QUEEN'S UNIVERSITY
POLYMERS RESEARCH GROUP
19 Division Street, Kingston, ON, K7L 3N6 Canada

NONLINEAR CORE DEFLECTION IN INJECTION MOLDING

P. Pongthong^{1,2}, A.J. Giacomini^{2,3,*}, C. Saengow²,
C. Kositawong¹, H.-C. Liao⁴ and S.-C. Tseng⁴

¹Polymer Research Center
Mechanical and Aerospace Engineering Department
King Mongkut's University of Technology North Bangkok
Bangkok, THAILAND 10800

²Polymers Research Group, Chemical Engineering Department

³Mechanical and Materials Engineering Department
Queen's University, Kingston, Ontario, CANADA K7L 3N6

⁴Mechanical Engineering Department
National Yunlin University of Science and Technology
Douliu, Yunlin, Taiwan 640

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*Corresponding author (giacomini@queensu.ca)

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ABSTRACT

Injection molding of thin slender parts is often complicated by core deflection. This deflection is caused by molten plastics *race tracking* through the slit between the core and the rigid cavity wall. The pressure of this liquid exerts a lateral force on the slender core, causing the core to bend, and this bending is governed by a nonlinear fifth order ordinary differential equation for the deflection that is not directly in the position along the core. Here we subject this differential equation to 6 sets of boundary conditions, corresponding to 6 commercial core constraints. For each such set of boundary conditions, we develop an explicit approximate analytical solution, including both a linear term, and a nonlinear term. By comparison with finite difference solutions, we find our new analytical solutions to be accurate. We then use these solutions to derive explicit analytical approximations for maximum deflections, and for the core position of these maximum deflections. Our experiments on the base-gated / free-tip boundary condition agree closely with our new explicit approximate analytical solution.

Keywords: Injection molding, thin-wall injection molding, core deflection, core bending, race tracking, core deflectability.

*Corresponding author (giacomini@queensu.ca)

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I. INTRODUCTION

Injection molding long hollow plastic products with a closed end (e.g., test tubes, medicinal vials, bottle pre-forms for stretch blow molding, tubular ink containers for pens, thermometer probe covers, and ferrules for fiber optic connections [1,2]) involve molds with slender cores. For fiber optic ferrules, the glass fiber tip itself forms the deflecting core. For these, the glass fiber tip must thus be inserted into the mold, before overmolding, and this is thus sometimes called *core-insert moulding*. Core deflection is thus a central concern whenever a slender core-insert is overmould. Good example of this includes overmolding the plastic handle on a metal chopsticks, of the overmolding onto the free ends of eyeglass arms. Injection molding with such cores poses a special challenge. Specifically, when polymer races ahead along one side of the core (sometimes called *race tracking*) [3,4], the melt pressure applies a side load to the core. Slender cores can then deflect enough to misshape products. Normally elastic, this core bending is called *core deflection*. If the core deflects enough to touch the cavity wall, a hole is left in the product [5]. Product solidified onto a bent core bows, making it difficult to pull off the core. Poorly designed cores may even break during molding [19]. An asymmetric core [16], or asymmetric gating of a symmetric core (see Section 6.3.1-3 of Chapter 6 and Fig. 6-39 of [6]) can provoke polymer to race down just one side of the core [7]. Even if gated symmetrically, core misalignment can cause polymer to race track. With improper locking [8], the side load itself can worsen core misalignment. This occurs through a sliding motion of the core with respect to the receiving mold part and along the parting surfaces. This sliding motion is called *core shift*. By *core shift*, we mean this solid body motion, but some define *core shift* as *any* core motion, including core deflection (see Subsection “Deflection of Slender Core Pins” of Chapter VII of [16,17], [20]). Asymmetric cooling [9] can also cause polymer to race ahead along the hotter side of a core, even if perfectly aligned, locked and gated symmetrically. In other words, an asymmetric temperature profile across the gate can cause race tracking [3]. Asymmetric venting can also cause race tracking [5]. For a more detailed review of the literature on core deflection, see [30].

Table I summarizes the literature on core deflection analysis. Using finite element modeling (FEM), the IKV (Germany) have calculated the effects of core length, core diameter, core misalignment, gating type, and gate position on core deflection for a particular solidifying, shear thinning molten plastic [10,18,20]. Giacomini *et al.* further studied this problem analytically by following Bird *et al.* (see Example 1.2-1 of [11]). Specifically, they derived the fifth order nonlinear governing equation, with the special complication that it is not directly in the variable of differentiation, X , for deflected core profile (combine Eqs. (1) and (3) of [30]):

$$\frac{d^5 y}{dx^5} = \frac{-12\mu Q}{EIB^3} \quad (1)$$

which can be nondimensionalized to (Eq. (6) of [30]):

$$\frac{d^5 Y}{dX^5} = \frac{-12\mathcal{Q}}{(1+Y)^3} \quad (2)$$

where:

$$\mathcal{Q} \equiv \frac{\mu QL}{EI} \mathcal{L}^4 \quad (3)$$

from which Eq. (2), we learn that core deflection depends on just one dimensionless group called *core deflectability*, $\mathcal{Q}$, and that this scales with the fourth power of the flow length ratio, $\mathcal{L} \equiv L/B_0$. In this paper, Table III and Table IV define all of the dimensional and dimensionless variables or groups, respectively. A summary of core injection molding experiments, categorized by core cross-section shape, round, square, rectangular or arbitrary, can be found in Table II of [30].

In this paper, we consider only the rectangular cross-section. We arrive at the maximum core deflection, that is, when the fluid reaches to the core tip by neglecting solidification, which narrows and eventually freezes the flow channels in injection molding.

II. ANALYTICAL SOLUTIONS

Table II summarizes the six most commercially relevant core conditions (from Table II of [30]). Figure 1 gives a free body diagram for each of these core conditions. Case I is the classic and worst case of a free tip with tip-gated injection. Case II is also of a free tip but base-gating. The core tip is sometimes pinned during most of the filling, and Case III represents this method with tip-gating [12], and Case IV, base-gating [13]. Sometimes, a small amount of plastic is first injected and frozen around the core tip before the rest can be injected, and Cases V and VI represent this approach. Case V employs the cantilevered tip with tip-gating, and Case VI, base-gating.

To solve Eq. (2), we expand its right side in a Taylor series about zero core deflection:

$$\frac{d^5 Y}{dX^5} = -12\mathcal{Q} \{1 - 3Y + 6Y^2 - 10Y^3 + 15Y^4 - 21Y^5 + \dots\} \quad (4)$$

which contracts to:

$$\frac{d^5 Y}{dX^5} = 6\mathcal{Q} \sum_{j=1}^{\infty} (j^2 + j) (-1)^j Y^{j-1} \quad (5)$$

To simplify this problem, we truncate Eq. (4) to the second term:

$$\frac{d^5 Y}{dX^5} = -12\mathcal{Q} \{1 - 3Y\} \quad (6)$$

keeping “1” for the linear response, and to approximate the nonlinear response, “ $-3Y$ ”.

Solving Eq. (6), we get:

$$Y(X) = \frac{1}{3} + C_1 \exp(a\mathcal{Q}X) + C_2 \exp\left(\frac{1}{4}ab\mathcal{Q}X\right) + C_3 \exp\left(-\frac{1}{4}ac\mathcal{Q}X\right) \\ + C_4 \exp\left(-\frac{1}{4}ad\mathcal{Q}X\right) + C_5 \exp\left(\frac{1}{4}ae\mathcal{Q}X\right) \quad (7)$$

where:

$$a \equiv \sqrt[5]{6^2}, \quad b \equiv i\sqrt{2}\sqrt{5-\sqrt{5}} - \sqrt{5} - 1, \quad c \equiv i\sqrt{2}\sqrt{5-\sqrt{5}} + \sqrt{5} + 1, \quad (8)$$

$$d \equiv i\sqrt{2}\sqrt{5+\sqrt{5}} - \sqrt{5} + 1, \quad e \equiv i\sqrt{2}\sqrt{5+\sqrt{5}} + \sqrt{5} - 1$$

and where $C_1, \dots, C_5$ are integration constants.

Differentiating Eq. (7) with respect to X five times gives:

$$Y'(X) = \frac{dY}{dX} = a \mathcal{Q} C_1 \exp(a \mathcal{Q} X) + \frac{1}{4} ab \mathcal{Q} C_2 \exp\left(\frac{1}{4} ab \mathcal{Q} X\right) - \frac{1}{4} ac \mathcal{Q} C_3 \exp\left(-\frac{1}{4} ac \mathcal{Q} X\right) \\ - \frac{1}{4} ad \mathcal{Q} C_4 \exp\left(-\frac{1}{4} ad \mathcal{Q} X\right) + \frac{1}{4} ae \mathcal{Q} C_5 \exp\left(\frac{1}{4} ae \mathcal{Q} X\right) \quad (9)$$

$$Y''(X) = \frac{d^2Y}{dX^2} = a^2 \mathcal{Q}^2 C_1 \exp(a \mathcal{Q} X) + \frac{1}{16} a^2 b^2 \mathcal{Q}^2 C_2 \exp\left(\frac{1}{4} ab \mathcal{Q} X\right) + \frac{1}{16} a^2 c^2 \mathcal{Q}^2 C_3 \exp\left(-\frac{1}{4} ac \mathcal{Q} X\right) \\ + \frac{1}{16} a^2 d^2 \mathcal{Q}^2 C_4 \exp\left(-\frac{1}{4} ad \mathcal{Q} X\right) + \frac{1}{16} a^2 e^2 \mathcal{Q}^2 C_5 \exp\left(\frac{1}{4} ae \mathcal{Q} X\right) \quad (10)$$

$$Y^{(3)}(X) = \frac{d^3Y}{dX^3} = a^3 \mathcal{Q}^3 C_1 \exp(a \mathcal{Q} X) + \frac{1}{64} a^3 b^3 \mathcal{Q}^3 C_2 \exp\left(\frac{1}{4} ab \mathcal{Q} X\right) - \frac{1}{64} a^3 c^3 \mathcal{Q}^3 C_3 \exp\left(-\frac{1}{4} ac \mathcal{Q} X\right) \\ - \frac{1}{64} a^3 d^3 \mathcal{Q}^3 C_4 \exp\left(-\frac{1}{4} ad \mathcal{Q} X\right) + \frac{1}{64} a^3 e^3 \mathcal{Q}^3 C_5 \exp\left(\frac{1}{4} ae \mathcal{Q} X\right) \quad (11)$$

$$Y^{(4)}(X) = \frac{d^4Y}{dX^4} = a^4 \mathcal{Q}^4 C_1 \exp(a \mathcal{Q} X) + \frac{1}{256} a^4 b^4 \mathcal{Q}^4 C_2 \exp\left(\frac{1}{4} ab \mathcal{Q} X\right) + \frac{1}{256} a^4 c^4 \mathcal{Q}^4 C_3 \exp\left(-\frac{1}{4} ac \mathcal{Q} X\right) \\ + \frac{1}{256} a^4 d^4 \mathcal{Q}^4 C_4 \exp\left(-\frac{1}{4} ad \mathcal{Q} X\right) + \frac{1}{256} a^4 e^4 \mathcal{Q}^4 C_5 \exp\left(\frac{1}{4} ae \mathcal{Q} X\right) \quad (12)$$

Solving Eqs. (7) and (9)-(12) subjected to each of the six sets of commercially relevant boundary conditions of Table II yields the integration constants, $C_1, \dots, C_5$. Appendices VIII summarizes these results.

Commercially, our main interest is in the maximum core deflection as it can cause serious damage to the products. As the core profile has been introduced in Eq. (7), to analyze this maximum, we replace Y with the maximum core deflection, $\mathcal{Y} \equiv Y(\mathcal{X})$, to get:

$$\mathcal{Y} = \frac{1}{3} + C_1 \exp(a \mathcal{Q} \mathcal{X}) + C_2 \exp\left(\frac{1}{4} ab \mathcal{Q} \mathcal{X}\right) + C_3 \exp\left(-\frac{1}{4} ac \mathcal{Q} \mathcal{X}\right) \\ + C_4 \exp\left(-\frac{1}{4} ad \mathcal{Q} \mathcal{X}\right) + C_5 \exp\left(\frac{1}{4} ae \mathcal{Q} \mathcal{X}\right) \quad (13)$$

To evaluate for $\mathcal{Y}$, we need the position at which the maximum core deflection takes place, $\mathcal{X}$. Geometrically, the core profile gradient at such position should be:

$$Y'(\mathcal{X}, \mathcal{Q}) = 0 \quad (14)$$

We will use Eq. (13), the main result of this paper, to depict the maximum core deflection in Section IV for all six cases, and to compare with our core-injection molding experiment in Section V for Case II.

a. Cases I and II

For Cases I and II (illustrated in Figure 1), we have the maximum core deflection at the core tip, $\mathcal{R} = 0$. Substituting $\mathcal{R} = 0$ into Eq. (13) gives:

$$\mathcal{Y} = \frac{1}{3} + C_1 + C_2 + C_3 + C_4 + C_5 \quad (15)$$

where $C_1, \dots, C_5$ are given in Appendices VIII.a. for Case I and Appendices VIII.b. for Case II.

b. Cases III through VI

For Cases III through VI, the maximum core deflection is not at the core tip, $0 < \mathcal{R} < 1$. Combining Eqs. (9) and (14) thus yields:

$$0 = a\mathcal{Q}C_1 \exp(a\mathcal{Q}\mathcal{R}) + \frac{1}{4}ab\mathcal{Q}C_2 \exp\left(\frac{1}{4}ab\mathcal{Q}\mathcal{R}\right) - \frac{1}{4}ac\mathcal{Q}C_3 \exp\left(-\frac{1}{4}ac\mathcal{Q}\mathcal{R}\right) - \frac{1}{4}ad\mathcal{Q}C_4 \exp\left(-\frac{1}{4}ad\mathcal{Q}\mathcal{R}\right) + \frac{1}{4}ae\mathcal{Q}C_5 \exp\left(\frac{1}{4}ae\mathcal{Q}\mathcal{R}\right) \quad (16)$$

which rearranges to:

$$0 = 1 + \frac{bC_2}{4C_1} \exp\left[a\mathcal{Q}\mathcal{R}\left(\frac{b}{4} - 1\right)\right] - \frac{cC_3}{4C_1} \exp\left[-a\mathcal{Q}\mathcal{R}\left(\frac{c}{4} + 1\right)\right] - \frac{dC_4}{4C_1} \exp\left[-a\mathcal{Q}\mathcal{R}\left(\frac{d}{4} + 1\right)\right] + \frac{eC_5}{4C_1} \exp\left[a\mathcal{Q}\mathcal{R}\left(\frac{e}{4} - 1\right)\right] \quad (17)$$

from which we learn that $\mathcal{R}$ and thus $\mathcal{Y}$ each depend on $\mathcal{Q}$ only. For Cases I and II, $\mathcal{R} = 0$, but for the rest of the boundary conditions (Cases III through VI), we use Eq. (17) to find $\mathcal{R}(\mathcal{Q})$. We then substitute these $\mathcal{R}(\mathcal{Q})$ into Eq. (13) to get $\mathcal{Y}(\mathcal{Q})$, which we then plot in Figure 14 through Figure 17 for Cases III through VI respectively.

III. NUMERICAL SOLUTIONS

We can test our approximations for $\mathcal{Y}$ and $\mathcal{R}$, obtained by substituting $\mathcal{R}(\mathcal{Q})$ into Eq. (13) by solving Eq. (2) for Y by finite difference. For this, we employ the *dsolve* subroutine by coding Eq. (2) into Maple (Version 2016) [14] for the Case I through VI

boundary conditions. On a MacBook Pro (2.3 GHz Intel Core i5 processor with 16 GB 2133 MHz LPDDR3 memory) employing the OS X (Version 10.13.1) operating system, we find such an evaluation to consume less than 1 sec of CPU time for any of the curves in Figure 2 through Figure 21 and Figure 24 through Figure 26.

IV. RESULTS

Figure 2 through Figure 7 show the dimensionless core deflection along the core, $Y(X)$. From these figures, we see how Y increases with $|\mathcal{Q}|$, for every position along the core. From Figure 2 and Figure 3, we learn that, for Cases I and II, the maximum core deflection is at $\mathcal{R} = X = 0$, but for the other cases, the position of maximum deflection is elsewhere. From Figure 2 and Figure 3 we see that, for $|\mathcal{Q}| < \frac{1}{2}$, our new analytical approximations for Cases I and II fall only slightly below the finite difference solutions, including some of the nonlinear region, $|\mathcal{Q}| > \frac{1}{10}$. For Figure 4 through Figure 7, we see that, for $|\mathcal{Q}| \leq 1$, our new analytical approximations for Cases III through VI fall only slightly below the finite difference solutions, thus validating our analytical method.

From Figure 8 through Figure 13, for all six Cases, we see how the maximum deflection $\mathcal{Y}(\mathcal{R})$ increases with $|\mathcal{Q}|$, as Figure 2 through Figure 7 imply. From Figure 8 through Figure 13, we see great improvement of our new nonlinear approximation, Eq. (13), over our previous linear approximation in explaining core deflection. From Figure 8 and Figure 9 we see that, for $|\mathcal{Q}| < 1$, our new analytical approximations for Cases I and II fall only slightly below the finite difference solutions, including some of the nonlinear region, $|\mathcal{Q}| > \frac{1}{10}$. Figure 10 and Figure 11 we see that, for $|\mathcal{Q}| < 20$, our new analytical approximations for Cases III and IV fall only slightly below the finite difference solutions, including some of the nonlinear region, $|\mathcal{Q}| > 2$. From Figure 12 and Figure 13 we see that, for $|\mathcal{Q}| < 30$, our new analytical approximations for Cases V and VI fall only slightly below the finite difference solutions, including some of the nonlinear region, $|\mathcal{Q}| > 6$, thus validating our analytical method.

For Cases I and II, from Figure 1, the $\mathcal{Y}$ obviously happens at the core tip, $\mathcal{R} = 0$. Figure 14 through Figure 17 show how $\mathcal{R}$ varies with $|\mathcal{Q}|$. From Figure 14 and Figure 16, we see that for tip-gating Cases III and V, $\mathcal{R}$ increases with $|\mathcal{Q}|$. We also see that, for $|\mathcal{Q}| < 8$, our new analytical approximations for Cases III and V fall only slightly below the finite difference solutions. From Figure 15 and Figure 17, we see that for base-gating Cases III and V, $\mathcal{R}$ decreases with $|\mathcal{Q}|$. Furthermore, we see that, for

$|\mathcal{Q}| < 8$, our new analytical approximations for Cases IV and VI fall only slightly below the finite difference solutions, thus validating our analytical method.

For completeness, we close our discussion on how $\mathcal{Y}$ and $\mathcal{R}$ are related. For Cases I and II, $\mathcal{Y}$ does not depend on $\mathcal{R}$. However, from Figure 18 and Figure 20, we see that, for tip-gated Cases III and V, $\mathcal{Y}$ increases with $\mathcal{R}$. From Figure 19 and Figure 21 we see that, for base-gated Cases III and V, $\mathcal{Y}$ decreases with $\mathcal{R}$.

V. APPLICATIONS: FREE TIP WITH BASE-GATED INJECTION

In this section, we show how to apply our new nonlinear analytical approximation for Case II, Eq. (15) [with Eqs. (8) and (25)-(30)]. We validate this use by comparing with a finite difference solution to Eq. (2). Finally, we compare the results of these two solutions with our experimental data of a base-gated deflecting A6061 T6 aluminum core of rectangular cross-section, Case II of Figure 1.

A molten polypropylene (PP6331 PRO-FAX at 220°C [15]) was injected under the core in our experiment shown in Figure 22. Experimental conditions are described elsewhere (see Section 4. of [30]). Figure 23 includes videography of the core deflection, taken through a rugged quartz window. We also precisely measured the developing core deflection at three positions using proximity, through a borescope. Figure 24 shows good agreement between our new nonlinear analytical approximation and this proximity. Figure 24 also shows that our new nonlinear analytical approximation only slightly under-predicts the finite difference solution to Eq. (2) for our specific experimental conditions.

Figure 25 and Figure 26 further illustrate the differences in deflections of the six cases when $|\mathcal{Q}| = 1$. From these two figures, we rank each case on its maximum deflection as follows:

$$\text{Case I} > \text{Case II} > \text{Case III} > \text{Case IV} > \text{Cases V or VI} \quad (18)$$

From the symmetry of Cases V and VI (see Figure 1), the maximum deflections for Cases V and VI match (see Figure 26), as they must.

VI. CONCLUSION

This paper uncovers new explicit approximate analytical solutions for the central equation of core deflection in injection molding. By including both a linear term, and a nonlinear term, we solve this fifth order nonlinear ordinary differential equation that is not directly the variable of differentiation. Since core deflection arises commercially in six different configurations, in this paper we consider the boundary conditions

corresponding to each of these six configurations. By incorporating nonlinearity, our main results significantly improve upon prior analytical work for all six configurations.

Our analysis shows that only core deflectability, $|\mathcal{Q}|$, matters. Our main results for deflected core profiles, Eq. (7), maximum core deflection and its position, Eq. (13) and (17), agree well with our experimental measurements on core-injection molding of polypropylene. Our experiments pertain to the most common core deflection configuration: base-gated free-tip (Case II). We know of no core deflection measurements for any of the other five cases.

To our knowledge, ours is the first analytical work to include the nonlinearity of the core deflection. We include this for each and every commercially relevant configuration of core deflection. To our knowledge, our paper discloses the first videography of a deflecting core.

VII. ACKNOWLEDGMENT

Poungthong acknowledges support from the Royal Golden Jubilee Program of the Thailand Research Fund (Contract No. PHD/0178/2557). Kolutawong acknowledges funding from King Mongkut's University of Technology North Bangkok (Contract No. KMUTNB-KNOW-61-039). Giacomini is indebted to the Faculty of Applied Science and Engineering of Queen's University at Kingston, for its support through a Research Initiation Grant (RIG). This research was undertaken, in part, thanks to support from the Canada Research Chairs program of the Government of Canada for the Natural Sciences and Engineering Research Council of Canada (NSERC) Tier 1 Canada Research Chair in Rheology. This research was also undertaken, in part, thanks to support from the Discovery Grant program of the Natural Sciences and Engineering Research Council of Canada (NSERC).

VIII. APPENDICES: INTEGRATION CONSTANTS

a. Case I

$$C_1 = \frac{-bcde \left[\begin{aligned} &ce(b+d)(b^2-bd+d^2)(c+e)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) - bc(e+d)(d^2-de+e^2)(b+c)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &-ed(b+c)(b^2-bc+c^2)(e+d)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + bd(c+e)(c^2-ce+e^2)(b+d)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+(b-e)(c-d)\left[be(c^2+cd+d^2)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) + cd(b^2+be+e^2)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \right] \end{aligned} \right]}{f_1} \quad (19)$$

$$C_2 = \frac{-16cde \left[\begin{aligned} &\left((c^2+4c)d^3+ce^3(c+4) \right)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) + \left((-c^3-e^3)d^2+(-4c^3-4e^3)d \right)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &-\frac{1}{4}ce(d+4)(d^2-4d+16)(c+e)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + \left((-e^2+4e)d^3+c^3e(e-4) \right)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+\frac{1}{4}d \left(e(c+4)(c^2-4c+16)(e+d)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) + c(e-4)(e^2+4e+16)(c-d)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \right) \end{aligned} \right]}{f_1} \quad (20)$$

$$C_3 = \frac{16bde \left[\begin{aligned} &\left((e^2-4e)d^3+b^3e(e-4) \right)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + \left((-b^2+4b)d^3-be^3(-4+b) \right)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &+\frac{1}{4} \left[be(d+4)(d^2-4d+16)(b-e)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + de(-4+b)(b^2+4b+16)(e+d)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \right] \\ &+d \left((-4b^3+4e^3)d-16b^3+16e^3 \right)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) - bd(e-4)(e^2+4e+16)(b+d)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \end{aligned} \right]}{f_1} \quad (21)$$

$$C_4 = \frac{16bce \left[\begin{aligned} &\left((-e^2+4e)b^3-c^3e(e-4) \right)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + \left((c^3+e^3)b^2+(-4c^3-4e^3)b \right)\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) \\ &-\frac{1}{4}be(c+4)(c^2-4c+16)(b-e)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) \\ &+\frac{1}{4}c \left[-e(-4+b)(b^2+4b+16)(c+e)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) + ((4c+16)b^3-4e^3(c+4))\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \right] \\ &+b(e-4)(e^2+4e+16)(b+c)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \end{aligned} \right]}{f_1} \quad (22)$$

$$C_5 = \frac{16bcd \left[\begin{aligned} &\left((c^2+4c)d^3+b^3c(c+4) \right)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + \left((-b^3-c^3)d^2+(-4b^3-4c^3)d \right)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &-\frac{1}{4}bc(d+4)(d^2-4d+16)(b+c)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + \left((-b^2+4b)d^3+bc^3(-4+b) \right)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+\frac{1}{4}d \left(b(c+4)(c^2-4c+16)(b+d)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) + c(-4+b)(b^2+4b+16)(c-d)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \right) \end{aligned} \right]}{f_1} \quad (23)$$

$$\begin{aligned}
f_1 = & \left(\begin{aligned} & 48(c-d)(b+d)(d^2+(-b+c)d+b^2-bc+c^2)e^2(b+c)(e-4)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ & -3(d^2+(-e-4)d+e^3+4e+16)b^2c^2(e+d)(b+c)(d+4)(e-4)\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ & +3(-4+b)(c+e)c^2(d^2+(-b-4)d+b^2+4b+16)(b+d)e^2(d+4)\exp\left(\frac{1}{4}a(4+b-e)\mathcal{Q}\right) \\ & -48(b-e)c^2(c+4)(b+d)(e+d)(d^2+(-b-e)d+b^2+be+e^2)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ & -48(-4+b)(c+e)b^2(c-d)(e+d)(d^2+(c-e)d+c^2-ce)+e^2\exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ & +3(c+e)b^2(c+4)(b+d)(c^2+(-e-4)c+e^2+4e+16)d^2(e-4)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ & -3(b^2+(4-c)b+c^2-4c+16)(-4+b)(c+4)(e+d)d^2e^2(b+c)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ & +3(b-e) \left[\begin{aligned} & 16(c+e)(b^2+(e-c)b+c^2-ce+e^2)d^2(b+c)(d+4)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ & + (c-d) \left[\begin{aligned} & b^2(c+4)(d^2+(-4+c)d+c^2-4c+16)e^2(d+4)\exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ & + (-4+b)c^2d^2(b^2+(e+4)b+e^2+4e+16)(e-4)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \end{aligned} \right] \end{aligned} \right] \end{aligned} \right) \quad (24)
\end{aligned}$$

b. Case II

$$C_1 = \frac{bcde \left(\begin{aligned} & bce(c+e)(b-e)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) - cde(e+d)(c-d)(c+e)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ & + bd(b+d) \left(-e(e+d)(b-e)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) + c(c-d)(b+c)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \right) \end{aligned} \right)}{f_2} \quad (25)$$

$$C_2 = \frac{4cde \left(\begin{aligned} & 4ce(e-4)(c+4)(c+e)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ & + d \left[\begin{aligned} & -4e(d+4)(e-4)(e+d)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ & + c(c-d) \left(-4(d+4)(c+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) + e(e+d)(c+e)\exp(a\mathcal{Q}) \right) \end{aligned} \right] \end{aligned} \right)}{f_2} \quad (26)$$

$$C_3 = \frac{-4bde \left(\begin{aligned} & 4eb(e-4)(b-4)(b-e)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) + 4de(d+4)(e-4)(e+d)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ & + bd(b+d) \left(-4(d+4)(b-4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) + e(e+d)(b-e)\exp(a\mathcal{Q}) \right) \end{aligned} \right)}{f_2} \quad (27)$$

$$C_4 = \frac{\left(\begin{array}{l} 16bc^2e^2(e-4)(c+4)(c+e)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ +4b^2ce \left(\begin{array}{l} 4e(e-4)(b-4)(b-e)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ +c(b+c)\left(-4(c+4)(b-4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right)+e(c+e)(b-e)\exp(a\mathcal{Q})\right) \end{array} \right) \end{array} \right)}{f_2} \quad (28)$$

$$C_5 = \frac{-4bcd \left(\begin{array}{l} 4cb(c+4)(b-4)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right)-4cd(d+4)(c+4)(c-d)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ +bd(b+d)\left(-4(d+4)(b-4)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right)+c(c-d)(b+c)\exp(a\mathcal{Q})\right) \end{array} \right)}{f_2} \quad (29)$$

where:

$$f_2 = 3 \left(\begin{array}{l} \left(\begin{array}{l} -16c^2e^2(e-4)(c+4)(c+e)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) \\ +16b^2c^2(c+4)(b-4)(b+c)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ +16e^2d^2(d+4)(e-4)(e+d)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ -16b^2d^2(d+4)(b-4)(b+d)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ -b^2c^2e^2(c+e)(b-e)(b+c)(d+4)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) \\ -16b^2e^2(e-4)(b-4)(b-e)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ b^2d^2e^2(e+d)(b+d)(b-e)(c+4)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) \end{array} \right) \\ +c^2d^2(c-d) \left(\begin{array}{l} e^2(e+d)(c+e)(b-4)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \\ -b^2(e-4)(b+d)(b+c)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right) \\ -16(d+4)(c+4)(b-e)\exp(a(b+e)\mathcal{Q}) \end{array} \right) \end{array} \right) \quad (30)$$

c. Case III

$$C_1 = \frac{\left(\begin{aligned} & -\left(d^2 + (-b+c)d + b^2 - cb + c^2\right)(c-d)(b+d)e^2(b+c)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ & - (b+d)(e+d)c^2(b-e)\left(d^2 + (-b-e)d + b^2 + eb + e^2\right)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ & + (c-d)b^2\left(d^2 + (c-e)d + c^2 - ec + e^2\right)(c+e)(e+d)\exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ & + \left(b^2 + (e-c)b + c^2 - ec + e^2\right)d^2(c+e)(b-e)(b+c)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ & + \left(b(c^2 - e^2)d^4 + b^4(c^2 - e^2)d\right)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + \left((b^2e - c^2e)d^4 + (b^2e^4 - c^2e^4)d\right)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ & + \left((-b^4c - bc^4)d^2 + b^4ce^2 + bc^4e^2\right)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + \left((c^4e + ce^4)d^2 + (-c^4e - ce^4)b^2\right)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ & - (c-d)(b-e)\left(cd(c^2 + cd + d^2)(b+e)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) - be(c+d)(b^2 + be + e^2)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)\right) \end{aligned} \right) }{f_3} \quad (31)$$

$$C_2 = \frac{\left(\begin{aligned} & -(d+4)\left(d^2 + (-e-4)d + e^2 + 4e + 16\right)(e-4)(e+d)c^2\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ & + \left((16c + 16e)d^4 + (-16c^4 + 16e^4)d - 16c^4e - 16e^4c\right)\exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ & + (c+4)\left(c^2 + (-e-4)c + e^2 + 4e + 16\right)d^2(e-4)(c+e)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ & + \left((c^2e - 16e)d^4 + (c^2e^4 - 16e^4)d\right)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) + \left((-c^4e - ce^4)d^2 + 16c^4e + 16e^4c\right)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ & + (d+4)(c+4)(c-d)e^2\left(d^2 + (-4+c)d + c^2 - 4c + 16\right)\exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ & + \left(c(e^2 - 16)d^4 + (-e^2 + 16)c^4d\right)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) + \left((-4c^2 + 4e^2)d^4 + (-256c^2 + 256e^2)d\right)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) \\ & + \left((4c^4 + 256c)d^2 - 4c^4e^2 - 256ce^2\right)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) + 4e(e-4)(e^2 + 4e + 16)(c-d)(c+d)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right) \end{aligned} \right) }{f_3} \quad (32)$$

$$C_3 = \frac{\left(\begin{aligned} & (d+4)b^2\left(d^2 + (-e-4)d + e^2 + 4e + 16\right)(e-4)(e+d)\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ & - (d+4)(b+d)(-4+b)e^2\left(d^2 + (-b-4)d + b^2 + 4b + 16\right)\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ & + \left((16b - 16e)d^4 + (16b^4 - 16e^4)d + 16b^4e - 16be^4\right)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ & + \left(b(e^2 - 16)d^4 + b^4(e^2 - 16)d\right)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + \left((-b^2e + 16e)d^4 + (-b^2e^4 + 16e^4)d\right)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ & - (-4+b)d^2(e-4)(b-e)\left(b^2 + (4+e)b + e^2 + 4e + 16\right)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \\ & + \left((4b^2 - 4e^2)d^4 + (256b^2 - 256e^2)d\right)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) + \left((-4b^4 + 256b)d^2 + 4b^4e^2 - 256be^2\right)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \\ & + e\left(\left((4e^3 - 256)d^2 + (-4e^3 - 256)b^2\right)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right) + b(d-4)(d+4)(b-e)(b^2 + be + e^2)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)\right) \end{aligned} \right) }{f_3} \quad (33)$$

$$C_4 = \frac{\left(\begin{aligned} &-(c+4)b^2(c^2+(-e+4)c+e^2+4e+16)(e-4)(c+e)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ &+(c+4)(-4+b)e^2(b^2+(4-c)b+c^2-4c+16)(b+c)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &+((-16c-16e)b^4+(-16c^4+16e^4)b+16c^4e+16e^4c)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ &+((-e^2+16)cb^4+(-e^2+16)c^4b)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right)+((c^4e+ce^4)b^2-16c^4e-16e^4c)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+(-4+b)(e-4)c^2(b-e)(b^2+(4+e)b+e^2+4e+16)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \\ &+((-4c^4-256c)b^2+4c^4e^2+256ce^2)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right)+b((4c^2-4e^2)b^3+(-256c^2+256e^2))\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \\ &-e\left[\left((-4e^3+256)b^2+(4e^3-256)b^2+(4e^3-256)c^2\right)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right)\right. \\ &\quad \left.+b(-4+c)(c+4)(b-e)(b^2+be+e^2)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)\right] \end{aligned} \right) }{f_3} \quad (34)$$

$$C_5 = \frac{\left(\begin{aligned} &((-16b-16c)d^4+(-16b^4+16c^4)d+16b^4c+16bc^4)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ &+(d+4)(b+d)(-4+b)(d^2+(-b-4)d+b^2+4b+16)c^2\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ &-(c+4)(-4+b)(b^2+(4-c)b+c^2-4c+16)d^2(b+c)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &+((-c^2+16)bd^4+(-c^2+16)b^4d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right)+((b^4c+bc^4)d^2-16b^4c-16bc^4)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &-(d+4)(c+4)(c-d)b^2(d^2+(-4+c)d+c^2-4c+16)\exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ &+((-b^2c+16c)d^4+(b^2c^4-16c^4)d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right)+((-4b^2+4c^2)d^4+(-256b^2+256c^2)d)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) \\ &+((4c^4-256c)d^2+(4c^4+256c)b^2)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right)-4b(c-d)(c+d)(-4+b)(b^2+4b+16)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \end{aligned} \right) }{f_3} \quad (35)$$

$$f_3 = \left(\begin{aligned} &3(b^2 + (-c-d)b + c^2 + dc + d^2)(c-d)(b+d)(4+e)(e-4)(b+c)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ &+3(c+4)(c^2 + (-e-4)c + e^2 + 4e + 16)(b+d)(e-4)(c+e)(b-d)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ &-3(c+4)(-4+b)((b^2 + (4-c)b + c^2 - 4c + 16)(e+d)(b+c)(-d+e)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &-3(d+4)(b^2 + (e-c)b + c^2 - ec + e^2)(d-4)(c+e)(b-e)(b+c)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ &-3(d+4)(b-c)(e^2 + (-d+4)e + d^2 - d + 16)(e-4)(e+d)(b+c)\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ &-3(d+4)(c-e)(b+d)(-4+b)(c+e)(b^2 + (-d+4)b + d^2 - 4d + 16)\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ &+3(c+4)(b+d)(-4+c)(e+d)(b-e)(b^2 + (-d+e)b + e^2 - de + d^2)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ &-3(c-d)((4+b)(-4+b)(c^2 + (d-e)c + e^2 - de + d^2)(c+e)(e+d)\exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ &+(b-e)\left[\begin{aligned} &(c+d)(-4+b)(e-4)(b^2 + (4+e)b + e^2 + 4e + 16)\exp\left(\frac{1}{4}a(4-b+e)\mathcal{Q}\right) \\ &-(d+4)(c+4)(b+e)(c^2 + (d-4)c + d^2 - 4d + 16)\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \end{aligned} \right] \end{aligned} \right) \quad (36)$$

d. Case IV

$$C_1 = \frac{f_4}{f_4} \left(\begin{aligned} &c^2e^2(c-e)(c+e)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + b^2c^2(b-c)(b+c)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &e^2d^2(e-d)(e+d)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) - b^2d^2(b-d)(b+d)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+b^2e^2(b-e)(b+e)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) - c^2d^2(c-d)(c+d)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &-d(c-e)(c+e)(b-e)(b+e)(b-c)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &-b(e-d)(e+d)(c-d)(c+d)(c-e)(c+e)\exp\left(-\frac{1}{4}ab\mathcal{Q}\right) \\ &-(b+d)(b-d)\left[(e-d)(e+d)(b-e)(b+e)c\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) + e(c-d)(c+d)(b-c)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \right] \end{aligned} \right) \quad (37)$$

$$C_2 = \frac{\left(\begin{aligned} &16c^2(-4+c)(c+4)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) - 16d^2(-4+d)(d+4)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &- c^2e^2(c-e)(c+e)(d+4)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + 16e^2(e-4)(4+e)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &- e^2d^2(e-d)(e+d)(c+4)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) - c^2d^2(c-d)(c+d)(e-4)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right) \\ &+ d(e-4)(e+4)(c-4)(c+4)(c-e)(c+e)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &+ c(d-4)(d+4)(e-4)(e+4)(e-d)(e+d)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &+ 4(c-d)(c+d)\left(\frac{1}{4}e(d-4)(d+4)(c-4)(c+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) + e(e-d)(e+d)(c-e)(c+e)\exp(a\mathcal{Q})\right) \end{aligned} \right)}{f_4} \quad (38)$$

$$C_3 = \frac{\left(\begin{aligned} &16e^2(e-4)(e+4)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) - 16b^2(b-4)(b+4)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &+ b^2e^2(e-d)(e+d)(b-4)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) + b^2d^2(b-d)(b+d)(e-4)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \\ &- 16e^2(d-4)(d+4)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) - d(e-4)(e+4)(b-4)(b+4)(b-e)(b+e)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &+ b(d-4)(d+4)(e-4)(e+4)(e-d)(e+d)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &- 4(b+d)(b-d)\left(\frac{1}{4}e(d+4)(d-4)(b-4)(b+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) + (e-d)(e+d)(b-e)(b+e)\exp(a\mathcal{Q})\right) \end{aligned} \right)}{f_4} \quad (39)$$

$$C_4 = \frac{\left(\begin{aligned} &-16e^2(e-4)(e+4)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + 16b^2(b-4)(b+4)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &- b^2e^2(b-e)(b+e)(c+4)\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) - c^2e^2(c-e)(c+e)(b-4)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &- b^2c^2(b-c)(b+c)(e-4)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) + 16c^2(c-4)(c+4)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &+ b(e-4)(e+4)(c-4)(c+4)(c-e)(c+e)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &+ c(e-4)(e+4)(b-4)(b+4)(b-e)(b+e)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &+ (b-c)(b+c)\left(\begin{aligned} &e(c-4)(c+4)(b-4)(b+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \\ &- 4(c-e)(c+e)(b-e)(b+e)\exp(a\mathcal{Q}) \end{aligned} \right) \end{aligned} \right)}{f_4} \quad (40)$$

$$C_5 = \frac{\left(\begin{aligned} &-16c^2(c-4)(c+4)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + 16d^2(d-4)(d+4)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &-b^2c^2(b-c)(b+c)(d+4)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) - 16b^2(b-4)(b+4)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+b^2d^2(b-d)(b+d)(c+4)\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) + c^2d^2(c-d)(c+d)(b-4)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &+d(c-4)(c+4)(b-4)(b+4)(b-c)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &-b(d-4)(d+4)(c-4)(c+4)(c-d)(c+d)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &+4(b+d)(b-d)\left(-\frac{1}{4}(d-4)(d+4)(b-4)(b+4)c\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) + (c-d)(c+d)(b-c)(b+c)\exp(a\mathcal{Q})\right) \end{aligned} \right)}{f_4} \quad (41)$$

$$f_4 = \left(\begin{aligned} &-3(e-4)(4+e)(-4+c)(c+4)(c-e)(c+e)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) \\ &-3(-4+c)(c+4)(-4+b)(4+b)(b-c)(b+c)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &-3(-4+d)(d+4)(e-4)(4+e)(e-d)(e+d)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &+3(-4+d)(d+4)(-4+b)(4+b)(b-d)(b+d)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &-3(e-4)(4+e)(-4+b)(4+b)(b-e)(b+e)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+3(-4+d)(d+4)(-4+c)(c+4)(c-d)(c+d)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &+3(c-e)(c+e)(b-e)(b+e)(b-c)(b+c)(d+4)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) \\ &+3(e-d)(e+d)(b-d)(b+d)(b-e)(b+e)(c+4)\exp\left(\frac{1}{4}a(-4+c)\mathcal{Q}\right) \\ &+3(c-d)(c+d)\left(\begin{aligned} &(e-d)(e+d)(c-e)(c+e)(-4+b)\exp\left(\frac{1}{4}a(4+b)\mathcal{Q}\right) \\ &+(b-d)(b+d)(b-c)(b+c)(e-4)\exp\left(\frac{1}{4}a(4+e)\mathcal{Q}\right) \end{aligned} \right) \end{aligned} \right) \quad (42)$$

e. Case V

$$C_1 = \frac{\left(\begin{aligned} &-(b+c)(b+d)(c-d)(d^2+(-b+c)d+b^2-bc+c^2)e \exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ &+c(b+d)(d^2+(-b-e)d+b^2+be+e^2)(d+e)(b-e) \exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ &+(c+e)(c-d)b(d^2+(c-e)d+c^2-ce+e^2)(e+d) \exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ &-(c+e)(b+c)(b^2+(e-c)b+c^2-ce+e^2)d(b-e) \exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ &-bd(b+d)(b^2-bd+d^2)(c+e) \exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + ed(e+d)(d^2-de+e^2)(b+c) \exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &+cb(b+c)(b^2-bc+c^2)(e+d) \exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) - ec(c+e)(c^2-ce+e^2)(b+d) \exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &-(c-d)(b-e)\left(cd(c^2+cd+d^2) \exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) - be(b^2+be+e^2) \exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)\right) \end{aligned} \right)}{f_5} \quad (43)$$

$$C_2 = \frac{\left(\begin{aligned} &c(e-4)(d^2+(-e-4)d+e^2+4e+16)(d+4)(d+e) \exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ &+((4c+4e)d^4+(-4c^4+4e^4)d-4c^4e) \exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ &-(e-4)(c+e)(c^2+(-e-4)c+e^2+4e+16)(c+4)d \exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ &+(-d^4e(c+4)-de^4(c+4)) \exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) + ec(c+e)(c^2-ce+e^2)(d+4) \exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+(c-d)(c+4)(d+4)(d^2+(-4+c)d+c^2-4c+16)e \exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ &+4d(d+4)(d^2-4d+16)(c+e) \exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + (c(e-4)d^4-c^4(e-4)d) \exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &-4c(c+4)(c^2-4c+16)(e+d) \exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) - 4e(e-4)(e^2+4e+16)(c-d) \exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \end{aligned} \right)}{f_5} \quad (44)$$

$$C_3 = \frac{\left(\begin{aligned} &b(e-4)(d^2+(-e-4)d+e^2+4e+16)(d+4)(e+d)\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ &-e(b+d)(d^2+(-b-4)d+b^2+4b+16)(b-4)(d+4)\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ &+\left((4b-4e)d^4+(4b^4-4e^4)d+4b^4e-4be^4\right)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ &+bd(b+d)(b^2-bd+d^2)(e-4)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right)+(-e(b-4)d^4-de^4(b-4))\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &+(e-4)(b^2+(e+4)b+e^2+4e+16)(b-4)d(b-e)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \\ &+4d(d+4)(d^2-4d+16)(b-e)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right)+4b(b-4)(b^2+4b+16)(e+d)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &-e\left(\left((b^4-be^3)d+4b^3\right)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)+4e(b+d)(e-4)(e^2+4e+16)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right)\right) \end{aligned} \right)}{f_5} \quad (45)$$

$$C_4 = \frac{\left(\begin{aligned} &-(e-4)(c+e)(c^2+(-e-4)c+e^2+4e+16)b(c+4)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ &+(b+c)(c+4)(b-4)e(b^2+(4-c)b+c^2-4c+16)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &+\left((-4c-4e)b^4+(-4c^4+4e^4)b+4c^4e+4ce^4\right)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ &+(-c(e-4)b^4-c^4(e-4)b)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right)+ce(c+e)(c^2-ce+e^2)(b-4)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &-(e-4)c(b^2+(e+4)b+e^2+4e+16)(b-4)(b-e)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \\ &-4c(c+4)(c^2-4c+16)(b-e)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right)-4b(b-4)(b^2+4b+16)(c+e)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &4e\left(\left(\left(\frac{1}{4}c+1\right)b^4-\frac{1}{4}be^3(c+4)\right)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)+(e-4)(e^2+4e+16)(b+c)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right)\right) \end{aligned} \right)}{f_5} \quad (46)$$

$$C_5 = \frac{\left(\begin{aligned} &((-4b-4c)d^4 + (-4b^4+4c^4)d + 4b^4c + 4c^4b)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ &-c(b+d)(d^2+(-b-4)d+b^2+4b+16)(b-4)(d+4)\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ &+(b+c)(c+4)(b-4)d(b^2+(4-c)b+c^2-4c+16)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &+bd(b+d)(b^2-bd+d^2)(c+4)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + ((-b^4c-bc^4)d-4b^4c-4c^4b)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &-(c-d)b(c+4)(d+4)(d^2+(c-4)d+c^2-4c+16)\exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ &-4d(d+4)(d^2-4d+16)(b+c)\exp\left(-\frac{1}{4}a(-4+d)\mathcal{Q}\right) + cd(c-d)(c^2+cd+d^2)(b-4)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+4c(c+4)(c^2-4c+16)(b+d)\exp\left(-\frac{1}{4}a(-4+c)\mathcal{Q}\right) + 4b(b-4)(b^2+4b+16)(c-d)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \end{aligned} \right)}{f_5} \quad (47)$$

$$f_5 = d \left(\begin{aligned} &3(e-4)(b+c)(b+d)(c-d)(d^2+(-b+c)d+b^2-cb+c^2)\exp\left(\frac{1}{4}a(b-c-d)\mathcal{Q}\right) \\ &-3(b+d)(d^2+(-b-e)d+b^2+be+e^2)(c+4)(e+d)(b-e)\exp\left(\frac{1}{4}a(b-d+e)\mathcal{Q}\right) \\ &-3(c+e)(c-d)(d^2+(c-e)d+c^2-ce+e^2)(b-4)(d+e)\exp\left(-\frac{1}{4}a(c+d-e)\mathcal{Q}\right) \\ &+3(c+e)(b+c)(b^2+(e-c)b+c^2-ce+e^2)(d+4)(b-e)\exp\left(\frac{1}{4}a(b-c+e)\mathcal{Q}\right) \\ &-3(e-4)(b+c)(d^2+(-e-4)d+e^2+4e+16)(d+4)(e+d)\exp\left(-\frac{1}{4}a(-4+d-e)\mathcal{Q}\right) \\ &+3(c+e)(b+d)(d^2+(-b-4)d+b^2+4b+16)(b-4)(d+4)\exp\left(\frac{1}{4}a(4+b-d)\mathcal{Q}\right) \\ &+3(e-4)(c+e)(c^2+(-e-4)c+e^2+4e+16)(b+d)(c+4)\exp\left(-\frac{1}{4}a(-4+c-e)\mathcal{Q}\right) \\ &-3(b+c)(c+4)(b-4)(e+d)(b^2+(4-c)b+c^2-4c+16)\exp\left(\frac{1}{4}a(4+b-c)\mathcal{Q}\right) \\ &+3(c-d)(b-e) \left(\begin{aligned} &(c+4)(d+4)(d^2+(c-4)d+c^2-4c+16)\exp\left(-\frac{1}{4}a(-4+c+d)\mathcal{Q}\right) \\ &+(b-4)(e-4)(b^2+(e+4)b+e^2+4e+16)\exp\left(\frac{1}{4}a(4+b+e)\mathcal{Q}\right) \end{aligned} \right) \end{aligned} \right) \quad (48)$$

f. Case VI

$$C_1 = \frac{\left(\begin{aligned} &ce(c+e)(c^2-ce+e^2)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) - bc(b+c)(b^2-bc+c^2)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &-ed(e+d)(d^2-de+e^2)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + bd(b+d)(b^2-bd+d^2)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+be(b-e)(b^2+be+e^2)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) + cd(c-d)(c^2+cd+d^2)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &+d(b-e)(b^2+(e-c)b+c^2-ec+e^2)(c+e)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &-b(c-d)(e+d)(d^2+(c-e)d+c^2-ce+e^2)^*(c+e)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &-(b+d)\left(\begin{aligned} &c(b-e)(e+d)(d^2+(-b-e)d+b^2+eb+e^2)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &-e(b+c)(c-d)(d^2+(-b+c)d+b^2-bc+c^2)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \end{aligned} \right) \end{aligned} \right)}{f_6} \quad (49)$$

$$C_2 = \frac{\left(\begin{aligned} &4c(c+4)(c^2-4c+16)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) - 4d(d+4)(d^2-4d+16)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+((-c^4e-ce^4)d-4c^4e-4ce^4)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) + 4e(e-4)(e^2+4e+16)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+ed(e+d)(d^2-de+e^2)(c+4)\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) + cd(c-d)(c^2+cd+d^2)(e-4)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \\ &+d(e-4)(c^2+(-4-e)c+e^2+4e+16)(c+4)(c+e)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &-c(e-4)(e+d)(d^2+(-4-e)d+e^2+4e+16)(d+4)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &-(c-d)\left(\begin{aligned} &e(d^2+(-4+c)d+c^2-4c+16)(c+4)(d+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \\ &-4(e+d)(c+e)(d^2+(c-e)d+c^2-ec+e^2)\exp(a\mathcal{Q}) \end{aligned} \right) \end{aligned} \right)}{f_6} \quad (50)$$

$$C_3 = \frac{\left(\begin{aligned} &4e(e-4)(e^2+4e+16)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) - 4b(b-4)(b^2+4b+16)(e+d)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\ &+be(b-e)(b^2+be+e^2)(d+4)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) + ed(e+d)(d^2-de+e^2)(b-4)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &+(-bd^4(e-4)-b^4d(e-4))\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) - 4d(d+4)(d^2-4d+16)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &-d(b-4)(e-4)(b-e)(b^2+(4+e)b+e^2+4e+16)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &-b(e-4)(e+d)(d^2+(-4-e)d+e^2+4e+16)(d+4)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &+(b+d)\left(\begin{aligned} &e(b-4)(d+4)(d^2+(-b-4)d+b^2+4b+16)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \\ &-4(b-e)(e+d)(d^2+(-b-e)d+b^2+eb+e^2)\exp(a\mathcal{Q}) \end{aligned} \right) \end{aligned} \right)}{f_6} \quad (51)$$

$$C_4 = \frac{\left(\begin{aligned} &-4e(e-4)(e^2+4e+16)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) + 4b(b-4)(b^2+4b+16)(c+e)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\ &+ (-eb^4(c+4)+be^4(c+4))\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) + (b(-c^4e-ce^4)+4c^4e+4ce^4)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &+ bc((b+c)(b^2-bc+c^2)(e-4)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) + 4c(c+4)(c^2-4c+16)(b-e)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right) \\ &+ b(e-4)(c^2+(-4-e)c+e^2+4e+16)(c+4)(c+e)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &+ c(b-4)(e-4)(b-e)(b^2+(e+4)b+e^2+4e+16)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &- (b+c)\left(\begin{aligned} &e(b-4)(b^2+(4-c)b+c^2-4c+16)(c+4)\exp\left(\frac{1}{4}ae\mathcal{Q}\right) \\ &- 4(b-e)(b^2+(e-c)b+c^2-ec+e^2)(c+e)\exp\left(\frac{1}{4}a\mathcal{Q}\right) \end{aligned} \right) \end{aligned} \right)}{f_6} \quad (52)$$

$$C_5 = \frac{\left(\begin{aligned} &-4c(c+4)(c^2-4c+16)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) + 4d(d+4)(d^2-4d+16)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\ &+ bc(b+c)(b^2-bc+c^2)(d+4)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) - 4b(b-4)(b^2+4b+16)(c-d)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\ &+ (-bd^4(c+4)-db^4(c+4))\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) + (cd^4(b-d)-c^4d(b-4))\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\ &- d(b-4)(b^2+(4-c)b+c^2-4c+16)(c+4)(b+c)\exp\left(-\frac{1}{4}ad\mathcal{Q}\right) \\ &+ b(c-d)(d^2+(c-4)d+c^2-4c+16)(c+4)(d+4)\exp\left(\frac{1}{4}ab\mathcal{Q}\right) \\ &+ (b+d)\left(\begin{aligned} &c(b-4)(d^2+(-b-4)d+b^2+4b+16)(d+4)\exp\left(-\frac{1}{4}ac\mathcal{Q}\right) \\ &- (4(c-d)(b+c)(d^2+(-b+c)d+b^2-bc+c^2)\exp(a\mathcal{Q}) \end{aligned} \right) \end{aligned} \right)}{f_6} \quad (53)$$

$$\begin{aligned}
f_6 = & \left(\begin{aligned}
& -3(e-4)(c^2+(-4-e)c+e^2+4e+16)(c+4)(c+e)(b+d)\exp\left(\frac{1}{4}a(b-d)\mathcal{Q}\right) \\
& +3(e-4)(d+4)(d^2+(-4-e)d+e^2+4e+16)(e+d)(b+c)\exp\left(\frac{1}{4}a(b-c)\mathcal{Q}\right) \\
& +3(b-4)(e+d)(b^2+(4-c)b+C^2-4c+16)(c+4)(b+c)\exp\left(-\frac{1}{4}a(d-e)\mathcal{Q}\right) \\
& -3(b-4)(d+4)(d^2+(-b-4)d+b^2+4b+16)(c+e)(b+d)\exp\left(-\frac{1}{4}a(c-e)\mathcal{Q}\right) \\
& -3(d+4)(b-e)(b^2+(e-c)b+c^2-ec+e^2)(c+e)(b+c)\exp\left(-\frac{1}{4}a(d-4)\mathcal{Q}\right) \\
& -3(b-4)(e-4)(b-e)(c-d)(b^2+(e+4)b+e^2+4e+16)\exp\left(-\frac{1}{4}a(c+d)\mathcal{Q}\right) \\
& +3(b-e)(e+d)(c+4)(d^2+(-b-e)d+b^2+eb+e^2)(b+d)\exp\left(-\frac{1}{4}a(c-4)\mathcal{Q}\right) \\
& +3(c-d) \left(\begin{aligned}
& -(e-4)(b+c)(b+d)(d^2+(-b+c)d+b^2-bc+c^2)\exp\left(\frac{1}{4}a(e+4)\mathcal{Q}\right) \\
& +(b-4)(e+d)(d^2+(c-e)d+c^2-ec+e^2)(c+e)\exp\left(\frac{1}{4}a(b+4)\mathcal{Q}\right) \\
& -(d^2+(-4+c)d+c^2-4c+16)(d+4)(b-e)(c+4)\exp\left(\frac{1}{4}a(b+e)\mathcal{Q}\right)
\end{aligned} \right)
\end{aligned} \right) \tag{54}
\end{aligned}$$

Table I: Literature Review on Core Deflection Analysis

Name (year)	Core Cross-Section	Rules of Thumb	Coupling	Non-Newtonian	Solidification	Fluid Pressure Profile	Analytical Solution	Finite Element Simulation	Finite Difference Solution	Nondimensionalization	Core deflection profile	References
Glanvill and Denton (1963,1965)	AR	X										[16,17]
Menges and Mohren (1993)	○, □			u	X	X		X			X	[18]
Wüst <i>et al.</i> (1997)	AR			u	X	X		X			X	[19]
Menges <i>et al.</i> (2001)	○			u	X	X		X			X	[20]
Bakharev <i>et al.</i> (2004)	□		X	GN	X	X		X			X	[21]
Engelstein (2004)	○		X	u	X	X		X			X	[22]
Ling <i>et al.</i> (2006)	○		X	HB	X	X		X			X	[23]
Giacomin (2005)	□		X						I-II	X		[24]
Chou <i>et al.</i> (2008)	□							X		X	X	[25]
Chou <i>et al.</i> (2008)	□				X			X		X	X	[26]
Chou (2009)	□			CM	X			X		X	X	[27]
Liao (2009)	□			MC	X	X		X			X	[28]
Chen <i>et al.</i> (2011)	□			MC	X			X			X	[29]
Giacomin <i>et al.</i> (2011)	□		X			X	L		I-II	X		[30]
This paper	□						NL		I-VI	X	X	[31]

Legend: u ≡ unspecified; HB ≡ Hershel-Bulkley; GN ≡ generalized; CM ≡ upper convected Maxwell; MC ≡ modified Cross; L ≡ linear; NL ≡ Nonlinear;

I-VI ≡ Cases I through VI (see Figure 1); □ ≡ Square; □ ≡ Rectangular; ○ ≡ Round; AR ≡ arbitrary.

Table II: Boundary Values for Six Core Conditions.

Case	Tip	Base	$Y(X)$	$Y'(X)$	$Y''(X)$	$Y^{(3)}(X)$	$Y^{(4)}(X)$
I	FG	C	$Y(1) = 0$	$Y'(1) = 0$	$Y''(0) = 0$	$Y^{(3)}(0) = 0$	$Y^{(4)}(1) = 0$
II	F	CG	$Y(1) = 0$	$Y'(1) = 0$	$Y''(0) = 0$	$Y^{(3)}(0) = 0$	$Y^{(4)}(0) = 0$
III	PG	C	$Y(0,1) = 0$	$Y'(1) = 0$	$Y''(0) = 0$		$Y^{(4)}(1) = 0$
IV	P	CG	$Y(0,1) = 0$	$Y'(1) = 0$	$Y''(0) = 0$		$Y^{(4)}(0) = 0$
V	CG	C	$Y(0,1) = 0$	$Y'(0,1) = 0$			$Y^{(4)}(1) = 0$
VI	C	CG	$Y(0,1) = 0$	$Y'(0,1) = 0$			$Y^{(4)}(0) = 0$

Legend: C $\equiv$ cantilevered; F $\equiv$ free; G $\equiv$ gated; P $\equiv$ pinned.

Table III: Dimensional Variables

Name	Unit	Variable
Axial position along core [Figure 1]	L	x
Core deflection profile [Figure 1]	L	y
Core length [Figure 1]	L	L
Core moment of inertia about z axis [Eq. (1)]	L^4	I
Gap between core and mold wall [Eq. (1)]	L	B
Gap between core and mold wall at core base [Figure 1]	L	B_0
Newtonian viscosity [Eq. (1)]	M / Lt	μ
Volumetric flow rate [Eq. (1)]	L^3 / t	Q
Young's modulus of core [Eq. (1)]	M / Lt^2	E

Legend: $L \equiv$ length; $M \equiv$ mass; $t \equiv$ time

Table IV: Dimensionless Variables and Groups

Name	Symbol
Axial position along the core [Eq. (2)]	$X \equiv x/L$
Core deflectability [Eq. (3)]	$\mathcal{Q} \equiv \frac{\mu QL}{EI} \left(\frac{L}{B_0} \right)^4$
Core deflection profile [Eq. (2)]	$Y \equiv y/B_0$
Flow length ratio [Eq. (3)]	$\mathcal{L} \equiv L/B_0$
Integration parameters [Appendices VIII]	$C_1, \dots, C_5$
Maximum deflection [Eq. (13)]	$\mathcal{Y} \equiv Y(\mathcal{X}, \mathcal{Q})$
Position at maximum core deflection [Eq. (13)]	$\mathcal{X}$

Figure 1: Race tracking along slit beneath deflecting core for Cases I through VI.

Figure 2: Dimensionless core deflection *versus* dimensionless axial position along core for Case I. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (19)-(24), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|Q| = \frac{1}{100}, \frac{1}{10}, \frac{1}{2}$ and 1.

Figure 3: Dimensionless core deflection *versus* dimensionless axial position along core for Case II. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (25)-(30), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|\mathcal{Q}| = \frac{1}{100}, \frac{1}{10}, \frac{1}{2}$ and 1.

Figure 4: Dimensionless core deflection *versus* dimensionless axial position along core for Case III. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (31)-(36), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|Q| = \frac{1}{100}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1.

Figure 5: Dimensionless core deflection *versus* dimensionless axial position along core for Case IV. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (37)-(42), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|\mathcal{Q}| = \frac{1}{100}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1.

Figure 6: Dimensionless core deflection *versus* dimensionless axial position along core for Case V. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (43)-(48), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|\mathcal{Q}| = \frac{1}{100}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1.

Figure 7: Dimensionless core deflection *versus* dimensionless axial position along core for Case VI. Comparison of the analytical solution, Eq. (7) [with Eqs. (8) and (49)-(54), **black**], with the numerical solution to Eq. (2) [**red**], parameterized with $|\mathcal{Q}| = \frac{1}{100}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1.

Figure 8: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case I. **Black** and **blue** are our main results, Eq. (15) [with Eqs. (8) and (19)-(24)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 9: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case II. **Black** and **blue** are our main results, Eq. (15) [with Eqs. (8) and (25)-(30)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 10: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case III. **Black** and **blue** are our main results, Eq. (13) [with Eqs. (8) and (31)-(36)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 11: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case IV. **Black** and **blue** are our main results, Eq. (13) [with Eqs. (8) and (37)-(42)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 12: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case V. **Black** and **blue** are our main results, Eq. (13) [with Eqs. (8) and (43)-(48)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 13: Dimensionless maximum core deflection *versus* dimensionless core deflectability for Case VI. **Black** and **blue** are our main results, Eq. (13) [with Eqs. (8) and (49)-(54)], and previous analytical solutions, **red**, numerical solution to Eq. (2).

Figure 14: Dimensionless position at maximum core deflection *versus* dimensionless core deflectability for Case III. **Black** is our analytical solution, Eq. (17) [with Eqs. (8) and (31)-(36)], **red**, numerical solution to Eq. (2).

Figure 15: Dimensionless position at maximum core deflection *versus* dimensionless core deflectability for Case IV. **Black** is our analytical solution, Eq. (17) [with Eqs. (8) and (37)-(42)], **red**, numerical solution to Eq. (2).

Figure 16: Dimensionless position at maximum core deflection *versus* dimensionless core deflectability for Case V. **Black** is our analytical solution, Eq. (17) [with Eqs. (8) and (43)-(48)], **red**, numerical solution to Eq. (2).

Figure 17: Dimensionless position at maximum core deflection *versus* dimensionless core deflectability for Case VI. **Black** is our analytical solution, Eq. (17) [with Eqs. (8) and (49)-(54)], **red**, numerical solution to Eq. (2).

Figure 18: Dimensionless maximum core deflection *versus* dimensionless position at maximum deflection for Case III. **Black** is our analytical solution, Eq. (13) [with Eqs. (8) and (31)-(36)], **red**, numerical solution to Eq. (2).

Figure 19: Dimensionless maximum core deflection *versus* dimensionless position at maximum deflection for Case IV. **Black** is our analytical solution, Eq. (13) [with Eqs. (8) and (37)-(42)], **red**, numerical solution to Eq. (2).

Figure 20: Dimensionless maximum core deflection *versus* dimensionless position at maximum deflection for Case V. **Black** is our analytical solution, Eq. (13) [with Eqs. (8) and (43)-(48)], **red**, numerical solution to Eq. (2).

Figure 21: Dimensionless maximum core deflection *versus* dimensionless position at maximum deflection for Case VI. **Black** is our analytical solution, Eq. (13) [with Eqs. (8) and (49)-(54)], **red**, numerical solution to Eq. (2).

Figure 22: Rectangular aluminum core with (dark) edge of quartz window through which the proximetry is accomplished. Free-tip, base-gating (Case II).

Figure 23: Videography of molten polypropylene (PP6331 PRO-FAX at 220°C) injected beneath a base-gated deflecting aluminum core of rectangular cross-section with free-tip (Case II).

Figure 24: Measured dimensionless core deflection *versus* the dimensionless core deflectability for Case II (magenta) compared with our analytical and numerical solutions. **Black** and **blue** are the our new, Eq. (15) [with Eqs. (8) and (25)-(30)], and previous analytical solutions, **red**, the numerical solution to Eq. (2).

Figure 25: Deflected core profiles when $|\mathcal{Q}| = 1$ for Case I, Eq. (7) [with Eqs. (8) and (19)-(24), **black**], and Case II ,Eq. (7) [with Eqs. (8) and (25)-(30), **blue**].

Figure 26: Deflected core profiles when $|\mathcal{Q}| = 1$ for Case III, Eq. (7) [with Eqs. (8) and (37)-(42), **black**], Case IV, Eq. (7) [with Eqs. (8) and (37)-(42), **green**], Case V, Eq. (7) [with Eqs. (8) and (43)-(48), **blue**], and Case VI, Eq. (7) [with Eqs. (8) and (49)-(54), **red**].

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