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EXACT ANALYTICAL SOLUTION FOR LARGE-AMPLITUDE OSCILLATORY SHEAR FLOW FROM OLDROYD 8-CONSTANT FRAMEWORK: SHEAR STRESS

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ABSTRACT

The Oldroyd 8-constant model is a continuum framework containing, as special cases, many important constitutive equations for elastic liquids. When polymeric liquids undergo large-amplitude oscillatory shear flow, the shear stress responds as a Fourier series, the higher harmonics of which are caused by the fluid nonlinearity. We choose this framework for its rich diversity of special cases (we tabulate 14 of these). Deepening our understanding of this Oldroyd 8-constant framework thus at once deepens our understanding of every one of these special cases. Previously [*Macromol Theor Simul*, **24**, 352 (2015)], we arrived at an exact analytical solution for the corotational Maxwell model. Here, we derive the exact analytical expression for the Oldroyd 8-constant framework for the shear stress response in large-amplitude oscillatory shear flow. Our exact solution reduces to our previous solution for the special case of the corotational Maxwell model, as it should. Our worked example uses the special case of the corotational Jeffreys model to explore the role of η_∞ on the higher harmonics.

Keywords: Large-amplitude oscillatory shear; LAOS; Oldroyd 8-constant model; higher harmonics.

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I. INTRODUCTION

Large-amplitude oscillatory shear (LAOS) flow experiments have been used to investigate the physics of complex liquids (see [1]; Ch. 11 of [2]). Since its conception in 1935 [3,4,5] oscillatory shear flow has become by far the most popular laboratory method for exploring the physics of polymeric liquids. We generate oscillatory shear flow by confining the fluid to a simple shear apparatus, and then subject one solid-liquid boundary to a coplanar sinusoidal displacement (see Figure 1) generating a corresponding cosinusoidal shear rate:

$$\dot{\gamma}(t) = \dot{\gamma}^0 \cos \omega t \quad (1)$$

Using the characteristic relaxation time of the viscoelastic fluid, λ_1 , we can nondimensionalize Eq. (1):

$$\lambda_1 \dot{\gamma}(t) = Wi \cos De(t / \lambda_1) \quad (2)$$

where:

$$De \equiv \lambda_1 \omega \quad (3)$$

and:

$$Wi \equiv \lambda_1 \dot{\gamma}^0 \quad (4)$$

are the Deborah and Weissenberg numbers. In this paper, we define dimensional symbols in Table II, and dimensionless ones in Table III (which follow Tables 2 and 3 of [6] with adaptations for [7]).

Increasing either the Weissenberg number or the Deborah number in Eq. (2) can cause the fluid response to depart from Newtonian behavior. We can construct a complex dimensionless number from the ordered pair (De, Wi) thus (see Figure 2 or **Fig. 1.** of [6]):

$$Gn \equiv De + i Wi \quad (5)$$

which defines a vector with magnitude:

$$|Gn| \equiv \sqrt{De^2 + Wi^2} \quad (6)$$

and with angle:

$$\hat{\phi} \equiv \arctan(Wi/De) \quad (7)$$

We can associate behavior in steady shear flow with $De = 0$, where $Gn = i Wi$. We further associate *linear viscoelastic* behavior with $Wi = 0$, where $Gn = De$. The $|Gn|$ thus reflects how far the fluid behavior departs from Newtonian behavior. The value $\hat{\phi} = 0$ corresponds to linear viscoelastic behavior, and $\hat{\phi} = \pi/2$, to steady shear flow. The angle $\hat{\phi}$ thus reflects the type of departure from Newtonian behavior.

When higher harmonics are observed in the shear stress response, we call the oscillatory experiment *large-amplitude*. By *higher harmonics*, we mean contributions to the shear stress at odd multiples of the test frequency:

$$\frac{\tau_{yx}(\tau, \dot{\gamma}^0)}{\dot{\gamma}^0} = - \sum_{n=1,2}^{\infty} \eta_n''(\omega, \dot{\gamma}^0) \cos n\tau + \eta_n'(\omega, \dot{\gamma}^0) \sin n\tau \quad (8)$$

For polymeric liquids, these higher harmonics are commonly observed when:

$$Wi/De > 1 \quad (9)$$

and when:

$$|Gn| > 1 \quad (10)$$

Eqs. (9) or (10) are thus our working definition of large-amplitude oscillatory shear flow [1,8,9], and with recent advances in rheometry, conducting experiments satisfying Eqs. (9) or (10) ~~are~~ ^{is} now commonplace for exploring the physics of complex liquids [10]. Eqs. (9) or (10) are refinement on the previous definition of large-amplitude (see Eq. (8) of [6]). Many notations have been introduced for analyzing the higher harmonics in the shear stress (see Section 9 of [6]) or in the normal stress differences (see Section 10 of [6]). In this paper, we follow Dealy *et al.* (1973) in plotting loops of shear stress *versus* shear rate, since these best bring out material nonlinearities as distortions from ellipticity [11,12,13].

Table I classifies (chronologically according to first findings) the constitutive equations that have lead to analytical solutions to the stress responses in large-amplitude oscillatory shear flow. From second column from the last, we learn that we have just one exact solution for large-amplitude oscillatory shear flow, and this, for the corotational Maxwell model [9]. With some difficulty, this solution was arrived at using the method of Kovacic [14] and variation of variable [15]. From Table I, we glean that the literature contains one and only one exact solution for the shear stress response of a complex fluid in LAOS (see third row from the bottom of Table I). A method has also been proposed for arrive at approximate analytical solutions to K-BKZ integral models in LAOS (see Appendix B of [16] or Subsection 2.2.5 of Chapter 11 of [2]).

In this paper, we undertake the more ambitious project of arriving at analytical solution for large-amplitude oscillatory shear flow for an entire framework of constitutive equations. Specifically, we choose the Oldroyd 8-constant framework. We choose this framework for its rich diversity of important special cases which we will discuss in the Subhead of Subsection I.b.

a. Oscillatory Shear Flow

In the absence of fluid inertia [17], for isothermal [17,18,19] oscillatory simple shear flow, the rate of deformation tensor is given by:

$$\dot{\gamma} = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & \dot{\gamma}^0 \cos \omega t & 0 \\ \dot{\gamma}^0 \cos \omega t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (11)$$

and for *any* simple shear flow (see Eq. (19) of [20]):

$$\frac{1}{2}\{\boldsymbol{\omega} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}\} = \frac{1}{2} \begin{bmatrix} -2\tau_{yx} & \tau_{xx} - \tau_{yy} & 0 \\ \tau_{xx} - \tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma} \quad (12)$$

and:

$$\frac{1}{2}\{\boldsymbol{\omega} \cdot \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\omega}\} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}^2 \quad (13)$$

which we will use in Subsection VIII.a below.

b. Oldroyd 8-Constant Framework

The Oldroyd 8-constant framework is given by (see Eq. (8.1-2) in [21]; [22]):

$$\begin{aligned} \boldsymbol{\tau} + \lambda_1 \frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} + \frac{1}{2}\mu_0(\text{tr}\boldsymbol{\tau})\dot{\boldsymbol{\gamma}} - \frac{1}{2}\mu_1\{\boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\tau}\} + \frac{1}{2}\nu_1(\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \\ = -\eta_0 \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \frac{\mathcal{D}\dot{\boldsymbol{\gamma}}}{\mathcal{D}t} - \mu_2\{\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}\} + \frac{1}{2}\nu_2(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \right) \end{aligned} \quad (14)$$

Eq. (14) reduces exactly to a rich diversity of special cases. We tabulate 14 of these in Table IV, first in descending order of the number of material constants (column 2), and next according to relevance (column 3). By *relevance*, we mean the number of shear stress harmonics predicted. From columns 2 and 3, we find the corotational Maxwell to be the simplest relevant special case of the Oldroyd 8-constant framework. Deepening our understanding of the Oldroyd 8-constant framework thus deepens our understanding of every one of the important special cases in Table IV. Moreover, an exact solution for the Oldroyd 8-constant framework would thus, of course, provide an exact solution for each of the important special cases in Table IV.

The Oldroyd 8-constant framework has also been closely connected, albeit approximately, with macromolecular theory ([23,24,25]; see Table 1 of [26]; see Eqs. (32) of [27],[28]; see Tables 6.2-1 and 6.2-2 of [29]; Problems 11B.9 and 11B.10 of [30]; §IV and §V. of [31]; §9.5 of [21]). For instance, for rigid dumbbells suspended in a Newtonian solvent:

$$\begin{aligned} \eta_p = \bar{n}kT\lambda, \quad \lambda_1 = \lambda, \quad \lambda_2 = 2\lambda/5, \quad \mu_0 = -2\lambda/7, \\ \mu_1 = -\lambda/7, \quad \mu_2 = -26\lambda/35, \quad \nu_1 = \nu_2 = 0 \end{aligned} \quad (15)$$

where λ is the characteristic time (Table 1 of [26]). With Eqs. (15), the Oldroyd 8-constant framework thus provides a useful approximation to the polymer contribution to the stresses for a suspension of rigid dumbbells [26]. Further, for finitely extensible nonlinear elastic (FENE) dumbbells:

$$\begin{aligned}
\eta_p &= \frac{b\bar{n}kT\lambda_H}{b+5}, & \lambda_1 &= \frac{b(2b+11)\lambda_H}{(2b+7)(b+9)}, & \lambda_2 &= \frac{-14b\lambda_H}{(2b+7)(b+7)(b+9)}, \\
\mu_0 &= \frac{-6b\lambda_H}{(2b+7)(b+5)(b+9)}, & \mu_1 &= \frac{b(2b+3)\lambda_H}{(2b+7)(b+9)}, & \mu_2 &= \frac{-2b(4b+21)\lambda_H}{(2b+7)(b+7)(b+9)}, \\
\nu_1 &= \nu_2 = 0
\end{aligned} \tag{16}$$

where λ_H is the characteristic time (Table 1 of [26]). With Eqs. (16), the Oldroyd 8-constant framework thus provides a useful approximation to the polymer contribution to the stresses for FENE dumbbells [26].

The corotational Jeffreys model ($\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) is classified in row seven of Table IV, and we will use this special case below as a worked example to demonstrate the usefulness of our results (see Section V). The dashed-underlined terms in Eq. (14) represent *higher order nonlinear* contributions. In Eq. (14), the total stress tensor is given by:

$$\boldsymbol{\pi} \equiv \boldsymbol{\tau} + p\boldsymbol{\delta} \tag{17}$$

and the rate-of-deformation tensor:

$$\dot{\boldsymbol{\gamma}} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \tag{18}$$

and the vorticity tensor is defined by:

$$\boldsymbol{\omega} \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \tag{19}$$

The corotational derivative for stress tensor is given by:

$$\frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} \equiv \frac{D\boldsymbol{\tau}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}\} \tag{20}$$

and, for shear rate tensor, by:

$$\frac{\mathcal{D}\dot{\boldsymbol{\gamma}}}{\mathcal{D}t} \equiv \frac{D\dot{\boldsymbol{\gamma}}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\omega}\} \tag{21}$$

In Eq. (17), $\boldsymbol{\delta}$ is the *kronecker delta*. The Oldroyd 8-constant model can be remarkably useful in polymer processing when it leads to analytical solutions, as it does for wire coating (Case III in [32]), and to a corrugated wire coated through a corrugated die (see Section 3. of [33]; Section 2. of [34]), and in plastic pipe extrusion for elliptical pipe ([35,36]), and also to extrusion from an eccentric annular pipe die (Case II in [32],[37],[38;39]).

Eq. (14) can be written in integral form (see EXAMPLE 8.1-2 of [21]), or rewritten in terms of the upper convected derivative (see Eq. 7.3-2 of [30]). The footnote on page 354 of [30] relates the parameters of Eq. 7.3-2 of [30] to those of our Eq. (14).

We close this subhead by noting that Eq. (14) has also been approximately connected to the Rivlin-Ericksen, Spriggs 6-constant and Giesekus fluids (see Appendix of [40]).

i. Steady Shear

For steady shear flow, the viscometric functions for the Oldroyd 8-constant framework are given by (see Eqs. (12)–(14) of [22]; Eqs. (14), (22) and (23) of [41]):

$$\frac{\eta(\dot{\gamma})}{\eta_0} = -\frac{\Psi_1(\dot{\gamma})}{\Psi_{10}} \left(\frac{\lambda_2}{\lambda_1} - 1 \right) + \frac{\lambda_2}{\lambda_1} = \frac{\Psi_2(\dot{\gamma})}{\Psi_{20}} \left(\frac{\lambda_1 - \lambda_2 - \mu_1 + \mu_2}{\lambda_1 - \mu_1} \right) + \frac{\lambda_2 - \mu_2}{\lambda_1 - \mu_1} = \frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \quad (22)$$

where:

$$\Psi_{10} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_1 = 2\eta_0(\lambda_1 - \lambda_2) \quad (23)$$

$$\Psi_{20} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_2 = -\eta_0(\lambda_1 - \lambda_2 - \mu_1 + \mu_2) \quad (24)$$

and where:

$$\sigma_1 \equiv \lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2}v_1) - \mu_1(\mu_1 - v_1) \quad (25)$$

$$\sigma_2 \equiv \lambda_1\lambda_2 + \mu_0(\mu_2 - \frac{3}{2}v_2) - \mu_1(\mu_2 - v_2) \quad (26)$$

As $Wi \rightarrow \infty$, Eq. (22) becomes:

$$\frac{\eta_\infty}{\eta_0} = \frac{\sigma_2}{\sigma_1} \quad (27)$$

which for the corotational Jeffreys fluid:

$$\frac{\eta_\infty}{\eta_0} = \frac{\lambda_2}{\lambda_1} \quad (28)$$

We will learn the effect of η_∞ later on in this paper through the corotational Jeffreys fluid.

Rewriting Eq. (22) gives us the dimensionless shear stress:

$$\tilde{\tau}_{yx} \equiv \frac{\sqrt{\sigma_1} \tau_{yx}}{\eta_0} = \frac{1 + \sigma(\sqrt{\sigma_1} \dot{\gamma})^2}{1 + (\sqrt{\sigma_1} \dot{\gamma})^2} \sqrt{\sigma_1} \dot{\gamma} \quad (29)$$

where:

$$\sigma \equiv \frac{\sigma_2}{\sigma_1} = \frac{\lambda_1\lambda_2 + \mu_0(\mu_2 - \frac{3}{2}v_2) - \mu_1(\mu_2 - v_2)}{\lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2}v_1) - \mu_1(\mu_1 - v_1)} \quad (30)$$

from which we see that for the shear stress to increase monotonically with shear rate, we need:

$$\sigma \geq 1/9 \quad (31)$$

Figure 3 illustrates this.

The Oldroyd 8-constant framework [Eq. (22)] allows for shear thinning ($\sigma < 1$), shear thickening ($\sigma > 1$), and the behaviors of constant viscosity fluids ($\sigma = 1$). In plastic pipe extrusion, we find examples of shear thinning (polyolefins) [42], shear thickening (short glass fiber filled polyolefins) [43], and

constant viscosity fluids (tubing from condensation polymers such as nylon). Using the Oldroyd 8-constant framework thus allows us to deepen our understanding of this wide variety of time steady flow problems.

ii. *Small-Amplitude Oscillatory Shear (SAOS)*

For SAOS, for *all special cases* of the Oldroyd 8-constant framework, the real part of the complex viscosity is given by (Eq. (8.1-10) of [21]):

$$\eta' = \eta_0 \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2} \quad (32)$$

and minus the imaginary part (Eq. (8.1-11) of [21]):

$$\eta'' = \eta_0 \frac{(\lambda_1 - \lambda_2) \omega}{1 + (\lambda_1 \omega)^2} \quad (33)$$

These can be nondimensionalized to:

$$\frac{\eta'}{\eta_0} = \frac{1 + (\lambda_2/\lambda_1) \text{De}^2}{1 + \text{De}^2} \quad (34)$$

$$\frac{\eta''}{\eta_0} = \frac{(1 - \lambda_2/\lambda_1) \text{De}}{1 + \text{De}^2} \quad (35)$$

so that (see Eq. (7.3-29) of [21]):

$$\begin{aligned} \mathbb{S} \equiv \frac{\tau_{yx}}{\eta_0 \dot{\gamma}^0} &= - \left(\frac{\eta'}{\eta_0} \cos \tau + \frac{\eta''}{\eta_0} \sin \tau \right) \\ &= - \left(\frac{1 + (\lambda_2/\lambda_1) \text{De}^2}{1 + \text{De}^2} \cos \tau + \frac{(1 - \lambda_2/\lambda_1) \text{De}}{1 + \text{De}^2} \sin \tau \right) \end{aligned} \quad (36)$$

which we will use as a consistency check on our new exact solution to the Oldroyd 8-constant framework for LAOS in Subsection IV.

II. METHOD

For oscillatory shear flow, Eq. (14) gives (see Appendix VIII.a):

$$\begin{aligned} \frac{d}{dt} \tau_{yx} &= -\frac{1}{2} \left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1} \right) N_1 \dot{\gamma}^0 \cos \omega t + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1} \right) N_2 \dot{\gamma}^0 \cos \omega t \\ &\quad - \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1} \right) \tau_{zz} \dot{\gamma}^0 \cos \omega t - \frac{1}{\lambda_1} \tau_{yx} - \eta_0 \left(\frac{1}{\lambda_1} \dot{\gamma}^0 \cos \omega t - \frac{\lambda_2}{\lambda_1} \dot{\gamma}^0 \omega \sin \omega t \right) \end{aligned} \quad (37)$$

$$\frac{d}{dt} N_1 = -\frac{1}{\lambda_1} N_1 + 2 \tau_{yx} \dot{\gamma}^0 \cos \omega t + 2 \frac{\lambda_2}{\lambda_1} \eta_0 (\dot{\gamma}^0)^2 \cos^2 \omega t \quad (38)$$

$$\frac{d}{dt} N_2 = -\frac{1}{\lambda_1} N_2 - \left(1 - \frac{\mu_1}{\lambda_1} \right) \tau_{yx} \dot{\gamma}^0 \cos \omega t - \eta_0 \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) (\dot{\gamma}^0)^2 \cos^2 \omega t \quad (39)$$

$$\frac{d}{dt}\tau_{zz} = -\frac{1}{\lambda_1}\tau_{zz} - \frac{\nu_1}{\lambda_1}\tau_{yx}\dot{\gamma}^0 \cos \omega t - \eta_0 \frac{\nu_2}{\lambda_1}(\dot{\gamma}^0)^2 \cos^2 \omega t \quad (40)$$

which can be non-dimensionalized to:

$$\begin{aligned} \frac{d\mathbb{S}}{d\tau} = & -\frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\mathbb{N}_1 \cos \tau + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\mathbb{N}_2 \cos \tau \\ & - \left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\tilde{\tau}_{zz} \cos \tau - \frac{1}{\text{De}}\mathbb{S} - \left(\frac{1}{\text{De}}\cos \tau - \frac{\lambda_2}{\lambda_1}\sin \tau\right) \end{aligned} \quad (41)$$

$$\frac{d\mathbb{N}_1}{d\tau} = -\frac{1}{\text{De}}\mathbb{N}_1 + 2\frac{\text{Wi}}{\text{De}}\mathbb{S} \cos \tau + 2\frac{\lambda_2}{\lambda_1}\frac{\text{Wi}}{\text{De}}\cos^2 \tau \quad (42)$$

$$\frac{d\mathbb{N}_2}{d\tau} = -\frac{1}{\text{De}}\mathbb{N}_2 - \frac{\text{Wi}}{\text{De}}\left(1 - \frac{\mu_1}{\lambda_1}\right)\mathbb{S} \cos \tau - \frac{\text{Wi}}{\text{De}}\left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1}\right)\cos^2 \tau \quad (43)$$

$$\frac{d\tilde{\tau}_{zz}}{d\tau} = -\frac{1}{\text{De}}\tilde{\tau}_{zz} - \frac{\nu_1}{\lambda_1}\frac{\text{Wi}}{\text{De}}\mathbb{S} \cos \tau - \frac{\nu_2}{\lambda_1}\frac{\text{Wi}}{\text{De}}\cos^2 \tau \quad (44)$$

In this paper, we will solve Eqs. (41)–(44) for $\mathbb{S}$, $\mathbb{N}_1$, $\mathbb{N}_2$ and $\tilde{\tau}_{zz}$ simultaneously and exactly.

The shear stress response has the form of:

$$\mathbb{S} = \mathbb{S}_h + \mathbb{S}_p \quad (45)$$

where $\mathbb{S}_h$ and $\mathbb{S}_p$ are the transient and particular parts of shear stress response.

We will elaborate $\mathbb{S}_h$ in Subsection II.a, and $\mathbb{S}_p$ in Subsection II.b.

a. Transient Parts of the Stress Responses

As intermediate results for the particular parts of the stress responses in Subsection II.b, we need the transient responses of $\mathbb{S}_h$, $\mathbb{N}_{1,h}$, $\mathbb{N}_{2,h}$ and $\tilde{\tau}_{zz,h}$. Specifically, we will use these to craft the fundamental matrix, Φ , for calculating $\mathbb{S}_p$.

We first consider the *homogeneous* system by setting the last term in Eqs. (41)–(44) to zero:

$$\frac{d\mathbb{N}_{1,h}}{d\tau} = -\frac{1}{\text{De}}\mathbb{N}_{1,h} + 2\frac{\text{Wi}}{\text{De}}\mathbb{S}_h \cos \tau \quad (46)$$

$$\frac{d\mathbb{N}_{2,h}}{d\tau} = -\frac{1}{\text{De}}\mathbb{N}_{2,h} - \left(1 - \frac{\mu_1}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\mathbb{S}_h \cos \tau \quad (47)$$

$$\frac{d\tilde{\tau}_{zz,h}}{d\tau} = -\frac{1}{\text{De}}\tilde{\tau}_{zz,h} - \frac{\nu_1}{\lambda_1}\frac{\text{Wi}}{\text{De}}\mathbb{S}_h \cos \tau \quad (48)$$

$$\begin{aligned} \frac{d\mathbb{S}_h}{d\tau} = & -\frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\mathbb{N}_{1,h}\cos\tau + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\mathbb{N}_{2,h}\cos\tau \\ & - \left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{\text{Wi}}{\text{De}}\tilde{\tau}_{zz,h}\cos\tau - \frac{1}{\text{De}}\mathbb{S}_h \end{aligned} \quad (49)$$

Solving Eqs. (46)–(49) simultaneously gives (see Appendix VIII.b):

$$\mathbb{S}_h(\tau) = C_1 e^{-\tau/\text{De}} \mathbb{S} + C_2 e^{-\tau/\text{De}} \mathbb{C} \quad (50)$$

$$\mathbb{N}_{1,h}(\tau) = \frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} e^{-\tau/\text{De}} [-C_1 \mathbb{C} + C_2 \mathbb{S}] + C_3 e^{-\tau/\text{De}} \quad (51)$$

$$\mathbb{N}_{2,h}(\tau) = \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} e^{-\tau/\text{De}} [C_1 \mathbb{C} - C_2 \mathbb{S}] + C_3 e^{-\tau/\text{De}} \quad (52)$$

$$\tilde{\tau}_{zz,h}(\tau) = \frac{1}{\alpha} \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} e^{-\tau/\text{De}} [C_1 \mathbb{C} - C_2 \mathbb{S}] - 2 \frac{\mu_0 - \mu_1}{3\mu_0 - 2\mu_1} C_3 e^{-\tau/\text{De}} - \frac{\lambda_1 + \mu_0 - \mu_1}{3\mu_0 - 2\mu_1} C_4 e^{-\tau/\text{De}} \quad (53)$$

where:

$$\mathbb{S} \equiv \sin(\alpha \sin \tau) \quad (54)$$

$$\mathbb{C} \equiv \cos(\alpha \sin \tau) \quad (55)$$

$$\alpha \equiv \frac{\text{Wi}}{\text{De}} \tilde{\lambda} \equiv \frac{\text{Wi}}{\text{De}} \sqrt{1 + \frac{\mu_0 \mu_1}{\lambda_1^2} - \frac{\mu_1^2}{\lambda_1^2} - \frac{3}{2} \frac{\mu_0 \nu_1}{\lambda_1^2} + \frac{\mu_1 \nu_1}{\lambda_1^2}} \quad (56)$$

Rewriting Eqs. (50)–(53) in matrix form gives:

$$\begin{bmatrix} \mathbb{S}_h(\tau) \\ \mathbb{N}_{1,h}(\tau) \\ \mathbb{N}_{2,h}(\tau) \\ \tilde{\tau}_{zz,h}(\tau) \end{bmatrix} = e^{-\frac{\tau}{\text{De}}} \begin{bmatrix} \mathbb{S} & \mathbb{C} & 0 & 0 \\ -\frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} \mathbb{C} & \frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} \mathbb{S} & 0 & 1 \\ \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} \mathbb{C} & -\frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} \mathbb{S} & 1 & 0 \\ \frac{1}{\alpha} \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} \mathbb{C} & -\frac{1}{\alpha} \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} \mathbb{S} & -2 \frac{\mu_0 - \mu_1}{3\mu_0 - 2\mu_1} & -\frac{\lambda_1 + \mu_0 - \mu_1}{3\mu_0 - 2\mu_1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \quad (57)$$

from which the fundamental matrix:

$$\Phi = e^{-\frac{\tau}{\text{De}}} \begin{bmatrix} \mathbb{S} & \mathbb{C} & 0 & 0 \\ -\frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} \mathbb{C} & \frac{2}{\alpha} \frac{\text{Wi}}{\text{De}} \mathbb{S} & 0 & 1 \\ \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} \mathbb{C} & -\frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1}\right) \frac{\text{Wi}}{\text{De}} \mathbb{S} & 1 & 0 \\ \frac{1}{\alpha} \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} \mathbb{C} & -\frac{1}{\alpha} \frac{\nu_1}{\lambda_1} \frac{\text{Wi}}{\text{De}} \mathbb{S} & -2 \frac{\mu_0 - \mu_1}{3\mu_0 - 2\mu_1} & -\frac{\lambda_1 + \mu_0 - \mu_1}{3\mu_0 - 2\mu_1} \end{bmatrix} \quad (58)$$

can be extracted. We will use this fundamental matrix to calculate the alternance responses, the *particular* part of our solutions, Eqs. (41)–(44), in the Subsection II.b.

b. Particular Part of the Stress Responses

The particular part of our stress response is given by (see Eq. (10) on p.711 of [15]):

$$\begin{bmatrix} \mathbb{S}_p(\tau) \\ \mathbb{N}_{1,p}(\tau) \\ \mathbb{N}_{2,p}(\tau) \\ \tilde{\tau}_{zz,p}(\tau) \end{bmatrix} = \Phi(\tau) \int_0^\tau \Phi^{-1}(\tau') \begin{bmatrix} 2 \frac{\lambda_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau \\ -\frac{\text{Wi}}{\text{De}} \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) \cos^2 \tau \\ -\frac{\nu_2}{\lambda_1} \frac{\text{Wi}}{\text{De}} \cos^2 \tau \\ -\left(\frac{1}{\text{De}} \cos \tau - \frac{\lambda_2}{\lambda_1} \sin \tau \right) \end{bmatrix} \quad (59)$$

where Φ is defined in Eq. (58). In this paper, we confine our attention to the exact evaluation of the shear stress, $\mathbb{S}_p(\tau)$, in Eq. (59), leaving the normal stress differences, $\mathbb{N}_{1,p}(\tau)$ and $\mathbb{N}_{2,p}(\tau)$, for another day.

III. RESULTS: SHEAR STRESS

This paper focuses on the alternant shear stress. By *alternant*, we mean after the transient due to startup of the oscillation has vanished.

a. Exact Solution

To get the alternant part of the shear stress response, where $\mathbb{S}_h = 0$ and $\mathbb{S} = \mathbb{S}_p$, we integrate the $\mathbb{S}_p(\tau)$ component of Eq. (59) to get:

$$\mathbb{S} = \frac{e^{-\tau/\text{De}}}{2\alpha \text{De}^2} \left[\sin(\alpha \sin \tau) I_{\mathbb{S}1} - \cos(\alpha \sin \tau) I_{\mathbb{S}2} \right] \quad (60)$$

where:

$$\begin{aligned} I_{\mathbb{S}1} = & \left(-2 \frac{\lambda_2}{\lambda_1} + 3 \frac{\mu_0 \nu_2}{\lambda_1^2} - 2 \frac{\mu_0 \mu_2}{\lambda_1^2} - 2 \frac{\mu_1 \nu_2}{\lambda_1^2} + 2 \frac{\mu_1 \mu_2}{\lambda_1^2} \right) \text{Wi} I_{\mathbb{S}1,1} \\ & - 2 I_{\mathbb{S}1,2} + 2 \frac{\lambda_2}{\lambda_1} \text{De} I_{\mathbb{S}1,3} + 2 \frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\mathbb{S}1,4} \end{aligned} \quad (61)$$

$$\begin{aligned} I_{\mathbb{S}2} = & \left(-2 \frac{\lambda_2}{\lambda_1} + 3 \frac{\mu_0 \nu_2}{\lambda_1^2} - 2 \frac{\mu_0 \mu_2}{\lambda_1^2} - 2 \frac{\mu_1 \nu_2}{\lambda_1^2} + 2 \frac{\mu_1 \mu_2}{\lambda_1^2} \right) \text{Wi} I_{\mathbb{S}2,1} \\ & - 2 I_{\mathbb{S}2,2} + 2 \frac{\lambda_2}{\lambda_1} \text{De} I_{\mathbb{S}2,3} + 2 \frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\mathbb{S}2,4} \end{aligned} \quad (62)$$

Evaluating the four integrals in Eq. (61) gives:

$$I_{S1,1} = \frac{J_0 \text{De}}{8\text{De}^2 + 2} \left[e^{\tau/\text{De}} (4\text{De}^2 + 2\text{De} \sin 2\tau + \cos 2\tau + 1) \right] \\ + \frac{\text{De}}{2} e^{\tau/\text{De}} \sum_{k=1}^{\infty} J_{2k} \left[\frac{2(2\text{De}k \sin 2k\tau + \cos 2k\tau)}{1 + 4\text{De}^2 k^2} + \frac{2\text{De}(k-1) \sin 2(k-1)\tau + \cos 2(k-1)\tau}{1 + 4\text{De}^2 (k-1)^2} \right. \\ \left. + \frac{2\text{De}(k+1) \sin 2(k+1)\tau + \cos 2(k+1)\tau}{1 + 4\text{De}^2 (k+1)^2} \right] \quad (63)$$

$$I_{S1,2} = J_0 \text{De} e^{\tau/\text{De}} + \sum_{k=1}^{\infty} \frac{2\text{De} J_{2k} e^{\tau/\text{De}}}{4\text{De}^2 k^2 + 1} (\cos 2k\tau + 2k \text{De} \sin 2k\tau) \quad (64)$$

$$I_{S1,3} = i\text{De} J_0 \left[-e^{\tau/\text{De}} + 2e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{2i\tau} \right) \right] \\ + 2\text{De} \sum_{k=1}^{\infty} J_{2k} \left[\frac{e^{\tau/\text{De}}}{1 + 4k^2 \text{De}^2} (-\sin 2k\tau + 2k \text{De} \cos 2k\tau) + i(-1)^{1+k} e^{\tau/\text{De}} \right. \\ \left. - \sum_{l=1}^k \left[\frac{(-1)^{1-l+k} 2e^{\tau/\text{De}}}{1 + 4l^2 \text{De}^2} (\sin 2l\tau - 2l \text{De} \cos 2l\tau) \right] \right. \\ \left. + 2i(-1)^k e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{2i\tau} \right) \right] \quad (65)$$

$$I_{S1,4} = 2iJ_0 \left[\frac{e^{\tau/\text{De}}}{e^{i2\tau} + 1} - e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{i2\tau} \right) \right] \\ + 2 \sum_{k=1}^{\infty} J_{2k} \left[(-1)^k 2k \text{De} e^{\tau/\text{De}} + (-1)^k i \left(\frac{2e^{\tau/\text{De}}}{e^{i2\tau} + 1} \right) \right. \\ \left. + 4e^{\tau/\text{De}} \text{De} \sum_{m=1}^k \frac{(-1)^{k-m} (k-m+1)}{1 + 4(m-1)^2 \text{De}^2} \left[\cos 2(m-1)\tau \right. \right. \\ \left. \left. + 2(m-1) \text{De} \sin 2(m-1)\tau \right] \right. \\ \left. - 2i(-1)^k e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{i2\tau} \right) \right] \quad (66)$$

For brevity, we just write J_m to mean $J_m(\alpha)$ where α is given by Eq. (56) (see after Eq. (5) of [44] or before Eq. (123) of [9]). The [2,1] hypergeometric function in Eqs. (65) and (66) is given by:

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\beta_1)_k} \frac{z^k}{k!} \quad (67)$$

where the hypergeometric function of argument z is given by (defined after Eq. (2) of [45]):

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \equiv \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_q)_k}{(\beta_1)_k \dots (\beta_s)_k} \frac{z^k}{k!} \quad (68)$$

The $[2,1]$ hypergeometric function, ${}_2F_1(\alpha_1, \alpha_2; \beta_1; z)$, drops out when λ_2 vanishes, as for the corotational Maxwell model, for instance [9]. The Pochhammer symbol in Eqs. (67) and (68) is defined as (see Eq. 6.1.22 of [46]):

$$(\xi)_k \equiv \xi(\xi+1)\dots(\xi+k-1) = \Gamma(\xi+k)/\Gamma(\xi) \quad (69)$$

Evaluating the four integrals in Eq. (62) gives:

$$I_{\mathbb{S}2,1} = 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\begin{aligned} & \frac{-(1+2k)\text{De}^2 e^{\tau/\text{De}} \cos(1+2k)\tau}{4(1+\text{De}^2(1+2k)^2)} + \frac{(3-2k)\text{De}^2 e^{\tau/\text{De}} \cos(3-2k)\tau}{4(1+\text{De}^2(3-2k)^2)} \\ & + \frac{\text{De}^2(1-2k)e^{\tau/\text{De}} \cos(1-2k)\tau}{2(1+\text{De}^2(1-2k)^2)} \\ & + \text{De}e^{\tau/\text{De}} \left[\frac{\sin(1+2k)\tau}{4(1+\text{De}^2(1+2k)^2)} - \frac{\sin(1-2k)\tau}{2(1+\text{De}^2(1-2k)^2)} - \frac{\sin(3-2k)\tau}{4(1+\text{De}^2(3-2k)^2)} \right] \end{aligned} \right] \quad (70)$$

$$I_{\mathbb{S}2,2} = 2\text{De} \sum_{k=1}^{\infty} J_{2k-1} \frac{e^{\tau/\text{De}} [(1-2k)\text{De} \cos(1-2k)\tau - \sin(1-2k)\tau]}{1+\text{De}^2(1-2k)^2} \quad (71)$$

$$I_{\mathbb{S}2,3} = 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\begin{aligned} & \frac{\text{De}e^{\tau/\text{De}} [\cos(2k-1)\tau + (2k-1)\text{De} \sin(2k-1)\tau]}{1+(2k-1)^2 \text{De}^2} \\ & + 2\text{De}e^{\tau/\text{De}} \sum_{n=1}^k \frac{(-1)^{k-n+1} [\cos(2n-1)\tau + (2n-1)\text{De} \sin(2n-1)\tau]}{1+(2n-1)^2 \text{De}^2} \\ & + \frac{(-1)^{k+1} 2\text{De}}{(1+i\text{De})} e^{\tau/\text{De}} e^{i\tau} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \end{aligned} \right] \quad (72)$$

$$I_{\mathbb{S}2,4} = 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\begin{aligned} & \sum_{m=1}^k \frac{(-1)^{k-m-1} 4\text{De}(k-m)e^{\tau/\text{De}} [\sin(2m-1)\tau]}{1+(2m-1)^2 \text{De}^2} \left[-(2m-1)\text{De} \cos(2m-1)\tau \right] \\ & + (-1)^{k+1} e^{\tau/\text{De}} \sec \tau - \frac{2i(-1)^{k+1}}{i-\text{De}} e^{(1/\text{De}+i)\tau} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \end{aligned} \right] \quad (73)$$

Eq. (60) [with Eqs. (61) and (62)] is the main result of this paper. We can report that our exact solution is both integrable and differentiable, and it thus provides a suitable starting point for exploring analytically and exactly many nonlinear problems in fluid physics, including especially the temperature rise [17,18,19]. Our exact solution takes neither of the usual forms of expansion supposed for approximate solutions, in even power of $\lambda\dot{\gamma}^0$, or of $\dot{\gamma}^0/\omega$ (see column 11 Table 1 of [9]). Instead, the exact solution takes the form of the difference between two trig functions of trig functions. We find this form to be intrinsically beautiful. Furthermore, we can distinguish the significance $\cos(\alpha \sin \tau)$ from that of $\sin(\alpha \sin \tau)$. For instance, in Subsection IV.a, we saw

that for steady shear flow $\sin(\alpha \sin \tau)$ drops out and that only $\cos(\alpha \sin \tau)$ contributes. Also, whereas only the $\cos(\alpha \sin \tau)$ term contributes to the rightmost and leftmost points on $\tau_{yx} - \dot{\gamma}$ loops [where $\tau = (0, \pi)$], only the $\sin(\alpha \sin \tau)$ term contributes to the ordinate intercepts [where $\tau = (\frac{1}{2}\pi, \frac{3}{2}\pi)$]. In other words, $\sin(\alpha \sin \tau)$ governs the purely elastic part of the response, and $\cos(\alpha \sin \tau)$, the purely viscous. Of course, for any other point on $\tau_{yx} - \dot{\gamma}$ loops, both $\sin(\alpha \sin \tau)$ and $\cos(\alpha \sin \tau)$ participate.

For the corotational Maxwell fluid, where $\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eqs. (60)–(62) reduces to:

$$\mathbb{S} = \frac{-e^{-\tau/\text{De}}}{\text{Wi De}} [SI_{s1,2} - CI_{s2,2}] \quad (74)$$

which we have yet to rewrite as Eq. (66) of [9]. However, Figure 6 shows that Eq. (74) and the exact solution for the corotational Maxwell model, Eq. (66) of [9], agree closely, as they should. In fact, comparing Eq. (74) with Eq. (66) of [9] for the loops calculated for in Figure 6, we find agreement to within at least 16 significant figures.

For the special case of the corotational Maxwell model, Eq. (60) has been rewritten as a Fourier series (see Eq. (66) with **Appendix: Fourier Analysis of Compact Forms** of [9]). We have yet to rewrite the main result of this paper, Eq. (60) with Eqs. (61) and (62), as a Fourier series.

b. $\tilde{\tau}_{zz}$ and $\tilde{\tau}_{zz}(0)$ Invariance

We require the predictions for any flow to be independent of the initial condition of any normal component of the extra stress tensor. In other words, in rheology, only differences in the normal components of the extra stress tensor matter. Thus, for any simple shear flow (for any constitutive equation), we require the predictions for $\tau_{yx}(t)$, $N_1(t)$ and $N_2(t)$ to be independent of the initial condition of $\tau_{zz}(0)$. For $\tau_{zz}(0)$ invariance, from Eqs. (37)–(40) [or (41)–(44)], we see that the Oldroyd 8-constant framework requires either that:

$$\nu_1 = \nu_2 = 0 \quad (75)$$

or:

$$\mu_0 = \frac{2}{3}\mu_1 \quad (76)$$

The former of these sufficient conditions defines the Oldroyd 6-constant model (see first row of Table IV). The results of this paper are thus restricted to either of these two and only two sufficient conditions. From Eqs. (75) and (76), we thus learn that the only way to include either or both of the higher order

nonlinear terms in the Oldroyd 8-constant model is by setting μ_0 equal to $\frac{2}{3}\mu_1$. In other words, if we imposed $\tau_{zz}(0)$ invariance, then the 8 constants in Oldroyd framework reduces to at most 7 constants.

The uninitiated may wish to set $\tilde{\tau}_{zz}$ to a constant. Though this is fine for steady shear flow, it is not fine for LAOS. In fact, letting $\tilde{\tau}_{zz}$ be constant in Eq. (44) gives:

$$\mathbb{S} = -\frac{v_2}{v_1} \cos \tau \quad (77)$$

which is not true [compare Eq. (77) with Eq. (60)]. We thus learn that if the flow is unsteady, $\tilde{\tau}_{zz}$ cannot be constant, unless $v_1 = v_2$.

IV. CONSISTENCY CHECKS

In Subsection III.a, we found that our main result [Eq. (60) with Eqs. (61) and (62)], and the exact solution for the corotational Maxwell model, Eq. (66) of [9], agree closely, as they should [see Eq. (74) and what follows]. In this section, we check our main result for consistency with well-known results (Subsections IV.a and IV.b), and with our new approximate solution for the corotational Jeffreys fluid (Subsection IV.c.i), and finally, with our finite difference solution for the corotational Jeffreys fluid (Subsection IV.c.ii).

a. SAOS

In the Introduction, we learnt that, for any case of the Oldroyd 8-constant framework, the fluid response in SAOS is given by Eqs. (32) and (33) (see Eqs. (8.1-10) and (8.1-11) of [21]). The following consistency check for SAOS thus applies for any case of the Oldroyd 8-constant framework, including for instance the 14 constitutive equations listed in Table IV.

In oscillatory shear flow, for small strain rate amplitude, where $Wi \ll 0$, the shear stress response for the Oldroyd 8-constant framework is given by Eqs. (32) and (33). We will next validate our main finding for any case with this limiting behaviour. Taking the limit as $Wi \rightarrow 0$ in Eq. (60) [with Eqs. (61) and (62)] gives (see derivation in Appendix VIII.c):

$$\begin{aligned}
\mathbb{S}_{p,ss} = & -\frac{\sin \tau}{De} + \frac{\lambda_2}{\lambda_1} i \sin \tau \left[-1 + {}_2F_1 \left(1, -\frac{i}{2De}; 1 - \frac{i}{2De}; -e^{2i\tau} \right) \right] \\
& + 2 \frac{\lambda_2}{\lambda_1} i \sin \tau \left[\frac{1}{e^{i2\tau} + 1} - {}_2F_1 \left(1, -\frac{i}{2De}; 1 - \frac{i}{2De}; -e^{i2\tau} \right) \right] - \frac{De \cos \tau - \sin \tau}{De(1 + De^2)} \\
& + \frac{\lambda_2}{\lambda_1} \left[\frac{\cos \tau + De \sin \tau}{1 + De^2} - \frac{2e^{i\tau}}{(1 + iDe)} {}_2F_1 \left(1, \frac{1}{2} - \frac{i}{2De}; \frac{3}{2} - \frac{i}{2De}; -e^{i2\tau} \right) \right] \\
& - \frac{\lambda_2}{\lambda_1} \left[\sec \tau - \frac{i2e^{i\tau}}{i - De} {}_2F_1 \left(1, \frac{1}{2} - \frac{i}{2De}; \frac{3}{2} - \frac{i}{2De}; -e^{i2\tau} \right) \right]
\end{aligned} \tag{78}$$

which we have yet to rewrite as Eqs. (32) and (33). Not surprisingly [see Eqs. (32) and (33)], we learn that only λ_1 and λ_2 affect the SAOS behavior.

Furthermore, for the corotational Maxwell fluid, where $\lambda_2 = 0$ and $\tilde{\lambda} = 1$, Eq. (78) reduces to:

$$\mathbb{S}_{p,ss} = - \left(\frac{\cos \tau + De \sin \tau}{1 + De^2} \right) \tag{79}$$

as it must (see Eq. (59) of [6]).

b. Steady Shear Flow

In steady shear flow, for the special case of the corotational Maxwell model, the viscosity function is given by (Eq. (84) of [6]):

$$\frac{\eta}{\eta_0} = \frac{1}{1 + (\lambda_1 \dot{\gamma})^2} \tag{80}$$

and thus, the shear stress, by:

$$\frac{\tau_{yx}}{\eta_0 \dot{\gamma}^0} = \frac{1}{1 + (\lambda_1 \dot{\gamma})^2} \frac{\lambda \dot{\gamma}}{Wi} \tag{81}$$

which we will next use to validate our main result, Eq. (60) [with Eqs. (61) and (62)]. We compare the limiting behaviour of our main result for steady shear flow, where $De \rightarrow 0$, with Eq. (81) in Figure 5. The close agreement in Figure 5, all within a pen width, confirms that our main result is consistent with the well-known result [Eqs. (80) and (81)] for steady shear flow of a corotational Maxwell fluid.

c. LAOS

Our main result, Eq. (60) [with Eqs. (61) and (62)], can also be checked for consistency in LAOS for the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. For the corotational Jeffreys fluid, our main result reduces to:

$$\mathbb{S} = \frac{-e^{-\tau/\text{De}}}{\text{Wi De}} \begin{bmatrix} \sin\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[\frac{\lambda_2}{\lambda_1} \text{Wi} I_{\mathbb{S}1,1} + I_{\mathbb{S}1,2} - \frac{\lambda_2}{\lambda_1} \text{De} I_{\mathbb{S}1,3} - \frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\mathbb{S}1,4} \right] \\ -\cos\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[\frac{\lambda_2}{\lambda_1} \text{Wi} I_{\mathbb{S}2,1} + I_{\mathbb{S}2,2} - \frac{\lambda_2}{\lambda_1} \text{De} I_{\mathbb{S}2,3} - \frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\mathbb{S}2,4} \right] \end{bmatrix} \quad (82)$$

where $I_{\mathbb{S}1,1} - I_{\mathbb{S}1,4}$ are given by Eqs. (63)–(66), and $I_{\mathbb{S}2,1} - I_{\mathbb{S}2,4}$, by Eqs. (70)–(73).

i. Goddard Integral Approximation

To validate our main result, we will need an approximate solution for the corotational Jeffreys model. Specifically, to get this approximation, we use the well-known Goddard integral expansion [6] employing the method of **Section 8.** of [6].

We begin with:

$$\mathbb{S} = \sum_{m=1}^{\infty} \left(1 - \frac{\lambda_2}{\lambda_1} \right) \mathbb{S}[ma] + \mathbb{S}[mb] \quad (83)$$

where $\mathbb{S}[ma]$ is the m th term of the *corotational Maxwell contribution* to $\mathbb{S}$, and where the *Dirac delta function contribution* is given by:

$$\mathbb{S}[mb] = \begin{cases} -\frac{\lambda_2}{\lambda_1} \cos \tau; & m = 1 \\ 0; & m = 2, 3, 4, \dots \end{cases} \quad (84)$$

To get $\mathbb{S}[ma]$, we follow the method of Section 3. of [6] to extend Eq. (58) of [6] to the next and sixth order of Wi :

$$\begin{aligned}
\sum_{m=1}^4 \mathbb{S}[ma] = & - \left(\frac{\cos \tau + \text{Desin } \tau}{1 + \text{De}^2} \right) \\
& + \frac{\text{Wi}^2}{4} \left[\frac{3\cos \tau + 6\text{Desin } \tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} + \frac{(1 - 11\text{De}^2)\cos 3\tau + 6\text{De}(1 - \text{De}^2)\sin 3\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)} \right] \\
& - \frac{\text{Wi}^4}{8} \left[\frac{5\cos \tau + 15\text{Desin } \tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)} \right. \\
& + \frac{(5 - 130\text{De}^2)\cos 3\tau + (45 - 120\text{De}^2)\text{Desin } 3\tau}{2(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)} \\
& + \frac{(1 - 85\text{De}^2 + 274\text{De}^4)\cos 5\tau + (15 - 225\text{De}^2 + 120\text{De}^4)\text{Desin } 5\tau}{2(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)} \left. \right] \\
& + \frac{\text{Wi}^6}{16} \left[\frac{35\cos \tau + 140\text{Desin } \tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)} \right. \\
& + \frac{21(1 - 47\text{De}^2)\cos 3\tau + 252(1 - 5\text{De}^2)\text{Desin } 3\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)} \\
& + \frac{7(1 - 155\text{De}^2 + 1044\text{De}^4)\cos 5\tau + 140(1 - 29\text{De}^2 + 36\text{De}^4)\sin 5\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)(1 + 36\text{De}^2)} \\
& + \frac{(1 - 322\text{De}^2 + 6769\text{De}^4 - 13068\text{De}^6)\cos 7\tau + 28(1 - 70\text{De}^2 + 469\text{De}^4 - 180\text{De}^6)\text{Desin } 7\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)(1 + 36\text{De}^2)(1 + 49\text{De}^2)} \left. \right]
\end{aligned} \tag{85}$$

which is consistent with the expressions for $S[1a]$ and $S[2a]$ in Eqs. (125) and (128) of [6]. Substituting Eq. (84) and (85) into Eq. (83) gives the Goddard integral approximation to Eq. (83):

$$\begin{aligned}
\mathbb{S} = & - \left(1 - \frac{\lambda_2}{\lambda_1} \right) \left(\frac{\cos \tau + \text{Desin } \tau}{1 + \text{De}^2} \right) - \frac{\lambda_2}{\lambda_1} \cos \tau \\
& + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \left[\frac{\text{Wi}^2}{4} \left[\frac{3\cos \tau + 6\text{Desin } \tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)} + \frac{(1 - 11\text{De}^2)\cos 3\tau + 6\text{De}(1 - \text{De}^2)\sin 3\tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)} \right] \right. \\
& - \frac{\text{Wi}^4}{8} \left[\frac{5\cos \tau + 15\text{Desin } \tau}{(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)} \right. \\
& + \frac{(5 - 130\text{De}^2)\cos 3\tau + (45 - 120\text{De}^2)\text{Desin } 3\tau}{2(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)} \\
& + \frac{(1 - 85\text{De}^2 + 274\text{De}^4)\cos 5\tau + (15 - 225\text{De}^2 + 120\text{De}^4)\text{Desin } 5\tau}{2(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)} \left. \right] \\
& + \frac{\text{Wi}^6}{16} \left[\frac{35\cos \tau + 140\text{Desin } \tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)} \right. \\
& + \frac{21(1 - 47\text{De}^2)\cos 3\tau + 252(1 - 5\text{De}^2)\text{Desin } 3\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)} \\
& + \frac{7(1 - 155\text{De}^2 + 1044\text{De}^4)\cos 5\tau + 140(1 - 29\text{De}^2 + 36\text{De}^4)\sin 5\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)(1 + 36\text{De}^2)} \\
& + \frac{(1 - 322\text{De}^2 + 6769\text{De}^4 - 13068\text{De}^6)\cos 7\tau + 28(1 - 70\text{De}^2 + 469\text{De}^4 - 180\text{De}^6)\text{Desin } 7\tau}{4(1 + \text{De}^2)(1 + 4\text{De}^2)(1 + 9\text{De}^2)(1 + 16\text{De}^2)(1 + 25\text{De}^2)(1 + 36\text{De}^2)(1 + 49\text{De}^2)} \left. \right] \\
& + \dots
\end{aligned} \tag{86}$$

which we will next use to validate our main result. The accuracies of Eqs. (85) and (86) can improve with the use of Padé approximants [47], though we found no need for this here (see Figure 6 through Figure 9).

When the Wi^6 term is omitted, Eq. (86) reduces to Eq. (58) of [6] as it must. To our knowledge, Eq. (86) is new. Since our exact solution does not take the form of a Fourier series, Eq. (86) is thus, to our knowledge, the first expression for each of the shear stress harmonics for the corotational Jeffreys fluid up to the seventh. Figure 6 through Figure 9 show that Eq. (86) agrees closely, well within a pen width, with our new exact solution, Eq. (60) [with Eqs. (61) and (62)].

ii. Finite Difference Solution

Viscoelastic models in large-amplitude oscillatory shear flow can also be solved approximately numerically by the finite element method [48,49], or by numerical integration [50,51,52,53,54], or by finite difference [55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81]. Here, as a consistency check on Eq. (60), we use the Runge-Kutta 5(4) finite difference scheme [82] to solve Eqs. (41)–(43) for the shear stress subject to the initial conditions:

$$\mathbb{S}(0) = \mathbb{N}_1(0) = \mathbb{N}_2(0) = 0 \quad (87)$$

Our computation is, of course, independent of $\tilde{\tau}_{zz}(0)$, since for the corotational Jeffreys model, $v_1 = v_2 = 0$, or since because for the corotational Jeffreys model $\mu_0 = \frac{2}{3}\mu_1 = 0$. The conditions $v_1 = v_2 = 0$ or $\mu_0 = \frac{2}{3}\mu_1 = 0$ satisfy Eqs. (75) and (76), our conditions for $\tilde{\tau}_{zz}$ invariance.

For our finite difference solution to Eqs. (41)–(43) subject to Eq. (87), we coded into MATLAB (Version R2012b) on a MacBook Air (1.3 GHz Intel Core i5 processor with 4 GB 1600 MHz DDR3 memory) employing the OS X (Version 10.10.2) operating system using the *ode45* scheme. For each point in Figure 10 through Figure 12, we find such an evaluation to consume less than 1 second of CPU time. Figure 10 through Figure 12 show the consistency between our numerical and exact solutions, well within a pen width, for the Oldroyd 8-constant framework in large-amplitude oscillatory shear flow. Indeed, we find that our finite difference and exact solutions agree to at least within 16 significant figures.

V. WORKED EXAMPLE: η_∞ AND THE HIGHER HARMONICS

In this section, we illustrate the use of our exact solution for the Oldroyd 8-constant framework in LAOS [Eq. (60) with Eqs. (61) and (62)]. Specifically, we use our exact solution to explore the role of η_∞ on the higher harmonics of the shear stress. For this exploration, we choose the special case of the corotational

Jeffreys model, the simplest relevant model containing η_∞ . By *relevant*, we still mean that the model at least predicts higher shear stress harmonics. More specifically, for the corotational Jeffreys model, $\eta_\infty/\eta_0 = \lambda_2/\lambda_1$ [see Eq. (28)].

Figure 6 through Figure 9 employ Eq. (60) with Eqs. (61) and (62) to illustrate the role of η_∞ on the loop shapes. Comparing Figure 6 with successively Figure 7 through Figure 9, we learn that increasing η_∞ narrows the shear stress *versus* shear rate loops, and thus diminishes their areas. In other words, increasing η_∞ decreases fluid elasticity, and increases viscous dissipation [17,18, 19]. From Figure 6 through Figure 9, we see that increasing η_∞ does not produce self-intersection, at least not over the ranges $\frac{1}{10} < Wi < \frac{5}{4}$ and $0.1 < De < 10$ [83].

VI. CONCLUSION

In this work, we arrive at the first exact analytical solution [Eq. (60) with Eqs. (61) and (62)] to a framework of constitutive equations for LAOS. We chose the Oldroyd 8-constant framework for a rich of diversity of popular constitutive equations. To our knowledge, our exact analytical solution [Eq. (60) with Eqs. (61) and (62)] is the first exact solution to a framework of constitutive equation in LAOS. Our new exact analytical solution reduces to our previous result for the corotational Maxwell fluid (Eq. (62) with Eqs. (127) and (134) of [9]), as it should.

To our knowledge, our Subsection IV.c.i provides the first integral expansion to the corotational Jeffreys fluid [Eq. (86)], by following the method in Section 8. of [6], and this up to and including the seventh harmonic. We use this Eq. (86) to validate our main result [Eq. (60) with Eqs. (61) and (62)] for the special case of the corotational Jeffreys fluid in LAOS.

To arrive at Eq. (86), obtained Eq. (85), the integral expansion to the corotational Maxwell fluid. Eq. (85) extends Eq. (58) of [6] to the next term, that is, to the sixth order of Wi .

Some special cases of the Oldroyd 8-constant framework (corotational Maxwell or corotational Jeffreys), are used with multiple relaxation times, λ_1 . Extending the results of this work to multiple λ_1 might thus be a useful next step, and for this, we would begin with the Spriggs relations (see Eqs. (6.1-14) and (6.1-15) of [21]; see also Appendix of [40]). Other special cases of the Oldroyd 8-constant framework (Johnson-Segalman or Gordon-Schowalter) are often useful without extension to multiple λ_1 .

In this paper, we have limited the scope to just the alternant part of the shear stress in LAOS. Previously, for the exact solution to the corotational Maxwell model, we solved exactly for both the shear stress and the normal stress

differences, and these for both start-up and alternance, using the Kovacic method [9]. We, of course, expect start-up and the normal stress differences for the Oldroyd 8-constant framework to yield to the Kovacic method. However, for the Oldroyd 8-constant framework, we leave the exact solution for start-up and for the normal stress differences for another day.

In our previous work, we succeeded in rewriting our exact solutions for the corotational Maxwell fluid (Eq. (62) with Eqs. (127) and (134) of [9]) as a Fourier series (Eq. (66) of [9]). However, in this work, we have yet to rewrite our main result [Eq. (60) with Eqs. (61) and (62)] as a Fourier series. This too we shall leave for another day.

We have a strong preference for the approach that we have taken in this paper, which produces the exact solution for an entire framework, over the approach in Ref. [9], which produces an exact solution for just one constitutive equation. We prefer the framework approach because it at once generates exact solutions in LAOS to whole sets of constitutive equations, rather than generating solutions for one constitutive equation at a time.

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VIII. APPENDICES

a. Deriving Four Differential Equations in $\mathbb{S}$, $\mathbb{N}_1$, $\mathbb{N}_2$ and $\tilde{\tau}_{zz}$

In this Appendix, we will derive four the differential equation in $\mathbb{S}$, $\mathbb{N}_1$, $\mathbb{N}_2$ and $\tilde{\tau}_{zz}$ from Eq. (14). To prepare for this derivation, we first substitute Eqs. (12) and (13) into Eqs. (20) and (21) to get:

$$\frac{\mathcal{D}\boldsymbol{\tau}}{\mathcal{D}t} \equiv \frac{\partial}{\partial t} \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} + \frac{1}{2}\dot{\gamma} \begin{bmatrix} -2\tau_{yx} & \tau_{xx} - \tau_{yy} & 0 \\ \tau_{xx} - \tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (88)$$

and:

$$\frac{\mathcal{D}\dot{\boldsymbol{\gamma}}}{\mathcal{D}t} \equiv \frac{\partial}{\partial t} \dot{\boldsymbol{\gamma}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dot{\gamma}^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (89)$$

We next use Eq. (11) to rewrite the third term on the left of Eq. (14) as:

$$\frac{1}{2} \mu_0 (\text{tr } \boldsymbol{\tau}) \dot{\boldsymbol{\gamma}} = \frac{1}{2} \mu_0 (\tau_{xx} + \tau_{yy} + \tau_{zz}) \dot{\boldsymbol{\gamma}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (90)$$

and the fourth:

$$-\frac{1}{2} \mu_1 \{ \boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\tau} \} = -\frac{1}{2} \mu_1 \dot{\boldsymbol{\gamma}} \begin{bmatrix} 2\tau_{yx} & \tau_{xx} + \tau_{yy} & \tau_{yz} \\ \tau_{xx} + \tau_{yy} & 2\tau_{yx} & \tau_{xz} \\ \tau_{yz} & \tau_{zx} & 0 \end{bmatrix} \quad (91)$$

and the last:

$$\frac{1}{2} \nu_1 (\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}}) \boldsymbol{\delta} = \nu_1 \tau_{yx} \dot{\boldsymbol{\gamma}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (92)$$

Using Eq. (11) to rewrite the third term on the right of Eq. (14), we get:

$$-\mu_2 \{ \dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} \} = -\mu_2 \dot{\gamma}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (93)$$

and to rewrite the last:

$$\frac{1}{2} \nu_2 (\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}) \boldsymbol{\delta} = \nu_2 \dot{\gamma}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (94)$$

Substituting Eqs. (88) through (94) into Eq. (14) gives:

$$\begin{aligned} & \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} + \lambda_1 \left(\frac{\partial}{\partial t} \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} + \frac{1}{2} \dot{\boldsymbol{\gamma}} \begin{bmatrix} -2\tau_{yx} & \tau_{xx} - \tau_{yy} & 0 \\ \tau_{xx} - \tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ & + \frac{1}{2} \mu_0 (\tau_{xx} + \tau_{yy} + \tau_{zz}) \dot{\boldsymbol{\gamma}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{2} \mu_1 \dot{\boldsymbol{\gamma}} \begin{bmatrix} 2\tau_{yx} & \tau_{xx} + \tau_{yy} & \tau_{yz} \\ \tau_{xx} + \tau_{yy} & 2\tau_{yx} & \tau_{xz} \\ \tau_{yz} & \tau_{zx} & 0 \end{bmatrix} + \nu_1 \tau_{yx} \dot{\boldsymbol{\gamma}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & = -\eta_0 \dot{\boldsymbol{\gamma}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \eta_0 \lambda_2 \left(\frac{\partial}{\partial t} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dot{\gamma}^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) + \eta_0 \mu_2 \dot{\gamma}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \eta_0 \nu_2 \dot{\gamma}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (95)$$

which we will use to get τ_{yx} , N_1 , N_2 and τ_{zz} (or, in dimensionless form, $\mathbb{S}$, N_1 , N_2 and $\tilde{\tau}_{zz}$).

The yx , xx , yy and zz -component of Eq. (95) are:

$$\begin{aligned} & \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau_{yx} + \frac{1}{2}(\lambda_1 - \mu_1 + \mu_0) \tau_{xx} \dot{\gamma} - \frac{1}{2}(\lambda_1 + \mu_1 - \mu_0) \tau_{yy} \dot{\gamma} + \frac{1}{2} \mu_0 \tau_{zz} \dot{\gamma} \\ & = -\eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \dot{\gamma} \end{aligned} \quad (96)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau_{xx} - (\lambda_1 + \mu_1 - \nu_1) \tau_{yx} \dot{\gamma} = \eta_0 (\lambda_2 + \mu_2 - \nu_2) \dot{\gamma}^2 \quad (97)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau_{yy} + (\lambda_1 - \mu_1 + \nu_1) \tau_{yx} \dot{\gamma} = -\eta_0 (\lambda_2 - \mu_2 + \nu_2) \dot{\gamma}^2 \quad (98)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau_{zz} + \nu_1 \tau_{yx} \dot{\gamma} = -\eta_0 \nu_2 \dot{\gamma}^2 \quad (99)$$

which match Eqs. (8.1-3)–(8.1-6) of [21]. We next rewrite these four differential equations in terms of the first and second normal stress differences as:

$$\begin{aligned} & \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau_{yx} + \frac{1}{2}(\lambda_1 - \mu_1 + \mu_0) N_1 \dot{\gamma} - (\mu_1 - \mu_0) N_2 \dot{\gamma} + \left(\frac{3}{2} \mu_0 - \mu_1\right) \tau_{zz} \dot{\gamma} \\ & = -\eta_0 \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \dot{\gamma} \end{aligned} \quad (100)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) N_1 - 2\lambda_1 \tau_{yx} \dot{\gamma} = 2\lambda_2 \eta_0 \dot{\gamma}^2 \quad (101)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) N_2 + (\lambda_1 - \mu_1) \tau_{yx} \dot{\gamma} = -\eta_0 (\lambda_2 - \mu_2) \dot{\gamma}^2 \quad (102)$$

Substituting the yx -component in Eq. (11) into Eqs. (99)–(102) gives Eqs. (37)–(44) above.

We next turn our attention to the two remaining components of extra stress tensor. The xz and yz -components of Eq. (95) are:

$$\tau_{xz} + \lambda_1 \frac{\partial}{\partial t} \tau_{xz} - \frac{1}{2} \mu_1 \tau_{yz} \dot{\gamma} = 0 \quad (103)$$

$$\tau_{yz} + \lambda_1 \frac{\partial}{\partial t} \tau_{yz} - \frac{1}{2} \mu_1 \tau_{xz} \dot{\gamma} = 0 \quad (104)$$

Solving Eqs. (103) and (104) subject to the initial conditions:

$$\tau_{yz}(0) = \tau_{xz}(0) = 0 \quad (105)$$

gives:

$$\tau_{xz}(t) = \tau_{yz}(t) = 0 \quad (106)$$

from which we confirm that $\tau_{xz}(t)$ and $\tau_{yz}(t)$ make no contribution to simple shear flow.

For the special case of corotational Maxwell fluid, where

$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eqs. (41)–(43) reduce to:

$$\frac{dN_1}{d\tau} = -\frac{1}{De}N_1 + 2\frac{Wi}{De}S\cos\tau \quad (107)$$

$$\frac{dN_2}{d\tau} = -\frac{1}{De}N_2 - \frac{Wi}{De}S\cos\tau \quad (108)$$

$$\frac{dS}{d\tau} = -\frac{1}{2}\frac{Wi}{De}N_1\cos\tau - \frac{1}{De}S - \frac{1}{De}\cos\tau \quad (109)$$

which match Eqs. (32), (33) and (36) in [9], which serves as an intermediate consistency check.

b. Solving Eqs. (46)–(49) by the Kovacic Method

Differentiating Eq. (49) once gives:

$$\begin{aligned} \frac{d^2S}{d\tau^2} = & -\frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}\left(\frac{dN_1}{d\tau}\cos\tau - N_1\sin\tau\right) + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}\left(\frac{dN_2}{d\tau}\cos\tau - N_2\sin\tau\right) \\ & - \left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{Wi}{De}\left(\frac{d\tilde{\tau}_{zz}}{d\tau}\cos\tau - \tilde{\tau}_{zz}\sin\tau\right) - \frac{1}{De}\frac{dS}{d\tau} \end{aligned} \quad (110)$$

Substituting Eq. (48) into Eq. (110) and rearranging gives:

$$\begin{aligned} \frac{d^2S}{d\tau^2} = & -\frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}\left(\frac{dN_1}{d\tau}\cos\tau - N_1\sin\tau\right) + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}\left(\frac{dN_2}{d\tau}\cos\tau - N_2\sin\tau\right) \\ & + \left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{Wi^2}{De^2}\frac{v_1}{\lambda_1}S\cos^2\tau - \frac{1}{De}\frac{dS}{d\tau} + \left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{Wi}{De}\left[\frac{1}{De}\cos\tau + \sin\tau\right]\tilde{\tau}_{zz} \end{aligned} \quad (111)$$

Solving Eq. (49) gives:

$$\left(\frac{3}{2}\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right)\frac{Wi}{De}\tilde{\tau}_{zz} = -\frac{1}{\cos\tau}\frac{dS}{d\tau} - \frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1} + \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}N_1 + \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_0}{\lambda_1}\right)\frac{Wi}{De}N_2 - \frac{1}{De}\frac{S}{\cos\tau} \quad (112)$$

Substituting Eq. (112) into Eq. (111), and then, Eqs. (46) and (47) into the result gives:

$$\frac{d^2S}{d\tau^2} + \left(\frac{2}{De} + \tan\tau\right)\frac{dS}{d\tau} + \left[\frac{1}{De^2} + \frac{\tan\tau}{De} + \left[1 + \frac{\mu_0}{\lambda_1}\left(\frac{\mu_1}{\lambda_1} - \frac{3}{2}\frac{v_1}{\lambda_1}\right) - \frac{\mu_1}{\lambda_1}\left(\frac{\mu_1}{\lambda_1} - \frac{v_1}{\lambda_1}\right)\right]\frac{Wi^2}{De^2}\cos^2\tau\right]S = 0 \quad (113)$$

Solving Eq. (113) using the Kovacic method (see Section 4.1 of [14]; Appendix: Fundamental Matrix of [9]), we first discover that a solution to Eq. (14) will both exist and be in closed form, and then that this solution will be unique. Proceeding then with the solution to Eq. (113) gives Eq. (50) above.

Substituting Eq. (50) into Eqs. (46) and (47) gives:

$$\frac{dN_1}{d\tau} = -\frac{1}{De}N_1 + 2\frac{Wi}{De}\left[C_1e^{-\tau/De}\sin(\alpha\sin\tau) + C_2e^{-\tau/De}\cos(\alpha\sin\tau)\right]\cos\tau \quad (114)$$

and:

$$\frac{dN_2}{d\tau} = -\frac{1}{De}N_2 - \left(1 - \frac{\mu_1}{\lambda_1}\right)\frac{Wi}{De}\left[C_1e^{-\tau/De}\sin(\alpha\sin\tau) + C_2e^{-\tau/De}\cos(\alpha\sin\tau)\right]\cos\tau \quad (115)$$

which can be solve to get Eqs. (51) and (52) above. Substituting Eq. (50), (51) and (52) into Eq. (49), and solving for $\tilde{\tau}_{zz}$ gives Eq. (53) above.

c. Evaluating $\lim_{Wi \rightarrow 0}$ of Eq. (60) for SAOS

In this Appendix, we detail our evaluation of the limit as $Wi \rightarrow 0$ in Eq. (60). Substituting Eqs. (61) and (62) into Eq. (60) gives:

$$\mathbb{S} = \frac{e^{-\tau/De}}{2\tilde{\lambda}De} \left[\begin{aligned} & \frac{1}{Wi} \sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) \left[\begin{aligned} & \left(-2\frac{\lambda_2}{\lambda_1} + 3\frac{\mu_0 v_2}{\lambda_1^2} - 2\frac{\mu_0 \mu_2}{\lambda_1^2} - 2\frac{\mu_1 v_2}{\lambda_1^2} + 2\frac{\mu_1 \mu_2}{\lambda_1^2}\right) Wi I_{S1,1} \\ & -2I_{S1,2} + 2\frac{\lambda_2}{\lambda_1} De I_{S1,3} + 2\frac{\lambda_2}{\lambda_1} De^2 I_{S1,4} \end{aligned} \right] \\ & -\frac{1}{Wi} \cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) \left[\begin{aligned} & \left(-2\frac{\lambda_2}{\lambda_1} + 3\frac{\mu_0 v_2}{\lambda_1^2} - 2\frac{\mu_0 \mu_2}{\lambda_1^2} - 2\frac{\mu_1 v_2}{\lambda_1^2} + 2\frac{\mu_1 \mu_2}{\lambda_1^2}\right) Wi I_{S2,1} \\ & -2I_{S2,2} + 2\frac{\lambda_2}{\lambda_1} De I_{S2,3} + 2\frac{\lambda_2}{\lambda_1} De^2 I_{S2,4} \end{aligned} \right] \end{aligned} \right] \quad (116)$$

$$\equiv \mathbb{S}^{(1)} + \mathbb{S}^{(2)} + \mathbb{S}^{(3)} + \mathbb{S}^{(4)} + \mathbb{S}^{(5)} + \mathbb{S}^{(6)} + \mathbb{S}^{(7)} + \mathbb{S}^{(8)}$$

To evaluate Eq. (116) term by term, we will need the following four identities:

$$\lim_{Wi \rightarrow 0} \sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_n\left(\frac{Wi}{De} \tilde{\lambda}\right) = 0 \quad (117)$$

$$\lim_{Wi \rightarrow 0} \cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_n\left(\frac{Wi}{De} \tilde{\lambda}\right) = 0 \quad (118)$$

$$\lim_{Wi \rightarrow 0} \frac{\sin\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_n\left(\frac{Wi}{De} \tilde{\lambda}\right)}{Wi} = \begin{cases} 0; & n = 2, 4, 6, \dots \\ (\tilde{\lambda} \sin \tau)/De; & n = 0 \end{cases} \quad (119)$$

$$\lim_{Wi \rightarrow 0} \frac{\cos\left(\frac{Wi}{De} \tilde{\lambda} \sin \tau\right) J_n\left(\frac{Wi}{De} \tilde{\lambda}\right)}{Wi} = \begin{cases} 0; & n = 3, 5, 7, \dots \\ \tilde{\lambda}/2De; & n = 1 \end{cases} \quad (120)$$

For the first term of Eq. (116), we get:

$$\mathbb{S}^{(1)} = \frac{e^{-\tau/De}}{2\tilde{\lambda}De} \left[\sin\left(\frac{Wi}{De} \sin \tau\right) \left(-2\frac{\lambda_2}{\lambda_1} + 3\frac{\mu_0 v_2}{\lambda_1^2} - 2\frac{\mu_0 \mu_2}{\lambda_1^2} - 2\frac{\mu_1 v_2}{\lambda_1^2} + 2\frac{\mu_1 \mu_2}{\lambda_1^2}\right) I_{S1,1} \right] \quad (121)$$

where:

$$I_{S1,1} = \frac{J_0 De}{8De^2 + 2} \left[e^{\tau/De} (4De^2 + 2De \sin 2\tau + \cos 2\tau + 1) \right] \\ + \frac{De}{2} e^{\tau/De} \sum_{k=1}^{\infty} J_{2k} \left[\frac{2(2De k \sin 2k\tau + \cos 2k\tau)}{1 + 4De^2 k^2} + \frac{2De(k-1) \sin 2(k-1)\tau + \cos 2(k-1)\tau}{1 + 4De^2 (k-1)^2} \right. \\ \left. + \frac{2De(k+1) \sin 2(k+1)\tau + \cos 2(k+1)\tau}{1 + 4De^2 (k+1)^2} \right] \quad (122)$$

where $J_m \equiv J_m(\alpha)$ is the m th order Bessel function of the first kind (see after Eq. (5) of [44] or before Eq. (123) of [9]). Using Eq. (117) to simplify Eq. (122) gives:

$$\mathbb{S}^{(1)} = 0 \quad (123)$$

For the second term of Eq. (116), we get:

$$\mathbb{S}^{(2)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda}\text{De}} \left[\frac{1}{\text{Wi}} \sin\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[-2I_{\text{S1},2} \right] \right] \quad (124)$$

where:

$$I_{\text{S1},2} = J_0 \text{De} e^{\tau/\text{De}} + \sum_{k=1}^{\infty} \frac{2\text{De} J_{2k} e^{\tau/\text{De}}}{4\text{De}^2 k^2 + 1} (\cos 2k\tau + 2k\text{De} \sin 2k\tau) \quad (125)$$

and using Eq. (119) then gives:

$$\mathbb{S}^{(2)} = -\frac{\sin \tau}{\text{De}} \quad (126)$$

For the third term of Eq. (116), we get:

$$\mathbb{S}^{(3)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda}\text{De}} \left[\frac{1}{\text{Wi}} \sin\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[2\frac{\lambda_2}{\lambda_1} \text{De} I_{\text{S1},3} \right] \right] \quad (127)$$

where:

$$\begin{aligned} I_{\text{S1},3} = i\text{De} J_0 & \left[-e^{\tau/\text{De}} + 2e^{\tau/\text{De}} {}_2F_1\left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{2i\tau}\right) \right] \\ & + 2\text{De} \sum_{k=1}^{\infty} J_{2k} \left[\frac{e^{\tau/\text{De}}}{1 + 4k^2 \text{De}^2} (-\sin 2k\tau + 2k\text{De} \cos 2k\tau) + i(-1)^{1+k} e^{\tau/\text{De}} \right. \\ & \left. - \sum_{l=1}^k \left[\frac{(-1)^{1-l+k} 2e^{\tau/\text{De}}}{1 + 4l^2 \text{De}^2} (\sin 2l\tau - 2l\text{De} \cos 2l\tau) \right] \right. \\ & \left. + 2i(-1)^k e^{\tau/\text{De}} {}_2F_1\left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{2i\tau}\right) \right] \end{aligned} \quad (128)$$

and using Eq. (119), we then get:

$$\mathbb{S}^{(3)} = \frac{\lambda_2}{\lambda_1} i \sin \tau \left[-1 + 2 {}_2F_1\left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{2i\tau}\right) \right] \quad (129)$$

For the fourth term of Eq. (116):

$$\mathbb{S}^{(4)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda}\text{De}} \left[\frac{1}{\text{Wi}} \sin\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[2\frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\text{S1},4} \right] \right] \quad (130)$$

where:

$$\begin{aligned}
I_{\mathbb{S}1,4} = & 2iJ_0 \left[\frac{e^{\tau/\text{De}}}{e^{i2\tau} + 1} - e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{i2\tau} \right) \right] \\
& + 2 \sum_{k=1}^{\infty} J_{2k} \left[\begin{aligned} & (-1)^k 2k \text{De} e^{\tau/\text{De}} + (-1)^k i \left(\frac{2e^{\tau/\text{De}}}{e^{i2\tau} + 1} \right) \\ & + 4e^{\tau/\text{De}} \text{De} \sum_{m=1}^k \frac{(-1)^{k-m} (k-m+1)}{1 + 4(m-1)^2 \text{De}^2} \left[\cos 2(m-1)\tau \right. \\ & \left. + 2(m-1) \text{De} \sin 2(m-1)\tau \right] \\ & - 2i(-1)^k e^{\tau/\text{De}} {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{i2\tau} \right) \end{aligned} \right] \quad (131)
\end{aligned}$$

and using Eq. (119) then gives:

$$\mathbb{S}^{(4)} = 2 \frac{\lambda_2}{\lambda_1} i \sin \tau \left[\frac{1}{e^{i2\tau} + 1} - {}_2F_1 \left(1, -\frac{i}{2\text{De}}; 1 - \frac{i}{2\text{De}}; -e^{i2\tau} \right) \right] \quad (132)$$

For the fifth term of Eq. (116), we get:

$$\mathbb{S}^{(5)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda} \text{De}} \left[-\cos \left(\frac{\text{Wi}}{\text{De}} \sin \tau \right) \left(-2 \frac{\lambda_2}{\lambda_1} + 3 \frac{\mu_0 \nu_2}{\lambda_1^2} - 2 \frac{\mu_0 \mu_2}{\lambda_1^2} - 2 \frac{\mu_1 \nu_2}{\lambda_1^2} + 2 \frac{\mu_1 \mu_2}{\lambda_1^2} \right) I_{\mathbb{S}2,1} \right] \quad (133)$$

where:

$$\begin{aligned}
I_{\mathbb{S}2,1} = & 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\begin{aligned} & \frac{-(1+2k) \text{De}^2 e^{\tau/\text{De}} \cos(1+2k)\tau}{4(1 + \text{De}^2(1+2k)^2)} + \frac{(3-2k) \text{De}^2 e^{\tau/\text{De}} \cos(3-2k)\tau}{4(1 + \text{De}^2(3-2k)^2)} \\ & + \frac{\text{De}^2(1-2k) e^{\tau/\text{De}} \cos(1-2k)\tau}{2(1 + \text{De}^2(1-2k)^2)} \\ & + \text{De} e^{\tau/\text{De}} \left[\frac{\sin(1+2k)\tau}{4(1 + \text{De}^2(1+2k)^2)} - \frac{\sin(1-2k)\tau}{2(1 + \text{De}^2(1-2k)^2)} - \frac{\sin(3-2k)\tau}{4(1 + \text{De}^2(3-2k)^2)} \right] \end{aligned} \right] \quad (134)
\end{aligned}$$

and using Eq. (118) then gives:

$$\mathbb{S}^{(5)} = 0 \quad (135)$$

For the sixth term of Eq. (116), we get:

$$\mathbb{S}^{(6)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda} \text{De}} \left[-\frac{1}{\text{Wi}} \cos \left(\frac{\text{Wi}}{\text{De}} \sin \tau \right) [-2I_{\mathbb{S}2,2}] \right] \quad (136)$$

where:

$$I_{\mathbb{S}2,2} = 2\text{De} \sum_{k=1}^{\infty} J_{2k-1} \frac{e^{\tau/\text{De}} [(1-2k) \text{De} \cos(1-2k)\tau - \sin(1-2k)\tau]}{1 + \text{De}^2(1-2k)^2} \quad (137)$$

and using Eq. (120) then gives:

$$\mathbb{S}^{(6)} = \frac{-\text{De} \cos \tau + \sin \tau}{\text{De}(1 + \text{De}^2)} \quad (138)$$

For the seventh term of Eq. (116), we get:

$$\mathbb{S}^{(7)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda}\text{De}} \left[-\frac{1}{\text{Wi}} \cos\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[2 \frac{\lambda_2}{\lambda_1} \text{De} I_{\mathbb{S}2,3} \right] \right] \quad (139)$$

where:

$$I_{\mathbb{S}2,3} = 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\frac{\text{De} e^{\tau/\text{De}} \left[\cos(2k-1)\tau + (2k-1)\text{De} \sin(2k-1)\tau \right]}{1 + (2k-1)^2 \text{De}^2} + 2 \text{De} e^{\tau/\text{De}} \sum_{n=1}^k \frac{(-1)^{k-n+1} \left[\cos(2n-1)\tau + (2n-1)\text{De} \sin(2n-1)\tau \right]}{1 + (2n-1)^2 \text{De}^2} + \frac{(-1)^{k+1} 2\text{De}}{(1+i\text{De})} e^{\tau/\text{De}} e^{i\tau} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \right] \quad (140)$$

and using Eq. (120) then gives:

$$\mathbb{S}^{(7)} = \frac{\lambda_2}{\lambda_1} \left[\frac{\cos \tau + \text{De} \sin \tau}{1 + \text{De}^2} - \frac{2e^{i\tau}}{(1+i\text{De})} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \right] \quad (141)$$

For the eighth term of Eq. (116), we get:

$$\mathbb{S}^{(8)} = \frac{e^{-\tau/\text{De}}}{2\tilde{\lambda}\text{De}} \left[-\frac{1}{\text{Wi}} \cos\left(\frac{\text{Wi}}{\text{De}} \sin \tau\right) \left[2 \frac{\lambda_2}{\lambda_1} \text{De}^2 I_{\mathbb{S}2,4} \right] \right] \quad (142)$$

where:

$$I_{\mathbb{S}2,4} = 2 \sum_{k=1}^{\infty} J_{2k-1} \left[\sum_{m=1}^k \frac{(-1)^{k-m-1} 4\text{De}(k-m) e^{\tau/\text{De}} \left[\sin(2m-1)\tau - (2m-1)\text{De} \cos(2m-1)\tau \right]}{1 + (2m-1)^2 \text{De}^2} + (-1)^{k+1} e^{\tau/\text{De}} \sec \tau - \frac{2i(-1)^{k+1}}{i - \text{De}} e^{(1/\text{De}+i)\tau} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \right] \quad (143)$$

and using Eq. (120) then gives:

$$\mathbb{S}^{(8)} = \frac{-\lambda_2}{\lambda_1} \left[\sec \tau - \frac{i2e^{i\tau}}{i - \text{De}} {}_2F_1\left(1, \frac{1}{2} - \frac{i}{2\text{De}}; \frac{3}{2} - \frac{i}{2\text{De}}; -e^{i2\tau}\right) \right] \quad (144)$$

Substituting Eqs. (123), (126), (129), (132), (135), (138), (141) and (144) back into Eq. (116) gives Eq. (78) above.

Table I: Literature on Analytical Solutions for Large-Amplitude Oscillatory Shear Flow

Authors (year)	Model	Shear Stress Harmonic			Normal Stress Difference Harmonic			Orientation	Startup	Form	[Ref.] (Correction to)
		First	Third	Fifth	Zeroth	Second	Fourth				
Kirkwood and Plock (1956, 1967); Plock (1957)	RD^h, SK^h	n						ψ		$\cong$	[84,85,86]
Lodge (1961, 1964)	$L^\dagger$	ℓ			N_1	N_1				$=$	[87,88]
Spriggs (1966)	NGJ	ℓ			N_1	N_1				$=$	[89]
Spriggs (1966)	CJ	ℓ			N_1 N_2	N_1 N_2				$=$	[89]
Williams and Bird (1962)	O_3	ℓ				N_1				$=$	[90]
Williams and Bird (1964)	O_3	ℓ			N_1	N_1				$=$	[91]
Spriggs (1965)	$O_3^\dagger$	ℓ			N_1 N_2	N_1 N_2				$=$	[92]
Akers and Williams (1969)	RZ				N_1	N_1				$=$	[93]
Paul (1969); Paul (1970); Bharadwaj (2012)	RD^h, SK^h	n	X					ψ		$\cong$	[94,95,96] (84,86)
Walters and Jones (1970); Walters (1975)	I_3	n	X							$\cong$	[97]; Section 6.2.3 of [98]
Paul and Mazo (1969), Paul (1970)	RR	n	X					ψ		$\cong$	[99,95]
MacDonald, Marsh and Ashare (1969)	$BC, OWFS^\dagger$	n								$\cong$	[50]
Bird, Warner and Evans (1971)	RD	n			N_1 N_2	N_1 N_2		ψ		$\cong$	[100]
Leal and Hinch (1972)	RSD	ℓ			N_1 N_2	N_1 N_2		ψ		$\cong$	Section 3.3 of [101]
Abdel-Khalik <i>et al.</i> (1974); Bird <i>et al.</i> (1974)	GE+SK	n			N_1 N_2	N_1 N_2				$\cong$	[23,24]
Bird <i>et al.</i> (1977)	BHS	ℓ	0	0				ψ		$=$	Table 11.4-2 [102]
Mou and Mazo (1977)	RR				N_1 N_2	N_1 N_2		ψ		$\cong$	[103](99)
Pearson and Rochefort (1982); Helfand and Rochefort (1982)	$R^\dagger$	n	X							$\cong$	[104,105]

Fan and Bird (1984)	CB [†]	n	X							$\cong$	[106](105)
Oakley (1992); Oakley and Giacomini (1994)	L [†]	ℓ			N_1	N_1				=	APPENDIX B of [107]; [108](87)
Phan-Thien <i>et al.</i> (2000)	Dough	ℓ								=	[109]
Yu <i>et al.</i> (2002); Zhou (2004)	SE	n	X		N_1	N_1	N_1			$\cong$	[110,111]
Cho <i>et al.</i> (2010)	K-BKZ										[16]
Hoyle (2010)	PP	n								$\cong$	Ch. 4 of [112]
Wagner <i>et al.</i> (2011)	R	n	X							$\cong$	[113]
Giacomini <i>et al.</i> (2011)	CM [†] , CJ	n	X	X	N_1 N_2	N_1 N_2	N_1 N_2		X	$\cong$	[6] (114)
Giacomini and Bird (2011)	ANSR	n	X	X	N_1 N_2	N_1 N_2	N_1 N_2		X	$\cong$	[115]
Gurnon and Wagner (2012)	G	ℓ	X		N_1 N_2	N_1 N_2				$\cong$	[116]
Bird <i>et al.</i> (2014)	RD	n	X					ψ		$\cong$	[117]
Schmalzer <i>et al.</i> (2014)	RD				N_1 N_2	N_1 N_2	N_1 N_2	ψ		$\cong$	[118,119](6,11 5); [120]
Thompson and de Souza Mendes (2015)	MSJ	n			0	0	0		0	$\cong$	See Section 5.2.2 of [121]
Bozorgi (2014); Bozorgi and Underhill (2014)	AS ^h	n	X					ψ		$\cong$	Ch. 8 of [122]; [123]
Blackwell (2014)	CJ ^ξ		X							$\cong$	[124]
Giacomini <i>et al.</i> (2015)	RD							ψ		$\cong$	[125]
Giacomini <i>et al.</i> (2015)	CM	n	X	X	N_1 N_2	N_1 N_2	N_1 N_2		X	$\cong$	[47]
Saengow <i>et al.</i> (2015)	CM [†]	n	X	X	N_1 N_2	N_1 N_2	N_1 N_2		X	=	[9]
Merger <i>et al.</i> (2016)	M ^{CM}	n	X							=	[126,127]
This paper	O ₈	n	X	X						=	[128]

Legend: ANSR $\equiv$ corotational arbitrary normal stress ratio; AS $\equiv$ active rod suspensions; BC $\equiv$ Bird-Carreau; BHS $\equiv$ Bead-Hookean spring; CB $\equiv$ Curtiss-Bird; CJ $\equiv$ corotational Jeffreys; CM $\equiv$ corotational Maxwell; G $\equiv$ Giesekus; GE $\equiv$ Goddard integral expansion; I₃ $\equiv$ third order integral; K-BKZ $\equiv$ Kaye-Bernstein, Kearsley, Zappas; L $\equiv$ Lodge rubberlike; M $\equiv$ Maxwell fluid; MSJ $\equiv$ material subjective Jeffreys; NGJ $\equiv$ nonlinear Generalized Jeffreys; O₃ $\equiv$ Oldroyd 3-constant; OWFS $\equiv$ modified Oldroyd-Walters-Fredrickson-Spriggs; O₈ $\equiv$ Oldroyd 8-constant; P $\equiv$ Padé approximants; PP $\equiv$ pom-pom; R $\equiv$ reptation; RD $\equiv$ rigid dumbbell; RR $\equiv$ planar rigid ring; RSD $\equiv$ rods, spheres or disks; RZ $\equiv$ Rouse-Zimm; SE $\equiv$ simple emulsion; SK $\equiv$ shish-kebab; $N_1, N_2 \equiv$ first and second normal stress differences; $n \equiv \eta^*(\omega, \gamma_0)$; $\ell \equiv \eta^*(\omega)$; [†] $\equiv$ multiple relaxation times; = $\equiv$ exact; $\cong \equiv$ approximate; ^h $\equiv$ with hydrodynamic interaction; $\psi \equiv$ orientation distribution. ^ξ $\equiv$ with thixotropic contribution. ^{CM} $\equiv$ with Cox-Merz modified viscosity.

Table II: Dimensional Variables

Angular frequency [Eq. (19)]	t^{-1}	ω
Fluid pressure [Eq. (17)]	M/Lt^2	p
Boltzmann constant	ML^2/t^2T	k
Cartesian coordinate	L	x, y, z
Characteristic time, rigid dumbbell model [Eq. (15)]	t	λ
Characteristic time, FENE dumbbell [Eq. (16)]	t	λ_H
Distance between two plates	L	h
Molar concentration of dumbbells	L^3	$\bar{n}$
Steady shear viscosity [Eq. (22)]	M/Lt	η
Steady shear viscosity, zero shear rate	M/Lt	η_0
Steady shear viscosity, infinite shear rate [Eq. (27)]	M/Lt	η_∞
Steady shear viscosity, polymer contribution [Eq. (15)]	M/Lt	η_p
Extra stress tensor* [Eq. (17)]	M/Lt^2	$\boldsymbol{\tau}$
Total stress tensor* [Eq. (17)]	M/Lt^2	$\boldsymbol{\pi}$
Extra stress, ij th component	M/Lt^2	τ_{ij}
First normal stress difference [Eq. (38)]	M/Lt^2	$N_1 \equiv \tau_{xx} - \tau_{yy}$
First normal stress coefficient [Eq. (22)]	M/L	$\Psi_1 \equiv -N_1/\dot{\gamma}^2$
First normal stress coefficient, zero shear rate [Eq. (23)]	M/L	$\Psi_{10} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_1$
Oldroyd constant [Eq. (14)]	t	μ_0
Oldroyd constant [Eq. (14)]	t	μ_1
Oldroyd constant [Eq. (14)]	t	μ_2
Oldroyd constant [Eq. (14)]	t	ν_1
Oldroyd constant [Eq. (14)]	t	ν_2
Oldroyd constant, relaxation time [Eq. (14)]	t	λ_1
Oldroyd constant, retardation time [Eq. (14)]	t	λ_2
Oldroyd shear thickening constant [Eq. (26)]	t^2	σ_2
Oldroyd shear thinning constant [Eq. (25)]	t^2	σ_1
Real part of complex viscosity [Eq. (32)]	M/Lt	η'
Minus the imaginary part of complex viscosity [Eq. (33)]	M/Lt	η''
Second normal stress difference [Eq. (39)]	M/Lt^2	$N_2 \equiv \tau_{yy} - \tau_{zz}$

Second normal stress coefficient [Eq. (22)]	M/L	$\Psi_2 \equiv -N_2/\dot{\gamma}^2$
Second normal stress coefficient, zero shear [Eq. (24)]	M/L	$\Psi_{20} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_2$
Storage and loss moduli, n th harmonic	M/Lt^2	η'_n, η''_n
Strain rate tensor [Eq. (18)]	t^{-1}	$\dot{\boldsymbol{\gamma}}$
Strain rate, yx -component [Eq. (1)]	t^{-1}	$\dot{\gamma}$
Shear rate, amplitude [Eq. (1)]	t^{-1}	$\dot{\gamma}^0$
Time	t	t
Temperature	T	T
Velocity, upper plate	L/t	V_0
Velocity vector	L/t	$\mathbf{v}$
Vorticity tensor [Eq. (19)]	t^{-1}	$\boldsymbol{\omega}$
Molar concentration of dumbbells	L^{-3}	$\bar{n}$

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

*Where τ_{ij} is the force exerted in the j th direction on a unit area of fluid surface of constant x_i by fluid in the region lesser x_i on fluid in the region greater x_i (see "Note on the Sign Convention for the Stress Tensor" on pp. 19–20 of [129], or pp. 24–25 of [130]).

Table III: Dimensionless Variables and Groups

Bessel function of first kind, m th order	$J_m(z) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k)!k!} \left(\frac{z}{2}\right)^{m+2k}$
Bessel function of first kind, m th order with argument α	$J_m \equiv J_m(\alpha)$
Combined fluid parameter [Eq. (56)]	$\tilde{\lambda}$
Constant in Eq. (56)	$\alpha \equiv (\text{Wi}/\text{De}) \tilde{\lambda}$
Function in Eq. (54)	$S \equiv \sin(\alpha \sin \tau)$
Function in Eq. (55)	$C \equiv \cos(\alpha \sin \tau)$
i th Constant in Eq. (57)	C_i
Constants in the hypergeometric function [Eq. (68)]	α_i, β_i
Deborah number [Eq. (3)]	$\text{De} \equiv \lambda_1 \omega$
Dimensionless parameter in elastic dumbbell [Eq. (16)]	b
First normal stress difference [Eq. (42)]	$\mathbb{N}_1 \equiv N_1 / \eta_0 \dot{\gamma}^0$
First normal stress difference, homogeneous part [Eq. (57)]	$\mathbb{N}_{1,h}$
First normal stress difference, particular part [Eq. (59)]	$\mathbb{N}_{1,p}$
Fundamental matrix [Eq. (58)]	Φ
Generalized non-Newtonianness [Eq. (5)]	$\text{Gn} \equiv \text{De} + i \text{Wi}$
Inclination of non-Newtonianness [Eq. (7)]	$\hat{\phi} \equiv \arctan(\text{Wi}/\text{De})$
Integral j term, i term of shear stress expression [Eq. (61) and (62)]	$I_{\text{Si},j}$
Integral, i term of shear stress expression [Eq. (61) and (62)]	I_{Si}
Kronecker delta	δ
Oldroyd constant ratio [Eq. (30)]	$\sigma \equiv \sigma_2 / \sigma_1$
Second normal stress difference [Eq. (43)]	$\mathbb{N}_2 \equiv N_2 / \eta_0 \dot{\gamma}^0$
Second normal stress difference, homogeneous part [Eq. (57)]	$\mathbb{N}_{2,h}$
Second normal stress difference, particular part [Eq. (59)]	$\mathbb{N}_{2,p}$
Shear strain amplitude	$\gamma_0 \equiv \dot{\gamma}^0 / \omega$
Shear stress [Eq. (41)]	$\mathbb{S} \equiv \tau_{yx} / \eta_0 \dot{\gamma}^0$
Shear stress, corotational Maxwell contribution, n th harmonic [Eq. (83)]	$\mathbb{S}[an]$
Shear stress, Dirac delta contribution, n th harmonic [Eq. (83)]	$\mathbb{S}[bn]$

Shear stress, homogeneous part [Eq. (57)]	$\mathbb{S}_h$
Shear stress, particular part [Eq. (59)]	$\mathbb{S}_p$
Shear stress, SAOS response, i -th term [Eq. (116)]	$\mathbb{S}^{(i)}$
Shear stress, steady [Eq. (29)]	$\check{\tau}_{yx} \equiv \sqrt{\sigma_1} \tau_{yx} / \eta_0$
Extra stress, zz -component [Eq. (44)]	$\check{\tau}_{zz} \equiv \tau_{zz} / \eta_0 \dot{\gamma}^0$
Extra stress, zz -component, homogeneous part [Eq. (57)]	$\check{\tau}_{zz,h}$
Extra stress, zz -component, particular part [Eq. (59)]	$\check{\tau}_{zz,p}$
Time	$\tau \equiv \omega t = \text{De}(t/\lambda_1)$
Weissenberg number	$\text{Wi} \equiv \lambda_1 \dot{\gamma}^0$

Table IV: Models Included in the Oldroyd 8-Constant Framework (see TABLE 8.1-1 of [21] and TABLE 7.3-2 of [29]).

Model	No. of Constant	Highest Harmonic	Reduce from Oldroyd 8-Constant Framework [Eq. (14)] by Assigning	Steady State Shear Flow Viscometric Functions
Oldroyd 6-constant [22]	6	∞	$\nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma}); \Psi_2(\dot{\gamma})$
Oldroyd 4-constant [22]	4	∞	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma}); \Psi_2 = 0$
Gordon-Schowalter*,† [131]	4	∞	$\lambda_2 = (\eta_s/\eta_0)\lambda_1; \mu_1 = (1-\xi)\lambda_1;$ $\mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1; \mu_0 = \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma});$ $\Psi_2(\dot{\gamma}) < 0$
Johnson-Segalman**,‡ [132,133]	4	∞	$\lambda_2 = (\eta_s/\eta_0)\lambda_1; \mu_1 = (1-\xi)\lambda_1;$ $\mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1; \mu_0 = \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma});$ $\Psi_2(\dot{\gamma}) < 0$
Oldroyd Fluid A (Lower Convected Jeffreys) (see Eq. (A) 9.1-9 of [21])	3	1	$\mu_1 = -\lambda_1; \mu_2 = -\lambda_2; \mu_0 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0;$ $\Psi_1 = 2\eta_0(\lambda_1 - \lambda_2);$ $\Psi_2 = -\Psi_1$
Oldroyd Fluid B (Upper Convected Jeffreys) (see Eq. (B) 9.1-9 of [21])	3	1	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \mu_0 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0;$ $\Psi_1 = 2\eta_0(\lambda_1 - \lambda_2);$ $\Psi_2 = 0$
Second-Order Fluid (see Eq. 8.4-3 of [21])	3	1	$\lambda_1 = \mu_0 = \mu_1 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0; \Psi_1 = -2\lambda_2\eta_0;$ $\Psi_2 = \eta_0(\lambda_2 - \mu_2)$
Arbitrary Normal Stress Ratio (see Eq. (10) of [134])	3	∞	$\lambda_2 = \mu_0 = \mu_2 = \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma});$ $\Psi_2 = -\frac{1}{2}\left(1 - \frac{\mu_1}{\lambda_1}\right)\Psi_1$
Corotational Jeffreys Model [135 or 136]	3	∞	$\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma});$ $\Psi_2 = -\frac{1}{2}\Psi_1$
Williams 3-constant Oldroyd Model [90,137]	3	∞	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \nu_1 = \frac{2}{3}\lambda_1; \nu_2 = \frac{2}{3}\lambda_2;$ $\mu_0 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma}); \Psi_2 = 0$
Denn Modified Convected Maxwell Model [138]	3	∞	$\mu_1 = \lambda_1; \lambda_2 = \mu_0 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0(1 - \lambda_1\mu_2\dot{\gamma}^2);$ $\Psi_1 = 2\lambda_1\eta; \Psi_2 = 0$
Lower Convected Maxwell Model (Eq. (3.23) of [139])	2	1	$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0; \Psi_1 = 2\lambda_1\eta_0;$ $\Psi_2 = -\Psi_1$
Upper Convected Maxwell Model [140]	2	1	$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	$\eta = \eta_0; \Psi_1 = 2\lambda_1\eta_0;$ $\Psi_2 = 0$
Corotational Maxwell Model [6]	2	∞	$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	$\eta(\dot{\gamma}); \Psi_1(\dot{\gamma});$ $\Psi_2 = -\frac{1}{2}\Psi_1$

Remarks: *corrects eighth row in TABLE 7.3-2 of [29]; † $\eta_0 \equiv (1-\xi)\eta_p + \eta_s$

**corrects last row in TABLE 7.3-2 of [29]; ‡ $\eta_0 \equiv \eta_p + \eta_s$

Figure 1: Liquid motion (Cartesian coordinates) in sliding-plate rheometer with oscillating upper plate. Particle trajectories, **red** for $0 < \omega t < \pi/2$, **blue** for $\pi < \omega t < 3\pi/2$ [Eq. (1)]. Newtonian velocity profile is $v_x/V_0 = (y/h)\cos\omega t$.

Figure 2: Diagram for LAOS illustrating the complex, dimensionless generalized non-Newtonianness, G_n . Newtonian behavior is observed well inside the **red** unit circle, where $|G_n| \ll 1$. The magnitude of G_n thus reflects how far the system departs from Newtonian behavior and the inclination $\hat{\phi}$ identifies the type of departure.

Figure 3: Dimensionless shear stress *versus* Weissenberg number with curves of constant $\sigma = \frac{1}{100}, \frac{1}{9}, \frac{1}{4}, 1$ from bottom to top [Eq. (29)]. **Blue** curve (second from the bottom) shows the critical value of $\sigma = 1/9$ for shear-stress monotonicity [Eq. (31)].

Figure 4: Illustration of consistency between SAOS predictions of Eq. (60) (**red**) [$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$] and Eq. (36) (**black**) [$\lambda_2/\lambda_1 = 0$]. Both use $Wi = \frac{1}{100}$, $De = 1$.

Figure 5: Illustration of consistency between our new exact solution for Oldroyd 8-constant framework (**black**) [Eq. (60) with $\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$] and the one from corotational Maxwell model (**red**) (Eq. (81); see also Figure 18 of [9]) for steady shear flow at very low Deborah number [$Wi = \frac{1}{2}$, $De = \frac{1}{100}$].

Figure 6: Comparison of analytical exact solution [Eq. (74) (**black**)] and our previous work [Eq. (66) of [9] (**red**)] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Maxwell fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 0$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 7: Comparison of analytical exact solution [Eq. (82) in **black**] and Goddard integral expansion solution [Eq. (86) in **red**] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/27$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 8: Comparison of analytical exact solution [Eq. (82) in **black**] and Goddard integral expansion solution [Eq. (86) in **red**] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/9$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 9: Comparison of analytical exact solution [Eq. (82) in **black**] and Goddard integral expansion solution [Eq. (86) in **red**] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 1/3$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 10: Comparison of analytical exact solution [Eq. (82) in **black**] and finite difference solution (**red**) for shear stress [solving Eqs. (41)–(43) with Eq. (87)] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 1/3$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 11: Comparison of analytical exact solution [Eq. (82) in **black**] and finite difference solution (**red**) for shear stress [solving Eqs. (41)–(43) with Eq. (87)] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 1/9$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$)..

Figure 12: Comparison of analytical exact solution [Eq. (82) in **black**] and finite difference solution (**red**) for shear stress [solving Eqs. (41)–(43) with Eq. (87)] for loops of minus dimensionless shear stress, *versus* dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 1/27$, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

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" $Z^* = x + iy$ " should be " $Z^* = x - iy$."; The first term on RHS of Eq. (23) " γ_2 " should be " γ^2 "; The expression for $F_1^{(4)}$ in Eq. (56), " r^{-4} " should be " $r - 4$ ".

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