What is the characteristic polynomial of a signal flow graph?*

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1. Problem statement

Suppose one is given signal flow graph $\mathcal{G}$ with n nodes whose branches have gains that are real rational functions (the open loop transfer functions). The gain of the branch connecting node i to node j is denoted R_{ji}, and we write $R_{ji} = \frac{N_{ji}}{D_{ji}}$ as a coprime fraction. The closed-loop transfer function from node i to node j for the closed-loop system is denoted T_{ji}.

The problem can then be stated as follows.

Is there an algorithmic procedure that takes a signal flow graph $\mathcal{G}$ and returns a "characteristic polynomial" $P_{\mathcal{G}}$ with the following properties:

1. *$P_{\mathcal{G}}$ is formed by products and sums of the polynomials N_{ji} and D_{ji}, $i, j = 1, \dots, n$;*
2. *all closed-loop transfer functions T_{ji}, $i, j = 1, \dots, n$, are analytic in the closed right half plane (CRHP) if and only if $P_{\mathcal{G}}$ is Hurwitz?*

The gist of condition 1 is that the construction of $P_{\mathcal{G}}$ depends only on the topology of the graph, and not on manipulations of the branch gains. That is, the definition of $P_{\mathcal{G}}$ should not depend on the choice of branch gains R_{ji}, $i, j = 1, \dots, n$. For example, one would be prohibited from factoring polynomials or from computing the GCD of polynomials. This excludes unhelpful solutions of the problem of the form "Let $P_{\mathcal{G}}$ be the product of the characteristic polynomials of the closed-loop transfer functions T_{ji}, $i, j = 1, \dots, n$."

2. Discussion

Signal flow graphs for modelling system interconnections are due to Mason [1953, 1956]. Of course, when making such interconnections, the stability of the interconnection is non-trivially related to the open-loop transfer functions that weight the branches of the signal flow graph. There are at least two things to consider in the course of making an interconnection: (1) is the interconnection *BIBO stable* in the sense that all closed-loop transfer functions between nodes have no poles in the CRHP? and (2) is the interconnection *well-posed* in the sense that all closed-loop transfer functions between nodes are proper? The problem stated above concerns only the first of these matters. Well-posedness when all branch gains R_{ji}, $i, j = 1, \dots, n$, are proper is known to be equivalent to the condition that the determinant of the graph be a biproper rational function. We comment that other forms of stability for signal flow graphs are possible. For example, Wang, Lee, and He [1999] consider *internal stabilty*, wherein not the transfer functions between signals are considered, but rather that all signals in the signal flow graph remain bounded when bounded inputs are provided. Internal stability as considered by Wang, Lee, and He and BIBO stability as considered here are different. The source of this difference accounts for the source of the open problem of our paper, since Wang, Lee, and He show that internal stability can be determined by an algorithmic procedure like that we ask for for BIBO stability. This is discussed a little further in Section 3.

As an illustration of what we are after, consider the single-loop configuration of Figure 1. As is well-known, if we write $R_i = \frac{N_i}{D_i}$, $i = 1, 2$, as coprime fractions, then all closed-loop transfer functions have no poles in the CRHP if and only if the polynomial $P_{\mathcal{G}} = D_1 D_2 + N_1 N_2$ is Hurwitz. Thus $P_{\mathcal{G}}$ serves as the characteristic polynomial in this case. The essential feature of $P_{\mathcal{G}}$ is that one computes it by looking at the topology of the graph, and the exact character of R_1 and R_2 are of no consequence. For example, pole/zero cancellations between R_1 and R_2 are accounted for in $P_{\mathcal{G}}$.

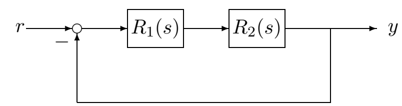

Figure 1: Single-loop interconnection

3. Approaches that do not solve the problem

Let us outline two approaches that yield solutions having one of properties 1 and 2, but not the other.

The problems of internal stability and well-posedness for signal flow graphs can be handled effectively using the polynomial matrix approach e.g., [Callier and Desoer 1982]. Such an approach will involve the determination of a coprime matrix fractional representation of a matrix of rational functions. This will certainly solve the problem of determining internal stability for any given example. That is, it is possible using matrix polynomial methods to provide an algorithmic construction of a polynomial satisfying property 2 above. However, the algorithmic procedure will involve computing GCD's of various of the polynomials N_{ji} and D_{ji}, $i, j = 1, \dots, n$. Thus the conditions developed in this manner have to do not only with the topology of the signal flow graph, but also the specific choices for the branch gains, thus violating condition 1 above. The problem we pose demands a simpler, more direct answer to the question of determining when an interconnection is BIBO stable.

Wang, Lee, and He [1999] provide a polynomial for a signal flow graph using the determinant of the graph which we denote by $\Delta_{\mathcal{G}}$ (see [Mason 1953, 1956]). Specifically, they define a polynomial

$$P = \prod_{(i,j) \in \{1,\dots,n\}^2} D_{ji} \Delta_{\mathcal{G}}. \tag{3.1}$$

Thus one multiplies the determinant by all denominators, arriving at a polynomial in the process. This polynomial has the property 1 above. However, while it is true that if this polynomial is Hurwitz then the system is BIBO stable, the converse is generally false. Thus

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property 2 is not satisfied by P. What is shown in Wang, Lee, and He is that all *signals* in the graph are bounded for bounded inputs if and only if P is Hurwitz. This is different from what we are asking here, i.e., that all closed-loop transfer functions have no CRHP poles. It is true that the polynomial P in (3.1) gives the desired characteristic polynomial for the interconnection of Figure 1. It is also true that if the signal flow graph has no loops (in this case $\Delta_{\mathcal{G}} = 1$) then the polynomial P of (3.1) satisfies condition 2. We comment that the condition of Wang, Lee, and He is the same condition one would obtain by converting (in a specific way) the signal flow graph to a polynomial matrix system, and then ascertaining when the resulting polynomial matrix system is internally stable.

The problem stated is very basic, one for which an inquisitive undergraduate would demand a solution. It was with some surprise that the author was unable to find its solution in the literature, and hopefully one of the readers of this article will be able to provide a solution, or point out an existing one.

References

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