

$$\begin{aligned}
\partial_t^2 g(t) &= (f(t), \partial_\nu \phi)_{L^2(\partial M)} - \lambda g(t) \quad t > 0, \\
g(0) &= 0, \\
\partial_t g(0) &= 0.
\end{aligned}$$

It is easy to verify that the solution to this differential equation is given by

$$g(t) = \int_0^t \int_{\partial M} \frac{\sin(\sqrt{\lambda}(t-\tau))}{\sqrt{\lambda}} f(\tau, x) \partial_\nu \phi(x) dS(x) d\tau.$$

Evaluating this function at $t = T$, we get

$$g(T) = (u^f(T), \phi)_{L^2(M)} = \int_0^T \int_{\partial M} s(t) f(t, x) \partial_\nu \phi(x) dS(x) dt.$$

□

We now prove a result on approximate controllability which is also required for our method to determine the Riemannian distance function. A necessary prerequisite is Tataru's unique continuation result, which we state without proof.

Lemma A.1.2 (Tataru's Unique Continuation, taken from [3])**.** *Let u be a solution to the wave equation*

$$(\partial_t^2 - \Delta_g) u(t, x) = 0 \quad (t, x) \in (0, \infty) \times M,$$

Suppose that, for $\Gamma \subset \partial M$ open and $\tau > 0$,

$$u(t, x)|_{(0,2\tau)\times\Gamma} = \partial_\nu u(t, x)|_{(0,2\tau)\times\Gamma} = 0.$$

Then,

$$u(\tau, x)|_{M(\Gamma,\tau)} = \partial_t u(\tau, x)|_{M(\Gamma,\tau)} = 0.$$

We will now use Tataru's unique continuation to prove approximate controllability. This method of proof is taken from [12]

Lemma A.1.3 (Approximate controllability, from [12])**.** *For $\Gamma \subset \partial M$ open and $\tau > 0$, the subspace $\{u^f(\tau) \mid f \in C_0^\infty((0, \tau) \times \Gamma)\}$ is dense in $L^2(M(\Gamma, \tau))$*

Proof. Set $A := \{u^f(\tau) \mid f \in C_0^\infty((0, \tau) \times \Gamma)\}$. We will prove the lemma by showing that $A^\perp = \{0\}$. Let $\psi \in A^\perp$ so that

$$(u^f(\tau), \psi)_{L^2(M)} = 0 \quad \text{for all} \quad f \in C_0^\infty((0, \tau) \times \Gamma).$$

Consider the following wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)\tilde{w}(t,x) &= 0 \quad \text{in } (0,\tau)\times M, \\ \tilde{w} &= 0 \quad \text{in } (0,\tau)\times \partial M, \\ \tilde{w}|_{t=0} = 0, \partial_t \tilde{w}|_{t=0} &= \psi \quad \text{in } M. \end{aligned}$$

By Theorem 2.30 of [12], there exists a unique weak solution $\tilde{w}$ of this PDE such that

$$\tilde{w} \in C^0([0,\tau]; H^1(M)) \cap C^1([0,\tau]; L^2(M)).$$

Furthermore, for such a solution, we have that $\partial_\nu \tilde{w}|_{(0,\tau)\times\partial M} \in L^2((0,\tau)\times\partial M)$. Note that this statement is not immediate from the existence of a weak solution, since it is not shown that $\tilde{w} \in C^0([0,\tau]; H^2(M))$, and so, *a priori*, $\partial_\nu \tilde{w}$ only exists in a distributional sense. For $t \in [0,\tau]$, let $w(t,x) := \tilde{w}(\tau - t, x)$. Then w solves the following PDE:

$$\begin{aligned} (\partial_t^2 - \Delta_g)w(t,x) &= 0 \quad \text{in } (0,\tau)\times M, \\ w &= 0 \quad \text{in } (0,\tau)\times \partial M, \\ w|_{t=\tau} = 0, \partial_t w|_{t=T} &= \psi \quad \text{in } M. \end{aligned}$$

Integrating by parts, we get:

$$\begin{aligned} (f, \partial_\nu w)_{L^2((0,\tau)\times\partial M)} &= (\Delta_g u^f, w)_{L^2((0,\tau)\times M)} - (u^f, \Delta_g w)_{L^2((0,\tau)\times M)}, \\ &= (\partial_t^2 u^f, w)_{L^2((0,\tau)\times M)} - (u^f, \partial_t^2 w)_{L^2((0,\tau)\times M)}, \\ &= (\partial_t u^f(\tau), w(\tau))_{L^2(M)} - (\partial_t u^f(0), w(0))_{L^2(M)} \\ &\quad - (u^f(\tau), \partial_t w(\tau))_{L^2(M)} + (u^f(0), \partial_t w(0))_{L^2(M)}, \\ &= -(u^f(\tau), \psi)_{L^2(M)}, \\ &= 0. \end{aligned}$$

The first line follows from Green's formula with an inward pointing normal and the boundary condition on w, and the third line follows from Green's formula applied in the time domain. Our initial and final conditions give us the fourth line, and the final line comes from the assumption that $\psi \in A^\perp$. This equality holds for all $f \in C_0^\infty((0,\tau)\times\Gamma)$, a dense subset of $L^2((0,\tau)\times\Gamma)$, and therefore $\partial_\nu w|_{L^2((0,\tau)\times\Gamma)} = 0$. We now extend w to the time interval $(0, 2\tau)$ as follows:

$$W(t,x) = \begin{cases} w(t,x) & t \leq \tau, \\ -w(2\tau - t, x) & t \geq \tau \end{cases}.$$

Then we have

$$W \in C^0([0,2\tau]; H^1(M)) \cap C^1([0,\tau]; L^2(M)).$$

In addition,

$$\begin{aligned} (\partial_t^2 - \Delta_g)W(t,x) &= 0 \quad \text{in } (0,2\tau) \times M, \\ W(t,x) &= 0 \quad \text{in } (0,2\tau) \times \Gamma, \\ \partial_\nu W(t,x) &= 0 \quad \text{in } (0,2\tau) \times \Gamma. \end{aligned}$$

As such, by Tataru's unique continuation, we get $\partial_t W(\tau,x)|_{M(\Gamma,\tau)} = 0$. But $\psi = \partial_t W(\tau,x)|_{M(\Gamma,\tau)}$, and therefore $\psi = 0$. □

We continue by proving an identity showing the relationship between $\Lambda_{\mathcal{S},\mathcal{R}}$ and $\Lambda_{\mathcal{R},\mathcal{S}}$

Lemma A.1.4 (From [13]). *Let $\mathcal{S},\mathcal{R} \subset \partial M$ be open sets, and let $T > 0$. Let $\Lambda_{\mathcal{S},\mathcal{R}} : L^2((0,T)\times\mathcal{S}) \to H^{-1}((0,T)\times\mathcal{R})$ be the Dirichlet to Neumann map from $\mathcal{S}$ to $\mathcal{R}$, and let $\Lambda_{\mathcal{R},\mathcal{S}}$ be defined similarly. Then*

$$\Lambda^*_{\mathcal{R},\mathcal{S}} = R\Lambda_{\mathcal{S},\mathcal{R}}R, \tag{A.1}$$

where $\Lambda^*_{\mathcal{R},\mathcal{S}}$ is the adjoint of $\Lambda_{\mathcal{R},\mathcal{S}}$, and $Rf(t) = f(T-t)$.

Proof. We prove the required identity for $\Lambda_{\mathcal{S},\mathcal{R}} : C_0^\infty((0,T)\times\mathcal{S}) \to C^\infty((0,T)\times\mathcal{R})$, which will prove the identity for the map in the lemma by a density argument. So, let $f \in C_0^\infty((0,T)\times\mathcal{S})$ and $\psi \in C_0^\infty((0,T)\times\mathcal{R})$. Set

$$C(f,\psi) := (\Lambda_{\mathcal{R},\mathcal{S}}\psi, f)_{L^2((0,T)\times\partial M)} - (\psi, R\Lambda_{\mathcal{S},\mathcal{R}}Rf)_{L^2((0,T)\times\partial M)}.$$

We aim to show that $C(f,\psi) = 0$ for all f,ψ. We have:

$$C(f,\psi) = \int_0^T \int_{\partial M} \partial_\nu u^\psi(t,x) f(t,x) - \psi(t,x)\partial_\nu R u^{Rf}(t,x) dS(x)dt. \tag{A.2}$$

Since $\partial_t^2 R u^{Rf} = \partial_t^2 u^{Rf}$, we see that Ru^{Rf} satisfies the following PDE:

$$\begin{aligned}(\partial_t^2 - \Delta_g)u(t,x) &= 0 \quad \text{in } (0,T)\times M,\\ u &= f \quad \text{in } (0,T)\times\partial M,\\ u|_{t=T} = \partial_t u|_{t=T} &= 0 \quad \text{in } M.\end{aligned}$$

Using this with equation (A.2) and Green's formula with an inward pointing normal, we get

$$\begin{aligned}C(f,\psi) &= \int_0^T\int_M u^\psi(t,x)\Delta_g Ru^{Rf}(t,x) - \Delta_g u^\psi(t,x)Ru^{Rf}(t,x)dV(x)dt,\\ &= \int_0^T\int_M u^\psi(t,x)\partial_t^2 Ru^{Rf}(t,x) - \partial_t^2 u^\psi(t,x)Ru^{Rf}(t,x)dV(x)dt,\\ &= \int_M u^\psi(t,x)\partial_t Ru^{Rf}(t,x) - \partial_t u^\psi(t,x)Ru^{Rf}(t,x)\Big|_0^T dV(x),\\ &= 0.\end{aligned}$$

where second to last equation follows from applying Green's formula in the time domain, and the last equation follows from the initial conditions on u^ψ and the conditions at $t = T$ on Ru^{Rf}. This proves the lemma. □

Blagovestchenskii's identity has been used extensively in research on reconstructing a Riemannian manifold from the hyperbolic Dirichlet to Neumann map ([3],[14], and [16]). Its primary utility comes from the fact that it shows how the hyperbolic Dirichlet to Neumann map can be used to determine the inner products of solutions to the Riemannian wave equation; in this way it makes data from the interior of the manifold visible on the boundary. It is also critical to the final step of our reconstruction, and so we reproduce the proof of this lemma from [12], Section 4.2 here.

Lemma A.1.5 (From [14]). *Let $T > 0$ and $\mathcal{S},\mathcal{R}\subset\partial M$ open. Let $f\in C_0^\infty((0,\infty)\times\mathcal{S})$ and $\psi\in C_0^\infty((0,\infty)\times\mathcal{R})$. Then*

$$(u^\psi(T),u^f(T))_{L^2(M)} = (\psi,(J\Lambda_{\mathcal{S},\mathcal{R}} - R\Lambda_{\mathcal{S},\mathcal{R}}RJ)f)_{L^2((0,T)\times\mathcal{R})}, \tag{A.3}$$

where $Jf(t) := \frac{1}{2}\int_t^{2T-t} f(s)ds$ and $Rf(t) = f(T-t)$.

Proof. Let $t,s>0$. We start by studying the map $(t,s)\mapsto(u^f(t),u^\psi(s))_{L^2(M)}$. In particular, we will show that this function satisfies a one dimensional wave equation over the domain $(0,\infty)\times(0,\infty)$. Indeed,

$$
\begin{aligned}
(\partial_t^2 - \partial_s^2)(u^\psi(t), u^f(s))_{L^2(M)} &= (\Delta_g u^\psi(t), u^f(s))_{L^2(M)} - (u^\psi(t), \Delta_g u^f(s))_{L^2(M)}, \\
&= (\psi(t), \partial_\nu u^f(s))_{L^2(\partial M)} - (\partial_\nu u^\psi(t), f(s))_{L^2(\partial M)}, \\
&= (\psi(t), \Lambda_{\mathcal{S},\mathcal{R}} f(s))_{L^2(\partial M)} - (\Lambda_{\mathcal{R},\mathcal{S}} \psi(t), f(s))_{L^2(\partial M)},
\end{aligned}
$$

where the first equality follows from the fact that u^f and u^ψ satisfy the Riemannian wave equation, the second follows from Green's theorem with an inward pointing normal, and the last follows from the definition of the Dirichlet to Neumann map. Setting $h(t,s) := (\psi(t), \Lambda_{\mathcal{S},\mathcal{R}} f(s))_{L^2(\partial M)} - (\Lambda_{\mathcal{R},\mathcal{S}} \psi(t), f(s))_{L^2(\partial M)}$, and $g(t,s) = (u^\psi(t), u^f(s))_{L^2(M)}$ we get

$$
\begin{aligned}
(\partial_t^2 - \partial_s^2) g(t,s) = h(t,s) &\quad \text{in } (0,\infty)\times(0,\infty), \\
g(t,0) = 0 &\quad t \in (0,\infty), \\
g(0,s) = \partial_t g(0,s) = 0 &\quad s \in (0,\infty).
\end{aligned}
$$

This is the one-dimensional wave equation for an infinitely long homogeneous vibrating string fixed at the origin. For fixed $s,t > 0$, integrate the PDE over the domain $\Omega(s,t)$, where

$$
\Omega(t,s) = \{(r,p) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid r \in (0,t), p \in (r+s-t, s+t-r)\}.
$$

Incidentally, this region is exactly the portion of $\mathbb{R}^+ \times \mathbb{R}^+$ which can affect the value of g at the point (t,s) (see e.g. [7]). We have

$$
\begin{aligned}
\int_{\Omega(t,s)} h(r,p)\,dr\,dp &= \int_{\Omega(t,s)} (\partial_r^2 - \partial_p^2) g(r,p)\,dr\,dp, \\
&= \int_{\Omega(t,s)} \partial_r(\partial_r g(r,p)) - \partial_p(\partial_p(g(r,p))\,dr\,dp, \\
&= \int_{\Omega(t,s)} \left(\nabla \times \begin{bmatrix} \partial_p g(r,p) \\ \partial_r g(r,p) \\ 0 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dr\,dp, \\
&= \int_{\partial\Omega(t,s)} \begin{bmatrix} \partial_p g(r,p) \\ \partial_r g(r,p) \end{bmatrix} \cdot d\gamma, \qquad \text{(A.4)}
\end{aligned}
$$

where the fourth line follows from Stokes theorem, and $d\gamma$ denotes the integral over the positively oriented contour $\partial\Omega(t,s)$. This integral can be broken up into three portions:

$$
\begin{aligned}
C_1 &:= \{(0,p) \mid p \in (s-t, s+t)\}, \\
C_2 &:= \{(r,p) \mid p \in (s-t, s), r = p - s + t\}, \\
C_3 &:= \{(r,p) \mid p \in (s, s+t), r = -p + s + t\}.
\end{aligned}
$$

Since $\partial_r g(0,p) = 0$ and $g(0,p) = 0$ for all $p > 0$, the integral over C_1 vanishes. We now calculate the integral over C_2 and C_3, keeping in mind that in Green's theorem the contour $\partial\Omega(t,s)$ is oriented positively.

$$
\begin{aligned}
\int_{C_2} \begin{bmatrix} \partial_p g(r,p) \\ \partial_r g(r,p) \end{bmatrix} \cdot d\gamma &= \int_{s-t}^{s} \begin{bmatrix} \partial_p g(p-s+t,p) \\ \partial_r g(p-s+t,p) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} dp, \\
&= \int_{s-t}^{s} \partial_p g(p-s+t,p) dp, \\
&= g(t,s) - g(0,p) = g(t,s),
\end{aligned}
$$

where the last equality follows from the initial conditions on g. Similarly,

$$
\begin{aligned}
\int_{C_3} \begin{bmatrix} \partial_p g(r,p) \\ \partial_r g(r,p) \end{bmatrix} \cdot d\gamma &= \int_{s}^{s+t} \begin{bmatrix} \partial_p g(-p+s+t,p) \\ \partial_r g(-p+s+t,p) \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} dp, \\
&= -\int_{s}^{s+t} \partial_p g(-p+s+t,p) dp, \\
&= -g(0,s+t) + g(t,s) = g(t,s).
\end{aligned}
$$

Recalling (A.4), we get

$$
g(t,s) = \frac{1}{2} \int_{\Omega(t,s)} h(r,p) dr dp.
$$

Recalling the definitions of g and h, we get

$$
\begin{aligned}
(u^f(T), u^\psi(T))_{L^2(M)} &= \frac{1}{2} \int_{\Omega(T,T)} (\psi(r), \Lambda_{\mathcal{S},\mathcal{R}} f(p))_{L^2(\partial M} - (\Lambda_{\mathcal{R},\mathcal{S}} \psi(r), f(p))_{L^2(\partial M)} dr dp, \\
&= \int_0^T (\psi(r), J\Lambda_{\mathcal{S},\mathcal{R}} f(r))_{L^2(\partial M)} - (\Lambda_{\mathcal{R},\mathcal{S}} \psi(r), Jf(r))_{L^2(\partial M)} dr, \\
&= (\psi, (J\Lambda_{\mathcal{S},\mathcal{R}} - R\Lambda_{\mathcal{S},\mathcal{R}} RJ) f)_{L^2((0,T) \times \partial M)},
\end{aligned}
$$

where in the final equality we used Lemma A.1.4. □