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EXTRUDING PLASTIC PIPE FROM ECCENTRIC DIES

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ABSTRACT

To solidify plastic pipe, the pipe is transported through a long cooling chamber. Inside this chamber, inside the pipe, the plastic remains molten, and this inner surface solidifies last. The flow due to the self-weight of the molten plastic then causes the product to thicken on bottom (and to thin on top). This is why plastic pipe is normally extruded from an eccentric die, and specifically, from a die where the mandrel (also called centerpiece [61] or core [12]) is decentered downward. This paper focuses on the consequences of this decentering in eccentric cylindrical coordinates. Specifically, when the molten polymer is viscoelastic, as is normally the case, ~~an upward~~ lateral force, F_x , is exerted on the mandrel. The die eccentricity also affects the positive axial force on the mandrel, F_z . These forces govern how rigidly the mandrel must be attached (normally, on a *spider die*, by eight bolts). We use the method of Jones (1964), called *polymer process partitioning*, designed for the Oldroyd 8-constant constitutive model. We produce a method for estimating F_x and F_z . We also obtain an expression for the shape of the extruded pipe, whose thickness scales with average velocity at each angular positions $\langle \tilde{v}_z \rangle_\theta$, by integrating the velocity profile, $\tilde{v}_z(\xi, \theta)$ through the thickness of the eccentric annulus (with respect to ξ). We further include expressions for the stresses in the extruded polymer melt. These expressions can be used to estimate the upper-bound for the stress that is frozen into the outermost layer of the plastic pipe (since this layer is quenched first, shortly after extrusion). We include detailed dimensional worked examples to help process engineers with their pipe die designs.

Keywords: plastic pipe; eccentric annular flow; extrusion die design; straw extrusion; catheter extrusion.

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I. INTRODUCTION

When plastic pipe is solidified, it proceeds through a long cooling chamber. Inside the chamber, inside the pipe, the plastic is molten, and the inner surface solidifies last. If the pipe is extruded from a concentric die, the flow due to the self-weight of the molten plastic causes the product to thicken on the bottom (and to thin at the top). This is why plastic pipe is normally extruded from an eccentric die, and specifically, from a die where the mandrel is below its centered position [1,2,3,4,5,6,7,8,9,10,11] (see this illustrated in **Figure 1** and **Figure 2**). We call this pipe extrusion practice *sag compensation*.

This paper focuses on the consequences of this decentering. Specifically, when the molten polymer is viscoelastic, as is normally the case, an upward lateral force, F_x , is exerted on the mandrel. The die eccentricity also affects the positive axial force on the mandrel, F_z (see **Figure 1** through **Figure 4**, **Table III** and **Table IV** for symbol definitions, both dimensional and dimensionless; see also **Table II**). These forces govern the required rigidity for the mandrel attachment (on a *spider die*, normally a set of eight bolts).

To compensate for sag, engineers must decenter the mandrel laboriously *by trial*. This paper can help shorten these trials, by providing an equations [Eq. (126) with Eqs. (124)-(125)] for predicting the pipe shape at the cooling chamber entrance.

John R. Jones of Wales [12] developed a method for solving polymer processing problems, specifically for the Oldroyd 8-constant fluid, by splitting the velocity field into its Newtonian and non-Newtonian parts. The method yields answers for the relevant polymer process features, including velocity, extrudate shape, stresses and forces. Each of these answers, split naturally into their Newtonian and non-Newtonian parts. These parts, in turn, arise with coefficients that just depend of the processing problem geometry. We call this method *polymer process partitioning* (see this discussed in Section V below). Curiously, we know of no polymer processing textbook treatment of the polymer process partitioning method. Jones [12] This method has been applied to the viscoelastic coating of an eccentric wire (with (Cases III and IV in [12]) and without (Cases I in [12]) pressure gradient), the extrusion of elliptical pipe [13,45], and also to extrusion from an eccentric annular pipe die (Cases II in [12]; Section 2. of [14]; [45]), and even to a corrugated wire coated through a corrugated die (see Section 3. of [14]; Section 2. of [15]). In the latter, we discovered an important opportunity to improve upon the method of Jones [12], an improvement that we call the rapid convergence approximation [see Eq. (70)]. For extrusion through curved eccentric annuli (circular [16] or elliptical [16]) or for flows along cylindrical boundaries [17], Jones and his coworkers abandoned polymer process partitioning.

Mindful of our errata to Ref. [12], we use polymer process partitioning to create a graphical procedure to estimate velocity, extrudate shape, stresses and forces in an eccentric pipe die. We accomplish this by adimensionalizing the problem, and then applying the method of Jones [12] which uses eccentric cylindrical coordinates (illustrated and defined in **Figure 3**) and which first

yields an expression for the velocity profile. From this profile, we obtain (i) an expression for the volumetric flow rate (called *throughput*). We then obtain (ii) an expression for the shape of the extruded pipe, which scales with $\langle \tilde{v}_z \rangle_\theta$, by integrating the velocity profile, $v_z(\xi, \theta)$, with respect to ξ , through the thickness of the eccentric annulus. Next, we get (iii) expressions for the stresses in the extruded polymer melt. These expressions for the stresses can be used to estimate the upper-bound, for the stress that is frozen into the outermost layer of the plastic pipe (since this layer is quenched first and normally shortly after extrusion). Of course, for semicrystalline polymers, shrinkage will superpose additional stresses upon the frozen pipe [18,19,20,21,22,23]. Finally, we get (iv) the forces on the mandrel, F_x and F_z .

As did Jones (1964), we choose the 8-constant model due to Oldroyd (see Section 4.9 of [24]) for our analysis because (1) it yields analytical solutions for Items (i) through (iv) above, and (2) because of its versatility. By *versatility* we mean that this model contains many simpler models as special cases: (a) Newtonian, (b) generalized Newtonian (including shear-thinning or thickening), (c) corotational Maxwell and (d) corotational Jeffreys. In other words, our analytical solutions for Items (i) through (iv) above can be used to explore the role of the Newtonian viscosity for nearly Newtonian material such as commercial grades of condensation polymers. By contrast, our work is equally useful for exploring the roles of shear-thinning or shear-thickening, with (corotational Maxwell), or without (generalized Newtonian) melt elasticity. We can even compare the roles of retardation without relaxation, to relaxation without retardation (corotational Maxwell), to relaxation with retardation (corotational Jeffreys). All of these cases can be explored under the one common framework of the Oldroyd 8-constant model.

We include two detailed dimensional worked examples to teach the reader how to use our work. Table I summarizes and classifies previous analytical work on pressure-driven flow through eccentric annuli, and compares the present paper to these contributions. With reference to column 2 of Table I, only the Bingham and power-law entries are not special cases of the Oldroyd 8-constant model. Further to Table I, for Newtonian fluids, secondary flows in the coating of elliptical wires (Section 2. [25]) and in the extrusion of the elliptical pipe (Section 3. [25]) and pressure-driven flow through curved eccentric annuli have also been investigated.

II. ECCENTRIC CYLINDRICAL COORDINATES

Eccentric cylindrical coordinates (illustrated in Figure 3) consist of two orthogonal sets of cylindrical shells. The first set is defined by the *eccentric radial coordinate*, ξ , and another, by the *eccentric angular coordinate*, θ . The transformations from Cartesian to eccentric cylindrical coordinates are given by:

$$x = a \frac{1 - \xi \cos \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (1)$$

$$y = a \frac{\xi \sin \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (2)$$

$$z_c = z \quad (3)$$

where the z coordinate is the flow direction and we call:

$$a \equiv \delta^{-1} (R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2)^{1/2} \quad (4)$$

the *dimensional confocal length*, and thus we define the *dimensionless confocal length* as:

$$f \equiv \frac{a}{R_o} \equiv \epsilon^{-1} (1 + \kappa^4 + \epsilon^4 - 2\kappa^2 - 2\kappa^2 \epsilon^2 - 2\epsilon^2)^{1/2} \quad (5)$$

where R_i , R_o and δ are the mandrel radius, the extrusion die radius and the dimension eccentricity (see Figure 1 and Table IV for variable definitions). The circles of constant ξ are given by:

$$\left(x - \frac{a}{1 - \xi^2} \right)^2 + y^2 = \left(\frac{a\xi}{1 - \xi^2} \right)^2 \quad (6)$$

and circles of constant θ , by:

$$x^2 + \left(y - \frac{1}{2} a \cot \theta \right)^2 = \left(\frac{a}{2 \sin \theta} \right)^2 \quad (7)$$

To illustrate our eccentric cylindrical coordinates, we use Eqs. (6) and (7) to draw the circular cross-sections of cylindrical shells of constant ξ (**black** circles) and of constant θ (**green** circles) in Figure 3. Inside the eccentric annulus, the shells of constant ξ start through the origin of the eccentric cylindrical coordinate system (not to be confused with the origin of our corresponding Cartesian system). This origin is on the line $\xi = 0$, that is, the line through positive focus, $x = a$. The shell of infinite radius $\xi = 1$ is the plane $\xi = 1$. **Inside the eccentric annulus, the cylindrical shells of constant θ start with the vertical half-plane ($\theta = 0$) immediately above the line $\xi = 0$, and the shell end with the vertical planar strip ($\theta = \pi$).** Eccentric cylindrical coordinates are not to be confused with bipolar cylindrical coordinates, which are also used for analyzing flow in eccentric annuli (see FIGURE A.7-1. In [26] or in [27], or Fig. 3. In [28], or **Fig. P3.16** in [29] where ξ and θ have other meanings; [30,31,61,32,33]).

We next illustrate our eccentric cylindrical differential volume element in Figure 4. Employing Eqs. (A.7-13) of [34] gives us the *scale factors* for coordinates ξ and θ :

$$h_\xi \equiv \frac{a}{1 + \xi^2 - 2\xi \cos \theta} \quad (8)$$

$$h_\theta \equiv \frac{a\xi}{1+\xi^2-2\xi\cos\theta} \quad (9)$$

The differential area relevant to the axial flow rate is given by:

$$dA_{\xi\theta} = \frac{a^2\xi}{(1+\xi^2-2\xi\cos\theta)^2} d\xi d\theta \quad (10)$$

and the differential area relevant to the axial force on the mandrel, by:

$$dA_{\theta z} = \frac{a\xi}{1+\xi^2-2\xi\cos\theta} d\theta dz \quad (11)$$

We will use Eq. (10) in Subsection. V.b to get the axial flow rate, and Eq. (11) in Subsection. V.d to get the axial force on mandrel.

III. OLDROYD CONSTITUTIVE EQUATION

In this paper, we choose the Oldroyd 8-constant constitutive model for the extruded plastics. We choose this model for its rich diversity of special cases (see Section I above). Like Jones [12] before us and for simplicity, we follow the notation and symbols of Oldroyd [35], with just a few modernizations. We include a short table of useful correspondences in Table II.

a. Oldroyd Model

We write the Oldroyd 8-constant model as:

$$\begin{aligned} \tau_{ik} + \lambda_1 \frac{\mathcal{D}\tau_{ik}}{\mathcal{D}t} + \mu_0 \tau_{ij} \dot{\gamma}_{ik} - \mu_1 (\tau_{ij} \dot{\gamma}_{ik} + \tau_{jk} \dot{\gamma}_{ij}) + \nu_1 \tau_{jl} \dot{\gamma}_{jl} \delta_{ik} \\ = 2\eta_0 \left(\dot{\gamma}_{ik} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}_{ik}}{\mathcal{D}t} - 2\mu_2 \dot{\gamma}_{ij} \dot{\gamma}_{jk} + \nu_2 \dot{\gamma}_{il} \dot{\gamma}_{jl} \delta_{ik} \right) \end{aligned} \quad (12)$$

where:

$$\pi_{ij} \equiv \tau_{ik} + p\delta_{ik} \quad (13)$$

and the rate-of-strain tensor is given by:

$$\dot{\gamma}_{ik} \equiv \frac{1}{2} \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \quad (14)$$

and:

$$\frac{\mathcal{D}b_{ik}}{\mathcal{D}t} \equiv \frac{\partial b_{ik}}{\partial t} + v_j \frac{\partial b_{ik}}{\partial x_j} + \omega_{ij} b_{jk} + \omega_{kj} b_{ij} \quad (15)$$

where the vorticity tensor is given by:

$$\omega_{ij} \equiv \frac{1}{2} \left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k} \right) \quad (16)$$

In Eq. (13), δ_{ik} is the *kronecker delta*. For convenience, we define every dimensional symbol in Table III, and dimensionless ones in Table IV.

b. Oldroyd Model Constants

To deepen our understandings of the Oldroyd model constants, we first examine Eq. (12) in steady shear flow:

$$\begin{aligned} \tau_{ik} + \mu_0 \tau_{ij} \dot{\gamma}_{jk} - \mu_1 (\tau_{ij} \dot{\gamma}_{jk} + \tau_{jk} \dot{\gamma}_{ij}) + \nu_1 \tau_{jl} \dot{\gamma}_{li} \delta_{ik} \\ = 2\eta_0 (\dot{\gamma}_{ik} - 2\mu_2 \dot{\gamma}_{ij} \dot{\gamma}_{jk} + \nu_2 \dot{\gamma}_{il} \dot{\gamma}_{lj} \delta_{ik}) \end{aligned} \quad (17)$$

which leads to Eq. (19) below. Substituting Eqs. (33a) of [12] and (33b) of [12] into Eq. (43) of [12] yields the magnitude of the rate of deformation tensor:

$$\dot{\gamma} \equiv \frac{(1 - 2\xi \cos \theta + \xi^2)}{a\xi} \sqrt{\left(\xi \frac{\partial v_z}{\partial \xi} \right)^2 + \left(\frac{\partial v_z}{\partial \theta} \right)^2} \quad (18)$$

From Eq. (8) of [5], we get the steady shear viscosity function as:

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \quad (19)$$

where:

$$\sigma_1 \equiv \lambda_1^2 + \mu_0 (\mu_1 - \frac{3}{2} \nu_1) - \mu_1 (\mu_1 - \nu_1) \quad (20)$$

$$\sigma_2 \equiv \lambda_1 \lambda_2 + \mu_0 (\mu_2 - \frac{3}{2} \nu_2) - \mu_1 (\mu_2 - \nu_2) \quad (21)$$

Introducing the Weissenberg:

$$\text{Wi} \equiv \sqrt{\sigma_1} \dot{\gamma} \quad (22)$$

into Eq. (19) gives:

$$\frac{\eta}{\eta_0} = \frac{1 + \sigma \text{Wi}^2}{1 + \text{Wi}^2} \quad (23)$$

where:

$$\sigma \equiv \sigma_2 / \sigma_1 \quad (24)$$

Taking the limit as strain rate, $\dot{\gamma}$, goes to infinity yields:

$$\eta_\infty / \eta_0 = \sigma \quad (25)$$

From Eq. (23) we learn that, for the shear stress to increase monotonically with shear rate:

$$\sigma \geq 1/9 \quad (26)$$

and that η inflects with $\dot{\gamma}$ at:

$$(\text{Wi}, \eta) = \left(\frac{1}{\sqrt{3}}, \frac{3 + \sigma}{4} \right) \quad (27)$$

and thus, more importantly, the *viscosity curves* will inflect at:

$$(\text{Wi}_i, \eta_i) = (1/\sigma^{1/4}, \sqrt{\sigma}) \quad (28)$$

By viscosity curves we mean curves of $\ln \eta$ versus $\ln \text{Wi}$. Further:

$$\left. \frac{d \ln \eta}{d \ln \text{Wi}} \right|_{\text{Wi}_i} \equiv (n - 1) = \frac{1}{\sigma^{1/4}} \quad (29)$$

where n is called the *power-law index*. By power-law index, we mean one plus the slope of the viscosity curve at its inflection point. Using Eq. (23) to plot

Figure 5 from which we see that when $\sigma > 1$, the fluid is shear thickening, and when $\sigma = 1$, Newtonian, and otherwise, shear thinning.

For the normal stress difference coefficients, in steady shear flow, we get (Eqs. (12) of [35]):

$$\Psi_1(\dot{\gamma}) \equiv -2\lambda_1\eta(\dot{\gamma}) + 2(\mu_2 - \nu_2)\eta_0 \quad (30)$$

$$\Psi_2(\dot{\gamma}) \equiv (\lambda_1 - \mu_1 + 2\nu_1)\eta(\dot{\gamma}) - (\lambda_2 - \mu_2)\eta_0 \quad (31)$$

where $\eta(\dot{\gamma})$ is given by Eq. (19). Hence:

$$\frac{-\Psi_2(\dot{\gamma})}{\Psi_1(\dot{\gamma})} \equiv \frac{1}{2} \left[\frac{(\lambda_1 - \mu_1 + 2\nu_1)\eta(\dot{\gamma}) - (\lambda_2 - \mu_2)\eta_0}{\lambda_1\eta(\dot{\gamma}) - (\mu_2 - \nu_2)\eta_0} \right] \quad (32)$$

Here, we demonstrate the versatility of Eq. (12) by identifying the special cases of (a) the Newtonian fluid:

$$\lambda_1 = \mu_0 = \mu_1 = \nu_1 = \lambda_2 = \mu_2 = \nu_2 = 0 \quad (33)$$

$$\tau_{ik} = 2\eta_0\dot{\gamma}_{ik} \quad (34)$$

(b) the generalized Newtonian fluid:

$$\lambda_1 = \lambda_2 = \mu_1 = \nu_1 = \nu_2 = 0 \quad (35)$$

$$\tau_{ik} + \mu_0\tau_{jj}\dot{\gamma}_{ik} = 2\eta_0(\dot{\gamma}_{ik} + \nu_2\dot{\gamma}_{il}\dot{\gamma}_{jl}\delta_{ik}) \quad (36)$$

(c) the corotational Maxwell fluid (see Section 2. of [36]; [37]):

$$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0 \quad (37)$$

$$\tau_{ik} + \lambda_1 \frac{\mathcal{D}\tau_{ik}}{\mathcal{D}t} = 2\eta_0\dot{\gamma}_{ik} \quad (38)$$

and finally, (d) the corotational Jeffreys fluid (see Section 8. of [36]):

$$\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0 \quad (39)$$

$$\tau_{ik} + \lambda_1 \frac{\mathcal{D}\tau_{ik}}{\mathcal{D}t} = 2\eta_0\dot{\gamma}_{ik} + 2\eta_0\lambda_2 \frac{\mathcal{D}\dot{\gamma}_{ik}}{\mathcal{D}t} \quad (40)$$

Eq. (12) has been treated in some detail in (see Chapters 2 and 3 in [38], §8.1 of [26], §7.3 in [27], Section 4.2. in [39], Section 6.3-2 in 40, Section 6.1-7 in [41] and Section 3.4.5 in [42]). Eq. (12) can be written in integral form (see EXAMPLE 8.1-2 of [26]) Tables 8.1-1 in [26] or 7.3-2 in [27] identify eleven special cases of Eq. (12) (see also Tables 8.1-8.3 in [43]).

IV. ANALYTICAL SOLUTION METHOD

Here, we begin our analysis of pressure-driven flow through the eccentric annulus illustrated in eccentric cylindrical coordinates in Figure 3 (blue region). We begin by assuming the flow to be rectilinear:

$$\tilde{\mathbf{v}} = [0, 0, \tilde{v}_z(\xi, \theta)] \quad (41)$$

This approximation neglects a helical secondary flow, where (Eqs. (12a) and (12b) of [14]):

$$\tilde{\mathbf{v}} = [0, \tilde{v}_\theta(\xi, \theta), \tilde{v}_z(\xi, \theta)] \quad (42)$$

We follow the method of Jones [12] who expands the dimensionless axial velocity in power of $(1-\sigma)S$:

$$\tilde{v}_z = v_0 + (1-\sigma)Sv_1 + (1-\sigma)^2 S^2 v_2 + (1-\sigma)^3 S^3 v_3 + O(S^4) \quad (43)$$

where $O(S^4)$ represents terms of fourth order and higher, and where:

$$\tilde{v}_z \equiv \frac{4\eta_0}{a^2 P} v_z \quad (44)$$

and the dimensionless pressure gradient squared:

$$S \equiv \frac{\sigma_1 a^2 P^2}{16\eta_0^2} \quad (45)$$

and where:

$$P \equiv dp/dz \quad (46)$$

which we truncate to:

$$\tilde{v}_z = v_0 + (1-\sigma)Sv_1 \quad (47)$$

Eq. (43) corrects two typos in Eq. (45) of [12]. When treating the helical secondary flow [Eq. (42) above], the Newtonian and non-Newtonian contributions to $\tilde{v}_z$ have been partitioned differently (see Eqs. (12a) and (12b) of [14]). For a Newtonian fluid where $\sigma = 1$, Eq. (47) thus reduces to $\tilde{v}_z = v_0$.

We begin with the z component of the equation of motion, in Cartesian coordinates, evaluated for steady shear flow:

$$\frac{\partial}{\partial x} \left[\eta(\dot{\gamma}) \frac{\partial v_z}{\partial x} \right] + \frac{\partial}{\partial y} \left[\eta(\dot{\gamma}) \frac{\partial v_z}{\partial y} \right] = -P \quad (48)$$

which, after changing variables (see Eq. (19) of [44] or Eq. (19) of [45]), Jones [12] rewrites, for an Oldroyd 8-constant fluid, as (Eq. (41) of [12]):

$$\left[1 + (1+\sigma)S\dot{\gamma}^2 + \sigma S^2\dot{\gamma}^4 \right] \frac{\partial^2 v_z}{\partial \zeta \partial \zeta^*} - \frac{(1-\sigma)S}{2} \left(\frac{\partial v_z}{\partial \zeta} \frac{\partial \dot{\gamma}^2}{\partial \zeta^*} + \frac{\partial v_z}{\partial \zeta^*} \frac{\partial \dot{\gamma}^2}{\partial \zeta} \right) = \frac{-(1+S\dot{\gamma}^2)^2}{(1-\zeta)^2(1-\zeta^*)^2} \quad (49)$$

Substituting Eq. (47) into Eq. (49) and then neglecting the terms of degree of S greater than two, gives:

$$\begin{aligned} \frac{\partial^2 v_0}{\partial \zeta \partial \zeta^*} + (1+\sigma)S\dot{\gamma}^2 \frac{\partial^2 v_0}{\partial \zeta \partial \zeta^*} + (1-\sigma)S \frac{\partial^2 v_1}{\partial \zeta \partial \zeta^*} - \frac{1}{2}(1-\sigma)S \left(\frac{\partial v_0}{\partial \zeta} \frac{\partial \dot{\gamma}^2}{\partial \zeta^*} + \frac{\partial v_0}{\partial \zeta^*} \frac{\partial \dot{\gamma}^2}{\partial \zeta} \right) - \frac{1}{2}(1-\sigma)S \left(\frac{\partial v_0}{\partial \zeta^*} \frac{\partial \dot{\gamma}^2}{\partial \zeta} \right) \\ = -(1-\zeta)^{-2}(1-\zeta^*)^{-2} - 2(1-\zeta)^{-2}(1-\zeta^*)^{-2} S\dot{\gamma}^2 \end{aligned} \quad (50)$$

where for simplicity, we define:

$$\zeta \equiv \xi e^{i\theta} \quad (51)$$

and:

$$\zeta^* \equiv \xi e^{-i\theta} \quad (52)$$

Equating coefficients in Eq. (50) for S^0 , and then simplifying gives:

$$\frac{\partial^2 v_0}{\partial \zeta \partial \zeta^*} = -(1-\zeta)^{-2} (1-\zeta^*)^{-2} \quad (53)$$

and for S^1 :

$$\begin{aligned} \square (1+\sigma) \tilde{\gamma}^2 \frac{\partial^2 v_0}{\partial \zeta \partial \zeta^*} + (1-\sigma) \frac{\partial^2 v_1}{\partial \zeta \partial \zeta^*} - \frac{1}{2} (1-\sigma) \left(\frac{\partial v_0}{\partial \zeta} \right) \frac{\partial \tilde{\gamma}^2}{\partial \zeta^*} - \frac{1}{2} (1-\sigma) \left(\frac{\partial v_0}{\partial \zeta^*} \right) \frac{\partial \tilde{\gamma}^2}{\partial \zeta} \\ = -2(1-\zeta)^{-2} (1-\zeta^*)^{-2} \tilde{\gamma}^2 \end{aligned} \quad (54)$$

Substituting Eq. (43) of [12] into Eq. (54) gives:

$$\frac{\partial^2 v_1}{\partial \zeta \partial \zeta^*} = 2 \frac{\partial}{\partial \zeta} \left[(1-\zeta)^2 (1-\zeta^*)^2 \frac{\partial v_0}{\partial \zeta} \left(\frac{\partial v_0}{\partial \zeta^*} \right)^2 \right] + 2 \frac{\partial}{\partial \zeta^*} \left[(1-\zeta)^2 (1-\zeta^*)^2 \left(\frac{\partial v_0}{\partial \zeta} \right)^2 \frac{\partial v_0}{\partial \zeta^*} \right] \quad (55)$$

Eqs. (53) and (55) are the Poisson equation in the ζ and ζ^* coordinates. We will employ the method of Poisson (see Section 7.6 in [46]) to solve Eqs. (53) and (55) for the zeroth and first axial velocity coefficients in Subsection V.a below.

V. RESULTS

In this Section V, we first tackle the velocity profile. From this profile, we then obtain (i) the volumetric flow rate, then (ii) the shape of the extruded pipe, then (iii) the stresses in the melt, and finally, (iv) the forces on the mandrel, F_x and F_z . We cast all of our results in dimensionless terms, and specifically in the form:

$$\mathcal{P} = \mathcal{P}_0 + (1-\sigma) S \mathcal{P}_1 \quad (56)$$

which is a truncation of an expansion in $(1-\sigma)S$ (see Eq.(47)):

$$\mathcal{P} = \mathcal{P}_0 + (1-\sigma) S \mathcal{P}_1 + O(S^2) \quad (57)$$

where $O(S^2)$ is the truncation, and where $\mathcal{P}_0$ and $\mathcal{P}_1$ are function of geometry only, and specifically $\mathcal{P}_0(\xi_i, \xi_o)$ and $\mathcal{P}_1(\xi_i, \xi_o)$. For items (i) through (iv) defined above, $\mathcal{P}$ is the result:

$$(i) \tilde{Q}, (ii) \left(\langle \tilde{v}_z \rangle_{\theta=0}, \langle \tilde{v}_z \rangle_{\theta=\pi} \right), (iii) \mathbb{S}_m \text{ and } (iv) \mathbb{F}, \chi,$$

$\mathcal{P}_0$ is the Newtonian contribution:

$$(i) \tilde{Q}_0, (ii) \left(\tilde{Q}_{\theta,0}(0), \tilde{Q}_{\theta,0}(\pi) \right), (iii) \mathbb{S}_0 \text{ and } (iv) \mathbb{F}_0, \chi_0$$

and $\mathcal{P}_1$ is the *non-Newtonian variable*:

$$(i) \tilde{Q}_1, (ii) \left(\tilde{Q}_{\theta,1}(0), \tilde{Q}_{\theta,1}(\pi) \right), (iii) \mathbb{S}_1 \text{ and } (iv) \mathbb{F}_1, \chi_1.$$

By non-Newtonian variable, we mean the non-Newtonian contribution divided by $(1-\sigma)S$. Below, we give dimensionless graphs for the Newtonian

contributions $\tilde{Q}_0$, $\left(\tilde{Q}_{\theta,0}(0), \tilde{Q}_{\theta,0}(\pi) \right)$, $\mathbb{S}_0$, $\mathbb{F}_0$ and χ_0 (in Figure 15, (Figure 17,

Figure 18), Figure 21, Figure 23 and Figure 25), and for the non-Newtonian variables $\tilde{Q}_1$, $(\tilde{Q}_{\theta,1}(0), \tilde{Q}_{\theta,1}(\pi))$, $\mathbb{S}_1$, $\mathbb{F}_1$ and χ_1 (in Figure 16, (Figure 19, Figure 20), Figure 22, Figure 24 and Figure 26). Practitioners can use these plots to design extrusion dies graphically. Herein lies the great advantage of expansions in $(1-\sigma)S$, and thus of the form of Eq. (57).

From Eq. (47), we glean that all of our results are subject to the constraint:

$$S^2 < 1 \quad (58)$$

where S is given by Eq. (116) from which we learn that S scales roughly with $1/a^6$, where the confocal length a decreases with eccentricity [see Eq. (4)].

From Eq. (57) we can expect our truncated expansion to be accurate when:

$$\frac{(1-\sigma)^2 S^2 \mathcal{P}_2}{\mathcal{P}_0} \ll 1 \quad (59)$$

Since obtaining $\mathcal{P}_1$ has been laborious, we would expect that obtain $\mathcal{P}_2$ would be more so.

~~To our solutions for χ_0 and χ_1 , to get analytical solutions [Eqs. (157) and (158)], we add the additional constraint:~~

$$\cancel{S^2 < 1} \quad (60)$$

~~after Eq. (153).~~

a. VELOCITY PROFILE

Solving Eq. (53) for the Newtonian part, we get:

$$v_0 = \sum_{n=0}^{\infty} \phi_n \cos n\theta - \frac{1}{1 - 2\xi \cos \theta + \xi^2} \quad (61)$$

since the particular part of Eq. (50) of [12] converges to a **closed form**. We call Eq. (61) *partly closed*. By partly closed, we mean that the particular part of Eq. (61) is in **closed form**. In Eq. (61):

$$\phi_n = \begin{cases} \alpha_0 + \beta_0 \ln \xi^2 & ; n = 0 \\ 2\alpha_n \xi^n + 2\beta_n \xi^{-n} & ; n \geq 1 \end{cases} \quad (62)$$

with α_n , β_n , ξ_i and ξ_o given by:

$$\alpha_n = \begin{cases} \frac{\xi_i^2 \ln \xi_i^2 + \ln \xi_o^2 - \xi_o^2 \ln \xi_o^2 - \ln \xi_i^2}{(-\ln \xi_o^2 + \xi_o^2 \ln \xi_o^2 + \ln \xi_i^2 - \xi_o^2 \ln \xi_i^2)(-1 + \xi_i^2)} & ; n = 0 \\ \frac{\xi_i^2 \xi_o^{2n} - \xi_o^{2n} + \xi_i^{2n} - \xi_o^2 \xi_i^{2n}}{(\xi_o^{2n} - \xi_o^{2n+2} - \xi_i^{2n} + \xi_o^2 \xi_i^{2n})(-1 + \xi_i^2)} & ; n \geq 1 \end{cases} \quad (63)$$

$$\beta_n = \begin{cases} \frac{\xi_o^2 - \xi_i^2}{(-\ln \xi_o^2 + \xi_o^2 \ln \xi_o^2 + \ln \xi_i^2 - \xi_o^2 \ln \xi_i^2)(-1 + \xi_i^2)} & ; n = 0 \\ \frac{-\xi_i^n \xi_o^n (\xi_o^2 - \xi_i^2)}{(-\xi_o^n \xi_i^{-n} + \xi_o^{n+2} \xi_i^{-n} + \xi_o^{-n} \xi_i^n - \xi_o^{-n+2} \xi_i^n)(-1 + \xi_i^2)} & ; n \geq 1 \end{cases} \quad (64)$$

where:

$$\xi_i \equiv \frac{(R_o^2 - R_i^2 - \delta^2) - (R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2)^{1/2}}{2\delta R_i}; \delta > (R_o - R_i) \quad (65)$$

or in dimensionless terms:

$$\xi_i \equiv \frac{(1 - \kappa^2 - \epsilon^2) - (1 + \kappa^4 + \epsilon^4 - 2\kappa^2 - 2\kappa^2 \epsilon^2 - 2\epsilon^2)^{1/2}}{2\epsilon \kappa}; \epsilon > (1 - \kappa) \quad (66)$$

and:

$$\xi_o \equiv \frac{(R_o^2 - R_i^2 + \delta^2) - (R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2)^{1/2}}{2\delta R_o}; \delta > (R_o - R_i) \quad (67)$$

or:

$$\xi_o \equiv \frac{(1 - \kappa^2 + \epsilon^2) - (1 + \kappa^4 + \epsilon^4 - 2\kappa^2 - 2\kappa^2 \epsilon^2 - 2\epsilon^2)^{1/2}}{2\epsilon}; \epsilon > (1 - \kappa) \quad (68)$$

where the dimensionless eccentricity, ϵ , and the aspect ratio, κ , are defined in Table IV. Our Eqs. (61) and (62) differ from Eqs. (49) and (50) of [12] in that Eq. (61) is partly closed.

To solve for v_1 , Jones truncated Eq. (49) of [12] to the second harmonic, $n = 2$, but we find that, for large eccentricity, this truncation gives inaccurate velocity (see Figure 11). To correct this, we begin by noting that in Eqs. (63) and (64), α_n and β_n converge rapidly to α_∞ and 0. We now thus depart from Jones [12], with our own approximation to Eq. (61), by assuming that:

$$\alpha_3 = \alpha_4 = \alpha_5 = \dots = \alpha_\infty \quad (69)$$

then Eq. (61) becomes:

$$v_0 = \sum_{n=0}^2 \phi_n \cos n\theta + 2\alpha_\infty \frac{(4\xi^3 \cos^3 \theta - 3\xi^3 \cos \theta + \xi^4 - 2\xi^4 \cos^2 \theta)}{1 - 2\xi \cos \theta + \xi^2} - \frac{1}{1 - 2\xi \cos \theta + \xi^2} \quad (70)$$

where:

$$\alpha_\infty \equiv \lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \frac{\xi_i^2 \xi_o^{2n} - \xi_o^{2n} + \xi_i^{2n} - \xi_o^2 \xi_i^{2n}}{(\xi_o^{2n} - \xi_o^{2n+2} - \xi_i^{2n} + \xi_o^2 \xi_i^{2n})(-1 + \xi_i^2)} = \frac{1}{1 - \xi_o^2} \quad (71)$$

Eq. (70) differs from Eq. (61) by the **closed form** for the portion of the homogeneous part of ϕ_n where $n \geq 3$. We thus call our Eq. (70) the *rapid convergence approximation*. This approximation is accurate when:

$$\left[\frac{\alpha_3}{\alpha_\infty} - 1 \right] \ll 1 \quad (72)$$

which, for normal sag compensation, is always the case. Evaluating Eq. (63) for $n = 3$ and substituting this with Eq. (71) into Eq. (72) gives:

$$\frac{\xi_i^6}{(1 - \xi_i^2)(\xi_o^4 + \xi_i^2 \xi_o^2 + \xi_i^4)} \ll 1 \quad (73)$$

Substituting Eq. (70) into Eq. (55), and then solving for the non-Newtonian part, v_1 , gives:

$$v_1 = \sum_{n=0}^{\infty} \psi_n \cos n\theta \quad (74)$$

where:

$$\psi_n = \beta_0^3 \psi_n^{(1)} + \beta_0^2 \psi_n^{(2)} + \beta_0 \psi_n^{(3)} + \psi_n^{(4)} \quad (75)$$

with:

$$\psi_n^{(m)} = \begin{cases} A_0^{(m)} + B_0^{(m)} \ln \xi^2 + v_{1p,0}^{(m)} & ; n = 0 \quad ; m = 1, 2, 3, 4 \\ A_n^{(m)} \xi^n + B_n^{(m)} \xi^{-n} + v_{1p,n}^{(m)} & ; n \geq 1 \quad ; m = 1, 2, 3, 4 \end{cases} \quad (76)$$

where the constants $A_n^{(m)}$ and $B_n^{(m)}$ in Eq. (76) are subject to:

$$\psi_n^{(m)}(\xi_i) = 0 \quad (77)$$

and:

$$\psi_n^{(m)}(\xi_o) = 0 \quad (78)$$

Eq. (75) separates ψ_n into four groups: $\psi_n^{(1)}$, $\psi_n^{(2)}$, $\psi_n^{(3)}$ and $\psi_n^{(4)}$. In Eq. (76), we neglect all non-zero harmonics higher than the fourth ($n > 4$).

The particular parts of the first group $\psi_n^{(1)}$ defined in Eq. (76) are given by:

$$v_{1p,0}^{(1)} = \xi^2 - \xi^{-2} \quad (79)$$

$$v_{1p,1}^{(1)} = 2\xi - 4\xi \ln \xi - 4\xi^{-1} \ln \xi \quad (80)$$

and the particular parts of the second group $\psi_n^{(2)}$ in Eq. (76):

$$v_{1p,0}^{(2)} = (6\alpha_2 - 6\alpha_1)\xi^2 + (6\beta_2 - 6\beta_1)\xi^{-2} \quad (81)$$

$$v_{1p,1}^{(2)} = C_{12,1}\xi^3 + C_{12,2}\xi + C_{12,3}\xi \ln \xi + C_{12,4}\xi^{-1} \ln \xi + C_{12,5}\xi^{-3} \quad (82)$$

$$v_{1p,2}^{(2)} = C_{22,1}\xi^4 + C_{22,2}\xi^2 + C_{22,3}\xi^2 \ln \xi + C_{22,4} + C_{22,5}\xi^{-2} + C_{22,6}\xi^{-2} \ln \xi + \frac{10}{3}\beta_2\xi^{-4} \quad (83)$$

$$v_{1p,3}^{(2)} = C_{32,1}\xi^3 + C_{32,2}\xi^3 \ln \xi + 8\beta_2\xi^{-3} \ln \xi + \frac{4}{3}\beta_2\xi^{-3} \quad (84)$$

$$v_{1p,4}^{(2)} = \left(-\frac{2}{3}\alpha_2 + \frac{2}{3}\alpha_\infty\right)\xi^2 - \frac{2}{3}\beta_2\xi^{-2} \quad (85)$$

where:

$$C_{12,\ell} = \left(\begin{matrix} (-8\alpha_2 + 6\alpha_\infty + 2\alpha_1), (2\beta_1 - 6\alpha_1 + 4\alpha_2), (-4\beta_1 - 8\alpha_2 + 12\alpha_1), \\ (-4 + 4\alpha_1 - 12\beta_1 + 8\beta_2), (-8\beta_2 + 2\beta_1) \end{matrix} \right) \quad (86)$$

$$C_{22,\ell} \in \left\{ \left(-\frac{10}{3}\alpha_\infty + \frac{10}{3}\alpha_2 \right), (\alpha_1 - 4\alpha_2 + 3\alpha_\infty), (-4\alpha_1 + 16\alpha_2 - 12\alpha_\infty), \right. \\ \left. (-2\alpha_1 + 2\alpha_2 - 2\beta_1 + 2\beta_2), (\beta_1 - 4\beta_2), (4\beta_1 - 16\beta_2) \right\} \quad (87)$$

$$C_{32,\ell} \in \left\{ \left(\frac{4}{3}\alpha_2 - \frac{4}{3}\alpha_\infty \right), (-8\alpha_2 + 8\alpha_\infty) \right\}; n=1,2 \quad (88)$$

and the particular parts of the third group $\psi_n^{(3)}$ in Eq. (76):

$$v_{1p,0}^{(3)} = C_{03,1}\xi^6 + C_{03,2}\xi^4 + C_{03,3}\xi^2 + 4\ln\xi - \ln(\xi^2 - 1)^2 + C_{03,4}\xi^{-2} + C_{03,5}\xi^{-4} - \frac{8}{3}\beta_2^2\xi^{-6} \quad (89)$$

$$v_{1p,1}^{(3)} = C_{13,1}\xi^5 - \xi^{-1}\ln(\xi^2 - 1)^2 - \xi\ln(\xi^2 - 1)^2 + C_{13,2}\xi^3 + C_{13,3}\xi \\ + C_{13,4}\xi\ln\xi - 6\xi^{-1} + C_{13,5}\xi^{-1}\ln\xi + C_{13,6}\xi^{-3} + C_{13,7}\xi^{-5} \quad (90)$$

$$v_{1p,2}^{(3)} = C_{23,1}\xi^4 + C_{23,2}\xi^{-4} + C_{23,3}\xi^2 - \xi^2\ln(\xi^2 - 1)^2 + C_{23,4}\xi^2\ln\xi \\ + C_{23,5} + C_{23,6}\xi^{-2} - \xi^{-2}\ln(\xi^2 - 1)^2 + C_{23,7}\xi^{-2}\ln\xi \quad (91)$$

$$v_{1p,3}^{(3)} = C_{33,1}\xi^5 - \xi^3\ln(\xi^2 - 1)^2 + C_{33,2}\xi^3 - \xi^{-3}\ln(\xi^2 - 1)^2 + C_{33,3}\xi^3\ln\xi \\ + C_{33,4}\xi + C_{33,5}\xi^{-1} + C_{33,6}\xi^{-5} + C_{33,7}\xi^{-3} + C_{33,8}\xi^{-3}\ln\xi \quad (92)$$

$$v_{1p,4}^{(3)} = -1 + C_{43,1}\xi^6 + \frac{1}{2}\alpha_\infty\xi^4 + (4 - 4\alpha_\infty)\xi^4\ln\xi - \xi^4\ln(\xi^2 - 1)^2 \\ + C_{43,2}\xi^2 + C_{43,3}\xi^{-2} - \xi^{-4} - \xi^{-4}\ln(\xi^2 - 1)^2 - \frac{4}{5}\beta_2^2\xi^{-6} \quad (93)$$

$$v_{1p,5}^{(3)} = \alpha_\infty^2\xi^7 + \frac{2}{5}\alpha_\infty\xi^5 - \xi^5\ln(\xi^2 - 1)^2 + (4 - 4\alpha_\infty)\xi^5\ln\xi - \xi^{-5}\ln(\xi^2 - 1)^2 - \frac{4}{5}\xi^{-5} \\ + C_{53,1}\xi^3 + C_{53,2}\xi^{-3} + C_{53,3}\xi + C_{53,4}\xi^{-1} \quad (94)$$

where:

$$C_{03,\ell} \in \left\{ \left(\frac{8}{3}\alpha_2^2 - \frac{16}{3}\alpha_2\alpha_\infty + \frac{8}{3}\alpha_\infty^2 \right), (6\alpha_1\alpha_\infty + 16\alpha_2^2 + \alpha_1^2 + 9\alpha_\infty^2 - 8\alpha_1\alpha_2 - 24\alpha_2\alpha_\infty), \right. \\ \left. (-6\alpha_\infty\beta_1 + 8\alpha_2\beta_1 + 8\alpha_2^2 - 16\alpha_1\alpha_2 + 9\alpha_1^2 + 8\alpha_\infty\beta_2 - 2\alpha_1\beta_1 - 8\alpha_2\beta_2), \right. \\ \left. (8\alpha_2\beta_2 - 2\beta_1 - 8\beta_2^2 + 16\beta_1\beta_2 + 2\alpha_1\beta_1 - 9\beta_1^2 - 8\alpha_1\beta_2), (8\beta_1\beta_2 - 16\beta_2^2 - \beta_1^2) \right\} \quad (95)$$

$$C_{13,\ell} = \left\{ \left(-10\alpha_\infty^2 - \frac{40}{3}\alpha_2^2 + \frac{10}{3}\alpha_1\alpha_2 - \frac{10}{3}\alpha_1\alpha_\infty + \frac{70}{3}\alpha_2\alpha_\infty \right), \right. \\ \left(-7\alpha_1^2 - 24\alpha_2^2 + 32\alpha_1\alpha_2 - 18\alpha_1\alpha_\infty + 18\alpha_2\alpha_\infty - 2\alpha_2\beta_1 + 2\alpha_\infty\beta_1 \right), \\ \left(-2\alpha_1 + 4\alpha_1^2 + 4\alpha_2 - 4\alpha_1\alpha_2 - 16\alpha_2\beta_2 + 4\alpha_1\beta_2 - 8\alpha_1\beta_1 + 12\alpha_\infty\beta_2 + 8\alpha_2\beta_1 \right), \\ \left(-16\alpha_2\beta_1 - 8\alpha_1\beta_2 + 32\alpha_2\beta_2 - 24\alpha_\infty\beta_2 + 16\alpha_1\beta_1 + 8\alpha_1\alpha_2 + 4 + 4\alpha_1 - 8\alpha_2 - 8\alpha_1^2 \right), \\ \left(-16\alpha_1\beta_2 + 32\alpha_2\beta_2 - 24\alpha_\infty\beta_2 - 8\alpha_2\beta_1 + 16\alpha_1\beta_1 - 8\beta_1^2 - 4\beta_1 + 8\beta_1\beta_2 \right), \\ \left(24\beta_2^2 + 7\beta_1^2 + 2\alpha_1\beta_2 - 32\beta_1\beta_2 - 2\beta_2 \right), \left(\frac{40}{3}\beta_2^2 - \frac{10}{3}\beta_1\beta_2 \right) \right\} \quad (96)$$

$$C_{23,\ell} \in \left\{ \begin{aligned} & \left(\frac{1}{3}\alpha_1^2 + \frac{38}{3}\alpha_1\alpha_\infty - \frac{40}{3}\alpha_1\alpha_2 + 12\alpha_2^2 - \frac{32}{3}\alpha_2\alpha_\infty \right), \left(-\frac{1}{3}\beta_1^2 - 12\beta_2^2 + \frac{40}{3}\beta_1\beta_2 \right), \\ & (4\alpha_1\alpha_2 + 3\alpha_\infty\beta_1 - 3\alpha_1\alpha_\infty - 4\alpha_2\beta_1 + 3\alpha_\infty + \alpha_1\beta_1 - 2\alpha_2 - \alpha_1^2), \\ & (-16\alpha_1\alpha_2 - 12\alpha_\infty + 4 - 12\alpha_\infty\beta_1 + 8\alpha_2 + 12\alpha_\infty\alpha_1 + 16\alpha_2\beta_1 + 4\alpha_1^2 - 4\alpha_1\beta_1), \\ & (-\beta_1^2 + 8\alpha_\infty\beta_2 - 2\alpha_1 + 8\alpha_2\beta_1 - 1 + \alpha_1^2 - 8\alpha_1\beta_2 - 6\alpha_\infty\beta_1), \\ & (\beta_1^2 + 4\alpha_1\beta_2 - 2 - \alpha_1\beta_1 - 4\beta_1\beta_2 - 2\beta_2 + \beta_1), \\ & (16\alpha_1\beta_2 + 4\beta_1^2 - 16\beta_1\beta_2 - 8\beta_2 + 4\beta_1 - 4\alpha_1\beta_1) \end{aligned} \right\} \quad (97)$$

$$C_{33,\ell} \in \left\{ \begin{aligned} & \left(-2\alpha_2^2 + \alpha_1\alpha_2 + 3\alpha_2\alpha_\infty - \alpha_1\alpha_\infty \right), \left(\frac{4}{3}\alpha_2\beta_1 + \frac{4}{3}\alpha_1\alpha_\infty - \frac{4}{3}\alpha_1\alpha_2 - \frac{4}{3}\alpha_\infty\beta_1 + \frac{2}{3}\alpha_\infty \right), \\ & (8\alpha_1\alpha_2 - 8\alpha_1\alpha_\infty + 4 - 8\alpha_2\beta_1 - 4\alpha_\infty + 8\alpha_\infty\beta_1), \\ & (-4\alpha_\infty\beta_1 - 8\alpha_2\beta_2 + 2\alpha_1\beta_2 - \alpha_1^2 - 2\alpha_1\beta_1 + 6\alpha_\infty\beta_2 - 2\alpha_2 + 6\alpha_2\beta_1 + 2\alpha_1\alpha_2), \\ & (\beta_1^2 - 2 + 8\alpha_2\beta_2 - 2\beta_1\beta_2 - 2\alpha_2\beta_1 - 6\alpha_1\beta_2 + 2\alpha_1\beta_1), (-\beta_1\beta_2 + 2\beta_2^2), \\ & \left(\frac{4}{3}\beta_2 - \frac{4}{3}\alpha_1\beta_2 + \frac{4}{3}\beta_1\beta_2 - \frac{4}{3} \right), (-8\alpha_1\beta_2 + 8\beta_2 + 8\beta_1\beta_2) \end{aligned} \right\} \quad (98)$$

$$C_{43,\ell} \in \left\{ \begin{aligned} & \left(\frac{9}{5}\alpha_\infty^2 - \frac{8}{5}\alpha_\infty\alpha_2 + \frac{4}{5}\alpha_2^2 \right), \left(\frac{1}{3} - \frac{16}{3}\alpha_\infty\beta_2 + \frac{1}{3}\alpha_1^2 + \frac{16}{3}\alpha_2\beta_2 - 2\alpha_\infty + \frac{16}{3}\alpha_\infty\beta_1 \right), \\ & \left(-\frac{8}{3}\alpha_1\alpha_2 + 2\alpha_1\alpha_\infty - \frac{16}{3}\alpha_2\beta_1 + \frac{4}{3}\alpha_2^2 \right), \\ & \left(-\frac{4}{3}\beta_2^2 - \frac{16}{3}\alpha_2\beta_2 - \frac{1}{3}\beta_1^2 - 2 + \frac{16}{3}\alpha_1\beta_2 + \frac{8}{3}\beta_1\beta_2 \right) \end{aligned} \right\} \quad (99)$$

$$C_{53,\ell} \in \left\{ \begin{aligned} & (\alpha_1\alpha_2 - 2\alpha_\infty + \frac{1}{2} + 3\alpha_\infty\alpha_2 - \alpha_1\alpha_\infty - 2\alpha_2^2), (2\beta_2^2 - 2 - \beta_1\beta_2), \\ & \left(\frac{2}{3}\alpha_2\beta_1 - \frac{2}{3} - \frac{2}{3}\alpha_\infty\beta_1 - \frac{8}{3}\alpha_2\beta_2 + \frac{8}{3}\alpha_\infty\beta_2 \right), (-1 - 2\alpha_\infty\beta_2 - \frac{2}{3}\alpha_1\beta_2 + \frac{8}{3}\alpha_2\beta_2) \end{aligned} \right\} \quad (100)$$

and the particular parts of the fourth group $\psi_n^{(4)}$ in Eq. (76):

$$v_{1p,0}^{(4)} = C_{04,1}\xi^6 + C_{04,2}\xi^4 + C_{04,3}\xi^2 + C_{04,4}\ln\xi + C_{04,5}\ln(\xi^2 - 1)^2 + C_{04,6}\xi^{-2} \\ + C_{04,7}\xi^{-4} + C_{04,8}\xi^{-6} + \frac{1}{2}(1 + \xi^2 - 4\alpha_\infty + 4\alpha_\infty\xi^2 + 4\alpha_\infty^2\xi^4 + 4\alpha_\infty^2 - 8\alpha_\infty^2\xi^2)(\xi^2 - 1)^{-3} \quad (101)$$

$$v_{1p,1}^{(4)} = C_{14,1}\xi^7 + C_{14,2}\xi^5 + C_{14,3}\xi^3 + C_{14,4}\xi + C_{14,5}\xi\ln\xi + C_{14,6}\xi\ln(\xi^2 - 1)^2 \\ + C_{14,7}\xi^{-1} + C_{14,8}\xi^{-1}\ln(\xi^2 - 1)^2 + C_{14,9}\xi^{-1}\ln\xi + C_{14,10}\xi^{-3} + C_{14,11}\xi^{-5} \\ + C_{14,12}\xi^{-7} + C_{14,13}\xi^{-1}(\xi^2 - 1)^{-3} + C_{14,14}\xi(\xi^2 - 1)^{-3} + C_{14,15}\xi^3(\xi^2 - 1)^{-3} \quad (102)$$

$$v_{1p,2}^{(4)} = C_{24,1}\xi^8 + C_{24,2}\xi^6 + C_{24,3}\xi^4 + C_{24,4}\xi^2 + C_{24,5}\xi^2\ln\xi + C_{24,6}\xi^2\ln(\xi^2 - 1)^2 + C_{24,7} \\ + C_{24,8}\xi^{-2} + C_{24,9}\xi^{-2}\ln\xi + C_{24,10}\xi^{-2}\ln(\xi^2 - 1)^2 + C_{24,11}\xi^{-4} + C_{24,12}\xi^{-6} + \frac{8}{5}\beta_2^3\xi^{-8} \\ + C_{24,13}\xi^6(\xi^2 - 1)^{-3} + C_{24,14}\xi^4(\xi^2 - 1)^{-3} + C_{24,15}\xi^2(\xi^2 - 1)^{-3} + C_{24,16}(\xi^2 - 1)^{-3} \\ + C_{24,17}\xi^{-2}(\xi^2 - 1)^{-3} \quad (103)$$

$$\begin{aligned}
v_{1p,3}^{(4)} = & C_{34,1}\xi^9 + C_{34,2}\xi^7 + C_{34,3}\xi^5 + C_{34,4}\xi^3 + C_{34,5}\xi^3 \ln \xi + C_{34,6}\xi^3 \ln(\xi^2 - 1)^2 + C_{34,7}\xi \\
& + C_{34,8}\xi^{-1} + C_{34,9}\xi^{-3} + C_{34,10}\xi^{-3} \ln \xi + C_{34,11}\xi^{-3} \ln(\xi^2 - 1)^2 + C_{34,12}\xi^{-5} + C_{34,13}\xi^{-7} \\
& + C_{34,14}\xi^7(\xi^2 - 1)^{-3} + C_{34,15}\xi^5(\xi^2 - 1)^{-3} + C_{34,16}\xi^3(\xi^2 - 1)^{-3} + C_{34,17}\xi(\xi^2 - 1)^{-3} \\
& + C_{34,18}\xi^{-1}(\xi^2 - 1)^{-3} + C_{34,19}\xi^{-3}(\xi^2 - 1)^{-3}
\end{aligned} \tag{104}$$

$$\begin{aligned}
v_{1p,4}^{(4)} = & C_{44,1}\xi^{10} + C_{44,2}\xi^8 + C_{44,3}\xi^6 + C_{44,4}\xi^4 + C_{44,5}\xi^4 \ln \xi + C_{44,6}\xi^4 \ln(\xi^2 - 1)^2 \\
& + C_{44,7}\xi^2 + C_{44,8} + C_{44,9}\xi^{-2} + C_{44,10}\xi^{-4} \ln(\xi^2 - 1)^2 + C_{44,11}\xi^{-6} + C_{44,12}\xi^8(\xi^2 - 1)^{-3} \\
& + C_{44,13}\xi^6(\xi^2 - 1)^{-3} + C_{44,14}\xi^4(\xi^2 - 1)^{-3} - 3\xi^2(\xi^2 - 1)^{-3} + C_{44,15}(\xi^2 - 1)^{-3} \\
& + C_{44,16}\xi^{-2}(\xi^2 - 1)^{-3} + C_{44,17}\xi^{-4}(\xi^2 - 1)^{-3}
\end{aligned} \tag{105}$$

$$\begin{aligned}
v_{1p,5}^{(4)} = & C_{54,1}\xi^{11} + C_{54,2}\xi^9 + C_{54,3}\xi^7 + C_{54,4}\xi^5 + C_{54,5}\xi^5 \ln \xi + C_{54,6}\xi^5 \ln(\xi^2 - 1)^2 + C_{54,7}\xi^3 \\
& + C_{54,8}\xi + C_{54,9}\xi^{-1} + C_{54,10}\xi^{-3} + C_{54,11}\xi^{-5} \ln(\xi^2 - 1)^2 + C_{54,12}\xi^9(\xi^2 - 1)^{-3} \\
& + C_{54,13}\xi^7(\xi^2 - 1)^{-3} + C_{54,14}\xi^5(\xi^2 - 1)^{-3} - \frac{6}{5}\xi^3(\xi^2 - 1)^{-3} - \frac{6}{5}\xi(\xi^2 - 1)^{-3} \\
& + C_{54,15}\xi^{-1}(\xi^2 - 1)^{-3} + C_{54,16}\xi^{-3}(\xi^2 - 1)^{-3} + C_{54,17}\xi^{-5}(\xi^2 - 1)^{-3}
\end{aligned} \tag{106}$$

where:

$$C_{04,\ell} \in \left\{ \begin{aligned} & \left(-\frac{16}{3}\alpha_1\alpha_2^2 + \frac{28}{3}\alpha_\infty\alpha_1\alpha_2 - \frac{8}{3}\alpha_\infty\alpha_2^2 + \frac{2}{3}\alpha_1^2\alpha_2 - 4\alpha_\infty^2\alpha_1 + \frac{8}{3}\alpha_2^3 - \frac{2}{3}\alpha_\infty\alpha_1^2 \right), \\ & \left(-2\alpha_1\alpha_2\beta_1 - 14\alpha_\infty\alpha_2\beta_1 + 2\alpha_\infty^2 + 8\alpha_2^2\beta_1 - 8\alpha_1\alpha_2^2 - 6\alpha_\infty\alpha_2 - \alpha_1^3 \right), \\ & \left(+6\alpha_\infty^2\beta_1 + 2\alpha_\infty\alpha_1\beta_1 + 6\alpha_1^2\alpha_2 + 6\alpha_1\alpha_2\alpha_\infty - 3\alpha_1^2\alpha_\infty + 4\alpha_2^2 \right), \\ & \left(-16\alpha_2^2\beta_2 + 2\alpha_1^2 + 2\alpha_1^2\alpha_2 + 16\alpha_1\alpha_2\beta_2 + 12\alpha_1\alpha_\infty\beta_1 + 16\alpha_2\alpha_\infty\beta_2 - 2\alpha_1^3 - 4\alpha_1\alpha_2 \right), \\ & \left(+2\alpha_\infty + 2\alpha_\infty^2 + 16\alpha_2^2\beta_1 - 20\alpha_1\alpha_2\beta_1 - 2\alpha_2 + 4\alpha_1^2\beta_1 - 16\alpha_1\alpha_\infty\beta_2 - 12\alpha_2\alpha_\infty\beta_1 \right), \\ & (4\alpha_1 - 8\beta_2 - 12\alpha_\infty + 8\alpha_2 - 4\beta_1), (\beta_1 + 3\alpha_\infty - 2\alpha_2 - \alpha_1 + 2\beta_2), \\ & (4\alpha_1\beta_1^2 - 16\alpha_2\beta_2^2 - 2\beta_1^2 + 2\beta_2\beta_1^2 + 16\alpha_1\beta_2^2 - 2\beta_1^3 - 20\alpha_1\beta_1\beta_2 + 16\alpha_2\beta_1\beta_2), \\ & (2\beta_1\beta_2 - \beta_1^3 + 8\alpha_1\beta_2^2 + 6\beta_1^2\beta_2 - 2\alpha_1\beta_1\beta_2 - 4\beta_2^2 - 8\beta_1\beta_2^2), \\ & \left(\frac{2}{3}\beta_1^2\beta_2 - \frac{16}{3}\beta_1\beta_2^2 + \frac{8}{3}\beta_2^3 \right) \end{aligned} \right\} \tag{107}$$

$$C_{14,\ell} = \left\{ \begin{array}{l} \left(-4\alpha_2^3 - 4\alpha_1\alpha_2\alpha_\infty + 2\alpha_1\alpha_2^2 + 10\alpha_2^2\alpha_\infty + 2\alpha_1\alpha_\infty^2 - 6\alpha_2\alpha_\infty^2 \right), \\ \left(16\alpha_1\alpha_2^2 + 2\alpha_2\alpha_\infty + 5\alpha_1^2\alpha_\infty - \frac{16}{3}\alpha_2^3 + 4\alpha_2^2\alpha_\infty + 8\alpha_2\alpha_\infty\beta_1 + \frac{1}{3}\alpha_1^3 \right) \\ \left(+6\alpha_1\alpha_\infty^2 - 4\alpha_\infty^2\beta_1 - 20\alpha_1\alpha_2\alpha_\infty - 6\alpha_1^2\alpha_2 - 2\alpha_\infty^2 - 4\alpha_2^2\beta_1 \right) \end{array} \right\}, \\ \left(\begin{array}{l} 4\alpha_1\alpha_2^2 - 10\alpha_1^2\alpha_2 + 2\alpha_\infty - 4\alpha_2^2 + 52\alpha_2\alpha_\infty\beta_1 - 16\alpha_1\alpha_\infty\beta_1 + 24\alpha_\infty^2\beta_2 \\ +3\alpha_1^3 - 6\alpha_1\alpha_\infty + 6\alpha_1\alpha_2 - 18\alpha_\infty^2\beta_1 - 56\alpha_2\alpha_\infty\beta_2 - 8\alpha_1\alpha_2\beta_2 - 36\alpha_2^2\beta_1 \\ +4\alpha_\infty^2 + 3\alpha_1^2\alpha_\infty - 2\alpha_1^2\beta_1 + 8\alpha_1\alpha_\infty\beta_2 + 32\alpha_2^2\beta_2 - 2\alpha_2 \end{array} \right), \\ \left(\begin{array}{l} -6\alpha_\infty + 4\alpha_2 - 24\alpha_1\alpha_\infty\beta_2 + 40\alpha_1\alpha_2\beta_2 + 2\alpha_1 - 16\alpha_1\alpha_2\beta_1 + 8\alpha_1^2\beta_1 \\ +8\alpha_2^2\beta_1 + 24\alpha_2\alpha_\infty\beta_2 - 8\alpha_1^2\beta_2 - 32\alpha_2^2\beta_2 \end{array} \right), \\ \left(\begin{array}{l} -4\beta_1 - 8\beta_2 - 48\alpha_2\alpha_\infty\beta_2 + 32\alpha_1\alpha_2\beta_1 - 80\alpha_1\alpha_2\beta_2 \\ +48\alpha_1\alpha_\infty\beta_2 + 16\alpha_1^2\beta_2 + 64\alpha_2^2\beta_2 - 16\alpha_2^2\beta_1 - 16\alpha_1^2\beta_1 \end{array} \right), \\ (\beta_1 + 3\alpha_\infty - 2\alpha_2 - \alpha_1 + 2\beta_2), (18\alpha_\infty + 6\beta_1 - 6\alpha_1 - 12\alpha_2 + 12\beta_2), \\ (\beta_1 + 3\alpha_\infty - 2\alpha_2 - \alpha_1 + 2\beta_2), \left(\begin{array}{l} -16\alpha_2\beta_1^2 + 48\alpha_\infty\beta_2^2 + 80\alpha_2\beta_1\beta_2 - 32\alpha_1\beta_1\beta_2 \\ -64\alpha_2\beta_2^2 + 16\alpha_1\beta_2^2 - 48\alpha_\infty\beta_1\beta_2 + 16\alpha_1\beta_1^2 \end{array} \right), \\ \left(\begin{array}{l} -2\alpha_1\beta_1^2 - 36\alpha_1\beta_2^2 + 3\beta_1^3 + 2\beta_1^2 + 32\alpha_2\beta_2^2 + 4\beta_1\beta_2^2 \\ -8\alpha_2\beta_1\beta_2 - 6\beta_1\beta_2 - 10\beta_1^2\beta_2 + 20\alpha_1\beta_1\beta_2 \end{array} \right), \\ \left(\frac{1}{3}\beta_1^3 - \frac{16}{3}\beta_2^3 + 16\beta_2^2\beta_1 - 6\beta_1^2\beta_2 + 4\beta_2^2 - 4\alpha_1\beta_2^2 \right), (2\beta_1\beta_2^2 - 4\beta_2^3), \\ (4\alpha_\infty^2 - 2\alpha_\infty), (-8\alpha_\infty^2 + 2), (2\alpha_\infty + 4\alpha_\infty^2) \end{array} \right\} \quad (108)$$

$$C_{24,\ell} = \left\{ \begin{aligned}
& \left(\frac{34}{5}\alpha_2\alpha_\infty^2 - \frac{18}{5}\alpha_\infty^3 + \frac{8}{5}\alpha_2^3 - \frac{24}{5}\alpha_2^2\alpha_\infty \right), \\
& \left(\alpha_1^2\alpha_2 + 8\alpha_2^3 + 10\alpha_1\alpha_2\alpha_\infty - \alpha_1^2\alpha_\infty - 3\alpha_1\alpha_\infty^3 - 2\alpha_\infty^2 - 18\alpha_2^2\alpha_\infty + 9\alpha_2\alpha_\infty^2 \right), \\
& \left(-6\alpha_1\alpha_2^2 + 2\alpha_2\alpha_\infty \right), \\
& \left(-6\alpha_1^2\alpha_\infty - 2\alpha_2 - 16\alpha_2^2\beta_2 - 8\alpha_1\alpha_2^2 + 8\alpha_1^2\alpha_2 + 32\alpha_2\alpha_\infty\beta_2 + 4\alpha_\infty^2 + 4\alpha_1\alpha_2\alpha_\infty \right), \\
& \left(-28\alpha_2\alpha_\infty\beta_1 - 4\alpha_1\alpha_2\beta_1 + 12\alpha_\infty^2\beta_1 - 4\alpha_2\alpha_\infty + 2\alpha_\infty - \frac{2}{3}\alpha_1^3 + 16\alpha_2^2\beta_1 - 16\alpha_\infty^2\beta_2 \right), \\
& \left(+4\alpha_1\alpha_\infty\beta_1 + \frac{8}{3}\alpha_2^3 \right), \\
& \left(-16\alpha_1\alpha_2\beta_2 + 2\alpha_1^2\beta_2 + \alpha_1 + 2\alpha_2 - 2\alpha_1^2\beta_1 + 32\alpha_2^2\beta_2 - 3\alpha_\infty + 10\alpha_1\alpha_2\beta_1 \right), \\
& \left(-8\alpha_2^2\beta_1 - 6\alpha_1\beta_1\alpha_\infty + 6\alpha_2\alpha_\infty\beta_1 + 12\alpha_1\alpha_\infty\beta_2 + 18\alpha_\infty^2\beta_2 \right), \\
& \left(-48\alpha_2\alpha_\infty\beta_2 \right), \\
& \left(-8\beta_2 - 4\beta_1 + 24\alpha_1\alpha_\infty\beta_1 + 64\alpha_1\alpha_2\beta_2 - 40\alpha_1\alpha_2\beta_1 + 192\alpha_2\alpha_\infty\beta_2 \right), \\
& \left(-48\alpha_1\alpha_\infty\beta_2 + 32\alpha_2^2\beta_1 - 128\alpha_2^2\beta_2 + 8\alpha_1^2\beta_1 - 8\alpha_1^2\beta_2 - 72\alpha_\infty^2\beta_2 \right), \\
& \left(-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1 \right), \\
& \left(14\alpha_2\beta_1^2 + 2\alpha_\infty^2 - 32\alpha_1\alpha_2\beta_2 + 4\alpha_1\beta_1\beta_2 - 32\alpha_2\beta_1\beta_2 + 28\alpha_\infty\beta_1\beta_2 \right), \\
& \left(+14\alpha_1^2\beta_2 - 4\alpha_2\beta_1 + 16\alpha_2\beta_2^2 - 16\alpha_\infty\beta_2^2 + 4\beta_2 + 2\beta_1 - 2\alpha_1\beta_1^2 \right), \\
& \left(-2\alpha_1^2\beta_1 + 16\alpha_2^2\beta_2 - 2\alpha_\infty - 12\alpha_\infty\beta_1^2 + 4\alpha_1\alpha_2\beta_1 \right), \\
& \left(2\alpha_2\beta_1^2 + 32\alpha_2\beta_2^2 - 24\alpha_\infty\beta_2^2 - 2\alpha_1\beta_1^2 - 8\alpha_1\beta_2^2 + 6\alpha_\infty\beta_1\beta_2 + 10\alpha_1\beta_1\beta_2 \right), \\
& \left(-16\alpha_2\beta_1\beta_2 \right), \\
& \left(128\alpha_2\beta_2^2 - 8\alpha_1\beta_1^2 - 96\alpha_\infty\beta_2^2 - 32\alpha_1\beta_2^2 + 24\alpha_\infty\beta_1\beta_2 - 64\alpha_2\beta_1\beta_2 + 40\alpha_1\beta_1\beta_2 \right), \\
& \left(+8\alpha_2\beta_1^2 \right), \\
& \left(-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1 \right) + \left(-4\alpha_1\beta_1\beta_2 + 8\beta_1^2\beta_2 + \frac{8}{3}\beta_2^3 + 16\alpha_1\beta_2^2 \right), \\
& \left(-\frac{2}{3}\beta_1^3 + 4\beta_1\beta_2 - 16\alpha_2\beta_2^2 - 8\beta_1\beta_2^2 \right), \\
& \left(-6\beta_1\beta_2^2 + \beta_1^2\beta_2 + 8\beta_2^3 \right), \left(-4\alpha_2 + 4\beta_2 + 4\alpha_\infty + 2\beta_1 - 2\alpha_1 + 2\alpha_\infty^2 \right), \\
& \left(4\alpha_1 - 8\beta_2 - 6\alpha_\infty^2 + 8\alpha_2 - 4\beta_1 - 6\alpha_\infty \right), \left(10\alpha_\infty^2 - 2\alpha_\infty \right), \\
& \left(3 - 4\alpha_1 + 4\beta_1 + 10\alpha_\infty - 8\alpha_2 - 10\alpha_\infty^2 + 8\beta_2 \right), \\
& \left(-1 + 4\alpha_2 - 2\beta_1 - 6\alpha_\infty - 4\beta_2 + 2\alpha_1 + 4\alpha_\infty^2 \right)
\end{aligned} \right\} \quad (109)$$

$$C_{34,\ell} = \left\{ \begin{aligned} & \left(2\alpha_2\alpha_\infty^2 - 2\alpha_\infty^3 \right), \left(\frac{44}{5}\alpha_\infty\alpha_2^2 + \frac{4}{5}\alpha_1\alpha_2^2 + \frac{27}{5}\alpha_\infty^3 + \frac{9}{5}\alpha_1\alpha_\infty^2 - 12\alpha_2\alpha_\infty^2 \right), \\ & \left(-4\alpha_2\alpha_\infty + 2\alpha_\infty - 4\alpha_2^3 - 2\alpha_2 + 6\alpha_2^2\alpha_\infty \right), \\ & \left(+2\alpha_1^2\alpha_\infty + 6\alpha_1\alpha_2^2 - 8\alpha_1\alpha_2\alpha_\infty - 2\alpha_1^2\alpha_2 + 4\alpha_\infty^2 \right), \\ & \left(\frac{8}{3}\alpha_1\alpha_\infty\beta_1 - \frac{8}{3}\alpha_1\beta_2\alpha_\infty - \frac{8}{3}\alpha_2\alpha_\infty\beta_1 + \frac{56}{3}\alpha_2\alpha_\infty\beta_2 + \frac{4}{3}\alpha_2 \right. \\ & \left. + \frac{8}{3}\alpha_1\alpha_2\beta_2 - \frac{8}{3}\alpha_1\alpha_2\beta_1 - 2\alpha_\infty - 8\alpha_\infty^2\beta_2 + \frac{2}{3}\alpha_1 - \frac{32}{3}\alpha_2^2\beta_2 + \frac{8}{3}\alpha_2^2\beta_1 \right), \\ & \left(-16\alpha_2^2\beta_1 + 16\alpha_1\alpha_\infty\beta_2 - 16\alpha_1\alpha_\infty\beta_1 - 16\alpha_1\alpha_2\beta_2 + 16\alpha_1\alpha_2\beta_1 \right) \\ & \left(+64\alpha_2^2\beta_2 + 16\alpha_2\alpha_\infty\beta_1 - 8\beta_2 - 112\alpha_2\alpha_\infty\beta_2 - 4\beta_1 + 48\alpha_\infty^2\beta_2 \right), \\ & (-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1), \\ & \left(\frac{5}{3}\beta_1 + \alpha_\infty\beta_1^2 - 3\alpha_\infty + 2\alpha_2\beta_1 - 6\alpha_1^2\beta_2 + 36\alpha_1\alpha_2\beta_2 - 24\alpha_1\alpha_\infty\beta_2 + \frac{10}{3}\beta_2 \right. \\ & \left. + \frac{4}{3}\alpha_\infty^2 + \frac{1}{3}\alpha_1 - \alpha_1\beta_1^2 - 6\alpha_\infty\beta_1 + 6\alpha_\infty\alpha_1\beta_1 - 4\alpha_\infty\beta_1\beta_2 + \frac{2}{3}\alpha_2 - 4\alpha_1\alpha_2\beta_1 \right. \\ & \left. + 4\alpha_2\beta_1\beta_2 - 32\alpha_2^2\beta_2 + 24\alpha_2\alpha_\infty\beta_2 \right), \\ & \left(\frac{8}{3}\alpha_\infty^2 - 2\alpha_1\beta_2 - 32\alpha_2\beta_2^2 + 4\alpha_1\alpha_2\beta_2 + 36\alpha_2\beta_1\beta_2 + 32\alpha_\infty\beta_2^2 - \frac{2}{3}\alpha_1 \right. \\ & \left. - 4\alpha_2\beta_2 - 6\alpha_2\beta_1^2 + \frac{10}{3}\beta_2 - \frac{4}{3} - \frac{1}{3}\beta_1 - 32\alpha_\infty\beta_1\beta_2 + 2\alpha_1\beta_1 - 4\alpha_1\beta_1\beta_2 \right. \\ & \left. - \frac{4}{3}\alpha_2 - \alpha_1^2\beta_1 + 6\alpha_\infty\beta_1^2 + \frac{2}{3}\alpha_\infty \right), \\ & \left(-\frac{32}{3}\alpha_2\beta_2^2 + 8\alpha_\infty\beta_2^2 - \frac{8}{3}\alpha_1\beta_1\beta_2 + \frac{8}{3}\alpha_2\beta_1\beta_2 + \frac{8}{3}\alpha_1\beta_2^2 \right), \\ & (16\alpha_2\beta_1\beta_2 + 48\alpha_\infty\beta_2^2 - 64\alpha_2\beta_2^2 + 16\alpha_1\beta_2^2 - 16\alpha_1\beta_1\beta_2), \\ & (-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1), \\ & \left(-2\beta_1^2\beta_2 + 6\beta_1\beta_2^2 - 4\beta_2^3 \right), \left(\frac{4}{5}\beta_1\beta_2^2 - \frac{16}{5}\beta_2^3 \right), \frac{2}{3}(2\beta_1 - 2\alpha_1 + 4\alpha_\infty^2 + 4\beta_2 - 4\alpha_2), \\ & \frac{2}{3}(4\alpha_1 + 8\alpha_2 - 1 - 10\alpha_\infty^2 + 5\alpha_\infty - 4\beta_1 - 8\beta_2), \\ & \frac{2}{3}(4\beta_2 + 2\beta_1 - 9\alpha_\infty + 6\alpha_\infty^2 + 3 - 4\alpha_2 - 2\alpha_1), \\ & \frac{2}{3}(-3 + 6\alpha_\infty + 8\alpha_\infty^2 + 4\alpha_2 - 2\beta_1 + 2\alpha_1 - 4\beta_2), \\ & \frac{2}{3}(7 + 4\beta_1 - 14\alpha_\infty^2 + \alpha_\infty - 4\alpha_1 - 8\alpha_2 + 8\beta_2), \\ & \frac{2}{3}(-3 + 4\alpha_2 - 2\beta_1 - 3\alpha_\infty - 4\beta_2 + 2\alpha_1 + 6\alpha_\infty^2) \end{aligned} \right\} \quad (110)$$

$$C_{44,\ell} = \left\{ \begin{aligned} & \left(2\alpha_2\alpha_\infty^2 - 2\alpha_\infty^3 \right), \left(\alpha_1\alpha_\infty^2 + 2\alpha_2\alpha_\infty - 2\alpha_\infty^2 - 4\alpha_2\alpha_\infty^2 + 3\alpha_\infty^3 \right), \\ & \left(2\alpha_\infty - 2\alpha_2 - 4\alpha_2\alpha_\infty + \frac{16}{5}\alpha_1\alpha_2\alpha_\infty + \frac{8}{5}\alpha_2^3 + \frac{18}{5}\alpha_2\alpha_\infty^2 - \frac{8}{5}\alpha_1\alpha_2^2 \right), \\ & \left(+4\alpha_\infty^2 - \frac{16}{5}\alpha_2^2\alpha_\infty - \frac{18}{5}\alpha_1\alpha_\infty^2 \right), \\ & \left(\frac{1}{2}\alpha_1 - \frac{3}{2}\alpha_\infty + \alpha_2 \right), \left(-4\beta_1 - 8\beta_2 \right), \left(-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1 \right), \\ & \left(-\frac{2}{3}\alpha_1^2\beta_2 + \frac{8}{3}\alpha_2^2\beta_1 + \frac{10}{3}\beta_2 + \frac{2}{3}\alpha_1^2\beta_1 + \frac{40}{3}\alpha_2^2\beta_2 + \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 \right. \\ & \left. - \frac{4}{3}\alpha_1\alpha_2\beta_1 - 4\alpha_\infty\beta_1 + \alpha_\infty^2 + 2\alpha_\infty\beta_1^2 - 2\alpha_2\beta_1^2 - 3\alpha_\infty - 16\alpha_2\alpha_\infty\beta_2 \right. \\ & \left. + 12\alpha_1\alpha_\infty\beta_2 - \frac{32}{3}\alpha_1\alpha_2\beta_2 + \frac{5}{3}\beta_1 \right), \\ & \left(-2\alpha_2\beta_1^2 + 2\alpha_\infty^2 - 2\alpha_1\beta_1\beta_2 + 2\alpha_\infty\beta_1\beta_2 - 2\alpha_1^2\beta_2 + 2\alpha_2\beta_1 + 2\beta_2 \right) \\ & \left(-\beta_1 - \frac{3}{2} + \alpha_1\beta_1^2 + \alpha_1^2\beta_1 - 2\alpha_\infty + \alpha_\infty\beta_1^2 - 2\alpha_1\alpha_2\beta_1 - 6\alpha_\infty\beta_2 \right. \\ & \left. + 6\alpha_1\alpha_\infty\beta_2 \right), \\ & \left(-3 + \frac{8}{3}\alpha_1\beta_2^2 + \frac{40}{3}\alpha_2\beta_2^2 - 16\alpha_\infty\beta_2^2 - \frac{4}{3}\alpha_1\beta_1\beta_2 + 3\alpha_\infty^2 - 2\alpha_1^2\beta_2 + \beta_1 \right. \\ & \left(-\frac{32}{3}\alpha_2\beta_1\beta_2 + 4\alpha_1\beta_2 - \frac{2}{3}\alpha_2\beta_1^2 - 2\alpha_2 + \frac{2}{3}\alpha_1\beta_1^2 - \alpha_1 + 3\alpha_\infty + 12\alpha_\infty\beta_1\beta_2 \right), \\ & \left(-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1 \right), \left(-\frac{8}{5}\beta_1\beta_2^2 + \frac{8}{5}\beta_2^3 \right), \\ & \frac{1}{2} \left(-6\alpha_\infty + 6\alpha_\infty^2 + 2\beta_1 - 2\alpha_1 - 4\alpha_2 + 4\beta_2 \right), \\ & \frac{1}{2} \left(4\alpha_1 - 4\beta_1 - 14\alpha_\infty^2 - 3 - 8\beta_2 + 8\alpha_2 + 20\alpha_\infty \right), \\ & \frac{1}{2} \left(7 - 18\alpha_\infty + 2\beta_1 + 8\alpha_\infty^2 - 4\alpha_2 - 2\alpha_1 + 4\beta_2 \right), \\ & \frac{1}{2} \left(-3 + 2\alpha_1 + 14\alpha_\infty + 10\alpha_\infty^2 - 4\beta_2 + 4\alpha_2 - 2\beta_1 \right), \\ & \frac{1}{2} \left(12 - 8\alpha_2 + 8\beta_2 - 18\alpha_\infty^2 - 4\alpha_1 + 4\beta_1 - 12\alpha_\infty \right), \\ & \frac{1}{2} \left(-6 - 2\beta_1 + 8\alpha_\infty^2 - 4\beta_2 + 4\alpha_2 + 2\alpha_1 + 2\alpha_\infty \right) \end{aligned} \right\}, \quad (111)$$

$$C_{54,\ell} = \left\{ \begin{array}{l} (2\alpha_2\alpha_\infty^2 - 2\alpha_\infty^3), (\alpha_1\alpha_\infty^2 + 2\alpha_2\alpha_\infty - 2\alpha_\infty^2 - 4\alpha_2\alpha_\infty^2 + 3\alpha_\infty^3), \\ (-2\alpha_2 + 2\alpha_2\alpha_\infty^2 + 2\alpha_\infty - 4\alpha_2\alpha_\infty - 2\alpha_\infty^2\alpha_1 + 4\alpha_\infty^2), (\frac{2}{5}\alpha_1 - \frac{6}{5}\alpha_\infty + \frac{4}{5}\alpha_2), \\ (-4\beta_1 - 8\beta_2), (-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1), \\ \left(-2\alpha_1\alpha_2\beta_2 + 6\alpha_2\alpha_\infty\beta_1 - 4\alpha_\infty\beta_1 + 4\alpha_2^2\beta_2 + \frac{3}{10}\alpha_1 + \frac{17}{5}\beta_2 + 2\alpha_1\alpha_\infty\beta_2 - 4\alpha_2^2\beta_1 \right) \\ \left(+\frac{4}{5}\alpha_\infty^2 + \frac{17}{10}\beta_1 + 2\alpha_1\alpha_2\beta_1 - 6\alpha_2\alpha_\infty\beta_2 + \frac{3}{5}\alpha_2 - 2\alpha_1\alpha_\infty\beta_1 - \frac{29}{10}\alpha_\infty \right. \\ \left. -\frac{14}{5}\alpha_\infty - \frac{20}{3}\alpha_1\alpha_2\beta_2 + \frac{26}{15}\beta_2 + \frac{8}{5}\alpha_\infty^2 - 6\alpha_\infty\beta_2 - 4\alpha_2\beta_1\beta_2 - 2\alpha_\infty\beta_1^2 - \frac{1}{3}\alpha_1^2\beta_1 \right) \\ \left. + 2\alpha_\infty\beta_1 + 4\alpha_\infty\beta_1\beta_2 + \frac{8}{3}\alpha_1\alpha_2\beta_1 + 4\alpha_1\alpha_\infty\beta_2 - 2\alpha_1\alpha_\infty\beta_1 + \frac{16}{3}\alpha_2^2\beta_2 + 2\alpha_2\beta_1^2 \right. \\ \left. + \frac{2}{15}\alpha_1 - \frac{17}{15}\beta_1 + \frac{4}{15}\alpha_2 - \frac{4}{3}\alpha_2^2\beta_1 - \frac{8}{5} + \frac{4}{3}\alpha_1^2\beta_2 \right) \\ \left(\frac{4}{3}\alpha_2\beta_1^2 - \frac{2}{5}\alpha_2 - \frac{18}{5} + \frac{16}{3}\alpha_2\beta_2^2 + \frac{1}{5}\beta_1 - \alpha_\infty\beta_1^2 + 4\alpha_2\beta_2 + \frac{12}{5}\alpha_\infty^2 - \frac{4}{3}\alpha_1\beta_2^2 \right. \\ \left. + 2\alpha_1^2\beta_2 - \frac{1}{3}\alpha_1\beta_1^2 - \frac{3}{5}\alpha_\infty - \frac{1}{5}\alpha_1 + 4\alpha_\infty\beta_1\beta_2 - 4\alpha_1\alpha_2\beta_2 + \frac{8}{3}\alpha_1\beta_1\beta_2 - 4\alpha_\infty\beta_2^2 \right. \\ \left. - \frac{8}{5}\beta_2 - \frac{20}{3}\alpha_2\beta_1\beta_2 \right) \\ \left(-\frac{6}{5}\alpha_1 + 4\alpha_2\beta_2^2 + \frac{16}{5}\alpha_\infty^2 + \frac{6}{5}\beta_1 - \frac{24}{5} + 2\alpha_1\beta_1\beta_2 - 2\alpha_2\beta_1\beta_2 + \frac{26}{5}\alpha_\infty - 4\alpha_1\beta_2^2 \right) \\ \left. + \frac{12}{5}\beta_2 - \frac{12}{5}\alpha_2 \right) \\ (-\alpha_1 + 3\alpha_\infty + 2\beta_2 - 2\alpha_2 + \beta_1), \frac{2}{5}(-2\alpha_1 + 2\beta_1 - 14\alpha_\infty + 4\beta_2 - 4\alpha_2 + 8\alpha_\infty^2), \\ \frac{2}{5}(-6 - 18\alpha_\infty^2 - 4\beta_1 + 39\alpha_\infty - 8\beta_2 + 4\alpha_1 + 8\alpha_2), \\ \frac{2}{5}(2\beta_1 - 2\alpha_1 + 4\beta_2 - 4\alpha_2 + 10\alpha_\infty^2 + 12 - 29\alpha_\infty), \\ \frac{2}{5}(-4\beta_2 + 2\alpha_1 + 4\alpha_2 - 3 - 2\beta_1 + 12\alpha_\infty^2 + 24\alpha_\infty), \\ \frac{2}{5}(-22\alpha_\infty^2 - 8\alpha_2 + 8\beta_2 - 29\alpha_\infty - 4\alpha_1 + 18 + 4\beta_1), \\ \frac{2}{5}(-10 - 2\beta_1 + 9\alpha_\infty + 10\alpha_\infty^2 + 4\alpha_2 + 2\alpha_1 - 4\beta_2) \end{array} \right\}, \quad (112)$$

We next use our final expression for the velocity profile, Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112) to draw Figure 6 through Figure 13. Our velocity profile differs only from that of Jones (1964) in its non-Newtonian part, v_1 , and specifically we departed from Jones with our rapid convergence approximation in Eq. (70). In Figure 6 through Figure 8, we compare, for small eccentricity, our velocity profiles with the corresponding ones from Jones (Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]). Figure 7 shows close agreement for Newtonian behavior, and by contrast, the discrepancies in Figure 6 and Figure 8 illustrate the improvement from our rapid convergence approximation [Eq. (70)]. However, for large eccentricity, Figure 9 through Figure 11 show that, for non-Newtonian behaviors, this same Eq. (58) in [12] gives a very different results from ours (Eqs. (47) ...). Figure 9 through Figure 11 also show the improvement effected by our rapid convergence approximation [Eq. (70)]. Figure 12 and Figure 13 (see also Fig. 2 of [12]), show the constant speed contours, from which we learn that the fluid, be it Newtonian or not, flows fastest through the thicker gap. By contrast, whereas both the Newtonian ($\sigma = 1$) and the shear-thickening ($\sigma = \frac{1}{2}$)

fluid flow most slowly through the thinnest gap, the shear-thinning ($\sigma = \frac{3}{2}$) fluid most slowly elsewhere.

As a consistency check, we compare our Newtonian velocity profile, Eq. (61) with Eqs. (62) through (64), with the well-known solution of Kolutawong and Giacomini, Eq. (25) of [61]. Figure 14 shows the close agreement. We attribute the slight discrepancy to the lubrication approximation used in the well-known solution (see **APPENDIX B.** of [61]).

b. EXTRUDATE SHAPE

The uninitiated may think that the shape of the extrudate will match the shape of the eccentric annulus. By *extrudate shape*, we mean the ratio of the average velocity from the top to the bottom of the die. However, the extrudate thickness at any given angular position θ , of course depends, on the integral of the velocity profile at that same position θ . In Subsection V.a above, we learnt that the extruded pipe thickness can be represented by the average velocity at that angular position θ . To calculate the extruded pipe shape, neglecting die swell, we begin with the definition of the volumetric flow rate of the whole cross sectional area of the extrusion die:

$$Q = \int_A v_z(\xi, \theta) dA_{\xi\theta} \quad (113)$$

Eq. (113), and thus the result of this paper are generally unaffected by die swell [47,48]. We are, however, neglecting any θ motion associated with die swell, a quantity that, to our knowledge, has yet to be investigated. Substituting Eq. (10) in Eq. (113), and then adimensionalizing gives:

$$\bar{Q} = \int_0^{2\pi} \int_{\xi_i}^{\xi_o} \frac{\xi \tilde{v}_z}{(1 + \xi^2 - 2\xi \cos \theta)^2} d\xi d\theta \quad (114)$$

where $\tilde{v}_z(\xi, \theta)$ is given by Eq. (47) and:

$$\bar{Q} \equiv \frac{4\eta_0 Q}{a^4 P} \quad (115)$$

which, with Eq. (45), implies that:

$$S \equiv \frac{\sigma_1 a^2 P^2}{16\eta_0^2} = \frac{\sigma_1 Q^2}{a^6 \bar{Q}^2} \quad (116)$$

Substituting Eq. (47) into Eq. (114) gives:

$$\bar{Q} = \bar{Q}_0 + (1 - \sigma) S \bar{Q}_1 \quad (117)$$

where:

$$\bar{Q}_0 \equiv \int_0^{2\pi} \int_{\xi_i}^{\xi_o} \frac{\xi v_0}{(1 + \xi^2 - 2\xi \cos \theta)^2} d\xi d\theta \quad (118)$$

and:

$$\bar{Q}_1 \equiv \int_0^{2\pi} \int_{\xi_i}^{\xi_o} \frac{\xi v_1}{(1 + \xi^2 - 2\xi \cos \theta)^2} d\xi d\theta \quad (119)$$

However, Eqs. (118) and (119) yield no analytical solution, so we integrate numerically, then plot the results in Figure 15 and Figure 16.

For curved hose manufacture, after the hose emerged from the die, the hose is free to bend [49,50]. We thus calculate the radius of curvature of the hose. First, we calculate the average velocity for each wedge $d\theta$:

$$\langle \tilde{v}_z \rangle_\theta = \frac{f^2 \tilde{Q}_\theta}{\pi(1-\kappa^2)} \quad (120)$$

where f is a dimensionless confocal length.

The dimensionless volumetric flow rate at the die exit, for each differential wedge $d\theta$, is defined as:

$$\tilde{Q}_\theta \equiv \frac{4\eta_0 Q_\theta}{a^4 P} \quad (121)$$

For angular position θ , Eq. (114) reduces to:

$$\tilde{Q}_\theta = \int_{\xi_i}^{\xi_o} \frac{\xi \tilde{v}_z}{(1 + \xi^2 - 2\xi \cos \theta)^2} d\xi \quad (122)$$

Substituting Eq. (47) into Eq. (122) gives:

$$\tilde{Q}_\theta = \tilde{Q}_{\theta,0} + (1-\sigma)S\tilde{Q}_{\theta,1} \quad (123)$$

where:

$$\tilde{Q}_{\theta,0} = \int_{\xi_i}^{\xi_o} \frac{\xi}{(1 + \xi^2 - 2\xi \cos \theta)^2} v_0 d\xi \quad (124)$$

and:

$$\tilde{Q}_{\theta,1} = \int_{\xi_i}^{\xi_o} \frac{\xi}{(1 + \xi^2 - 2\xi \cos \theta)^2} v_1 d\xi \quad (125)$$

We can then take the ratio of the average velocity at the top, $\langle \tilde{v}_z \rangle_{\theta=0}$, to the bottom, $\langle \tilde{v}_z \rangle_{\theta=\pi}$, of the hose, by substituting Eqs. (124) and (125) into Eq. (123), and then this result, into Eq. (120):

$$\frac{\langle \tilde{v}_z \rangle_{\theta=0}}{\langle \tilde{v}_z \rangle_{\theta=\pi}} = \frac{\tilde{Q}_{\theta,0}(0) + (1-\sigma)S\tilde{Q}_{\theta,1}(0)}{\tilde{Q}_{\theta,0}(\pi) + (1-\sigma)S\tilde{Q}_{\theta,1}(\pi)} \quad (126)$$

To relate hose curvature to the extrudate shape [given by Eq. (126)], we use Eq. (47) from [61]:

$$\frac{r_0}{(R_i + R_o)/2} = \frac{\langle \tilde{v}_z \rangle_{\theta=0} / \langle \tilde{v}_z \rangle_{\theta=\pi} + 1}{\langle \tilde{v}_z \rangle_{\theta=0} / \langle \tilde{v}_z \rangle_{\theta=\pi} - 1} \quad (127)$$

where r_0 is the radius of curvature of the hose (see Figure 2 of [61]), R_i is the radius of the mandrel and R_o is the inner radius of the die.

We next use Eq. (126) with Eqs. (124) and (125) and with Eqs. (61) through (64), and also with Eqs. (74) through (112) to draw Figure 17 through Figure 20. Together with Eq. (126), Figure 17 through Figure 20 can help plastics engineers predict the shape of the pipe right after emerges from the extrusion

die. Together with Eqs. (126) and (127), Figure 17 through Figure 20, can be used by hose engineer to predict extrudate curvature.

c. SHEAR STRESS IN DIE ANNULUS

When extruded at high throughput, the surface of most polymer extrudates roughen visibly. This wavy skin is called *sharkskin*. To prevent sharkskin, some suggest that the highest shear stress cannot exceed roughly 0.1 Mpa (see [51]).

The general expression for shear stress in the extrudate is given by:

$$\tau_{\xi z} = \eta(\dot{\gamma})\dot{\gamma} \quad (128)$$

where the viscosity function, $\eta(\dot{\gamma})$ is defined in Eq. (23). Adimensionalizing Eq. (128), and then substituting Eq. (23) into it gives:

$$\mathbb{S} = \frac{1 + \sigma \text{Wi}^2}{1 + \text{Wi}^2} \text{Wi} \quad (129)$$

where:

$$\mathbb{S} \equiv \frac{\sqrt{\sigma_1} \tau_{\xi z}}{\eta_0} \quad (130)$$

or, for a low shear rate, Eq. (129) reduces to:

$$\mathbb{S} = \text{Wi} \quad (131)$$

In pipe extrusion, we freeze the pipe from the outside. We care about the maximum shear stress at the outer surface of the extrudate, which, for a sag compensating die, is at the top $[(\xi, \theta) = (\xi_o, 0)]$ (see Figure 1) [61].

Adimensionalizing Eq. (18) gives:

$$\text{Wi} = \sqrt{S} \frac{(1 + \xi^2 - 2\xi \cos \theta)}{\xi} \sqrt{\left(\xi \frac{\partial \tilde{v}_z}{\partial \xi} \right)^2 + \left(\frac{\partial \tilde{v}_z}{\partial \theta} \right)^2} \quad (132)$$

Substituting Eqs. (61) through (64), and (74) through (112) into Eq. (132) then evaluating at $\xi = \xi_o$ and $\theta = 0$ gives the maximum Weissenberg number:

$$\text{Wi}_m = \sqrt{S} (1 + \xi_o^2 - 2\xi_o) [\mathbb{S}_0 + (1 - \sigma) \mathbb{S}_1] \quad (133)$$

where:

$$\mathbb{S}_0 \equiv \sum_{n=0}^{\infty} \phi'(\xi_o) \frac{2\xi_o - 2}{(1 - 2\xi_o + \xi_o^2)^2} \quad (134)$$

$$\mathbb{S}_1 \equiv \sum_{n=0}^{\infty} \psi'(\xi_o) \quad (135)$$

Substituting Eq. (133) into Eq. (131) then gives the maximum dimensionless shear stress:

$$\mathbb{S}_m = \sqrt{S} (1 + \xi_o^2 - 2\xi_o) [\mathbb{S}_0 + (1 - \sigma) \mathbb{S}_1] \quad (136)$$

Eq. (136) and Figure 21 and Figure 22 can help plastics engineers find the maximum shear stress, and thus, can help them operate without melt fracture.

Our expression for maximum shear stress is also the upper-bound, for the stress that is frozen into the outermost layer of the plastic pipe (since this layer is quenched first and normally shortly after extrusion). Of course, for semicrystalline polymers, shrinkage will superpose additional stresses upon the frozen pipe. Eq. (136) and Figure 21 and Figure 22 can thus help plastics engineers estimate the maximum stress that is frozen into the outer surfaces of pipe and hose.

d. AXIAL FORCE ON MANDREL

The axial force on the mandrel per unit length is given by:

$$F_z \equiv \frac{1}{L} \int_A \tau_{\xi z}(\xi_i, \theta) dA_{\theta z} \quad (137)$$

where L is the extrusion die length (see Figure 2). Substituting Eq. (11) into Eq. (137), and integrating with respect to z gives:

$$F_z = \int_0^{2\pi} \tau_{\xi z}(\xi_i, \theta) \frac{a\xi_i}{1 + \xi_i^2 - 2\xi_i \cos \theta} d\theta \quad (138)$$

Adimensionalizing Eq. (138) gives:

$$\mathbb{F} = \int_0^{2\pi} S_i \frac{\xi_i}{1 + \xi_i^2 - 2\xi_i \cos \theta} d\theta \quad (139)$$

where:

$$\mathbb{F} \equiv \frac{\sqrt{\sigma_1} F_z}{\eta_0 a} \quad (140)$$

Evaluating Eq. (139) [with Eqs. (129) and (133)] at the inner contour, $\xi = \xi_i$, and then substituting into Eq. (139) gives:

$$\mathbb{F} = \int_0^{2\pi} \xi_i \sqrt{S} \left[\frac{1 + S(1 + \xi_i^2 - 2\xi_i \cos \theta)^2 \frac{\partial \tilde{v}_z}{\partial \xi} \Big|_{\xi=\xi_i}^2}{1 + \sigma S(1 + \xi_i^2 - 2\xi_i \cos \theta)^2 \frac{\partial \tilde{v}_z}{\partial \xi} \Big|_{\xi=\xi_i}^2} \right] \frac{(1 + \xi_i^2 - 2\xi_i \cos \theta) \frac{\partial \tilde{v}_z}{\partial \xi} \Big|_{\xi=\xi_i}}{1 + \xi_i^2 - 2\xi_i \cos \theta} d\theta \quad (141)$$

which yields no analytical solution. However, for small Wi , substituting Eq. (131) [with Eq. (133)] into Eq. (139), and integrating gives:

$$\mathbb{F} = 2\pi \xi_i \sqrt{S} \left[K_0 - \frac{2\xi_i}{1 - 2\xi_i^2 + \xi_i^4} + (1 - \sigma) S L_0 \right] \quad (142)$$

where:

$$\mathbb{F}_0 \equiv K_0 = \phi'_0(\xi_i) \quad (143)$$

and:

$$\mathbb{F}_1 \equiv L_0 = \psi'_0(\xi_i) \quad (144)$$

We use Eqs. (143) and (144) to plot Figure 23 and Figure 24. These two figures with Eq. (142) can help plastics engineers design mandrel attachments to withstand the axial force exerted by molten plastic flowing through the die annulus.

e. LATERAL FORCE ON MANDREL

To calculate the lateral force on the mandrel, we use Eqs. (24) and (59) from [12], subject to our correction to the Jones [12] velocity profile [see Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112) above]:

$$F_x = \Re \left\{ i \oint \tau_{\xi\xi}(\dot{\gamma}) \Big|_{\xi=\xi_i} dZ \right\} \quad (145)$$

where we quarter the normal component of the extra stress tensor (evaluated on the mandrel) into:

$$\begin{aligned} \tau_{\xi\xi}(\dot{\gamma}) \Big|_{\xi=\xi_i} \equiv & \left[\frac{(\lambda_1 - \mu_1)\eta_0(1 - \sigma\gamma^2)}{1 - \gamma^2} - (\lambda_2 - \mu_2)\eta_0 \right] \gamma^2 \\ & - \frac{1}{2}(\lambda_1 - \mu_1 - \lambda_2 + \mu_2) \left[\sigma\gamma^2 + \ln(1 + \gamma^2) - \sigma \ln(1 + \gamma^2) \right] \end{aligned} \quad (146)$$

and:

$$Z \equiv \frac{a}{1 - \xi_i \cos \theta - i \xi_i \sin \theta} \quad (147)$$

Substituting Eq. (147) into Eq. (145) then gives:

$$F_x = -i \xi_i a \int_{-\pi}^{\pi} \frac{\sin \theta - i \cos \theta}{(1 - \xi_i \cos \theta - i \xi_i \sin \theta)^2} \tau_{\xi\xi}(\dot{\gamma}) \Big|_{\xi=\xi_i} d\theta \quad (148)$$

for small Wi , Eq. (146) becomes:

$$\tau_{\xi\xi}(\dot{\gamma}) \Big|_{\xi=\xi_i} = (\lambda_1 - \mu_1 - \lambda_2 + \mu_2) \left(1 - \frac{1}{2}\sigma\right) \eta_0 \dot{\gamma}(\xi_i, \theta)^2 \quad (149)$$

Evaluating Eq. (18) at $\xi = \xi_i$, then squaring gives:

$$\dot{\gamma}(\xi_i, \theta)^2 = \frac{(1 - \xi_i \cos \theta - i \xi_i \sin \theta)^2 (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2}{a^2} \left(\frac{\partial v_z}{\partial \xi} \Big|_{\xi=\xi_i} \right)^2 \quad (150)$$

Substituting Eq. (44) into Eq. (150) gives:

$$\dot{\gamma}(\xi_i, \theta)^2 = \frac{(1 - \xi_i \cos \theta - i \xi_i \sin \theta)^2 (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2}{a^2} W^2 \left(\sum_{n=0}^{\infty} (K_n + (1 - \sigma)SL_n) \cos(n\theta) \right)^2 \quad (151)$$

and Eqs. (149) and (151) into Eq. (148) gives:

$$F_x = -i\xi_i \frac{1}{a} (\lambda_1 - \mu_1 - \lambda_2 + \mu_2) (1 - \frac{1}{2}\sigma) \eta_0 W^2 \quad (152)$$

$$\times \int_{-\pi}^{\pi} (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left(\sum_{n=0}^{\infty} K_n \cos n\theta + \sum_{n=0}^{\infty} (1 - \sigma) S L_n \cos n\theta \right)^2 d\theta$$

where $W \equiv a^2 P / 4 \eta_0$ and which expands as:

$$F_x = -i\xi_i \frac{1}{a} (\lambda_1 - \mu_1 - \lambda_2 + \mu_2) (1 - \frac{1}{2}\sigma) \eta_0 W^2 \quad (153)$$

$$\times \left[\int_{-\pi}^{\pi} (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left(\sum_{n=0}^{\infty} K_n \cos(n\theta) \right)^2 d\theta \right.$$

$$\left. + 2 \int_{-\pi}^{\pi} (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left(\sum_{n=0}^{\infty} K_n \cos(n\theta) \right) \times \sum_{n=0}^{\infty} (1 - \sigma) S L_n \cos(n\theta) d\theta \right.$$

$$\left. + (1 - \sigma)^2 S^2 \int_{-\pi}^{\pi} (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left(\sum_{n=0}^{\infty} L_n \cos(n\theta) \right)^2 d\theta \right]$$

or:

$$\chi = \chi_0 + (1 - \sigma) S \chi_1 + (1 - \sigma)^2 S^2 \chi_2 \quad (154)$$

which we truncate to:

$$\chi = \chi_0 + (1 - \sigma) S \chi_1 \quad (155)$$

where:

$$\chi \equiv \frac{F_x}{\xi_i a^3 P^2 (\lambda_1 - \mu_1 - \lambda_2 + \mu_2) (1 - \frac{1}{2}\sigma)} \quad (156)$$

$$\chi_0 \equiv \int_{-\pi}^{\pi} \left[-i (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left[\sum_{n=0}^{\infty} K_n \cos n\theta \right] \right] d\theta \quad (157)$$

$$\chi_1 \equiv \int_{-\pi}^{\pi} \left[-2i (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \sum_{n=0}^{\infty} K_n \cos n\theta \left[\sum_{n=0}^{\infty} L_n \cos n\theta \right] \right] d\theta \quad (158)$$

$$\chi_2 \equiv \int_{-\pi}^{\pi} (\sin \theta - i \cos \theta) (1 - \xi_i \cos \theta + i \xi_i \sin \theta)^2 \left(\sum_{n=0}^{\infty} L_n \cos(n\theta) \right)^2 d\theta + \dots \quad (159)$$

and where:

$$K_n \equiv \phi'_n(\xi_i) \quad (160)$$

$$L_n \equiv \psi'_n(\xi_i) \quad (161)$$

Since Eqs. (157) and (158) do not yield to analytical solution, we evaluate these numerically (see Figure 25 and Figure 26).

From Figure 25 and Figure 26, we see that, for Newtonian or shear-thinning fluids ($\sigma \leq 1$), χ is always negative. Hence, from Eq. (156), we see that the lateral force, F_x , will be negative when the coefficient $(\lambda_1 - \mu_1 - \lambda_2 + \mu_2)(1 - \frac{1}{2}\sigma)$ is positive. Since this coefficient, for Newtonian or shear-thinning fluids, will normally be positive, the lateral force on a sag compensating die will be downward.

VI. CONCLUSION

We report a new expression for the lateral force exerted on a pipe extrusion die by an elastic liquid obeying the Oldroyd 8-constant model [Eq. (155) with Eqs. (157)-(158)] which corrects the previous contribution [see rapid convergence approximation in Eq. (70)] due to Jones [12]. For Newtonian or shear-thinning fluids ($\sigma \leq 1$), we find the sign of this lateral force to be decentering.

Each quantity calculated in this paper, including the (i) throughput, (ii) extrudate shape, (iii) stresses in the melt, and (iv) forces on the mandrel, F_x and F_z , consists of both Newtonian and non-Newtonian contributions [see subhead to Section V above]. Furthermore, each of these contributions takes the form of the product of a coefficient depending only on the problem geometry, and a power of $(1 - \sigma)S$. This power is zero for each Newtonian contribution, and one for the non-Newtonian. This form of the answer is useful because the geometry depending coefficients can be summarized in convenient graphs that practitioners can use to solve polymer processing problems graphically. In this paper, we have constructed such figures for extruding plastic pipe through eccentric dies, and specifically for (i) throughput [Figure 15 and Figure 16], (ii) extrudate shape [Figure 17, Figure 18, Figure 19 and Figure 20], (iii) stresses in the melt [Figure 21 and Figure 22], and (iv) forces on the mandrel, F_x [Figure 23 and Figure 24] and F_z [Figure 25 and Figure 26]. This useful form of the answers arises by virtue of our choice of the polymer process partitioning method, a method perfectly suited for the rich diversity of constitutive equations encompassed by the Oldroyd 8-constant model (see Section III above). The polymer process partitioning method involves expanding the velocity field into powers of $(1 - \sigma)S$, so that the zeroth power term gives the Newtonian contribution, and the next, the non-Newtonian.

Though the polymer process partitioning method has been called “complex mathematically” (see last paragraph, p. 125 of [28]), we conclude that the especially useful form of the results make its complexity worthwhile. We produce this paper with detail sufficient for graduate students to teach themselves the polymer process partitioning method, and to deepen their understanding of polymer flow through straight eccentric annular dies. Further, we provide the following worked example to teach practitioners how to use our results to help them extrude plastic pipe from eccentric dies.

VII. WORKED EXAMPLE

An engineer gathers the following process details for her high density polyethylene pipe extrusion line, where the pipe dimensions are $R_o = 0.32\text{m}$, $R_i = 0.22\text{m}$, $\delta = 0.01\text{m}$ with the throughput of $\rho Q = 0.01\text{kg/s}$, and the material

properties at the operating temperature are $\eta_0 = 1.1 \times 10^7 \text{ Pa}\cdot\text{s}$ and $\rho = 900 \text{ kg/m}^3$. From her measured crossover frequency, ω_c , of her $\eta'(\omega)$ and $\eta''(\omega)$ curves at the operating temperature, she estimates a characteristic relaxation time of $\lambda_1 = 1/\omega_c = 1.38 \text{ s}$.

She wants to calculate the (i) pressure drop, (ii) extrudate shape, (iii) stresses in the melt, and finally, the (iv) forces on the mandrel, F_x and F_z .

To get the pressure drop per unit length, P , she begins by calculating $\xi_i = 0.0409$, $\xi_o = 0.0594$, $a = 5.3720 \text{ m}$, $\sigma_1 = 1.9 \text{ s}^2$ and $\sigma = 0$ by inserting the given information into, in order, Eqs. (65), (67), (4), (20) and (24). Solving Eq. (115) yields the pressure gradient:

$$P = \frac{4\eta_0 Q}{a^4 \bar{Q}} \quad (162)$$

Substituting Eq. (117) into Eq. (162), and since $\sigma = 0$, gives:

$$P = \frac{4\eta_0 Q}{a^4 (\bar{Q}_0 + S\bar{Q}_1)} \quad (163)$$

Substituting Eq. (45) into Eq. (163) then gives:

$$P = \frac{4\eta_0 Q}{a^4 \left(\bar{Q}_0 + \frac{\sigma_1 a^2 P^2}{16\eta_0^2} \bar{Q}_1 \right)} \quad (164)$$

Solving the cubic Eq. (164) yields:

$$P = \frac{c}{6} - \frac{2B}{c} \quad (165)$$

where:

$$B \equiv \frac{16\eta_0^2 \bar{Q}_0}{\sigma_1 a^2 \bar{Q}_1} \quad (166)$$

$$c \equiv \left(108A + 12\sqrt{12B^3 + 81A^2} \right)^{1/3} \quad (167)$$

and:

$$A \equiv \frac{64\eta_0^3 Q}{\sigma_1 a^6 \bar{Q}_1} \quad (168)$$

To evaluate Eqs. (168), (166) and (167), we need $\bar{Q}_0$ and $\bar{Q}_1$. Interpolating Figure 15 and Figure 16 for $\xi_i = 0.0409$ and $\xi_o = 0.0594$ gives $\bar{Q}_0 = 6.94 \times 10^{-7}$ and $\bar{Q}_1 = 2.34 \times 10^{-10}$. Evaluating Eqs. (168), (166) and (167):

$$A = \frac{64 \left(1.1 \times 10^7 \text{ Pa}\cdot\text{s} \right)^3 \frac{0.01 \text{ kg/s}}{900 \text{ kg/m}^3}}{1.9 \text{ s}^2 (5.372 \text{ m})^6 2.34 \times 10^{-10}} = 8.837 \times 10^{22} \frac{\text{Pa}^3}{\text{m}^3} \quad (169)$$

$$B = \frac{16(1.1 \times 10^7 \text{ Pa}\cdot\text{s})^2 6.94 \times 10^{-7}}{1.9 \text{ s}^2 (5.372 \text{ m})^2 2.34 \times 10^{-10}} = 1.045 \times 10^{17} \frac{\text{Pa}^2}{\text{m}^2} \quad (170)$$

$$c = \left(108 \left(8.837 \times 10^{22} \frac{\text{Pa}^3}{\text{m}^3} \right) + 12 \sqrt{12 \left(1.045 \times 10^{17} \frac{\text{Pa}^2}{\text{m}^2} \right)^3 + 81 \left(8.837 \times 10^{22} \frac{\text{Pa}^3}{\text{m}^3} \right)^2} \right)^{1/3} \quad (171)$$

$$= 1.122 \times 10^9 \frac{\text{Pa}}{\text{m}}$$

and inserting the values in Eqs. (170) and (171) into Eq. (165) yields the required pressure drop per unit die land length:

$$P = \frac{1.122 \times 10^9 \frac{\text{Pa}}{\text{m}}}{6} - \frac{2 \left(1.045 \times 10^{17} \frac{\text{Pa}^2}{\text{m}^2} \right)}{1.122 \times 10^9 \frac{\text{Pa}}{\text{m}}} = 8.46 \times 10^5 \frac{\text{Pa}}{\text{m}} \quad (172)$$

(ii) To get the extrudate shape, $\langle \tilde{v}_z \rangle_{\theta=0} / \langle \tilde{v}_z \rangle_{\theta=\pi}$, she will need five ingredients, in addition to the given information, for Eq. (126) which, since $\sigma = 0$, simplifies to:

$$\frac{\langle \tilde{v}_z \rangle_{\theta=0}}{\langle \tilde{v}_z \rangle_{\theta=\pi}} = \frac{\bar{Q}_{\theta,0}(0) + S\bar{Q}_{\theta,1}(0)}{\bar{Q}_{\theta,0}(\pi) + S\bar{Q}_{\theta,1}(\pi)} \quad (173)$$

Using Eq. (45), she first evaluates the dimensionless pressure gradient squared:

$$S \equiv \frac{\sigma_1 a^2 P^2}{16\eta_0^2} = \frac{1.9 \text{ s}^2 (5.372 \text{ m})^2 \left(8.46 \times 10^5 \frac{\text{Pa}}{\text{m}} \right)^2}{16(1.1 \times 10^7 \text{ Pa}\cdot\text{s})^2} = 0.02031 \quad (174)$$

and thus S^2 is well below one, as it should be [see Eq. (58)]. Interpolating Figure 17 through Figure 20 for $\xi_i = 0.0409$ and $\xi_o = 0.0594$ gives

$\bar{Q}_{\theta,0}(0) = 1.6 \times 10^{-7}$, $\bar{Q}_{\theta,0}(\pi) = 7.181 \times 10^{-8}$, $\bar{Q}_{\theta,1}(0) = 4.8136 \times 10^{-7}$ and $\bar{Q}_{\theta,1}(\pi) = 3.214 \times 10^{-7}$. Evaluating Eq. (173) yields the shape of the extrudate that will enter her cooling chamber:

$$\frac{\langle \tilde{v}_z \rangle_{\theta=0}}{\langle \tilde{v}_z \rangle_{\theta=\pi}} = \frac{1.6 \times 10^{-7} + (0.02)4.8136 \times 10^{-7}}{7.181 \times 10^{-8} + (0.02)3.214 \times 10^{-7}} = 2.168 \quad (175)$$

She will use this shape in her simulation of sag.

(iii) To calculate the maximum shear stress in the molten plastic, $\mathbb{S}_m$, she uses Eq. (136) which, since $\sigma = 0$, simplifies to:

$$\mathbb{S}_m = \sqrt{S} (1 + \xi_o^2 - 2\xi_o) [\mathbb{S}_0 + S\mathbb{S}_1] \quad (176)$$

Interpolating Figure 21 and Figure 22 for $\xi_i = 0.0409$ and $\xi_o = 0.0594$ gives $\mathbb{S}_0 = -0.04361$ and $\mathbb{S}_1 = -0.1361$, and substituting these into Eq. (176) gives:

$$\begin{aligned}\mathbb{S}_m &= \sqrt{0.02} \left\{ 1 + (0.0594)^2 - 2(0.0594) \right\} [-0.04361 + 0.02(-0.1361)] \\ &= -0.005848\end{aligned}\quad (177)$$

Solving Eq. (130) for the maximum dimensional shear stress gives:

$$\tau_{\xi z} = \frac{\mathbb{S}_m \eta_0}{\sqrt{\sigma_1}} = \frac{-0.005848(1.1 \times 10^7 \text{ Pa}\cdot\text{s})}{\sqrt{1.9 \text{ s}^2}} = -46.6 \text{ kPa} \quad (178)$$

which she understands to be the upper-bound for the shear stress that will be frozen in to the outer skin of the pipe product.

(iv) For the axial forces on the mandrel, she uses Eq. (142) which, since $\sigma = 0$, reduces to:

$$\mathbb{F} = 2\pi\xi_i\sqrt{S}\left[K_0 - \frac{2\xi_i}{1-2\xi_i^2+\xi_i^4} + SL_0\right] \quad (179)$$

Interpolating Figure 23 and Figure 24 for $\xi_i = 0.0409$ and $\xi_o = 0.0594$ gives $K_0 = 0.122$ and $L_0 = 5.71 \times 10^{-6}$, and substituting these into Eq. (142) gives:

$$\begin{aligned}\mathbb{F} &= 2\pi(0.0409)\sqrt{0.02}\left[0.122 - \frac{2(0.0409)}{1-2(0.0409)^2+(0.0409)^4} + 0.02(5.71 \times 10^{-6})\right] \\ &= 0.001462\end{aligned}\quad (180)$$

Solving Eq. (140) for dimensional axial force gives:

$$F_z = \frac{a\eta_0\mathbb{F}}{\sqrt{\sigma_1}} = \frac{(5.372 \text{ m})(1.1 \times 10^7 \text{ Pa}\cdot\text{s})0.001462}{\sqrt{1.9 \text{ s}^2}} = 62.6 \text{ kN} \quad (181)$$

Finally, she evaluates Eq. (155) for the lateral force, which, since $\sigma = 0$, reduces to:

$$\chi = \chi_0 + S\chi_1 \quad (182)$$

Interpolating Figure 25 and Figure 26 for $\xi_i = 0.0409$ and $\xi_o = 0.0594$ gives $\chi_0 = -1.542$ and $\chi_1 = -4.959 \times 10^{-3}$, and substituting these into Eq. (182) gives:

$$\chi = -1.542 + 0.02(-4.959 \times 10^{-3}) = -1.542 \quad (183)$$

Solving Eq. (156), and then, since $\sigma = 0$, this simplifies to:

$$\begin{aligned}F_x &= \frac{\lambda_1\xi_i a^3 P^2}{16\eta_0}\chi = \frac{-1.38 \text{ s}(5.3720 \text{ m})^3 0.0409(8.46 \times 10^5)^2}{16(1.1 \times 10^7 \text{ Pa}\cdot\text{s})}1.542 \\ &= -54.8 \text{ kN}\end{aligned}\quad (184)$$

If the cooling chamber were removed, the now freely emerging extrudate would now curve with a radius of curvature given by Eq. (127). Substituting the value from Eq. (175) into Eq. (127) gives the dimensionless radius of curvature:

$$\frac{r_0}{(R_i + R_o)/2} = \frac{2.168 + 1}{2.168 - 1} = 2.712 \quad (185)$$

Solving Eq. (185) for the dimensional radius of curvature gives:

$$r_0 = \frac{(0.22\text{m} + 0.32\text{m})}{2} 2.712 = 0.732\text{m} \quad (186)$$

Of course, curved conduit of other radii of curvature can be produced by adjusting the extrudate shape. For this, we solve Eq. (185):

$$\frac{\langle \tilde{v}_z \rangle_{\theta=0}}{\langle \tilde{v}_z \rangle_{\theta=\pi}} = \frac{\left[\frac{r_0}{(R_i + R_o)/2} \right] + 1}{\left[\frac{r_0}{(R_i + R_o)/2} \right] - 1} \quad (187)$$

which gives the required shape for a given extrudate curvature. This specific worked example happens to be the one we show to illustrated in Figure 2.

VIII.ACKNOWLEDGMENT

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Table I: Literature on Analytical Solutions for Flow between Straight Eccentric Cylinders

	Constitutive Model	Coordinates System	Velocity Profile	Extrudate Shape	Shear Stress	Forces	[Ref.] (Correction to)
Caldwell (1930)	N	bi	$\mathbb{C}$	Σ			[52]
Piercy <i>et al.</i> (1933)	N	bi	Σ	Σ			[53]
Oldroyd (1947)	B	ec	Σ				[54]
Heyda (1959)	N	bi	Σ				[55]
Berker (1963)	N	bi					Sect. 23.9 [56]
Jones (1964); Jones and Jones (1966)	O_8	ec	Σ			χ	[12,14]
Snyder <i>et al.</i> (1965)	N	bi	Σ		Σ		[57]
Camilleri and Jones (1966)	O_8	ec	Σ				[45]
Guckes (1975)	P	bi		$\mathfrak{N}$			[58]
Ballal and Rivlin (1975)	RE	bi	Σ				[59]
Zidan and Hassan (1985)	$O^{(2)}$	bi	Σ				[60]
Kolitawong and Giacomini (2001)	P	bi	$\mathbb{C}$	$\mathbb{C}\mathfrak{N}$	Σ	$\mathbb{F}$	[61]
This paper	O_8	ec	Σ	$\mathbb{C}\mathfrak{N}$	$\mathbb{C}\mathfrak{N}$	$\mathbb{F}, \chi$	(12)

Legend: B $\equiv$ Bingham; O_8 $\equiv$ 8-constant Oldroyd; P $\equiv$ power-law; N $\equiv$ Newtonian; $O^{(2)}$ $\equiv$ Second-order; RE $\equiv$ Rivlin-Ericksen; ec $\equiv$ eccentric cylindrical coordinates; bi $\equiv$ bipolar cylindrical coordinates; $\mathbb{F}$ $\equiv$ axial force; χ $\equiv$ lateral force; Σ $\equiv$ series; $\mathbb{C}$ $\equiv$ closed form; $\mathfrak{N}$ $\equiv$ numerical; $\mathbb{C}\mathfrak{N}$ $\equiv$ closed form with integral defined function;

Table II: Symbol Correspondence between This Paper and Jones [12]

Variable name	This paper	Jones [12]
Axial velocity	v_z	ω
Axial velocity, dimensionless	$\tilde{v}_z$	ω'
Axial velocity, Newtonian contribution	v_0	ω_0
Axial velocity, non-Newtonian variable	v_1	ω_1
Confocal length	a	d
Eccentric angular cylindrical coordinate	θ	φ
Eccentric radial cylindrical coordinate	ξ	r
Eccentricity	δ	h
Extra stress tensor	τ_{ij}	p'_{ij}
Inner contour	ξ_i	a
Magnitude of rate of strain tensor	$\dot{\gamma}$	γ
Magnitude of rate of strain tensor, dimensionless	$\tilde{\gamma}$	γ'
Outer contour	ξ_o	b
Steady shear viscosity function	$\eta(\dot{\gamma})$	$F(\gamma)$

Table III: Dimensional Variables

Name	Unit	Symbol	Eq.
Any coordinates, i th	L	x_i	
Any tensor, ik th component		b_{ik}	(15)
Area on mandrel per unit length	L	Z	(147)
Area, $\xi\theta$ – plane	L^2	$A_{\xi\theta}$	(10)
Area, θz – plane	L^2	$A_{\theta z}$	(11)
Axial force	ML/t^2	F_z	(137)
Axial velocity	L/t	v_z	
Cartesian coordinates	L	x, y, z_c	
Characteristic velocity	L/t	$W \equiv a^2 P / 4\eta_0$	
Confocal length	L	a	(4)
Corotational derivative	t^{-1}	$\frac{\mathcal{D}}{\mathcal{D}t}$	(15)
Crossover angular frequency	t^{-1}	ω_c	
Density	M/L^3	ρ	
Die land length	L	L	
Eccentric cylindrical coordinate, axial direction	L	z	
Eccentricity	L	δ	
Extra stress tensor	M/Lt^2	τ_{ik}	
First normal stress coefficient	M/L	$\Psi_1 \equiv (\tau_{11} - \tau_{22})/\dot{\gamma}^2$	(30)
Infinite shear rate viscosity	M/Lt	η_∞	(25)
Lateral force	ML/t^2	F_x	(145)
Magnitude of the rate of deformation tensor	t^{-1}	$\dot{\gamma}$	(18)
Magnitude of the rate of deformation tensor, inflection point	t^{-1}	$\dot{\gamma}_c$	(27)
Minus the imaginary part of complex viscosity	M/Lt	η''	
Oldroyd coefficient	t	μ_0	(12)
Oldroyd coefficient	t	μ_1	(12)
Oldroyd coefficient	t	μ_2	(12)
Oldroyd coefficient	t	ν_1	(12)
Oldroyd coefficient	t	ν_2	(12)

Oldroyd coefficient, relaxation time	t	λ_1	(12)
Oldroyd coefficient, retardation time	t	λ_2	(12)
Oldroyd shear thickening constant	t^2	σ_2	(21)
Oldroyd shear thinning constant	t^2	σ_1	(20)
Pressure gradient in die land	M/Lt^2	$P \equiv -dp/dz$	(46)
Radius of curvature of the free extrudate	L	r_0	(127)
Radius, die barrel	L	R_o	
Radius, die mandrel	L	R_i	
Rate of deformation tensor	t^{-1}	$\dot{\gamma}_{ik}$	(14)
Real part of complex viscosity	M/Lt	η'	
Scale factor, eccentric angular cylindrical coordinate	L	h_θ	(9)
Scale factor, eccentric radial cylindrical coordinate	L	h_ξ	(8)
Second normal stress coefficient	M/L	$\Psi_1 \equiv (\tau_{22} - \tau_{33})/\dot{\gamma}^2$	(31)
Steady shear viscosity function	M/Lt	η	(19)
Steady shear viscosity, viscosity curve inflection	M/Lt	η_i	(28)
Total stress tensor	M/Lt^2	π_{ij}	(13)
Volumetric flow rate	L^3/t	Q	(113)
Vorticity tensor	t^{-1}	ω_{ij}	(16)
Zero shear viscosity	M/Lt	η_0	

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

Table IV: Dimensionless Variables and Groups

Name	Symbol	Eq.
Aspect ratio	$\kappa \equiv R_i/R_o$	
Average velocity at angular position θ	$\bar{Q}_\theta(\theta)$	(122)
Average velocity at angular position θ	$\langle \bar{v}_z \rangle_\theta$	(120)
Axial force	$\mathbb{F}$	(140)
Axial force, Newtonian contribution	$\mathbb{F}_0$	(143)
Axial force, non-Newtonian variable	$\mathbb{F}_1$	(144)
Axial velocity	$\bar{v}_z \equiv \frac{4\eta_0}{P\xi_i^2} v_z$	(44)
Axial velocity, Newtonian contribution	v_0	(61)
Axial velocity, non-Newtonian variable	v_1	(74)
Axial velocity, non-Newtonian variable, particular solution, m th group, n th harmonic	$v_{1p,n}^{(m)}$	(79)-(85), (89)-(94), (101)-(106)
Complex conjugate of zeta	$\zeta^* \equiv \xi e^{-i\theta}$	(52)
Complex zeta	$\zeta \equiv \xi e^{i\theta}$	(51)
Constant, m th group, n th harmonic, ℓ th element	$C_{n,\ell}^{(m)}$	(86)-(88), (95)-(100), (107)-(112)
Eccentric angular cylindrical coordinate	θ	
Eccentric radial cylindrical coordinate	ξ	
Eccentric radial cylindrical coordinate , inner contour	ξ_i	(66)
Eccentric radial cylindrical coordinate , outer contour	ξ_o	(67)
Eccentricity	$\epsilon \equiv \delta/R_o$	
Integration coefficient, non-Newtonian variable, n th harmonic	A_n	(76)
Integration coefficient, non-Newtonian variable, n th harmonic	B_n	(76)
Integration coefficient, Newtonian contribution, n th harmonic	α_n	(63)
Integration coefficient, Newtonian contribution, n th harmonic	β_n	(64)
Kronecker delta	δ_{ik}	

Lateral force	χ	(156)
Lateral force, Newtonian contribution	χ_0	(157)
Lateral force, non-Newtonian variable	χ_1	(158)
Non-Newtonian coefficient	$(1 - \sigma)S$	
Oldroyd constant ratio	$\sigma \equiv \sigma_2 / \sigma_1 = \eta_\infty / \eta_0$	(24), (25)
Oldroyd model constant ratio	$\sigma \equiv \sigma_2 / \sigma_1$	(24)
Power-law index	n	(29)
Pressure gradient squared	$S \equiv \frac{\sigma_1 a^2 P^2}{16 \eta_0^2}$	(45)
Shear stress	$\mathbb{S} \equiv \sqrt{\sigma_1} \tau_{\xi z} / \eta_0$	(130)
Shear stress, maximum	$\mathbb{S}_m \equiv \sqrt{\sigma_1} \tau_{\xi z, m} / \eta_0$	(130)
Shear stress, maximum, Newtonian contribution	$\mathbb{S}_0$	(134)
Shear stress, maximum, non-Newtonian variable	$\mathbb{S}_1$	(135)
Spatial function of axial velocity, Newtonian contribution, n th harmonic	ϕ_n	(62)
Spatial function of axial velocity, non-Newtonian variable, n th harmonic	ψ_n	(75)
Spatial function of axial velocity, non-Newtonian variable, n th harmonic	$\psi_n^{(m)}$	(76)
Velocity gradient, n th harmonic, Newtonian contribution	$K_n \equiv \phi_n'(\xi_i)$	(143)
Velocity gradient, n th harmonic, non-Newtonian variable	$L_n \equiv \psi_n'(\xi_i)$	(144)
Velocity vector	$\check{\mathbf{v}}$	
Volumetric flow rate	$\check{Q}$	(115)
Volumetric flow rate of wedge $d\theta$ at angular position θ , Newtonian contribution	$\check{Q}_{\theta,0}(\theta)$	(124)
Volumetric flow rate of wedge $d\theta$ at angular position θ , non-Newtonian variable	$\check{Q}_{\theta,1}(\theta)$	(125)
Volumetric flow rate, Newtonian contribution	$\check{Q}_0$	(118)
Volumetric flow rate, non-Newtonian variable	$\check{Q}_1$	(119)
Weissenberg number	$Wi \equiv \sqrt{\sigma_1} \dot{\gamma}$	(22)

Weissenberg number, viscosity curve inflection	Wi_i	(28)
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Figure 1: End view of pipe extrusion die defining the die dimensions.

Figure 2: Side view of pipe extrusion die defining the die dimensions. The surfaces of the mandrel and the barrel correspond to $\xi = \xi_i$ and $\xi = \xi_o$.

Figure 3: Eccentric cylindrical coordinates showing the circles of constant **eccentric radial coordinate**, ξ , and the circles of constant **eccentric angular coordinate**, θ . **Cyan** example between $\xi_i = 0.2$ and $\xi_o = 0.4$.

Figure 4: Differential volume element of the molten plastic in eccentric cylindrical coordinates (**cyan**). Edges of differential surface element through which melt flows labeled $\frac{a}{1-2\xi \cos \theta + \xi^2} d\xi$ and $\frac{a}{1-2\xi \cos \theta + \xi^2} d\theta$. Cylindrical shell of constant eccentric cylindrical radial coordinate ξ parallels z -axis, and intersects x -axis at $x = \frac{a}{1-\xi^2}$. This cylindrical shell contains surface element whose edges are dashed and labeled $\frac{a}{1-2\xi \cos \theta + \xi^2} d\theta$ and dz .

Figure 5: Dimensionless viscosity curves for shear thinning ($\sigma = \frac{1}{2}, \frac{3}{4}$), Newtonian ($\sigma = 1$) and shear thickening ($\sigma = \frac{5}{4}, \frac{3}{2}$) behaviors. **Blue** dots indicate inflections given by Eq. (28).

Figure 6: For shear-thinning fluid ($\sigma = \frac{1}{2}$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\tilde{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{2}{5}$. **Black** indicates ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] and **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]]. Discrepancies illustrate improvement from rapid convergence approximation [Eq. (70)].

Figure 7: For Newtonian fluid ($\sigma = 1$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\tilde{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{2}{5}$. **Black**, ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] closely matched **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]].

Figure 8: For shear-thickening fluid ($\sigma = \frac{3}{2}$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\tilde{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{2}{5}$. **Black** indicates ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] and **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]]. Discrepancies illustrate improvement from rapid convergence approximation [Eq. (70)].

Figure 9: For shear-thinning fluid ($\sigma = \frac{1}{2}$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\tilde{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{3}{5}$. **Black** indicates ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] and **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]]. Discrepancies illustrate improvement from rapid convergence approximation [Eq. (70)].

Figure 10: For Newtonian fluid ($\sigma = 1$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\check{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{3}{5}$. **Black**, ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] closely matched **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]].

Figure 11: For shear-thickening fluid ($\sigma = \frac{3}{2}$, $S = \frac{1}{5}$), dimensionless axial velocity profiles, $\check{v}_z$ versus ξ , around the eccentric annulus (curves of constant eccentric angular coordinate $\theta = 0, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{5}{6}\pi, \pi$ from top to bottom) for the particular die shape $\xi_i = \frac{1}{5}$ and $\xi_o = \frac{3}{5}$. **Black** indicates ours [Eq. (47) with Eqs. (61) through (64) and with Eqs. (74) through (112)] and **red**, Jones's [Eq. (58) in [12] with Eqs. (49) through (51) and (53) through (57) in [12]]. Discrepancies illustrate improvement from rapid convergence approximation [Eq. (70)].

Figure 12: Dimensionless speed, $\tilde{v}_z$, over the pipe cross section (30 contours of constant $\tilde{v}_z$). Oldroyd 8-constant fluids: shear-thinning, $\sigma = \frac{1}{2}$ (left) *versus* Newtonian, $\sigma = 1$ (right). $S = \frac{1}{20}$, $\xi_i = \frac{1}{10}$ and $\xi_o = \frac{1}{5}$. The **black** near-wall regions are nearly motionless, and the **red**, highest speed ($\tilde{v}_{z,\max} = 9.77 \times 10^{-3}$ for $\sigma = \frac{1}{2}$, $\tilde{v}_{z,\max} = 9.45 \times 10^{-3}$ for $\sigma = 1$).

Figure 13: Dimensionless speed, $\tilde{v}_z$, over the pipe cross section (30 contours of constant $\tilde{v}_z$). Oldroyd 8-constant fluids: shear thickening, $\sigma = \frac{3}{2}$ (left) *versus* Newtonian, $\sigma = 1$ (right). $S = \frac{1}{20}$, $\xi_i = \frac{1}{10}$ and $\xi_o = \frac{1}{5}$. The **black** near-wall regions are nearly motionless, and the **red**, highest speed ($\tilde{v}_z = 9.14 \times 10^{-3}$ for $\sigma = \frac{3}{2}$, $\tilde{v}_z = 9.45 \times 10^{-3}$ for $\sigma = 1$).

Figure 14: Comparison between our dimensionless axial velocity profiles and well-known solution at evaluates at $\theta = 0$ (corresponding to $\beta = \pi$ in Eq. (25) of [61]) for die shape $R_i = 80\text{cm}$, $R_o = 100\text{cm}$ and $\delta = 3.5\text{cm}$ for Newtonian fluid $\sigma = 1$. **Black**, ours and **red**, well-known solution. Slight discrepancy due to our improvement upon the lubrication approximation.

Figure 15: Newtonian contribution to volumetric flow rate, $\tilde{Q}_0$, versus ξ_i with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (118)].

Figure 16: Non-Newtonian variable for volumetric flow rate, $\bar{Q}_1$, versus ξ_i with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (119)].

Figure 17: Newtonian contribution to volumetric flow rate evaluated at $\theta = 0$, $\bar{Q}_{\theta,0}(0)$, versus ξ_i , with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (124)].

Figure 18: Newtonian contribution to volumetric flow rate evaluating at $\theta = \pi$, $\bar{Q}_{\theta,0}(\pi)$, versus ξ_i , with $\xi_0 = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (124)].

Figure 19: Non-Newtonian variable for volumetric flow rate evaluating at $\theta = 0$, $\check{Q}_{\theta,1}(0)$, versus ξ_i , with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (125)].

Figure 20: Non-Newtonian variable for volumetric flow rate evaluating at $\theta = \pi$, $\check{Q}_{\theta,1}(\pi)$, versus ξ_i , with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (125)].

Figure 21: Newtonian contribution to maximum strain rate, S_0 , versus inner contour, ξ_i , parametrized with curves of $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ [Eq. (134)]

Figure 22: Non-Newtonian variable for maximum strain rate, S_1 , versus inner contour, ξ_i , parametrized with curves of $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ [Eq. (135)].

Figure 23: Newtonian contribution to axial force, $\mathbb{F}_0$, versus inner contour, ξ_i , parametrized with curves of $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ [Eq. (143)].

Figure 24: Non-Newtonian variable for axial force, $\mathbb{F}_1$, *versus* inner contour, ξ_i , parametrized with curves of $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ [Eq. (144)].

Figure 25: Newtonian contribution to lateral force coefficient, χ_0 , versus ξ_i , with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (157)].

Figure 26: Non-Newtonian variable for lateral force coefficient, χ_1 , versus ξ_i , with $\xi_o = 0.02, 0.04, 0.06, 0.08, 0.10$ isopleths [Eq. (158)].

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