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TRANSPORT PHENOMENA IN ECCENTRIC CYLINDRICAL COORDINATES

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ABSTRACT

Studies in transport phenomena have been limited to a select few coordinate systems. Specifically, Cartesian, cylindrical, spherical, Dijkstra toroidal, and bipolar cylindrical coordinates have been the primary focus of transport work. The lack of diverse coordinate systems, for which the equations of change have been worked out, limits the diversity of transport phenomena problem solutions. Here, we introduce eccentric cylindrical coordinates and develop the corresponding equations of change (continuity, motion, and energy). This new coordinate system is unique, distinct from bipolar cylindrical coordinates, and does not contain cylindrical coordinates as a special case. We find eccentric cylindrical coordinates to be more intuitive for solving transport problems than bipolar cylindrical coordinates. We complete our analysis of eccentric cylindrical coordinates by using the new equations to solve one momentum, one energy, and one mass transport problem exactly.

Keywords: Coordinate systems, eccentric cylinders, transport phenomena, equations of change

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I. INTRODUCTION

In undergraduate chemical engineering education, for the transport phenomena, our textbooks limit themselves to the relevant equations of change in three coordinate systems: Cartesian, cylindrical, and spherical (Appendix B of [1] or [2]). By *relevant*, we mean continuity, motion in terms of the extra stress, energy in terms of heat flux, and continuity for a chemical species α :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (1)$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g} \quad (2)$$

$$\rho \hat{C}_p \frac{DT}{Dt} = -(\nabla \cdot \mathbf{q}) - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (3)$$

$$\rho \frac{D\omega_\alpha}{Dt} = -(\nabla \cdot \mathbf{j}_\alpha) + r_\alpha \quad (4)$$

where the dimensional and dimensionless variables are summarized in Table I and Table II. More advanced treatments for continuity, motion (Table 1 of [3]; Table P-13.15 of [4] or Table P3.16 of [5]; see also Table A.7-4 of [6] or [7]; Eq. (23) of [8]) and energy (Eq. (45) of [9]; Eq. (49) of [10]) in bipolar cylindrical coordinates [11,12,13,14,15]. One other treatment gives continuity and motion in Dijkstra toroidal coordinates (see Appendices I and II of [16]). Bipolar cylindrical coordinates yield exact solutions for axial flows through eccentric annuli ([3,9]; Example 6.4-1 of [7]) or the rotational eccentric annular flow arising in journal bearings, or the squeezing flow between cylinders of time-varying eccentricity [17,18]. Dijkstra toroidal coordinates yield exact solutions for flows between cylinders and cones (as sometimes arises in wire coating [16]), between coaxial eccentric cones (as sometimes arises in plastic pipe extrusion [16]), or through curved pipes. An approximate mapping of the annulus between eccentric cylinders into cylindrical coordinates has also been used for journal bearings (Example 7.5-1 of [7]; [19]).

With the exceptions of $[\nabla \cdot \boldsymbol{\tau}]$ and $(\boldsymbol{\tau} : \nabla \mathbf{v})$, the terms in Eqs. (1)-(4) are easily written in the seventeen standard coordinate systems (following [15]; [11,12,13,14,20]). Table III reviews the available literature and identifies where to find the equations of change, and thus $[\nabla \cdot \boldsymbol{\tau}]$ and $(\boldsymbol{\tau} : \nabla \mathbf{v})$, for some coordinate systems. Problems involving eccentric cylinders can be attacked with bipolar cylindrical coordinates, but some prefer the nonstandard *eccentric cylindrical coordinates* [21,22,23,24]. Indeed, we find eccentric cylindrical coordinates easier to teach than bipolar cylindrical coordinates. Figure 1 shows the $\xi\theta$ -plane of eccentric cylindrical coordinates, and Figure 2 provides a three-dimensional depiction of a volume element.

From Table III, we see that progress in transport phenomena might accelerate with the availability of the equations of change in additional coordinate systems. In this paper, we derive the equations of change in eccentric cylindrical coordinates and then provide three worked examples of their usefulness for obtaining new exact solutions. For this derivation, we begin by following the method of Section A.7 of [1] for getting the differential operations in eccentric cylindrical coordinates.

II. METHOD

The characteristic quantity defining eccentric cylindrical coordinates is the eccentricity, δ , which describes the distance between the centers of two cylinders at the same pole of the coordinate system. Figure 3 shows the eccentricity of two cylinders forming an annulus through which a Newtonian fluid is pumped. We relate the eccentricity to the confocal length, or distance between poles of the coordinate system:

$$a \equiv \delta^{-1} \left(R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2 \right)^{1/2} \quad (5)$$

which appears in the equations of change and in the relations between Cartesian and eccentric cylindrical coordinates. Geometric constraints require $0 < \delta < R_2 - R_1$, which restricts the inner cylinder to the interior of the larger cylinder. The geometric constraint also excludes cylindrical coordinates (where $\delta = 0$) as a special case.

We develop eccentric cylindrical coordinates by beginning with the Cartesian representation:

$$x + iy = \frac{a}{1 - \xi e^{i\theta}} \quad (6)$$

for which, after rearrangement, we evaluate the real and imaginary parts:

$$\Re(\xi \cos \theta + i \xi \sin \theta) = \xi \cos \theta = \frac{(x^2 + y^2) - ax}{(x^2 + y^2)} \quad (7)$$

$$\Im(\xi \cos \theta + i \xi \sin \theta) = \xi \sin \theta = \frac{ay}{(x^2 + y^2)} \quad (8)$$

We use Eqs. (7) and (8) to find explicit expressions for the *radial eccentric cylindrical coordinate*:

$$\xi = \sqrt{1 + \frac{a(a - 2x)}{x^2 + y^2}} \quad (9)$$

and the *circumferential eccentric cylindrical coordinate*:

$$\theta = \tan^{-1} \frac{ay}{(x^2 + y^2) - ax} \quad (10)$$

The *axial eccentric cylindrical coordinate*, z , is of course equivalent to the z - Cartesian coordinate:

$$z = z_c \quad (11)$$

Inverting Eqs. (9)-(11) yields:

$$x = a \frac{1 - \xi \cos \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (12)$$

$$y = a \frac{\xi \sin \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (13)$$

$$z_c = z \quad (14)$$

We will use Eqs. (12)-(14) for getting the differential operations in eccentric cylindrical coordinates (see Section A.7 of [1]). Appendix A details the derivation of eccentric cylindrical coordinates.

III. RESULTS

We provide the derivation of the differential operations in Appendix B and the development of the equations of change from these operations in Appendix C. We derive the equation of continuity for eccentric cylindrical coordinates by substituting the expression for $(\nabla \cdot \rho \mathbf{v})$ related to Eq. (218) of Table IV into Eq. (1):

$$\rho \frac{\partial \rho}{\partial t} + \frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi}{X} \rho v_\xi \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{\rho v_\theta}{X} \right) + \frac{\partial}{\partial z} \rho v_z = 0 \quad (15)$$

where $X = 1 - 2\xi \cos \theta + \xi^2$.

We can also find the ξ , θ and z components of the equation of motion in terms of the extra stress by inserting the expressions for $D\mathbf{v}/Dt$, ∇p and $[\nabla \cdot \boldsymbol{\tau}]$ found in Eqs. (121), (126), (128), (130) and (133) of Appendix C into Eq. (2):

$$\rho \left(\frac{\partial v_\xi}{\partial t} + v_\xi \frac{X}{a} \frac{\partial v_\xi}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_z \frac{\partial v_\xi}{\partial z} - 2 \frac{v_\xi v_\theta \sin \theta}{a} + \frac{v_\theta^2 (\xi^2 - 1)}{a\xi} \right) = - \frac{X}{a} \frac{\partial p}{\partial \xi} - \left[\frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} + \frac{\partial \tau_{z\xi}}{\partial z} + \frac{\tau_{\theta\theta} - \tau_{\xi\xi}}{a\xi} (\xi^2 - 1) - 2 \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \sin \theta \right] + \rho g_\xi \quad (16)$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_\xi \frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} - \frac{v_\xi v_\theta (\xi^2 - 1)}{a\xi} + 2 \frac{v_\theta^2 \sin \theta}{a} \right) = - \frac{X}{a\xi} \frac{\partial p}{\partial \theta} - \left[\frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + 2 \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \sin \theta - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a\xi} (\xi^2 - 1) \right] + \rho g_\theta \quad (17)$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_\xi \frac{X}{a} \frac{\partial v_z}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} - \left[\frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - 2 \frac{\tau_{\theta z}}{a} \sin \theta - \frac{\tau_{\xi z}}{a\xi} (\xi^2 - 1) \right] + \rho g_z \quad (18)$$

For a Newtonian fluid, we replace each τ_{ij} term in Eqs. (16)-(18) with Newton's Law found in Eqs. (138)-(143) of Appendix C. We can therefore, with moderate effort, determine velocity profiles for eccentric cylindrical flows exactly, where the center of the interior cylinder is at $\xi = 0$. We illustrate the use of Eqs. (16)-(18) to determine the velocity profile and subsequent flow rate through an eccentric annulus in Section IV.

The equation of energy in eccentric cylindrical coordinates is:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_\xi \frac{X}{a} \frac{\partial T}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{q_\xi}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{q_\theta}{X} \right) + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (19)$$

where the rightmost term (viscous dissipation) is found in Eq. (151) in Appendix C. Any other volumetric heat source or sink must be added as additional source term to the end of Eq. (19). The heat fluxes, q_i , in Eq. (19) are commonly replaced with Fourier's law of heat conduction, which we provide in Eqs. (153)-(155) of Appendix C for eccentric cylindrical coordinates. We use Eq. (19) in Section V to find the temperature profile in a polyethylene wire coating.

Similarly, we find the equation of continuity for species α :

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_\xi \frac{X}{a} \frac{\partial \omega_\alpha}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial \omega_\alpha}{\partial \theta} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{j_{a\xi}}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{j_{a\theta}}{X} \right) + \frac{\partial j_{az}}{\partial z} \right] + r_\alpha \quad (20)$$

which is in terms of mass fraction, ω_α , and mass flux, j_{ai} . Because many chemical engineering problems are better represented by molar continuity, the mass-based quantities in Eq. (20) can be replaced by mole-based variables.

Specifically, ω_α , j_{ai} , ρ , r_α and v_i can be substituted by x_α , J_{ai}^* , c ,

$R_\alpha - x_\alpha \sum_{\beta=1}^N R_\beta$ and v_i^* to generate the equations of continuity in terms of moles and mole-average quantities. These variable substitutions are summarized at the end of Section B.10 of [1]. Tables 17.8-1 and 17.8-2 of [1] provide necessary definitions for mole and mass average velocities in terms of material fluxes and species concentrations or mass fractions. The material fluxes in Eq. (20) can, for many mass transfer problems, be replaced with Fick's law of diffusion. We supply calculations of Fick's law for eccentric cylindrical coordinates in Eqs. (159)-(161) of Appendix C. We illustrate the utility of Eq. (20) with a mass transfer example in Section VI.

IV. WORKED EXAMPLE: Momentum Transfer

A chemical engineer pumps a Newtonian fluid through a straight horizontal eccentric annulus. She is interested in the connection between eccentricity and the volumetric flow rate. She develops the transport phenomena diagram provided in Figure 3. She begins her analysis of the annular flow by simplifying the equation of continuity [Eq. (15)] for constant-density axial flow to get:

$$\frac{\partial v_z}{\partial z} = 0 \quad (21)$$

which tells her that velocity does not depend on the z -coordinate.

To develop the velocity profile, she simplifies Eq. (18) for steady state highly-viscous flow along the z -axis:

$$\frac{\partial p}{\partial z} = - \left[\frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - 2 \frac{\tau_{\theta z}}{a} \sin \theta - \frac{\tau_{\xi z}}{a\xi} (\xi^2 - 1) \right] \quad (22)$$

She next evaluates each of the stresses for a Newtonian fluid using Eqs. (140), (142) and (143):

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) = 0 \quad (23)$$

$$\tau_{\theta z} = -\mu \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right) = -\frac{\mu X}{a\xi} \frac{\partial v_z}{\partial \theta} \quad (24)$$

$$\tau_{\xi z} = -\mu \left(\frac{X}{a} \frac{\partial v_z}{\partial \xi} + \frac{\partial v_\xi}{\partial z} \right) = -\frac{\mu X}{a} \frac{\partial v_z}{\partial \xi} \quad (25)$$

Substituting Eqs. (23)-(25) into Eq. (22), she gets:

$$\frac{\partial p}{\partial z} = \frac{\mu X^2}{a^2 \xi} \left(\frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_z}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 v_z}{\partial \theta^2} \right) \quad (26)$$

She integrates Eq. (26) with respect to z using the pressure boundary conditions:

$$\frac{p_L - p_0}{L} = \frac{\Delta p}{L} = \frac{\mu X^2}{a^2 \xi} \left(\frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_z}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 v_z}{\partial \theta^2} \right) \quad (27)$$

She next assumes that the solution to this nonhomogeneous PDE will take the form:

$$v_z = v_{z,h} + v_{z,p} \quad (28)$$

where $v_{z,h}$ is the homogeneous solution and $v_{z,p}$ is the particular solution to the nonhomogeneous problem.

She first finds the particular solution to Eq. (27):

$$v_{z,p} = \frac{V}{4(1 - 2\xi \cos \theta + \xi^2)}; V = \frac{a^2 \Delta p}{\mu L} \quad (29)$$

for which she provides the detailed derivation in Appendix D [see Eq. (184)]. She next turns her attention to the homogenous problem:

$$0 = \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_{z,h}}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 v_{z,h}}{\partial \theta^2} \quad (30)$$

Using separation of variables, she finds:

$$v_{z,h} = \sum_{n=0}^{\infty} f_n(\xi) \cos n\theta \quad (31)$$

where:

$$f_n = \begin{cases} \alpha_0 + \beta_0 \ln \xi & ; n = 0 \\ \alpha_n \xi^n + \beta_n \xi^{-n} & ; n \geq 1 \end{cases} \quad (32)$$

which is the solution to Eq. (30) and is derived fully in Appendix D.

She then combines the particular, Eq. (29), and homogenous, Eq. (31), solutions to obtain the velocity profile:

$$v_z(\xi, \theta) = \alpha_0 + \beta_0 \ln \xi + \sum_{n=1}^{\infty} (\alpha_n \xi^n + \beta_n \xi^{-n}) \cos n\theta + \frac{V}{4(1 - 2\xi \cos \theta + \xi^2)} \quad (33)$$

where:

$$\alpha_n \equiv \begin{cases} V \frac{\xi_i^2 \ln \xi_i - \xi_o^2 \ln \xi_o - \ln \frac{\xi_i}{\xi_o}}{4(\xi_i^2 - 1)(\xi_o^2 - 1) \ln \frac{\xi_i}{\xi_o}} & ; n = 0 \\ V \frac{\xi_i^{2n}(\xi_o^2 - 1) - \xi_o^{2n}(\xi_i^2 - 1)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} & ; n \geq 1 \end{cases} \quad (34)$$

$$\beta_n \equiv \begin{cases} \frac{V(\xi_o^2 - \xi_i^2)}{4(\xi_i^2 - 1)(\xi_o^2 - 1) \ln \frac{\xi_1}{\xi_2}} & ; n = 0 \\ \frac{V \xi_i^{2n} \xi_o^{2n} (\xi_i^2 - \xi_o^2)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} & ; n \geq 1 \end{cases} \quad (35)$$

and:

$$\xi_i \equiv \frac{(R_o^2 - R_i^2 - \delta^2) - (R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2)^{1/2}}{2\delta R_i}; \delta < (R_o - R_i) \quad (36)$$

$$\xi_o \equiv \frac{(R_o^2 - R_i^2 + \delta^2) - (R_o^4 + R_i^4 + \delta^4 - 2R_i^2 R_o^2 - 2R_i^2 \delta^2 - 2R_o^2 \delta^2)^{1/2}}{2\delta R_o}; \delta < (R_o - R_i) \quad (37)$$

Appendix D details the derivation of the coefficients α_n and β_n [see Eqs. (208), (207), (213) and (215)]. Her results match Eqs. (65)-(67) of [22]. Thus, Eq. (33) with Eqs. (34) and (35) is the velocity profile for pressure driven flow through an eccentric annulus.

She sketches the velocity profile in Figure 4 and finds that the velocity increases with increasing distance between inner and outer cylinders. In other words, the velocity is maximized at the largest annular gap. She also recognizes that the velocity profile, Eq. (33), cannot be integrated analytically to obtain a flow rate.

V. WORKED EXAMPLE: Heat Transfer

A chemical engineer is interested in specifying a tolerance for an imperfectly coated wire, where the polyethylene coating and the copper core are eccentric. She wants an expression for the maximum temperature in the polyethylene coating at steady state. The copper core provides a constant heat flux ($q_e \equiv \frac{S_e}{2\pi R_i L}$) to the inner surface of the coating, and the outer boundary of the coating is held at constant temperature, T_0 . Figure 5 shows the relevant transport diagram.

She reduces the equation of energy for steady state radial conduction in eccentric cylindrical coordinates [Eq. (19)] to get:

$$0 = -\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{q_\xi}{X} \right) \quad (38)$$

since she expects constant T along the θ -coordinate and she ignores end effects in the z direction. Simplifying and integrating once, she gets:

$$q_\xi = \frac{C_3 X}{\xi} = \frac{C_3 (1 - 2\xi \cos \theta + \xi^2)}{\xi} \quad (39)$$

She thus finds that the flux depends on the θ direction, which agrees with her physical intuition about the asymmetry of the wire coating thickness. Applying the constant flux boundary condition at the wire surface:

$$q_\xi (\xi = \xi_i) = q_e \equiv \frac{S_e}{2\pi R_i L} \quad (40)$$

she finds:

$$q_{\xi} = \frac{S_e \xi_i (1 - 2\xi \cos \theta + \xi^2)}{2\pi R_i L \xi (1 - 2\xi_i \cos \theta + \xi_i^2)} \quad (41)$$

where S_e is the total heat generation caused by the electrical heating of the wire. She develops plots for the ξ and θ dependence of the dimensionless heat flux (Figure 6):

$$H = \frac{2\pi R_i L q_{\xi}}{S_e} = \frac{\xi_i (1 - 2\xi \cos \theta + \xi^2)}{\xi (1 - 2\xi_i \cos \theta + \xi_i^2)} \quad (42)$$

Interestingly, the heat flux does not depend on the confocal length. As expected, the largest flux occurs where the coating is thinnest ($\theta = \pi$).

She next substitutes Fourier's law for heat flux into Eq. (41) and integrates to get:

$$T = -\frac{aS_e \xi_i \ln \xi}{2\pi R_i L k (1 - 2\xi_i \cos \theta + \xi_i^2)} + C_4 \quad (43)$$

Evaluating Eq. (43) for the isothermal wire exterior:

$$T(\xi = \xi_o) = T_0 \quad (44)$$

she gets:

$$T - T_0 = -\frac{aS_e \xi_i \ln(\xi/\xi_o)}{2\pi R_i L k (1 - 2\xi_i \cos \theta + \xi_i^2)} \quad (45)$$

for the temperature profile. She then nondimensionalizes Eq. (45) as:

$$\Theta = \frac{2\pi L k (T - T_0)}{S_e} = -\frac{a \xi_i \ln(\xi/\xi_o)}{R_i (1 - 2\xi_i \cos \theta + \xi_i^2)} \quad (46)$$

and evaluates the temperature for different values of θ to create Figure 7. She finds that the temperature is greatest where the wire coating is thickest and that increasing the eccentricity increases the maximum temperature.

However, increasing eccentricity also reduces the temperature in the thinnest part of the coating, as expected. In other words, larger eccentricities lead to a wider range of temperatures near the heated wire. Her solution is invalid when $\frac{\delta}{R_o - R_i} \ll 1$, since eccentric cylindrical coordinates do not include

cylindrical coordinates as a special case. Systems having dimensionless eccentricities near zero must be approximated with cylindrical coordinates.

The solution for the dimensionless temperature rise in Eq. (46) resembles the solution obtained for a concentrically coated wire in cylindrical coordinates:

$$\Theta = \frac{2\pi R_i k L (T - T_0)}{S_e} = -\ln(r/R_o) \quad (47)$$

which applies to coated wires having negligible or nearly negligible dimensionless eccentricity. Figure 7 shows that the concentric wire coating profile is approximately an average of the eccentric solutions. This is expected since the thickness of the concentric coating is an average of the eccentric thicknesses.

Evaluating Eq. (46), we find the hottest point on the coating inner surface, $\xi = \xi_i$, and at its thickest, $\theta = 0$:

$$\Theta_{\max} = \frac{2\pi L k (T_{\max} - T_0)}{S_e} = \frac{a \xi_i \ln(\xi_o/\xi_i)}{R_i (1 - \xi_i)^2} \quad (48)$$

As a practical matter, $T_{\max}$ must not exceed the degradation point of the wire coating.

VI. WORKED EXAMPLE: Mass Transfer

A chemical engineer is venting compound A through a curved and expanding rectangular duct, initially filled with air (species B). A tank of liquid A that is constantly replenished is located at $\theta = \pi$, where the concentration of A is held at the saturation value of c_{A0} . The vent opens to the atmosphere at $\theta = 0$, where compound A is constantly swept away to keep the concentration of A at zero at the vent exit. She is interested in determining the steady state concentration profile of A along the exhaust vent. She also wants to examine the effect of eccentricity on molar flux. Figure 8 illustrates the relevant transport diagram.

She begins her analysis with the equation of continuity in terms of molar quantities by rewriting Eq. (20) [see after Eq. (20)]:

$$c \left(\frac{\partial x_A}{\partial t} + v_{\xi}^* \frac{X}{a} \frac{\partial x_A}{\partial \xi} + v_{\theta}^* \frac{X}{a\xi} \frac{\partial x_A}{\partial \theta} + v_z^* \frac{\partial x_A}{\partial z} \right) = - \left[\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{J_{A\xi}^*}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{J_{A\theta}^*}{X} \right) + \frac{\partial J_{Az}^*}{\partial z} \right] + x_B R_A - x_A R_B \quad (49)$$

She then simplifies Eq. (49) for one-dimensional unimolecular diffusion, in the absence of reaction, to get:

$$c v_{\theta}^* \frac{\partial x_A}{\partial \theta} = -X \frac{\partial}{\partial \theta} \left(\frac{J_{A\theta}^*}{X} \right) \quad (50)$$

For Fickian diffusion, Eq. (50) becomes:

$$v_{\theta}^* \frac{\partial x_A}{\partial \theta} = \frac{\mathcal{D}_{AB} X}{a\xi} \frac{\partial^2 x_A}{\partial \theta^2} \quad (51)$$

She then relates the average molar velocity to the molar flux by using Eqs. (M) and (V) in Table 17.8-1 of [1]:

$$v_{\theta}^* = \frac{J_{A\theta}^*}{c(1-x_A)} \quad (52)$$

Substituting Eq. (52) into Eq. (51) and inserting Fick's law [Eq. (160)], she finds:

$$-\frac{1}{1-x_A} \left(\frac{\partial x_A}{\partial \theta} \right)^2 = \frac{\partial^2 x_A}{\partial \theta^2} \quad (53)$$

$$\frac{\partial}{\partial \theta} \left(\frac{1}{1-x_A} \frac{\partial x_A}{\partial \theta} \right) = 0 \quad (54)$$

Integrating:

$$\frac{1}{1-x_A} \frac{\partial x_A}{\partial \theta} = C_1 \quad (55)$$

$$-\ln(1-x_A) = C_1 \theta + C_2 \quad (56)$$

and solving for the constants using the boundary conditions:

$$x_A(\pi) = x_{A0} \quad (57)$$

$$x_A(0) = 0 \quad (58)$$

she obtains:

$$x_A = 1 - (1 - x_{A0})^{\theta/\pi} \quad (59)$$

which is the concentration profile along the vent. She develops an illustration of the θ dependence of concentration, provided in Figure 9. She finds that the concentration in the vent is independent of diffusivity and of eccentricity. The concentration profile in Eq. (59) is identical to the profile for a concentric cylindrical vent, solved for using cylindrical coordinates:

$$x_A = 1 - (1 - x_{A0})^{\theta/\pi} \quad (60)$$

which emphasizes the independence of angular concentration from eccentricity.

To determine the molar flux, she derives Eq. (59) to get:

$$J_{A\theta}^* = \frac{c\mathcal{D}_{AB}X}{a\pi\xi} \ln(1 - x_{A0})(1 - x_{A0})^{\theta/\pi} \quad (61)$$

which she makes dimensionless to find:

$$\Psi_{A\theta} = \frac{J_{A\theta}^* R_i}{c\mathcal{D}_{AB}} = \frac{XR_i}{a\pi\xi} \ln(1 - x_{A0})(1 - x_{A0})^{\theta/\pi} \quad (62)$$

As expected, the molar flux depends on the radial coordinate, the angular coordinate, and the eccentricity. The corresponding molar flux profile is provided in Figure 10. The solution for the molar flux only applies when $\frac{\delta}{R_o - R_i} \ll 1$, as shown in Figure 11. She integrates Eq. (61) over the ξz - plane to find the molar flow rate:

$$W = \int_0^L \int_{\xi_i}^{\xi_o} N_{A\theta} \frac{a}{X} d\xi dz = \frac{c\mathcal{D}_{AB}L \ln(\xi_o/\xi_i)}{\pi} \ln(1 - x_{A0}) \quad (63)$$

where:

$$N_{A\theta} = \frac{J_{A\theta}^*}{1 - x_A} = -\frac{c\mathcal{D}_{AB}X}{a\pi\xi} \ln(1 - x_{A0}) \quad (64)$$

is the combined flux. She can use Eq. (63) to determine the molar flow rate of a compound vented through the curved expanding rectangular duct. Eq. (63) also allows her to determine if the venting adheres to regulations for maximum concentration of compound A released to the environment.

VII. CONCLUSION

We have introduced the equations of change for another coordinate system, specifically eccentric cylindrical coordinates, thus, unlocking opportunities for transport phenomena to expand into the worlds of eccentric annuli and curved expanding channels. Our worked examples illustrate how to use our main result: (1) equation of continuity [Eq. (15)], (2) equation of motion [Eqs. (16)–(18)], (3) equation of energy [Eq. (19)] and (4) equation of continuity of species α [Eq. (20)].

Heretofore, for exact solutions, we have been limited to only a few coordinate systems restraining study of many common transport problems. Developing the equations of change for more coordinate systems is one of the pressing and tractable frontiers in transport phenomena.

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IX. APPENDICES

Appendix A: Eccentric Cylindrical Coordinates

The Cartesian representation of eccentric cylindrical coordinates is:

$$x + iy = \frac{a}{1 - \xi e^{i\theta}} \quad (65)$$

We can easily rearrange Eq. (65) to separate the real and imaginary parts:

$$\xi \cos \theta + i\xi \sin \theta = \frac{(x^2 + y^2) - ax}{(x^2 + y^2)} + \frac{ia y}{(x^2 + y^2)} \quad (66)$$

Taking the real part:

$$\Re(\xi \cos \theta + i\xi \sin \theta) = \xi \cos \theta = \frac{(x^2 + y^2) - ax}{(x^2 + y^2)} \quad (67)$$

and imaginary part:

$$\Im(\xi \cos \theta + i\xi \sin \theta) = \xi \sin \theta = \frac{ay}{(x^2 + y^2)} \quad (68)$$

we develop relationships between the eccentric cylindrical and Cartesian coordinates. Dividing Eq. (68) by Eq. (67), we get:

$$\tan \theta = \frac{ay}{(x^2 + y^2) - ax} \quad (69)$$

Rearranging, we find:

$$\theta = \tan^{-1} \frac{ay}{(x^2 + y^2) - ax} \quad (70)$$

which explicitly defines θ in terms of x and y . We substitute Eq. (70) into Eq. (67) and rearrange to get:

$$\xi = \sqrt{1 + \frac{a(a - 2x)}{x^2 + y^2}} \quad (71)$$

The axial eccentric coordinate is simply related to Cartesian coordinates:

$$z = z_C \quad (72)$$

We invert Eqs. (70)-(72) to find:

$$x = a \frac{1 - \xi \cos \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (73)$$

$$y = a \frac{\xi \sin \theta}{1 - 2\xi \cos \theta + \xi^2} \quad (74)$$

$$z_c = z \quad (75)$$

which we will use in Appendix B to develop the differential operations for eccentric cylindrical coordinates.

Appendix B: Eccentric Cylindrical Coordinate Differential Operations

We begin by relating the Cartesian derivative operators to the eccentric cylindrical derivative operators. To do this, we first construct the partial derivative of each Cartesian coordinate relative to each of the eccentric cylindrical coordinates. For the x - coordinate, we get:

$$\frac{\partial}{\partial x} = -a \frac{2\xi - (1 + \xi^2) \cos \theta}{(1 - 2\xi \cos \theta + \xi^2)^2} \frac{\partial}{\partial \xi} + a \frac{\xi (\xi^2 - 1) \sin \theta}{(1 - 2\xi \cos \theta + \xi^2)^2} \frac{\partial}{\partial \theta} \quad (76)$$

Similarly, we find the partial derivative for the y - coordinate:

$$\frac{\partial}{\partial y} = -a \frac{(\xi^2 - 1) \sin \theta}{(1 - 2\xi \cos \theta + \xi^2)^2} \frac{\partial}{\partial \xi} - a \frac{\xi (2\xi - (1 + \xi^2) \cos \theta)}{(1 - 2\xi \cos \theta + \xi^2)^2} \frac{\partial}{\partial \theta} \quad (77)$$

and z_c - coordinate:

$$\frac{\partial}{\partial z_c} = \frac{\partial}{\partial z} \quad (78)$$

We next develop the scale factors that we will use to construct the unit vectors for eccentric cylindrical coordinates. For a summary of vector and tensor operations in terms of scale factors, h_i , see the tabulation in [25]. The radial shape factor is:

$$h_\xi = \frac{a}{1 - 2\xi \cos \theta + \xi^2} \quad (79)$$

and the circumferential shape factor is:

$$h_\theta = \frac{a\xi}{1 - 2\xi \cos \theta + \xi^2} \quad (80)$$

The general expression for a curvilinear unit vector, δ_i , in terms of Cartesian unit vectors ($\delta_x, \delta_y, \delta_z$) is:

$$\delta_i = \frac{1}{h_i} \left(\frac{\partial x}{\partial n_i} \right) \delta_x + \frac{1}{h_i} \left(\frac{\partial y}{\partial n_i} \right) \delta_y + \frac{1}{h_i} \left(\frac{\partial z_c}{\partial n_i} \right) \delta_{z_c} \quad (81)$$

where n_i is a curvilinear coordinate. We construct the eccentric cylindrical unit vectors ($\delta_\xi, \delta_\theta, \delta_z$) using Eq. (81) to get:

$$\delta_{\xi} = -\frac{2\xi - (1 + \xi^2)\cos\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_x - \frac{(\xi^2 - 1)\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_y \quad (82)$$

$$\delta_{\theta} = \frac{(\xi^2 - 1)\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_x - \frac{2\xi - (1 + \xi^2)\cos\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_y \quad (83)$$

$$\delta_z = \delta_{z_c} \quad (84)$$

Solving for the Cartesian unit vectors $(\delta_x, \delta_y, \delta_{z_c})$, we get:

$$\delta_x = -\frac{2\xi - (1 + \xi^2)\cos\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\xi} + \frac{(\xi^2 - 1)\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\theta} \quad (85)$$

$$\delta_y = -\frac{(\xi^2 - 1)\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\xi} - \frac{2\xi - (1 + \xi^2)\cos\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\theta} \quad (86)$$

$$\delta_{z_c} = \delta_z \quad (87)$$

which we will use to construct our del operator, ∇ . We will also need the derivatives of the eccentric cylindrical coordinates:

$$\frac{\partial}{\partial \xi}\delta_{\xi} = \frac{2\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\theta} \quad (88)$$

$$\frac{\partial}{\partial \xi}\delta_{\theta} = -\frac{2\sin\theta}{1 - 2\xi\cos\theta + \xi^2}\delta_{\xi} \quad (89)$$

$$\frac{\partial}{\partial \xi}\delta_z = 0 \quad (90)$$

$$\frac{\partial}{\partial \theta}\delta_{\theta} = \frac{(\xi^2 - 1)}{1 - 2\xi\cos\theta + \xi^2}\delta_{\xi} \quad (91)$$

$$\frac{\partial}{\partial \theta}\delta_z = 0 \quad (92)$$

$$\frac{\partial}{\partial z}\delta_{\xi} = 0 \quad (93)$$

$$\frac{\partial}{\partial z}\delta_{\theta} = 0 \quad (94)$$

$$\frac{\partial}{\partial z}\delta_z = 0 \quad (95)$$

to create the equations of change in eccentric cylindrical coordinates.

We begin the derivation of the eccentric cylindrical ∇ -operator with the general expression for ∇ in terms of scale factors:

$$\nabla = \sum_i \frac{\delta_i}{h_i} \frac{\partial}{\partial n_i} \quad (96)$$

Substituting the eccentric cylindrical unit vectors and scale factors, Eqs. (79) and (80), into Eq. (96), we get:

$$\nabla = \delta_\xi \frac{1-2\xi \cos \theta + \xi^2}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{1-2\xi \cos \theta + \xi^2}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \quad (97)$$

which we can simplify to:

$$\nabla = \delta_\xi \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{X}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \quad (98)$$

where $X = 1 - 2\xi \cos \theta + \xi^2$. We use Eq. (98) to develop the equations of change in Appendix C.

We continue our analysis by developing additional derivative operations that are not used in the equations of change but are nonetheless useful. The first of these is the curl of velocity:

$$[\nabla \times \mathbf{v}] = \left(\delta_\xi \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{X}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right) \times (\delta_\xi v_\xi + \delta_\theta v_\theta + \delta_z v_z) \quad (99)$$

$$[\nabla \times \mathbf{v}] = \delta_\xi \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) + \delta_\theta \left(\frac{\partial v_\xi}{\partial z} - \frac{X}{a} \frac{\partial v_z}{\partial \xi} \right) + \delta_z \left(\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi v_\theta}{X} \right) - \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{v_\xi}{X} \right) \right) \quad (100)$$

We next develop the Laplacian of velocity. Recall that the gradient of velocity is:

$$\{\nabla \mathbf{v}\} = \begin{Bmatrix} \frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_\theta \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_\xi}{\partial z} \\ \frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\xi \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_\theta}{\partial z} \\ \frac{X}{a} \frac{\partial v_z}{\partial \xi} & \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{Bmatrix} \quad (101)$$

and the definition of the Laplacian is:

$$[\nabla \cdot \nabla \mathbf{v}] = \left[\delta_\xi \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{X}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right] \cdot \left\{ \begin{aligned} & \left[\delta_\xi \left(\frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} \right) + \delta_\theta \left(\frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_\theta \frac{\xi^2 - 1}{a\xi} \right) + \delta_z \frac{\partial v_\xi}{\partial z} \right] \delta_\xi \\ & + \left[\delta_\xi \left(\frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\xi \frac{2 \sin \theta}{a} \right) + \delta_\theta \left(\frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} \right) + \delta_z \frac{\partial v_\theta}{\partial z} \right] \delta_\theta \\ & + \left[\delta_\xi \left(\frac{X}{a} \frac{\partial v_z}{\partial \xi} \right) + \delta_\theta \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} \right) + \delta_z \frac{\partial v_z}{\partial z} \right] \delta_z \end{aligned} \right\} \quad (102)$$

After significant simplification, we find:

$$[\nabla^2 \mathbf{v}] = \left[\begin{aligned} & \left(\frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_\xi}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_\xi}{\partial \theta^2} + \frac{\partial^2 v_\xi}{\partial z^2} - \frac{v_\xi}{a^2 \xi^2} \left((\xi^2 - 1)^2 + 4\xi^2 \sin^2 \theta \right) - \frac{2X}{a^2 \xi^2} \left((1 - \xi^2) \frac{\partial v_\theta}{\partial \theta} + 2\xi^2 \sin \theta \frac{\partial v_\theta}{\partial \xi} \right) \right) \delta_\xi \\ & + \left(\frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_\theta}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{a^2 \xi^2} \left(4\xi^2 \sin^2 \theta + (\xi^2 - 1)^2 \right) + \frac{2X}{a^2 \xi^2} \left(2\xi^2 \sin \theta \frac{\partial v_\xi}{\partial \xi} + (1 - \xi^2) \frac{\partial v_\xi}{\partial \theta} \right) \right) \delta_\theta \\ & + \left(\frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_z}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \delta_z \end{aligned} \right] \quad (103)$$

We complete our analysis of differential operations by calculating $[\mathbf{v} \cdot \nabla \boldsymbol{\tau}]$ which is used in some constitutive equations and substantial derivatives (Eq. (8.5-2) of [1]). We begin with the gradient of the extra stress tensor:

$$\{\nabla \boldsymbol{\tau}\} = \left(\delta_\xi \frac{1-2\xi \cos \theta + \xi^2}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{1-2\xi \cos \theta + \xi^2}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right) \left\{ \begin{aligned} & \tau_{\xi\xi} \delta_\xi \delta_\xi + \tau_{\theta\xi} \delta_\theta \delta_\xi + \tau_{z\xi} \delta_z \delta_\xi \\ & + \tau_{\xi\theta} \delta_\xi \delta_\theta + \tau_{\theta\theta} \delta_\theta \delta_\theta + \tau_{z\theta} \delta_z \delta_\theta \\ & + \tau_{\xi z} \delta_\xi \delta_z + \tau_{\theta z} \delta_\theta \delta_z + \tau_{zz} \delta_z \delta_z \end{aligned} \right\} \quad (104)$$

which results in a third order tensor:

$$\{\nabla \mathbf{\tau}\} = \left\{ \begin{aligned} & \left(\frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} - \frac{2\sin\theta}{a} \tau_{\theta\xi} - \frac{2\sin\theta}{a} \tau_{\xi\theta} \right) \delta_\xi \otimes \delta_\xi \otimes \delta_\xi + \left(\frac{X}{a\xi} \frac{\partial \tau_{\xi\xi}}{\partial \theta} + \frac{\xi^2-1}{a\xi} \tau_{\theta\xi} + \frac{\xi^2-1}{a\xi} \tau_{\xi\theta} \right) \delta_\theta \otimes \delta_\xi \otimes \delta_\xi \\ & + \left(\frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{2\sin\theta}{a} \tau_{\xi\xi} - \frac{2\sin\theta}{a} \tau_{\theta\theta} \right) \delta_\xi \otimes \delta_\theta \otimes \delta_\xi + \left(\frac{X}{a\xi} \frac{\partial \tau_{\xi\theta}}{\partial \theta} + \frac{\xi^2-1}{a\xi} \tau_{\theta\theta} - \frac{\xi^2-1}{a\xi} \tau_{\xi\xi} \right) \delta_\theta \otimes \delta_\theta \otimes \delta_\xi \\ & + \left(\frac{X}{a} \frac{\partial \tau_{\theta\xi}}{\partial \xi} + \frac{2\sin\theta}{a} \tau_{\xi\xi} - \frac{2\sin\theta}{a} \tau_{\theta\theta} \right) \delta_\xi \otimes \delta_\xi \otimes \delta_\theta + \left(\frac{X}{a\xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} - \frac{\xi^2-1}{a\xi} \tau_{\xi\xi} + \frac{\xi^2-1}{a\xi} \tau_{\theta\theta} \right) \delta_\theta \otimes \delta_\xi \otimes \delta_\theta \\ & + \left(\frac{X}{a} \frac{\partial \tau_{\theta\theta}}{\partial \xi} + \frac{2\sin\theta}{a} \tau_{\theta\xi} + \frac{2\sin\theta}{a} \tau_{\xi\theta} \right) \delta_\xi \otimes \delta_\theta \otimes \delta_\theta + \left(\frac{X}{a\xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} - \frac{\xi^2-1}{a\xi} \tau_{\xi\theta} - \frac{\xi^2-1}{a\xi} \tau_{\theta\xi} \right) \delta_\theta \otimes \delta_\theta \otimes \delta_\theta \\ & + \left(\frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} - \frac{2\sin\theta}{a} \tau_{\theta z} \right) \delta_\xi \otimes \delta_z \otimes \delta_\xi + \left(\frac{X}{a\xi} \frac{\partial \tau_{\xi z}}{\partial \theta} + \frac{\xi^2-1}{a\xi} \tau_{\theta z} \right) \delta_\theta \otimes \delta_z \otimes \delta_\xi \\ & + \left(\frac{X}{a} \frac{\partial \tau_{\theta z}}{\partial \xi} + \frac{2\sin\theta}{a} \tau_{\xi z} \right) \delta_\xi \otimes \delta_z \otimes \delta_\theta + \left(\frac{X}{a\xi} \frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\xi^2-1}{a\xi} \tau_{\xi z} \right) \delta_\theta \otimes \delta_z \otimes \delta_\theta \\ & + \left(\frac{X}{a} \frac{\partial \tau_{z\xi}}{\partial \xi} - \frac{2\sin\theta}{a} \tau_{z\theta} \right) \delta_\xi \otimes \delta_\xi \otimes \delta_z + \left(\frac{X}{a\xi} \frac{\partial \tau_{z\xi}}{\partial \theta} + \frac{\xi^2-1}{a\xi} \tau_{z\theta} \right) \delta_\theta \otimes \delta_\xi \otimes \delta_z \\ & + \left(\frac{X}{a} \frac{\partial \tau_{z\theta}}{\partial \xi} + \frac{2\sin\theta}{a} \tau_{z\xi} \right) \delta_\xi \otimes \delta_\theta \otimes \delta_z + \left(\frac{X}{a\xi} \frac{\partial \tau_{z\theta}}{\partial \theta} - \frac{\xi^2-1}{a\xi} \tau_{z\xi} \right) \delta_\theta \otimes \delta_\theta \otimes \delta_z \\ & + \frac{\partial \tau_{\xi\xi}}{\partial z} \delta_z \otimes \delta_\xi \otimes \delta_\xi + \frac{\partial \tau_{\theta\theta}}{\partial z} \delta_z \otimes \delta_\theta \otimes \delta_\theta + \frac{\partial \tau_{\xi\theta}}{\partial z} \delta_z \otimes \delta_\theta \otimes \delta_\xi + \frac{\partial \tau_{\theta\xi}}{\partial z} \delta_z \otimes \delta_\xi \otimes \delta_\theta \\ & + \frac{\partial \tau_{\xi z}}{\partial z} \delta_z \otimes \delta_z \otimes \delta_\xi + \frac{\partial \tau_{z\xi}}{\partial z} \delta_z \otimes \delta_\xi \otimes \delta_z + \frac{\partial \tau_{\theta z}}{\partial z} \delta_z \otimes \delta_z \otimes \delta_\theta + \frac{\partial \tau_{z\theta}}{\partial z} \delta_z \otimes \delta_\theta \otimes \delta_z \\ & + \frac{X}{a} \frac{\partial \tau_{zz}}{\partial \xi} \delta_\xi \otimes \delta_z \otimes \delta_z + \frac{X}{a\xi} \frac{\partial \tau_{zz}}{\partial \theta} \delta_\theta \otimes \delta_z \otimes \delta_z + \frac{\partial \tau_{zz}}{\partial z} \delta_z \otimes \delta_z \otimes \delta_z \end{aligned} \right\} \quad (105)$$

We then dot the velocity into the gradient of the extra stress tensor to get:

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\xi\xi} = (\mathbf{v} \cdot \nabla) \tau_{\xi\xi} - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (106)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\xi\theta} = (\mathbf{v} \cdot \nabla) \tau_{\xi\theta} + \frac{(\tau_{\xi\xi} - \tau_{\theta\theta})}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (107)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\xi z} = (\mathbf{v} \cdot \nabla) \tau_{\xi z} - \frac{\tau_{\theta z}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (108)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\theta\xi} = (\mathbf{v} \cdot \nabla) \tau_{\theta\xi} + \frac{(\tau_{\xi\xi} - \tau_{\theta\theta})}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (109)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\theta\theta} = (\mathbf{v} \cdot \nabla) \tau_{\theta\theta} + \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (110)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{\theta z} = (\mathbf{v} \cdot \nabla) \tau_{\theta z} + \frac{\tau_{\xi z}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (111)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{z\xi} = (\mathbf{v} \cdot \nabla) \tau_{z\xi} - \frac{\tau_{z\theta}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (112)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{z\theta} = (\mathbf{v} \cdot \nabla) \tau_{z\theta} + \frac{\tau_{z\xi}}{a} \left(2v_\xi \sin\theta - v_\theta \frac{\xi^2-1}{\xi} \right) \quad (113)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{\tau}\}_{zz} = (\mathbf{v} \cdot \nabla) \tau_{zz} \quad (114)$$

where:

$$(\mathbf{v} \cdot \nabla) = \left[v_{\xi} \frac{X}{a} \frac{\partial}{\partial \xi} + v_{\theta} \frac{X}{a\xi} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right] \quad (115)$$

and:

$$(\mathbf{v} \cdot \nabla) \tau_{ij} = \left[v_{\xi} \frac{X}{a} \frac{\partial \tau_{ij}}{\partial \xi} + v_{\theta} \frac{X}{a\xi} \frac{\partial \tau_{ij}}{\partial \theta} + v_z \frac{\partial \tau_{ij}}{\partial z} \right] \quad (116)$$

Table IV summarizes all differential operations in eccentric cylindrical coordinates.

Appendix C: Eccentric Cylindrical Coordinate Equations of Change

The equation of continuity is written generally as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (117)$$

Substituting Eq. (98) into Eq. (117) and expressing $\mathbf{v}$ in unit vector notation, we get:

$$\frac{\partial \rho}{\partial t} + \frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi}{X} \rho v_{\xi} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{\rho v_{\theta}}{X} \right) + \frac{\partial}{\partial z} \rho v_z = 0 \quad (118)$$

which is the equation of continuity in eccentric cylindrical coordinates.

We next calculate the equation of motion for eccentric cylindrical coordinates. We begin with the definition of the equation of motion:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g} \quad (119)$$

and develop each term separately. For the pressure gradient, we get:

$$[\nabla p] = \frac{\delta_{\xi}}{h_{\xi}} \frac{\partial p}{\partial \xi} + \frac{\delta_{\theta}}{h_{\theta}} \frac{\partial p}{\partial \theta} + \frac{\delta_z}{h_z} \frac{\partial p}{\partial z} \quad (120)$$

$$[\nabla p] = \delta_{\xi} \frac{X}{a} \frac{\partial p}{\partial \xi} + \delta_{\theta} \frac{X}{a\xi} \frac{\partial p}{\partial \theta} + \delta_z \frac{\partial p}{\partial z} \quad (121)$$

We next find the dyadic product of the gradient and velocity vectors:

$$\{\nabla \mathbf{v}\} = \left\{ \left[\delta_{\xi} \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_{\theta} \frac{X}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right] [v_{\xi} \delta_{\xi} + v_{\theta} \delta_{\theta} + v_z \delta_z] \right\} \quad (122)$$

$$\{\nabla \mathbf{v}\} = \left\{ \begin{array}{ccc} \frac{X}{a} \frac{\partial v_{\xi}}{\partial \xi} - v_{\theta} \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_{\xi}}{\partial \theta} + v_{\theta} \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_{\xi}}{\partial z} \\ \frac{X}{a} \frac{\partial v_{\theta}}{\partial \xi} + v_{\xi} \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_{\theta}}{\partial \theta} - v_{\xi} \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_{\theta}}{\partial z} \\ \frac{X}{a} \frac{\partial v_z}{\partial \xi} & \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{array} \right\} \quad (123)$$

for each component of the equation of motion. We next develop the substantial derivative on the left side of Eq. (119) for the radial component, ξ :

$$\rho \frac{Dv_{\xi}}{Dt} = \rho \left(\frac{\partial v_{\xi}}{\partial t} + (\mathbf{v} \cdot \nabla v_{\xi}) \right) \quad (124)$$

where:

$$\{\nabla v_{\xi}\} = \delta_{\xi} \frac{X}{a} \frac{\partial v_{\xi}}{\partial \xi} - \delta_{\xi} v_{\theta} \frac{2 \sin \theta}{a} + \delta_{\theta} \frac{X}{a\xi} \frac{\partial v_{\xi}}{\partial \theta} + \delta_{\theta} v_{\xi} \frac{\xi^2 - 1}{a\xi} + \delta_z \frac{\partial v_{\xi}}{\partial z} \quad (125)$$

and

$$\rho \frac{Dv_{\xi}}{Dt} = \rho \left(\frac{\partial v_{\xi}}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_{\xi}}{\partial \xi} - v_{\theta} v_{\xi} \frac{2 \sin \theta}{a} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_{\xi}}{\partial \theta} + v_{\theta}^2 \frac{\xi^2 - 1}{a \xi} + v_z \frac{\partial v_{\xi}}{\partial z} \right) \quad (126)$$

We repeat this analysis for the other velocity components:

$$\{\nabla v\}_{\theta} = \delta_{\xi} \left(\frac{X}{a} \frac{\partial v_{\theta}}{\partial \xi} + v_{\xi} \frac{2 \sin \theta}{a} \right) + \delta_{\theta} \left(\frac{X}{a \xi} \frac{\partial v_{\theta}}{\partial \theta} - v_{\xi} \frac{\xi^2 - 1}{a \xi} \right) + \delta_z \frac{\partial v_{\theta}}{\partial z} \quad (127)$$

$$\rho \frac{Dv_{\theta}}{Dt} = \rho \left(\frac{\partial v_{\theta}}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_{\theta}}{\partial \xi} + v_{\theta}^2 \frac{2 \sin \theta}{a} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_{\theta}}{\partial \theta} - v_{\xi} v_{\theta} \frac{\xi^2 - 1}{a \xi} + v_z \frac{\partial v_{\theta}}{\partial z} \right) \quad (128)$$

$$\{\nabla v\}_z = \delta_{\xi} \frac{X}{a} \frac{\partial v_z}{\partial \xi} + \delta_{\theta} \frac{X}{a \xi} \frac{\partial v_z}{\partial \theta} + \delta_z \frac{\partial v_z}{\partial z} \quad (129)$$

$$\rho \frac{Dv_z}{Dt} = \rho \left(\frac{\partial v_z}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_z}{\partial \xi} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \quad (130)$$

Next we evaluate the divergence of the extra stress tensor:

$$[\nabla \cdot \boldsymbol{\tau}] = \left[\delta_{\xi} \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_{\theta} \frac{X}{a \xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right] \cdot \left\{ \begin{aligned} &\tau_{\xi\xi} \delta_{\xi} \delta_{\xi} + \tau_{\theta\xi} \delta_{\theta} \delta_{\xi} + \tau_{z\xi} \delta_z \delta_{\xi} \\ &+ \tau_{\xi\theta} \delta_{\xi} \delta_{\theta} + \tau_{\theta\theta} \delta_{\theta} \delta_{\theta} + \tau_{z\theta} \delta_z \delta_{\theta} \\ &+ \tau_{\xi z} \delta_{\xi} \delta_z + \tau_{\theta z} \delta_{\theta} \delta_z + \tau_{zz} \delta_z \delta_z \end{aligned} \right\} \quad (131)$$

and after careful simplification, we get:

$$[\nabla \cdot \boldsymbol{\tau}] = \left[\begin{aligned} &\frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} \otimes \delta_{\xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} \otimes \delta_{\xi} + \frac{\partial \tau_{z\xi}}{\partial z} \otimes \delta_{\xi} + \frac{\tau_{\theta\theta} - \tau_{\xi\xi}}{a \xi} (\xi^2 - 1) \otimes \delta_{\xi} - 2 \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \sin \theta \otimes \delta_{\xi} \\ &+ \frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} \otimes \delta_{\theta} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} \otimes \delta_{\theta} + \frac{\partial \tau_{z\theta}}{\partial z} \otimes \delta_{\theta} + 2 \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \sin \theta \otimes \delta_{\theta} - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a \xi} (\xi^2 - 1) \otimes \delta_{\theta} \\ &+ \frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} \otimes \delta_z + \frac{X}{a \xi} \frac{\partial \tau_{\theta z}}{\partial \theta} \otimes \delta_z + \frac{\partial \tau_{zz}}{\partial z} \otimes \delta_z - 2 \frac{\tau_{\theta z}}{a} \sin \theta \otimes \delta_z - \frac{\tau_{\xi z}}{a \xi} (\xi^2 - 1) \otimes \delta_z \end{aligned} \right] \quad (132)$$

We can express Eq. (132) as a column vector:

$$[\nabla \cdot \boldsymbol{\tau}] = \left[\begin{aligned} &\frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} + \frac{\partial \tau_{z\xi}}{\partial z} + \frac{\tau_{\theta\theta} - \tau_{\xi\xi}}{a \xi} (\xi^2 - 1) - 2 \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \sin \theta \\ &\frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + 2 \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \sin \theta - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a \xi} (\xi^2 - 1) \\ &\frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - 2 \frac{\tau_{\theta z}}{a} \sin \theta - \frac{\tau_{\xi z}}{a \xi} (\xi^2 - 1) \end{aligned} \right] \quad (133)$$

We can thus write down the equation of motion for each component of velocity by substituting Eqs. (121), (126), (128), (130) and (133) into Eq. (119):

$$\rho \left(\frac{\partial v_{\xi}}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_{\xi}}{\partial \xi} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_{\xi}}{\partial \theta} + v_z \frac{\partial v_{\xi}}{\partial z} - 2 \frac{v_{\xi} v_{\theta} \sin \theta}{a} + \frac{v_{\theta}^2 (\xi^2 - 1)}{a \xi} \right) = - \frac{X}{a} \frac{\partial p}{\partial \xi} - \left[\frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} + \frac{\partial \tau_{z\xi}}{\partial z} + \frac{\tau_{\theta\theta} - \tau_{\xi\xi}}{a \xi} (\xi^2 - 1) - 2 \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \sin \theta \right] + \rho g_{\xi} \quad (134)$$

$$\rho \left(\frac{\partial v_{\theta}}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_{\theta}}{\partial \xi} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_{\theta}}{\partial \theta} + v_z \frac{\partial v_{\theta}}{\partial z} - \frac{v_{\xi} v_{\theta} (\xi^2 - 1)}{a \xi} + 2 \frac{v_{\theta}^2 \sin \theta}{a} \right) = - \frac{X}{a \xi} \frac{\partial p}{\partial \theta} - \left[\frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + 2 \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \sin \theta - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a \xi} (\xi^2 - 1) \right] + \rho g_{\theta} \quad (135)$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_{\xi} \frac{X}{a} \frac{\partial v_z}{\partial \xi} + v_{\theta} \frac{X}{a \xi} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} - \left[\frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} + \frac{X}{a \xi} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - 2 \frac{\tau_{\theta z}}{a} \sin \theta - \frac{\tau_{\xi z}}{a \xi} (\xi^2 - 1) \right] + \rho g_z \quad (136)$$

For a Newtonian fluid, the extra stress components are expressed generally as:

$$\boldsymbol{\tau} = -\mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \right) + \left(\frac{2}{3} \mu - \kappa \right) (\nabla \cdot \mathbf{v}) \delta \quad (137)$$

which we evaluate in eccentric cylindrical coordinates to get:

$$\tau_{\xi\xi} = -2\mu \left(\frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} \right) + \left(\frac{2}{3} \mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (138)$$

$$\tau_{\theta\theta} = -2\mu \left(\frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} \right) + \left(\frac{2}{3} \mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (139)$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \left(\frac{2}{3} \mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (140)$$

$$\tau_{\xi\theta} = \tau_{\theta\xi} = -\mu \left(\xi \frac{\partial}{\partial \xi} \left(\frac{X}{a\xi} v_\theta \right) + \frac{1}{a\xi} \frac{\partial}{\partial \theta} (X v_\xi) \right) \quad (141)$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right) \quad (142)$$

$$\tau_{z\xi} = \tau_{\xi z} = -\mu \left(\frac{X}{a} \frac{\partial v_z}{\partial \xi} + \frac{\partial v_\xi}{\partial z} \right) \quad (143)$$

We can use Eqs. (138)-(143) to replace the extra stress components in Eqs. (134)-(136) for a Newtonian fluid.

We now turn our attention to developing the equation of energy and begin with the general expression:

$$\rho \hat{C}_p \frac{DT}{Dt} = -(\nabla \cdot \mathbf{q}) - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (144)$$

and insert expressions for the substantial derivative of temperature:

$$\rho \hat{C}_p \frac{DT}{Dt} = \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + [\delta_\xi v_\xi + \delta_\theta v_\theta + \delta_z v_z] \cdot \nabla T \right) \quad (145)$$

$$\rho \hat{C}_p \frac{DT}{Dt} = \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_\xi \frac{X}{a} \frac{\partial T}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) \quad (146)$$

and the divergence of heat flux:

$$(\nabla \cdot \mathbf{q}) = \left[\delta_\xi \frac{X}{a} \frac{\partial}{\partial \xi} + \delta_\theta \frac{X}{a\xi} \frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \right] \cdot [q_\xi \delta_\xi + q_\theta \delta_\theta + q_z \delta_z] \quad (147)$$

$$(\nabla \cdot \mathbf{q}) = \frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{q_\xi}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{q_\theta}{X} \right) + \frac{\partial q_z}{\partial z} \quad (148)$$

to get:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_\xi \frac{X}{a} \frac{\partial T}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{q_\xi}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{q_\theta}{X} \right) + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (149)$$

where:

$$(\boldsymbol{\tau} \cdot \nabla \mathbf{v}) = \begin{Bmatrix} \tau_{\xi\xi} & \tau_{\xi\theta} & \tau_{\xi z} \\ \tau_{\xi\theta} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{\xi z} & \tau_{\theta z} & \tau_{zz} \end{Bmatrix} : \begin{Bmatrix} \frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_\theta \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_\xi}{\partial z} \\ \frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\xi \frac{2 \sin \theta}{a} & \frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} & \frac{\partial v_\theta}{\partial z} \\ \frac{X}{a} \frac{\partial v_z}{\partial \xi} & \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{Bmatrix} \quad (150)$$

$$(\boldsymbol{\tau} \cdot \nabla \mathbf{v}) = \tau_{\xi\xi} \left(\frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} \right) + \tau_{\xi\theta} \left(\frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_\theta \right) + \tau_{\xi z} \left(\frac{\partial v_\xi}{\partial z} \right) + \tau_{\theta\xi} \left(\frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\xi \frac{2 \sin \theta}{a} \right) + \tau_{\theta\theta} \left(\frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} \right) + \tau_{\theta z} \left(\frac{\partial v_\theta}{\partial z} \right) + \tau_{z\xi} \left(\frac{X}{a} \frac{\partial v_z}{\partial \xi} \right) + \tau_{z\theta} \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) \quad (151)$$

The heat fluxes in Eq. (149) can be replaced by Fourier's law of heat conduction, which is written generally as:

$$\mathbf{q} = -k \nabla T \quad (152)$$

We thus get:

$$q_\xi = -k \frac{X}{a} \frac{\partial T}{\partial \xi} \quad (153)$$

$$q_\theta = -k \frac{X}{a\xi} \frac{\partial T}{\partial \theta} \quad (154)$$

$$q_z = -k \frac{\partial T}{\partial z} \quad (155)$$

the components of Fourier's law in eccentric cylindrical coordinates.

We finally develop the equation of continuity for species α :

$$\rho \frac{D\omega_\alpha}{Dt} = -(\nabla \cdot \mathbf{j}_\alpha) + r_\alpha \quad (156)$$

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_\xi \frac{X}{a} \frac{\partial \omega_\alpha}{\partial \xi} + v_\theta \frac{X}{a\xi} \frac{\partial \omega_\alpha}{\partial \theta} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{j_{a\xi}}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{j_{a\theta}}{X} \right) + \frac{\partial j_{az}}{\partial z} \right] + r_\alpha \quad (157)$$

for which the standard substitutions to convert from mass to moles can be found at the bottom of Section B.11 in [1]. The mass flux can be replaced by Fick's law of diffusion for a binary system, which is stated generally as:

$$\mathbf{j}_A = -\rho \mathcal{D}_{AB} \nabla \omega_A \quad (158)$$

We thus get:

$$j_{A\xi} = -\rho \mathcal{D}_{AB} \frac{X}{a} \frac{\partial \omega_A}{\partial \xi} \quad (159)$$

$$j_{A\theta} = -\rho \mathcal{D}_{AB} \frac{X}{a\xi} \frac{\partial \omega_A}{\partial \theta} \quad (160)$$

$$j_{Az} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial z} \quad (161)$$

the components of Fick's law in eccentric cylindrical coordinates.

Appendix D: Supplementary Calculations for Section IV

She first considers the particular solution to Eq. (27) and introduces the combined variables:

$$\chi = \xi e^{i\theta} \quad (162)$$

$$\chi^* = \xi e^{-i\theta} \quad (163)$$

or:

$$\xi = \sqrt{\chi\chi^*} \quad (164)$$

$$\theta = \frac{1}{2i} \ln \frac{\chi}{\chi^*} \quad (165)$$

She rewrites the footnote equation on p. 826 of [1] for eccentric cylindrical coordinates to get:

$$\frac{\partial}{\partial \xi} = \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial \chi} + \frac{\partial \chi^*}{\partial \xi} \frac{\partial}{\partial \chi^*} \quad (166)$$

$$\frac{\partial}{\partial \theta} = \frac{\partial \chi}{\partial \theta} \frac{\partial}{\partial \chi} + \frac{\partial \chi^*}{\partial \theta} \frac{\partial}{\partial \chi^*} \quad (167)$$

She then substitutes Eqs. (162)-(165) into Eqs. (166)-(167) to find:

$$\frac{\partial}{\partial \xi} = e^{i\theta} \frac{\partial}{\partial \chi} + e^{-i\theta} \frac{\partial}{\partial \chi^*} = \sqrt{\frac{\chi}{\chi^*}} \frac{\partial}{\partial \chi} + \sqrt{\frac{\chi^*}{\chi}} \frac{\partial}{\partial \chi^*} \quad (168)$$

$$\frac{\partial}{\partial \theta} = i\chi \frac{\partial}{\partial \chi} - i\chi^* \frac{\partial}{\partial \chi^*} \quad (169)$$

$$\frac{\partial^2}{\partial \xi^2} = \frac{\chi}{\chi^*} \frac{\partial^2}{\partial \chi^2} + 2 \frac{\partial^2}{\partial \chi \partial \chi^*} + \frac{\chi^*}{\chi} \frac{\partial^2}{\partial \chi^{*2}} \quad (170)$$

$$\frac{\partial^2}{\partial \theta^2} = -\chi \frac{\partial}{\partial \chi} - \chi^2 \frac{\partial^2}{\partial \chi^2} + 2\chi\chi^* \frac{\partial^2}{\partial \chi \partial \chi^*} - \chi^* \frac{\partial}{\partial \chi^*} - \chi^{*2} \frac{\partial^2}{\partial \chi^{*2}} \quad (171)$$

Inserting Eqs. (168)-(171) into Eq. (27):

$$\frac{V}{X^2} = \frac{1}{\xi} \left(\frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_z}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 v_z}{\partial \theta^2} \right); V \equiv \frac{a^2 \Delta p}{\mu L} \quad (172)$$

she gets:

$$\frac{V}{X^2} = \left(\frac{1}{\xi} \frac{\partial v_{z,p}}{\partial \xi} + \frac{\partial^2 v_{z,p}}{\partial \xi^2} + \frac{1}{\xi^2} \left(-\chi \frac{\partial v_{z,p}}{\partial \chi} - \chi^2 \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2\chi\chi^* \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} - \chi^* \frac{\partial v_{z,p}}{\partial \chi^*} - \chi^{*2} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) \right) \quad (173)$$

$$\frac{V}{X^2} = \left(\frac{1}{\xi} \left(\sqrt{\frac{\chi}{\chi^*}} \frac{\partial v_{z,p}}{\partial \chi} + \sqrt{\frac{\chi^*}{\chi}} \frac{\partial v_{z,p}}{\partial \chi^*} \right) + \frac{\chi}{\chi^*} \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2 \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} + \frac{\chi^*}{\chi} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right. \\ \left. + \frac{1}{\xi^2} \left(-\chi \frac{\partial v_{z,p}}{\partial \chi} - \chi^2 \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2\chi\chi^* \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} - \chi^* \frac{\partial v_{z,p}}{\partial \chi^*} - \chi^{*2} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) \right) \quad (174)$$

She uses Eq. (164) to obtain:

$$\frac{V}{X^2} = \left(\frac{1}{\sqrt{\chi\chi^*}} \left(\sqrt{\frac{\chi}{\chi^*}} \frac{\partial v_{z,p}}{\partial \chi} + \sqrt{\frac{\chi^*}{\chi}} \frac{\partial v_{z,p}}{\partial \chi^*} \right) + \frac{\chi}{\chi^*} \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2 \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} + \frac{\chi^*}{\chi} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) + \frac{1}{\chi\chi^*} \left(-\chi \frac{\partial v_{z,p}}{\partial \chi} - \chi^2 \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2\chi\chi^* \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} - \chi^* \frac{\partial v_{z,p}}{\partial \chi^*} - \chi^{*2} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) \quad (175)$$

$$\frac{V}{X^2} = \left(\frac{1}{\chi^*} \frac{\partial v_{z,p}}{\partial \chi} + \frac{1}{\chi} \frac{\partial v_{z,p}}{\partial \chi^*} + \frac{\chi}{\chi^*} \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2 \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} + \frac{\chi^*}{\chi} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) - \left(\frac{1}{\chi^*} \frac{\partial v_{z,p}}{\partial \chi} - \frac{\chi}{\chi^*} \frac{\partial^2 v_{z,p}}{\partial \chi^2} + 2 \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} - \frac{1}{\chi} \frac{\partial v_{z,p}}{\partial \chi^*} - \frac{\chi^*}{\chi} \frac{\partial^2 v_{z,p}}{\partial \chi^{*2}} \right) \quad (176)$$

$$\frac{V}{X^2} = 4 \frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} \quad (177)$$

She recalls that:

$$X = 1 - 2\xi \cos \theta + \xi^2 \quad (178)$$

and uses Eqs. (164)-(165) to solve Eq. (177):

$$\frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} = \frac{V}{4 \left(1 - 2\sqrt{\chi\chi^*} \cos \frac{1}{2i} \ln \frac{\chi}{\chi^*} + \chi\chi^* \right)^2} \quad (179)$$

$$\frac{\partial^2 v_{z,p}}{\partial \chi \partial \chi^*} = \frac{V}{4 \left(1 - (\chi + \chi^*) + \chi\chi^* \right)^2} \quad (180)$$

$$\frac{\partial v_{z,p}}{\partial \chi^*} = - \frac{V}{4(\chi - 1)(\chi^* - 1)^2} \quad (181)$$

$$\frac{\partial v_{z,p}}{\partial \chi^*} = - \frac{V}{4(\chi - 1)(\chi^* - 1)^2} \quad (182)$$

$$v_{z,p} = \frac{V}{4(\chi - 1)(\chi^* - 1)} \quad (183)$$

$$v_{z,p} = \frac{V}{4(1 - 2\xi \cos \theta + \xi^2)} \quad (184)$$

Eq. (184) is the particular solution to Eq. (172).

To obtain the homogenous solution, she begins with the homogenous differential equation:

$$0 = \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_{z,h}}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 v_{z,h}}{\partial \theta^2} \quad (185)$$

She then performs separation of variables by assuming:

$$v_{z,h} = f(\xi)g(\theta) \quad (186)$$

to get:

$$0 = \frac{\partial}{\partial \xi} \left(\xi \frac{\partial f(\xi)g(\theta)}{\partial \xi} \right) + \frac{1}{\xi} \frac{\partial^2 f(\xi)g(\theta)}{\partial \theta^2} \quad (187)$$

Applying orthogonality, she finds:

$$-\frac{\xi}{f(\xi)} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial f(\xi)}{\partial \xi} \right) = -\lambda^2 = \frac{1}{g(\theta)} \frac{\partial^2 g(\theta)}{\partial \theta^2} \quad (188)$$

She first solves the rightmost equality of Eq. (188):

$$0 = \frac{\partial^2 g(\theta)}{\partial \theta^2} + \lambda^2 g(\theta) \quad (189)$$

Integrating, she finds:

$$g(\theta) = C_1 \cos \lambda \theta + C_2 \sin \lambda \theta \quad (190)$$

$$g'(\theta) = -C_1 \lambda \sin \lambda \theta + C_2 \lambda \cos \lambda \theta \quad (191)$$

She then applies symmetry boundary conditions at $\theta = n\pi$:

$$g'(0) = 0 = C_2 \quad (192)$$

$$g'(\pi) = 0 = -C_1 \lambda \sin \lambda \pi \quad (193)$$

to find:

$$\lambda = n \quad (194)$$

and thus:

$$g(\theta) = C_1 \cos n\theta \quad (195)$$

where n is an integer.

Substituting Eq. (195) into Eq. (186) and summing over all n , she gets:

$$\phi_h = \sum_{n=0}^{\infty} f_n(\xi) \cos n\theta \quad (196)$$

Solving the leftmost equality in Eq. (188) using Eq. (194):

$$\frac{\partial^2 f_n}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial f_n}{\partial \xi} - \frac{n^2}{\xi^2} f_n = 0 \quad (197)$$

she finds the general solutions:

$$f_n = \begin{cases} \alpha_0 + \beta_0 \ln \xi & ; n = 0 \\ \alpha_n \xi^n + \beta_n \xi^{-n} & ; n \geq 1 \end{cases} \quad (198)$$

for $n = 0$ and $n \geq 1$.

Substituting Eq. (198) into Eq. (196) and inserting the result into Eq. (28) recalling Eq. (184), she finds:

$$v_z(\xi, \theta) = \alpha_0 + \beta_0 \ln \xi + \sum_{n=1}^{\infty} (\alpha_n \xi^n + \beta_n \xi^{-n}) \cos n\theta + \frac{V}{4(1 - 2\xi \cos \theta + \xi^2)} \quad (199)$$

the general expression for the velocity profile. She approximates the particular solution as a cosine series:

$$\frac{V}{4(1-2\xi\cos\theta+\xi^2)} = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos n\theta \quad (200)$$

where:

$$b_n = \frac{-V\xi^n}{2(\xi^2-1)} \quad (201)$$

Replacing the particular solution with this expansion:

$$v_z(\xi, \theta) = \alpha_0 + \beta_0 \ln \xi + \sum_{n=1}^{\infty} (\alpha_n \xi^n + \beta_n \xi^{-n}) \cos n\theta - \frac{V}{4(\xi^2-1)} + \sum_{n=1}^{\infty} \frac{-V\xi^n}{2(\xi^2-1)} \cos n\theta \quad (202)$$

allows her to solve for the constants in Eq. (198). She begins by inserting the no slip boundary conditions at the inner and outer cylinders:

$$v_z(\xi_o, \theta) = 0 = \alpha_0 + \beta_0 \ln \xi_o + \sum_{n=1}^{\infty} (\alpha_n \xi_o^n + \beta_n \xi_o^{-n}) \cos n\theta - \frac{V}{4(\xi_o^2-1)} + \sum_{n=1}^{\infty} \frac{-V\xi_o^n}{2(\xi_o^2-1)} \cos n\theta \quad (203)$$

$$v_z(\xi_i, \theta) = 0 = \alpha_0 + \beta_0 \ln \xi_i + \sum_{n=1}^{\infty} (\alpha_n \xi_i^n + \beta_n \xi_i^{-n}) \cos n\theta - \frac{V}{4(\xi_i^2-1)} + \sum_{n=1}^{\infty} \frac{-V\xi_i^n}{2(\xi_i^2-1)} \cos n\theta \quad (204)$$

She then focuses on the case where $n=0$ first to get:

$$v_z(\xi_o, \theta) = 0 = \alpha_0 + \beta_0 \ln \xi_o - \frac{V}{4(\xi_o^2-1)} \quad (205)$$

$$\frac{V}{4(\xi_o^2-1)} - \beta_0 \ln \xi_o = \alpha_0 \quad (206)$$

$$\alpha_0 = V \frac{\xi_i^2 \ln \xi_i - \xi_o^2 \ln \xi_o - \ln \frac{\xi_i}{\xi_o}}{4(\xi_i^2-1)(\xi_o^2-1) \ln \frac{\xi_i}{\xi_o}} \quad (207)$$

for the first integration constant and:

$$\beta_0 = \frac{V(\xi_o^2 - \xi_i^2)}{4(\xi_i^2-1)(\xi_o^2-1) \ln \frac{\xi_i}{\xi_o}} \quad (208)$$

for the second.

Turning her attention to $n \geq 1$, she finds:

$$0 = \alpha_n \xi_o^n + \beta_n \xi_o^{-n} - \frac{V\xi_o^n}{2(\xi_o^2-1)} \quad (209)$$

$$\frac{V\xi_o^{2n}}{2(\xi_o^2-1)} - \alpha_n \xi_o^{2n} = \beta_n \quad (210)$$

$$0 = \alpha_n \xi_i^n + \beta_n \xi_i^{-n} - \frac{V\xi_i^n}{2(\xi_i^2-1)} \quad (211)$$

$$0 = \alpha_n \xi_i^n + \frac{V \xi_o^{2n} \xi_i^{-n}}{2(\xi_o^2 - 1)} - \alpha_n \xi_o^{2n} \xi_i^{-n} - \frac{V \xi_i^n}{2(\xi_i^2 - 1)} \quad (212)$$

$$V \frac{\xi_i^{2n} (\xi_o^2 - 1) - \xi_o^{2n} (\xi_i^2 - 1)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} = \alpha_n \quad (213)$$

for the third integration constant and:

$$\frac{V \xi_o^{2n}}{2(\xi_o^2 - 1)} - V \frac{\xi_i^{2n} (\xi_o^2 - 1) - \xi_o^{2n} (\xi_i^2 - 1)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} \xi_o^{2n} = \beta_n \quad (214)$$

$$\frac{V \xi_i^{2n} \xi_o^{2n} (\xi_i^2 - \xi_o^2)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} = \beta_n \quad (215)$$

for the final integration constant. These constants are briefly summarized as:

$$\alpha_n \equiv \begin{cases} V \frac{\xi_i^2 \ln \xi_i - \xi_o^2 \ln \xi_o - \ln \frac{\xi_i}{\xi_o}}{4(\xi_i^2 - 1)(\xi_o^2 - 1) \ln \frac{\xi_i}{\xi_o}} & ; n = 0 \\ V \frac{\xi_i^{2n} (\xi_o^2 - 1) - \xi_o^{2n} (\xi_i^2 - 1)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} & ; n \geq 1 \end{cases} \quad (216)$$

and:

$$\beta_n \equiv \begin{cases} \frac{V (\xi_o^2 - \xi_i^2)}{4(\xi_i^2 - 1)(\xi_o^2 - 1) \ln \frac{\xi_i}{\xi_o}} & ; n = 0 \\ \frac{V \xi_i^{2n} \xi_o^{2n} (\xi_i^2 - \xi_o^2)}{2(\xi_i^2 - 1)(\xi_o^2 - 1)(\xi_i^{2n} - \xi_o^{2n})} & ; n \geq 1 \end{cases} \quad (217)$$

which she uses with Eq. (33) to determine the velocity profile for pressure-driven flow through an eccentric annulus of a Newtonian fluid.

Table I: Dimensional Variables

Binary diffusivity for system $A - B$	L^2/t	$\mathcal{D}_{AB}$
Cartesian coordinates	L	x, y, z_C
Combined molar flux ($\mathbf{N}_A = \mathbf{J}_A^* + c_A \mathbf{v}^*$)	mol/L^2t	$\mathbf{N}_A$
Concentration	mol/L^3	c
Confocal length [Eq. (5)]	L	a
Density	M/L^3	ρ
Eccentric cylindrical coordinate: Axial [Eq. (11)]	L	z
Eccentric cylindrical coordinate: Circumferential [Eq. (10)]	rad	θ
Eccentricity	L	δ
Extra stress tensor	M/Lt^2	$\boldsymbol{\tau}$
Gravity	L/t^2	$\mathbf{g}$
Heat capacity	ML^2/t^2T	$\hat{C}_p$
Heat flux	M/t^3	$\mathbf{q}$
Heat source: electrical	ML^2/t^3	S_e
Inner cylinder radius	L	R_i
Length (cylinder, wire, vent width)	L	L
Mass average velocity	L/t	$\mathbf{v}, \mathbf{w}$
Mass flux of species α	M/L^2t	$\mathbf{j}_\alpha$
Mass rate of production of species α by homogeneous chemical reaction	M/L^3t	r_α
Molar flow rate	mol/t	W
Molar flux of species α	mol/L^2t	$\mathbf{J}_\alpha^*$
Mole average velocity	L/t	$\mathbf{v}^*$
Mole rate of production of species α by homogeneous chemical reaction	mol/L^3t	R_α
Nondimensionalizing velocity coefficient [Eq. (29)]	L/t	V
Outer cylinder radius	L	R_o

Pressure	M/Lt^2	p
Temperature	T	T
Temperature	T	T
ij – th shear stress component*	M/Lt^2	τ_{ij}
Thermal conductivity	ML/t^3T	k
Time	t	t

Legend: M = mass, L = length, t = time, T = temperature, mol = mole;
 Boldface symbols are vectors or tensors. * ij – th component refers to the shear stress acting in the j – direction on surfaces of constant i from surfaces of lesser i to surfaces of greater i .

Table II: Dimensionless Variables and Groups

Combined variables [Eqs. (162) and (163)]	χ, χ^*
Del operator (nabla)	∇
Dimensionless heat flux	H
Dimensionless molar flux of species A	Ψ_A
Dimensionless temperature	Θ
Eccentric cylindrical coordinate: Radial [Eq. (9)]	ξ
General coordinate	n_i
Imaginary part operator	$\Im$
Imaginary unity	i
Mass fraction of species α	ω_α
Modified shape factor	X
Mole fraction of species α	x_α
Partial derivative operator	∂
Real part operator	$\Re$
Substantial derivative operator	D
i – th direction shape factor	h_i
i – th direction unit vector	δ_i

Table III: Summary of coordinate systems and equations of change

Coordinate System	Equations of Change	Contributors
Cartesian	C M E C _A	[1,2,6,7]
<i>Cylindrical</i>		
Circular Cylindrical	C M E C _A	[1,2,6,7]
Bipolar Cylindrical	C M E	[3,4,5,6,7,8,9,10]
Cardioid Cylindrical	X	-
Cassinian Oval	X	-
Elliptic Cylindrical	X	-
Logarithmic Cylindrical	X	-
Maxwell Cylindrical	X	-
Parabolic Cylindrical	X	-
Rose	X	-
Tangent Cylindrical	X	-
Zeta	X	-
<i>Rotational</i>		
Spherical	C M E C _A	[1,2,6,7]
Dijksman Toroidal	M	[16]
Bispherical	X	-
Cardioid	X	-
Conical	X	-
Ellipsoidal	X	-
Oblate Spheroidal	X	-
Parabolic	X	-
Paraboloidal	X	-
Prolate Spheroidal	X	-
Tangent Spherical	X	-
Toroidal	X	-
Eccentric Cylindrical	C M E C _A	This Paper

Legend: X – No known equations of change developed, C – Equation of continuity, M – Equation of motion, E – Equation of energy, C_A – Equation of continuity for a species

Table IV: Summary of Differential Operations in Eccentric Cylindrical Coordinates

$$(\nabla \cdot \mathbf{v}) = \frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi v_\xi}{X} \right) + \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{v_\theta}{X} \right) + \frac{\partial}{\partial z} v_z \quad (218)$$

$$\nabla^2 s = \left(\frac{X}{a} \right)^2 \frac{\partial^2 s}{\partial \xi^2} + \left(\frac{X}{a\xi} \right)^2 \frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \quad (219)$$

$$(\boldsymbol{\tau} \cdot \nabla \mathbf{v}) = \tau_{\xi\xi} \left(\frac{X}{a} \frac{\partial v_\xi}{\partial \xi} - v_\theta \frac{2 \sin \theta}{a} \right) + \tau_{\xi\theta} \left(\frac{X}{a\xi} \frac{\partial v_\xi}{\partial \theta} + v_\theta \right) + \tau_{\xi z} \left(\frac{\partial v_\xi}{\partial z} \right) + \tau_{\theta\xi} \left(\frac{X}{a} \frac{\partial v_\theta}{\partial \xi} + v_\xi \frac{2 \sin \theta}{a} \right) \\ + \tau_{\theta\theta} \left(\frac{X}{a\xi} \frac{\partial v_\theta}{\partial \theta} - v_\xi \frac{\xi^2 - 1}{a\xi} \right) + \tau_{\theta z} \left(\frac{\partial v_\theta}{\partial z} \right) + \tau_{z\xi} \left(\frac{X}{a} \frac{\partial v_z}{\partial \xi} \right) + \tau_{z\theta} \left(\frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) \quad (220)$$

$$[\nabla s]_\xi = \frac{X}{a} \frac{\partial s}{\partial \xi} \quad (221)$$

$$[\nabla s]_\theta = \frac{X}{a\xi} \frac{\partial s}{\partial \theta} \quad (222)$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \quad (223)$$

$$[\nabla \times \mathbf{v}]_\xi = \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad (224)$$

$$[\nabla \times \mathbf{v}]_\theta = \frac{\partial v_\xi}{\partial z} - \frac{X}{a} \frac{\partial v_z}{\partial \xi} \quad (225)$$

$$[\nabla \times \mathbf{v}]_z = \frac{X^2}{a\xi} \frac{\partial}{\partial \xi} \left(\frac{\xi v_\theta}{X} \right) - \frac{X^2}{a\xi} \frac{\partial}{\partial \theta} \left(\frac{v_\xi}{X} \right) \quad (226)$$

$$[\nabla \cdot \boldsymbol{\tau}]_\xi = \frac{X}{a} \frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta\xi}}{\partial \theta} + \frac{\partial \tau_{z\xi}}{\partial z} + \frac{\tau_{\theta\theta} - \tau_{\xi\xi}}{a\xi} (\xi^2 - 1) - 2 \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \sin \theta \quad (227)$$

$$[\nabla \cdot \boldsymbol{\tau}]_\theta = \frac{X}{a} \frac{\partial \tau_{\xi\theta}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + 2 \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \sin \theta - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a\xi} (\xi^2 - 1) \quad (228)$$

$$[\nabla \cdot \boldsymbol{\tau}]_z = \frac{X}{a} \frac{\partial \tau_{\xi z}}{\partial \xi} + \frac{X}{a\xi} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} - 2 \frac{\tau_{\theta z}}{a} \sin \theta - \frac{\tau_{\xi z}}{a\xi} (\xi^2 - 1) \quad (229)$$

$$[\nabla^2 \mathbf{v}]_\xi = \frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_\xi}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_\xi}{\partial \theta^2} + \frac{\partial^2 v_\xi}{\partial z^2} - \frac{v_\xi}{a^2 \xi^2} \left((\xi^2 - 1)^2 + 4 \xi^2 \sin^2 \theta \right) - \frac{2X}{a^2 \xi^2} \left((1 - \xi^2) \frac{\partial v_\theta}{\partial \theta} + 2 \xi^2 \sin \theta \frac{\partial v_\theta}{\partial \xi} \right) \quad (230)$$

$$[\nabla^2 \mathbf{v}]_\theta = \frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_\theta}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{a^2 \xi^2} \left(4 \xi^2 \sin^2 \theta + (\xi^2 - 1)^2 \right) + \frac{2X}{a^2 \xi^2} \left(2 \xi^2 \sin \theta \frac{\partial v_\xi}{\partial \xi} + (1 - \xi^2) \frac{\partial v_\xi}{\partial \theta} \right) \quad (231)$$

$$[\nabla^2 \mathbf{v}]_z = \frac{X^2}{a^2 \xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial v_z}{\partial \xi} \right) + \frac{X^2}{a^2 \xi^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \quad (232)$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_\xi = v_\xi \left(\frac{X}{a} \frac{\partial w_\xi}{\partial \xi} - \frac{2w_\theta}{a} \sin \theta \right) + v_\theta \left(\frac{X}{a\xi} \frac{\partial w_\xi}{\partial \theta} + \frac{w_\theta}{a\xi} (\xi^2 - 1) \right) + v_z \frac{\partial w_\xi}{\partial z} \quad (233)$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_\theta = v_\xi \left(\frac{X}{a} \frac{\partial w_\theta}{\partial \xi} - \frac{w_\theta}{a\xi} (\xi^2 - 1) \right) + v_\theta \left(\frac{X}{a\xi} \frac{\partial w_\theta}{\partial \theta} + \frac{2w_\theta}{a} \sin \theta \right) + v_z \frac{\partial w_\theta}{\partial z} \quad (234)$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_z = v_{\xi} \frac{X}{a} \frac{\partial w_z}{\partial \xi} + v_{\theta} \frac{X}{a\xi} \frac{\partial w_z}{\partial \theta} + v_z \frac{\partial w_z}{\partial z} \quad (235)$$

$$\{\nabla \mathbf{v}\}_{\xi\xi} = \frac{X}{a} \frac{\partial v_{\xi}}{\partial \xi} - \frac{2v_{\theta}}{a} \sin \theta \quad (236)$$

$$\{\nabla \mathbf{v}\}_{\xi\theta} = \frac{X}{a} \frac{\partial v_{\theta}}{\partial \xi} + v_{\xi} \frac{2\sin \theta}{a} \quad (237)$$

$$\{\nabla \mathbf{v}\}_{\xi z} = \frac{X}{a} \frac{\partial v_z}{\partial \xi} \quad (238)$$

$$\{\nabla \mathbf{v}\}_{\theta\xi} = \frac{X}{a\xi} \frac{\partial v_{\xi}}{\partial \theta} + v_{\theta} \frac{\xi^2 - 1}{a\xi} \quad (239)$$

$$\{\nabla \mathbf{v}\}_{\theta\theta} = \frac{X}{a\xi} \frac{\partial v_{\theta}}{\partial \theta} - v_{\xi} \frac{\xi^2 - 1}{a\xi} \quad (240)$$

$$\{\nabla \mathbf{v}\}_{\theta z} = \frac{X}{a\xi} \frac{\partial v_z}{\partial \theta} \quad (241)$$

$$\{\nabla \mathbf{v}\}_{z\xi} = \frac{\partial v_{\xi}}{\partial z} \quad (242)$$

$$\{\nabla \mathbf{v}\}_{z\theta} = \frac{\partial v_{\theta}}{\partial z} \quad (243)$$

$$\{\nabla \mathbf{v}\}_{zz} = \frac{\partial v_z}{\partial z} \quad (244)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\xi\xi} = (\mathbf{v} \cdot \nabla) \tau_{\xi\xi} - \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (245)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\xi\theta} = (\mathbf{v} \cdot \nabla) \tau_{\xi\theta} + \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (246)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\xi z} = (\mathbf{v} \cdot \nabla) \tau_{\xi z} - \frac{\tau_{\theta z}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (247)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\theta\xi} = (\mathbf{v} \cdot \nabla) \tau_{\theta\xi} + \frac{\tau_{\xi\xi} - \tau_{\theta\theta}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (248)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\theta\theta} = (\mathbf{v} \cdot \nabla) \tau_{\theta\theta} + \frac{\tau_{\xi\theta} + \tau_{\theta\xi}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (249)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{\theta z} = (\mathbf{v} \cdot \nabla) \tau_{\theta z} + \frac{\tau_{\xi z}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (250)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{z\xi} = (\mathbf{v} \cdot \nabla) \tau_{z\xi} - \frac{\tau_{z\theta}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (251)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{z\theta} = (\mathbf{v} \cdot \nabla) \tau_{z\theta} + \frac{\tau_{z\xi}}{a} \left(2v_{\xi} \sin \theta - v_{\theta} \frac{\xi^2 - 1}{\xi} \right) \quad (252)$$

$$\{\mathbf{v} \cdot \nabla \mathbf{t}\}_{zz} = (\mathbf{v} \cdot \nabla) \tau_{zz} \quad (253)$$

$$\text{where } (\mathbf{v} \cdot \nabla) = v_{\xi} \frac{X}{a} \frac{\partial}{\partial \xi} + v_{\theta} \frac{X}{a \xi} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

Figure 1: Eccentric cylindrical coordinates showing the circles of constant eccentric radial coordinate (black), ξ , and the circles of constant **eccentric angular coordinate** (green), θ . **Cyan** example between $\xi_i = 0.2$ and $\xi_o = 0.4$.

Figure 2: Differential volume element of the molten plastic in eccentric cylindrical coordinates (cyan). Edges of differential surface element through which melt flows labeled $\left[\frac{a}{1 - 2\xi \cos \theta + \xi^2} \right] d\xi$ and $\left[\frac{a\xi}{1 - 2\xi \cos \theta + \xi^2} \right] d\theta$. Cylindrical shell of constant eccentric cylindrical radial coordinate ξ parallels z -axis, and intersects x -axis at $x = a/(1 - \xi^2)$. This cylindrical shell contains surface element whose edges are dashed and labeled $\left[\frac{a\xi}{1 - 2\xi \cos \theta + \xi^2} \right] d\theta$ and dz .

Figure 3: Transport phenomena diagram for pressure driven flow through an eccentric annulus having eccentricity δ , inner radius R_i , outer radius R_o , and length L .

Figure 4: Velocity as a function of radial, ξ , and angular, θ , coordinates showing the effect of eccentricity on pressure driven flow through an annular die using Eq. (33) with Eqs. (34) and (35).

Figure 5: Transport phenomena diagram for heat conduction radially through a polyethylene wire coating, where the wire is eccentric with the coating having eccentricity δ , inner radius R_i , and outer radius R_o .

Figure 6: Heat flux as a function of radial, ξ , and angular, θ , coordinates showing the effect of eccentricity on temperature gradient using Eq. (42).

Figure 7: Eccentric wire coating temperature profile at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$, $R_i = 0.2$ and $R_o = 0.4$ for $\delta = 0.1$ (dashed lines) and $\delta = 0.15$ (solid lines) using Eq. (46) compared with the concentric wire coating temperature profile (dotted line) using Eq. (47).

Figure 8: Transport phenomena diagram for diffusion of compound A through a curved and expanding vent having eccentricity δ , inner radius R_i , outer radius R_o , and length L .

Figure 9: Concentration profile of compound A in the vent using Eq. (59).

Figure 10: θ dependence of molar flux of compound A for different values of ξ between the inner and outer vent walls ($\xi = \xi_i, \xi_o$) and for $\delta = 0.1$ using Eq. (61).

Figure 11: Effect of eccentricity on molar flux of compound A using Eq. (59).

References

- ¹ Bird, R.B., W.E. Stewart, and E.N. Lightfoot, "Transport phenomena," Revised 2nd ed., Wiley, New York (2007).
- ² Bird, R.B., W.E. Stewart, E.N. Lightfoot, and D.J. Klingenberg, "Introductory transport phenomena," Wiley, New York (2015).
- ³ Tadmor, Z., and R.B. Bird, "Rheological analysis of stabilizing forces in wire-coating dies," *Polymer Engineering & Science*, **14**(2), 124-136 (1974).
- ⁴ Tadmor, Z., and C.G. Gogos, "Principles of polymer processing," 1st ed., Wiley (1979).
- ⁵ Tadmor, Z., and C.G. Gogos, "Principles of polymer processing," 2nd ed., Wiley (2006).
- ⁶ Bird, R.B., R.C. Armstrong, and O. Hassager, "Dynamics of polymeric liquids. Vol. 1: Fluid mechanics," 1st ed., Wiley, New York (1987).
- ⁷ Bird, R.B., R.C. Armstrong, and O. Hassager, "Dynamics of polymeric liquids. Vol. 1: Fluid mechanics," 2nd ed., Wiley, New York (1987).
- ⁸ Kolitawong, C. and A.J. Giacomin, "Axial Flow Between Eccentric Cylinders," *Polymer-Plastics Technology and Engineering* **40**(3), 373-394 (2001).
- ⁹ Kolitawong, C., A.J. Giacomin and U. Nontakaew, "Viscous dissipation in plastic pipe extrusion," *Polymer Engineering and Science*, **53**(10), 2205-2218 (October, 2013).
- ¹⁰ Kolitawong, C., N. Kananai, A.J. Giacomin and U. Nontakaew, "Viscous Dissipation of a Power Law Fluid in Axial Flow Between Isothermal Eccentric Cylinders," *Journal of Non-Newtonian Fluid Mechanics*, **166**, 133-144 (2011); Errata: In Table 1 and Ref. 9, "Buckes" should be "Guckes".
- ¹¹ Moon, P. and D.E. Spencer, "Cylindrical and rotational coordinate systems," *Journal of the Franklin Institute*, **252**(4), 327-344 (1951).
- ¹² Moon, P. and D.E. Spencer, "Separability in a class of coordinate systems," *Journal of the Franklin Institute*, **254**(3), 227-242 (1952).
- ¹³ Moon, P. and D.E. Spencer, "Field Theory Handbook," Springer, Berlin (1961).
- ¹⁴ Moon, P. and D.E. Spencer, "Field Theory Handbook," 2nd ed., Springer, Berlin (1988).
- ¹⁵ Moon, P. and D.E. Spencer, "Field Theory for Engineers," Van Nostrand, Princeton, NJ (1961).

- ¹⁶ Dijksman, J.F., and E. P.W. Savenije, "The flow of Newtonian and non-Newtonian liquids through annular converging regions," *Rheologica Acta*, **24**(2), 105-118 (1985).
- ¹⁷ Kolitawong, C., A.J. Giacomini and L.M. Johnson, "Assumed Periodicity and Dynamic Shear Stress Transduction in Rheometry," *Journal of Rheology*, **54**(4), 835-858 (July / August, 2010).
- ¹⁸ Kolitawong, C., "Local Shear Stress Transduction in Sliding Plate Rheometry," PhD Thesis, University of Wisconsin, Mechanical Engineering Dept., University of Wisconsin, Madison, WI (December, 2002).
- ¹⁹ Ehrlich, R. and J.C. Slattery, "Evaluation of power-model lubricants in an infinite journal bearing," *Industrial & Engineering Chemistry Fundamentals*, **7**(2) 239-246 (1968).
- ²⁰ Moon, P. and D.E. Spencer, "Some coordinate systems associated with elliptic functions," *Journal of the Franklin Institute* **255**(6) 531-543 (1953).
- ²¹ Jones, J.R., "Flow of elastico-viscous liquids in pipes with cores (part one)," *Journal de Mécanique*, **3**(1) 79-99 (1964). See errata after Reference [12] in [22].~~pedi~~
- ²² Saengow, C., A.J. Giacomini and C. Kolitawong, "Extruding plastic pipe from eccentric dies," *Journal of Non-Newtonian Fluid Mechanics*, **223**, 176-199 (2015).
- ²³ Saengow, C., "Polymer Process Partitioning: Extruding Plastic Pipe," PhD Thesis, Polymers Research Group, Chemical Engineering Department, Queen's University, Kingston, CANADA (September 12, 2016).
- ²⁴ Saengow, C., "Polymer Process Partitioning Approach: Plastic Pipe Extrusion," PhD Thesis, Mechanical and Aerospace Engineering Department, King Mongkut's University of Technology North Bangkok, Bangkok, THAILAND (September 12, 2016).
- ²⁵ Morse, P.M., and H. Feshbach, "Methods of theoretical physics," McGraw-Hill, New York, 24-27, 115-116 (1953).