Topological and Algebraic Lower Bounds on the Chromatic Number of Graphs

by

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Abstract

In this paper we present a survey of various lower bounds on graph chromatic number. We present lower bounds derived both from elementary graph invariants, as well as invariants of topological spaces derived from the combinatorial structure of graphs. We discuss the graph $U(5, 3)$ as an example where some of these bounds differ.
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Abstract

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Chapter 1

Introduction

1.1 Preliminaries

Given a set $S$ let $\binom{S}{k}$ be the set of subsets of $S$ of size $k$, and we write $2^S$ to denote the power set of $S$. We write $[k]$ to represent the set $\{1, \ldots, k\}$. A simple graph $G$ is an ordered pair of sets $(V(G), E(G))$, where $E(G) \subseteq \binom{V(G)}{2}$. We call $V(G)$ the vertex set of $G$ and we call $E(G)$ the edge set of $G$. We say that a pair of vertices $u, v \in V(G)$ are adjacent in $G$ when $\{u, v\} \in E(G)$, and we sometimes write $u \sim v$ to indicate adjacency. A complete graph on $n$ vertices, written $K_n$, is a graph such that $\#V(K_n) = n$ and $E(K_n) = \binom{V(K_n)}{2}$. A complete bipartite graph, $K_{m,n}$, is a graph such that the vertex set may be partitioned into two parts $S$ and $T$ where $\#S = m$ and $\#T = n$, and that $\{u, v\} \in E(K_{m,n})$ if and only if $u \in S$ and $v \in T$. A subgraph $H$ of a graph $G$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; if $E(H) = \binom{V(H)}{2} \cap E(G)$, $H$ is called an induced subgraph of $G$, and we say that $V(H)$ induces $H$.

An $n$-cycle of a graph $G$ is a sequence of $n+1$ vertices $v_1, v_2, \ldots, v_{n+1}$ of $G$, such
that $v_1, v_2, \ldots, v_n$ all distinct, $v_1 = v_{n+1}$, and $v_i \sim v_{i+1}$ for $1 \leq i \leq n$. An $n$-cycle graph, $C_n$, is a graph on $n$ vertices which contains a single cycle through all the vertices.

A proper coloring of a graph $G$ is a set map $c : V(G) \rightarrow C$ such that $\{x, y\} \in E(G)$ implies that $c(x) \neq c(y)$. The chromatic number of a graph $\chi(G)$ is the minimum cardinality of a set $C$ such that there exists a proper coloring $c : V(G) \rightarrow C$. A graph with chromatic number $k$ is said to be $k$-chromatic.

Let $G$ and $H$ be graphs. A graph homomorphism is a map $f : V(G) \rightarrow V(H)$ such that $u \sim v$ implies that $f(u) \sim f(v)$.

A proper coloring can also equivalently be defined as a graph homomorphism $f : V(G) \rightarrow K_n$. Additionally the chromatic number of a graph can be defined as the minimum number $n$ such that there exists a graph homomorphism $f : V(G) \rightarrow K_n$.

A clique in a graph $G$ is a subset $C$ of the vertex set $V(G)$ such that the subgraph induced by $C$ is a complete graph. An independent set in a graph is a subset $I$ of the vertex set $V(G)$ such that the induced subgraph of the set has no edges. The clique number, written $\omega(G)$, is the size of the largest clique in $G$. The independence number, $\alpha(G)$, is the size of the largest independent set in $G$.

In discussion of the graph coloring problem and the various lower bounds below we will reference various notions of computational difficulty namely NP-completeness, co-NP-completeness, and uncomputability. A detailed exposition of these topics can be found in [14]. For the purposes of this paper it suffices to know first that computational problems which are NP-complete or co-NP-complete have algorithms which will determine a solution to the problem, but these problems are thought require a great deal of computational resources in the worst case, and second that a problem
which is uncomputable is a problem for which no single algorithm exists which solves the problem for all possible inputs.

1.2 Graph Coloring

There are two hurdles to overcome to verify that a graph is $k$-chromatic. First we must show that a $k$-coloring of the graph in question exists, and second we must show that a proper coloring with fewer colors than $k$ does not exist. The most obvious way to do the latter for an arbitrary graph is to check that every set map $f : V(G) \to [k-1]$ is not a proper coloring. This naive algorithm for determining $k$-chromaticity of a graph requires examining $(k-1)^{|V(G)|}$ set maps in addition to finding a $k$-coloring. To formalize the difficulty of the problem of determining chromatic number, it is known that determining whether a graph has a $k$-coloring is NP-complete for $k \geq 3$, and the problem of verifying that no $(k-1)$-coloring exists is co-NP-complete for $k \geq 4$. For the other cases a graph $G$ is one-chromatic if and only if $E(G) = \emptyset$ and is two-chromatic if and only if the graph is bipartite, a graph property which is polynomially computable.

Due to the computational difficulty of the problem of determining the chromatic number, we instead look for lower bounds on the chromatic number of a graph in the hope that it will yield a lower bound equal to a known upper bound and thus determine the chromatic number of the graph. There are a few elementary lower bounds on the chromatic number of a graph. Knowledge of the independence number and the clique number of a graph yields lower bounds on the chromatic number of a graph. For the clique number $\omega(G)$ it is clear that there can be no proper coloring of size less than $\omega(G)$ on a graph $G$, otherwise we would have a coloring of a complete
subgraph of size $m$ with fewer than $m$ colors, which is clearly impossible. For the
independence number we first make the observation that each color class (the subset
of the vertices that are mapped to a particular color) forms an independent set, and
together they partition $V(G)$. This is clear from the definitions of proper coloring
and independent set. This implies that $\#V(G)/\alpha(G)$ bounds the chromatic number
of a graph from below. Thus we have the following bound:

$$\chi(G) \geq \max\{\omega(G), \#V(G)/\alpha(G)\}$$  \hspace{1cm} (1.1)

While the clique number and the independence number provide lower bounds on
the chromatic number, these lower bounds do not reduce the difficulty of the problem
significantly. This is due to the fact that determining that a clique or independent set
of a certain size exists is NP-complete, and thus that determining that no clique or
independent set exists of a certain size is co-NP-complete. We must also note that in
order to use the clique number lower bound it suffices to just find a clique, whereas to
use the bound derived from the independence number we must have an upper bound
on the independence number.

Another lower bound on the chromatic number of a graph is the fractional chro-
matic number. Scheinermann and Ullman provide a detailed discussion regarding
fractional chromatic number in [10], from there we present the following definitions.

For a set $C$, a $b$-fold coloring of a graph $G$ is a set map $c : V(G) \to \binom{C}{b}$, such
that $u, v \in V(G)$ and $u \sim v$ implies that $c(u) \cap c(v) = \emptyset$. The $b$-fold chromatic
number $\chi_b(G)$ is the minimum cardinality of a set $C$ such that a $b$-fold coloring of $G$
exists. The fractional chromatic number of a graph $G$, written $\chi_f(G)$, is defined to
be $\inf_b \left\{ \frac{\chi_b(G)}{b} \right\}$. An important fact is that there exists a $b$ such that $\chi_f(G) = \frac{\chi_b(G)}{b}$, and
hence the infimum may be taken to be a minimum. The fractional chromatic number
may equivalently be defined as:

$$\inf \{ \frac{n}{k} : G \text{ admits a homomorphism to } KG_{n,k} \}$$

where a Kneser graph $KG_{n,k}$ is defined as:

$$V(KG_{n,k}) = \binom{n}{k}$$

$$E(KG_{n,k}) = \{ \{S,T\} : S, T \in V(KG_{n,k}) \text{ and } S \cap T = \emptyset \}$$

Noting that $\chi(G) = \chi_1(G)$ we have that the fractional chromatic number is another lower bound on the chromatic number of a graph. Additionally $\omega(G)$ and $\#V(G)/\alpha(G)$ are lower bounds on the fractional chromatic number of a graph [4]. Computing the fractional chromatic number of an arbitrary graph remains difficult though, for every real number $r > 2$, the problem of determining whether a graph $G$ satisfies $\chi_f(G) \leq r$ is NP-complete [10].

Graph homomorphisms may also yield upper and lower bounds on the chromatic number of a graph. Given two graphs $G$ and $H$ and a graph homomorphism $\phi : G \rightarrow H$, we have that $\chi(G) \leq \chi(H)$. To see this observe first that by definition of chromatic number there is a graph homomorphism $\psi : H \rightarrow K_{\chi(H)}$. Now $\psi \circ \phi$ is again a graph homomorphism, from $G$ to $K_{\chi(H)}$, which shows the inequality. In this paper we will consider lower bounds derived from homomorphisms from graphs derived by applying the Generalized Mycielskian construction, which will be defined below.

While the computation of the lower bounds discussed so far are in general difficult, the bounds they yield may not even be particularly good lower bounds. There are a great number of graphs where these elementary lower bounds on the chromatic number are insufficient for determining the chromatic numbers of graphs. To demonstrate
the existence of graphs where the clique number and independence number bounds are loose we first introduce the graph construction called Generalized Mycielskian.

The categorical product of two graphs $G$ and $G'$ is written $G \times G'$ and defined by:

$$V(G \times G') = V(G) \times V(G')$$

$$E(G \times G') = \{\{(u, u'), (v, v')\} : \{u, v\} \in E(G) \text{ and } \{u', v'\} \in G'\}$$

We also define a graph quotient. Given a graph $G$ and an equivalence relation $R$ defined over the vertices of $G$ we define the graph quotient $G/R$ as follows

$$V(G/R) = \{\bar{u} : u \in V(G)\}$$

$$E(G/R) = \{\{\bar{u}, \bar{v}\} : \{u, v\} \in E(G)\}$$

where $\bar{u}$ denotes the equivalence class of $u \in V(G)$ with respect to the equivalence relation $R$.

For $n \in \mathbb{N}$ let $P_n$ denote the graph with vertices $0, 1, \ldots, n$ where consecutive vertices are connected by an edge, and which has a loop at the 0 vertex. For a graph $G$ the $n^{th}$ Generalized Mycielskian over $G$, written $M_n(G)$ is the graph $(G \times P_n)/\sim_n$, where $\sim_n$ is the equivalence relation which identifies all vertices whose second coordinate is $n$. An example of the Mycielskian construction is illustrated in Figure 1.1.

The Mycielskian construction $M_2(\bullet)$ can be used to construct examples of graphs with clique number two, and arbitrarily high chromatic number. It is known [16] that for a graph $G$ we have $\omega(M_2(G)) = \omega(G)$, and $\chi(M_2(G)) = \chi(G) + 1$. Repeatedly applying the Mycielskian construction (we will write $M_2^k(\bullet)$ to indicate composition of $M_2(\bullet)$ with itself $k$ times), thus yields $\omega(M_2^k(G)) = \omega(G)$, and $\chi(M_2^k(G)) = \chi(G) + k$. Thus the lower bound on chromatic number given by the clique number on a graph can be arbitrarily loose. The Mycielskian can also be used to construct graphs for
which the lower bound on chromatic number yielded from the independence number is arbitrarily loose. To see this observe that for a graph $G$ we have that $V(M_2(G)) = (V(G) \times \{1, 2\}) \cup \{n\}$ and that $V(G) \times \{2\}$ forms an independent set. Thus we have

$$\#V(M_2(G))/\alpha(M_2(G)) \leq (2 \times \#V(G) + 1)/\#V(G) \leq 3$$

And hence iterative application of the Mycielskian construction shows that the bound on the chromatic number of a graph due to the independence number may also be arbitrarily loose.

The Generalized Mycielskian construction has some additional properties that are worth mentioning. We have that $M_k(\bullet)$ for $k \geq 3$ still maps a graph with clique number two to a graph with clique number two, but applying $M_k(\bullet)$ does necessarily imply an increase in chromatic number, as $M_2(\bullet)$ does. A counter example is the circulant graph $K_{7/2}$. Exhibited in Figures 1.2 and 1.3 are 4-colorings of $K_{7/2}$ and $M_3(K_{7/2})$. To see that the chromatic number of both $K_{7/2}$ and $M_3(K_{7/2})$ is four observe that $K_{7/2}$ is a subgraph of $M_3(K_{7/2})$, namely the induced subgraph of $V(M_3(K_{7/2})) \times \{1\}$. Now also observe that $K_{7/2}$ can have an independent set of size
at most two, yielding the necessary lower bound to complete the demonstration that $\chi(K_{7/2}) = \chi(M_3(K_{7/2})) = 4$.

Finally to see that knowledge of the fractional chromatic number may yield a loose bound we will consider the chromatic number of Kneser graphs. Lovász showed in [6] that $\chi(KG_{n,k}) = n - 2k + 2$. From the alternate definition of fractional chromatic number from Kneser graphs it is clear that $\chi_f(KG_{n,k}) \leq \frac{n}{k}$ and in fact equality holds [4]. Thus it follows that there exist Kneser graphs where the fractional chromatic number yields an arbitrarily loose lower bound on the chromatic number of a graph.

![Figure 1.2: Four-coloring of $K_{7/2}$](image)

---

1Note that the edges between $V(K_{7/2}) \times \{1\}$ and $V(K_{7/2}) \times \{2\}$ and the edges between $V(K_{7/2}) \times \{2\}$ and $V(K_{7/2}) \times \{3\}$ are not included in the figure, and are left implicit.
Figure 1.3: Four-coloring of $M_5(K_{7/2})$
Chapter 2

Topological Bounds

In this section we introduce two functors $B_0(\bullet)$ and $H(\bullet)$ which carry a graph to a $\mathbb{Z}_2$-space, a topological space with some additional useful properties. We will demonstrate that the functoriality of $B_0(\bullet)$ and $H(\bullet)$ can be used to generate non-trivial lower bounds to the chromatic number of a graph. To do this we will use topological obstructions to the existence of certain maps, called $\mathbb{Z}_2$-maps. The functoriality of $B_0(\bullet)$ and $H(\bullet)$ will then imply the non-existence of graph homomorphisms between the graphs carried by the functors, possibly generating a lower bound on the chromatic number.

We will now more formally introduce these ideas. We introduce basic topological concepts and notation that are needed, using the notation found in [8], where a more detailed exposition on this material may also be found.
2.1 \( \mathbb{Z}_2 \)-Spaces

A \( \mathbb{Z}_2 \)-space is a pair \((X, \nu)\), where \( X \) is a topological space and \( \nu : X \to X \) is a continuous map, such that \( \nu \circ \nu = \text{id}_X \), the identity map on \( X \). If for all \( x \in X \) we have that \( \nu(x) \neq x \) then \( \nu \) is said to be a free \( \mathbb{Z}_2 \)-action. The points \( x \in X \) and \( \nu(x) \) are called antipodal. A continuous map \( f : X \to Y \) between two \( \mathbb{Z}_2 \)-spaces \((X, \nu)\) and \((Y, \omega)\) is called a \( \mathbb{Z}_2 \)-map if it respects the \( \mathbb{Z}_2 \)-actions of both spaces, specifically if \( f \circ \nu = \omega \circ f \). If there exists a \( \mathbb{Z}_2 \)-map between two \( \mathbb{Z}_2 \)-spaces we write \( (X, \nu) \xrightarrow{\mathbb{Z}_2} (Y, \omega) \), and if no such map exists we write \( (X, \nu) \xleftarrow{\mathbb{Z}_2} (Y, \omega) \). We may sometimes use the topological space of the \( \mathbb{Z}_2 \)-space to denote the \( \mathbb{Z}_2 \)-space if the involution is clear from the context. We say that two \( \mathbb{Z}_2 \)-spaces \( X \) and \( Y \) are \( \mathbb{Z}_2 \)-equivalent if we have \( X \xrightarrow{\mathbb{Z}_2} Y \) and \( Y \xrightarrow{\mathbb{Z}_2} X \), and we write \( X \leftrightarrow \mathbb{Z}_2 Y \).

An important example of a free \( \mathbb{Z}_2 \)-space is the unit sphere \( S^n \). The set defining the topology of the unit sphere is the set

\[
S^n = \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \}
\]

the topology of this space is the subspace topology with \( \mathbb{R}^{n+1} \). Specifically this means that \( A \subseteq S^n \) is an open set of \( S^n \) if an only if there is some open set \( B \) of \( \mathbb{R}^{n+1} \) such that \( A = B \cap S^n \). The free \( \mathbb{Z}_2 \)-action on \( S^n \) is given by \( x \mapsto -x \) for \( x \in S^n \).

To use the structure of \( \mathbb{Z}_2 \)-spaces to derive lower bounds on the chromatic number of a graph we will consider two invariants of a \( \mathbb{Z}_2 \)-space, the \( \mathbb{Z}_2 \)-index and the \( \mathbb{Z}_2 \)-coindex. The \( \mathbb{Z}_2 \)-index of a \( \mathbb{Z}_2 \)-space \( X \), is defined as

\[
\text{ind}_{\mathbb{Z}_2}(X) := \min \{ n \in \mathbb{N} : X \xrightarrow{\mathbb{Z}_2} S^n \}
\]

And the \( \mathbb{Z}_2 \)-coindex of a \( \mathbb{Z}_2 \)-space \( X \), is defined as

\[
\text{coind}_{\mathbb{Z}_2}(X) := \max \{ n \in \mathbb{N} : S^n \xrightarrow{\mathbb{Z}_2} X \}
\]
These invariants will only be useful for free $\mathbb{Z}_2$-spaces though. Suppose that $(Y, \omega)$ is a $\mathbb{Z}_2$-space which is not free, and $y \in Y$ such that $y = \nu(y)$. Then $X \xrightarrow{\mathbb{Z}_2} Y$ holds for all $\mathbb{Z}_2$-spaces $(X, \nu)$. Explicitly we can just take the map mapping all of $X$ to $y$. This is a $\mathbb{Z}_2$-map as:

$$f(\nu(X)) = f(X) = y = \omega(y) = \omega(f(X))$$

Specifically this implies that $\mathbb{Z}_2$-coindex of any non-free $\mathbb{Z}_2$-space is infinite. This further implies that the $\mathbb{Z}_2$-index is infinite as the $\mathbb{Z}_2$-coindex of a $\mathbb{Z}_2$-space is a lower bound on the $\mathbb{Z}_2$-index, a fact which is now to be seen.

We will now show that $\mathbb{Z}_2$-index and the $\mathbb{Z}_2$-coindex are non-trivial invariants of $\mathbb{Z}_2$-spaces through the Borsuk-Ulam theorem. The Borsuk-Ulam theorem has many different expressions but the most common are the following.

**Theorem.** The following are equivalent expressions of the Borsuk-Ulam theorem:

1. For every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists some $x \in S^n$ such that $f(x) = f(-x)$

2. There is no antipodal map $f : S^n \to S^{n-1}$. (i.e. $S^n \xrightarrow{\mathbb{Z}_2} S^{n-1}$)

The Borsuk-Ulam theorem also gives us that $coind_{\mathbb{Z}_2}(X) \leq ind_{\mathbb{Z}_2}(X)$, as composing the two maps given by the index and coindex yields a $\mathbb{Z}_2$-map $f : S^{coind_{\mathbb{Z}_2}(X)} \xrightarrow{\mathbb{Z}_2} S^{ind_{\mathbb{Z}_2}(X)}$. By the Borsuk-Ulam theorem statement 2 the existence of such a $\mathbb{Z}_2$ map implies $coind_{\mathbb{Z}_2}(X) \leq ind_{\mathbb{Z}_2}(X)$. 


2.2 Simplicial Complexes

Simplicial complexes are structures which allow topological structures to be constructed combinatorially. We will use these combinatorial constructions to carry graphs to $\mathbb{Z}_2$-spaces. To do this we must first discuss some preliminary concepts.

The convex hull of a set $X = \{x_1, \ldots, x_n : x_i \in \mathbb{R}^m\}$ is the set of points

$$conv\{X\} = \{\alpha_1 x_1 + \ldots + \alpha_n x_n : \sum_{i=1}^n \alpha_i = 1, 0 \leq \alpha_i \leq 1\}$$

$X$ is said to be affinely dependent if there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that the following hold:

$$\sum_{i=1}^n \alpha_i = 0$$

$$\sum_{i=1}^n \alpha_i x_i = 0$$

If there is no such set of scalars then the set is said to be affinely independent. If we have a set of affinely independent points all points in the convex hull can be uniquely identified by the coefficients $\alpha_i$, these coefficients are called the barycentric coordinates. To see that barycentric coordinates are unique suppose that for $x \in conv\{X\}$ we have:

$$0 \leq \alpha_i, \beta_i \leq 1$$

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$$

$$x = \sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \beta_i x_i$$

This yields

$$\sum_{i=1}^n (\alpha_i - \beta_i) = 0$$
\[
\sum_{i=1}^{n} (\alpha_i - \beta_i) x_i = 0
\]
which implies that for all \(i\) that \(\alpha_i = \beta_i\) since \(X\) is assumed to be an affinely independent set.

An abstract simplicial complex is a pair \((V, K)\) where \(V\) is a set and \(K \subseteq 2^V\), and \(\sigma \in K\) implies that for all \(\sigma' \subseteq \sigma\) that \(\sigma' \in K\). The sets in \(K\) are called (abstract) simplices. We will write just the set system \(K\) to represent the simplicial complex, with the set \(V\) implicit. An abstract simplicial complex can be geometrically realized as a topological space by first selecting an affinely independent set of size \(#V\), call it \(X\). The set defining the topological space is the convex hull \(X\), with the topology being the subspace topology from \(\mathbb{R}^{#V-1}\). We write \(|K|\) to denote this geometric realization of \(K\). As each point of \(|K|\) is uniquely determined by its barycentric coordinates, a \(\mathbb{Z}_2\)-action defined on \(K\) can be uniquely extended from the vertices of \(K\) to \(|K|\) as follows. Let \(\nu\) be a \(\mathbb{Z}_2\)-action on \(K\), and

\[
x = \sum_{i=1}^{n} \alpha_i x_i \in \text{conv}(X)
\]

We extend \(\nu\) by defining

\[
|\nu|(x) = \sum_{i=1}^{n} \alpha_i \nu(x_i)
\]

Clearly if \(\nu\) is a \(\mathbb{Z}_2\)-action on \(K\), then \(|\nu| \circ |\nu| = id_{|K|}\), and hence \(|K|\) is a \(\mathbb{Z}_2\)-space.

Not only can \(\mathbb{Z}_2\)-spaces be induced by simplicial complexes, but \(\mathbb{Z}_2\)-maps may also be induced by special maps, called simplicial maps, between simplicial complexes. A set map \(f : V(K) \to V(L)\) between the vertex sets of two simplicial complexes \(K\) and \(L\), is called a simplicial map if for all simplices \(\sigma \in K\) we have that \(f(\sigma)\) is a simplex in \(L\). A simplicial map induces a continuous map \(|f| : |K| \to |L|\) by extending the map on the vertices to mapping the barycentric coordinates. Specifically for a
x ∈ ∥K∥ we define ∥f∥ : x = ∑\(i=1\)\(^n\) \(α_i x_i \mapsto ∑\(i=1\)\(^n\) \(α_i f(x_i)\). This map is well defined as the barycentric coordinates of a point in ∥K∥ uniquely determine that point. Further if there is a Z\(_2\)-action defined on two abstract simplicial complexes K and L, and the respective Z\(_2\) actions \(ν\) and \(ω\) respect a simplicial map \(f : V(K) → V(L)\), then ∥f∥ is a Z\(_2\)-map. \(∥f∥ \circ ∥ν∥(x) = ∥f∥ \circ ∥ν∥(∑\(i=1\)\(^n\) \(α_i x_i)) = ∑\(i=1\)\(^n\) \(α_i (f \circ ν)(x_i)) = ∑\(i=1\)\(^n\) \(α_i (ω \circ f)(x_i) = ∥ω∥ \circ ∥f∥(x)\)

Having defined simplicial complexes, we will now present some simplicial complexes induced by graphs, with associated Z\(_2\)-actions which will help derive lower bounds on the chromatic number of a graph. These simplicial complexes are the box complexes \(B_{\text{chain}}(G)\) and \(B_0(G)\), defined for a graph G. We will see that the lower bounds provided by these two complexes are better than the bound provided by the clique number of a graph, but there is no simple relation to the fractional chromatic number or the lower bound yielded from the independence number.

Before defining the Box Complex and the Hom Complex let A and B be sets, then we define \(A \uplus B := \{(a, 1) : a ∈ A\} \cup \{(b, 2) : b ∈ B\}\).

The Box Complex \(B_0(G)\) is a simplicial complex with vertex set \(V(G) \times \{1, 2\}\). For \(S, T ⊆ V(G)\), \(S \uplus T\) is a simplex of \(B_0(G)\) if and only if \(S \cap T = \emptyset\) and the subgraph induced by \(S \cup T\) is a complete bipartite graph with parts \(S\) and \(T\). We note that \(V(G) \uplus \emptyset\) and \(\emptyset \uplus V(G)\) are included in the simplices of \(B_0(G)\). The Box Complex also has a Z\(_2\)-action defined on it namely \(S \uplus T \mapsto T \uplus S\) for \(S \uplus T ∈ B_0(G)\). This is a well defined Z\(_2\)-action as \(S\) and \(T\) induce the same subgraph of \(G\) independently of their order, and hence \(S \uplus T\) is a simplex of \(B_0(G)\) if and only if \(T \uplus S\) is a simplex.

The simplicial complex \(B_{\text{chain}}(G)\) is a subcomplex of \(B_0(G)\), where:

\[B_0(G) \setminus B_{\text{chain}}(G) = \{S \uplus T ∈ B_0(G) : S = \emptyset \text{ or } T = \emptyset\}\]
In other words $B_{\text{chain}}(G)$ is the subcomplex where simplices are $S \cup T$ where $S$ and $T$ are parts of a complete bipartite graph, and both parts are non-empty. Observe that the restriction of $\mathbb{Z}_2$-action defined on $B_0(G)$ to $B_{\text{chain}}(G)$ is still a $\mathbb{Z}_2$-action. We can now define the Hom Space $H(G)$ as the geometric realization of $B_{\text{chain}}(G)$, and note that it is a $\mathbb{Z}_2$-space. The Hom Space also has a few other $\mathbb{Z}_2$-equivalent definitions. Matoušek and Ziegler [9] showed that $H(G)$ is $\mathbb{Z}_2$-equivalent to $B(G)$ another Box Complex topological construction.

An important property of the functors $B_0(\bullet)$ and $B_{\text{chain}}(\bullet)$ is that they map the complete graph to $\mathbb{Z}_2$-spaces for which there is a $\mathbb{Z}_2$-homeomorphism from the $\mathbb{Z}_2$-spaces to spheres. Specializing this implies the spaces are $\mathbb{Z}_2$-equivalent to spheres. Explicitly we have that $H(K_m) \xrightarrow{\mathbb{Z}_2} S^{m-2}$ and $\|B_0(K_m)\| \xrightarrow{\mathbb{Z}_2} S^{m-1}$ [13]. This allows us to use the Borsuk-Ulam theorem, which defines obstructions to the existence of $\mathbb{Z}_2$-maps into spheres to derive obstructions to the existence of graph homomorphisms into complete graphs, thereby providing a lower bound on the chromatic number of a graph. These obstructions will be constructed using knowledge of the $\mathbb{Z}_2$-index or $\mathbb{Z}_2$-coindex of $\|B_0(G)\|$ or $\|B_{\text{chain}}(G)\|$ for a graph $G$.

We will now define how $B_{\text{chain}}(\bullet)$ and $B_0(\bullet)$ act functorially. First let us consider the simplicial map induced by a graph homomorphism $f : G \to H$, for graphs $G$ and $H$, via $B_{\text{chain}}(\bullet)$. Let the induced map $B(f)$ be defined on the vertices of $B_{\text{chain}}(G)$ as follows. $B_{\text{chain}}(f)(v, i) \mapsto (f(v), i)$. To see that this is indeed a simplicial map observe that $A \cup B$ a simplex of $B_{\text{chain}}(G)$ implies that $A$ and $B$ are partite sets of a complete bipartite subgraph of $G$. So for $a \in A$ and $b \in B$ we have that $a \sim b$. The definition of graph homomorphism implies that then $f(a) \sim f(b)$, and hence $f(A) \cup f(B)$ is a simplex of $B_{\text{chain}}(H)$. The existence of a $\mathbb{Z}_2$-simplicial map implies
the existence of a $\mathbb{Z}_2$-map from $\|B_{\text{chain}}(G)\|$ to $\|B_{\text{chain}}(H)\|$.

The definition for $B_0(G)$ is similar, but we must also account for simplices $A \sqcup B$ with $A$ or $B$ the empty set. Assume that $A$ is empty, then $B$ may be arbitrary by the construction of $B_0(G)$ since $f(\emptyset) = \emptyset$ for set maps, $B_0(f)(A \sqcup B) := f(A) \sqcup f(B)$ is a simplex of $B_0(H)$.

We now see we can use knowledge use the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex to show that there is no $\mathbb{Z}_2$-map from a $\mathbb{Z}_2$-space $H(G)$ or $\|B_0(G)\|$. The fact that a graph homomorphism $f : G \to K_n$ for a graph $G$ induces $\mathbb{Z}_2$-maps $B_{\text{chain}}(f)$ and $B_0(f)$ implies that the non-existence of $\mathbb{Z}_2$-maps between $B_0(G)$ and $B_0(K_n)$ or $B_{\text{chain}}(G)$ and $B_{\text{chain}}(K_n)$ implies the non-existence of a graph homomorphism $h : G \to K_n$. Specifically knowledge of the $\mathbb{Z}_2$-index of $B_0(G)$ and $B_{\text{chain}}(G)$ implies the non-existence of a graph homomorphism $f : G \to K_{\text{ind}_{\mathbb{Z}_2}(H(G)) + 1}$ and $f : G \to K_{\text{ind}_{\mathbb{Z}_2}(\|B_0(G)\|)}$, due to the fact that $H(K_m) \xrightarrow{\mathbb{Z}_2} S^{m-2}$ and $\|B_0(K_m)\| \xrightarrow{\mathbb{Z}_2} S^{m-1}$. This completes the demonstration that these invariants are lower bounds on $\chi(G)$. More comprehensively the following inequalities hold [13]:

$$\chi(G) \geq \text{ind}_{\mathbb{Z}_2}(H(G)) + 2 \geq \text{ind}_{\mathbb{Z}_2}(\|B_0(G)\|) + 1$$

$$\geq \text{coind}_{\mathbb{Z}_2}(\|B_0(G)\|) + 1 \geq \text{coind}_{\mathbb{Z}_2}(H(G)) + 2 \quad (2.1)$$

These inequalities show that the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex of $\|B_0(G)\|$ and $H(G)$ provide lower bounds on the chromatic number. For convenience we say that a graph $G$ is topologically $t$-chromatic if:

$$\text{coind}_{\mathbb{Z}_2}(\|B_0(G)\|) \geq t - 1$$
And we say that a graph $G$ is strongly topologically $t$-chromatic if:

$$\text{coind}_{\mathbb{Z}_2}(H(G)) \geq t - 2$$

The fact that a graph homomorphism induces $\mathbb{Z}_2$-maps implies that these topological constructions provide better lower bounds on the chromatic number than the clique number. The graph homomorphism $f : K_{\omega(G)} \to G$ induces a $\mathbb{Z}_2$-map $B_0(f) : S^{\omega(G)-2} \to H(G)$, implying that $\text{coind}_{\mathbb{Z}_2}(H(G)) + 2 \geq \omega(G)$, as desired. The inequalities in equation 2.1 further imply that $\text{coind}_{\mathbb{Z}_2}(|B_0(G)|) + 1 \geq \omega(G)$.

While the bounds derived from the topological constructions which have been discussed so far may be better lower bounds than those derived from elementary properties of graphs, decidability is unknown for the problem of computing the index of a $\mathbb{Z}_2$-space, or even for the geometric realizations of Box Complexes in particular. However in [15] explicit algorithms were presented which compute the homotopy equivalence classes of continuous maps between two simplicial complexes. The hope is that such an algorithm may lead to algorithms for computing the $\mathbb{Z}_2$-index of simplicial complexes with a $\mathbb{Z}_2$-action defined on them.

Despite the current difficulty of computing the $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex, there are other lower bounds derived from topological invariants which are known to be computable. The Hom Space $H(G)$ can also be defined as the geometric realization of the Hom complex $Hom(K_2, G)$. Koslov [5] discussed another topological invariant of the Hom complex $Hom(K_2, G)$, which yields lower bounds on the chromatic number of graphs, the so-called Stiefel-Whitney height, $h(Hom(K_2, G))$. We will not discuss any of the details of this invariant however it is worth noting that for any $\mathbb{Z}_2$-space, $X h(X)$ is computable. Further we have that $\text{coind}_{\mathbb{Z}_2}(X) \leq h(X) \leq ind_{\mathbb{Z}_2}(X)$. Thus the Stiefel-Whitney height yields a better lower bound than the strong topological
chromatic number, but can be no better that the bounds derived from the knowledge of the $\mathbb{Z}_2$-index of $\|H(G)\|$.

While the difficulty of computation of these topological invariants is high, they may yield poor lower bounds, as did the elementary methods. This is partly due to the fact that any graph $G$ without 4-cycles satisfies $\text{ind}_{\mathbb{Z}_2}(H(G)) + 2 \leq 3$ [13]. This implies that for all of topologically based bounds discussed so far that the absence of a 4-cycle upper bounds the lower bound that they provide. This fact along with Erdős’ result [2], that for any $n, k \in \mathbb{Z}$, there is some graph $G_{n,k}$ such that $\chi(G) \geq k$ and $G$ contains no $m$-cycles where $m < n$. Specifically this result shows that the bound given by $\mathbb{Z}_2$-index and $\mathbb{Z}_2$-coindex may be arbitrarily loose, as there are graphs with arbitrarily high chromatic number and no 4-cycles.

### 2.3 Examples

We define the Borsuk Graph $B_{n,\epsilon}$ of parameters $n$ and $0 < \epsilon < 1$. $B_{n,\epsilon}$ is the infinite graph whose vertices are the points of the $(n-2)$-sphere $S^{n-2}$, and whose edges join the pair of points $x, y$ such that $\|x - y\| \geq 2 - \epsilon$, where $\|v\|$ is the norm of $v$. Erdős and Hajnal used the Borsuk-Ulam theorem to show that $\chi(B_{n,\epsilon}) = n$, and showed that this equality is actually equivalent to the Borsuk-Ulam theorem.

The Borsuk graph allows a purely graph theoretic characterization of strong topological chromatic number. Simonyi and Tardos showed in [12] for any graph $G$, $\text{coind}_{\mathbb{Z}_2}(H(G)) + 2 \geq n$ if and only if there is some $\epsilon > 0$ for which $B_{n,\epsilon}$ admits a graph homomorphism to $G$.

In order to provide a graph theoretic characterization of strong topological chromatic number using only finite graphs we must first define the family of graphs $K_k$. 
Let $K_2 := \{K_2\}$, for $k \geq 3$ let

$$K_k := \{M_n(G) : G \in K_{k-1}, n \in \{1, 2, 3, \ldots\}\}$$

It is also known that $\chi(G) = k$ for $G \in K_k$ [11]. Using the characterization of strong topological chromatic number through Borsuk graphs Simons, Tardif, and Wehlau [11] showed that there exists a graph homomorphism $f : G \to H$ from $G \in K_n$ if and only if $\text{coind}_{Z_2}(B(H)) + 2 \geq n$.

The Generalized Mycielskian has another interesting property relating to topological chromatic number. For a graph $G$ it holds that

$$\text{coind}_{Z_2}(|B_0(M_n(G))|) \leq \text{coind}_{Z_2}(|B_0(G)|) + 1$$

showing that repeated application of the Generalized Mycielskian construction yields bounded growth on the topological chromatic number[13].

### 2.4 The Algebraic Bounds on Chromatic Number

Simons, Tardif, and Wehlau [11] gave the following definitions to yield the forthcoming algebraic bound on chromatic number. Let $H$ be a graph. Let $A(H)$ denote the set of arcs of the graph $H$. Specifically for each edge $\{u, v\} \in E(H)$ we have that $A(H)$ contains the pairs $(u, v)$ and $(v, u)$. Let $\mathcal{F}(A(H)) = \mathbb{Z}^{A(H)}$, the free abelian group with basis $A(H)$. Now let $\theta$ be the subgroup of $\mathcal{F}(A(H))$ generated by the set:

$$\{(u, v) - (w, v) + (w, x) - (u, x) \in \mathcal{F}(A(H)) : (u, v, w, x) \text{ is a 4-cycle of } H\}$$

Let $\mathcal{G}(H) = \mathcal{F}(H)/\theta$, and let $f : C_{2n+1} \to H$ be a graph homomorphism from an odd cycle graph $C_{2n+1}$ to $H$, where

$$V(C_{2n+1}) = \mathbb{Z}_{2n+1}$$
E(C_{2n+1}) = \{\{\bar{u}, \bar{v}\} : \bar{u} \in \{\bar{v} + 1, \bar{v} - 1\}\}

where \( \bar{k} \in \mathbb{Z}_{2n+1} \) is the equivalence class of \( k \) in \( \mathbb{Z}_{2n+1} \). We now define the \( G(H) \)-signature of \( f \) to be:

\[
\sigma_{G(H)}(f) := \left( \sum_{i=0}^{2n} (f(2i), f(2i+1)) - (f(2i+2), f(2i+1)) \right)/\theta
\]

Simons, Tardif, and Wehlau [11] showed the following bounds. For a graph \( H \), if there exists an odd cycle graph \( C_{2n+1} \) and a graph homomorphism \( f : C_{2n+1} \to H \) such that \( \sigma_{G(H)}(f) = 0_{G(H)} \) then \( \chi(H) \geq 4 \), and if no such homomorphism exists then \( \text{coind}_{\mathbb{Z}_2}(H(H)) + 2 \leq 3 \). We make the observation that if \( H \) has no 4-cycle then no such graph homomorphism exists, so the hurdle presented for the \( \mathbb{Z}_2 \)-index lower bounds for the chromatic number of a graph and the Stiefel-Whitney bounds are not surpassed by this method.

Further Simons, Tardif, and Wehlau give the following system of equations with terms in \( \mathbb{Z} \) and \( G(H) \), for which there is a solution if and only if the algebraic conditions above are satisfied. Let \( H \) be a connected graph, with variables \( X_{u,v} \in \mathbb{Z} \) for \( (u,v) \in A(H) \) and a positive integer \( N \).

\[
\sum_{v \in N_H(u)} X_{u,v} - X_{v,u} = 0 \text{ for all } u \in V(H)
\]

\[
\sum_{(u,v) \in A(H)} X_{u,v} - 2N = 1
\]

\[
\left( \sum_{(u,v) \in A(H)} (X_{u,v} - X_{v,u}) \cdot (u,v) \right)/\theta = 0_{G(H)}
\]

Simons, Tardif, and Wehlau also define the structure \( G^*(H) \) given a graph \( H \). This will be an object of study in this paper. Let \( F^*(A(H)) \) be the free group freely generated by \( A(H) \). Now let \( \theta^* \) be the normal subgroup of \( F^*(A(H)) \) generated by
the set
\[ \{(u,v)(w,v)^{-1}(w,x)(u,x)^{-1} \in F^*(A(H)) : (u,v,w,x) \text{ is a 4-cycle of } H\} \]

We now define:
\[ G^*(H) := F^*(A(H))/\theta^* \]

This object can also be used to define \( G(H) \). Observe that \( G(H) \) is the quotient of \( G^*(H) \) with its commutator subgroup.
Chapter 3

The Graph U(5,3)

3.1 Definition

The graph $U(5,3)$ is a graph with a few interesting properties, it provides a concrete example which allows us to investigate some relationships between the various lower bounds on the chromatic number of graphs discussed so far.

Local chromatic number, written $\psi(G)$, is defined as

$$\psi(G) := \min_{c} \max_{v \in V(G)} \left| \{c(u) : u \in N(v)\} \right| + 1$$

where the minimum is taken over all proper colorings $c$ of $G$, and $N(v)$ is the set of all neighbors of $v$ in $G$, or the set of all vertices adjacent to $v$.

The graph $U(m,r)$ is defined as follows:

$$V(U(m,r)) := \{(i, A) : i \in [m], A \in \binom{[m]}{r-1}, i \notin A\}$$

$$E(U(m,r)) := \{\{(i, A), (i', A')\} : i \in A', i' \in A\}$$

The family of graphs $U(m,r)$ arise in the context of defining local chromatic
number via graph homomorphisms. Specifically the family of graphs $U(m, r)$ have
the property that, for a graph $G$, there is a homomorphism from $G$ to $U(m, r)$ if and
only if $\psi(G) \leq r$ and this local coloring can be attained with at most $m$ colors [3].

To provide a greater description of the graphs $U(m, r)$ we now present the number
of vertices and edges of $U(m, r)$, which can be found through combinatorial argu-
ments. To count the number of vertices, $(i, A)$, of $U(m, r)$ first observe that there are
$m$ choices for the first coordinate $i$ of a vertex, and having chosen the first coordinate,
there are $\binom{m-1}{r-1}$ choices for the second coordinate set, as we are choosing a set of size $r$
from $[m] \setminus \{i\}$. To count the number of edges we observe that every vertex in $U(m, r)$
has the same number incident edges, and we will apply the handshake lemma [16],
which states that the number of edges in a graph is one-half the sum of the number
of incident edges to each vertex in the graph. To count the number of edges incident
to a vertex $(i, A)$, we will count the number of vertices $(i', A')$ adjacent to $(i, A)$. We
note that there are $r - 1$ choices for $i'$ of an adjacent vertex to $(i, A)$, as there are
$r - 1$ distinct elements in $A$. Having chosen a first coordinate there are $\binom{m-2}{r-2}$
choices for the $A'$ set as we are looking for a subset of $[m] \setminus i'$ of size $r - 1$ which contains
$i$. This shows that the number of incident edges to each vertex is $(r - 1)\binom{m-2}{r-2}$. To
summarize we have that:

$$
\#V(U(m, r)) = m \binom{m-1}{r-1}
$$

$$
\#E(U(m, r)) = \frac{1}{2} m(r - 1) \binom{m-1}{r-1} \binom{m-2}{r-2}
$$
3.2 Discussion

Simonyi et. al. [13] showed that $U(2r + 1, r + 1)$ is a topologically $2r$-chromatic graph, but is not strong topologically $2r$-chromatic [13]. This property shows that strong topological chromaticity and topological chromaticity provide different lower bounds on the chromatic number of a graph.

Specializing the result of [13] we have that $U(5, 3)$ is topologically 4-chromatic, but strong topologically 3-chromatic. The algebraic conditions presented in Section 2.4 find that the graph is at least four chromatic, in that a solution was found to the system of equations. This provides an example where the algebraic conditions discussed in the previous section provide better bounds than strong topological chromatic number.

**Proposition 1.** The fractional chromatic number of $U(5, 3)$ is three.

*Proof.* The result follows by first claiming that $U(5, 3)$ is vertex transitive, the independence number of $U(5, 3)$ is ten, and using the theorem that the fractional chromatic number of a vertex transitive graph is graph is $\#V(G)/\alpha(G)$ [4]. Specializing to $U(5, 3)$ this implies that $\chi_f(U(5, 3)) = \#V(U(5, 3))/10 = 3$.

To see that $U(5, 3)$ is vertex transitive we note that any permutation $\pi$ on [5] induces a graph automorphism on $U(5, 3)$. To induce a graph automorphism we take the map $\hat{\pi} : (i, \{j, k\}) \mapsto (\pi(i), \{\pi(j), \pi(k)\})$. Clearly $\hat{\pi} \circ \hat{\pi}^{-1} = \hat{\pi}^{-1} \circ \hat{\pi} = id$, and hence $\hat{\pi}$ is a bijection. It is also clear that $\hat{\pi}$ is a graph homomorphism, and hence is a graph automorphism. Now for any two vertices $(i, \{j, k\})$ and $(a, \{b, c\})$ in $U(5, 3)$, there is a permutation $\pi$ for which $\pi(i) = a$, $\pi(j) = b$, and $\pi(k) = c$. This is due to the fact that $i, j, k$ are distinct and $a, b, c$ are distinct, and hence we have thus far
defined an injective partial function which can be extended to a permutation. We
now have that $U(5, 3)$ is vertex transitive.

To see that $\alpha(U(5, 3)) = 10$ we present an independent set of size ten, and an
argument which shows that there can be no independent set of size greater than ten
in $U(5, 3)$. Observe that the following is an independent set of size ten:

1. $(3, \{1, 5\})$
2. $(4, \{2, 5\})$
3. $(4, \{1, 5\})$
4. $(3, \{1, 2\})$
5. $(3, \{1, 4\})$
6. $(3, \{2, 5\})$
7. $(5, \{1, 2\})$
8. $(3, \{2, 4\})$
9. $(4, \{1, 2\})$
10. $(3, \{4, 5\})$

Further there can be no independent set of a larger size as the vertices of $U(5, 3)$
may be partitioned into sets of size three, where each partite set consists of three
vertices whose induced subgraph from $U(5, 3)$ is $K_3$. Each partite set is as follows. For
$i, j, k \in [5]$ and $i, j, k$ distinct, then $(i, \{j, k\}), (j, \{i, k\}), (k, \{i, j\})$ form a clique, by the
definition of $U(5, 3)$. This partition shows that any independent set of $U(5, 3)$ must
contain at most one vertex from each set in this partition, and hence $\alpha(U(5, 3)) \leq
\#V(U(5, 3))/3 = 10$. Further here is an independent set of size 10, which shows 10
is a lower as well as an upper bound on the independence number of $U(5, 3)$, so the
theorem from Godsil and Royle completes the proof that the fractional chromatic
number of $U(5, 3)$ is 3.

To the best knowledge of the author there is no known closed form expression for
the independence number, the clique number, or the fractional chromatic number of
a graph $U(m, r)$. However knowledge of the independence number, as observed in
the previous proposition, would also provide knowledge of the fractional chromatic
number, and the reverse is true as well.
We now have that the fractional chromatic number is also a strictly worse bound than the bound yielded from the algebraic bound in Section 2.4, in the case of $U(5,3)$.

A question which we tried to address was whether more could be inferred from the algebraic conditions presented in Section 2.4 if the condition $\sigma_{G(H)}(f) = 0_{G(H)}$ for some graph homomorphism $f$ from an odd cycle to $G$ is strengthened to the condition that $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$. Determining whether this was the case for any graphs however proves difficult in general. This follows from the fact that determining if $\sigma_{G^*(H)}(f) = 1_{G^*(H)}$ amounts to solving the word problem for the finitely presented group $G^*(H)$. Solving the word problem for an arbitrary finitely presented group is an uncomputable problem. We were however able to find some graphs for which the word problem is solvable. Trivially for any graph $H$ without a 4-cycle, the word problem for $G^*(H)$ is solvable as $G^*(H)$ is a free group. Another example was determined using the Magma computer algebra package [1] we studied simplified presentations of $G^*(U(5,3))$ and found that $G^*(U(5,3))$ has a presentation as a group with 71 generators and a single relation. (For the explicit presentation, and isomorphism with the presentation from the definition of $G^*(U(5,3))$ see Appendix A). Such groups are called one-relator groups. Magnus proved that the word problem for one-relator groups is decidable [7], and thus the word problem for $G^*(U(5,3))$ is decidable. Additionally $G(H)$ is a free abelian group of rank 71, and hence the generating set of $G^*(H)$, which is of size 71 is also a generating set and hence a basis of $G(H)$, as it is a generating set with size equal to the rank of $G(H)$.
Chapter 4

Conclusion

4.1 Contributions

In this paper we have provided examples of graphs where the algebraic conditions presented in Section 2.4 provide a lower bound of the chromatic number is better than the bound provided by fractional chromatic number and strong topological chromatic number. Hopefully this will help further motivate the study of these algebraic conditions as useful computable lower bounds on the chromatic number of graphs. We have demonstrated $G(U(5, 3))$ is a non-trivial example of a graph for which the word problem is decidable.

4.2 Further Questions

We have presented examples where the algebraic conditions presented in Section 2.4 provide better lower bounds to the chromatic number of a graph than some of the lower bounds presented earlier in the paper, a further area of study is to investigate
whether there is a relationship with any of the other lower bounds, namely the $\mathbb{Z}_2$-index of $B_{\text{chain}}(G)$ and $B_0(G)$ for any graph $G$, or the Stiefel-Whitney height of $H(K_2,G)$. Another avenue of investigation is into possible strengthenings of the algebraic conditions presented in Section 2.4. Specifically whether more information may be inferred if the condition $\sigma_{G\langle H \rangle}(f) = 0$ for some graph homomorphism $f$ from an odd cycle to $G$ is strengthened to the condition that $\sigma_{G^\ast\langle H \rangle}(f) = 1$. In answering this question it may also be necessary to investigate the decidability of the word problem for the group $G^\ast(G)$ for all graphs $G$. If this is the case and there exists efficient algorithm for solving the word problem would aid in the study of the strengthening of the algebraic considerations.
Bibliography


Appendix A

$U(5, 3)$ Details

The following is a list of 71 elements of $A(H)$, and a relation in $G^*(U(5, 3))$. These elements form a basis for $G(U(5, 3))$, and these elements along with the relation form a presentation of $G^*(U(5, 3))$. The set and relation were produced by the Magma [1] code in Appendix B.

- $((3, \{1, 5\}), (5, \{1, 3\}))$
- $((5, \{1, 3\}), (3, \{1, 5\}))$
- $((3, \{1, 5\}), (1, \{3, 5\}))$
- $((1, \{3, 5\}), (3, \{1, 5\}))$
- $((3, \{1, 5\}), (5, \{2, 3\}))$
- $((5, \{2, 3\}), (3, \{1, 5\}))$
- $((3, \{1, 5\}), (5, \{3, 4\}))$
- $((5, \{3, 4\}), (3, \{1, 5\}))$
- $((5, \{3, 4\}), (3, \{1, 5\}))$
- $((1, \{3, 4\}), (3, \{1, 5\}))$
• $(5, \{1, 3\}), (3, \{2, 5\})$

• $(3, \{2, 5\}), (5, \{1, 3\})$

• $(5, \{1, 3\}), (3, \{4, 5\})$

• $(3, \{4, 5\}), (5, \{1, 3\})$

• $(5, \{1, 3\}), (1, \{2, 5\})$

• $(1, \{2, 5\}), (5, \{1, 3\})$

• $(1, \{3, 5\}), (5, \{1, 4\})$

• $(5, \{1, 4\}), (1, \{3, 5\})$

• $(1, \{3, 5\}), (3, \{1, 2\})$

• $(3, \{1, 2\}), (1, \{3, 5\})$

• $(1, \{3, 5\}), (3, \{1, 4\})$

• $(3, \{1, 4\}), (1, \{3, 5\})$

• $(1, \{3, 5\}), (5, \{1, 2\})$

• $(5, \{1, 2\}), (1, \{3, 5\})$

• $(4, \{1, 5\}), (5, \{1, 4\})$

• $(5, \{1, 4\}), (4, \{1, 5\})$

• $(4, \{1, 5\}), (5, \{2, 4\})$

• $(5, \{2, 4\}), (4, \{1, 5\})$

• $(5, \{1, 5\}), (5, \{2, 4\})$

• $(5, \{2, 4\}), (4, \{1, 5\})$

• $(5, \{1, 5\}), (4, \{1, 5\})$

• $(5, \{2, 4\}), (4, \{1, 5\})$

• $(5, \{1, 5\}), (1, \{2, 5\})$

• $(1, \{2, 5\}), (5, \{1, 5\})$

• $(5, \{1, 5\}), (4, \{1, 5\})$

• $(5, \{2, 4\}), (4, \{1, 5\})$

• $(5, \{1, 5\}), (1, \{2, 5\})$

• $(1, \{2, 5\}), (5, \{1, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$

• $(5, \{1, 4\}), (4, \{3, 5\})$
\begin{itemize}
\item $(5, \{1, 2\}), (2, \{3, 5\})$
\item $(3, \{2, 4\}), (4, \{2, 3\})$
\item $(4, \{2, 3\}), (3, \{2, 4\})$
\item $(3, \{2, 4\}), (4, \{3, 5\})$
\item $(4, \{3, 5\}), (3, \{2, 4\})$
\item $(4, \{1, 3\}), (3, \{2, 4\})$
\item $(4, \{2, 3\}), (2, \{3, 4\})$
\item $(2, \{3, 4\}), (4, \{2, 3\})$
\item $(2, \{4, 5\}), (4, \{2, 3\})$
\item $(4, \{2, 3\}), (2, \{1, 4\})$
\end{itemize}

As noted above $\mathcal{G} \ast (U(5, 3))$ is a one relator group, its single relation is:

\[(5, \{1, 4\}), (4, \{1, 5\})^{-1} \ast (1, \{3, 4\}), (4, \{1, 5\}) \ast (4, \{1, 2\}), (1, \{4, 5\})^{-1} \ast (1, \{4, 5\}), (4, \{1, 3\}) \ast \]

\[(5, \{1, 4\}), (4, \{2, 5\})^{-1} \ast (5, \{2, 3\}), (3, \{1, 5\}) \ast (1, \{3, 5\}), (3, \{1, 5\})^{-1} \ast (1, \{3, 5\}), (5, \{1, 4\}) \ast \]

\[(4, \{1, 5\}), (1, \{2, 4\})^{-1} \ast (2, \{1, 3\}), (3, \{2, 5\}) \ast (5, \{1, 3\}), (3, \{4, 5\})^{-1} \ast (5, \{1, 3\}), (3, \{1, 5\}) \ast \]

\[(5, \{2, 3\}), (3, \{1, 5\})^{-1} \ast (5, \{1, 4\}), (4, \{2, 5\}) \ast (1, \{4, 5\}), (5, \{1, 2\})^{-1} \ast (5, \{1, 2\}), (1, \{3, 5\}) \ast \]

\[(5, \{1, 3\}), (1, \{3, 5\})^{-1} \ast (5, \{1, 3\}), (1, \{4, 5\}) \ast (4, \{1, 5\}), (1, \{4, 5\})^{-1} \ast (4, \{1, 5\}), (5, \{3, 4\}) \ast \]

\[(5, \{1, 3\}), (5, \{3, 4\})^{-1} \ast (3, \{1, 5\}), (1, \{3, 5\}) \ast (5, \{1, 2\}), (1, \{3, 5\})^{-1} \ast (1, \{4, 5\}), (5, \{1, 2\}) \ast \]

\[(1, \{4, 5\}), (5, \{1, 4\})^{-1} \ast (3, \{2, 5\}), (5, \{1, 3\}) \ast (3, \{1, 5\}), (5, \{1, 3\})^{-1} \ast (3, \{1, 5\}), (5, \{3, 4\}) \ast \]

\[(4, \{1, 5\}), (5, \{3, 4\})^{-1} \ast (4, \{1, 5\}), (5, \{1, 4\}) \ast (1, \{3, 5\}), (5, \{1, 4\})^{-1} \ast (1, \{3, 5\}), (5, \{1, 3\}) \ast \]

\[(3, \{2, 5\}), (5, \{1, 3\})^{-1} \ast (1, \{4, 5\}), (5, \{1, 4\}) \ast (4, \{1, 5\}), (4, \{1, 3\})^{-1} \ast (4, \{1, 2\}), (1, \{4, 5\}) \ast \]

\[(1, \{3, 4\}), (4, \{1, 5\})^{-1} \ast (5, \{1, 4\}), (1, \{3, 5\}) \ast (3, \{1, 5\}), (1, \{3, 5\})^{-1} \ast (3, \{1, 5\}), (1, \{2, 3\}) \ast \]
((2, \{3, 5\}), (5, \{2, 3\}))^{-1} * ((5, \{2, 3\}), (2, \{4, 5\})) * ((5, \{1, 3\}), (1, \{2, 5\}))^{-1} * ((5, \{1, 3\}), (1, \{3, 5\})) *
((5, \{1, 4\}), (1, \{3, 5\}))^{-1} * ((5, \{1, 4\}), (4, \{1, 5\})) * ((5, \{3, 4\}), (4, \{1, 5\}))^{-1} * ((5, \{3, 4\}), (3, \{1, 5\})) *
((5, \{1, 3\}), (3, \{1, 5\}))^{-1} * ((5, \{1, 3\}), (1, \{2, 5\})) * ((5, \{2, 3\}), (2, \{4, 5\}))^{-1} * ((2, \{3, 5\}), (5, \{2, 3\})) *
((3, \{1, 5\}), (1, \{2, 3\}))^{-1} * ((3, \{1, 5\}), (5, \{2, 3\})) * ((4, \{2, 5\}), (5, \{1, 4\}))^{-1} * ((1, \{2, 5\}), (5, \{1, 4\})) *
((3, \{2, 5\}), (5, \{2, 3\}))^{-1} * ((4, \{1, 5\}), (1, \{2, 4\})) * ((4, \{1, 5\}), (5, \{1, 4\}))^{-1} * ((4, \{1, 5\}), (1, \{4, 5\})) *
((5, \{1, 3\}), (1, \{4, 5\}))^{-1} * ((5, \{1, 3\}), (3, \{4, 5\})) * ((2, \{1, 3\}), (3, \{2, 5\}))^{-1} * ((3, \{2, 5\}), (5, \{2, 3\})) *
((1, \{2, 5\}), (5, \{1, 4\}))^{-1} * ((4, \{1, 3\}), (1, \{4, 5\})) * ((1, \{3, 5\}), (5, \{1, 2\}))^{-1} * ((1, \{3, 5\}), (3, \{1, 5\})) *
((5, \{3, 4\}), (3, \{1, 5\}))^{-1} * ((5, \{3, 4\}), (4, \{1, 5\})) * ((1, \{4, 5\}), (4, \{1, 5\}))^{-1} * ((1, \{4, 5\}), (5, \{1, 3\})) *
((1, \{3, 5\}), (5, \{1, 3\}))^{-1} * ((1, \{3, 5\}), (5, \{1, 2\})) * ((4, \{1, 3\}), (1, \{4, 5\}))^{-1} * ((4, \{2, 5\}), (5, \{1, 4\})) *
((3, \{1, 5\}), (5, \{2, 3\}))^{-1} * ((3, \{1, 5\}), (5, \{1, 3\})) * ((1, \{4, 5\}), (5, \{1, 3\}))^{-1} * ((1, \{4, 5\}), (4, \{1, 5\})) =
1
Appendix B

Magma Code

function GetGraphU(m, r)
local M, S, V, E, A, B, v, w;
M := {1..m};
S := Subsets(M, r-1);
V := {<i, A> : i in M, A in S | i notin A};
return Graph<V|E>;
end function;

edge_index := function(e,EG)
local i;
for i:=1 to #EG do
    if EG.i eq e then return i; end if;
end for;
return 0; //e notin EG;
end function;

arc_index := function(u,v,EG); //u and v are vertices of G
    if u lt v then return 2*edge_index(EG!{u,v},EG) - 1;
    else return 2*edge_index(EG!{u,v},EG); end if;
end function;

find_4_cycles := function(G)
local u,v,x,y,e1,e2,four_cycles;
  four_cycles := [];
  for e1 in Edges(G) do
    u := Minimum(EndVertices(e1));
    v := Maximum(EndVertices(e1));
    for y in Neighbours(u) do
      if y gt v then for x in Neighbours(y) meet Neighbours(v) do
        if x gt u then Append(~four_cycles,[u,v,x,y]); end if;
      // u < v < y and u = Min{u,v,x,y}
        end for;
      end if;
    end for;
  end for;
  return four_cycles;
end function;

FreeGroupQuotient := function(G)
local EG, F, four_cycles, rels, e1, e2, e3, e4, Fquo, psi;
  EG := EdgeSet(G); //edge set of G as an enumerated set
  F := FreeGroup(2*#EG);
  four_cycles := find_4_cycles(G);
  rels := [];
  for quad in four_cycles do //quad := {u,v,x,y Θ}
    e1 := arc_index(quad[1],quad[2],EG); // (u -> v)
    e2 := arc_index(quad[3],quad[2],EG); // -(x -> v)
    e3 := arc_index(quad[3],quad[4],EG); // (x -> y)
    e4 := arc_index(quad[1],quad[4],EG); // -(u -> y)
    Append(~rels,F.e1 * F.e2^-1 * F.e3 * F.e4^-1);
    e1 := arc_index(quad[2],quad[3],EG); // (v -> x)
    e2 := arc_index(quad[4],quad[3],EG); // -(y -> x)
    e3 := arc_index(quad[4],quad[1],EG); // (y -> u)
    e4 := arc_index(quad[2],quad[1],EG); // -(v -> u)
    Append(~rels,F.e1 * F.e2^-1 * F.e3 * F.e4^-1);
  end for;
  Fquo, psi := quo <F|rels>;
  return F, Fquo, psi;
end function;
LatexString := function(x)
string := Sprint(x);
E := Eltseq(string);

for i in [1..#E] do
  if E[i] eq "{" then
    E[i] := ";";
  elif E[i] eq "}" then
    E[i] := "}";
  elif E[i] eq "<" then
    E[i] := "(";
  elif E[i] eq ">" then
    E[i] := ")";
  end if;
end for;
return &cat E;
end function;

// code to produce Basis and relation of G*(U(5,3))

U := GetGraphU(5,3);
freeGroup, F, psi := FreeGroupQuotient(U);

//f is an isomorphism between F and S
S,f := Simplify(F: Preserve:=[1..38]);

f_inv := Inverse(f);

// recover the arcs whose images generate G*(U(5,3))
SetOutputFile("Basis.txt": Overwrite := true);
print "Basis:";
basis := [];
for k in [1..71] do
  arcIndex := GeneratorNumber(f_inv(S.k));
basisElement := InverseArcIndex(U, arcIndex);
basis := basis cat [basisElement];
print LatexString(Sprint(basisElement));
end for;

R := Relations(S)[1];
E := Eltseq(LHS(R));

relation := [];
for e in E do
    arcIndex := Abs(e);
    relation cat:= [LatexString(Sprint(InverseArcIndex(U, arcIndex)))];
    if e lt 0 then
        relation cat:= ["^{-1}"];
    end if;
    relation cat:= ["*"];
end for;

Prune(~relation);
relation cat:= ["= 1"]; string := &cat relation;
print string;
UnsetOutputFile();