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EXACT SOLUTIONS FOR OSCILLATORY SHEAR SWEEP BEHAVIORS OF COMPLEX FLUIDS FROM THE OLDROYD 8-CONSTANT FRAMEWORK

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ABSTRACT

Large-amplitude oscillatory shear flow (LAOS) is a popular for studying the nonlinear physics of complex fluids. Specifically, the strain rate sweep (also called the strain sweep) is used routinely to identify the onset of nonlinearity. In this paper, we give exact expressions for the nonlinear complex viscosity and the corresponding nonlinear complex normal stress coefficients for the Oldroyd 8-constant framework for oscillatory shear sweeps. We choose the Oldroyd 8-constant framework for its rich diversity of popular special cases (we list 18 of these). We evaluate the Fourier integrals of our previous exact solution to get exact expressions for the real and imaginary parts of the complex viscosity, and for the complex normal stress coefficients, as functions of both test frequency and shear rate amplitude. For our comparisons with data, we use the Spriggs relations to generalize the Oldroyd 8-constant framework to multimode. We explore the role of infinite shear rate viscosity on strain rate sweep responses for the special case of the corotational Jeffreys fluid. We find that raising η_{∞} , raises the real part of the complex viscosity, and lowers the imaginary. In our worked examples, we thus first use the corotational Jeffreys fluid, and then, for greater accuracy, we use the Johnson-Segalman fluid, to describe the strain rate sweep response of molten atactic polystyrene. Our generalization yields unequivocally, a longest fluid relaxation time, used to assign Weissenberg and Deborah numbers to each oscillatory shear flow experiment. We then locate each experiment in Pipkin space.

Keywords: Strain rate sweep; strain sweep; frequency sweep; nonlinear complex viscosity; Oldroyd 8-constant framework; exact solutions; large-amplitude oscillatory shear flow; LAOS; lost modulus; storage modulus; finite difference.

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I. INTRODUCTION

Since its conception in 1935 [1,2,3] oscillatory shear flow has become by far the most popular laboratory method for exploring the physics of polymeric liquids. We generate oscillatory shear flow by confining the fluid to a simple shear apparatus, and then subject one solid-liquid boundary to a coplanar sinusoidal displacement generating a corresponding cosinusoidal shear rate:

$$\dot{\gamma}(t) = \dot{\gamma}^0 \cos \omega t \quad (1)$$

Using the characteristic relaxation time of the viscoelastic fluid, λ_1 , we can nondimensionalize Eq. (1):

$$\lambda_1 \dot{\gamma}(t) = \text{Wi} \cos \text{De}(t / \lambda_1) \quad (2)$$

where:

$$\text{De} \equiv \lambda_1 \omega, \quad \text{Wi} \equiv \lambda_1 \dot{\gamma}^0 \quad (3), (4)$$

are the Deborah and Weissenberg numbers. Here we focus on large-amplitude oscillatory shear flow (LAOS), an experimental flow protocol used to investigate the physics of complex liquids (see [4]; Ch. 11 of [5]; [6]). In the absence of fluid inertia [7], for isothermal [7,8,9] oscillatory simple shear flow, the rate of strain tensor is given by:

$$\dot{\boldsymbol{\gamma}} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\gamma}^0 \cos \omega t & 0 \\ \dot{\gamma}^0 \cos \omega t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5)$$

which we will use below. Table I gathers all of errata to cited articles in this paper.

When higher harmonics arise in the stress responses, we call the oscillatory experiment *large-amplitude*, and the shear stress responds as the Fourier series:

$$\frac{\tau_{yx}(\omega, \dot{\gamma}^0, \tau)}{\dot{\gamma}^0} = -\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \eta'_n(\omega, \dot{\gamma}^0) \cos n\tau + \eta''_n(\omega, \dot{\gamma}^0) \sin n\tau \quad (6)$$

and the first normal stress difference responses:

$$\frac{N_1(\omega, \dot{\gamma}^0, \tau)}{(\dot{\gamma}^0)^2} \equiv \frac{\tau_{xx} - \tau_{yy}}{(\dot{\gamma}^0)^2} = -\sum_{\substack{n=0 \\ \text{even}}}^{\infty} \Psi'_{1,n}(\omega, \dot{\gamma}^0) \cos n\tau + \Psi''_{1,n}(\omega, \dot{\gamma}^0) \sin n\tau \quad (7)$$

and the second:

$$\frac{N_2(\omega, \dot{\gamma}^0, \tau)}{(\dot{\gamma}^0)^2} \equiv \frac{\tau_{yy} - \tau_{zz}}{(\dot{\gamma}^0)^2} = -\sum_{\substack{n=0 \\ \text{even}}}^{\infty} \Psi'_{2,n}(\omega, \dot{\gamma}^0) \cos n\tau + \Psi''_{2,n}(\omega, \dot{\gamma}^0) \sin n\tau \quad (8)$$

In Eqs. (6)–(8), the Fourier coefficients (η'_n, η''_n) , $(\Psi'_{1,n}, \Psi''_{1,n})$ and $(\Psi'_{2,n}, \Psi''_{2,n})$ are the real and minus [10] the imaginary parts of the nonlinear complex viscosities, η_n^* , and first and second normal stress coefficients, $\Psi_{1,n}^*$ and $\Psi_{2,n}^*$. Some prefer to write Eq. (6) as (see Eq. (3) of [11]):

$$\frac{\tau_{yx}(\omega t, \dot{\gamma}^0)}{\gamma_0} = - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} G_n''(\omega, \dot{\gamma}^0) \cos n\omega t + G_n'(\omega, \dot{\gamma}^0) \sin n\omega t \quad (9)$$

where (G_n', G_n'') are the real and the imaginary parts of the nonlinear complex shear moduli, G_n^* , so that:

$$G_n'(\omega, \dot{\gamma}^0) \equiv \omega \eta_n''(\omega, \dot{\gamma}^0), \quad G_n''(\omega, \dot{\gamma}^0) \equiv \omega \eta_n'(\omega, \dot{\gamma}^0) \quad (10)$$

For the shear stress response, by *higher harmonics*, we mean contributions at multiples of the test frequency, ω , that are higher than the test frequency. By contrast, for the normal stress difference responses, by *higher harmonics*, we mean contributions at multiples of the test frequency, ω , that are higher than twice the test frequency. For brevity, for the lowest harmonics in the *sweep material functions*, we use the standardized symbols $\eta_1'(\omega, \dot{\gamma}^0)$, $\eta_1''(\omega, \dot{\gamma}^0)$, $\Psi_{i,0}(\omega, \dot{\gamma}^0)$, $\Psi_{i,2}'(\omega, \dot{\gamma}^0)$ and $\Psi_{i,2}''(\omega, \dot{\gamma}^0)$ throughout this paper, where $i=1,2$ (see Table IV of [12]). We distinguish these sweep material functions with SAOS material functions, where the standardized symbols are $\eta'(\omega) \equiv \lim_{\dot{\gamma}^0 \rightarrow 0} \eta_1'(\omega, \dot{\gamma}^0)$, $\eta''(\omega) \equiv \lim_{\dot{\gamma}^0 \rightarrow 0} \eta_1''(\omega, \dot{\gamma}^0)$, $\Psi_i^d \equiv \lim_{\dot{\gamma}^0 \rightarrow 0} \Psi_{i,0}(\omega, \dot{\gamma}^0)$, $\Psi_i'(\omega) \equiv \lim_{\dot{\gamma}^0 \rightarrow 0} \Psi_{i,2}'(\omega, \dot{\gamma}^0)$ and $\Psi_i''(\omega) \equiv \lim_{\dot{\gamma}^0 \rightarrow 0} \Psi_{i,2}''(\omega, \dot{\gamma}^0)$.

We associate the real part of the shear stress n th harmonic, $\dot{\gamma}^0 \eta_1'$, with viscous dissipation, and minus the imaginary parts, $\dot{\gamma}^0 \eta_1''$, with elasticity [13]. For polymeric liquids, these higher harmonics are commonly observed when:

$$[\text{Wi} > 1] \cap \left[\frac{\text{Wi}}{\text{De}} > 1 \right] \quad (11)$$

which is thus our working definition of *large-amplitude* oscillatory shear flow, and which improves upon previous definitions ([4,14,104]; Eq. (8) of [15]; Eq. (9) of [19]). We will illustrate Eq. (11) as a crosshatched region on a Pipkin map in Figure 14. Thus:

$$[\text{Wi} < 1] \cup \left[\frac{\text{Wi}}{\text{De}} < 1 \right] \quad (12)$$

is thus our working definition of *small-amplitude* oscillatory shear flow. With recent advances [16], rheometers for experiments satisfying Eq. (11) are now available. Many important notations, alternative to Eqs. (6) through (8), have been introduced for analyzing the higher harmonics in the shear stress (see Sections 9 and 10 of [15]).

The Oldroyd 8-constant framework is given by (see Eq. (8.1-2) in [17];[18]):

$$\begin{aligned} \tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{1}{2} \mu_0 (\text{tr } \tau) \dot{\gamma} - \frac{1}{2} \mu_1 \{ \tau \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau \} + \frac{1}{2} \nu_1 (\tau : \dot{\gamma}) \delta \\ = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} - \mu_2 \{ \dot{\gamma} \cdot \dot{\gamma} \} + \frac{1}{2} \nu_2 (\dot{\gamma} : \dot{\gamma}) \delta \right) \end{aligned} \quad (13)$$

Eq. (13) reduces exactly to the rich diversity of 14 special cases listed in Table IV of [19]. In addition to these 14 tabulated cases, the Oldroyd 8-constant framework also reduces to (i) constant viscosity fluids ($\sigma_1 = \sigma_2$), (ii) Boger fluids having $\Psi_2(\dot{\gamma}) = 0$ ($\sigma_1 = \sigma_2$; $\lambda_1 > \lambda_2$; $\lambda_1 - \mu_1 = \lambda_2 - \mu_2$), (iii) Boger fluids having $\Psi_2(\dot{\gamma}) < 0$ ($\sigma_1 = \sigma_2$; $\lambda_1 > \lambda_2$; $\lambda_1 - \mu_1 > \lambda_2 - \mu_2$), and (iv) Newtonian fluids ($\lambda_1 = \lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

For small amplitude oscillatory shear flow (SAOS), for the Oldroyd 8-constant framework, and thus, for all of its special cases, the expression for the real part and minus the imaginary part of the complex viscosity are given by (Eqs. (8.1-10) and (8.1-11) of [17]):

$$\eta' = \eta_0 \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2}, \quad \eta'' = \eta_0 \frac{(\lambda_1 - \lambda_2) \omega}{1 + (\lambda_1 \omega)^2} \quad (14), (15)$$

For SAOS, for the Oldroyd 8-constant framework, and thus, for all of its special cases, the expression for the real part and minus the imaginary part of both complex normal stress coefficients are given by (Eqs. (32)–(34) of [39]):

$$\Psi'_1 = -2\Psi'_2 = \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{\eta_0 \lambda_1 (1 - 2\text{De}^2)}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \quad (16)$$

$$\Psi''_1 = -2\Psi''_2 = \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{3\eta_0 \lambda_1 \text{De}}{(1 + \text{De}^2)(1 + 4\text{De}^2)} \quad (17)$$

and the companion expression for the displacement term is:

$$\Psi_1^d = -2\Psi_2^d = \left(1 - \frac{\lambda_2}{\lambda_1}\right) \frac{\eta_0 \lambda_1}{1 + \text{De}^2} \quad (18)$$

The Oldroyd 8-constant framework has also been closely connected, though approximately, with macromolecular theory ([20,21,22]; see Table 1 of [23]; see Eqs. (32) of [24],[25]; see Tables 6.2-1 and 6.2-2 of [26]; Problems 11B.9 and 11B.10 of [27]; §IV and §V. of [28]; §9.5 of [17]). With Eqs. (13), the Oldroyd 8-constant framework thus provides a useful approximation to the polymer contribution to the stresses for a suspension of rigid dumbbells, and for FENE dumbbells [23].

The total stress tensor is defined by:

$$\boldsymbol{\pi} \equiv \boldsymbol{\tau} + p\boldsymbol{\delta} \quad (19)$$

the rate of strain tensor, by Eq. (5), the vorticity tensor by:

$$\boldsymbol{\omega} \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \quad (20)$$

and the corotational derivative of any tensor $\mathbf{b}$, by:

$$\frac{\mathcal{D}\mathbf{b}}{\mathcal{D}t} \equiv \frac{D\mathbf{b}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \mathbf{b} - \mathbf{b} \cdot \boldsymbol{\omega}\} \quad (21)$$

In Eq. (19), $\boldsymbol{\delta}$ is the *Kronecker delta* and p is the hydrodynamic pressure.

When it leads to analytical solutions, the Oldroyd 8-constant model can be remarkably useful for polymer processing, as it is for wire coating (Case III in [29]), and for corrugated wire coated through a corrugated die (see Section 3. of [30]; Section 2. of [31]), and for plastic pipe extrusion for elliptical pipe ([32,33]), and also for extrusion from an axially eccentric annular pipe die (Case II in [29];[34];[35,36];[37]).

In this work, we focus on both the strain rate and frequency sweeps. By *strain rate sweep*, we mean a sequence of LAOS experiments conducted at the same ω , performed one after another, with increasing $\dot{\gamma}^0$. Each experiment of the sweep, of course, proceeds to alternance, before continuing to the next $\dot{\gamma}^0$. Strain rate sweeps are often called *strain sweeps*. However, in light of Eq. (11), the defining intersection for large-amplitude, we prefer the term strain rate sweeps. By contrast, *frequency sweeps of the first kind* are sequences of oscillatory shear experiments conducted at the same $\dot{\gamma}^0/\omega$, performed one after another, with increasing ω . Further, *frequency sweeps of the second kind* are sequences of oscillatory shear experiments conducted at the same $\dot{\gamma}^0$, performed one after another, with increasing ω . Whereas frequency sweeps of the first kind are ubiquitous for $\eta^*(\omega)$ measurements, we know of no reports of frequency sweeps of the second kind. We illustrate these three protocols on the Pipkin diagram in Figure 1.

In this paper, we derive exact solutions for the *sweep material functions*, $\eta'_1(\omega, \dot{\gamma}^0)$, $\eta''_1(\omega, \dot{\gamma}^0)$, $\Psi'_i(\omega, \dot{\gamma}^0)$ and $\Psi''_i(\omega, \dot{\gamma}^0)$ for the Oldroyd 8-constant framework. In Section IV, we will use our results to explore the role of η_∞ on these sweep responses, for the special case of the corotational Jeffreys fluid, for which $\eta_\infty/\eta_0 = \lambda_2/\lambda_1$ (see Eqs. (20) and (103) in [38]). In Section VI, we will apply our main results to the sweep responses, $|\eta^*(\omega, \dot{\gamma}^0)|$, of molten polystyrene in LAOS. For this application, we employ the Spriggs relations, Eqs. (61), to introduce analytically, an infinity of relaxation and retardation times.

II. EXACT SOLUTIONS

In this section, we recall our results for the exact analytical solution for the Oldroyd 8-constant framework in LAOS. For the shear stress, the exact solution is given by (Eq. (65) of [19]).

$$\mathbb{S} \equiv \frac{\tau_{yx}}{\eta_0 \dot{\gamma}^0} = e^{\frac{-\tau}{\text{De}}} \left\{ \begin{array}{l} \sin(\alpha \sin \tau) \left[-\frac{1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 + \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_5 \right] \\ + \cos(\alpha \sin \tau) \left[-\frac{1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_6 \right] \end{array} \right\} \quad (22)$$

and for the first normal stress difference, by (Eq. (54) of [39]):

$$\mathbb{N}_1 \equiv \frac{\tau_{xx} - \tau_{yy}}{\eta_0 \dot{\gamma}^0} = 2 \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \left\{ \begin{aligned} & -\frac{1}{\alpha} \cos(\alpha \sin \tau) \left[\frac{-1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 + \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_5 \right] \\ & + \frac{1}{\alpha} \sin(\alpha \sin \tau) \left[\frac{-1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_6 \right] \\ & + \frac{\psi}{\tilde{\lambda}^2} I_7 \end{aligned} \right\} \quad (23)$$

and for the second, by (Eq. (55) of [39]):

$$\mathbb{N}_2 \equiv \frac{\tau_{yy} - \tau_{zz}}{\eta_0 \dot{\gamma}^0} = \frac{\text{Wi}}{\text{De}} e^{\frac{-\tau}{\text{De}}} \left\{ \begin{aligned} & \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \cos(\alpha \sin \tau) \left[\frac{-1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 + \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_5 \right] \\ & - \frac{1}{\alpha} \left(1 - \frac{\mu_1}{\lambda_1} \right) \sin(\alpha \sin \tau) \left[\frac{-1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{\phi}{\tilde{\lambda}} I_6 \right] \\ & + \frac{\Omega}{\tilde{\lambda}^2} I_7 \end{aligned} \right\} \quad (24)$$

where:

$$\alpha \equiv \frac{\text{Wi}}{\text{De}} \tilde{\lambda} \equiv \frac{\text{Wi}}{\text{De}} \sqrt{1 + \frac{\mu_0 \mu_1}{\lambda_1^2} - \frac{\mu_1^2}{\lambda_1^2} - \frac{3}{2} \frac{\mu_0 v_1}{\lambda_1^2} + \frac{\mu_1 v_1}{\lambda_1^2}} \quad (25)$$

$$\phi \equiv -\frac{\lambda_2}{\lambda_1} - \frac{\mu_0 \mu_2}{\lambda_1^2} + \frac{3}{2} \frac{\mu_0 v_2}{\lambda_1^2} + \frac{\mu_1 \mu_2}{\lambda_1^2} - \frac{\mu_1 v_2}{\lambda_1^2} \quad (26)$$

$$\psi \equiv -\left(1 - \tilde{\lambda}^2\right) \frac{\lambda_2}{\lambda_1} - \left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\mu_2}{\lambda_1} + \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{v_2}{\lambda_1} \quad (27)$$

$$\Omega \equiv \left(1 - \frac{\mu_1}{\lambda_1}\right) \left[\left(\frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{\lambda_2}{\lambda_1} - \left(\frac{3}{2} \frac{\mu_0}{\lambda_1} - \frac{\mu_1}{\lambda_1}\right) \frac{v_2}{\lambda_1} + \frac{\mu_2}{\lambda_1} \right] - \left[\frac{\mu_0}{\lambda_1} - \frac{3}{2} \frac{\mu_0 v_1}{\lambda_1^2} + \frac{\mu_1 v_1}{\lambda_1^2} \right] \left(\frac{\lambda_2}{\lambda_1} - \frac{\mu_2}{\lambda_1} \right) \quad (28)$$

and where $I_1 - I_7$ are given by (Eqs. (56)–(62) in [39]):

$$I_1 = \frac{4 \text{De}}{\alpha} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[-\frac{2k^2 J_{2k} \text{De}}{1 + 4k^2 \text{De}^2} \cos 2k\tau + \frac{k J_{2k}}{1 + 4k^2 \text{De}^2} \sin 2k\tau \right] \quad (29)$$

$$I_2 = \frac{2 \text{De}}{\alpha} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{(2k-1) J_{2k-1}}{1 + (2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{\text{De}(2k-1)^2 J_{2k-1}}{1 + (2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (30)$$

$$I_3 = \text{De} e^{\frac{\tau}{\text{De}}} \left(J_1 - \sum_{k=1}^{\infty} \left[\frac{J_{2k-1} - J_{2k+1}}{1 + 4k^2 \text{De}^2} \cos 2k\tau + \frac{2k(J_{2k-1} - J_{2k+1}) \text{De}}{1 + 4k^2 \text{De}^2} \sin 2k\tau \right] \right) \quad (31)$$

$$I_4 = \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-(2k-1)(J_{2k-2} - J_{2k}) \text{De}}{1 + (2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{J_{2k-2} - J_{2k}}{1 + (2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (32)$$

$$I_5 = \text{De} e^{\frac{\tau}{\text{De}}} \left(\frac{J_1}{\alpha} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{2(1 + 4\text{De}^2 k^2)} \cos 2k\tau + \frac{k(J_{2k-2} + 2J_{2k} + J_{2k+2})\text{De}}{1 + 4\text{De}^2 k^2} \sin 2k\tau \right] \right) \quad (33)$$

$$I_6 = \frac{\text{De}}{2} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})\text{De}}{1 + (2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{J_{2k-3} + 2J_{2k-1} + J_{2k+1}}{1 + (2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (34)$$

$$I_7 = \frac{\text{De} e^{\frac{\tau}{\text{De}}}}{2} \left[1 + \frac{1}{1 + 4\text{De}^2} \cos 2\tau + \frac{2\text{De}}{1 + 4\text{De}^2} \sin 2\tau \right] \quad (35)$$

where J_m means $J_m(\alpha)$. Eqs. (29)–(35) are much more compact than Eqs. (66)–(70) of [19] (or than (56)–(62) of [39]). We find Eqs. (22)–(24) to be intrinsically beautiful. In Section III, we extract the sweep material functions, η'_1 , η''_1 , $\Psi_{i,0}$, $\Psi'_{i,2}$, $\Psi''_{i,2}$, from Eqs. (22)–(24). Whereas the right sides of Eqs. (22)–(24) all share the common factor $e^{-\tau/\text{De}}$, all of the integrals I_1 – I_7 share the common factor $e^{+\tau/\text{De}}$. For the alternant parts of the stress responses, these of course cancel, but for start-up of LAOS, they do not. We are thus resisting these cancelations, since we are saving the start-up problem for another day.

III. STRAIN RATE SWEEP MATERIAL FUNCTIONS

In this section, we arrive at sweep responses for both the shear stress and the normal stress differences in LAOS for the Oldroyd 8-constant framework. For the shear stress response, we extract the real part and minus the imaginary part of the nonlinear complex viscosity by performing the following Fourier integration (see Eqs. (23) and (24) of [40]):

$$\frac{\eta'_1}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{S} \cos \tau d\tau, \quad \frac{\eta''_1}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{S} \sin \tau d\tau \quad (36), (37)$$

where $\mathbb{S}$ is given by Eq. (22). For the first normal stress difference response, we extract the displacement term, the real part and minus the imaginary part of the nonlinear complex first normal stress coefficient by performing the following Fourier integration:

$$\frac{\Psi'_{1,0}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{2\pi} \int_0^{2\pi} \mathbb{N}_1 d\tau, \quad \frac{\Psi'_{1,2}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{N}_1 \cos 2\tau d\tau, \quad \frac{\Psi''_{1,2}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{N}_1 \sin 2\tau d\tau \quad (38), (39), (40)$$

where $\mathbb{N}_1$ is given by Eq. (23). Similarly, for the second normal stress difference:

$$\frac{\Psi'_{2,0}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{2\pi} \int_0^{2\pi} \mathbb{N}_2 d\tau, \quad \frac{\Psi'_{2,2}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{N}_2 \cos 2\tau d\tau, \quad \frac{\Psi''_{2,2}\dot{\gamma}^0}{\eta_0} \equiv \frac{-1}{\pi} \int_0^{2\pi} \mathbb{N}_2 \sin 2\tau d\tau \quad (41), (42), (43)$$

where $\mathbb{N}_2$ is given by Eq. (24).

We will evaluate Eqs. (36) and (37) [with Eq. (22) along with Eqs. (29)–(34)] in Subsection III.a, Eqs. (38)–(43) [with Eq. (23) and (24) along with Eqs. (29)–(35)] in Subsection III.b.

a. Shear Stress

For the nonlinear complex viscosity, substituting Eqs. (22) [with Eqs. (29)–(35)] into Eq. (36), gives:

$$\begin{aligned} \frac{\eta'_1}{\eta_0} = & \frac{2\text{De}}{\text{Wi}\tilde{\lambda}} \left(\frac{J_0 J_1}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{2(k-1)J_{2k-2}J_{2k-1}}{1 + 4(k-1)^2 \text{De}^2} + \frac{2kJ_{2k-1}J_{2k}}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k-1}J_{2k}}{1 + (2k-1)^2 \text{De}^2} + \frac{(2k+1)J_{2k}J_{2k+1}}{1 + (2k+1)^2 \text{De}^2} \right] \right) \\ & + \text{De}^2 \frac{\lambda_2}{\lambda_1} \left(\frac{J_0(J_0 - J_2)}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{-2(k-1)J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 \text{De}^2} - \frac{2kJ_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} + \frac{(2k+1)J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 \text{De}^2} \right] \right) \\ & - \frac{\text{WiDe}\phi}{\tilde{\lambda}} \left(\frac{J_0(J_1 + J_3)}{2(1 + \text{De}^2)} + \sum_{k=1}^{\infty} \left[\frac{(k-1)J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 \text{De}^2} + \frac{kJ_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. + \frac{J_{2k}(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{2(1 + (2k-1)^2 \text{De}^2)} + \frac{(2k+1)J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{2(1 + (2k+1)^2 \text{De}^2)} \right] \right) \end{aligned} \quad (44)$$

and into Eq. (37):

$$\begin{aligned} \frac{\eta''_1}{\eta_0} = & \frac{2\text{De}^2}{\text{Wi}\tilde{\lambda}} \left(\frac{J_0 J_1}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{-4(k-1)^2 J_{2k-1}J_{2k-2}}{1 + 4(k-1)^2 \text{De}^2} + \frac{4k^2 J_{2k-1}J_{2k}}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. + \frac{-(2k-1)^2 J_{2k-1}J_{2k}}{1 + (2k-1)^2 \text{De}^2} + \frac{(2k+1)^2 J_{2k}J_{2k+1}}{1 + (2k+1)^2 \text{De}^2} \right] \right) \\ & - \text{De}^2 \frac{\lambda_2}{\lambda_1} \left(\frac{J_0^2 - J_0 J_2}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 \text{De}^2} - \frac{J_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. + \frac{-J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} + \frac{J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 \text{De}^2} \right] \right) \\ & + \frac{\text{Wi}\phi}{2\tilde{\lambda}} \left(\frac{J_0(J_1 + J_3)}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 \text{De}^2} + \frac{J_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \right. \\ & \left. \left. - \frac{J_{2k}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} + \frac{J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{1 + (2k+1)^2 \text{De}^2} \right] \right) \end{aligned} \quad (45)$$

Eqs. (44) and (45) are main results of this paper.

For the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eq. (44) reduces to:

$$\begin{aligned} \frac{\eta'_1}{\eta_0} = & \frac{2\text{De}}{\text{Wi}} \left[\frac{J_0 J_1}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{2(k-1)J_{2k-2}J_{2k-1} + \frac{2kJ_{2k-1}J_{2k}}{1+4k^2\text{De}^2}}{1+4(k-1)^2\text{De}^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k-1}J_{2k}}{1+(2k-1)^2\text{De}^2} + \frac{(2k+1)J_{2k}J_{2k+1}}{1+\text{De}^2(2k+1)^2} \right] \right] \\ & + \text{De}^2 \frac{\lambda_2}{\lambda_1} \left[\frac{J_0(J_0 - J_2)}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{-2(k-1)J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1+4(k-1)^2\text{De}^2} - \frac{2kJ_{2k-1}(-J_{2k-1} + J_{2k+1})}{1+4k^2\text{De}^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k}(J_{2k-2} - J_{2k})}{1+(2k-1)^2\text{De}^2} + \frac{(2k+1)J_{2k}(J_{2k} - J_{2k+2})}{1+(2k+1)^2\text{De}^2} \right] \right] \\ & + \text{WiDe} \frac{\lambda_2}{\lambda_1} \left[\frac{J_0(J_1 + J_3)}{2(1 + \text{De}^2)} + \sum_{k=1}^{\infty} \left[\frac{(k-1)J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1+4(k-1)^2\text{De}^2} + \frac{kJ_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1+4k^2\text{De}^2} \right. \right. \\ & \left. \left. + \frac{J_{2k}(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{2(1+(2k-1)^2\text{De}^2)} + \frac{(2k+1)J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{2(1+(2k+1)^2\text{De}^2)} \right] \right] \end{aligned} \quad (46)$$

and Eq. (45), to:

$$\begin{aligned} \frac{\eta''_1}{\eta_0} = & \frac{2\text{De}^2}{\text{Wi}} \left[\frac{J_0 J_1}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{-4(k-1)^2 J_{2k-1}J_{2k-2} + \frac{4k^2 J_{2k-1}J_{2k}}{1+4k^2\text{De}^2}}{1+4(k-1)^2\text{De}^2} \right. \right. \\ & \left. \left. + \frac{-(2k-1)^2 J_{2k-1}J_{2k}}{1+(2k-1)^2\text{De}^2} + \frac{(2k+1)^2 J_{2k}J_{2k+1}}{1+(2k+1)^2\text{De}^2} \right] \right] \\ & - \text{De} \frac{\lambda_2}{\lambda_1} \left[\frac{J_0^2 - J_0 J_2}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1+4(k-1)^2\text{De}^2} - \frac{J_{2k-1}(-J_{2k-1} + J_{2k+1})}{1+4k^2\text{De}^2} \right. \right. \\ & \left. \left. + \frac{-J_{2k}(J_{2k-2} - J_{2k})}{1+(2k-1)^2\text{De}^2} + \frac{J_{2k}(J_{2k} - J_{2k+2})}{1+(2k+1)^2\text{De}^2} \right] \right] \\ & - \frac{\text{Wi}}{2} \frac{\lambda_2}{\lambda_1} \left[\frac{J_0(J_1 + J_3)}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{-J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1+4(k-1)^2\text{De}^2} + \frac{J_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1+4k^2\text{De}^2} \right. \right. \\ & \left. \left. - \frac{J_{2k}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1+(2k-1)^2\text{De}^2} + \frac{J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{1+(2k+1)^2\text{De}^2} \right] \right] \end{aligned} \quad (47)$$

Taking the limits as $\text{Wi} \rightarrow 0$ in Eqs. (46) and (47) gives Eqs. (14) and (15), which are the well-known results for SAOS responses to any and all special cases of the Oldroyd 8-constant framework as they must. We will use in Eqs. (46) and (47) in Section VI to illustrate the usefulness of our main results.

Further, for the special case of the corotational Maxwell fluid, where $\lambda_2 = 0$, Eqs. (46) and (47) reduce to:

$$\frac{\eta'_1}{\eta_0} = \frac{2\text{De}}{\text{Wi}} \left(\left\{ \frac{J_0 J_1}{1 + \text{De}^2} \right\} + \sum_{k=1}^{\infty} \left[\frac{2(k-1)J_{2k-2}J_{2k-1}}{1 + 4(k-1)^2 \text{De}^2} + \frac{2kJ_{2k-1}J_{2k}}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)J_{2k-1}J_{2k}}{1 + (2k-1)^2 \text{De}^2} + \frac{(2k+1)J_{2k}J_{2k+1}}{1 + \text{De}^2(2k+1)^2} \right] \right) \quad (48)$$

and:

$$\frac{\eta''_1}{\eta_0} = \frac{2\text{De}^2}{\text{Wi}} \left(\left\{ \frac{J_0 J_1}{1 + \text{De}^2} \right\} + \sum_{k=1}^{\infty} \left[\frac{-4(k-1)^2 J_{2k-1}J_{2k-2}}{1 + 4(k-1)^2 \text{De}^2} + \frac{4k^2 J_{2k-1}J_{2k}}{1 + 4k^2 \text{De}^2} + \frac{-(2k-1)^2 J_{2k-1}J_{2k}}{1 + (2k-1)^2 \text{De}^2} + \frac{(2k+1)^2 J_{2k}J_{2k+1}}{1 + (2k+1)^2 \text{De}^2} \right] \right) \quad (49)$$

which agree with the coefficients of $\cos(2n-1)\tau$ and $\sin(2n-1)\tau$ [evaluated at $n=1$] in Eq. (66) of [104]. Taking the limit as $\text{Wi} \rightarrow 0$ in Eqs. (48) and (49) gives Eqs. (14) and (15) (evaluated at $\lambda_2 = 0$) as it must. In the process of taking these limits, we learned that the bracketed terms in Eqs. (48) and (49) make no contribution to the linear viscoelastic part of the material response.

b. Normal Stress Differences

We now turn our attention to the complex first normal stress coefficient. Substituting Eq. (23) [Eqs. (29)–(35)] into Eq. (38) gives:

$$\begin{aligned} \Psi_{1,0} = & \frac{4\text{De}^2}{\text{Wi} \tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k}^2}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1}^2}{1 + (2k-1)^2 \text{De}^2} \right] \\ & + \frac{2\text{De}}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(J_0 J_1 + \sum_{k=1}^{\infty} \left[\frac{J_{2k}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 \text{De}^2} - \frac{J_{2k-1}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\ & + \frac{\phi}{\tilde{\lambda}^2} \left(\frac{2\text{De} J_0 J_1}{\tilde{\lambda}} + \text{Wi} \sum_{k=1}^{\infty} \left[\frac{J_{2k}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{J_{2k-1}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) - \frac{\psi \text{Wi}}{\tilde{\lambda}^2} \end{aligned} \quad (50)$$

and into Eq. (39):

$$\begin{aligned}
\Psi'_{1,2} = & \frac{4\text{De}^2}{\text{Wi}\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k} (J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1} (J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + \frac{2\text{De}}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(2J_1 J_2 + \sum_{k=1}^{\infty} \left[\frac{(-J_{2k-1} + J_{2k+1})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} - \frac{(J_{2k-2} - J_{2k})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& + \frac{\phi}{\tilde{\lambda}^2} \left(\frac{4\text{De}}{\tilde{\lambda}} J_1 J_2 + \text{Wi} \sum_{k=1}^{\infty} \left[\frac{(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) - \frac{\psi \text{Wi}}{\tilde{\lambda}^2 (1 + 4\text{De}^2)}
\end{aligned} \tag{51}$$

and finally, into Eq. (40):

$$\begin{aligned}
\Psi''_{1,2} = & \frac{-4\text{De}}{\text{Wi}\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{2kJ_{2k} (J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)J_{2k-1} (J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + \frac{-2\text{De}^2}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \sum_{k=1}^{\infty} \left[\frac{2k(J_{2k-1} - J_{2k+1})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)(J_{2k-2} - J_{2k})(J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + \frac{\phi \text{WiDe}}{\tilde{\lambda}^2} \sum_{k=1}^{\infty} \left[\frac{2k(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] - \frac{2\psi \text{WiDe}}{\tilde{\lambda}^2 (1 + 4\text{De}^2)}
\end{aligned} \tag{52}$$

For the complex second normal stress coefficient, substituting Eq. (23) [Eqs. (29)–(35)] into Eq. (41) gives:

$$\begin{aligned}
\Psi_{2,0} = & \frac{-2\text{De}^2}{\text{Wi}\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1} \right) \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k}^2}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1}^2}{1 + (2k-1)^2 \text{De}^2} \right] \\
& - \frac{\text{De}}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left(J_0 J_1 + \sum_{k=1}^{\infty} \left[\frac{J_{2k} (-J_{2k-1} + J_{2k+1})}{1 + 4k^2 \text{De}^2} - \frac{J_{2k-1} (J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& \frac{\phi}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1} \right) \left(-\frac{\text{De} J_0 J_1}{\tilde{\lambda}} - \frac{\text{Wi}}{2} \sum_{k=1}^{\infty} \left[\frac{J_{2k} (J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{J_{2k-1} (J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& + \frac{-\Omega \text{Wi}}{2\tilde{\lambda}^2}
\end{aligned} \tag{53}$$

and into Eq. (42):

$$\begin{aligned}
\Psi'_{2,2} = & \frac{-2\text{De}^2}{\text{Wi}\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k} (J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1} (J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + \frac{\text{De}}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(1 - \frac{\mu_1}{\lambda_1}\right) \left(-2J_1 J_2 + \sum_{k=1}^{\infty} \left[\frac{(J_{2k-1} - J_{2k+1})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(J_{2k-2} - J_{2k})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& - \frac{\phi}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \left(\frac{2\text{De}}{\tilde{\lambda}} J_1 J_2 + \frac{\text{Wi}}{2} \sum_{k=1}^{\infty} \left[\frac{(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \right. \\
& \left. \left. + \frac{(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& - \frac{\Omega \text{Wi}}{2\tilde{\lambda}^2(1 + 4\text{De}^2)}
\end{aligned} \tag{54}$$

and finally, into Eq. (43):

$$\begin{aligned}
\Psi''_{2,2} = & \frac{2\text{De}}{\text{Wi}\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \sum_{k=1}^{\infty} \left[\frac{2kJ_{2k} (J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)J_{2k-1} (J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& \frac{\text{De}^2}{\tilde{\lambda}} \frac{\lambda_2}{\lambda_1} \left(1 - \frac{\mu_1}{\lambda_1}\right) \sum_{k=1}^{\infty} \left[\frac{2k(J_{2k-1} - J_{2k+1})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)(J_{2k-2} - J_{2k})(J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& - \frac{\phi \text{WiDe}}{\tilde{\lambda}^2} \left(1 - \frac{\mu_1}{\lambda_1}\right) \sum_{k=1}^{\infty} \left[\frac{k(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \\
& \left. + \frac{(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} - J_{2k+1})}{2(1 + (2k-1)^2 \text{De}^2)} \right] \\
& - \frac{\Omega \text{WiDe}}{\tilde{\lambda}^2(1 + 4\text{De}^2)}
\end{aligned} \tag{55}$$

Eqs. (50)–(52) and Eqs. (53)–(55) are main results of this paper.

For the special case of the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eq. (50) [or Eq. (53)] reduces to:

$$\begin{aligned}
\Psi_{1,0} = -2\Psi_{2,0} = & \frac{4\text{De}^2}{\text{Wi}} \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k}^2}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1}^2}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + 2\text{De} \frac{\lambda_2}{\lambda_1} \left(J_0 J_1 + \sum_{k=1}^{\infty} \left[\frac{J_{2k} (-J_{2k-1} + J_{2k+1})}{1 + 4k^2 \text{De}^2} - \frac{J_{2k-1} (J_{2k-2} - J_{2k})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& - \frac{\lambda_2}{\lambda_1} \left(\frac{2\text{De} J_0 J_1}{\tilde{\lambda}} + \text{Wi} \sum_{k=1}^{\infty} \left[\frac{J_{2k} (J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{J_{2k-1} (J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right)
\end{aligned} \tag{56}$$

Eq. (51) [or Eq. (54)], to:

$$\begin{aligned}
\Psi'_{1,2} = -2\Psi'_{2,2} = & \frac{4\text{De}^2}{\text{Wi}} \sum_{k=1}^{\infty} \left[\frac{4k^2 J_{2k} (J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)^2 J_{2k-1} (J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& + 2\text{De} \frac{\lambda_2}{\lambda_1} \left(2J_1 J_2 + \sum_{k=1}^{\infty} \left[\frac{(-J_{2k-1} + J_{2k+1})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} - \frac{(J_{2k-2} - J_{2k})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right) \\
& - \frac{\lambda_2}{\lambda_1} \left(4\text{De} J_1 J_2 + \text{Wi} \sum_{k=1}^{\infty} \left[\frac{(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} + J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \right. \\
& \left. \left. + \frac{(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} + J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \right)
\end{aligned} \tag{57}$$

and Eq. (52) [or Eq. (55)], to:

$$\begin{aligned}
\Psi''_{1,2} = -2\Psi''_{2,2} = & \frac{-4\text{De}}{\text{Wi}} \sum_{k=1}^{\infty} \left[\frac{2kJ_{2k} (J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)J_{2k-1} (J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& - 2\text{De}^2 \frac{\lambda_2}{\lambda_1} \sum_{k=1}^{\infty} \left[\frac{2k(J_{2k-1} - J_{2k+1})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} + \frac{(2k-1)(J_{2k-2} - J_{2k})(J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right] \\
& - \text{WiDe} \frac{\lambda_2}{\lambda_1} \sum_{k=1}^{\infty} \left[\frac{2k(J_{2k-2} + 2J_{2k} + J_{2k+2})(J_{2k-2} - J_{2k+2})}{1 + 4k^2 \text{De}^2} \right. \\
& \left. + \frac{(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})(J_{2k-3} - J_{2k+1})}{1 + (2k-1)^2 \text{De}^2} \right]
\end{aligned} \tag{58}$$

We will use Eqs. (56)–(58) to plot Figure 9 through Figure 11 in Subsection IV to explore the role of η_{∞} . We will verify Eqs. (56)–(58) in Section V by finite difference.

IV. DISCUSSIONS

We judge the corotational Jeffreys fluid ($\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) to be the simplest relevant model for exploring the role of η_{∞} in LAOS. Since, for the corotational Jeffreys fluid $\eta_{\infty}/\eta_0 = \lambda_2/\lambda_1$, we can use this special case of the Oldroyd 8-constant framework to deepen our understanding of the effect of η_{∞} on the sweep material functions, $[\eta'_1, \eta''_1]$, $[\Psi_{1,0}, \Psi'_{1,2}, \Psi''_{1,2}]$ and $[\Psi_{2,0}, \Psi'_{2,2}, \Psi''_{2,2}]$, by increasing λ_2 (holding η_0 and λ_1 constant).

Figure 2 and Figure 3 illustrate the behavior of the strain rate sweep material functions for the shear stress response, given by Eqs. (46) and (47) for the special case of the corotational Jeffrey fluid. From Figure 2, we learn that when both Wi and De approach zero, $\eta'_1 = \eta_0$, as it must. From Figure 2, we also learn that increasing η_{∞} (by increasing λ_2) raises the limit $\eta'_1(\infty, \dot{\gamma}^0)$, and thus increasing η_{∞} viscosifies the fluid.

This finding is consistent with our examination of the shear stress *versus* shear rate loops of the corotational Jeffrey fluid in FIG. 5 through FIG. 8 of [19].

Figure 3 through Figure 6 illustrate all aspects of the fluid elastic responses. Specifically, Figure 4 through Figure 6 illustrate the behavior of the strain rate sweep material functions for both normal stress difference responses, given by Eqs. (56)–(58). From Figure 3 through Figure 6, we learn that increasing η_∞ reduces all aspects of the fluid elastic responses: η''_1 , $\Psi_{1,0}$, $\Psi'_{1,2}$ and $\Psi''_{1,2}$. These findings conform with our examination of the normal stress differences *versus* shear rate loops in Figure 3 through Figure 6 of [39].

V. CONSISTENCY CHECK: FINITE DIFFERENCE

In Section III, we found that our main results [Eqs. (44) and (45), Eqs. (50)–(52), and Eqs. (53)–(55)] reduce to the exact solution for the corotational Maxwell fluid as they should. Viscoelastic models can also be evaluated and checked approximately, numerically. In LAOS, many viscoelastic models have been evaluated by the finite element method [41,42], or by numerical integration [43,44,45,46,47], or by finite difference [19,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74]. In this section, we check our main results for consistency with our finite difference solution for the specific case of the corotational Jeffreys fluid ($\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$) in LAOS.

For our finite difference solution, we coded Eqs. (36)–(43) [with Eqs. (22)–(24) along with Eqs. (29)–(34)] using the *quadgk* scheme into MATLAB (Version R2012b) on a MacBook Air (1.3 GHz Intel Core i5 processor with 4 GB 1600 MHz DDR3 memory) employing the OS X (Version 10.12.3) operating system. For each point in Figure 7 through Figure 11, we find such an evaluation to consume less than 5 seconds of CPU time. Figure 7 through Figure 11 show good agreement between our numerical and exact solutions, well within a pen width.

VI. APPLICATIONS: POLYSTYRENE MELTS

We next consider the commercial grade of molten atactic polystyrene made by Dow Chemical Canada Ltd., Styron 683, studied extensively by Tee and Dealy [111,112]. Tee and Dealy report one frequency sweep, comprising ten SAOS measurements taken over the ranges $0.38 < \omega < 24 \text{ rad/s}$ and $0.076 < \dot{\gamma}^0 < 4.8 \text{ s}^{-1}$, performed using oscillatory cone-plate flow [75,111,112] (the $\frac{Wi}{De} = 0.2$ set of points in Figure 12). Tee and Dealy also report a set of thirty LAOS measurements, comprising 5 frequency sweeps over the ranges $0.96 < \omega < 25 \text{ rad/s}$ and $2.6 < \dot{\gamma}^0 < 250 \text{ s}^{-1}$, performed on a concentric cylinder rheometer specially designed for LAOS [75,111,112] (the $2.67 \leq \frac{Wi}{De} \leq 10$ sets of points in

Figure 12). Specifically, they report frequency sweeps of $\tau_0(\omega, \dot{\gamma}^0)/\dot{\gamma}^0$, where τ_0 is the peak shear stress in LAOS. Since the higher harmonics normally affect loop shapes while hardly affecting τ_0 :

$$\frac{\tau_0(\omega, \dot{\gamma}^0)}{\dot{\gamma}^0} \equiv |\eta^*(\omega, \dot{\gamma}^0)| \quad (59)$$

where (see Eqs. (4.4–4) and (4.4–5) of [17]):

$$|\eta^*(\omega, \dot{\gamma}^0)| \equiv \sqrt{\eta_1'^2(\omega, \dot{\gamma}^0) + \eta_1''^2(\omega, \dot{\gamma}^0)} \quad (60)$$

For this application, for the LAOS measurements, we will thus compare $\tau_0(\omega, \dot{\gamma}^0)/\dot{\gamma}^0$ measurements with the $|\eta^*(\omega, \dot{\gamma}^0)|$ predictions from our main results, for the special case of the corotational Jeffreys fluid in Subsection VI.a, and for the special case of the Johnson-Segalman fluid in Subsection VI.b.

a. Corotational Jeffreys Fluid

We begin with the simplest model for exploring the role of η_∞ , the corotational Jeffreys fluid, where $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. We proceed as follows, by first *generalizing* to multiple relaxation and retardation times, then fitting all but one of the model parameters to the linear viscoelastic behavior, and then finally, we fit one remaining undetermined model parameter, λ_2 , to the nonlinear behavior:

(1) We generalize our results for the corotational Jeffreys fluid, Eqs. (46) and (47), to infinities of relaxation and retardation times, by replacing η_0 , λ_1 and λ_2 with the Spriggs relations (Eqs. (6.1-14) and (6.1-15) of [17]):

$$\eta_k = \eta_0 \lambda_{1k} / \sum_{k=1}^{\infty} \lambda_{1k}, \quad \lambda_{1k} = \lambda_1 / k^{\alpha_1}, \quad \lambda_{2k} = \lambda_2 / k^{\alpha_1} \quad (61), (62), (63)$$

and then from $\tau = \sum_k \tau_k$ and Eq. (6), we see that:

$$\eta_{1k}'(\omega, \dot{\gamma}^0) = \sum_{k=1}^{\infty} \eta_k'(\omega, \dot{\gamma}^0), \quad \eta_{1k}''(\omega, \dot{\gamma}^0) = \sum_{k=1}^{\infty} \eta_k''(\omega, \dot{\gamma}^0) \quad (64)$$

(2) We then fit η_0, λ_1 and α_1 to the $|\eta^*(\omega)|$ measurements of Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

(3) Mindful of Eq. (59), we then fit λ_2 to the $|\eta^*(\omega, \dot{\gamma}^0)|$ measurements from the nonlinear frequency sweeps of Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

(4) Still mindful of Eq. (59), we then check our best fit against the shear rate sweep interpolations from Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

In Eqs. (61)–(63), $\lambda_{11} \equiv \lambda_1$ and $\lambda_{21} \equiv \lambda_2$ are the longest relaxation and retardation times. Typically, for concentrated solutions and melts $2 \leq \alpha_1 \leq 4$ (see after Eq. (6.1-14,

15) of [17]). We use the value of λ_1 fitted from Eqs. (61) and (62), to determine De and Wi from Eqs. (3) and (4). In our experience, fitting from Eqs. (61) and (62) yields the characteristic longest relaxation time of the fluid, λ_1 , unequivocally. This fitted value, in turn, allows us to use our defining relation for *large-amplitude*, Eq. (11). For more on the Spriggs relations, we direct the reader to §6.1 of [17], Appendix of [76], or §8.5 of [77]). Following Step (1), we get:

$$\begin{aligned} \eta'_{1k} = & 2\eta_k \frac{\omega}{\dot{\gamma}^0} \left(\frac{J_0 J_1}{1 + (\lambda_{1k} \omega)^2} + \sum_{k=1}^{\infty} \left[\frac{2(k-1)J_{2k-2}J_{2k-1}}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} + \frac{2kJ_{2k-1}J_{2k}}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k-1}J_{2k}}{1 + (2k-1)^2 (\lambda_{1k} \omega)^2} + \frac{(2k+1)J_{2k}J_{2k+1}}{1 + (2k+1)^2 (\lambda_{1k} \omega)^2} \right] \right) \\ & + \eta_k \lambda_{1k} \lambda_{2k} \omega^2 \left(\frac{J_0(J_0 - J_2)}{1 + (\lambda_{1k} \omega)^2} + \sum_{k=1}^{\infty} \left[\frac{-2(k-1)J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} - \frac{2kJ_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 (\lambda_{1k} \omega)^2} + \frac{(2k+1)J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 (\lambda_{1k} \omega)^2} \right] \right) \\ & + \eta_k \lambda_{1k} \lambda_{2k} \dot{\gamma}^0 \omega \left(\frac{J_0(J_1 + J_3)}{2(1 + (\lambda_{1k} \omega)^2)} + \sum_{k=1}^{\infty} \left[\frac{(k-1)J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} + \frac{kJ_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. + \frac{(2k-1)J_{2k}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{2(1 + (2k-1)^2 (\lambda_{1k} \omega)^2)} + \frac{(2k+1)J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{2(1 + (2k+1)^2 (\lambda_{1k} \omega)^2)} \right] \right) \end{aligned} \quad (65)$$

and:

$$\begin{aligned} \eta''_{1k} = & 2\eta_k \frac{\omega^2}{\dot{\gamma}^0} \lambda_{1k} \left(\frac{J_0 J_1}{1 + (\lambda_{1k} \omega)^2} + \sum_{k=1}^{\infty} \left[\frac{-4(k-1)^2 J_{2k-1}J_{2k-2}}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} + \frac{4k^2 J_{2k-1}J_{2k}}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. + \frac{-(2k-1)^2 J_{2k-1}J_{2k}}{1 + (2k-1)^2 (\lambda_{1k} \omega)^2} + \frac{(2k+1)^2 J_{2k}J_{2k+1}}{1 + (2k+1)^2 (\lambda_{1k} \omega)^2} \right] \right) \\ & - \eta_k \omega \lambda_{2k} \left(\frac{J_0^2 - J_0 J_2}{1 + (\lambda_{1k} \omega)^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} - \frac{J_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. + \frac{-J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 (\lambda_{1k} \omega)^2} + \frac{J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 (\lambda_{1k} \omega)^2} \right] \right) \\ & - \frac{1}{2} \eta_k \dot{\gamma}^0 \lambda_{2k} \left(\frac{J_0(J_1 + J_3)}{1 + (\lambda_{1k} \omega)^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 (\lambda_{1k} \omega)^2} + \frac{J_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 (\lambda_{1k} \omega)^2} \right. \right. \\ & \left. \left. - \frac{J_{2k}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 (\lambda_{1k} \omega)^2} + \frac{J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{1 + (2k+1)^2 (\lambda_{1k} \omega)^2} \right] \right) \end{aligned} \quad (66)$$

Following Step (2), we next get the best fit of η_0 , λ_1 and α_1 to the data of Fig. 9. of [111] (or of FIGURE 7-12 of [112]). By *best fit*, we mean the values (or ranges of values) that minimize the following sum of the squares of residuals for the SAOS measurements:

$$S_R^{(\omega)} = \sum_{m=1}^{10} \left(\frac{|\eta^*(\omega)|_m}{\eta_0} - \frac{|\eta^*(\omega)|}{\eta_0} \right)^2 \quad (67)$$

where $|\eta^*(\omega)|_m$ is the m th measurement, and:

$$|\eta^*(\omega)| \equiv \sqrt{\eta'^2(\omega) + \eta''^2(\omega)} \quad (68)$$

and:

$$\frac{\eta'_*(\omega)}{\eta_0} = \frac{1}{\zeta(\alpha_1)} \sum_{k=1}^{\infty} \frac{k^{\alpha_1} + \lambda_1 \lambda_2 k^{-\alpha_1} \omega^2}{k^{2\alpha_1} + \lambda_1^2 \omega^2}, \quad \frac{\eta''_*(\omega)}{\eta_0} = \frac{1}{\zeta(\alpha_1)} \sum_{k=1}^{\infty} \frac{(\lambda_1 - \lambda_2) \omega}{k^{2\alpha_1} + \lambda_1^2 \omega^2} \quad (69), (70)$$

in which $\zeta(\alpha_1) = \sum_{k=1}^{\infty} k^{-\alpha_1}$ is the Riemann zeta function. For the special case of the corotational Maxwell fluid, $\lambda_2 = 0$, and Eqs. (69) and (70) then reduce to Eqs. (6.2-10) and (6.2-11) in [17], as they should. Our best fit yields $\eta_0 = 0.169 \text{ MPa}\cdot\text{s}$, $\lambda_1 = 0.775 \text{ s}$, $\lambda_2 \leq 0.0186 \text{ s}$, $\alpha_1 = 3.27$ ($\sigma^{(\omega)} \equiv \sqrt{S_R^{(\omega)}/10} = 0.0358$). Figure 12 compares this fit (upper most curve) with the ten SAOS measurements (**triangles**).

Since Step (2) yielded an inequality for λ_2 , we must next use the thirty LAOS measurements, Step (3), to arrive at the best fit for λ_2 , by minimizing the following sum of the squares of residuals for the LAOS measurements:

$$S_R^{(\omega, \dot{\gamma}^0)} = \sum_{m=1}^{30} \left(\frac{|\eta^*(\omega, \dot{\gamma}^0)|_m}{\eta_0} - \frac{|\eta^*(\omega, \dot{\gamma}^0)|}{\eta_0} \right)^2 \quad (71)$$

where $|\eta^*(\omega, \dot{\gamma}^0)|_m$ is the m th measurement, and where $|\eta^*(\omega, \dot{\gamma}^0)|$ is given by Eq. (60)

[with Eqs. (65) and (66)]. This step yields $\lambda_2 = 0.0186 \text{ s}$ ($\sigma^{(\omega, \dot{\gamma}^0)} \equiv \sqrt{S_R^{(\omega, \dot{\gamma}^0)}/30} = 0.191$).

Figure 12 compares the $|\eta^*(\omega, \dot{\gamma}^0)|$ predictions with the frequency sweep measurements in Fig. 9. of [111] (or of FIGURE 7-12 of [112]). Step (4) yields Figure 13, which shows the $|\eta^*(\omega, \dot{\gamma}^0)|$ predictions *versus* the strain rate sweep interpolations from Fig. 9. of [111] (or of FIGURE 7-12 of [112]). From the disagreements in Figure 12 and Figure 13, we learn that the corotational Jeffreys fluid describes poorly the behavior of molten atactic polystyrene in oscillatory shear sweeps.

We can next use our longest relaxation time, λ_1 , to deepen our understanding of the experimental measurements of Tee and Dealy in Figure 12 by nondimensionalizing the experimental ranges for SAOS as:

$$0.3 < \lambda_1 \omega < 20, \quad 0.06 < \lambda_1 \dot{\gamma}^0 < 4 \quad (72), (73)$$

and for LAOS, as:

$$0.7 < \lambda_1 \omega < 20, \quad 2 < \lambda_1 \dot{\gamma}^0 < 200 \quad (74), (75)$$

and by plotting the experimental conditions, (De, Wi) , for each of the 40 experiments. Figure 14 illustrates our working definition of large-amplitude oscillatory shear flow. The **45° hatching** in Figure 14 illustrates the left side of the intersection Eq. (11), and the **135° hatching**, the right. From Figure 14, we learn that all ten $|\eta^*(\omega)|$ measurements taken over the ranges $0.38 < \omega < 24 \text{ rad/s}$ and $0.076 < \dot{\gamma}^0 < 4.8 \text{ s}^{-1}$ are indeed *small-amplitude*. From Figure 14, we also learn that all thirty $|\eta^*(\omega, \dot{\gamma}^0)|$ measurements, comprising the 5 frequency sweeps, taken over the ranges $0.96 < \omega < 25 \text{ rad/s}$ and $2.6 < \dot{\gamma}^0 < 250 \text{ s}^{-1}$ are indeed *large-amplitude*.

b. Johnson-Segalman Fluid

Whereas for the corotational Jeffreys fluid, using oscillatory shear flow, all of the parameters are identifiable by fitting small-amplitude experiments [Eq. (12)], for the Johnson-Segalman fluid [78], ξ is only identifiable by fitting large-amplitude [Eq. (11)]. We thus now turn to the Johnson-Segalman fluid, where $\lambda_2 = (\eta_s/\eta_0)\lambda_1$, $\mu_1 = (1-\xi)\lambda_1$, $\mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1$ and $\mu_0 = \nu_1 = \nu_2 = 0$ (see Table IV of [19]). We proceed as follows, by first generalizing to multiple relaxation times, then fitting all but one of the model parameters to the linear viscoelastic behavior, and then finally, we fit one remaining undetermined model parameter, ξ , to the nonlinear behavior:

(1) We generalize our results, Eqs. (44) and (45), for the special case of the Johnson-Segalman fluid by replacing η_0 and λ_1 with Eqs. (61) and (62), so that:

$$\lambda_{2k} = \frac{\eta_s}{\eta_0} \lambda_{1k}, \quad \mu_{1k} = (1-\xi)\lambda_{1k}, \quad \mu_{2k} = (1-\xi)\frac{\eta_s}{\eta_0} \lambda_{1k} \quad (76),(77),(78)$$

(2) We then fit η_0 , η_s/η_0 , λ_1 , and α_1 to the $|\eta^*(\omega)|$ measurements of Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

(3) Mindful of Eq. (59), we then fit ξ to the $|\eta^*(\omega, \dot{\gamma}^0)|$ measurements from the nonlinear frequency sweeps of Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

(4) Still mindful of Eq. (59), we then check our best fit against the shear rate sweep interpolations from Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

Step (1) yields:

$$\begin{aligned}
\eta'_{1k} = & \frac{2\eta_k(\lambda_{1k}\omega)}{(\lambda_{1k}\dot{\gamma}^0)\tilde{\lambda}_k} \left(\frac{J_0 J_1}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{2(k-1)J_{2k-2}J_{2k-1}}{1 + 4(k-1)^2 \text{De}^2} + \frac{2kJ_{2k-1}J_{2k}}{1 + 4k^2 \text{De}^2} \right. \right. \\
& + \frac{(2k-1)J_{2k-1}J_{2k}}{1 + (2k-1)^2 \text{De}^2} + \left. \left. \frac{(2k+1)J_{2k}J_{2k+1}}{1 + (2k+1)^2 \text{De}^2} \right] \right) \\
& + (\lambda_{1k}\omega)^2 \frac{\eta_s}{\eta_0} \left(\frac{J_0(J_0 - J_2)}{1 + (\lambda_{1k}\omega)^2} + \sum_{k=1}^{\infty} \left[\frac{-2(k-1)J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 (\lambda_{1k}\omega)^2} - \frac{2kJ_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 (\lambda_{1k}\omega)^2} \right. \right. \\
& + \left. \left. \frac{(2k-1)J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 (\lambda_{1k}\omega)^2} + \frac{(2k+1)J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 (\lambda_{1k}\omega)^2} \right] \right) \\
& - \frac{\lambda_{1k}^2 \omega \dot{\gamma}^0 \phi_k}{\tilde{\lambda}_k} \left(\frac{J_0(J_1 + J_3)}{2(1 + \text{De}^2)} + \sum_{k=1}^{\infty} \left[\frac{(k-1)J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 (\lambda_{1k}\omega)^2} + \frac{kJ_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 (\lambda_{1k}\omega)^2} \right. \right. \\
& + \left. \left. \frac{J_{2k}(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{2(1 + (2k-1)^2 (\lambda_{1k}\omega)^2)} + \frac{(2k+1)J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{2(1 + (2k+1)^2 (\lambda_{1k}\omega)^2)} \right] \right), \tag{79}
\end{aligned}$$

and:

$$\begin{aligned}
\eta''_{1k} = & \frac{2\eta_k \lambda_{1k} \omega^2}{\dot{\gamma}^0 \tilde{\lambda}_k} \left(\frac{J_0 J_1}{1 + (\lambda_{1k}\omega)^2} + \sum_{k=1}^{\infty} \left[\frac{-4(k-1)^2 J_{2k-1}J_{2k-2}}{1 + 4(k-1)^2 (\lambda_{1k}\omega)^2} + \frac{4k^2 J_{2k-1}J_{2k}}{1 + 4k^2 (\lambda_{1k}\omega)^2} \right. \right. \\
& + \left. \left. \frac{-(2k-1)^2 J_{2k-1}J_{2k}}{1 + (2k-1)^2 (\lambda_{1k}\omega)^2} + \frac{(2k+1)^2 J_{2k}J_{2k+1}}{1 + (2k+1)^2 (\lambda_{1k}\omega)^2} \right] \right) \\
& - \lambda_{1k} \omega \frac{\eta_s}{\eta_0} \left(\frac{J_0^2 - J_0 J_2}{1 + \text{De}^2} + \sum_{k=1}^{\infty} \left[\frac{J_{2k-1}(-J_{2k-3} + J_{2k-1})}{1 + 4(k-1)^2 (\lambda_{1k}\omega)^2} - \frac{J_{2k-1}(-J_{2k-1} + J_{2k+1})}{1 + 4k^2 (\lambda_{1k}\omega)^2} \right. \right. \\
& + \left. \left. \frac{-J_{2k}(J_{2k-2} - J_{2k})}{1 + (2k-1)^2 (\lambda_{1k}\omega)^2} + \frac{J_{2k}(J_{2k} - J_{2k+2})}{1 + (2k+1)^2 (\lambda_{1k}\omega)^2} \right] \right) \\
& + \frac{\lambda_{1k} \dot{\gamma}^0 \phi_k}{2\tilde{\lambda}_k} \left(\frac{J_0(J_1 + J_3)}{1 + (\lambda_{1k}\omega)^2} + \sum_{k=1}^{\infty} \left[\frac{-J_{2k-1}(J_{2k-4} + 2J_{2k-2} + J_{2k})}{1 + 4(k-1)^2 (\lambda_{1k}\omega)^2} + \frac{J_{2k-1}(J_{2k-2} + 2J_{2k} + J_{2k+2})}{1 + 4k^2 (\lambda_{1k}\omega)^2} \right. \right. \\
& + \left. \left. \frac{-J_{2k}(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1 + (2k-1)^2 (\lambda_{1k}\omega)^2} + \frac{J_{2k}(J_{2k-1} + 2J_{2k+1} + J_{2k+3})}{1 + (2k+1)^2 (\lambda_{1k}\omega)^2} \right] \right) \tag{80}
\end{aligned}$$

Following Step (2), we next get the best fit of η_0 , η_s/η_0 , λ_1 and α_1 to the data of Fig. 9. of [111] (or of FIGURE 7-12 of [112]) using Eqs. (67) and (68) with:

$$\frac{\eta'_k(\omega)}{\eta_0} = \frac{1}{\zeta(\alpha_1)} \sum_{k=1}^{\infty} \frac{k^{\alpha_1} + (\eta_s/\eta_0) \lambda_1^2 k^{-\alpha_1} \omega^2}{k^{2\alpha_1} + \lambda_1^2 \omega^2}, \quad \frac{\eta''_k(\omega)}{\eta_0} = \frac{1}{\zeta(\alpha_1)} \sum_{k=1}^{\infty} \frac{[1 - (\eta_s/\eta_0)] \lambda_1 \omega}{k^{2\alpha_1} + \lambda_1^2 \omega^2} \tag{81}, (82)$$

yielding $\eta_0 = 0.169 \text{ MPa}\cdot\text{s}$, $\eta_s/\eta_0 = 0.001$, $\lambda_1 = 0.775 \text{ s}$, and $\alpha_1 = 3.27$

($\sigma^{(\omega)} \equiv \sqrt{S_R^{(\omega)}/10} = 0.0366$). Figure 15 compares this fit (upper most curve) with the ten SAOS measurements (**triangles**).

Following Step (3), we obtain the remaining undetermined model parameter, ξ , with the best fit to the thirty LAOS measurements using Eqs. (60), (71), (79) and (80) yielding $\xi = 0.050$ (or $\xi = 1.95$). Both $\xi = 0.050$ and $\xi = 1.95$ give us equally good fits

($\sigma^{(\omega, \dot{\gamma}^0)} \equiv \sqrt{S_R^{(\omega, \dot{\gamma}^0)}/30} = 0.0395$). We choose $\xi = 0.05$ over $\xi = 1.95$ since it gives $-\Psi_{2,0}/\Psi_{1,0} = 0.025$ rather than $-\Psi_{2,0}/\Psi_{1,0} = 0.975$; the value 0.025 is nearer the observed range $\frac{1}{10} \leq -\Psi_{2,0}/\Psi_{1,0} \leq \frac{1}{7}$. Figure 15 and Figure 16 illustrate the good agreement between the $|\eta^*(\omega, \dot{\gamma}^0)|$ predictions and the frequency sweep measurements and strain rate sweep interpolations from Fig. 9. of [111] (or of FIGURE 7-12 of [112]).

VII. CONCLUSION

We consider the sweep material functions for the Oldroyd 8-constant framework subject to large-amplitude oscillatory shear flow. We arrive at exact analytical expressions for the sweep material functions, Eqs. (44) and (45), Eqs. (50)–(52), and Eqs. (53)–(55), by extracting the leading terms from our previous work [19]. By *extracting*, we mean that we evaluated the Fourier integrals of our exact solutions for the first harmonic for the shear stress, and for both the zeroth and second harmonics for each normal stress difference. We check our main results, Eqs. (44) and (45), Eqs. (50)–(52), and Eqs. (53)–(55), with some finite difference calculations. We illustrate the usefulness of our results for $\eta'_1(\omega, \dot{\gamma}^0)$ and $\eta''_1(\omega, \dot{\gamma}^0)$ for both frequency and strain rate sweeps using the special case of the corotational Jeffreys fluid, generalized with the Spriggs relations. We find that the corotational Jeffreys fluid does not describe adequately the behavior of molten atactic polystyrene in oscillatory shear sweeps. We also illustrate the usefulness of our result for $\eta'_1(\omega, \dot{\gamma}^0)$ and $\eta''_1(\omega, \dot{\gamma}^0)$ for both frequency and strain rate sweeps using the special case of the Johnson-Segalman fluid, also generalized with the Spriggs relations. We find that the $|\eta^*(\omega, \dot{\gamma}^0)|$ predictions using the generalized Johnson-Segalman fluid agree well with both frequency and strain rate sweep measurements.

We find our new exact solutions for the sweep material functions [Eqs. (44) and (45), Eqs. (50)–(52), and Eqs. (53)–(55)], when generalized with the Spriggs relations, to be a playground for interpreting large-amplitude oscillatory shear flow measurements. We can thus use our main results for such interpretations in terms of any of the rich diversity of special cases of the Oldroyd 8-constant framework.

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Table I: Errata to cited articles

Ref.	Errata
[2]	See the following errata of Ref. 4 of [19].
[104]	In Eqs. (47) and (48), each τ in the integrands should be τ' .
[15]	See the following errata of Ref. 6 of [19].
[19]	In column 4 of rows 12 and 13, $\lambda_2 = \mu_0 = \mu_1$ should be $\mu_1 = -\lambda_1; \lambda_2 = \mu_0$ and $\mu_1 = \lambda_1; \lambda_2 = \mu_0$.
[23]	See the following errata of Ref. 26 of [19].
[29]	See the following errata of Ref. 32 of [19].
[38]	In Eq. (12), both occurrences of " τ " should be " $\mathbf{b}$ ".
[42]	In Eq. (5), the rightmost term should be multiplied by $\sin n\omega t$; ordinate labels in FIGURE1(a)–(d) should be W_i .
[80]	On the left side of Eq. (1) on p. 113, ϵ should be ϵ . See also Eq. (1) of [79]; In Eq. (2a), G' should be G'' , and in Eq. (2b), G'' should be G' . See Eqs. (117a) and (117b) of [83]
[81]	In Eqs. (2.4a), G' should be G'' , and in Eq. (2.4b), G'' should be G' . See Eqs. (117a) and (117b) of [83].
[90]	On p. 95, $e^{i\omega s}$ should be $e^{-i\omega s}$ in Eq. (A2); after Eq. (A.10), α should be $\sqrt{\omega \tau_d/2}$; and in Eq. (A.11), $\cos x$ should be $\cosh x$.
[98]	In Eqs. (47) and (48), " 20De^2 " and " $10\text{De}^2 - 50\text{De}^4$ " should be " 20De " and " $10\text{De} - 50\text{De}^3$ "; columns 10 and 11 of Table I in [39] are corrections to columns 6 and 7 of Table 1; eighteenth row of Table 1, "(2008)" should be "(2010)"; last row of Table 1, "(2007)" should be "(2008)".
[99]	See the following errata of Ref. 116 of [19].
[103]	In Eq. (35), " $\mathbb{S}(\tau)$ " should be " $-\mathbb{S}(\tau)$ ".
[107]	See the following errata of Ref. 126 of [19].

Table II: Literature on analytical solutions for $\eta'_1(\omega, \dot{\gamma})$ and $\eta''_1(\omega, \dot{\gamma})$ in large-amplitude oscillatory shear flow.

Authors (year)	Model	Form	[Ref.] (Correction to)
Kirkwood and Plock (1956, 1967); Plock (1957)	RD ^h , SK ^h	$\cong$	[79,80,81]
Paul (1969); Paul (1970); Bharadwaj (2012)	RD ^h , SK ^h	$\cong$	[82,83,84] (79,81)
Walters and Jones (1970); Walters (1975)	I ₃	$\cong$	[85]; Section 6.2.3 of [86]
Paul and Mazo (1969), Paul (1970)	RR	$\cong$	[87,83]
MacDonald, Marsh and Ashare (1969)	BC, OWFS [†]	$\cong$	[88]
Bird, Warner and Evans (1971)	RD	$\cong$	[89]
Abdel-Khalik <i>et al.</i> (1974); Bird <i>et al.</i> (1974)	GE+SK	$\cong$	[20,21]
Pearson and Rochefort (1982); Helfand and Rochefort (1982)	R [†]	$\cong$	[90,91]
Fan and Bird (1984)	CB [†]	$\cong$	[92](91)
Yu <i>et al.</i> (2002); Zhou (2004)	SE	$\cong$	[93,94]
Hoyle (2010)	PP	$\cong$	Ch. 4 of [95]
Wagner <i>et al.</i> (2011)	R	$\cong$	[96]
Giacomin <i>et al.</i> (2011)	CM [†] , CJ	$\cong$	[15](97)
Giacomin and Bird (2011)	ANSR	$\cong$	[98]
Bird <i>et al.</i> (2014)	RD	$\cong$	[99]
Thompson and de Souza Mendes (2015)	MSJ	$\cong$	See Section 5.2.2 of [100]
Bozorgi (2014); Bozorgi and Underhill (2014)	AS ^h	$\cong$	Ch. 8 of [101]; [102]
Giacomin <i>et al.</i> (2015)	CM ^p	$\cong$	[103]
Saengow <i>et al.</i> (2015)	CM [†]	=	[104]
Merger <i>et al.</i> (2016)	M ^{CM}	=	[105,106]
Layal <i>et al.</i> (2016)	RD	$\cong$	[107]
This paper	O ₈	=	[108]

Legend: ANSR $\equiv$ corotational arbitrary normal stress ratio; AS $\equiv$ active rod suspensions; BC $\equiv$ Bird-Carreau; CB $\equiv$ Curtiss-Bird; CJ $\equiv$ corotational Jeffreys; CM $\equiv$ corotational Maxwell; GE $\equiv$ Goddard integral expansion; I₃ $\equiv$ third order integral; L $\equiv$ Lodge rubberlike; M $\equiv$ Maxwell fluid; MSJ $\equiv$ material subjective Jeffreys; NGJ $\equiv$ nonlinear Generalized Jeffreys; OWFS $\equiv$ modified Oldroyd-Walters-Fredrickson-Spriggs; ^p $\equiv$ Padé approximants; PP $\equiv$ pompom; R $\equiv$ reptation; RD $\equiv$ rigid dumbbell; SE $\equiv$ simple emulsion; SK $\equiv$ shish-kebab; [†] $\equiv$ multiple relaxation times; = $\equiv$ exact; $\cong$ $\equiv$ approximate; ^h $\equiv$ with hydrodynamic interaction, ^{CM} $\equiv$ with Cox-Merz modified viscosity.

Table III: Dimensional variables

Name	Unit	Symbol
Angular frequency [Eq. (1)]	t^{-1}	ω
Complex first and second normal stress coefficient, the real and minus the imaginary parts, n th harmonic [Eq. (7)]	M/L	$\Psi'_{1,n}, \Psi''_{1,n}, \Psi'_{2,n}, \Psi''_{2,n}$
Extra stress tensor [Eq. (19)]	M/Lt^2	$\boldsymbol{\tau}$
Extra stress, ij th component*	M/Lt^2	τ_{ij}
Hydrodynamic pressure [Eq. (19)]	M/Lt^2	p
Nonlinear complex shear moduli, real and imaginary parts, n th harmonic [Eqs. (9), (10)]	M/Lt^2	G'_n, G''_n
Normal stress difference, first [Eq. (7)]	M/Lt^2	$N_1 = \tau_{xx} - \tau_{yy}$
Normal stress difference, second [Eq. (8)]	M/Lt^2	$N_2 = \tau_{yy} - \tau_{zz}$
Oldroyd constant [Eq. (13)]	t	$\mu_0, \mu_1, \mu_2, \nu_1, \nu_2$
Oldroyd constant, k th spectrum [Eq. (13)]	t	μ_{1k}, μ_{2k}
Oldroyd constant, relaxation time [Eq. (2)]	t	λ_1
Oldroyd constant, relaxation time, k th spectrum [Eq. (62)]	t	λ_{1k}
Oldroyd constant, retardation time [Eq. (13)]	t	λ_2
Oldroyd constant, retardation time, k th spectrum [Eq. (63)]	t	λ_{2k}
Peak value of the shear stress response in LAOS	M/Lt^2	τ_0
Rate of strain tensor [Eq. (5)]	t^{-1}	$\dot{\boldsymbol{\gamma}}$
Shear rate, yx component [Eq. (1)]	t^{-1}	$\dot{\gamma}_{yx} \equiv \dot{\gamma}$
Shear rate, amplitude [Eq. (1)]	t^{-1}	$\dot{\gamma}^0$
Sum of the squares of residuals, LAOS [Eq. (71)]	M/Lt	$S_R^{(\omega, \dot{\gamma}^0)}$
Sum of the squares of residuals, SAOS [Eq. (67)]	M/Lt	$S_R^{(\omega)}$
Velocity [Eq. (5)]	L/t	$\mathbf{v}$
Viscosity, complex [Eq. (60)]	M/Lt	$ \eta^* $
Viscosity, complex, m th measurement [Eqs. (67) and (71)]	M/Lt	$ \eta^* _m$
Viscosity, complex, shifting constant [Figure 13, Figure 16]	M/Lt	η_+
Viscosity, infinite shear	M/Lt	$\eta_\infty = \lim_{\dot{\gamma}^0 \rightarrow \infty} \eta(\dot{\gamma})$
Viscosity, loss and storage, n th harmonic [Eq. (6)]	M/Lt	η'_n, η''_n
Viscosity, loss and storage, 1st harmonic, k th spectrum [Eq. (64)]	M/Lt	η'_k, η''_k
Viscosity, solvent	M/Lt	η_s
Viscosity, zero shear [Eq. (13)]	M/Lt	$\eta_0 = \lim_{\dot{\gamma}^0 \rightarrow 0} \eta(\dot{\gamma})$
Vorticity tensor [Eq. (20)]	t^{-1}	$\boldsymbol{\omega}$

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

*Where τ_{ij} is the force exerted in the j th direction on a unit area of fluid surface of constant x_i by fluid in the region lesser x_i on fluid in the region greater x_i (see “Note on the Sign Convention for the Stress Tensor” on pp. 19–20 of [109], or pp. 24–25 of [110]).

Table IV: Dimensionless variables and groups

Name	Symbol
Coefficient defined in Eq. (26), (27) and (28)	ϕ, ψ, Ω
Combined fluid parameter [Eq. (25)]	$\tilde{\lambda}$
Constant defined in Eq. (25)	$\alpha \equiv (\text{Wi}/\text{De}) \tilde{\lambda}$
Deborah number [Eq. (3)]	De
First and second normal stress difference [Eqs. (23) and (24)]	$\mathbb{N}_1, \mathbb{N}_2$
Integral-defined function [Eq. (29)–(35)]	$I_1, I_2, \dots, I_7$
Johnson-Segalman fluid parameter [Eq. (77)]	ξ
Oldroyd constant, spectrum dispersibility [Eq. (62)]	α_1
Shear stress [Eq. (22)]	$\mathbb{S}$
Standard deviation, LAOS	$\sigma^{(\omega, \dot{\gamma}^0)}$
Standard deviation, SAOS	$\sigma^{(\omega)}$
Time	$\tau \equiv \omega t$
Weissenberg number [Eq. (3)]	Wi

Table V: Order-pairs of $\dot{\gamma}$ and $|\eta^*(\omega, \dot{\gamma}^0)|$ measurements of Polystyrene at 190°C using Weissenberg rheogoniometer at six different strain amplitudes (see Fig. 9. of [111] or FIGURE 7-12 of [112])).

$\omega [s^{-1}], \eta^*(\omega, \dot{\gamma}^0) \times 10^4 [Pa \cdot s]$					
$\gamma_0 = 0.2$	$\gamma_0 = 2.67$	$\gamma_0 = 5.00$	$\gamma_0 = 6.67$	$\gamma_0 = 8.00$	$\gamma_0 = 10.0$
0.3796, 15.71	0.9550, 11.46	0.6362, 11.07	0.6367, 9.889	0.6313, 8.978	0.6437, 7.168
0.6074, 13.97	1.572, 8.820	0.9565, 8.932	1.268, 6.131	0.9583, 6.370	0.9592, 5.380
0.9632, 12.73	2.495, 6.840	1.255, 7.805	1.960, 4.316	1.550, 4.274	1.260, 4.067
1.501, 10.61	4.376, 4.666	2.477, 4.957	-	3.144, 2.485	2.487, 2.403
2.404, 8.228	6.288, 3.675	4.385, 3.197	-	5.672, 1.444	4.406, 1.362
3.746, 6.483	9.456, 2.714	6.247, 2.342	-	-	-
6.001, 4.908	12.41, 2.188	9.484, 1.622	-	-	-
9.527, 3.627	15.43, 1.881	-	-	-	-
15.13, 2.637	18.50, 1.591	-	-	-	-
23.59, 1.871	24.73, 1.303	-	-	-	-

Table VI: Order-pairs of ω and $|\eta^*(\omega, \dot{\gamma}^0)|$ interpolations from points in Figure 12.

Polystyrene at 190°C at six frequencies, ω [rad/s] (see Fig. 9. of [111] or FIGURE 7-12 of [112]).

	$\dot{\gamma}^0$ [s ⁻¹], $ \eta^*(\omega, \dot{\gamma}^0) \times 10^4$ [Pa·s]					
ω [rad/s]	0.9568	1.256	1.940	4.007	5.701	9.929
	9.568, 5.252	12.56, 4.067	19.40, 2.817	40.07, 1.464	-	-
	7.654, 6.474	10.05, 5.136	15.52, 3.645	32.06, 1.940	45.61, 1.432	-
	6.382, 7.543	8.378, 6.230	12.94, 4.316	-	-	-
	4.784, 8.932	6.280, 7.805	9.700, 5.813	20.04, 3.380	28.51, 2.516	49.65, 1.571
	2.555, 11.46	3.354, 10.02	5.180, 7.891	10.70, 5.054	15.22, 3.916	26.51, 2.608
	0, 12.52	0, 11.30	0, 9.344	0, 6.281	0, 5.067	0, 3.485

Figure 1: Pipkin map. Newtonian behavior falls within the quarter unit circle, viscometric flow where $\lambda_1 \omega \ll 1$, and linear viscoelasticity where $\lambda_1 \dot{\gamma}^0 \ll 1$. Sweeps of strain rate ($\bullet$), and of frequency of the first ($\bullet$) and second ($\bullet$) kinds.

Figure 2: 30 contours of η'_1/η_0 using our new exact solutions [Eqs. (46)] for the special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). The maxima (darkest **red**) for all of λ_2/λ_1 are all 1. The minima (darkest **blue**) for each increasing value of λ_2/λ_1 are 0., 0.0370, 0.1111, 0.3333.

Figure 3: 20 contours of η_1''/η_0 using our new exact solutions [Eqs. (47)] for the special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). The maxima (darkest **red**) for each increasing value of λ_2/λ_1 are 0.4983, 0.4799, 0.4429, 0.3322, and the minima (darkest **blue**), -0.0150, -0.0144, -0.0133, -0.0100.

Figure 4: 15 contours of $\Psi_{1,0}\dot{\gamma}^0/\eta_0 = -2\Psi_{2,0}\dot{\gamma}^0/\eta_0$ using our new exact solutions [Eq. (50) or Eq. (53)] for the special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). The maxima (darkest **red**) for each increasing value of λ_2/λ_1 are 0.5999, 0.5777, 0.5333, 0.3999. The minima (darkest **blue**) are all 0..

Figure 5: 25 contours of $\Psi'_{1,2}\dot{\gamma}^0/\eta_0 = -2\Psi'_{2,2}\dot{\gamma}^0/\eta_0$ using our new exact solutions [Eq. (51) or Eq. (54)] for the special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). The maxima (darkest **red**) for each increasing value of λ_2/λ_1 are 0.4755, 0.4579, 0.4227, 0.3170, and the minima (darkest **blue**), -0.1138, -0.1096, -0.1011, -0.0759.

Figure 6: 15 contours of $\Psi''_{1,2}\dot{\gamma}^0/\eta_0 = -2\Psi''_{2,2}\dot{\gamma}^0/\eta_0$ using our new exact solutions [Eq. (52) or Eq. (55)] for the special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). The maxima (darkest **red**) for each increasing value of λ_2/λ_1 are 0.3705, 0.3568, 0.3293, 0.2470. The minima (darkest **blue**) are all 0.

Figure 7: Behavior of η'_i/η_0 at different values of $De \equiv \lambda_1 \omega = 0.1, 1, 10, 100$. Close agreement shows consistency between our new exact solutions [Eqs. (46)] (**black** curves) and our finite difference calculations using Eqs. (36) and (37) [with Eq. (22) along with Eqs. (29)–(34)] (**red** curves). Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 8: Behavior of η''/η_0 at different values of $De \equiv \lambda_1 \omega = 0.1, 1, 10, 100$. Close agreement shows consistency between our new exact solutions [Eq. (47)] (**black** curves) and our finite difference calculations using Eqs. (36) and (37) [with Eq. (22) along with Eqs. (29)–(34)] (**red** curves). Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 9: Behavior of $\Psi_{1,0}\dot{\gamma}^0/\eta_0 = -2\Psi_{2,0}\dot{\gamma}^0/\eta_0$ at different values of $De \equiv \lambda_1\omega = 0.1, 1, 10, 100$. Close agreement shows consistency between our new exact solutions [Eq. (50) or Eq. (53), **black** curves) and our finite difference calculations using Eq. (38), or Eq. (41) [with Eq. (23), or Eq. (24) along with Eqs. (29)–(35)] (**red** curves). Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 10: Behavior of $\Psi'_{1,2}\dot{\gamma}^0/\eta_0 = -2\Psi'_{2,2}\dot{\gamma}^0/\eta_0$ at different values of $De \equiv \lambda_1\omega = 0.1, 1, 10, 100$. Close agreement shows consistency between our new exact solutions [Eq. (51) or Eq. (54), **black** curves) and our finite difference calculations using Eq. (39), or Eq. (42) [with Eq. (23), or Eq. (24) along with Eqs. (29)–(35)] (**red** curves). Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 11: Behavior of $\Psi''_{1,2}\dot{\gamma}^0/\eta_0 = -2\Psi''_{2,2}\dot{\gamma}^0/\eta_0$ at different values of $De \equiv \lambda_1\omega = 0.1, 1, 10, 100$. Close agreement shows consistency between our new exact solutions [Eq. (52) or Eq. (55), **black** curves) and our finite difference calculations using Eq. (40), or Eq. (43) [with Eq. (23), or Eq. (24) along with Eqs. (29)–(35)] (**red** curves). Special case of the corotational Jeffreys fluid ($\lambda_2/\lambda_1 = 0, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ and $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

Figure 12: Best fit of generalized corotational Jeffrey fluid, Eq. (60) [with Eq. (64)–(66)], to the $|\eta^*(\omega, \gamma^0)|$ frequency sweep measurements on molten atactic polystyrene at 190°C at six different strain amplitudes $\frac{Wi}{De} = 0.2(\Delta), 2.67(\circ), 5(\square), 6.67(\diamond), 8(+), 10(\times)$: $\lambda_1 = 0.775\text{s}, \lambda_2 = 0.0186\text{s}, \eta_0 = 0.169\text{MPa}\cdot\text{s}$ and $\alpha_1 = 3.27$. **Red** curve is $|\eta^*(\omega)|$ from Eq. (68) [with Eqs. (69) and (70)]. Measurements are in Table V.

Figure 13: $|\eta^*(\omega, \dot{\gamma}^0)|$ for strain rate sweeps from Eq. (60) [with Eq. (64)–(66)] using best fit parameters from Figure 12, *versus* interpolations from Table V (which we reported in Table VI). Inset, from top to bottom, $|\eta^*(\omega, \dot{\gamma}^0)|$ is offset by $\eta_+ = 2.5, 2, 0.15, 0.1, 0.05, 0$ MPa·s.

Figure 14: Pipkin map of experimental conditions for $|\eta^*(\omega)|$ and $|\eta^*(\omega, \dot{\gamma}^0)|$ for the frequency sweeps of Fig. 9. of [111] (or of FIGURE 7-12 of [112]). Atactic polystyrene melt at 190°C taken over six different strain amplitudes. For $|\eta^*(\omega)|$, $\frac{Wi}{De} = 0.2 (\Delta)$, and $|\eta^*(\omega, \dot{\gamma}^0)|$, $\frac{Wi}{De} = 2.67 (\circ), 5 (\square), 6.67 (\diamond), 8 (+), 10 (\times)$. Measurements are in Table V.

Figure 15: Best fit of generalized Johnson-Segalman, Eq. (60) [with Eqs. (64), Eqs. (79) and (80)], to the $|\eta^*(\omega, \dot{\gamma}^0)|$ frequency sweep measurements on molten atactic polystyrene at 190°C at six strain amplitudes $\frac{Wi}{De} = 0.2(\Delta)$, $2.67(\circ)$, $5(\square)$, $6.67(\diamond)$, $8(+)$, $10(\times)$: $\eta_0 = 0.169 \text{ MPa}\cdot\text{s}$, $\eta_s/\eta_0 = 0.001$, $\lambda_1 = 0.775 \text{ s}$, $\xi = 0.05$, and $\alpha_1 = 3.27$. **Red** curve is $|\eta^*(\omega)|$ from Eq. (68) [with Eqs. (81) and (82)]. Measurements are in Table V.

Figure 16: $|\eta^*(\omega, \dot{\gamma}^0)|$ for strain rate sweeps from Eq. (60) [with Eqs. (64), Eqs. (79) and (80)] using best fit parameters from Figure 15, *versus* interpolations from Table V (which we reported in Table VI). Inset, from top to bottom, $|\eta^*(\omega, \dot{\gamma}^0)|$ is offset by $\eta_+ = 2.5, 2, 0.15, 0.1, 0.05, 0$ MPa·s.

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