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**SERIES EXPANSION FOR SHEAR STRESS
IN LARGE-AMPLITUDE OSCILLATORY SHEAR FLOW
FROM OLDROYD 8-CONSTANT FRAMEWORK**

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ABSTRACT

When polymeric liquids undergo large-amplitude oscillatory shear flow (LAOS), the shear stress responds as a Fourier series, the higher harmonics of which are caused by the fluid nonlinearity, and the first harmonic of which is a nonlinear function of both the frequency and the shear rate amplitude. The Oldroyd 8-constant framework for continuum constitutive theory contains a rich diversity of popular special cases for polymeric liquids. The shear stress response for the Oldroyd 8-constant framework has recently yielded to exact analytical solution. However, in its closed form, Bessel functions appear 24 times, each within summations to infinity. In this paper, to simplify the exact solution, we expand it in a Taylor series of the dimensionless shear rate amplitude. We truncate the power series expansion after the 16th power of the shear rate amplitude. We find our main result reduces to the well-known expression for the special cases of the corotational Jeffreys and corotational Maxwell fluids. Whereas these special cases yielded to the Goddard integral expansion (GIE), the more general Oldroyd 8-constant framework does not. We use Ewoldt grids to show our main result to be highly accurate, for the corotational Jeffreys and corotational Maxwell fluids. Our solutions agree closely with the exact solutions so long as $Wi/De < \frac{7}{2}$, for the special cases of the corotational Jeffreys and corotational Maxwell fluids. We apply our main result for the special case of the Johnson-Segalman fluid, to describe the measured frequency sweep and shear rate amplitude sweep responses of molten atactic polystyrene. For this, we use the Spriggs relations to generalize our main result to multimode, which then agrees closely with the measured responses.

Keywords: Large-amplitude oscillatory shear; LAOS; Oldroyd 8-constant framework; shear stress; Goddard integral expansion; shear rate amplitude sweep.

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I INTRODUCTION

Chemical engineers use large-amplitude oscillatory shear (LAOS) flow experiments to investigate the physics of complex liquids (see [1]; Ch. 11 of [2]; see Eq. (2.42) of [3]). Since its conception in 1935 (see [4,5] and, mindful of the erratum following Ref. 4 of [6], see also Ref. 7) oscillatory shear flow has become by far the most popular laboratory method for exploring the physics of polymeric liquids.

We generate oscillatory shear flow by confining the fluid to a simple shear apparatus and then by displacing one solid-liquid boundary sinusoidally in its own plane (see FIG. 1. of [6]). The corresponding cosinusoidal shear rate is given by:

$$\dot{\gamma}(t) = \dot{\gamma}^0 \cos \omega t \quad (1)$$

Using the characteristic relaxation time of the viscoelastic fluid, λ_1 , to deepen our understanding of the flow, we normally nondimensionalize Eq. (1) to:

$$\lambda_1 \dot{\gamma}(t) = Wi \cos De(t / \lambda_1) \quad (2)$$

where the dimensionless Deborah and Weissenberg numbers are given by:

$$De \equiv \lambda_1 \omega, \quad Wi \equiv \lambda_1 \dot{\gamma}^0 \quad (3),(4)$$

the former of which reflects elasticity in both the fluid and the flow, and the latter, reflects nonlinearity. In this paper, we define dimensional symbols in Table I, and dimensionless ones in Table II.

When higher harmonics arrive in the shear stress response (see Eq. (21) of [8]):

$$\frac{\tau_{yx}(\omega, \dot{\gamma}^0; t)}{\dot{\gamma}^0} = - \sum_{n=1, \text{odd}}^{\infty} \eta'_n(\omega, \dot{\gamma}^0) \cos n\omega t + \eta''_n(\omega, \dot{\gamma}^0) \sin n\omega t \quad (5)$$

we call the oscillatory experiment *large-amplitude*. Many important notations, alternative to Eq. (5), have been introduced for analyzing the higher harmonics in the shear stress (see Eq. (3) of [9]); §9 and §10 of [10]). For polymeric liquids, these higher harmonics are commonly observed when [11,12,13]:

$$Wi > 1 \quad (6)$$

which is sometimes further complicated to (see Eq. (11) and Figure 14 of [14]):

$$[Wi > 1] \cap [Wi/De > 1] \quad (7)$$

but, for this paper, Eq. (6) shall serve as our working definition of large-amplitude.

In this paper, we arrive at the approximate analytical solution in power of shear rate amplitude for shear stress for large-amplitude oscillatory shear flow for an entire framework of constitutive equations by using Taylor series expansion into the exact analytical solution. Specifically, we choose the Oldroyd 8-constant framework. We choose this framework for its rich diversity of important and popular special cases (TABLE IV of [6] lists 14 of these). The Oldroyd 8-constant framework is not to be confused with an 8-constant model. In fact, we know of no chemical engineering application of the Oldroyd 8-constant framework employing all of its 8 constants. Instead, the 8-constant framework reflects Oldroyd's remarkable vision [25] for the many popular special cases that would follow.

II OLDROYD 8-CONSTANT FRAMEWORK

Chemical engineers can use the Oldroyd 8-constant framework in polymer processing, as for wire coating (mindful of the errata following Ref. 32 of [6], see Case III in [15]), and for corrugated wire coated through a corrugated die (see Section 3. of [16]; Section 2. of [17]), and for plastic pipe extrusion for elliptical pipe ([18,19]), and also for extrusion from an eccentric annular pipe die (Case II in [15];[20];[21,22];[23]).

The Oldroyd 8-constant framework is given by (see Eq. (8.1-2) in [24];[25]):

$$\begin{aligned} \tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{1}{2}\mu_0(\text{tr}\tau)\dot{\gamma} - \frac{1}{2}\mu_1\{\tau \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau\} + \frac{1}{2}v_1(\tau : \dot{\gamma})\delta \\ = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} - \mu_2\{\dot{\gamma} \cdot \dot{\gamma}\} + \frac{1}{2}v_2(\dot{\gamma} : \dot{\gamma})\delta \right) \end{aligned} \quad (8)$$

in which the higher order nonlinear terms are underlined, and:

$$\dot{\gamma} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger, \quad \omega \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger, \quad \frac{\mathcal{D}\mathbf{b}}{\mathcal{D}t} \equiv \frac{D\mathbf{b}}{Dt} + \frac{1}{2}\{\omega \cdot \mathbf{b} - \mathbf{b} \cdot \omega\} \quad (9),(10),(11)$$

Eq. (8) reduces exactly to a rich diversity of special cases, 14 of which are tabulated in TABLE IV of [6], and reduces approximately to suspensions of rigid or finitely extensible nonlinear elastic dumbbells, in Table 1 of [26].

The exact solution for the dimensionless shear stress response is (see Eq. (65) of [6]):

$$\begin{aligned} \mathbb{S} = e^{\frac{-\tau}{\text{De}}} \sin(\alpha \sin \tau) \left[-\frac{1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 - \frac{\text{Wi}}{\text{De}} \frac{\sigma_2}{\tilde{\lambda} \lambda_1^2} I_5 \right] \\ + e^{\frac{-\tau}{\text{De}}} \cos(\alpha \sin \tau) \left[-\frac{1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{\sigma_2}{\tilde{\lambda} \lambda_1^2} I_6 \right] \end{aligned} \quad (12)$$

where:

$$\tau \equiv \omega t \quad (13)$$

$$\alpha \equiv \frac{\text{Wi}}{\text{De}} \tilde{\lambda} \equiv \frac{\text{Wi}}{\text{De}} \frac{\sqrt{\sigma_1}}{\lambda_1} \quad (14)$$

$$\sigma_1 \equiv \lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2}v_1) - \mu_1(\mu_1 - v_1) \quad (15)$$

and:

$$\sigma_2 \equiv \lambda_1 \lambda_2 + \mu_0(\mu_2 - \frac{3}{2}v_2) - \mu_1(\mu_2 - v_2) \quad (16)$$

Substituting Eqs. (14) through (16) into Eq. (12) gives:

$$\begin{aligned} \mathbb{S} = e^{\frac{-\tau}{\text{De}}} \sin \left(\frac{\text{Wi}}{\text{De}} \frac{\sqrt{\sigma_1}}{\lambda_1} \sin \tau \right) \left[-\frac{1}{\text{De}} I_1 + \frac{\lambda_2}{\lambda_1} I_3 - \frac{\text{Wi}}{\text{De}} \frac{\sigma_2}{\lambda_1 \sqrt{\sigma_1}} I_5 \right] \\ + e^{\frac{-\tau}{\text{De}}} \cos \left(\frac{\text{Wi}}{\text{De}} \frac{\sqrt{\sigma_1}}{\lambda_1} \sin \tau \right) \left[-\frac{1}{\text{De}} I_2 + \frac{\lambda_2}{\lambda_1} I_4 - \frac{\text{Wi}}{\text{De}} \frac{\sigma_2}{\lambda_1 \sqrt{\sigma_1}} I_6 \right] \end{aligned} \quad (17)$$

where I_1 through I_6 are defined by (Eqs. (66) through (71) of [71]):

$$I_1 = \frac{4\lambda_1 \text{De}^2}{\text{Wi} \sqrt{\sigma_1}} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[-\frac{2\text{De}k^2 J_{2k}}{1+4k^2 \text{De}^2} \cos 2k\tau + \frac{k J_{2k}}{1+4k^2 \text{De}^2} \sin 2k\tau \right] \quad (18)$$

$$I_2 = \frac{2\lambda_1 \text{De}^2}{\text{Wi}\sqrt{\sigma_1}} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{(2k-1)J_{2k-1}}{1+(2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{\text{De}(2k-1)^2 J_{2k-1}}{1+(2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (19)$$

$$I_3 = \text{De} e^{\frac{\tau}{\text{De}}} J_1 + \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 \text{De}^2} \cos 2k\tau + (2\text{De}k) \frac{-J_{2k-1} + J_{2k+1}}{1+4k^2 \text{De}^2} \sin 2k\tau \right] \quad (20)$$

$$I_4 = \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-(2k-1)\text{De}(J_{2k-2} - J_{2k})}{1+(2k-1)^2 \text{De}^2} \cos(2k-1)\tau + \frac{J_{2k-2} - J_{2k}}{1+(2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (21)$$

$$I_5 = \frac{\lambda_1 \text{De}^2 e^{\frac{\tau}{\text{De}}}}{\text{Wi}\sqrt{\sigma_1}} J_1 + \text{De} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{2(1+4k^2 \text{De}^2)} \cos 2k\tau + (\text{De}k) \frac{J_{2k-2} + 2J_{2k} + J_{2k+2}}{1+4k^2 \text{De}^2} \sin 2k\tau \right] \quad (22)$$

$$I_6 = \frac{\text{De}}{2} e^{\frac{\tau}{\text{De}}} \sum_{k=1}^{\infty} \left[\frac{-\text{De}(2k-1)(J_{2k-3} + 2J_{2k-1} + J_{2k+1})}{1+(2k-1)^2 \text{De}^2} \cos(2k-1)\tau \right. \\ \left. + \frac{J_{2k-3} + 2J_{2k-1} + J_{2k+1}}{1+(2k-1)^2 \text{De}^2} \sin(2k-1)\tau \right] \quad (23)$$

For steady shear flow, the viscometric functions for the Oldroyd 8-constant framework are given by (see Eqs. (12)–(14) of [25]; Eq. (14) of [27]):

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \quad (24)$$

Taking limit $\dot{\gamma} \rightarrow \infty$ into Eq. (24) yields:

$$\eta_{\infty}/\eta_0 = \sigma \quad (25)$$

where the Oldroyd constant ratio is defined by:

$$\sigma \equiv \sigma_2/\sigma_1 \quad (26)$$

where, for the special case of the corotational Jeffreys fluid, $\sigma = \lambda_2/\lambda_1$, and for the special case of the corotational Maxwell fluid, $\sigma = 0$. Eq. (25) thus assigns clear physical significance to σ . Further, in the limit as $\text{De} \rightarrow 0$, Eqs. (17) through (23) yield Eq. (24). The Oldroyd 8-constant framework thus predicts viscosity curve shapes with just three parameters: σ_1 , η_0 and σ .

Introducing the characteristic time of the Oldroyd 8-constant fluid, $\sqrt{\sigma_1}$ (Eq. (17) of [27]), into Eq. (24) gives:

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \sigma(\sqrt{\sigma_1}\dot{\gamma})^2}{1 + (\sqrt{\sigma_1}\dot{\gamma})^2} \quad (27)$$

Since we require the characteristic time to be real, Eq. (15) requires that:

$$\lambda_1^2 + \mu_1(\mu_0 - \mu_1) > \nu_1\left(\frac{3}{2}\mu_0 + \mu_1\right) \quad (28)$$

Further, from Eq. (27), the shear stress for steady shear flow is given by:

$$\mathbb{S} \equiv \frac{\tau_{yx}}{\eta_0 \dot{\gamma}^0} = \frac{1 + \sigma (\sqrt{\sigma_1} \dot{\gamma})^2}{1 + (\sqrt{\sigma_1} \dot{\gamma})^2} \frac{\sqrt{\sigma_1} \dot{\gamma}}{\tilde{\lambda} \text{Wi}} \quad (29)$$

where Eq. (14) defines $\tilde{\lambda}$.

For small amplitude oscillatory shear flow (SAOS), in the limit as $\text{Wi} \rightarrow 0$, holding the frequency constant, the shear stress response is given by (Eq. (40) of [6]; Eq. (8.1-10) of [24]):

$$\mathbb{S} = - \left(\frac{1 + (\lambda_2/\lambda_1) \text{De}^2}{1 + \text{De}^2} \cos \tau + \frac{(1 - \lambda_2/\lambda_1) \text{De}}{1 + \text{De}^2} \sin \tau \right) \quad (30)$$

We will use Eqs. (29) and (30) as consistency checks on our main results in Section IV and also to plot Figure 5 through Figure 8.

III ANALYSIS

We expand the exact solution for the shear stress responses for any special cases of the Oldroyd 8-constant framework, Eq. (17) [with Eqs. (18) through (23)] in a Taylor series about $\text{Wi} = 0$ (in even powers up to Wi^{16} , holding De constant). We get:

$$\mathbb{S}_{\mathbb{T}}(\tau) = - \sum_{m=0}^{\infty} \mathbb{S}_{\mathbb{T},m} = - \frac{\lambda_2}{\lambda_1} C_1 - \left(1 - \frac{\lambda_2}{\lambda_1} \right) \sum_{m=0}^{\infty} \frac{\text{Wi}^{2m}}{(-1)^m 2^{2m}} \left(\frac{\sigma_1}{\lambda_1^2} \right)^{m-1} \left(\frac{(\kappa_m C_n + \varphi_m S_n)}{\prod_{k=1}^{m+(n+1)/2} 1 + k^2 \text{De}^2} \right) \quad (31)$$

where:

$$C_n \equiv \cos n\tau, \quad S_n \equiv \sin n\tau \quad (32),(33)$$

where m is the power of each term in the series, and where $\kappa_m C_n$ and $\sigma_m S_n$ are given up to Wi^{16} in Table III and Table IV. Eq. (31) [with Eqs. (32) and (33), Table III and Table IV (up to Wi^{16})] is the main result of this paper. To our knowledge, Eq. (31) is new. We will compare our main result for the special cases of corotational Jeffreys and corotational Maxwell fluids with the exact solution (Figure 1 through Figure 4).

From Eq. (31) [with Eqs. (32) and (33), Table III and Table IV (up to Wi^{16})], for nonlinear viscosities, the loss and storage viscosities for first harmonic are:

$$\frac{\eta'_1}{\eta_0} = - \frac{\lambda_2}{\lambda_1} - \left(1 - \frac{\lambda_2}{\lambda_1} \right) \sum_{m=0}^{\infty} \frac{\text{Wi}^{2m}}{(-1)^m 2^{2m}} \left(\frac{\sigma_1}{\lambda_1^2} \right)^{m-1} \frac{\kappa_m}{\prod_{k=1}^{m+(n+1)/2} 1 + k^2 \text{De}^2} \quad (34)$$

and:

$$\frac{\eta''_1}{\eta_0} = - \left(1 - \frac{\lambda_2}{\lambda_1} \right) \sum_{m=0}^{\infty} \frac{\text{Wi}^{2m}}{(-1)^m 2^{2m}} \left(\frac{\sigma_1}{\lambda_1^2} \right)^{m-1} \frac{\varphi_m}{\prod_{k=1}^{m+(n+1)/2} 1 + k^2 \text{De}^2} \quad (35)$$

where κ_m and φ_m are the coefficients of C_1 and S_1 of the m th rows in Table III and Table IV. We will use Eqs. (34) and (35) in the worked example below.

Figure 1 through Figure 4 employ our main result, Eq. (31) [with Eqs. (32) and (33),

Table III and Table IV (up to Wi^{16}) to illustrate the role of η_∞ on the loop shapes. From these figures, we learn that increasing η_∞ narrows the shear stress *versus* shear rate loops and thus diminishes their areas. In other words, increasing η_∞ decreases fluid elasticity and increases viscous dissipation.

IV CONSISTENCY CHECKS

In this section, we will check our main result for consistency with well-known previous series for special cases, and with two well-known results for the Oldroyd 8-constant framework.

a Previous series for two special cases

For consistency, we first compare our new series, up to Wi^6 , to the well-known solutions for the corotational Jeffreys and corotational Maxwell fluids (Eqs. (82) and (83) of [6]). Specifically, for the corotational Jeffreys fluid (up to Wi^6), truncating Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, $\sigma_2 = \lambda_1\lambda_2$ and $\sigma_1 = \lambda_1^2$] after N terms, we get:

$$\mathbb{S}_T(\tau) \equiv -\frac{\lambda_2}{\lambda_1}C_1 - \left(1 - \frac{\lambda_2}{\lambda_1}\right) \sum_{m=0}^N \frac{Wi^{2m}}{(-1)^m 2^{2m}} \left(\frac{(\kappa_m C_n + \varphi_m S_n)}{\prod_{k=1}^{m+(n+1)/2} 1+k^2 De^2} \right) \quad (36)$$

which, for $N=3$, matches Eq. (83) of [6], as it must. Further, for the corotational Maxwell fluid (up to Wi^6), truncating Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, $\sigma_2 = \lambda_2 = 0$ and $\sigma_1 = \lambda_1^2$] after N terms, we get:

$$\mathbb{S}_T(\tau) \equiv -\sum_{m=0}^N \frac{Wi^{2m}}{(-1)^m 2^{2m}} \left(\frac{(\kappa_m C_n + \varphi_m S_n)}{\prod_{k=1}^{m+(n+1)/2} 1+k^2 De^2} \right) \quad (37)$$

which, for $N=3$, matches Eq. (82) of [6], as it must. The expressions for $\kappa_m C_n$ and $\varphi_m S_n$ in Eqs. (36) and (37) are given in Tables I and II of Supplementary Material II of [28].

We also compare our main result with our previous power series, up to Wi^{16} , for the corotational Jeffreys and corotational Maxwell fluids (Eqs. (20) and (39) of [28]). For the corotational Jeffreys fluid, $\sigma_2 = \lambda_1\lambda_2$ and $\sigma_1 = \lambda_1^2$, Eq. (31) becomes Eq. (36) with $N=8$ which matches Eq. (39) of [28], as it must. We will use Eq. (36) with $N=8$ to plot Figure 5 through Figure 7 below. For the corotational Maxwell fluid, $\sigma_2 = \lambda_2 = 0$ and $\sigma_1 = \lambda_1^2$, Eq. (31) becomes Eq. (37) with $N=8$ which matches Eq. (20) of [28], as it must. We will use Eq. (37) with $N=8$ to plot Figure 8 below.

b Steady shear flow

In the low frequency limit (vanishingly small De), the main result, Eq. (31) [with Table III and Table IV] reduces to:

$$\begin{aligned}\frac{\eta_{\mathbb{T}}}{\eta_0} &= \left(1 + (1-\sigma) \sum_{m=1}^{\infty} (-1)^m (\sqrt{\sigma_1} \dot{\gamma})^{2m} \right) = \sigma + (1-\sigma) \sum_{m=0}^{\infty} (-1)^m (\sqrt{\sigma_1} \dot{\gamma})^{2m} \\ &= \frac{1 + \sigma (\sqrt{\sigma_1} \dot{\gamma})^2}{1 + (\sqrt{\sigma_1} \dot{\gamma})^2}\end{aligned}\quad (38)$$

which matches the result for steady shear flow, Eq. (27), as it should. Thus, the shear stress for steady shear flow is given by:

$$\mathbb{S}_{\mathbb{T}} = \frac{\sqrt{\sigma_1} \dot{\gamma}}{\tilde{\lambda} \text{Wi}} \left[\sigma + (1-\sigma) \sum_{m=0}^{\infty} (-1)^m (\sqrt{\sigma_1} \dot{\gamma})^{2m} \right] \quad (39)$$

which, for the special case of the corotational Jeffreys fluid, $\sigma = \lambda_2/\lambda_1$ and $\sigma_1 = \lambda_1^2$, Eq. (39), reduces to:

$$\mathbb{S}_{\mathbb{T}} = \frac{\lambda_1 \dot{\gamma}}{\text{Wi}} \left[\frac{\lambda_2}{\lambda_1} + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \sum_{m=0}^{\infty} (-1)^m (\lambda_1 \dot{\gamma})^{2m} \right] \quad (40)$$

which we will use to plot the cells in the leftmost column of Figure 5 through Figure 8 below.

c SAOS

In the limit of small strain rate amplitude, holding the frequency constant, Eq. (31) reduces to the single harmonic:

$$\begin{aligned}\mathbb{S}_{\mathbb{T}}(\tau) &= - \left[\frac{\lambda_2}{\lambda_1} C_1 + \left(1 - \frac{\lambda_2}{\lambda_1} \right) \left(\frac{\sigma_1}{\lambda_1^2} \right)^{-1} \left(\frac{\kappa_0 C_1 + \phi_0 S_1}{1 + \text{De}^2} \right) \right] \\ &= - \left(\frac{1 + (\lambda_2/\lambda_1) \text{De}^2}{1 + \text{De}^2} \cos \tau + \frac{(1 - \lambda_2/\lambda_1)}{1 + \text{De}^2} \text{De} \sin \tau \right)\end{aligned}\quad (41)$$

which matches Eq. (30), as it should. We will use this to plot the cells in the bottom row of Figure 5 through Figure 8 below. From Eq. (41), the expression for the real part and minus the imaginary part of the complex viscosities are:

$$\frac{\eta'}{\eta_0} = \frac{1 + (\lambda_2/\lambda_1) \text{De}^2}{1 + \text{De}^2} = \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + (\lambda_1 \omega)^2} \quad (42)$$

and:

$$\frac{\eta''}{\eta_0} = \frac{(1 - \lambda_2/\lambda_1) \text{De}}{1 + \text{De}^2} = \frac{(\lambda_1 - \lambda_2) \omega}{1 + (\lambda_1 \omega)^2} \quad (43)$$

where De is defined in Eq. (3).

V CONVERGENCE

From our previous work on the corotational Maxwell fluid [28], we learn that the special case of our main result, Eq. (31) [with Eqs. (32) and (33), Table III, Table IV, $\eta_{\infty}/\eta_0 = \lambda_2/\lambda_1 = 0$, $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_2 = 0$], for oscillatory shear, has a radius of

convergence that appears to be infinite. We also learn that, for steady shear flow, the same special case of our main result has a radius of convergence of just 1.

Our main result, Eq. (31), agrees exactly with the exact solution up to a certain range of Wi . We call such agreement *convergence*. For both the result from steady shear flow, Eq. (39), and from LAOS, Eq. (31), we assess this range of Wi in our power series solutions by employing the ratio test (see Subsection 3.34 of [29]):

$$\lim_{m \rightarrow \infty} \left| \frac{S_{T,m}}{S_{T,m-1}} \right| < 1 \quad (44)$$

where m is defined after Eq. (33).

We begin with steady shear flow, by applying Eq. (44) to Eq.(39). We find that S_T passes the ratio test when:

$$0 < Wi < 1 \quad (45)$$

in which, for steady shear flow, $Wi = \lambda \dot{\gamma}$. Hence, the radius of convergence for the Oldroyd 8-constant framework in steady shear flow is $Wi = 1$. By *radius of convergence*, we mean the largest possible Wi that satisfies the ratio test, Eq. (44).

By contrast, for the shear stress response in oscillatory shear, be its amplitude large or small, we first rearrange Eq. (31) to get:

$$S_T(\tau) = - \sum_{m=0}^{\infty} S_{T,m} = - \frac{\lambda_2}{\lambda_1} C_1 - \left(1 - \frac{\lambda_2}{\lambda_1} \right) \sum_{m=0}^{\infty} \frac{A_m Wi^{2m}}{(-1)^m 2^{2m}} \quad (46)$$

where (with Table III and Table IV):

$$A_m \equiv \left(\frac{\sigma_1}{\lambda_1^2} \right)^{m-1} \left(\frac{(\kappa_m C_n + \phi_m S_n)}{\prod_{k=1}^{m+(n+1)/2} 1 + k^2 De^2} \right) \quad (47)$$

from which, mindful of Eq. (28), we learn that A_m is always positive. Applying the ratio test, Eq. (44), to Eq. (46) gives:

$$\lim_{m \rightarrow \infty} \left| \frac{\frac{A_m Wi^{2m}}{(-1)^m 2^{2m}}}{\frac{A_{m-1} Wi^{2(m-1)}}{(-1)^{m-1} 2^{2(m-1)}}} \right| = \frac{1}{2} Wi^2 \lim_{m \rightarrow \infty} \frac{A_m}{A_{m-1}} < 1 \quad (48)$$

and substituting Eq. (47) into Eq. (48) gives:

$$\frac{\sigma_1 Wi^2}{2 \lambda_1^2} \lim_{m \rightarrow \infty} \frac{(\kappa_m C_n + \phi_m S_n)}{(\kappa_{m-1} C_n + \phi_{m-1} S_n)} \frac{\prod_{k=m+(n+1)/2}^{m+(n+1)/2} 1 + k^2 De^2}{\prod_{k=m+(n-1)/2}^{m+(n-1)/2} 1 + k^2 De^2} < 1 \quad (49)$$

and, for series truncated after N terms:

$$\frac{\sigma_1 Wi^2}{2 \lambda_1^2} \lim_{m \rightarrow N} \frac{(\kappa_m C_n + \phi_m S_n)}{(\kappa_{m-1} C_n + \phi_{m-1} S_n)} \frac{\prod_{k=m+(n+1)/2}^{m+(n+1)/2} 1 + k^2 De^2}{\prod_{k=m+(n-1)/2}^{m+(n-1)/2} 1 + k^2 De^2} < 1 \quad (50)$$

which is subject to Eq. (28), and which is the sufficient condition for infinite radius of convergence for our power series in Wi , truncated after N terms, for the Oldroyd 8-constant fluid in LAOS.

VI EWOLDT GRIDS

We next explore the accuracy of our main result for the special cases of corotational Jeffreys ($\lambda_2/\lambda_1 > 0$) and corotational Maxwell fluids ($\lambda_2/\lambda_1 = 0$). Figure 5 through Figure 8 illustrate each of these λ_2/λ_1 conditions using an Ewoldt grid of shear stress *versus* shear rate loops (following [30,31,32]). Each of the dimensionless loops is constructed for a specific ordered pair of conditions (De, Wi) . When miniaturized and mapped into rows and columns onto Pipkin space, that is, onto a map of Wi *versus* De (see FIG. 2. of [33]), we call the result an *Ewoldt grid* (see Fig. 4. and Fig. 8. of [34]; Figures 4-4 through 4-6 in [35]; Fig 1. in [36], Figure 2 of [37]).

The general expression for each Ewoldt grid cell is given by (Eqs. (29) and (33) of [37]):

$$\left[\frac{\mathbb{S}(De, Wi)}{\mathbb{S}_{\max}} \right]_{i,j} = \frac{\mathbb{S}(De + 2(i-1), Wi)}{\mathbb{S}_{\max}} + 2(j-1) \quad (51)$$

where $i = 1$ and $j = 1$ is the lowest leftmost cell. $\mathbb{S}_{\max}$ is the peak shear stress given by the exact solution, Eq. (17) [with Eqs. (18) through (23) where $\sigma_2 = \lambda_1\lambda_2$ and $\sigma_1 = \lambda_1^2$], for each (De, Wi) .

Figure 5 through Figure 8 compare our main results, Eq. (31) [with Eqs. (32) and (33) and Table III and Table IV], with the exact solution, Eq. (17) [with Eqs. (18) through (23)]. From these figures, we observe that our main results for special cases of the corotational Jeffreys and corotational Maxwell fluids agree with the exact solution, to within a penwidth, so long as $Wi/De < \frac{7}{2}$.

VII APPLICATION

We consider the commercial grade of molten atactic polystyrene made by Dow Chemical Canada Ltd., Styron 683, studied extensively by Tee and Dealy [31,32]. Tee and Dealy report one frequency sweep, comprising ten SAOS measurements over the ranges $0.38 < \omega < 24 \text{ rad/s}$ and $0.076 < \dot{\gamma}^0 < 4.8 \text{ s}^{-1}$, performed using oscillatory cone-plate flow [30,31,32] (the $\frac{Wi}{De} = 0.2$ set of points in FIGURE 12 of [14]). Tee and Dealy also report a set of thirty LAOS measurements, comprising 5 frequency sweeps over the ranges $0.96 < \omega < 25 \text{ rad/s}$ and $2.6 < \dot{\gamma}^0 < 250 \text{ s}^{-1}$, performed on a concentric cylinder rheometer specially designed for LAOS [30,31,32] (the $2.67 \leq \frac{Wi}{De} \leq 10$ sets of points in FIGURE 12 of [14]). Specifically, they report frequency sweeps of $\tau_0(\omega, \dot{\gamma}^0)/\dot{\gamma}^0$, where τ_0 is the peak shear stress in LAOS. Since the higher harmonics normally affect loop shapes while hardly affecting τ_0 :

$$\frac{\tau_0(\omega, \dot{\gamma}^0)}{\dot{\gamma}^0} \equiv |\eta_1^*(\omega, \dot{\gamma}^0)| \quad (52)$$

where (see Eqs. (4.4–4) and (4.4–5) of [24]):

$$|\eta_1^*(\omega, \dot{\gamma}^0)| \equiv \sqrt{\eta_1'^2(\omega, \dot{\gamma}^0) + \eta_1''^2(\omega, \dot{\gamma}^0)} \quad (53)$$

For this application, for the LAOS measurements, we will thus compare $\tau_0(\omega, \dot{\gamma}^0)/\dot{\gamma}^0$ measurements with the $|\eta_1^*(\omega, \dot{\gamma}^0)|$ predictions from our results, for the special case of the Johnson-Segalman fluid by following Subsection VI.B of [14].

Here we apply our results, Eqs. (34) and (35) [with Table III and Table IV], for the special case of the Johnson-Segalman fluid for which (see Table IV of [6]):

$$\lambda_2 = (\eta_s/\eta_0)\lambda_1, \quad \mu_1 = (1-\xi)\lambda_1, \quad \mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1, \quad \mu_0 = \nu_1 = \nu_2 = 0 \quad (54),(55),(56),(57)$$

Substituting Eqs. (54) through (57) into Eqs. (15) and (16), we get:

$$\sigma_1 = \lambda_1^2 \xi (2 - \xi) \quad (58)$$

$$\sigma_2 = (\eta_s/\eta_0) \lambda_1^2 \xi (2 - \xi) \quad (59)$$

For the Johnson-Segalman fluid, we then get the SAOS measurements (Eq. (65) of [14]):

$$|\eta^*(\omega)| \equiv \sqrt{\eta'^2(\omega) + \eta''^2(\omega)} \quad (60)$$

where (Eqs. (84) and (85) of [14]):

$$\eta'(\omega) = \frac{\eta_0}{\zeta(\alpha_1)} \sum_{j=1}^{\infty} \frac{k^{\alpha_1} + (\eta_s/\eta_0) \lambda_1^2 k^{-\alpha_1} \omega^2}{k^{2\alpha_1} + \lambda_1^2 \omega^2}, \quad \eta''(\omega) = \frac{\eta_0}{\zeta(\alpha_1)} \sum_{j=1}^{\infty} \frac{[1 - (\eta_s/\eta_0)] \lambda_1 \omega}{k^{2\alpha_1} + \lambda_1^2 \omega^2} \quad (61),(62)$$

in which $\zeta(\alpha) \equiv \sum_{j=1}^{\infty} k^{-\alpha}$ is the Riemann zeta function.

We first generalize our results, Eqs. (34) and (35) [with Eqs. (58) and (59)], for the special case of the Johnson-Segalman fluid to multimode by replacing η_0 and λ_1 with the Spriggs relations (Eqs. (6.1-14) and (6.1-15) of [24]):

$$\eta_k = \eta_0 \lambda_{1k} / \sum_{k=1}^{\infty} \lambda_{1k}, \quad \lambda_{1k} = \lambda_1 / k^{\alpha_1} \quad (63),(64)$$

so that (Eqs. (79) through (81) of [14]):

$$\lambda_{2k} = \frac{\eta_s}{\eta_0} \lambda_{1k}, \quad \mu_{1k} = (1-\xi) \lambda_{1k}, \quad \mu_{2k} = (1-\xi) \frac{\eta_s}{\eta_0} \lambda_{1k} \quad (65),(66),(67)$$

We get the generalized results for the special case of the Johnson-Segalman fluid are:

$$\eta'_{1k} = \eta_k \left(-(\eta_s/\eta_0) - \left[\sqrt{\xi(2-\xi)} - (\eta_s/\eta_0) \right] \sum_{m=0}^{\infty} \frac{(\lambda_{1k} \dot{\gamma}^0)^{2m}}{(-1)^m 2^{2m}} [\xi(2-\xi)]^{m-1} \frac{\kappa_m}{\prod_{z=1}^{m+(n+1)/2} 1 + z^2 (\lambda_{1k} \omega)^2} \right) \quad (68)$$

and:

$$\eta_{1k}'' = \eta_k \left(- \left[\sqrt{\xi(2-\xi)} - (\eta_s/\eta_0) \right] \sum_{m=0}^{\infty} \frac{(\lambda_{1k}\dot{\gamma}^0)^{2m}}{(-1)^m 2^{2m}} [\xi(2-\xi)]^{m-1} \frac{\varphi_m}{\prod_{z=1}^{m+(n+1)/2} 1+z^2(\lambda_{1k}\omega)^2} \right) \quad (69)$$

where $\lambda_{11} = \lambda_1$ and $\lambda_{22} = \lambda_2$ are the longest relaxation and retardation times, and then from $\tau = \sum_k \tau_k$ and Eq. (5), we see that:

$$\eta_1'(\omega, \dot{\gamma}^0) = \sum_{k=1}^{\infty} \eta_{1k}'(\omega, \dot{\gamma}^0), \quad \eta_1''(\omega, \dot{\gamma}^0) = \sum_{k=1}^{\infty} \eta_{1k}''(\omega, \dot{\gamma}^0) \quad (70), (71)$$

We then use the well-known fitted values for this molten atactic polystyrene reported in Subsection VI.B of [14]:

$$\eta_0 = 0.169 \text{ MPa}\cdot\text{s}, \quad \eta_s/\eta_0 = 0.169 \text{ MPa}\cdot\text{s}, \quad \lambda_1 = 0.775 \text{ s}, \quad \alpha_1 = 3.27, \quad \xi = 0.050 \quad (72)$$

Figure 9 compares this fit (upper most curve) with the ten SAOS measurements (**triangles**). Figure 9 and Figure 10 illustrate the good agreement between the shear thinning predictions of $|\eta_1^*(\omega, \dot{\gamma}^0)|$ of our main result, Eq. (31) [with Eqs. (32) and (33),

Table III and Table IV (up to Wi^{16})], and the frequency sweep measurements and strain rate sweep interpolations from Fig. 9. of [31] (or of FIGURE 7-12 of [32]). We can compare this worked example with the one of Subsection VI.B of [14] which relies on the exact solution, Eqs. (17) through (23). As a practical matter, we find our main result to be within a line-width of the exact solution so long as $\text{Wi}/\text{De} \leq 13$.

VIII CONCLUSION

We arrive at a power series in the shear rate amplitude for the shear stress responses of the Oldroyd 8-constant framework, Eq. (31) [with Eqs. (32) and (33), Table III and Table IV (up to Wi^{16})]. Here, we truncate the power series expansion after the 16th power of the shear rate amplitude. This series includes shear stress harmonics up to the 17th. We arrive at an expression for the region of usefulness so long as $\text{Wi}/\text{De} < \frac{7}{2}$, for the special cases of the corotational Jeffreys and corotational Maxwell fluids. We also arrive at a power series in the shear rate amplitude for $\eta_1'(\omega, \dot{\gamma}^0)$ and $\eta_1''(\omega, \dot{\gamma}^0)$, the sweep material functions for the first harmonic, Eqs. (34) and (35), by extracting the leading terms from our main result, Eq. (31).

We illustrate the usefulness of our results for $\eta_1'(\omega, \dot{\gamma}^0)$ and $\eta_1''(\omega, \dot{\gamma}^0)$ for both frequency and strain rate sweeps using the special case of the Johnson-Segalman fluid, generalized to multimode with the Spriggs relations. The $|\eta_1^*(\omega, \dot{\gamma}^0)|$ predictions agree closely with the measured behaviors of molten atactic polystyrene. We find our main result to be much easier to use than the exact solution from whence it came.

Following this same method, starting with [38], we can arrive at power series for the normal stress difference responses for the Oldroyd 8-constant framework in LAOS. We leave this for another day.

IX ACKNOWLEDGMENT

Poungthong acknowledges support from the Royal Golden Jubilee Program of the Thailand Research Fund (Contract No. PHD/0178/2557). Giacomini is indebted to the Faculty of Applied Science and Engineering of Queen's University at Kingston, for its support through a Research Initiation Grant (RIG). This research was undertaken, in part, thanks to support from the Canada Research Chairs program of the Government of Canada for the Natural Sciences and Engineering Research Council of Canada (NSERC) Tier 1 Canada Research Chair in Rheology. This research was also undertaken, in part, thanks to support from the Discovery Grant program of the Natural Sciences and Engineering Research Council of Canada (NSERC).

Table I: Dimensional Variables

Name	Unit	Symbol
Angular frequency [Eq. (1)]	t^{-1}	ω
Cartesian coordinates	L	x, y, z
Corotational derivative [Eq. (11)]	t^{-1}	$\mathcal{D}/\mathcal{D}t$
Extra stress tensor [Eq. (8)]	M/Lt^2	$\boldsymbol{\tau}$
Extra stress, yx component* [Eq. (5)]	M/Lt^2	τ_{yx}
Oldroyd constant [(8)]	t	$\mu_0, \mu_1, \mu_2, \nu_1, \nu_2$
Oldroyd constant, k th spectrum [Eqs. (66) and (67)]	t	μ_{1k}, μ_{2k}
Oldroyd constant, relaxation and retardation times [Eq. (64)], k th spectrum [Eq. (65)]	t	$\lambda_{1k}, \lambda_{2k}$
Oldroyd constant, relaxation time [Eq. (2)] and retardation time [Eq. (8)]	t	λ_1, λ_2
Oldroyd steady shear constant [Eqs. (15) and (16)]	t^2	σ_1, σ_2
Peak value of the shear stress response in LAOS [Eq. (52)]	M/Lt^2	τ_0
Rate of strain tensor [Eq. (9)]	t^{-1}	$\dot{\gamma}$
Shear rate, amplitude [Eq. (1)]	t^{-1}	$\dot{\gamma}^0$
Time [Eq. (1)]	t	t
Viscosity function, steady shear, exact [Eq. (24)] and power series [Eq. (38)]	M/Lt	η, η_T
Viscosity, complex [Eq. (60)]	M/Lt	$ \eta^* $
Viscosity, complex, shifting constant [Figure 10]	M/Lt	η_+
Viscosity, loss and storage, n th harmonic [Eq. (5)]	M/Lt	η'_n, η''_n
Viscosity, loss and storage, 1st harmonic, k th spectrum [Eqs. (70) and (71)]	M/Lt	η'_{1k}, η''_{1k}
Viscosity, solvent [Eq. (54)]	M/Lt	η_s
Viscosity, zero shear [Eq. (38)]	M/Lt	$\eta_0 = \lim_{\dot{\gamma}^0 \rightarrow 0} \eta(\dot{\gamma})$
Vorticity tensor [Eq. (10)]	t^{-1}	$\boldsymbol{\omega}$

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

*Where τ_{ij} is the force exerted in the j th direction on a unit area of fluid surface of constant x_i by fluid in the region lesser x_i on fluid in the region greater x_i (see “Note on the Sign Convention for the Stress Tensor” on pp. 19–20 of [39], or pp. 24–25 of [40]).

Table II: Dimensionless Variables and Groups

Name	Symbol
Bessel function of first kind, m th order, argument Wi/De [Eq. (18)]	$J_m \equiv J_m\left(\frac{Wi}{De}\right) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k)!k!} \left(\frac{Wi}{2De}\right)^{m+2k}$
Combined fluid parameter [Eq. (14)]	$\tilde{\lambda}$
Constant defined in Eq. (14)	$\alpha \equiv (Wi/De)\tilde{\lambda}$
Deborah number [Eq. (3)]	$De \equiv \lambda_1 \omega$
Function in Eq. (47)	A_m
Integral function [Eqs. (18) through (23)]	$I_1, I_2, \dots, I_6$
Johnson-Segalman fluid parameter [Eq. (55)]	ξ
Riemann zeta function of α [below Eq. (62)]	$\zeta(\alpha)$
Shear stress response, exact [Eq. (12)]	$\mathbb{S} \equiv \tau_{yx}/\eta_0 \dot{\gamma}^0$
Shear stress response, power series [Eq. (31)]	$\mathbb{S}_{\mathbb{T}}$
Shear stress response, power series, m th spectrum [Eq. (31)]	$\mathbb{S}_{\mathbb{T},m}$
Time [Eq. (13)]	$\tau \equiv \omega t$
Weissenberg number [Eq. (4)]	$Wi \equiv \lambda_1 \dot{\gamma}^0$

Table III: Expressions of $\kappa_m C_n$

m	$\kappa_m C_n$
0	$(\sigma_1/\lambda_1^2)C_1$
1	$3(\varpi - 2\vartheta \text{De}^2)C_1 + (\varpi - (11\varpi + 6\vartheta)\text{De}^2 + 6\vartheta \text{De}^4)C_3$
2	$10(\varpi - 3\vartheta \text{De}^2)C_1 + 5(\varpi - (26\varpi + 7\vartheta)\text{De}^2 + 56\vartheta \text{De}^4)C_3$ $+ (\varpi - (85\varpi + 15\vartheta)\text{De}^2 + (274\varpi + 225\vartheta)\text{De}^4 - 120\vartheta \text{De}^6)C_5$
3	$35(\varpi - 4\vartheta \text{De}^2)C_1 + 21(\varpi - (47\varpi + 8\vartheta)\text{De}^2 + 160\vartheta \text{De}^4)C_3$ $+ 7(\varpi - (155\varpi + 16\vartheta)\text{De}^2 + (1044\varpi + 680\vartheta)\text{De}^4 - 2304\vartheta \text{De}^6)C_5$ $+ (\varpi - (322\varpi + 28\vartheta)\text{De}^2 + (6769\varpi + 1960\vartheta)\text{De}^4 - (13068\varpi + 13132\vartheta)\text{De}^6 + 5040\vartheta \text{De}^8)C_7$
4	$126(\varpi - 5\vartheta \text{De}^2)C_1 + 84(\varpi - (74\varpi + 9\vartheta)\text{De}^2 + 336\vartheta \text{De}^4)C_3$ $+ 36(\varpi - (245\varpi + 17\vartheta)\text{De}^2 + (2754\varpi + 1415\vartheta)\text{De}^4 - 9968\vartheta \text{De}^6)C_5$ $+ 9(\varpi - (511\varpi + 29\vartheta)\text{De}^2 + (18424\varpi + 4039\vartheta)\text{De}^4 - (69264\varpi + 70756\vartheta)\text{De}^6 + 154496\vartheta \text{De}^8)C_7$ $+ \left(\varpi - (870\varpi + 45\vartheta)\text{De}^2 + (63273\varpi + 9450\vartheta)\text{De}^4 - (723680\varpi + 269325\vartheta)\text{De}^6 \right. \\ \left. + (1026576\varpi + 1172700\vartheta)\text{De}^8 - 362880\vartheta \text{De}^{10} \right) C_9$
5	$462(\varpi - 6\vartheta \text{De}^2)C_1 + 330(\varpi - (107\varpi + 10\vartheta)\text{De}^2 + 602\vartheta \text{De}^4)C_3$ $+ 165(\varpi - (355\varpi + 18\vartheta)\text{De}^2 + (5944\varpi + 2490\vartheta)\text{De}^4 - 28992\vartheta \text{De}^6)C_5$ $+ 55 \left(\varpi - (742\varpi + 30\vartheta)\text{De}^2 + (40369\varpi + 6972\vartheta)\text{De}^4 - (241128\varpi + 217350\vartheta)\text{De}^6 \right. \\ \left. + 904608\vartheta \text{De}^8 \right) C_7$ $+ 11 \left(\varpi - (1266\varpi + 46\vartheta)\text{De}^2 + (140889\varpi + 16116\vartheta)\text{De}^4 - (2655764\varpi + 878934\vartheta)\text{De}^6 \right. \\ \left. + (6999840\varpi + 9536264\vartheta)\text{De}^8 - 15696000\vartheta \text{De}^{10} \right) C_9$ $+ \left(\varpi - (1925\varpi + 66\vartheta)\text{De}^2 + (357423\varpi + 32670\vartheta)\text{De}^4 - (13339535\varpi + 2637558\vartheta)\text{De}^6 \right. \\ \left. + (105258076\varpi + 45995730\vartheta)\text{De}^8 - (120543840\varpi + 150917976\vartheta)\text{De}^{10} + 39916800\vartheta \text{De}^{12} \right) C_{11}$
6	$1716(\varpi - 7\vartheta \text{De}^2)C_1 + 1287(\varpi - (146\varpi + 11\vartheta)\text{De}^2 + 976\vartheta \text{De}^4)C_3$ $+ 715(\varpi - (485\varpi + 19\vartheta)\text{De}^2 + (11274\varpi + 3965\vartheta)\text{De}^4 - 68256\vartheta \text{De}^6)C_5$ $+ 286 \left(\varpi - (1015\varpi + 31\vartheta)\text{De}^2 + (77224\varpi + 10885\vartheta)\text{De}^4 - (662640\varpi + 521164\vartheta)\text{De}^6 \right. \\ \left. + 3384000\vartheta \text{De}^8 \right) C_7$ $+ 78 \left(\varpi - (1734\varpi + 47\vartheta)\text{De}^2 + (271929\varpi + 24798\vartheta)\text{De}^4 - (7494416\varpi + 2143743\vartheta)\text{De}^6 \right. \\ \left. + (30334320\varpi + 39196852\vartheta)\text{De}^8 - 116487360\vartheta \text{De}^{10} \right) C_9$ $+ 13 \left(\varpi - (2640\varpi + 67\vartheta)\text{De}^2 + (696333\varpi + 49830\vartheta)\text{De}^4 - (38759930\varpi + 6557331\vartheta)\text{De}^6 \right. \\ \left. + (489896616\varpi + 213126980\vartheta)\text{De}^8 - (1007441280\varpi + 1686370752\vartheta)\text{De}^{10} \right. \\ \left. + 2265200640\vartheta \text{De}^{12} \right) C_{11}$ $+ \left(\varpi - (3731\varpi + 91\vartheta)\text{De}^2 + (1474473\varpi + 91091\vartheta)\text{De}^4 - (135036473\varpi + 16669653\vartheta)\text{De}^6 \right. \\ \left. + (3336118786\varpi + 790943153\vartheta)\text{De}^8 - (20313753096\varpi + 9957703756\vartheta)\text{De}^{10} \right. \\ \left. + (19802759040\varpi + 26596717056\vartheta)\text{De}^{12} - 6227020800\vartheta \text{De}^{14} \right) C_{13}$

7	$ \begin{aligned} & 6435(\varpi - 8\vartheta \text{De}^2)C_1 + 5005(\varpi - (191\varpi + 12\vartheta)\text{De}^2 + 1476\vartheta \text{De}^4)C_3 \\ & + 3003(\varpi - (635\varpi + 20\vartheta)\text{De}^2 + (19524\varpi + 5900\vartheta)\text{De}^4 - 140240\vartheta \text{De}^6)C_5 \\ & + 1365 \left(\varpi - (1330\varpi + 32\vartheta)\text{De}^2 + (134449\varpi + 15904\vartheta)\text{De}^4 \right. \\ & \quad \left. - (1557660\varpi + 1076992\vartheta)\text{De}^6 + 9914432\vartheta \text{De}^8 \right) C_7 \\ & + 455 \left(\varpi - (2274\varpi + 48\vartheta)\text{De}^2 + (476049\varpi + 35712\vartheta)\text{De}^4 \right. \\ & \quad \left. - (17893196\varpi + 4446288\vartheta)\text{De}^6 + (101378880\varpi + 119626368\vartheta)\text{De}^8 \right. \\ & \quad \left. - 534518784\vartheta \text{De}^{10} \right) C_9 \\ & + 105 \left(\varpi - (3465\varpi + 68\vartheta)\text{De}^2 + (1225983\varpi + 71060\vartheta)\text{De}^4 \right. \\ & \quad \left. - (94092515\varpi + 13659228\vartheta)\text{De}^6 + (1698920916\varpi + 680390524\vartheta)\text{De}^8 \right. \\ & \quad \left. - (5231113920\varpi + 8745876304\vartheta)\text{De}^{10} + 20417769216\vartheta \text{De}^{12} \right) C_{11} \\ & + 15 \left(\varpi - (4901\varpi + 92\vartheta)\text{De}^2 + (2611323\varpi + 129116\vartheta)\text{De}^4 \right. \\ & \quad \left. - (333710663\varpi + 34948628\vartheta)\text{De}^6 \right. \\ & \quad \left. + (12061579816\varpi + 2657637268\vartheta)\text{De}^8 \right. \\ & \quad \left. - (115119818736\varpi + 61169831008\vartheta)\text{De}^{10} \right. \\ & \quad \left. + (196287356160\varpi + 384643844416\vartheta)\text{De}^{12} - 441957583872\vartheta \text{De}^{14} \right) C_{13} \\ & + \left(\varpi - (6580\varpi + 120\vartheta)\text{De}^2 - (4899622\varpi + 218400\vartheta)\text{De}^4 \right. \\ & \quad \left. - (928095740\varpi + 78558480\vartheta)\text{De}^6 + (54631129553\varpi + 8207628000\vartheta)\text{De}^8 \right. \\ & \quad \left. - (1009672107080\varpi + 272803210680\vartheta)\text{De}^{10} \right. \\ & \quad \left. + (5056995703824\varpi + 2706813345600\vartheta)\text{De}^{12} \right. \\ & \quad \left. - (4339163001600\varpi + 6165817614720\vartheta)\text{De}^{14} + 1307674368000\vartheta \text{De}^{16} \right) C_{15} \end{aligned} $
8	$ \begin{aligned} & 24310(\varpi - 9\vartheta \text{De}^2)C_1 + 19448(\varpi - (242\varpi + 13\vartheta)\text{De}^2 + 2120\vartheta \text{De}^4)C_3 \\ & + 12376(\varpi - (805\varpi + 21\vartheta)\text{De}^2 + (31594\varpi + 8355\vartheta)\text{De}^4 - 261624\vartheta \text{De}^6)C_5 \\ & + 6118 \left(\varpi - (1687\varpi + 33\vartheta)\text{De}^2 + (218344\varpi + 22155\vartheta)\text{De}^4 \right. \\ & \quad \left. - (3272688\varpi + 2010372\vartheta)\text{De}^6 + 24744960\vartheta \text{De}^8 \right) C_7 \\ & + 2308 \left(\varpi - (2886\varpi + 49\vartheta)\text{De}^2 + (775929\varpi + 49074\vartheta)\text{De}^4 - (37972304\varpi + 8287041\vartheta)\text{De}^6 \right. \\ & \quad \left. + (284574960\varpi + 304806236\vartheta)\text{De}^8 - 1878289920\vartheta \text{De}^{10} \right) C_9 \\ & + 680 \left(\varpi - (4400\varpi + 69\vartheta)\text{De}^2 + (2005773\varpi + 96690\vartheta)\text{De}^4 \right. \\ & \quad \left. - (201790490\varpi + 25425477\vartheta)\text{De}^6 + (4876196776\varpi + 1770271140\vartheta)\text{De}^8 \right. \\ & \quad \left. - (20606463360\varpi + 32539065504\vartheta)\text{De}^{10} + 111221752320\vartheta \text{De}^{12} \right) C_{11} \\ & + 136 \left(\varpi - (6227\varpi + 93\vartheta)\text{De}^2 + (4288713\varpi + 174525\vartheta)\text{De}^4 \right. \\ & \quad \left. - (723638201\varpi + 65066859\vartheta)\text{De}^6 + (35465473186\varpi + 7070466975\vartheta)\text{De}^8 \right. \\ & \quad \left. - (474957547272\varpi + 243391610748\vartheta)\text{De}^{10} \right. \\ & \quad \left. + (1188825724800\varpi + 2420189647200\vartheta)\text{De}^{12} - 4695848640000\vartheta \text{De}^{14} \right) C_{13} \end{aligned} $

+ 17	$ \begin{aligned} & \left(\begin{aligned} & \varpi - (8365\varpi + 121\vartheta)\text{De}^2 + (8078707\varpi + 293965\vartheta)\text{De}^4 \\ & - (2036157695\varpi + 146566147\vartheta)\text{De}^6 + (164932175408\varpi + 22380710495\vartheta)\text{De}^8 \\ & - (4392554369840\varpi + 1176903896168\vartheta)\text{De}^{10} \\ & + (33896996544384\varpi + 21035113539440\vartheta)\text{De}^{12} \\ & - (49811492505600\varpi + 110730304376064\vartheta)\text{De}^{14} + 112224094617600\vartheta\text{De}^{16} \end{aligned} \right) C_{15} \\ & + \left(\begin{aligned} & \varpi - (10812\varpi + 153\vartheta)\text{De}^2 + (13896582\varpi + 468180\vartheta)\text{De}^4 \\ & - (4853222764\varpi + 299650806\vartheta)\text{De}^6 + (577924894833\varpi + 60202693980\vartheta)\text{De}^8 \\ & - (24871845297936\varpi + 4308105301929\vartheta)\text{De}^{10} \\ & + (369012649234384\varpi + 110228466184200\vartheta)\text{De}^{12} \\ & - (1583313975727488\varpi + 909299905844112\vartheta)\text{De}^{14} \\ & + (1223405590579200\varpi + 1821602444624640\vartheta)\text{De}^{16} - 355687428096000\vartheta\text{De}^{18} \end{aligned} \right) C_{17} \end{aligned} $	
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Legend: $\tilde{\lambda} \equiv \sqrt{\sigma_1}/\lambda_1$; $\varpi \equiv (\lambda_1^2 \tilde{\lambda}^3 - \sigma_2)/\lambda_1(\lambda_1 - \lambda_2)$; $\vartheta \equiv (\sigma_2 - \lambda_1 \lambda_2 \tilde{\lambda}^2)/\lambda_1(\lambda_1 - \lambda_2)$.

Table IV: Expressions of $\varphi_m S_n$

m	$\varphi_m S_n$
0	$(\sigma_1/\lambda_1^2)\text{De}S_1$
1	$(2\zeta/\text{De}^2 + 6\varpi + \vartheta + 10\zeta + 8(\zeta - \vartheta)\text{De}^2)\text{De}S_1 + (6\varpi + \vartheta - (6\varpi + 11\vartheta)\text{De}^2)\text{De}S_3$
2	$2(\zeta/\text{De}^4 + 14\zeta/\text{De}^2 + 15\varpi + \vartheta + 49\zeta + 36(\zeta - \vartheta)\text{De}^2)\text{De}S_1$ $+ (1/3)(-\zeta/\text{De}^4 - 30\zeta/\text{De}^2 + 135\varpi + 9\vartheta - 273\zeta - 10(36\varpi + 45\vartheta + 82\zeta)\text{De}^2 - 576(\zeta - \vartheta)\text{De}^4)\text{De}S_3$ $+ (15\varpi + \vartheta - (225\varpi + 85\vartheta)\text{De}^2 + (120\varpi + 274\vartheta)\text{De}^4)\text{De}S_5$
3	$(5/6)(\zeta/\text{De}^6 + 30\zeta/\text{De}^4 + 273\zeta/\text{De}^2 + 168\varpi + 6\vartheta + 820\zeta + 576(\zeta - \vartheta)\text{De}^2)\text{De}S_1$ $+ \frac{1}{4}\left(-\zeta/\text{De}^6 - 55\zeta/\text{De}^4 - 1023\zeta/\text{De}^2 + 1008\varpi + 36\vartheta - 7645\zeta - (5040\varpi + 4572\vartheta + 21076\zeta)\text{De}^2 - 14400(\zeta - \vartheta)\text{De}^4\right)\text{De}S_3$ $+ \frac{1}{60}\left(\zeta/\text{De}^6 + 91\zeta/\text{De}^4 + 3003\zeta/\text{De}^2 + 8400\varpi + 300\vartheta + 44473\zeta - (243600\varpi + 60900\vartheta - 296296\zeta)\text{De}^2 + (302400\varpi + 730800\vartheta + 773136\zeta)\text{De}^4 + 518400(\zeta - \vartheta)\text{De}^6\right)\text{De}S_5$ $+ (28\varpi + \vartheta - (1960\varpi + 322\vartheta)\text{De}^2 + (13132\varpi + 6769\vartheta)\text{De}^4 - (5040\varpi + 13068\vartheta)\text{De}^6)\text{De}S_7$
4	$\frac{7}{36}\left(\zeta/\text{De}^8 + 55\zeta/\text{De}^6 + 1023\zeta/\text{De}^4 + 7645\zeta/\text{De}^2 + 3240\varpi + 72\vartheta + 21076\zeta + 14400(\zeta - \vartheta)\text{De}^2\right)\text{De}S_1$ $+ \frac{7}{90}\left(-\zeta/\text{De}^8 - 91\zeta/\text{De}^6 - 3003\zeta/\text{De}^4 - 44473\zeta/\text{De}^2 + 16200\varpi + 360\vartheta - 296296\zeta - (129600\varpi + 91440\vartheta + 773136\zeta)\text{De}^2 - 518400(\zeta - \vartheta)\text{De}^4\right)\text{De}S_3$ $+ \frac{1}{90}\left(\zeta/\text{De}^8 + 140\zeta/\text{De}^6 + 7462\zeta/\text{De}^4 + 191620\zeta/\text{De}^2 + 81000\varpi + 1800\vartheta + 2475473\zeta - (3807000\varpi + 693000\vartheta - 15291640\zeta)\text{De}^2 + (8164800\varpi + 16801200\vartheta + 38402064\zeta)\text{De}^4 + 25401600(\zeta - \vartheta)\text{De}^6\right)\text{De}S_5$ $+ \frac{1}{2520}\left(-\zeta/\text{De}^8 - 204\zeta/\text{De}^6 - 16422\zeta/\text{De}^4 - 669188\zeta/\text{De}^2 + 793800\varpi + 17640\vartheta - 14739153\zeta - (91287000\varpi + 10425240\vartheta + 173721912\zeta)\text{De}^2 + (1108144800\varpi + 487287360\vartheta - 1017067024\zeta)\text{De}^4 - (914457600\varpi + 3191852160\vartheta + 2483133696\zeta)\text{De}^6 - 1625702400(\zeta - \vartheta)\text{De}^8\right)\text{De}S_7$ $+ \left(45\varpi + \vartheta - (9450\varpi + 870\vartheta)\text{De}^2 + (269325\varpi + 63273\vartheta)\text{De}^4 - (1172700\varpi + 723680\vartheta)\text{De}^6 + (362880\varpi + 1026576\vartheta)\text{De}^8\right)\text{De}S_9$
5	$\frac{7}{240}\left(\zeta/\text{De}^{10} + 91\zeta/\text{De}^8 + 3003\zeta/\text{De}^6 + 44473\zeta/\text{De}^4 + 296296\zeta/\text{De}^2 + 95040\varpi + 1440\vartheta + 773136\zeta + 518400(\zeta - \vartheta)\text{De}^2\right)\text{De}S_1$ $+ \frac{1}{72}\left(-\zeta/\text{De}^{10} - 140\zeta/\text{De}^8 - 7462\zeta/\text{De}^6 - 191620\zeta/\text{De}^4 - 247573\zeta/\text{De}^2 + 427680\varpi + 6480\vartheta - 15291640\zeta - (4989600\varpi + 2870640\vartheta + 38402064\zeta)\text{De}^2 - 25401600(\zeta - \vartheta)\text{De}^4\right)\text{De}S_3$ $+ \frac{1}{336}\left(\zeta/\text{De}^{10} + 204\zeta/\text{De}^8 + 16522\zeta/\text{De}^6 + 669188\zeta/\text{De}^4 + 1473915\zeta/\text{De}^2 + 173721912\zeta + 1663200\varpi + 25200\vartheta - (114760800\varpi + 16203600\vartheta - 1017067024\zeta)\text{De}^2 + (372556800\varpi + 650563200\vartheta + 2483133696\zeta)\text{De}^4 + 1625702400(\zeta - \vartheta)\text{De}^6\right)\text{De}S_5$

	$ \begin{aligned} & + \frac{1}{4032} \left(\begin{aligned} & -\zeta/\text{De}^{10} - 285\zeta/\text{De}^8 - 32946\zeta/\text{De}^6 - 1999370\zeta/\text{De}^4 - 68943381\zeta/\text{De}^2 + 14110\vartheta \\ & + 9313920\varpi - 1367593305\zeta - (1583366400\varpi + 135192960\vartheta + 15088541896\zeta)\text{De}^2 \\ & + (29702090880\varpi + 10878799680\vartheta - 84865562640\zeta)\text{De}^4 \\ & - (40236134400\varpi + 131234826240\vartheta + 202759531776\zeta)\text{De}^6 \\ & - 131681894400(\zeta - \vartheta)\text{De}^8 \end{aligned} \right) \text{De}S_7 \\ & + \frac{1}{181440} \left(\begin{aligned} & \zeta/\text{De}^{10} + 385\zeta/\text{De}^8 + 61446\zeta/\text{De}^6 + 5293970\zeta/\text{De}^4 \\ & + 268880381\zeta/\text{De}^2 + 107775360\varpi + 1632960\vartheta + 8261931405\zeta \\ & - (33697762560\varpi + 2263282560\vartheta - 151847872396\zeta)\text{De}^2 \\ & + (1519165549440\varpi + 291334760640\vartheta + 1593719752240\zeta)\text{De}^4 \\ & - (11483536458240\varpi + 7098875562240\vartheta - 8689315795776\zeta)\text{De}^6 \\ & + (7242504192000\varpi + 32309615923200\vartheta + 20407635072000\zeta)\text{De}^8 \\ & + 13168189440000(\zeta - \vartheta)\text{De}^{10} \end{aligned} \right) \text{De}S_9 \\ & + \left(\begin{aligned} & 66\varpi + \vartheta - (32670\varpi + 1925\vartheta)\text{De}^2 + (2637558\varpi + 357423\vartheta)\text{De}^4 \\ & - (45995730\varpi + 13339535\vartheta)\text{De}^6 + (150917976\varpi + 105258076\vartheta)\text{De}^8 \\ & - (39916800\varpi + 120543840\vartheta)\text{De}^{10} \end{aligned} \right) \text{De}S_{11} \end{aligned} $
6	$ \begin{aligned} & \frac{11}{3600} \left(\zeta/\text{De}^{12} + 140\zeta/\text{De}^{10} + 7462\zeta/\text{De}^8 + 191620\zeta/\text{De}^6 + 2475473\zeta/\text{De}^4 \right. \\ & \left. + 15291640\zeta/\text{De}^2 + 3931200\varpi + 43200\vartheta + 38402064\zeta + 25401600(\zeta - \vartheta)\text{De}^2 \right) \text{De}S_1 \\ & + \frac{11}{6720} \left(\begin{aligned} & -\zeta/\text{De}^{12} - 204\zeta/\text{De}^{10} - 16422\zeta/\text{De}^8 - 669188\zeta/\text{De}^6 - 14739153\zeta/\text{De}^4 \\ & - 173721912\zeta/\text{De}^2 + 16511040\varpi + 181440\vartheta - 1017067024\zeta \\ & - (264176640\varpi + 128096640\vartheta + 2483133696\zeta)\text{De}^2 - 1625702400(\zeta - \vartheta)\text{De}^{16} \end{aligned} \right) \text{De}S_3 \\ & + \frac{11}{24192} \left(\begin{aligned} & \zeta/\text{De}^{12} + 285\zeta/\text{De}^{10} + 32946\zeta/\text{De}^8 + 1999370\zeta/\text{De}^6 + 68943381\zeta/\text{De}^4 \\ & + 1367593305\zeta/\text{De}^2 + 55036800\varpi + 604800\vartheta + 15088541896\zeta \\ & - (5228496000\varpi + 598147200\vartheta - 84865562640\zeta)\text{De}^2 \\ & + (23775897600\varpi + 35776339200\vartheta + 202759531776\zeta)\text{De}^4 \\ & + 131681894400(\zeta - \vartheta)\text{De}^6 \end{aligned} \right) \text{De}S_5 \\ & + \frac{11}{181440} \left(\begin{aligned} & -\zeta/\text{De}^{12} - 385\zeta/\text{De}^{10} - 61446\zeta/\text{De}^8 - 5293970\zeta/\text{De}^6 - 268880381\zeta/\text{De}^4 \\ & - 8261931405\zeta/\text{De}^2 + 231154560\varpi + 2540160\vartheta - 151847872396\zeta \\ & - (54321321600\varpi + 3645129600\vartheta + 1593719752240\zeta)\text{De}^2 \\ & + (1443329072640\varpi + 446875107840\vartheta - 8689315795776\zeta)\text{De}^4 \\ & - (2853107712000\varpi + 8344730419200\vartheta + 20407635072000\zeta)\text{De}^6 \\ & - 13168189440000(\zeta - \vartheta)\text{De}^8 \end{aligned} \right) \text{De}S_7 \end{aligned} $

	$ \begin{aligned} & \left(\begin{aligned} & \zeta/\text{De}^{12} + 506\zeta/\text{De}^{10} + 108031\zeta/\text{De}^8 + 12728936\zeta/\text{De}^6 + 909450751\zeta/\text{De}^4 \\ & + 40796457506\zeta/\text{De}^2 + 1485993600\varpi + 16329600\vartheta + 1151541572401\zeta \\ & - (644921222400\varpi + 33345043200\vartheta - 19967312312156\zeta)\text{De}^2 \\ & + (41680634486400\varpi + 6623302089600\vartheta + 201529405816816\zeta)\text{De}^4 \\ & - (471232346457600\varpi + 263453732236800\vartheta + 1071814846360896\zeta)\text{De}^6 \\ & + (470762772480000\varpi + 2090287561420800\vartheta + 2482492033152000\zeta)\text{De}^8 \\ & + 1593350922240000(\zeta - \vartheta)\text{De}^{10} \end{aligned} \right) \text{De}S_9 \\ & + \frac{1}{302400} \left(\begin{aligned} & -\zeta/\text{De}^{12} - 650\zeta/\text{De}^{10} - 180895\zeta/\text{De}^8 - 28285400\zeta/\text{De}^6 - 2742417535\zeta/\text{De}^4 \\ & - 171757365650\zeta/\text{De}^2 + 19978358400\varpi + 219542400\vartheta - 7026231453265\zeta \\ & - (13785067296000\varpi + 616475059200\vartheta + 185789298737900\zeta)\text{De}^2 \\ & + (1616508913219200\varpi + 178323973027200\vartheta - 3076822378767280\zeta)\text{De}^4 \\ & - (43410175833024000\varpi + 11493772203513600\vartheta + 30092049283982400\zeta)\text{De}^6 \\ & + \left(\begin{aligned} & 239770987558502400\varpi + 187694941904870400\vartheta \\ & - 156823829909121024\zeta \end{aligned} \right) \text{De}^8 \\ & - \left(\begin{aligned} & 124281371934720000\varpi + 663830207347507200\vartheta \\ & + 359072203696128000\zeta \end{aligned} \right) \text{De}^{10} \\ & - 229442532802560000(\zeta - \vartheta)\text{De}^{12} \end{aligned} \right) \text{De}S_{11} \\ & + \left(\begin{aligned} & (91\varpi + \vartheta - (91091\varpi + 3731\vartheta)\text{De}^2 + (16669653\varpi + 1474473\vartheta)\text{De}^4 \\ & - (790943153\varpi + 135036473\vartheta)\text{De}^6 + (9957703756\varpi + 3336118786\vartheta)\text{De}^8 \\ & - (26596717056\varpi + 20313753096\vartheta)\text{De}^{10} + (6227020800\varpi + 19802759040\vartheta)\text{De}^{12} \\ & + (\varpi + \vartheta)\text{De}^{14} \end{aligned} \right) \text{De}S_{13} \end{aligned} $
7	$ \begin{aligned} & \frac{143}{604800} \left(\begin{aligned} & \zeta/\text{De}^{14} + 204\zeta/\text{De}^{12} + 16422\zeta/\text{De}^{10} + 669188\zeta/\text{De}^8 \\ & + 14739153\zeta/\text{De}^6 + 173721912\zeta/\text{De}^4 + 1017067024\zeta/\text{De}^2 \\ & + 217728000\varpi + 1814400\vartheta + 2483133696\zeta + 1625702400(\zeta - \vartheta)\text{De}^2 \end{aligned} \right) \text{De}S_1 \\ & + \frac{143}{1036800} \left(\begin{aligned} & -\zeta/\text{De}^{14} - 285\zeta/\text{De}^{12} - 32946\zeta/\text{De}^{10} - 1999370\zeta/\text{De}^8 - 68943381\zeta/\text{De}^6 \\ & - 1367593305\zeta/\text{De}^4 - 15088541896\zeta/\text{De}^2 + 870912000\varpi + 7257600\vartheta \\ & - 84865562640\zeta - (18289152000\varpi + 7656768000\vartheta + 202759531776\zeta)\text{De}^2 \\ & - 131681894400(\zeta - \vartheta)\text{De}^2 \end{aligned} \right) \text{De}S_3 \\ & + \frac{143}{3110400} \left(\begin{aligned} & \zeta/\text{De}^{14} + 385\zeta/\text{De}^{12} + 61446\zeta/\text{De}^{10} + 5293970\zeta/\text{De}^8 + 268880381\zeta/\text{De}^6 \\ & + 8261931405\zeta/\text{De}^4 + 151847872396\zeta/\text{De}^2 + 2612736000\varpi + 21772800\vartheta \\ & + 1593719752240\zeta - (326592000000\varpi + 31243968000\vartheta - 8689315795776\zeta)\text{De}^2 \\ & + (1975228416000\varpi + 2602372147200\vartheta + 20407635072000\zeta)\text{De}^4 \\ & + 13168189440000(\zeta - \vartheta)\text{De}^6 \end{aligned} \right) \text{De}S_5 \end{aligned} $

	$ \begin{aligned} & + \frac{13}{1555200} \left(\begin{aligned} & -\zeta/\text{De}^{14} - 506\zeta/\text{De}^{12} - 108031\zeta/\text{De}^{10} - 12728936\zeta/\text{De}^8 \\ & -909450751\zeta/\text{De}^6 - 40796457506\zeta/\text{De}^4 - 1151541572401\zeta/\text{De}^2 \\ & +9144576000\varpi + 76204800\vartheta - 19967312312156\zeta \\ & - (2834818560000\varpi + 155000563200\vartheta + 201529405816816\zeta)\text{De}^2 \\ & - (100855528704000\varpi + 26876594707200\vartheta - 1071814846360896\zeta)\text{De}^4 \\ & - (271593907200000\varpi + 710386937164800\vartheta + 2482492033152000\zeta)\text{De}^6 \\ & - 1593350922240000(\zeta - \vartheta)\text{De}^8 \end{aligned} \right) \text{De} S_7 \end{aligned} $
	$ \begin{aligned} & + \frac{13}{17107200} \left(\begin{aligned} & \zeta/\text{De}^{14} + 650\zeta/\text{De}^{12} + 180895\zeta/\text{De}^{10} + 28285400\zeta/\text{De}^8 \\ & + 2742417535\zeta/\text{De}^6 + 171757365650\zeta/\text{De}^4 + 7026231453265\zeta/\text{De}^2 \\ & + 43110144000\varpi + 359251200\vartheta + 185789298737900\zeta \\ & - (24745222656000\varpi + 1023865920000\vartheta - 3076822378767280\zeta)\text{De}^2 \\ & + (2156757394176000\varpi + 289798243257600\vartheta + 30092049283982400\zeta)\text{De}^4 \\ & - \left(\begin{aligned} & 33668850023424000\varpi + 16780587626880000\vartheta \\ & - 156823829909121024\zeta \end{aligned} \right) \text{De}^6 \\ & + \left(\begin{aligned} & 47800527667200000\varpi + 198030964407091200\vartheta \\ & + 359072203696128000\zeta \end{aligned} \right) \text{De}^8 \\ & + 229442532802560000(\zeta - \vartheta)\text{De}^{10} \end{aligned} \right) \text{De} S_9 \end{aligned} $
	$ \begin{aligned} & + \frac{1}{34214400} \left(\begin{aligned} & -\zeta/\text{De}^{14} - 819\zeta/\text{De}^{12} - 290745\zeta/\text{De}^{10} - 58856655\zeta/\text{De}^8 \\ & - 7522650135\zeta/\text{De}^6 - 635225929065\zeta/\text{De}^4 - 36053226248115\zeta/\text{De}^2 \\ & + 316141056000\varpi + 2634508800\vartheta - 1373222414339685\zeta \\ & - (289269066240000\varpi + 10224528652800\vartheta + 34475213865472380\zeta)\text{De}^2 \\ & + \left(\begin{aligned} & 46094314387968000\varpi + 4232662431782400\vartheta \\ & - 550075031295652720\zeta \end{aligned} \right) \text{De}^4 \\ & - \left(\begin{aligned} & 1735198671951360000\varpi + 407681181993254400\vartheta \\ & + 5242380158902146624\zeta \end{aligned} \right) \text{De}^6 \\ & + \left(\begin{aligned} & 13992591558629376000\varpi + 10491177499804108800\vartheta \\ & - 26862299458337581056\zeta \end{aligned} \right) \text{De}^8 \\ & - \left(\begin{aligned} & 11185323474124800000\varpi + 62289066392624332800\vartheta \\ & + 60912644957448192000\zeta \end{aligned} \right) \text{De}^{10} \\ & - 38775788043632640000(\zeta - \vartheta)\text{De}^{12} \end{aligned} \right) \text{De} S_{11} \end{aligned} $

	$+ \frac{1}{3113510400}$	$ \begin{aligned} & \left(\zeta / \text{De}^{14} + 1015 \zeta / \text{De}^{12} + 451269 \zeta / \text{De}^{10} + 115842675 \zeta / \text{De}^8 \right. \\ & + 19058554515 \zeta / \text{De}^6 + 2109665355525 \zeta / \text{De}^4 + 160557508344855 \zeta / \text{De}^2 \\ & + 4857076224000 \varpi + 40475635200 \vartheta + 8439654758970225 \zeta \\ & - (6464768454144000 \varpi + 207437630400000 \vartheta - 303626807076050640 \zeta) \text{De}^2 \\ & + \left(1620626624128512000 \varpi + 117762524918438400 \vartheta \right) \text{De}^4 \\ & + \left(7307216948928239200 \zeta \right. \\ & - \left(109645838936105472000 \varpi + 16532320756311360000 \vartheta \right) \text{De}^6 \\ & - \left(113057086292850079744 \zeta \right. \\ & + \left(2083021912630923264000 \varpi + 692872337248829337600 \vartheta \right) \text{De}^8 \\ & + \left(1054368810603158319360 \zeta \right. \\ & - \left(9147623154945859584000 \varpi + 8547855357692851200000 \vartheta \right) \text{De}^{10} \\ & - \left(5325923338791614078976 \zeta \right. \\ & + \left(4071457744581427200000 \varpi + 25020418644870237388800 \vartheta \right) \text{De}^{12} \\ & + \left(11977654199703478272000 \zeta \right. \\ & \left. \left. + 7600054456551997440000 (\zeta - \vartheta) \text{De}^{14} \right) \right) \end{aligned} $	$\text{De} S_{13}$
	$+$	$ \begin{aligned} & \left(120 \varpi + \vartheta - (218400 \varpi + 6580 \vartheta) \text{De}^2 + (78558480 \varpi + 4899622 \vartheta) \text{De}^4 \right) \\ & - (8207628000 \varpi + 928095740 \vartheta) \text{De}^6 \\ & + (272803210680 \varpi + 54631129553 \vartheta) \text{De}^8 \\ & - (2706813345600 \varpi + 1009672107080 \vartheta) \text{De}^{10} \\ & + (6165817614720 \varpi + 5056995703824 \vartheta) \text{De}^{12} \\ & - (1307674368000 \varpi + 4339163001600 \vartheta) \text{De}^{14} \end{aligned} $	$\text{De} S_{15}$

8	$\frac{143}{10160640} \left(\begin{aligned} &\zeta/\text{De}^{16} + 285\zeta/\text{De}^{14} + 32946\zeta/\text{De}^{12} + 1999370\zeta/\text{De}^{10} + 68943381\zeta/\text{De}^8 \\ &+ 1367593305\zeta/\text{De}^6 + 15088541896\zeta/\text{De}^4 + 84865562640\zeta/\text{De}^2 \\ &+ 15545779200\varpi + 101606400\vartheta + 202759531776\zeta \\ &+ 131681894400(\zeta - \vartheta)\text{De}^2 \end{aligned} \right) \text{De}S_1$
	$+ \frac{143}{16329600} \left(\begin{aligned} &-\zeta/\text{De}^{16} - 385\zeta/\text{De}^{14} - 61446\zeta/\text{De}^{12} - 5293970\zeta/\text{De}^{10} \\ &- 268880381\zeta/\text{De}^8 - 8261931405\zeta/\text{De}^6 - 151847872396\zeta/\text{De}^4 \\ &- 1593719752240\zeta/\text{De}^2 + 59962291200\varpi + 391910400\vartheta - 8689315795776\zeta \\ &- (1598994432000\varpi + 588649420800\vartheta + 20407635072000\zeta)\text{De}^2 \\ &- 13168189440000(\zeta - \vartheta)\text{De}^4 \end{aligned} \right) \text{De}S_3$
	$+ \frac{13}{3888000} \left(\begin{aligned} &\zeta/\text{De}^{16} + 506\zeta/\text{De}^{14} + 108031\zeta/\text{De}^{12} + 12728936\zeta/\text{De}^{10} + 909450751\zeta/\text{De}^8 \\ &+ 40796457506\zeta/\text{De}^6 + 1151541572401\zeta/\text{De}^4 + 19967312312156\zeta/\text{De}^2 \\ &+ 166561920000\varpi + 1088640000\vartheta + 201529405816816\zeta \\ &- (26483345280000\varpi + 2169659520000\vartheta - 1071814846360896\zeta)\text{De}^2 \\ &+ (205204285440000\varpi + 240029879040000\vartheta + 2482492033152000\zeta)\text{De}^4 \\ &+ 1593350922240000(\zeta - \vartheta)\text{De}^6 \end{aligned} \right) \text{De}S_5$
	$+ \frac{13}{17107200} \left(\begin{aligned} &-\zeta/\text{De}^{16} - 650\zeta/\text{De}^{14} - 180895\zeta/\text{De}^{12} - 28285400\zeta/\text{De}^{10} - 2742417535\zeta/\text{De}^8 \\ &- 171757365650\zeta/\text{De}^6 - 7026231453265\zeta/\text{De}^4 - 185789298737900\zeta/\text{De}^2 \\ &+ 513010713600\varpi + 3353011200\vartheta - 3076822378767280\zeta \\ &- (202639231872000\varpi + 9277781990400\vartheta + 30092049283982400\zeta)\text{De}^2 \\ &+ (9277285744742400\varpi + 2162504455372800\vartheta - 156823829909121024\zeta)\text{De}^4 \\ &- (32504358813696000\varpi + 76460082422169600\vartheta + 359072203696128000\zeta)\text{De}^6 \\ &- 229442532802560000(\zeta - \vartheta)\text{De}^8 \end{aligned} \right) \text{De}S_7$
	$+ \frac{1}{10264320} \left(\begin{aligned} &\zeta/\text{De}^{16} + 819\zeta/\text{De}^{14} + 290745\zeta/\text{De}^{12} + 58856655\zeta/\text{De}^{10} + 7522650135\zeta/\text{De}^8 \\ &+ 635225929065\zeta/\text{De}^6 + 36053226248115\zeta/\text{De}^4 + 1373222414339685\zeta/\text{De}^2 \\ &+ 1978755609600\varpi + 12933043200\vartheta + 34475213865472380\zeta \\ &- (1449768276633600\varpi + 551475369481175920\vartheta - 550075031295652720\zeta)\text{De}^2 \\ &+ (163357488520934400\varpi + 18904293910656000\vartheta + 5242380158902146624\zeta)\text{De}^4 \\ &- \left(3345661516325990400\varpi + 1490460907222425600\vartheta \right) \text{De}^6 \\ &+ \left(6338349968670720000\varpi + 24147996586489036800\vartheta \right) \text{De}^8 \\ &+ 38775788043632640000(\zeta - \vartheta)\text{De}^{10} \end{aligned} \right) \text{De}S_9$

	$ \begin{aligned} & \left(\begin{aligned} & -\zeta/\text{De}^{16} - 1015\zeta/\text{De}^{14} - 451269\zeta/\text{De}^{12} - 115842675\zeta/\text{De}^{10} \\ & -19058554515\zeta/\text{De}^8 - 2109665355525\zeta/\text{De}^6 \\ & -160557508344855\zeta/\text{De}^4 - 8439654758970225\zeta/\text{De}^2 \\ & +10480076006400\vartheta + 68497228800\vartheta - 303626807076050640\zeta \\ & -(12261688927488000\vartheta + 353171711692800\vartheta + 7307216948928239200\zeta)\text{De}^2 \\ & + \left(\begin{aligned} & 2538201048218035200\vartheta + 197977060920038400\vartheta \\ & +113057086292850079744\zeta \end{aligned} \right) \text{De}^4 \\ & - \left(\begin{aligned} & 126494307795727872000\vartheta + 26363788660271462400\vartheta \\ & +1054368810603158319360\zeta \end{aligned} \right) \text{De}^6 \\ & + \left(\begin{aligned} & 1381595294853572198400\vartheta + 959036663583090892800\vartheta \\ & -5325923338791614078976\zeta \end{aligned} \right) \text{De}^8 \\ & - \left(\begin{aligned} & 1538106259064094720000\vartheta + 8238191798334622924800\vartheta \\ & +11977654199703478272000\zeta \end{aligned} \right) \text{De}^{10} \\ & -7600054456551997440000(\zeta - \vartheta)\text{De}^{12} \end{aligned} \right) \end{aligned} $	De S ₁₁
	$ \begin{aligned} & \left(\begin{aligned} & \zeta/\text{De}^{16} + 1240\zeta/\text{De}^{14} + 679644\zeta/\text{De}^{12} + 217378200\zeta/\text{De}^{10} \\ & +45123156390\zeta/\text{De}^8 + 6397840121400\zeta/\text{De}^6 \\ & +635232213337980\zeta/\text{De}^4 + 44565094136562600\zeta/\text{De}^2 \\ & +86698810598400\vartheta + 566658892800\vartheta + 2202549127844351265\zeta \\ & - \left(\begin{aligned} & 147821472070272000\vartheta + 3834580727577600\vartheta \\ & -75623248541039633200\zeta \end{aligned} \right) \text{De}^2 \\ & + \left(\begin{aligned} & 48383398338974899200\vartheta + 2951960202717926400\vartheta \\ & +1757180899801703899744\zeta \end{aligned} \right) \text{De}^4 \\ & - \left(\begin{aligned} & 4376531249846211456000\vartheta + 580820957080472908800\vartheta \\ & -26492213226494426261760\zeta \end{aligned} \right) \text{De}^6 \\ & + \left(\begin{aligned} & 114556944900083548262400\vartheta + 35543406650017006540800\vartheta \\ & +242558905724502235934976\zeta \end{aligned} \right) \text{De}^8 \\ & - \left(\begin{aligned} & 720824009992556101632000\vartheta + 673457546923273586073600\vartheta \\ & -1210310405427816646041600\zeta \end{aligned} \right) \text{De}^{10} \\ & + \left(\begin{aligned} & 484503471605189836800000\vartheta + 3217743410097523507200000\vartheta \\ & +2702572249389834608640000\zeta \end{aligned} \right) \text{De}^{12} \\ & +1710012252724199424000000(\zeta - \vartheta)\text{De}^{14} \end{aligned} \right) \end{aligned} $	De S ₁₃

	$ \begin{aligned} & \left(-\varsigma/\text{De}^{16} - 1496\varsigma/\text{De}^{14} - 997084\varsigma/\text{De}^{12} - 391367064\varsigma/\text{De}^{10} \right. \\ & - 100771975590\varsigma/\text{De}^8 - 17949368157240\varsigma/\text{De}^6 \\ & - 2273079284416380\varsigma/\text{De}^4 - 207184540751085480\varsigma/\text{De}^2 \\ & + 1500556337280000\varpi + 9807557760000\vartheta - 13611213226804376865\varsigma \\ & - \left(3504799418440320000\varpi + 84864797297280000\vartheta \right) \text{De}^2 \\ & + \left(1654766011135036800000\varpi + 85829655022151040000\vartheta \right) \text{De}^4 \\ & - \left(233653318404860914560000\varpi + 23084587870199856000000\vartheta \right) \text{De}^6 \\ & - \left(476330523575730624596224\varsigma \right) \\ & + \frac{1}{653837184000} + \left(10914814579278929679360000\varpi \right) \text{De}^8 \\ & + \left(2057399847911443875840000\vartheta - 7024565491707075358945536\varsigma \right) \text{De}^{10} \\ & - \left(160826434701657081477120000\varpi \right) \text{De}^{12} \\ & + \left(591116406906522011074560000\varpi \right) \text{De}^{14} \\ & + \left(591116406906522011074560000\vartheta - 312542036038910895995289600\varsigma \right) \text{De}^{16} \\ & - \left(232561666370491121664000000\varpi \right) \text{De}^{18} \\ & - \left(1601218796978638203125760000\vartheta + 693568508096521859235840000\varsigma \right) \text{De}^{20} \\ & \left. - 437763136697395052544000000(\varsigma - \vartheta)\text{De}^{22} \right) \end{aligned} $	DeS ₁₅
+	$ \begin{aligned} & \left(153\varpi + \vartheta - (468180\varpi + 10812\vartheta)\text{De}^2 + (299650806\varpi + 13896582\vartheta)\text{De}^4 \right. \\ & - (60202693980\varpi + 4853222764\vartheta)\text{De}^6 + (4308105301929\varpi + 577924894833\vartheta)\text{De}^8 \\ & - (110228466184200\varpi + 24871845297936\vartheta)\text{De}^{10} \\ & + (909299905844112\varpi + 369012649234384\vartheta)\text{De}^{12} \\ & - (1821602444624640\varpi + 1583313975727488\vartheta)\text{De}^{14} \\ & \left. + (355687428096000\varpi + 1223405590579200\vartheta)\text{De}^{16} \right) \end{aligned} $	DeS ₁₇

Legend: $\tilde{\lambda} \equiv \sqrt{\sigma_1}/\lambda_1$; $\varpi \equiv \frac{\lambda_1^2 \tilde{\lambda}^3 - \sigma_2}{\lambda_1(\lambda_1 - \lambda_2)}$; $\vartheta \equiv \frac{\sigma_2 - \lambda_1 \lambda_2 \tilde{\lambda}^2}{\lambda_1(\lambda_1 - \lambda_2)}$; $\varsigma \equiv \frac{\sigma_2(1 - \tilde{\lambda})}{\lambda_1(\lambda_1 - \lambda_2)}$.

Figure 1: Comparison of our main result, Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, **red**], with the exact solution, Eq. (17) [with Eqs. (18) through (23), **black**], for loops of minus dimensionless shear stress versus dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Maxwell fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = 0$, $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_2 = 0$).

Figure 2: Comparison of our main result, Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, **red**], with the exact solution, Eq. (17) [with Eqs. (18) through (23), **black**], for loops of minus dimensionless shear stress versus dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = \frac{1}{27}$, $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_1\lambda_2$).

Figure 3: Comparison of our main result, Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, **red**], with the exact solution, Eq. (17) [with Eqs. (18) through (23), **black**], for loops of minus dimensionless shear stress versus dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_{\infty}/\eta_0 = \lambda_2/\lambda_1 = \frac{1}{9}$, $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_1\lambda_2$).

Figure 4: Comparison of our main result, Eq. (31) [with Eqs. (32) and (33), Table III and Table IV, **red**], with the exact solution, Eq. (17) [with Eqs. (18) through (23), **black**], for loops of minus dimensionless shear stress versus dimensionless shear rate for $Wi/De = \frac{1}{10}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$ for each De . Special case of the corotational Jeffreys fluid ($\eta_\infty/\eta_0 = \lambda_2/\lambda_1 = \frac{1}{3}$, $\sigma_1 = \lambda_1^2$, $\sigma_2 = \lambda_1\lambda_2$).

Figure 5: 11x6 Ewoldt grids of our main result [Eq. (31) with Eqs. (32) and (33), Table III and Table IV, **red**] *versus* exact solutions [Eq. (17) with Eqs. (18) and (23) for LAOS, Eq. (29) for steady shear flow and Eq. (30) for SAOS, **black**]. Counterclockwise loops of minus dimensionless shear stress, $-\mathbb{S}$, *versus* $\dot{\gamma}/\dot{\gamma}^0 = \cos \tau$, calculated for the special case of corotational Maxwell fluid, $\sigma_2 = \lambda_2 = 0$ and $\sigma_1 = \lambda_1^2$.

Figure 6: 11x6 Ewoldt grids of our main result [Eq. (31) with Eqs. (32) and (33), Table III and Table IV, **red**] *versus* exact solutions [Eq. (17) with Eqs. (18) and (23) for LAOS, Eq. (29) for steady shear flow and Eq. (30) for SAOS, **black**]. Counterclockwise loops of minus dimensionless shear stress, $-\mathbb{S}$, *versus* $\dot{\gamma}/\dot{\gamma}^0 = \cos \tau$, calculated for the special case of corotational Jeffreys fluid, $\sigma_2 = \lambda_1 \lambda_2$, $\sigma_1 = \lambda_1^2$ and $\lambda_2/\lambda_1 = \frac{1}{27}$.

Figure 7: 11x6 Ewoldt grids of our main result [Eq. (31) with Eqs. (32) and (33), Table III and Table IV, **red**] *versus* exact solutions [Eq. (17) with Eqs. (18) and (23) for LAOS, Eq. (29) for steady shear flow and Eq. (30) for SAOS, **black**]. Counterclockwise loops of minus dimensionless shear stress, $-\mathbb{S}$, *versus* $\dot{\gamma}/\dot{\gamma}^0 = \cos \tau$, calculated for the special case of corotational Jeffreys fluid, $\sigma_2 = \lambda_1 \lambda_2$, $\sigma_1 = \lambda_1^2$ and $\lambda_2/\lambda_1 = \frac{1}{9}$.

Figure 8: 11x6 Ewoldt grids of our main result [Eq. (31) with Eqs. (32) and (33), Table III and Table IV, **red**] *versus* exact solutions [Eq. (17) with Eqs. (18) and (23) for LAOS, Eq. (29) for steady shear flow and Eq. (30) for SAOS, **black**]. Counterclockwise loops of minus dimensionless shear stress, $-\mathbb{S}$, *versus* $\dot{\gamma}/\dot{\gamma}^0 = \cos \tau$, calculated for the special case of corotational Jeffreys fluid, $\sigma_2 = \lambda_1 \lambda_2$, $\sigma_1 = \lambda_1^2$ and $\lambda_2/\lambda_1 = \frac{1}{3}$

Figure 9: Comparison of generalized Johnson-Segalman, Eq. (53) [with Eqs.(68) through (71)], by using well-known fitted values, Eq. (72), with the $|\eta_1^*(\omega, \dot{\gamma}^0)|$ frequency sweep measurements on molten atactic polystyrene at 190°C at six strain amplitudes $\frac{Wi}{De} = 0.2(\Delta)$, $2.67(\circ)$, $5(\square)$, $6.67(\diamond)$, $8(+)$, $10(\times)$. **Red** curve is $|\eta^*(\omega)|$ from Eq. (60) [with Eqs. (61) and (62)]. Measurements are in TABLE V of [14].

Figure 10: $|\eta_1^*(\omega, \dot{\gamma}^0)|$ for strain rate sweeps from Eq. (53) [with Eqs. (68) through (71)] using well-known fitted values, Eq. (72), *versus* interpolations from TABLE VI of [14]. Inset, from top to bottom, $|\eta_1^*(\omega, \dot{\gamma}^0)|$ is offset by $\eta_+ = 2.5, 2, 0.15, 0.1, 0.05, 0$ MPa·s. Measurements are in Table VI of [14].

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