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EXACT-SOLUTION FOR CONE-PLATE VISCOMETRY

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ABSTRACT

The viscosity of a Newtonian fluid is often measured by confining the fluid to the gap between a rotating cone that is perpendicular to a fixed disk. We call this experiment *cone-plate viscometry*. When the cone angle approaches $\pi / 2$, the viscometer gap is called *narrow*. The shear stress in the fluid, throughout a narrow gap, hardly departs from the shear stress exerted on the plate, and we thus call cone-plate flow *nearly homogeneous*. In this paper, we derive an exact solution for this slight heterogeneity, and from this we derive the correction factors for the shear rate on the cone and plate, for the torque, and thus, for the measured Newtonian viscosity. These factors thus allow the cone-plate viscometer to be used more accurately, and with cone-angles well below $\pi / 2$. We find cone-plate flow field heterogeneity to be far slighter than previously thought. We next use our exact solution for the velocity to arrive at the exact solution for the temperature rise, due to viscous dissipation, in cone-plate flow subject to isothermal boundaries. Since Newtonian viscosity is a strong function of temperature, we expect our new exact solution for the temperature rise be useful to those measuring Newtonian viscosity, and especially so, to those using wide gaps. We include two worked examples to teach practitioners how to use our main results.

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CONTENTS

I. INTRODUCTION.....	4
II. METHOD.....	5
a. Physical Intuition.....	5
b. Equation of Continuity	6
c. Equation of Motion.....	6
d. Eliminate stresses.....	7
III. NARROW GAP	7
IV. ARBITRARY GAP	9
a. Velocity Profile.....	9
b. Shear Rate Profile	10
c. Shear Rate Variation.....	12
d. Shear Rate Correction	12
e. Torque Correction	13
f. Viscosity Correction	13
g. Worked Example: Viscosity Correction.....	14
V. VISCOUS HEATING.....	14
a. Physical intuition.....	14
b. Equation of energy.....	15
c. Fluxes and Viscous Dissipation.....	15
d. Non-Dimensionalizing.....	17
e. Temperature Profile	18
f. Maximum Temperature Rise.....	18
g. Worked Example: Temperature Rise.....	19
VI. CONCLUSION	20
VII. APPENDIX: FLUID INERTIA.....	21
a. Narrow Gap	21
b. Arbitrary Gap	23
VIII. ACKNOWLEDGMENT	24
IX. REFERENCES	39

FIGURES

Figure 1: Physical intuition about the velocity profile in cone-plate flow.....	29
Figure 2: Velocity profile, $v_\phi(r, \theta) / \Omega r$ versus θ , for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash). [calculated from Eq. (56) with Eq. (3)].....	30
Figure 3: Shear rate profile, $\dot{\gamma}_{\theta\phi} / \dot{\gamma}_0$ versus θ [calculated from Eq. (58)].	31
Figure 4: Shear rate at the plate, $\dot{\gamma}_0 / \Omega$ versus gap angle, θ_0 [calculated from Eq. (59) with Eq. (3)].	32
Figure 5: Correction factor for the shear rate on the plate, f_γ , versus gap angle, θ_0 [calculated using Eq. (68) with Eq. (3)].	33
Figure 6: Correction factor for the torque on the plate, $f_\mathcal{T}$, versus gap angle, θ_0 [calculated using Eq. (73) with Eq. (3)].	34
Figure 7: Correction factor for the viscosity deduced from the measured torque on the plate, $f_\mathcal{T} / f_\gamma$, versus gap angle, θ_0 [calculated using Eq. (76) with Eq. (3)].	35
Figure 8: Physical intuition about the temperature rise in cone-plate flow.	36
Figure 9: Dimensional temperature rise profiles illustrating effect of viscous dissipation for the fluid in cone-plate flow (Worked Example of Subsection V.g) using Eq. (122) for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash).	37
Figure 10: Dimensionless temperature rise profiles illustrating effect of viscous dissipation for the fluid in cone-plate flow (Worked Example of Subsection V.g). Calculated from Eq. (113) along with Eqs. (116) and (117) for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash).	38

TABLES

Table I: Dimensional Variables	25
Table II: Dimensionless Variables and Groups.....	26
Table III: Variation, Errors and Corrections Factors versus Gap Angle	27
Table IV: Angle of maximum temperature rise for different cone-angles for Worked Example of Subsection V.g.....	28

I. INTRODUCTION

Weissenberg discovered that cone-plate flow can be nearly perfectly homogeneous, and this thus places it amongst the best geometries for viscometry. By *cone-plate flow*, we mean the deformation generated by confining a fluid between a shallow rotating cone and a fixed perpendicular disk, said cone and disk having identical radii, R , with cone apex and disk coplanar. The standard homogeneous flow field approximation for the velocity in this cone-plate gap is (Eq. (2B.11-1) of [1]):

$$\frac{v_\phi}{r\Omega} \equiv \frac{\pi - \theta}{2\theta_0} \quad (1)$$

and thus, for the shear rate [2]:

$$\dot{\gamma}_{\theta\phi} \equiv \frac{-2\Omega}{\pi - 2\theta_c} \equiv \frac{-\Omega}{\theta_0} \quad (2)$$

where the included angle θ_0 is called the *gap angle*, and θ_c , the *cone angle*, and thus:

$$\theta_0 \equiv \frac{\pi - 2\theta_c}{2} = \frac{\pi}{2} - \theta_c \quad (3)$$

Textbook treatments of this problem (EXAMPLES 1.2-5 and 4.5-1 of [3]; EXAMPLES 1.3-4 and 10.2-1 of [4]; Problem 2B.11 of [1]; Eq. (5.4.13) of [5]) arrive at Eq. (2) with the reasonable approximation:

$$\theta_0 \approx \sin\theta_0 \quad (4)$$

In this paper, we first focus on departures from Eq. (2), that is, on the flow field heterogeneity that arises when the equation of motion is solved exactly for cone-plate flow (Sections II, III and IV).

Previous work, in spherical coordinates, on the velocity field in cone-plate flow focused on the variation of the shear rate across the gap and the error in using the approximation (Table 2 of [6]; Table 4.1ff of [7]), yielding the Adams-Lodge equation (Eq. (A17) of [6]):

$$\frac{\dot{\gamma}_{\theta\phi}(\theta, \theta_c)}{\dot{\gamma}_0(\theta_c)} \equiv \csc^2\theta \equiv \frac{1}{\sin^2\theta} \quad (5)$$

where the shear rate in the fluid touching the plate is given approximately by:

$$\dot{\gamma}_0 \equiv \dot{\gamma}_{\theta\phi}(\pi/2, \theta_c) \equiv \frac{-\Omega}{\theta_0 + A\theta_0^3 - \dots} \quad (6)$$

where:

$$3 < A < 4; \theta_0 < 10^\circ \quad (7)$$

Eq. (5) implies that the flow field heterogeneity causes the shear rate away from the plate to *exceed* the shear rate at the plate:

$$\frac{\dot{\gamma}_{\theta\phi}}{\dot{\gamma}_0} \geq 1 \quad (8)$$

up to a maximum departure from $\dot{\gamma}_0$ at the cone:

$$\frac{\dot{\gamma}_c}{\dot{\gamma}_0} \equiv \frac{\dot{\gamma}(\theta_c, \theta_c)}{\dot{\gamma}_0} \equiv \csc^2 \theta_c \equiv \frac{1}{\sin^2 \theta_c} \quad (9)$$

Previous work, in spherical coordinates, on the velocity field in cone-plate flow has also focused on the variation of the shear stress (Eq. (9.56) of [8]):

$$\frac{\tau_{\theta\phi}(\theta_c) - \tau_{\theta\phi}(\frac{\pi}{2})}{\tau_{\theta\phi}(\frac{\pi}{2})} = \tan^2(\frac{\pi}{2} - \theta_c) = \cot^2 \theta_c \quad (10)$$

which is deducible from, and thus consistent with, Eq. (5).

In this paper, we next turn our attention to the corresponding temperature rise in the fluid that is caused by viscous heating (Section V). Previous treatments of this problem adapt solutions for sliding plate flow in Cartesian coordinates to cone-plate flow by identifying the proximity to the fixed plate in sliding plate flow with the arc length at the rim of cone-plate flow $R \sin(\frac{\pi}{2} - \theta)$ ([9]; Problem 10C.3 of [1]; EXAMPLE 4.4-4 of [4]; [10]). We call this simplification *neglecting curvature*. For isothermal boundaries, neglecting curvature yields the following *Cartesian approximation* (Eq. (4.5-9), Problem 4A.3 and §5.4 of [3]; Eq. (10.2-8) of [4]; [11]):

$$(T_{\max} - T_0) \approx \frac{3\mathcal{T}\Omega\theta_0}{16\pi kR} \quad (11)$$

Neglecting curvature has allowed exploration of the temperature dependencies of the fluid viscosity and of thermal conductivity [11], but it precludes exploring heterogeneity in cone-plate flow, hence the subject of this paper. To explore this heterogeneity, we use spherical coordinates to arrive at an exact solution for the velocity field, and thus for the shear rate profile through the cone-plate gap. We then use this velocity field to arrive at an exact solution for the temperature rise caused by viscous heating in the fluid deforming in the cone-plate gap (between isothermal boundaries).

In Section IV of this paper, we derive exact expressions for the velocity field, $v_\phi(\theta)$, and then use this to get $\dot{\gamma}_c$ and $\dot{\gamma}_0$. We next use this result for $\dot{\gamma}_0$ to solve exactly for A in Eq. (7). In Section V, we reuse $v_\phi(\theta)$ to explore the temperature rise in the fluid between an isothermal cone and plate.

II. METHOD

a. Physical Intuition

Figure 1 illustrates our physical intuition about the velocity profile in cone-plate flow, where:

$$v_\theta = v_r = 0 \quad (12)$$

and with no slip on the plate:

$$v_\phi\left(r, \frac{\pi}{2}\right) = 0 \quad (13)$$

or on the cone:

$$v_\phi(r, \theta_c) = \Omega r \sin \theta_c \quad (14)$$

which is consistent with the solid body rotation of a cone. Thus:

$$\underline{v} = [0, 0, v_\phi(r, \theta)] \quad (15)$$

where $v_\phi(r, \theta)/\Omega > 0$. For non-Newtonian fluids, departures from Eq. (12), called *secondary flows*, can arise [12,13,14].

b. Equation of Continuity

We begin with the equation of continuity in spherical coordinates (Eq. (B.4-3) of [1]):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \quad (16)$$

Using Eq. (12), we simplify Eq. (16) to get:

$$\frac{dv_\phi}{d\phi} = 0 \quad (17)$$

which confirms our physical intuition of the ϕ independence of v_ϕ in Eq. (15).

c. Equation of Motion

We start with the ϕ -component of equation of motion in spherical coordinates (Eq. (B.5-9) of [1]):

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) = \frac{-1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \left[\frac{1}{r^3} \frac{\partial}{\partial r}(r^3 \tau_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\tau_{\theta\phi} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{\phi\theta} \cot \theta}{r} \right] + \rho g_\phi \quad (18)$$

Applying Eqs. (12) and (17) along with:

$$g_\phi = 0 \quad (19)$$

we get:

$$0 = \frac{1}{r^3} \frac{\partial}{\partial r}(r^3 \tau_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\tau_{\theta\phi} \sin \theta) + \frac{\tau_{\phi\theta} \cot \theta}{r} \quad (20)$$

where:

$$\dot{\gamma}_{\theta\phi} = \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) \quad (21)$$

$$\dot{\gamma}_{r\phi} = r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right)$$

from which we learn that neither $\tau_{\theta\phi}$ nor $\tau_{r\phi}$ are zero. Eqs. (20) and (21) apply to all fluids. By comparing Eq. (18) with (20) we learn that, without neglecting it, fluid inertia [the left side of Eq. (18)] vanished. The only effect of fluid inertia in cone-plate flow, is to lower the pressure in the deforming liquid below its atmospheric value at the sample rim (see APPENDIX: FLUID INERTIA).

d. Eliminate stresses

For a *Newtonian fluid*, from Eqs. (21) and Appendix B of [1], we get:

$$\tau_{\theta\phi} = -\mu \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \quad (22)$$

$$\tau_{r\phi} = -\mu r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \quad (23)$$

and substituting these into Eq. (20), gives:

$$0 = \frac{1}{r^3} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) r^3 \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \sin\theta \right) + \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \frac{\cot\theta}{r} \quad (24)$$

from which we learn that, since μ dropped out, the flow is *viscometric*.

Simplifying further yields:

$$0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^4 \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\sin^2\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \right) + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \quad (25)$$

Eq. (14) suggests that:

$$v_\phi(r, \theta) = \Omega r \sin\theta_c f(\theta) \quad (26)$$

where:

$$f\left(\frac{\pi}{2}\right) = 0; f(\theta_c) = 1 \quad (27)$$

and substituting Eq. (26) into Eq. (25), gives:

$$0 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin^2\theta \frac{\partial}{\partial\theta} \left(\frac{f}{\sin\theta} \right) \right) + \cos\theta \frac{\partial}{\partial\theta} \left(\frac{f}{\sin\theta} \right) \quad (28)$$

or:

$$0 = \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin^2\theta \frac{dF}{d\theta} \right) + \cos\theta \frac{dF}{d\theta} \quad (29)$$

where:

$$F \equiv \frac{f}{\sin\theta} \quad (30)$$

Simplifying further yields a simple second order differential equation:

$$0 = 3\cos\theta \frac{dF}{d\theta} + \sin\theta \frac{d^2F}{d\theta^2} \quad (31)$$

for Newtonian fluids.

III. NARROW GAP

For the special case where the cone-angle, θ_c , approaches $\frac{\pi}{2}$ and the gap becomes narrow, Eq. (31) simplifies to:

$$\frac{d^2F}{d\theta^2} = 0 \quad (32)$$

which has the general solution:

$$F = C_1\theta + C_2 \quad (33)$$

Substituting Eq. (30) into Eq. (33), we find:

$$\frac{f}{\sin\theta} = C_1\theta + C_2 \quad (34)$$

which is subject to Eqs. (27), so that:

$$0 = \frac{\pi}{2}C_1 + C_2 \quad (35)$$

and

$$\frac{1}{\sin\theta_c} = C_1\theta_c + C_2 \quad (36)$$

We solve Eqs. (35) and (36) simultaneously to obtain:

$$C_1 = \frac{1}{\left(\theta_c - \frac{\pi}{2}\right)\sin\theta_c} \quad (37)$$

$$C_2 = -\frac{\pi}{2}C_1 = -\frac{\pi}{2} \frac{1}{\left(\theta_c - \frac{\pi}{2}\right)\sin\theta_c} = -\frac{\pi}{(2\theta_c - \pi)\sin\theta_c} \quad (38)$$

Thus:

$$\frac{f}{\sin\theta} = \frac{2\theta - \pi}{(2\theta_c - \pi)\sin\theta_c} \quad (39)$$

and:

$$f = \frac{2\theta - \pi}{2\theta_c - \pi} \frac{\sin\theta}{\sin\theta_c} \quad (40)$$

which when substituted into Eq. (26), gives:

$$v_\phi(r, \theta) = \Omega r \left[\frac{2\theta - \pi}{2\theta_c - \pi} \right] \sin\theta \quad (41)$$

so that, from Eq. (21)(a):

$$\dot{\gamma}_{\theta\phi} \cong \frac{2\Omega}{2\theta_c - \pi} = \frac{-\Omega}{\theta_0} \quad (42)$$

which matches Eq. (5-15) of [15] (or Eq. (B-5) of Table 4.5-1 and Eqs. (1.2-56) and (4.5-1) of [3], or Eq. (1.2-56) of [3], or Eq. (B-5) of Table 10.2-1 and Eq. (10.2-1) of [4]) from which we learn that the flow field is nearly homogeneous, and which implies the constant shear stress:

$$\begin{aligned} \tau_{\theta\phi} &= -\mu\dot{\gamma}_{\theta\phi} \\ &\cong -\frac{2\Omega\mu}{2\theta_c - \pi} \end{aligned} \quad (43)$$

throughout the gap (Eq. 2B.11-2) of [1]), and from which we can calculate the torque exerted by the fluid on the plate, using Fig. A.8-2 of [1]:

$$\begin{aligned}\mathcal{T} &= \int_0^{2\pi} \int_0^R r \tau_{\theta\phi} dr r \sin\theta d\phi \\ &= \int_0^{2\pi} \int_0^R r^2 \tau_{\theta\phi} dr d\phi\end{aligned}\tag{44}$$

and since the shear stress is nearly constant and given by Eq. (43), the torque exerted by the deforming fluid on the plate is given by (see Eq. (2B.11-3) of [1]):

$$\mathcal{T} = \frac{-4\pi R^3 \Omega \mu}{3(2\theta_c - \pi)} > 0\tag{45}$$

and thus:

$$\mu = \frac{-3(2\theta_c - \pi)}{4\pi R^3} \frac{\mathcal{T}}{\Omega} > 0\tag{46}$$

which matches Eq. (5-17) of [15]. However, substituting Eq. (41) into Eq. (23), yields:

$$\tau_{r\phi} = 0\tag{47}$$

Eqs. (42), (43) and (47) are thus consistent with *simple shear flow*.

IV. ARBITRARY GAP

We next repeat our analysis from the previous section for a gap of arbitrary angle.

a. Velocity Profile

We begin by solving Eq. (31) without further simplification, which has general solution:

$$F = C_1 \frac{\left(\ln\left(\frac{1 - \cos\theta}{\sin\theta}\right) \cos^2\theta - \ln\left(\frac{1 - \cos\theta}{\sin\theta}\right) + \cos\theta \right)}{\sin^2\theta} + C_2\tag{48}$$

Simplifying Eq. (48), we get:

$$F = C_1^* \left(\ln \frac{1 - \cos\theta}{1 + \cos\theta} - \frac{2\cos\theta}{\sin^2\theta} \right) + C_2\tag{49}$$

which, subject to:

$$f\left(\frac{\pi}{2}\right) = 0; f(\theta_c) = 1\tag{27}$$

implies that:

$$F\left(\frac{\pi}{2}\right) = 0 = C_1^* \ln 1 + C_2\tag{50}$$

and:

$$F(\theta_c) = \frac{1}{\sin\theta_c} = C_1^* \left(\ln \frac{1 - \cos\theta_c}{1 + \cos\theta_c} - \frac{2\cos\theta_c}{\sin^2\theta_c} \right) + C_2\tag{51}$$

Solving Eqs. (50) and (51) simultaneously gives:

$$C_2 = 0\tag{52}$$

and:

$$C_1^* = \frac{1}{\ln\left(\frac{1-\cos\theta_c}{1+\cos\theta_c}\right)\sin\theta_c - \frac{2\cos\theta_c}{\sin\theta_c}} \quad (53)$$

Substituting Eqs. (52) and (53) into Eq. (49), we get:

$$F = \frac{1}{\sin\theta_c} \frac{\ln\frac{1-\cos\theta}{1+\cos\theta} - \frac{2\cos\theta}{\sin^2\theta}}{\ln\frac{1-\cos\theta_c}{1+\cos\theta_c} - \frac{2\cos\theta_c}{\sin^2\theta_c}} \quad (54)$$

which implies:

$$f = \frac{\sin\theta}{\sin\theta_c} \frac{\ln\frac{1-\cos\theta}{1+\cos\theta} - \frac{2\cos\theta}{\sin^2\theta}}{\ln\frac{1-\cos\theta_c}{1+\cos\theta_c} - \frac{2\cos\theta_c}{\sin^2\theta_c}} \quad (55)$$

and thus:

$$\frac{v_\phi(r, \theta)}{\Omega r} = \sin\theta \frac{\ln\frac{1-\cos\theta}{1+\cos\theta} - \frac{2\cos\theta}{\sin^2\theta}}{\ln\frac{1-\cos\theta_c}{1+\cos\theta_c} - \frac{2\cos\theta_c}{\sin^2\theta_c}} \quad (56)$$

which confirms the suggestion in Eq. (26). Figure 2 illustrates this velocity field.

b. Shear Rate Profile

Recall that:

$$\dot{\gamma}_{\theta\phi} = \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} \left(\frac{v_\phi}{\sin\theta} \right) \quad (21)(a)$$

so that, substituting Eq. (56) into Eq. (21)(a) gives:

$$\frac{\dot{\gamma}_{\theta\phi}}{\Omega} = \frac{2\csc^2\theta}{\ln\sqrt{\frac{1-\cos\theta_c}{1+\cos\theta_c}} - \frac{\cos\theta_c}{\sin^2\theta_c}} \quad (57)$$

and thus:

$$\frac{\dot{\gamma}_{\theta\phi}}{\dot{\gamma}_0} = \csc^2\theta \quad (58)$$

where:

$$\frac{\dot{\gamma}_0}{\Omega} = 2 \left(\ln\sqrt{\frac{1-\cos\theta_c}{1+\cos\theta_c}} - \frac{\cos\theta_c}{\sin^2\theta_c} \right)^{-1} \quad (59)$$

Eqs. (58) and (59) are the main results of this work and Figure 3 and Figure 4 illustrate these results. Eq. (58) differs from Eq. (42), and this difference reflects the flow field heterogeneity. Eq. (58) matches the Adam-Lodge equation [Eq. (5) of Section I above] as it should. Eq. (59) is the exact expression for the shear rate in the fluid touching the stationary disk, $\dot{\gamma}_0$, and to our knowledge, this equation is

new. By inspection of Eq. (58) or of Figure 3, we see that the shear rate is highest on the cone, and lowest on the plate.

Expanding the denominator of $\dot{\gamma}_0$ in Eq. (58) with respect to θ_0 , about $\theta_0 = 0$, gives:

$$\dot{\gamma}_0 = \frac{-\Omega}{\theta_0 + \frac{1}{2}\theta_0^3 + \frac{11}{40}\theta_0^5 + \frac{241}{1680}\theta_0^7 + \dots} \quad (60)$$

from which we see that the use of the approximation Eq. (42) corresponds to the dropping all but the θ_0 term in the denominator of Eq. (60).

The Lagrange remainder is *minus* the error introduced by truncating after n terms:

$$R_n = \frac{f^{(n+1)}(x^*)}{(n+1)!} (x - x_0)^{n+1} \quad (61)$$

so that, for example:

$$\begin{aligned} R_1 &= \frac{\frac{d^2}{d\theta_0^2} \left(\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c}} - \frac{\cos \theta_c}{\sin^2 \theta_c} \right)}{2} \theta_0^2 \\ &= \frac{-3\theta_0^2 \sin \theta_0}{\cos^4 \theta_0} \end{aligned} \quad (62)$$

Remainders calculated from Eq. (62) are negative, and we tabulate these in column 4 of Table III, from which we learn, for example, that for $\theta_0 \leq 10^\circ$, a maximum fractional error in the shear rate of -0.00540 is incurred with the approximation Eq. (42).

From Eq. (60) we also learn that the coefficient of θ_0^3 is *one half* which lies outside the Adams-Lodge range given by Eq. (7). Evaluating the remainder in the denominator of Eq. (60) after the θ_0^3 term gives:

$$R_2 = \frac{\theta_0^3}{12} \frac{d^3}{d\theta_0^3} \left(\ln \sqrt{\frac{1 - \sin \theta_0}{1 + \sin \theta_0}} - \frac{\sin \theta_0}{\cos^2 \theta_0} \right) \quad (63)$$

so that $R_2(\pi/18) = -0.00313$ and hence:

$$\frac{1}{2} < A < \frac{1}{2} - R_2\left(\frac{\pi}{18}\right); \theta_0 < 10^\circ \quad (64)$$

or:

$$\frac{1}{2} < A < 0.50313; \theta_0 < 10^\circ \quad (65)$$

which corrects the Adams-Lodge range given by Eq. (7). In other words, the error in the approximation given by Eq. (42) is much less than the oft cited Adams-Lodge conclusion (see Subsection 5.4.2 of [5]), and thus, we find that there is far less flow field heterogeneity in cone-plate flow than once thought.

c. Shear Rate Variation

We can use Eq. (58) [with Eq. (59)] to estimate the fractional variation in shear rate across the gap:

$$v \equiv \frac{\dot{\gamma}_c - \dot{\gamma}_p}{\dot{\gamma}_p} = \frac{\csc^2 \theta_c - \csc^2 \theta_p}{\csc^2 \theta_p} = \frac{\csc^2 \theta_c}{\csc^2 \theta_p} - 1 = \frac{\sin^2 \theta_p}{\sin^2 \theta_c} - 1 = \frac{1}{\sin^2 \theta_c} - 1 \quad (66)$$

Table III gives values of v calculated using Eq. (66) for ten values of θ_0 .

d. Shear Rate Correction

Now, if we divide this $\dot{\gamma}_{\theta\phi}$ by the desired $\dot{\gamma}_{\theta\phi}$ [given by Eq. (42)] and then subtract the result from one, we get the *signed fractional error in the shear rate*:

$$\varepsilon = 1 - \frac{(2\theta_c - \pi)\csc^2 \theta}{\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{\cos \theta_c}{\sin^2 \theta_c}}} \quad (67)$$

which, for a torque transducer mounted on the plate ($\theta = \frac{\pi}{2}$) gives:

$$\varepsilon = 1 - \frac{2\theta_c - \pi}{\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{\cos \theta_c}{\sin^2 \theta_c}}} \quad (68)$$

from which we see that $\varepsilon > 0$. Table III gives values of ε calculated using Eq. (68) for seven values of θ_0 . Column 3 of Table III corrects Columns 3 of Table 2 of [6] and of Table 4.1 of [7]. In other words, to correct the narrow gap approximation given by Eq. (42), we must multiply its right side by

$$f_{\dot{\gamma}} = (1 - \varepsilon) :$$

$$\begin{aligned} \dot{\gamma}_{\theta\phi} &= \frac{\Omega}{\theta_0} f_{\dot{\gamma}} \\ &= \frac{\Omega}{\theta_0} (1 - \varepsilon) \end{aligned} \quad (69)$$

Table III gives values of $f_{\dot{\gamma}}$ calculated using Eq. (68) for seven values of θ_0 . Figure 5 provides a useful graph of the correction factor $f_{\dot{\gamma}}$ for practitioners.

Moving on to the $r\phi$ -component of the extra stress, substituting Eq. (56) into Eq. (21)(b):

$$\dot{\gamma}_{r\phi} = r \frac{\partial}{\partial r} \left(\Omega \sin \theta \frac{\ln \frac{1 - \cos \theta}{1 + \cos \theta} - \frac{2 \cos \theta}{\sin^2 \theta}}{\ln \frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{2 \cos \theta_c}{\sin^2 \theta_c}} \right) = 0 \quad (70)$$

Thus, since $v_{\phi} \propto r$, there is no $r\phi$ component of the rate of deformation tensor, and thus $\tau_{r\phi} = 0$. In other words, whereas widening the gap

compromises the θ – homogeneity of the flow field, it does not compromise its r – homogeneity.

e. Torque Correction

Substituting Eq. (59) into Eq. (58), and the result into Eq. (43)(a), and then evaluating at $\theta = \pi / 2$ gives the shear stress on the plate:

$$\tau_{\theta\phi} = -\mu\dot{\gamma}_{\theta\phi} = -4\mu\Omega \frac{1}{\sin^2\theta_c} \frac{1}{\ln \frac{1-\cos\theta_c}{1+\cos\theta_c} - \frac{2\cos\theta_c}{\sin^2\theta_c}} \quad (71)$$

from which we can calculate the torque on the plate, using Eq. (44):

$$\mathcal{T} = \frac{-4\pi\mu\Omega R^3 \csc^2\theta_c}{3 \left(\ln \sqrt{\frac{1-\cos\theta_c}{1+\cos\theta_c}} - \frac{\cos\theta_c}{\sin^2\theta_c} \right)} \quad (72)$$

If we divide this by our approximate expression for the torque on the plate, Eq. (45), we arrive at the correction factor for the torque on the plate:

$$f_{\mathcal{T}} = \frac{(2\theta_c - \pi) \csc^2\theta_c}{\ln \sqrt{\frac{1-\cos\theta_c}{1+\cos\theta_c}} - \frac{\cos\theta_c}{\sin^2\theta_c}} > 0 \quad (73)$$

In other words, to correct our approximate expression for the torque on the plate, we must multiply the right side of Eq. (45) by $f_{\mathcal{T}}$.

$$\begin{aligned} \mathcal{T} &= \frac{-4\pi R^3 \Omega \mu}{3(2\theta_c - \pi)} f_{\mathcal{T}} \\ &= \frac{-\frac{4}{3}\pi R^3 \Omega \mu \csc^2\theta_c}{\ln \sqrt{\frac{1-\cos\theta_c}{1+\cos\theta_c}} - \frac{\cos\theta_c}{\sin^2\theta_c}} \end{aligned} \quad (74)$$

Table III gives values of $f_{\mathcal{T}}$ calculated using Eq. (73) for seven values of θ_0 . Figure 6 provides a useful graph of the correction factor $f_{\mathcal{T}}$ for practitioners.

f. Viscosity Correction

Eq. (73) gives the correction factor for the torque on the plate, and Eq. (69) [with Eq. (68)], for the shear rate on the plate. We can thus use these results to correct Eq. (46) for the wide gap as follows:

$$\mu \equiv \frac{-\tau_{\theta\phi}}{\dot{\gamma}_{\theta\phi}} = \frac{-3(2\theta_c - \pi) \left(\frac{\mathcal{T}}{\Omega} \right) f_{\mathcal{T}}}{4\pi R^3 \left(\frac{\mathcal{T}}{\Omega} \right) f_{\dot{\gamma}}} \quad (75)$$

where:

$$\frac{f_{\mathcal{T}}}{f_{\dot{\gamma}}} = \frac{\frac{(2\theta_c - \pi) \csc^2 \theta_c}{\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{\cos \theta_c}{\sin^2 \theta_c}}}}{1 - \varepsilon} = \frac{\frac{(2\theta_c - \pi) \csc^2 \theta_c}{\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{\cos \theta_c}{\sin^2 \theta_c}}}}{\frac{2\theta_c - \pi}{\ln \sqrt{\frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{\cos \theta_c}{\sin^2 \theta_c}}}} = \csc^2 \theta_c \quad (76)$$

so that:

$$\mu = \frac{-3(2\theta_c - \pi)}{4\pi R^3} \left(\frac{\mathcal{T}}{\Omega} \right) \csc^2 \theta_c \quad (77)$$

Table III gives values of $f_{\mathcal{T}} / f_{\dot{\gamma}}$ calculated using Eq. (76) for seven values of θ_0 .

Figure 7 provides a useful graph of the correction factor $f_{\mathcal{T}} / f_{\dot{\gamma}}$ for practitioners.

Eq. (77) improves upon the standard method for deducing viscosity from a cone-plate viscometer (see Problem 1A.3 of [3]).

g. Worked Example: Viscosity Correction

An engineer measures the steady shear viscosity of a nearly Newtonian fluid using wide gap cone-plate flow. She rotates the cone having a cone-angle of $\theta_c = \pi/3$ rad and a diameter of 3.00 cm at angular velocity $\Omega = 6.0$ rad/s and records a torque of $\mathcal{T} = 0.100$ Nm. She first uses Eq. (46) to determine the viscosity of her fluid and finds $\mu = 1235$ Pa·s. She next applies the correction factor from Eq. (76), $f_{\mathcal{T}} / f_{\dot{\gamma}} = 1.33$, to get $\mu = 1646$ Pa·s.

The engineer wants to verify her results obtained by the wide gap cone-plate viscometer with a narrow gap viscometer. She thus selects a cone of $\theta_c = 17\pi/36$ rad and holds all other experimental parameters constant. While shearing her fluid, she measures a torque of $\mathcal{T} = 0.794$ Nm. She again applies Eq. (46) to find $\mu = 1634$ Pa·s. She then calculates the correction factor, $f_{\mathcal{T}} / f_{\dot{\gamma}} = 1.01$, from Eq. (76), which yields $\mu = 1650$ Pa·s, which fall just above the value of $\mu = 1646$ Pa·s measured in her wide gap cone-plate flow (see Subsection V.g below).

V. VISCOUS HEATING

a. Physical intuition

Figure 8 illustrates the following physical intuition about the temperature rise in cone-plate flow. We suppose that:

$$T = T(r, \theta) \quad (78)$$

with isothermal walls:

$$T\left(r, \frac{\pi}{2}\right) = T\left(r, \theta_c\right) = T_0 \quad (79)$$

and where $\theta_0 > 0$. We must not suppose mid-gap symmetry [see Eq. (121) below].

b. Equation of energy

We begin with the equation of energy in spherical coordinates (Eq. B.8-3 of [1]):

$$\begin{aligned} \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v}_r \frac{\partial T}{\partial r} + \frac{\underline{v}_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{\underline{v}_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) \\ = - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial q_\phi}{\partial \phi} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_P \frac{Dp}{Dt} - (\tau : \nabla v) \end{aligned} \quad (80)$$

which simplifies to:

$$0 = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \underline{q}_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) \right] + (\tau : \nabla v) \quad (81)$$

and since for a cone-plate viscometer:

$$\theta_0 R \ll R \quad (82)$$

or since:

$$\theta_0 \ll 1 \text{ radian} \quad (83)$$

then:

$$q_r \ll q_\theta \quad (84)$$

so that:

$$0 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) + (\tau : \nabla v) \quad (85)$$

which we will use to develop the temperature profile within the gap in the following sections.

c. Fluxes and Viscous Dissipation

Substituting Eq. B.2-8 of [1]:

$$q_\theta = \frac{-k}{r} \frac{\partial T}{\partial \theta} \quad (86)$$

into Eq. (85) yields:

$$0 = \frac{-k}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial T}{\partial \theta} \sin \theta \right) + (\tau : \nabla v) \quad (87)$$

where:

$$\begin{aligned} (\tau : \nabla v) = \tau_{rr} \left(\frac{\partial \underline{v}_r}{\partial r} \right) + \tau_{\theta\theta} \left(\frac{1}{r} \frac{\partial \underline{v}_\theta}{\partial \theta} + \frac{\underline{v}_r}{r} \right) + \tau_{\phi\phi} \left(\frac{1}{r \sin \theta} \frac{\partial \underline{v}_\phi}{\partial \phi} + \frac{\underline{v}_r}{r} + \frac{\underline{v}_\theta}{r} \cot \theta \right) \\ + \tau_{r\theta} \left(\frac{1}{r} \frac{\partial \underline{v}_r}{\partial \theta} - \frac{\underline{v}_\theta}{r} + \frac{\partial \underline{v}_\theta}{\partial r} \right) + \tau_{r\phi} \left(\frac{1}{r \sin \theta} \frac{\partial \underline{v}_r}{\partial \phi} - \frac{\underline{v}_\phi}{r} + \frac{\partial \underline{v}_\phi}{\partial r} \right) + \tau_{\theta\phi} \left(\frac{1}{r \sin \theta} \frac{\partial \underline{v}_\theta}{\partial \phi} - \frac{\underline{v}_\phi}{r} \cot \theta + \frac{1}{r} \frac{\partial \underline{v}_\phi}{\partial \theta} \right) \end{aligned} \quad (88)$$

We simplify Eq. (88) using our physical intuition from Eq. (15) to get:

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = \tau_{r\phi} \left(-\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r} \right) + \tau_{\theta\phi} \left(-\frac{v_\phi}{r} \cot\theta + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} \right) \quad (89)$$

where for a Newtonian fluid:

$$\tau_{r\phi} = -\mu \left[\frac{1}{r \sin\theta} \frac{\partial v_{\phi r}}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right] = -\mu r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \quad (90)$$

and:

$$\tau_{\theta\phi} = -\mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial v_{\theta\phi}}{\partial \phi} \right] = -\mu \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \quad (91)$$

Replacing the shear stresses in Eq. (89) using Eqs. (90) and (91) yields:

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \left(-\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r} \right) + \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \left(-\frac{v_\phi}{r} \cot\theta + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} \right) \right] \quad (92)$$

Substituting this into Eq. (87), we find:

$$\frac{k}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{\partial T}{\partial \theta} \sin\theta \right) = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \left(-\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r} \right) + \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \left(-\frac{v_\phi}{r} \cot\theta + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} \right) \right] \quad (93)$$

which simplifies to:

$$\frac{k}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{\partial T}{\partial \theta} \sin\theta \right) = -\mu \left[r^2 \left[\frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right]^2 + \frac{\sin^2\theta}{r^2} \left[\frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \right]^2 \right] \quad (94)$$

We are given that:

$$v_\phi(r, \theta) = A\Omega r \sin\theta \left(\ln \frac{1 - \cos\theta}{1 + \cos\theta} - \frac{2\cos\theta}{\sin^2\theta} \right) \quad (95)$$

which implies:

$$\frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) = 0 \quad (96)$$

allowing for further simplification of Eq. (94) to get:

$$\frac{k}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{\partial T}{\partial \theta} \sin\theta \right) = -\mu \left[\frac{\sin^2\theta}{r^2} \left[\frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \right]^2 \right] \quad (97)$$

From Appendix B of [1]:

$$\dot{\gamma}_{\theta\phi} = \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial v_{\theta\phi}}{\partial \phi} = \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \quad (98)$$

which when inserted into Eq. (97) yields:

$$\frac{k}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\frac{\partial T}{\partial \theta} \sin\theta \right) = -\mu \dot{\gamma}_{\theta\phi}^2 \quad (99)$$

Using Eq. (58):

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_0 \csc^2\theta \quad (100)$$

we find:

$$\frac{k}{r^2 \sin \theta} \frac{d}{d\theta} \left(\frac{dT}{d\theta} \sin \theta \right) = -\mu \dot{\gamma}_0^2 \csc^4 \theta = -\frac{\mu \dot{\gamma}_0^2}{\sin^4 \theta} \quad (101)$$

so that the viscous dissipation term is given by:

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = -\frac{\mu \dot{\gamma}_0^2}{\sin^4 \theta} \quad (102)$$

where $\dot{\gamma}_0$ is given by Eq. (59).

d. Non-Dimensionalizing

We rearrange Eq. (101) by grouping the material properties to get:

$$\frac{d}{d\theta} \left(\frac{dT}{d\theta} \sin \theta \right) \equiv \frac{\beta}{\sin^3 \theta}; \beta \equiv \frac{-\mu \dot{\gamma}_0^2 r^2}{k} \quad (103)$$

where $\beta [=] T$. Dividing through by the maximum temperature difference we find:

$$\frac{d}{d\theta} \left(\frac{d \left(\frac{T - T_{\max}}{T_0 - T_{\max}} \right)}{d\theta} \sin \theta \right) = \frac{\beta}{(T_0 - T_{\max}) \sin^3 \theta} \quad (104)$$

or:

$$\frac{d}{d\theta} \left(\frac{d\Theta}{d\theta} \sin \theta \right) \equiv \frac{\xi}{\sin^3 \theta} \quad (105)$$

where:

$$\xi \equiv \frac{\beta}{T_0 - T_{\max}} \equiv \frac{-\mu \dot{\gamma}_0^2 r^2}{k(T_0 - T_{\max})} = \frac{\mu \dot{\gamma}_0^2 r^2}{k(T_{\max} - T_0)} > 0 \quad (106)$$

$$\Theta \equiv \frac{T - T_{\max}}{T_0 - T_{\max}} \quad (107)$$

$$T_{\max} \equiv T(R, \theta_m) \quad (108)$$

and $\theta = \theta_m$ is the *hottest cone*, so that $T = T_{\max}$ and:

$$\left. \frac{\partial T}{\partial \theta} \right|_{\theta=\theta_m} = 0 \quad (109)$$

so that, from (106):

$$\xi_{\max} \equiv \frac{\mu \dot{\gamma}_0^2 R^2}{k(T_{\max} - T_0)} \quad (110)$$

From Eq. (107) we see that, by definition, $\Theta_{\max} \equiv 0$.

Nondimensionalizing the boundary conditions from Eqs. (79), we get:

$$\Theta \left(\frac{r}{R}, \frac{\pi}{2} \right) = \Theta \left(\frac{r}{R}, \theta_c \right) = 1 \quad (111)$$

where Eq. (105) is subject to Eqs. (111).

e. Temperature Profile

Integrating Eq. (105) gives:

$$\frac{d\Theta}{d\theta} = \frac{\xi}{2\sin\theta} \left\{ \frac{\cos\theta}{\cos^2\theta - 1} + \log(\csc\theta - \cot\theta) \right\} + C_1 \quad (112)$$

and again gives:

$$\Theta = \frac{\xi}{4} \left(\frac{1 + \cos^2\theta}{2\sin^2\theta} + \log^2\left(\frac{\sin\theta}{1 + \cos\theta}\right) \right) + C_1 \log\left(\frac{\sin\theta}{1 + \cos\theta}\right) + C_2 \quad (113)$$

which is subject to Eqs. (111) so that:

$$1 = \frac{\xi}{4} \left(\frac{1}{2} + \log^2(1) \right) + C_1 \log(1) + C_2 \quad (114)$$

and:

$$1 = \frac{\xi}{4} \left(\frac{1 + \cos^2\theta_c}{2\sin^2\theta_c} + \log^2\left(\frac{\sin\theta_c}{1 + \cos\theta_c}\right) \right) + C_1 \log\left(\frac{\sin\theta_c}{1 + \cos\theta_c}\right) + C_2 \quad (115)$$

From Eqs. (114) and (115), we learn that whereas $C_2 = C_2(\xi, \theta_c)$, $C_1 = C_1(\theta_c)$.

Solving Eqs. (114) and (115), we get:

$$C_2 = 1 - \frac{\xi}{8} \quad (116)$$

and:

$$C_1 = \frac{-\xi \left(\frac{\cos^2\theta_c}{\sin^2\theta_c} + \log^2\left(\frac{\sin\theta_c}{1 + \cos\theta_c}\right) \right)}{4 \log\left(\frac{\sin\theta_c}{1 + \cos\theta_c}\right)} \quad (117)$$

Eq. (113) [with its constants given by Eqs. (116) and (117)] is the main result of this work, in dimensionless form.

f. Maximum Temperature Rise

To find the angle of maximum temperature rise, we take the derivative of Eq. (113):

$$\frac{\partial\Theta}{\partial\theta} = \frac{\xi}{2\sin^3\theta} \left\{ \sin^2\theta \log\left(\frac{\sin\theta}{1 + \cos\theta}\right) - \cos\theta \right\} + \frac{C_1}{\sin\theta} \quad (118)$$

and set it equal to zero so that:

$$0 = \frac{\xi}{2\sin^3\theta_m} \left\{ \sin^2\theta_m \log\left(\frac{\sin\theta_m}{1 + \cos\theta_m}\right) - \cos\theta_m \right\} + \frac{C_1}{\sin\theta_m} \quad (119)$$

Solving Eq. (119) yields an implicit definition for the hottest cone, θ_m :

$$\frac{\cos\theta_m}{\sin^2\theta_m} = \log\left(\frac{\sin\theta_m}{1 + \cos\theta_m}\right) + 2 \frac{C_1}{\xi} \quad (120)$$

Recalling that the integration constant $C_1(\theta_c)$ is given by Eq. (117), we get:

$$\frac{\cos \theta_m}{\sin^2 \theta_m} = \log \left(\frac{\sin \theta_m}{1 + \cos \theta_m} \right) - \frac{\left(\frac{\cos^2 \theta_c}{\sin^2 \theta_c} + \log^2 \left(\frac{\sin \theta_c}{1 + \cos \theta_c} \right) \right)}{2 \log \left(\frac{\sin \theta_c}{1 + \cos \theta_c} \right)} \quad (121)$$

In the following example, we show how to use Eqs. (121) to get θ_m and then Eq. (113) [with Eqs. (116) and (117)] to predict the steady state temperature rise profile arising in the fluid during a viscosity measurement, $\Theta(\theta)$. We then evaluate $\Theta(\theta_m)$ to get the maximum temperature rise during the viscosity measurement.

g. Worked Example: Temperature Rise

An engineer wants to determine the temperature rise in her fluid of Subsection IV.g above (with thermal conductivity $k = 0.45 \text{ W/m/K}$) undergoing Newtonian viscosity measurements using the wide gap cone-plate viscometer with the experimental conditions of Subsection IV.g above (with walls isothermal at room temperature). Specifically, she seeks the temperature rise profiles in the fluid across the conical gap at the rim. She cares more deeply about the rim, since this is where the maximum temperature rise happens.

She begins by determining the dimensional temperature profile by rearranging Eq. (113), along with Eqs. (116) and (117), to get:

$$T - T_0 = \frac{\beta}{4} \left(\frac{1 + \cos^2 \theta}{2 \sin^2 \theta} + \log^2 \left(\frac{\sin \theta}{1 + \cos \theta} \right) \right) + \frac{-\beta \left(\frac{\cos^2 \theta_c}{\sin^2 \theta_c} + \log^2 \left(\frac{\sin \theta_c}{1 + \cos \theta_c} \right) \right)}{4 \log \left(\frac{\sin \theta_c}{1 + \cos \theta_c} \right)} \log \left(\frac{\sin \theta}{1 + \cos \theta} \right) - \frac{\beta}{8} \quad (122)$$

She calculates β using Eq. (103) and tabulates her results in column 3 of Table IV, where the shear rate on the plate, $\dot{\gamma}_0$, are determined from Eq. (59). Figure 9 illustrates her results for the dimensional temperature rise across gaps of varying angle.

The engineer must next determine precisely the maximum temperature rise. For this, she uses Eq. (121) to determine the angle of the hottest cone within the polymer (column 5 of Table IV) for several gap angles. She finds that the maximum temperature rise occurs near the mid-cone for gap angles below 5° and otherwise, nearer the cone (columns 4 and 5 of Table IV). She then inserts her results into Eq. (122) to find the maximum dimensional temperature rise for each cone-angle (rightmost column of Table IV). This maximum temperature rise increases as the gap narrows because the shear rate increases as the gap narrows. So, the hottest curve in Figure 9 corresponds to the smallest gap angle.

She uses the maximum temperature rise to determine ζ from Eq. (110) for each cone angle (divide column 3 by column 6 in Table IV) and puts this into Eq. (113), along with Eqs. (116) and (117), to get the dimensionless temperature rise profile. Figure 10 illustrates the dimensionless temperature rise profile showing the

shifting of the hottest cone angle toward the cone surface at larger gap angles. The engineer also finds that increasing the parameter, ξ , causes the maximum temperature to increase and the location to shift toward the mid-cone. Because the engineer used a constant angular velocity, Ω , the shear rate increased with decreasing gap angle. Thus, it is important for the practitioner to account for viscous heating when using large shear rates or ξ to avoid viscosity measurements that are lower than they ought to be.

Comparing the two rightmost columns of Table IV, we learn that the Cartesian approximation of Eq. (11) over predicts the temperature rise for wide gap angles. Eq. (11) matches the result for gap angles of 5° and smaller, and thus, it is an appropriate approximation for narrow gap angles.

VI. CONCLUSION

In this paper, we arrive at an exact solution for the velocity profile, and by differentiation, the corresponding shear rate profile, for a Newtonian fluid in cone-plate flow for arbitrarily wide gaps [Eqs. (58) and (59)]. We then use this exact solution to arrive at a second exact solution for the corresponding temperature rise for this same fluid in cone-plate flow with isothermal walls (Eq. (113) [with its constants given by Eqs. (116) and (117)]). Our two worked examples show practitioners how to use our results, one to correct measured viscosity for the flow field heterogeneity, and a second to calculate the temperature rise in cone-plate flow.

From our exact solution for the velocity profile we learn that the shear rate is not symmetric about the midcone, and that the shear rate peaks nearer the cone. From our exact solution for the corresponding temperature rise, we learn that the temperature is not symmetric about the midcone, and that the temperature peaks nearer the cone. This result is reminiscent of viscous dissipation in a pressure-driven flow between two coaxial copical cones, where the maximum is nearer the inner cone [16].

Although our exact solution for the flow field heterogeneity has been derived for Newtonian fluids, Adams and Lodge [6] argue “When the variation of shear rate across the gap is small, it is legitimate, in order to estimate the magnitude of this variation, to neglect the dependence of viscosity on shear rate.” To be sure of this, we would repeat the analysis by replacing μ in Eqs. (22) and (23) with a suitable non-Newtonian steady shear viscosity function as has been done for shear heating in sliding plate flow (see Subsection 5.4.6. of [5]; [9]).

We expect that our results will be equally useful to those using truncated cone and plate flow (see Table 10.2-1 and Problem 10B.7 of [3]; Table 10-2-1, Problem 10B.4 and 10B.7 of [4]). Our results can also be used to estimate end corrections in concentric cylinder viscometers with conical bottoms, since these bottoms generate cone-plate flow at the bottom (see Problem 4B.4 of [3]).

VII. APPENDIX: FLUID INERTIA

The uninitiated may expect fluid inertia to tend to throw fluid radially from the gap, to cause $v_r > 0$, which would in turn violate our physical intuition about the velocity, Eq. (12). In this appendix we learn otherwise, by checking Eq. (12) for consistency with the following r -component of the equation of motion:

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = \frac{-\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] + \rho g_r \quad (1)$$

in which $-\rho v_\phi^2/r$ is the centrifugal force term (see Problem 8C.1 of [1]; see also Problem 10B.3 of [4]). Applying Eq. (12), to Eq. (1) and since:

$$\frac{-g_r}{g} = \tan \left(\frac{\pi}{2} - \theta \right) \ll 1 \quad (2)$$

we get:

$$\frac{-\rho v_\phi^2}{r} = \frac{-\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] \quad (3)$$

and using Eq. (23):

$$\frac{-\rho v_\phi^2}{r} = \frac{-\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] \quad (4)$$

where:

$$\begin{aligned} \tau_{rr} &= -2\mu \frac{\partial v_r}{\partial r} = 0 \\ \tau_{\theta\theta} &= -2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) = 0 \\ \tau_{\phi\phi} &= -2\mu \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) = 0 \\ \tau_{r\theta} &= -\mu r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} = 0 \end{aligned} \quad (5)$$

so that:

$$\frac{\rho v_\phi^2}{r} = \frac{\partial p}{\partial r} \quad (6)$$

from which we learn that, for a Newtonian fluid, the only consequence of fluid inertia in steady cone-plate flow is the development of a positive pressure gradient within the fluid.

a. Narrow Gap

Substituting Eq. (41) into Eq. (6):

$$\frac{\rho}{r} \left(\Omega r \left[\frac{2\theta - \pi}{2\theta_c - \pi} \right] \sin\theta \right)^2 = \frac{\partial p}{\partial r} \quad (7)$$

and since $\theta \cong \theta_c$ and since $\sin\theta \cong 1$:

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r \quad (8)$$

so that:

$$p = \frac{\rho \Omega^2 r^2}{2} + f(\theta) \quad (9)$$

where f is at most a function of θ . At the outer edge:

$$p_a = \frac{\rho \Omega^2 R^2}{2} + f(\theta) \quad (10)$$

so that:

$$\frac{p - p_a}{\rho \Omega^2 R^2} = \frac{1}{2} \left[\left(\frac{r}{R} \right)^2 - 1 \right] < 0 \quad (11)$$

from which we learn that fluid inertia lowers the fluid pressure at the cone apex by:

$$\frac{p_0 - p_a}{\rho \Omega^2 R^2} = -\frac{1}{2} < 0 \quad (12)$$

See also EXAMPLE 4.5-1 and Problem 4B.3 of [3]. More importantly, from this appendix we learn that our Eq. (12), the physical intuition upon which this paper relies, is in keeping with the centrifugal force term of the equation of motion, and it should be.

For completeness, we next examine the θ – component of the equation of motion:

$$\begin{aligned} \rho \left(\frac{\partial \mathbf{v}_\theta}{\partial t} + \mathbf{v}_r \frac{\partial \mathbf{v}_\theta}{\partial r} + \frac{\mathbf{v}_\theta}{r} \frac{\partial \mathbf{v}_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial \mathbf{v}_\theta}{\partial \phi} - \frac{v_r \mathbf{v}_\theta - v_\phi^2 \cot\theta}{r} \right) = \frac{-1}{r} \frac{\partial p}{\partial \theta} \\ - \left[\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\theta}) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial \tau_{\phi\theta}}{\partial \phi} - \frac{\tau_{\phi\phi} \cot\theta}{r} \right] + \rho g_\theta \end{aligned} \quad (13)$$

Using Eq. (12) and Eq. (5) and:

$$\frac{g_\theta}{g} = \sin\left(\frac{\pi}{2} - \theta\right) \ll 1 \quad (14)$$

gives:

$$\rho \left(\frac{v_\phi^2 \cot\theta}{r} \right) = \frac{-1}{r} \frac{\partial p}{\partial \theta} - \frac{1}{r \sin\theta} \frac{\partial \tau_{\phi\theta}}{\partial \phi} \quad (15)$$

and next using Appendix B of [1]:

$$\tau_{\theta\phi} = -\mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial \mathbf{v}_\theta}{\partial \phi} \right] = -\mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) \right] \quad (16)$$

we find:

$$\rho v_\phi^2 \cot \theta = -\frac{\partial p}{\partial \theta} + \frac{\mu}{r} \frac{\partial}{\partial \phi} \left[\frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) \right] \quad (17)$$

and also substituting Eq. (41), we get:

$$\rho \left(\Omega r \left[\frac{2\theta - \pi}{2\theta_c - \pi} \right] \sin \theta \right)^2 \cot \theta = -\frac{\partial p}{\partial \theta} + \frac{\mu}{r} \frac{\partial}{\partial \phi} \left[\frac{\partial}{\partial \theta} \left(\Omega r \left[\frac{2\theta - \pi}{2\theta_c - \pi} \right] \right) \right] \quad (18)$$

$$\rho \Omega r \left[\frac{2\theta - \pi}{2\theta_c - \pi} \right] \sin \theta \cos \theta = -\frac{\partial p}{\partial \theta} \quad (19)$$

and since $\cos \theta \cong 0$:

$$\frac{\partial p}{\partial \theta} = 0 \quad (20)$$

and so, f is a constant, called the gauge pressure. When we assign the absolute atmospheric pressure to the fluid at the rim, $p(R) = p_a$, we then get $f = 0$.

b. Arbitrary Gap

Substituting Eq. (56) into Eq. (6):

$$\frac{\partial p}{\partial r} = r \rho \Omega^2 \left(\sin \theta \frac{\ln \frac{1 - \cos \theta}{1 + \cos \theta} - \frac{2 \cos \theta}{\sin^2 \theta}}{\ln \frac{1 - \cos \theta_c}{1 + \cos \theta_c} - \frac{2 \cos \theta_c}{\sin^2 \theta_c}} \right)^2 \equiv \alpha r \quad (21)$$

where $\alpha [=] p / R^2$. By contrast, from Eq. (8) we see that for a narrow gap, where $\theta \cong \theta_c \cong \pi / 2$, $\alpha = \rho \Omega^2$. Hence, the pressure profile is parabolic:

$$p = \frac{\alpha}{2} r^2 + C \quad (22)$$

where:

$$p_a = \frac{\alpha}{2} R^2 + C \quad (23)$$

and:

$$p - p_a = \frac{\alpha}{2} (r^2 - R^2) \quad (24)$$

so that the apex pressure is given by:

$$p(0) = p_a - \frac{\alpha}{2} R^2 \quad (25)$$

which is less than p_a . Making Eq. (24) dimensionless:

$$\frac{p - p_a}{\alpha R^2} = \frac{1}{2} \left[\left(\frac{r}{R} \right)^2 - 1 \right] \quad (26)$$

where α is defined by Eq. (21).

VIII. ACKNOWLEDGMENT

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Table I: Dimensional Variables

	Symbol	Dimensions	Range
Angular rate of rotation, cone	Ω	$\text{rad}t^{-1}$	$(-\infty, \infty)$
Cone angle	θ_c	t^{-1}	$(0, \pi/2)$
Cone and plate radii	R	L	$(0, \infty)$
Expansion coefficient, Eq. (6)	A	rad^{-2}	$(0, \infty)$
Extra stress, ij th component	τ_{ij}	$ML^{-1}t^{-2}$	$(-\infty, \infty)$
Gap angle	θ_0	t^{-1}	$(0, \pi/2)$
Gravity, acceleration due to, i th component	g_i	Lt^{-2}	$(0, \infty)$
Heat capacity, constant pressure	$\hat{C}_p$	$L^2t^{-2}T^{-1}$	$(0, \infty)$
Integration constant, Eq. (37)	C_1	rad^{-1}	
Integration constant, Eq. (38)	C_2	1	
Mass density, fluid	ρ	ML^{-3}	$(0, \infty)$
Rate of deformation, ij th component	$\dot{\gamma}_{ij}$	t^{-1}	$(-\infty, \infty)$
Rate of deformation, at cone	$\dot{\gamma}_c$	t^{-1}	$(-\infty, \infty)$
Rate of deformation, at plate	$\dot{\gamma}_0$	t^{-1}	$(-\infty, \infty)$
Spherical radial coordinate	r	L	$(0, \infty)$
Spherical latitudinal coordinate	θ	rad	$(0, \pi)$
Spherical longitudinal coordinate	ϕ	rad	$(0, 2\pi)$
Temperature, absolute	T	T	$(0, \infty)$
Temperature, absolute isothermal wall	T_0	T	$(0, \infty)$
Torque, exerted by fluid on plate	$\mathcal{T}$	ML^2t^{-2}	$(-\infty, \infty)$
Thermal conductivity, fluid	k	$MLt^{-3}T^{-1}$	$(0, \infty)$
Velocity, i th component	v_i	Lt^{-1}	$(0, \infty)$
Viscosity, Newtonian	μ	$ML^{-1}t^{-1}$	$(0, \infty)$

Table II: Dimensionless Variables and Groups

Name	Symbol	Range
Correction factor, shear rate	$f_{\dot{\gamma}}$	$(0, \infty)$
Correction factor, torque	$f_{\mathcal{T}}$	$(0, \infty)$
Correction factor, viscosity	f_{μ}	$(0, \infty)$
Fractional error, shear rate, Eq. (68)	ε	$(-\infty, \infty)$
Function of θ , Eqs. (26) and (40)	f	$(0, \infty)$
Function of θ , Eq. (30) and (54)	F	$(0, \infty)$
Lagrange remainder, n th, Eq. (61)	R_n	$(-\infty, \infty)$
Variation, shear rate, Eq. (66)	v	$(-\infty, \infty)$

Legend: $\mathbb{R} \equiv$ all real numbers

Table III: Variation, Errors and Corrections Factors *versus* Gap Angle

$\theta_0(^{\circ})$	v	ε	$-R_1$	$f_{\dot{\gamma}}$	$f_{\mathcal{T}}$	$f_{\mathcal{T}}/f_{\dot{\gamma}}$
	Eq. (66)	Eq. (68)	Eq. (62)	$(1-\varepsilon)$	Eq. (73)	
0.5	0.000076	0.000038	0.000002	0.999962	1.000038	1.00008
1	0.000305	0.000152	0.000016	0.999848	1.000152	1.00030
2	0.001219	0.000609	0.000128	0.999391	1.000609	1.00122
3	0.002747	0.001371	0.000432	0.998629	1.001372	1.00275
4	0.004890	0.002438	0.001030	0.997562	1.002440	1.00489
5	0.007654	0.003809	0.002022	0.996191	1.003816	1.00765
7	0.015076	0.007469	0.005623	0.992531	1.007495	1.01508
10	0.031091	0.015254	0.016871	0.984746	1.015363	1.03109
15	0.071797	0.034385	0.061134	0.965615	1.034943	1.07180
30	0.333333	0.138799	0.731082	0.861201	1.148269	1.33333

Table IV: Angle of maximum temperature rise for different cone-angles for Worked Example of Subsection V.g

$\theta_c(^{\circ})$	$\theta_0(^{\circ})$	$\beta \text{ (K)}$	$(\theta_m - \theta_c)/(\frac{\pi}{2} - \theta_c)$	$\theta_m(^{\circ})$	$T_{\max} - T_0 \text{ (K)}$	$T_{\max} - T_0 \text{ (K)}$
		Eq. (103)		Eq. (121)	Eq. (11)	Eq. (122)
30	60	-5.20	0.347	50.8	3.70	1.89
55	35	-52.3	0.457	71.0	3.70	3.16
75	15	-404	0.493	82.4	3.70	3.62
85	5	-3870	0.499	87.5	3.70	3.70

Figure 1: Physical intuition about the velocity profile in cone-plate flow.

Figure 2: Velocity profile, $v_\phi(r, \theta) / \Omega r$ versus θ , for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash). [calculated from Eq. (56) with Eq. (3)].

Figure 3: Shear rate profile, $\dot{\gamma}_{\theta\phi} / \dot{\gamma}_0$ versus θ [calculated from Eq. (58)].

Figure 4: Shear rate at the plate, $\dot{\gamma}_0 / \Omega$ *versus* gap angle, θ_0 [calculated from Eq. (59) with Eq. (3)].

Figure 5: Correction factor for the shear rate on the plate, $f_{\dot{\gamma}}$, versus gap angle, θ_0 [calculated using Eq. (68) with Eq. (3)].

Figure 6: Correction factor for the torque on the plate, $f_{\mathcal{T}}$, versus gap angle, θ_0 [calculated using Eq. (73) with Eq. (3)].

Figure 7: Correction factor for the viscosity deduced from the measured torque on the plate, $f_{\mathcal{T}} / f_{\dot{\gamma}}$, versus gap angle, θ_0 [calculated using Eq. (76) with Eq. (3)].

Figure 8: Physical intuition about the temperature rise in cone-plate flow.

Figure 9: Dimensional temperature rise profiles illustrating effect of viscous dissipation for the fluid in cone-plate flow (Worked Example of Subsection V.g) using Eq. (122) for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash).

Figure 10: Dimensionless temperature rise profiles illustrating effect of viscous dissipation for the fluid in cone-plate flow (Worked Example of Subsection V.g). Calculated from Eq. (113) along with Eqs. (116) and (117) for various cone angles, $\theta_c = 30^\circ$ (solid), 55° (dashed), 75° (dotted), 85° (dot-dash).

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