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**STARTUP STEADY SHEAR FLOW
FROM THE OLDROYD 8-CONSTANT FRAMEWORK**

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ABSTRACT

One good way to explore fluid microstructure, experimentally, is to suddenly subject the fluid to a large steady shearing deformation, and to then observe the evolving stress response. If the steady shear rate is high enough, the shear stress and also the normal stress differences can, overshoot, and then, they can even undershoot. We call such responses *nonlinear*, and this experiment *shear stress growth*. This paper is devoted to providing exact analytical solutions for interpreting measured nonlinear shear stress growth responses. Specifically, we arrive at the exact solutions for the Oldroyd 8-constant constitutive framework. We test our exact solution against the measured behaviours of two wormlike micellar solutions. At high shear rates, these solutions overshoot in stress growth without subsequent undershoot. The micellar solutions present linear behavior at low shear rate, and otherwise, nonlinear. Our framework provides slightly early underpredictions of the overshoots at high shear rate. The effect of salt concentration on the nonlinear parameters is explored.

Keywords: Shear stress growth; shear stress growth material function; Oldroyd 8-constant framework; wormlike micelle.

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CONTENTS

I	INTRODUCTION.....	5
II	EXACT SOLUTIONS.....	8
a	Overshoots and Undershoots.....	9
III	CONSISTENCY CHECKS.....	10
a	Goddard Integral Approach.....	10
b	Finite Difference Calculations	10
IV	MATERIALS	11
a	Preparation	11
b	Oscillatory Shear and Shear Stress Growth Measurements	11
c	Fitting Exact Solution	11
V	DISCUSSION.....	12
VI	CONCLUSION	13
VII	SUPPLEMENTARY MATERIAL	14
VIII	ACKNOWLEDGMENT	14
IX	APPENDIX: DERIVATION OF EQ. (6)	15
X	REFERENCES	32

TABLES

Table I: Exact solutions for startup of steady shear flow for special cases of the Oldroyd 8-constant framework.....	18
Table II: Literature review of $\eta^+(t)$ and $\Psi_1^+(t)$ measurements of wormlike micellar solutions of CPyCl	19
Table III: Dimensional variables and symbols.....	20
Table IV: Dimensionless variables and groups	21
Table V: Residual sum of squares for fitted curves in Figure 7, Figure 8 and Figure 10	21

FIGURES

Figure 1: Structure of wormlike micelle in aqueous solution. Penetrated and adsorbed counter-ions, NaSal and NaDiclo, shown in magnified cross section. Surfactant molecules (CPyCl) have hydrophilic heads and hydrophobic tails.	22
Figure 2: Flow profile of the startup of the steady shear flow in cone-plate rheometer. The WLM solution, CPyCl with two binding salts (NaSal+NaDiclo), is loaded into the shear fixture, shown in (a). A strand in the entangled network is illustrated in Figure 1. A constant shear rate is then suddenly applied, as shown in (b).	23
Figure 3: Four classic shapes of observed stress growth viscosity functions (top to bottom): (i) linear without overshoot, (ii) nonlinear without overshoot, (iii) nonlinear with overshoot and without undershoot, and (iv) nonlinear with both overshoot and single undershoot.	24
Figure 4: Exact shear stress-growth viscosity [green , Eq. (7)] <i>versus</i> finite difference calculations from Eq. (6) (black), and <i>versus</i> the well-known solution [red , Eq. (13)] for the special case of the corotational Jeffreys fluid, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Curves are parametrized with increasing $Wi \equiv \lambda_1 \dot{\gamma} = \frac{1}{10}, 1, 10$, from top to bottom.	25
Figure 5: Exact shear stress-growth first normal stress coefficient [green , Eq. (8)] <i>versus</i> finite difference calculations from Eq. (6) (black), and <i>versus</i> the well-known solution [red , Eq. (14)] for the special case of the corotational Jeffreys fluid, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Curves are parametrized with increasing $Wi \equiv \lambda_1 \dot{\gamma} = \frac{1}{10}, 1, 10$, from top to bottom.	26
Figure 6: Predictions of (a) first overshoot and (b) undershoot from Eq. (11). Dashed curves at low Wi indicate that the viscosity function has reached its steady state.	27
Figure 7: SAOS measurements of (a) 40D10S and (b) 20D30S <i>versus</i> predicted values from the Oldroyd 8-constant framework [Eq. (15)], with (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$, $\lambda_2 = 1.00 \times 10^{-2} \text{ s}$ and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$, $\lambda_2 = 5.00 \times 10^{-3} \text{ s}$. The RSS are reported in Table V.	28
Figure 8: Evolution of measured η^+ of (a) 40D10S and (b) 20D30S <i>versus</i> predicted values from the Oldroyd 8-constant framework [Eq. (7)], for (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$, $\lambda_2 = 1.00 \times 10^{-2} \text{ s}$, $\mu_0 = 0.205 \text{ s}$, $\mu_1 = \mu_2 = 0$, $\nu_1 = -2.05 \text{ s}$, $\nu_2 = -1.03 \text{ s}$, and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$, $\lambda_2 = 5.00 \times 10^{-3} \text{ s}$, $\mu_0 = 5.00 \times 10^{-3} \text{ s}$, $\nu_1 = -0.305 \text{ s}$,	

$v_2 = -1.13\text{s}$, $\mu_1 = \mu_2 = 0$. Experimental conditions: (a) $\lambda_1\dot{\gamma} = 0$ (LVE), 0.625, 1.25, 3.75, and (b) $\lambda_1\dot{\gamma} = 0$ (LVE), 0.608, 0.912, 1.52 (both from top to bottom). The RSS are reported in Table V. 29

Figure 9: η^+/η_0 versus t/λ_1 from the Oldroyd 8-constant fluid with $\lambda_2 = \mu_1 = \mu_2 = 0$ [Eq. (7)] at $Wi \equiv \lambda_1\dot{\gamma} = 1$. Three subfigures illustrate three different values of $\check{v}_2 \equiv v_2/\lambda_1 = -1, 1, 0.1$. Each subfigure shows four sets of three curves for $\check{\mu}_0 \equiv \mu_0/\lambda_1 = -0.2, 0.05, 0.2, 0.4$. Each set of curves is parametrized with $\check{v}_1 \equiv v_1/\lambda_1 = -1, 0.1, 1$ (from bottom to top). 30

Figure 10: Steady shear viscosity measurements of (a) 40D10S and (b) 20D30S, both from Figure 8, versus predicted values from the Oldroyd 8-constant framework [Eq. (17)], with (a) $\eta_0 = 45.0\text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25\text{ s}$, $\sigma_1 = 2.19\text{ s}^2$ and $\sigma_2 = 0.329\text{ s}^2$, and (b) $\eta_0 = 59.5\text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52\text{ s}$, $\sigma_1 = 2.33\text{ s}^2$ and $\sigma_2 = 0.0931\text{ s}^2$. The RSS are reported in Table V. 31

I INTRODUCTION

Large rapid transient deformations can bring out nonlinear behaviours of complex fluids. One popular laboratory experiment that we use to deepen our understanding of such nonlinear transient behaviours has been variously called *stress growth upon inception of steady shear flow* (see Row c. of FIGURE 4.3–1 of [1]; also §11. of [2]; §3.1.3 of [3]), or *stress growth in steady shear* (§4.6.1 of [4]; §10.6.1 of [6]), or *sudden-start of shear flow* or *sudden imposition of shearing* (§3.6. of [5]; §4.1.1 of [6]), or *start-up of shear flow* (§2.3.2 of [7]), or *start-up of steady simple-shear* (§2.10 and §5.3.1 of [8]; §3.4 [4]), or officially, *start-up shear (at fixed shear rate)* (see TABLE IV of [9]). The shear rate in this experiment is given as:

$$\dot{\gamma} = \begin{cases} 0 & t \leq 0 \\ \dot{\gamma} & t > 0 \end{cases} \quad (1)$$

which we call *startup of steady shear flow* (§3.2.1.1 of [10]) or simply *stress growth*. Using the characteristic time for a fluid, Eq. (1) nondimensionalizes as:

$$\lambda \dot{\gamma} = \begin{cases} 0 & t \leq 0 \\ \lambda \dot{\gamma} & t > 0 \end{cases} \quad (2)$$

where $Wi \equiv \lambda \dot{\gamma}^0$ is the Weissenberg number, and which reflects the amount of nonlinearity in the fluid and the flow.

Figure 2 shows the flow field associated with Eq. (1). When the shear stress response is divided by the fixed shear rate, we define the *stress growth viscosity function*, $\eta^+(t, \dot{\gamma})$, (TABLE IV of [9]). We cast all of the shear stress response results in this work in terms of this function. Whereas for *nonlinear* responses, the stress growth viscosity function depends of the fixed shear rate, $\eta^+(t, \dot{\gamma})$, for linear viscoelastic (LVE) responses, $\eta^+(t)$, no such dependence exists. Figure 3 illustrates *four classic shapes* of observed stress growth viscosity functions: (i) linear without overshoot, (ii) nonlinear without overshoot, (iii) nonlinear with overshoot and without undershoot, and (iv) nonlinear with both single overshoot and single undershoot, shown in Fig. 9 of [11] are examples of measured values for Case (iii) and (iv)). Not shown in Figure 3, is the rarer nonlinear response having multiple overshoots and undershoots, which has been attributed to flow instability (see Fig. 6 of [12] and its discussion).

To deepen our understanding of startup of steady shear flow, we further nondimensionalize the stress growth viscosity function with the zero-shear viscosity. We can then identify when to expect Newtonian, linear and nonlinear responses as follows:

$$\frac{\eta^+(\tau, Wi)}{\eta_0} = \begin{cases} 1 & Wi \ll 1 \\ \frac{\eta^+(\tau)}{\eta_0} & Wi < 1 \\ \frac{\eta^+(\tau, Wi)}{\eta_0} & Wi > 1 \end{cases} \quad (3)$$

where $\tau \equiv t/\lambda$. As Figure 3 shows, in startup of steady shear flow, we expect nonlinearity when $Wi > 1$. When to expect a particular classic shape of stress growth viscosity function, Shapes (ii) through (iv) for instance, is a more complicated story, which this paper tells in Section IIa.

Molecular origins have been attributed to the *classic shapes* of observed stress growth viscosity functions. For entangled polymeric liquids, overshoot in startup of steady shear flow has been attributed to polymer disentanglement [13,14]. By contrast, polymer orientation, without any interaction between polymers, also explains overshoot in startup of steady shear flow. For instance, stress growth viscosity functions for a dilute suspension of rigid dumbbells exhibits classic shapes (i) through (iii) (see Fig. 7. of [2]).

Wormlike micelle (WLM) solutions are used in several applications, from household goods, personal care products to drug delivery systems [15]. These solutions are made of surfactants that have hydrophilic heads and hydrophobic tails [16,17]. To minimize the exposure of water, these surfactant molecules agglomerate, forming one of many possible structures, depending on binding salt concentration (§2 of [18], [19]). Each of these structures, including WLM, requires a minimum surfactant concentration. To form a WLM, the hydrophobic tails must pack to balance the polarity of the entire wormlike structure (see Figure 1; Ch. 20 of [20]; [21]). This network of flexible worms entangles and disentangles, much like polymers. Therefore, many nonlinear rheological behaviors exhibited by polymeric liquids are also observed in WLM solutions (§1.4 of [16]; §2.2 of [22]).

In this paper, we use the surfactant cetylpyridinium chloride (CPyCl). Table II classifies this literature on startup of steady shear flow concerning the material function for shear and first normal stress difference. From Table II, we learn that most uses sodium salicylate (NaSal) as the lone binding salt, and only a few consider adding a second binding salt. Adding a little of sodium diclofenac (NaDiclo) increases zero shear viscosity (see Fig. 5 of [23] and discussion of Fig. 2 of [17]), not unlike a lone strong binding salt. This is an indication of the counter-ion, Diclo^- , penetrate the WLM structures deeply (see Figure 1). Thus, in this work, we explore the effect of NaDiclo as a second binding salt. We do so by changing the NaDiclo concentration, holding the total salt concentration constant. For a comprehensive comparison of topologically branched with linear WLM solutions, see [24,25]. This paper focuses only on topologically linear micelles.

The LVE response in shear stress growth, $\eta^+(\tau)$, of WLM solutions is known to follow the single relaxation time Maxwell fluid [66]. Further, after wormlike micelles rupture in steady shear flow, the WLM solution then exhibits a second Newtonian plateau, η_∞ . These WLM solution behaviors are both properly predicted by the single-mode corotational Jeffreys fluid, which is defined in Row 9 of Table I (see second paragraph of “Results and Discussion” section of [26]). Therefore, in this paper, we will model our WLM solutions with a single λ_1 and a single λ_2 .

There have been a few attempts in modeling nonlinear behaviors in stress growth, $\eta^+(\tau, Wi)$. We direct our readers to a comprehensive review on stress growth by Pipe *et al.* (see “Constitutive modeling” section of [27]). From this review we learn that, though there exist many empirical formulae for fitting $\eta^+(\tau, Wi)$, but none are three-dimensional constitutive model. In addition to this, the best available molecular theory for WLM solutions failed to predict nonlinear behaviors in stress growth (Fig. 8 of [27]), and hence motivates this paper.

This paper considers the single-mode Oldroyd 8-constant framework with a single λ_1 and a single λ_2 (Eq. (8.1–2) of [1], Eq. (4) of [28]):

$$\begin{aligned} \tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{1}{2}\mu_0(\text{tr}\tau)\dot{\gamma} - \frac{1}{2}\mu_1\{\tau \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau\} + \frac{1}{2}\nu_1(\tau : \dot{\gamma})\delta \\ = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} - \mu_2\{\dot{\gamma} \cdot \dot{\gamma}\} + \frac{1}{2}\nu_2(\dot{\gamma} : \dot{\gamma})\delta \right) \end{aligned} \quad (4)$$

where the corotational derivative of a tensor $\mathbf{b}$, $\mathcal{D}\mathbf{b}/\mathcal{D}t$, extra stress, τ , rate-of-strain, $\dot{\gamma}$, and vorticity tensors, ω , are defined in Table III. The underlined terms in Eq. (4) add nonlinearity to the model, where the three on the left are all possible products of τ and $\dot{\gamma}$, and the two on the right, all possible terms quadratic in $\dot{\gamma}$. This model has been studied extensively, both in theory [29,30,31] and in engineering practice [32,33,34]. We know of no other constitutive framework as versatile as the Oldroyd 8-constant, whose rich diversity of special cases includes the 21 defined in Table I (which improves Table IV of [30]). We know of just one special case of the Oldroyd 8-constant framework that has been tested in shear stress growth against experimental measurements on WLM solutions. Specifically, the Johnson-Segalman fluid (Row 16 of Table I) in shear stress growth gives (Eq. (4) of [35]):

$$\frac{\eta^+(\tau, Wi)}{\eta(Wi)} = 1 - e^{-\tau} \cos \sqrt{1 - \xi^2} Wi \tau - \frac{1}{\sqrt{1 - \xi^2}} e^{-\tau} \sin \sqrt{1 - \xi^2} Wi \tau \quad (5)$$

which gives only qualitative agreement with shear stress growth measurements on WLM solutions (see Figure 10. of [35]). In this work, we pursue the exact analytical solutions to the Oldroyd 8-constant framework for $\eta^+(\tau, Wi)$, $\Psi_1^+(\tau, Wi)$ and $\Psi_2^+(\tau, Wi)$.

This paper provides exact analytical solutions for interpreting measured nonlinear shear stress growth responses from the Oldroyd 8-constant framework for $\eta^+(\tau, Wi)$, $\Psi_1^+(\tau, Wi)$ and $\Psi_2^+(\tau, Wi)$. For consistency, for the special case of the corotational Jeffreys fluid, we check our newly discovered exact solutions for $\eta^+(\tau, Wi)$ and $\Psi_1^+(\tau, Wi) = -2\Psi_2^+(\tau, Wi)$ with the well-known analytical results, and with our finite difference calculations. We close this paper by comparing the nonlinear predictions of our main results for $\eta^+(\tau, Wi)$ with our newly measured behaviours of WLM solutions.

II EXACT SOLUTIONS

Evaluating Eq. (4) in startup of steady shear flow and nondimensionalizing yields (see detailed derivation in §IX):

$$\begin{bmatrix} d\tilde{\tau}_{yx}/d\tau \\ d\tilde{N}_1/d\tau \\ d\tilde{N}_2/d\tau \\ d\tilde{\tau}_{zz}/d\tau \end{bmatrix} = \begin{bmatrix} -\tilde{\tau}_{yx} - \frac{1}{2}(1 + \tilde{\mu}_0 - \tilde{\mu}_1)Wi\tilde{N}_1 - (\tilde{\mu}_0 - \tilde{\mu}_1)Wi\tilde{N}_2 - (\frac{3}{2}\tilde{\mu}_0 - \tilde{\mu}_1)Wi\tilde{\tau}_{zz} - Wi \\ 2Wi\tilde{\tau}_{yx} - \tilde{N}_1 + 2\tilde{\lambda}_2 Wi^2 \\ -(1 - \tilde{\mu}_1)Wi\tilde{\tau}_{yx} - \tilde{N}_2 - (\tilde{\lambda}_2 - \tilde{\mu}_2)Wi^2 \\ -\tilde{\nu}_1 Wi\tilde{\tau}_{yx} - \tilde{\tau}_{zz} - \tilde{\nu}_2 Wi^2 \end{bmatrix} \quad (6)$$

Table IV defines all symbols, and $\tilde{x}$ indicates the dimensionless x . Eq. (6) can be solved, both numerically and analytically, to get the stress responses in startup of steady shear flow.

Solving Eqs. (6) gives the following exact expressions for material functions in startup of steady shear flow, and are the highlight of this paper (see detailed derivation in §IX):

$$\tilde{\eta}^+ \equiv \frac{\eta^+}{\eta_0} = -e^{-\tau} \left[\frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} \cos \sqrt{\tilde{\sigma}_1} Wi \tau - \frac{1}{\sqrt{\tilde{\sigma}_1}} \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{1 + \tilde{\sigma}_1 Wi^2} Wi \sin \sqrt{\tilde{\sigma}_1} Wi \tau \right] + \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} \quad (7)$$

$$\tilde{\Psi}_1^+ \equiv \frac{\Psi_1^+}{\Psi_{1,0}} = \frac{-e^{-\tau}}{1 - \tilde{\lambda}_2} \left[\sigma - \tilde{\lambda}_2 + \frac{1 - \sigma}{1 + \tilde{\sigma}_1 Wi^2} \cos \sqrt{\tilde{\sigma}_1} Wi \tau + \frac{1}{Wi \sqrt{\tilde{\sigma}_1}} \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} \sin \sqrt{\tilde{\sigma}_1} Wi \tau \right] + \frac{1}{1 + \tilde{\sigma}_1 Wi^2} - \frac{\tilde{\lambda}_2 \tilde{\sigma}_1 - \tilde{\sigma}_2}{(1 - \tilde{\lambda}_2)(1 + \tilde{\sigma}_1 Wi^2)} Wi^2 \quad (8)$$

$$\begin{aligned} \tilde{\Psi}_2^+ \equiv \frac{\Psi_{2,0}^+}{\Psi_{2,0}} = & \frac{e^{-\tau}}{1 - \tilde{\mu}_1 - \tilde{\lambda}_2 + \tilde{\mu}_2} \left[\frac{\tilde{\lambda}_2 - \tilde{\mu}_2 - (1 - \tilde{\mu}_1)\tilde{\sigma} - \frac{(1 - \tilde{\mu}_1)(1 - \tilde{\sigma})}{1 + \tilde{\sigma}_1 \text{Wi}^2} \cos \sqrt{\tilde{\sigma}_1} \text{Wi} \tau}{-\frac{1 - \tilde{\mu}_1}{\sqrt{\tilde{\sigma}_1}} \frac{1 + \tilde{\sigma}_2 \text{Wi}^2}{1 + \tilde{\sigma}_1 \text{Wi}^2} \frac{1}{\text{Wi}} \sin \sqrt{\tilde{\sigma}_1} \text{Wi} \tau} \right] \\ & + \frac{1}{1 + \tilde{\sigma}_1 \text{Wi}^2} - \frac{(\tilde{\lambda}_2 - \tilde{\mu}_2)\tilde{\sigma}_1 - (1 - \tilde{\mu}_1)\tilde{\sigma}_2}{(1 - \tilde{\mu}_1 - \tilde{\lambda}_2 + \tilde{\mu}_2)(1 + \tilde{\sigma}_1 \text{Wi}^2)} \text{Wi}^2 \end{aligned} \quad (9)$$

Each of these three exact expressions is nondimensionalized with its own zero-shear values, η_0 , $\Psi_{1,0}$ and $\Psi_{2,0}$ (all are defined in Table III), and recover the well-known steady shear flow material functions (Eqs. (14), (22) and (23) of [36]), as they should. For the special case the Johnson-Segalman fluid, Eq. (7) reduces to:

$$\frac{\eta^+(\tau, \text{Wi})}{\eta(\text{Wi})} = 1 - e^{-\tau} \left[\cos \sqrt{\xi(2 - \xi)} \text{Wi} \tau + \frac{(1 - \tilde{\eta}_s) \sqrt{\xi(2 - \xi)} \text{Wi}}{1 + \tilde{\eta}_s \xi(2 - \xi) \text{Wi}^2} \sin \sqrt{\xi(2 - \xi)} \text{Wi} \tau \right] \quad (10)$$

which corrects Eq. (5). In practice, we see that steady shear flow is reached when $|d^2\tilde{\eta}^+ / d\tau^2| > 10^{-5}$ (to be discussed in Section IIa). Even though this paper reports the exact expressions of the stress-growth material functions for both normal stress differences (Ψ_1^+ and Ψ_2^+) and shear stress (η^+), only the latter, given in Eq. (7), is tested against experimental observations.

a Overshoots and Undershoots

Shear stress overshoots or undershoots happen when $d\tilde{\eta}^+ / d\tau = 0$ (see Figure 3). They cause shear-induced structure, which are reflected through transparency change [37], and later on, are elucidated using small-angle neutron scattering (SANS) [38]. Particle tracking velocimetry associates chaotic departures from simple shear flow with $\eta^+(\tau, \text{Wi})$ overshoots in molten styrene-butadiene rubber [39], and also in solutions of 1,4-polybutadiene [40,41], deoxyribonucleic acid (DNA) [42,43] and WLM [44,45].

To get an analytical expression of the times when $d\tilde{\eta}^+ / d\tau = 0$, we differentiate Eq. (7) and set the result to zero:

$$\tau = \frac{1}{\sqrt{\tilde{\sigma}_1} \text{Wi}} \left(n\pi - \text{arccot} \frac{\tilde{\sigma}_2 \text{Wi}}{\sqrt{\tilde{\sigma}_1}} \right); \quad n = \begin{cases} 1, 3, 5, \dots; & \text{overshoots} \\ 2, 4, 6, \dots; & \text{undershoots} \end{cases} \quad (11)$$

where $n = 1$ is the first overshoot, and $n = 2$, the first undershoot. For the case of linear viscoelasticity ($\text{Wi} \rightarrow 0$), Eq. (11) gives $\tau \rightarrow \infty$, as it should (see Shape (i) in Figure 3). For the case of $\text{Wi} \rightarrow \infty$, Eq. (11) suggests that the first overshoot happens instantly, once the shear rate is applied, which agrees with the observed behaviors (see FIG. 1c. and 1d. of [43]). Eq. (11) thus exhibits correct limiting behaviors.

To distinguish overshoots from undershoots, we consider the second derivative of the viscosity function, Eq. (7):

$$\frac{d^2\tilde{\eta}^+}{d\tau^2} = e^{-\tau} \left[(1 - \tilde{\sigma}_2 \text{Wi}^2) \cos(\sqrt{\tilde{\sigma}_1} \text{Wi} \tau) - \frac{\tilde{\sigma}_1 + \tilde{\sigma}_2}{\sqrt{\tilde{\sigma}_1}} \text{Wi} \sin(\sqrt{\tilde{\sigma}_1} \text{Wi} \tau) \right] \quad (12)$$

from which we learn that when $d^2\tilde{\eta}^+/d\tau^2 < -\epsilon_o$, overshoots emerge, and when $d^2\tilde{\eta}^+/d\tau^2 > \epsilon_u$, undershoots emerge. In contrast, when $|d^2\tilde{\eta}^+/d\tau^2| < \epsilon_{ss}$, the system plateaus to within a line-width, and thus, startup completes. From many observations of Eq. (7) in conjunction with Eq. (12), we find that $\epsilon_o = \epsilon_u = \epsilon_{ss} \cong 10^{-5}$.

III CONSISTENCY CHECKS

For the special case of the corotational Jeffreys fluid, we compare our main results with those that are arrived at by the Goddard integral approach, and by finite difference. We illustrate this work in Figure 4 for η^+ and Figure 5 for Ψ_1^+ .

a Goddard Integral Approach

The traditional analytical solution is by the Goddard approach applied to the integral form of the corotational Jeffreys fluid (Eq. 7.3-4 in conjunction with **EXAMPLE 7.3-1b** of [1]). This yields (Eqs. (7.3-33)–(7.3-34) of [1]):

$$\frac{\eta^+}{\eta_0} = \frac{1 + \tilde{\lambda}_2 \text{Wi}^2}{1 + \text{Wi}^2} + \frac{1 - \tilde{\lambda}_2}{1 + \text{Wi}^2} (\text{Wi} \sin \text{Wi} \tau - \cos \text{Wi} \tau) e^{-\tau} \quad (13)$$

$$\frac{\Psi_1^+}{2\eta_0\lambda_1} = \frac{1 - \tilde{\lambda}_2}{1 + \text{Wi}^2} \left[1 - \left(\frac{1}{\text{Wi}} \sin \text{Wi} \tau + \cos \text{Wi} \tau \right) e^{-\tau} \right] \quad (14)$$

From Figure 4 and Figure 5, we learn that when $\lambda_2/\lambda_1 \geq 1/9$, our main results disagree with Eqs. (13) and (14). This disagreement worsens with increasing λ_2/λ_1 . Further, Eqs. (13) and (14) exhibit non-zero initial stress, $\eta^+(0, \text{Wi}) > 0$, which our exact solution, Eq. (7) corrects.

b Finite Difference Calculations

We next apply an adaptive Runge-Kutta numerical scheme [46,47] to solve Eqs. (6) for the special case of the corotational Jeffreys fluid ($\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$). For our finite difference calculations, we coded Eq. (6), subject to the stress-free initial condition, using the *ode45* scheme in MATLAB (Version R2018a) on a Macbook Pro (3.1 GHz Intel Core i7 processor with 16GB 1867 MHz DDR3 memory) employing the High Sierra operating system (Version 10.13.6) operating system. For each point in Figure 4 and Figure 5, we find such an evaluation to consume less than a millisecond of CPU time. Figure 4 and Figure 5 show consistency between our exact solutions, Eqs. (7)–(9), and the finite difference calculations.

IV MATERIALS

a Preparation

Two WLM solutions were prepared by adding 100mM cetylpyridinium chloride (CPyCl–Applichem Panreac, Milan, Italy) in bi-distilled water with 50mM binding salts (surfactant to total salt molar ratio is 2). Two aromatic sodium binding salts were used, salicylate (Sal–Applichem Panreac, Chestershire, UK) and Diclofenac (Diclo–Farmalabor, Canosa di Puglia, Italy). All chemicals were used as received, without further purification. Samples were prepared by stirring surfactant and salts gently into bi-distilled water for a day at room temperature, followed by weeklong storage to equilibrate at room temperature. We prepared and used two samples of 50mM total salt concentration, one containing 40mM Diclo and 10mM Sal (40D10S) and the other, 20mM Diclo and 30mM Sal (20D30S).

b Oscillatory Shear and Shear Stress Growth Measurements

Rheological experiments were performed by using a strain-controlled rheometer ARES (TA Instruments) at room temperature with a solvent trap to control evaporation with a cone-plate fixture of cone radius 25mm and 0.017rad gap. Linear viscoelasticity was performed using small-amplitude oscillatory shear flow (SAOS) with a strain amplitude of 0.05 on both samples. Figure 7 shows that the complex viscosities of both solutions nearly match.

Startup steady shear flow was performed on both samples, by tuning the shear rate value in linear ($Wi < 1$) and non-linear regimes ($Wi > 1$), both of which are reported in Figure 8. For $Wi < 1$, η^+ curves increase smoothly to their steady shear viscosity values. When $Wi > 1$, η^+ curves start to exhibit overshoot before plateau.

c Fitting Exact Solution

To use η^+ , given in Eq. (7), to capture our startup experiments, we first get linear Oldroyd 8-constant parameters from SAOS. For the single-mode Oldroyd 8-constant framework, loss and storage viscosities are given by (Eqs. (17) and (18) of [31]):

$$\frac{\eta'(\omega)}{\eta_0} = \frac{1 + \lambda_1 \lambda_2 \omega^2}{1 + \lambda_1^2 \omega^2}; \quad \frac{\eta''(\omega)}{\eta_0} = \frac{(\lambda_1 - \lambda_2) \omega}{1 + \lambda_1^2 \omega^2} \quad (15)$$

which, fit to the data in Figure 7, gives (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$ and $\lambda_2 = 1.00 \times 10^{-2} \text{ s}$ for 40D10S and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$ and $\lambda_2 = 5.00 \times 10^{-3} \text{ s}$ for 20D30S. We use the method given in the “Results” section of [48], for minimizing the residual sum of squares (RSS) for SAOS fitting. Figure 7 shows our best fits of these two WLM solutions with the corresponding minimized RSS in Row 1 of Table V. From Figure 7, we learn that our WLM solutions really do obey the single-mode Oldroyd 8-constant framework, that is, they have a single λ_1 and a single λ_2 .

To capture nonlinear startup experiments, the nonlinear Oldroyd 8-constant parameters must be determined. We next fit our η^+ , given by Eq. (7), to the measured values. To capture the highly nonlinear measurements correctly, we find that we cannot do without nonlinear parameters, v_1 and v_2 . However, from our examination of the exact solution of η^+ , we find that also need μ_0 . Therefore, in addition to the linear parameters fitted to SAOS measurements ($\eta_0, \lambda_1, \lambda_2$), we further get (a) $\mu_0 = 0.205\text{s}$, $v_1 = -2.05\text{s}$, $v_2 = -1.03\text{s}$, $\mu_1 = \mu_2 = 0$ for 40D10S (so that $\sigma_1 = 2.19\text{s}^2$ and $\sigma_2 = 0.329\text{s}^2$) and (b) $\mu_0 = 0.53\text{s}$, $v_1 = -0.305\text{s}$, $v_2 = -1.35\text{s}$, $\mu_1 = \mu_2 = 0$ for 20D30S (so that $\sigma_1 = 2.33\text{s}^2$ and $\sigma_2 = 0.0931\text{s}^2$).

Figure 8 shows our best fits to the measured η^+ behaviors of these two WLM solutions with the corresponding minimized RSS in Rows 2–4 of Table V. From the good agreement in Figure 8, we discover that WLM solutions follow:

$$\tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{1}{2}\mu_0(\text{tr}\tau)\dot{\gamma} + \frac{1}{2}v_1(\tau : \dot{\gamma})\delta = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} + \frac{1}{2}v_2(\dot{\gamma} : \dot{\gamma})\delta \right) \quad (16)$$

which is a new special case of Eq. (4). In other words, for WLM solutions, we find that the v_i terms dwarf the μ_i terms. Eq. (16) also satisfies initial τ_{zz} -invariance (see §III.B. of [30]), as it must. We close our fitting section with Figure 10, showing the good agreement between Eq. (16) evaluated in steady shear flow:

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \left[1 - \frac{3}{2}\tilde{\mu}_0\tilde{v}_1\right]\lambda_1^2\dot{\gamma}^2}{1 + \left[\tilde{\lambda}_2 - \frac{3}{2}\tilde{\mu}_0\tilde{v}_2\right]\lambda_1^2\dot{\gamma}^2} \quad (17)$$

and our WLM solution viscosity functions.

When explicit expressions for complex viscosity and the steady shear material functions are available from macromolecular theory, continuum models such as our new Eq. (16) can be molecularized (see Table I of [49]; [50]). By *molecularized*, we mean assigning molecular dimensions to the model parameters. Since there is no macromolecular theory for WLM solutions giving explicit expressions for complex viscosity and the steady shear material functions, Eq. (16) cannot yet be molecularized.

V DISCUSSION

Figure 4 and Figure 5 demonstrate the growth of η^+ and Ψ_1^+ respectively for the special case of the corotational Jeffreys fluid, when $\mu_0 = \mu_1 = \mu_2 = v_1 = v_2 = 0$. For the growth of η^+ , we observe higher overshoots as λ_2/λ_1 increases. Unlike the growth of η^+ , the growth of Ψ_1^+ can start out negative when $\lambda_2/\lambda_1 > 1/9$. For the corotational Jeffreys fluid, we also observe ringing at high Wi.

To deepen our understanding of shear stress overshoots and undershoots, we construct Figure 6, which shows that the overshoot or undershoot times decrease with

Wi. Interestingly, from Figure 6(a) and Figure 6(b), the slopes of τ versus Wi are -1 at low shear rate. Departures occur at moderate Wi then back again to linear at high Wi. The behaviors at low and moderate Wi agree with the observed behavior of styrene-butadiene rubber (see Fig. 7 of [39]). However, the observed behavior at high Wi remains uncover.

The ratio $\check{\sigma}_2 / \check{\sigma}_1$ governs the steady shear behaviors of shear thinning, for $\check{\sigma}_2 / \check{\sigma}_1 < 1$, and shear thickening, for $\check{\sigma}_2 / \check{\sigma}_1 > 1$ (Fig. 1 of [36]). Figure 6(a) suggests that overshoots happen sooner, as the fluid gets thinner, by reducing $\check{\sigma}_2 / \check{\sigma}_1$. Certainly, Figure 6(a) also suggests that overshoots happen later, as the fluid thickens, by increasing $\check{\sigma}_2 / \check{\sigma}_1$.

Figure 9 illustrates the effects of μ_0, v_1 and v_2 on η^+ . Thus, we used Figure 9 to assist at our fitting of μ_0, v_1 and v_2 to η^+ in Figure 8. Decreasing v_2 fans out the η^+ curves upwardly, and *vice versa*. Therefore, when $v_1 \rightarrow 0$, all η^+ curves clump together, as shown in Figure 9(b). The effect of μ_0 depends on v_2 . Comparing Figure 9(b) with Figure 9(c), we learn that, for increasing v_2 , μ_0 locally distribute η^+ curves downward and *vice versa*. Increasing v_1 shift η^+ curves upwardly, it also makes overshoot less pronounced.

For WLM solutions, for a given surfactant, the primary factor that governs $\eta(\dot{\gamma}) \equiv \lim_{t \rightarrow \infty} \eta^+$ is salt concentration (cf. Figs. 2 and 6 of [51]; Fig. 7 of [26]). Hence, all three nonlinear parameters (μ_0, v_1, v_2) depend on salt concentration. Among these three nonlinear parameters, only v_1 is closely related to overshoots. These overshoots have been attributed to disentanglement [13,14,52,53], or straight-chain reptation (§7.6.1 of [54]), or pom-pom reptation (§11.6.1 of [6]), or macromolecular orientation (Fig. 7 of [2]; [55]), or to orientation with micellar scission [56].

VI CONCLUSION

We explore fluid nonlinearity experimentally by subjecting the fluid to a large steady shearing deformation [Eq. (1)], and to then observe the evolving stress responses. If the steady shear rate is high enough, these stress responses can overshoot, and then, they can even undershoot (see Figure 3 for 4 classic shapes). We call such responses *nonlinear*, and this experiment *shear stress growth*.

This paper provides exact analytical solutions for interpreting measured nonlinear stress growth responses. Specifically, we arrive at these exact solutions for the Oldroyd 8-constant constitutive framework [Eqs. (7)–(9)]. We test our $\eta^+(\tau, \text{Wi})$ against the measured behaviours of two wormlike micelle (WLM) solutions. At high shear rates, these solutions overshoot in stress growth without subsequent undershoot.

Our new exact solutions are consistent with the well-known solutions for steady shear flow ($\eta(\dot{\gamma})$, $\Psi_1(\dot{\gamma})$ and $\Psi_2(\dot{\gamma})$). Further, for the special case of the corotational Jeffreys fluid, we test for consistency with (i) published solutions arrived at by the

Goddard integral approach and (ii) with our finite difference calculations. Whereas for the latter, we find close agreement (Figure 4 and Figure 5), for the former, Figure 4 and Figure 5 show a heretofore unknown flaw in the textbook solution [Eqs. (13) and (14)].

The test of η^+ exact solution against our experiments uncovers a new special case, Eq. (16), for WLM solutions. This 6-constant special case distinguishes itself from all other special cases (see Table I) in that the v_i terms dwarf the μ_i terms. Comparing Eq. (16) to all other entries in Table I, except this work, we learn that both terms in Eq. (4) that contains double-dot products have always been omitted. This underscores the novelty of Eq. (16). Our new special case provides slightly early underpredictions of the overshoots at high shear rate (see Figure 8). To help readers use Eq. (16) we provide Figure 9 to guide the fitting of the three nonlinear parameters (μ_0, v_1 and v_2).

Our WLM solutions are made of cetylpyridinium chloride (CPyCl) dissolved in two aromatic sodium salts, salicylate (NaSal) and Diclofenac (NaDiclo). We find that all three of our nonlinear parameters μ_0, v_1 and v_2 depend upon the salt proportions, from which the WLM solution is made.

Though the single mode Oldroyd 8-constant framework is used for the WLM solutions in this work [Eqs. (7)–(9)], for other materials, such as polymers, the multimode versions must be used to capture nonlinear behaviors accurately (following the method in §II of Supplementary Material).

We close by contemplating the molecular underpinnings of the three nonlinear parameters. We have shown that these depend on the structure of the salt (holding the total molar salt concentration constant). It would be interesting to know how the structures of our two WLM solutions differ. We leave the detailed determination of the equilibrium structure of our WLM solutions including length [57], diameter, length distribution, and aggregation number [58] for another day. We also leave the experimental determination of the orientations of our WLM solutions in stress growth [38,55,59,60,61,62] for another day.

VII SUPPLEMENTARY MATERIAL

There are two sections in our Supplementary Material. §I provides the MATLAB code that we used to generate Figure 4 through Figure 10. We devote §II to our future work on generalizing our main results to the multimode Oldroyd 8-constant framework.

VIII ACKNOWLEDGMENT

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IX APPENDIX: DERIVATION OF EQ. (6)

In startup of the steady shear flow, the velocity profile is (Row c. of FIGURE 4.3-1 of [1]):

$$\mathbf{v} = \begin{bmatrix} \dot{\gamma}^0 y & 0 & 0 \end{bmatrix} \quad (18)$$

which gives the following strain-rate and vorticity tensors:

$$\dot{\gamma} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger = \dot{\gamma}^0 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (19)$$

$$\omega \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger = \dot{\gamma}^0 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (20)$$

To get Eq. (6), we begin with the Oldroyd 8-constant framework:

$$\begin{aligned} \tau + \lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{1}{2} \mu_0 (\text{tr } \tau) \dot{\gamma} - \frac{1}{2} \mu_1 \{ \tau \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau \} + \frac{1}{2} \nu_1 (\tau : \dot{\gamma}) \delta \\ = -\eta_0 \left(\dot{\gamma} + \lambda_2 \frac{\mathcal{D}\dot{\gamma}}{\mathcal{D}t} - \mu_2 \{ \dot{\gamma} \cdot \dot{\gamma} \} + \frac{1}{2} \nu_2 (\dot{\gamma} : \dot{\gamma}) \delta \right) \end{aligned} \quad (4)$$

into which, we insert the following:

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{bmatrix} \quad (21)$$

$$\begin{aligned} \frac{\mathcal{D}\tau}{\mathcal{D}t} &\equiv \frac{D\tau}{Dt} + \frac{1}{2} \{ \omega \cdot \tau - \tau \cdot \omega \} = \frac{\partial \tau}{\partial t} + \frac{1}{2} \{ \omega \cdot \tau - \tau \cdot \omega \} \\ &= \begin{bmatrix} \dot{\tau}_{xx} & \dot{\tau}_{yx} & 0 \\ \dot{\tau}_{yx} & \dot{\tau}_{yy} & 0 \\ 0 & 0 & \dot{\tau}_{zz} \end{bmatrix} + \frac{1}{2} \dot{\gamma}^0 \begin{bmatrix} -2\tau_{yx} & \tau_{xx} - \tau_{yy} & 0 \\ \tau_{xx} - \tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (22)$$

$$(\text{tr } \tau) \dot{\gamma} = (\tau_{xx} + \tau_{yy} + \tau_{zz}) \dot{\gamma}^0 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (23)$$

$$\{ \tau \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau \} = \dot{\gamma}^0 \begin{bmatrix} 2\tau_{yx} & \tau_{yy} + \tau_{xx} & 0 \\ \tau_{yy} + \tau_{xx} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

$$(\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} = 2\dot{\gamma}^0 \boldsymbol{\tau}_{yx} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (25)$$

$$\frac{\mathcal{D}\dot{\boldsymbol{\gamma}}}{\mathcal{D}t} \equiv \frac{D\dot{\boldsymbol{\gamma}}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\omega}\} = \frac{1}{2}\{\boldsymbol{\omega} \cdot \dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\omega}\} = (\dot{\gamma}^0)^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (26)$$

$$\{\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}\} = (\dot{\gamma}^0)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (27)$$

$$(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} = 2(\dot{\gamma}^0)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (28)$$

We then consider the xx , yy , zz and yx components separately:

$$\frac{d\tau_{xx}}{dt} = -\frac{1}{\lambda_1} \tau_{xx} + (1 + \bar{\mu}_1 - \bar{v}_1) \dot{\gamma}^0 \tau_{yx} + \eta_0 (\bar{\lambda}_2 + \bar{\mu}_2 - \bar{v}_2) (\dot{\gamma}^0)^2 \quad (29)$$

$$\frac{d\tau_{yy}}{dt} = -\frac{1}{\lambda_1} \tau_{yy} - (1 - \bar{\mu}_1 + \bar{v}_1) \dot{\gamma}^0 \tau_{yx} - \eta_0 (\bar{\lambda}_2 - \bar{\mu}_2 + \bar{v}_2) (\dot{\gamma}^0)^2 \quad (30)$$

$$\frac{d\tau_{zz}}{dt} = -\frac{1}{\lambda_1} \tau_{zz} - \bar{v}_1 \dot{\gamma}^0 \tau_{yx} - \eta_0 \bar{v}_2 (\dot{\gamma}^0)^2 \quad (31)$$

$$\frac{d\tau_{yx}}{dt} = -\frac{1}{2} (1 + \bar{\mu}_0 - \bar{\mu}_1) \dot{\gamma}^0 N_1 - (\bar{\mu}_0 - \bar{\mu}_1) \dot{\gamma}^0 N_2 - \left(\frac{3}{2} \bar{\mu}_0 - \bar{\mu}_1 \right) \dot{\gamma}^0 \tau_{zz} - \frac{1}{\lambda_1} \tau_{yx} - \frac{\eta_0 \dot{\gamma}^0}{\lambda_1} \quad (32)$$

To get the first normal stress difference, we subtract Eq. (30) from Eq. (29):

$$\frac{dN_1}{dt} = \frac{-1}{\lambda_1} N_1 + 2\dot{\gamma}^0 \tau_{yx} + 2\eta_0 \bar{\lambda}_2 (\dot{\gamma}^0)^2 \quad (33)$$

and to get the second normal stress difference, Eq. (31) from Eq. (30):

$$\frac{dN_2}{dt} = \frac{-N_2}{\lambda_1} + (-1 + \bar{\mu}_1) \dot{\gamma}^0 \tau_{yx} - \eta_0 (\bar{\lambda}_2 - \bar{\mu}_2) (\dot{\gamma}^0)^2 \quad (34)$$

Lastly, we nondimensionalize our governing equations, Eqs. (31)–(34):

$$\frac{d\bar{\tau}_{yx}}{d\tau} = -\frac{1}{2} (1 + \bar{\mu}_0 - \bar{\mu}_1) \text{Wi} \bar{N}_1 - (\bar{\mu}_0 - \bar{\mu}_1) \text{Wi} \bar{N}_2 - \left(\frac{3}{2} \bar{\mu}_0 - \bar{\mu}_1 \right) \text{Wi} \bar{\tau}_{zz} - \bar{\tau}_{yx} - \text{Wi} \quad (35)$$

$$\frac{d\bar{N}_1}{d\tau} = -\bar{N}_1 + 2 \text{Wi} \bar{\tau}_{yx} + 2 \bar{\lambda}_2 \text{Wi}^2 \quad (36)$$

$$\frac{d\bar{N}_2}{d\tau} = -\bar{N}_2 + (-1 + \bar{\mu}_1) \text{Wi} \bar{\tau}_{yx} - (\bar{\lambda}_2 - \bar{\mu}_2) \text{Wi}^2 \quad (37)$$

$$\frac{d\bar{\tau}_{zz}}{d\tau} = -\bar{\tau}_{zz} - \bar{v}_1 \text{Wi} \bar{\tau}_{yx} - \bar{v}_2 \text{Wi}^2 \quad (38)$$

Combining Eqs. (35)–(38) gives Eq. (6) above. Alternatively, Eq. (6) can also be derived similarly using Eqs. (A12)–(A15) of [30] by setting shear rate, $\dot{\gamma}$, to a constant.

Eq. (6) is a system of linear first-order ordinary differential equations, which yields the following exact solutions by the method of Kovacic [63]:

$$\begin{aligned} \tilde{\tau}_{yx} = & Wi e^{-\tau} \left(\frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} \cos(\sqrt{\tilde{\sigma}_1} Wi \tau) - \frac{1}{\sqrt{\tilde{\sigma}_1}} \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{1 + \tilde{\sigma}_1 Wi^2} Wi \sin(\sqrt{\tilde{\sigma}_1} Wi \tau) \right) \\ & - \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi \end{aligned} \quad (39)$$

$$\begin{aligned} \tilde{N}_1 = & 2e^{-\tau} \left[(\tilde{\sigma} - \tilde{\lambda}_2) Wi^2 + \frac{1 - \tilde{\sigma}}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \cos \sqrt{\tilde{\sigma}_1} Wi \tau + \frac{1}{\sqrt{\tilde{\sigma}_1}} \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi \sin \sqrt{\tilde{\sigma}_1} Wi \tau \right] \\ & - \frac{2(1 - \tilde{\lambda}_2 - (\tilde{\lambda}_2 \tilde{\sigma}_1 - \tilde{\sigma}_2) Wi^2)}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \end{aligned} \quad (40)$$

$$\begin{aligned} \tilde{N}_2 = & e^{-\tau} \left[(\tilde{\lambda}_2 - \tilde{\mu}_2 - (1 - \tilde{\mu}_1) \tilde{\sigma}) Wi^2 - \frac{(1 - \tilde{\mu}_1)(1 - \tilde{\sigma})}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \cos \sqrt{\tilde{\sigma}_1} Wi \tau \right. \\ & \left. - \frac{1 - \tilde{\mu}_1}{\sqrt{\tilde{\sigma}_1}} \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi \sin \sqrt{\tilde{\sigma}_1} Wi \tau \right] \\ & + \frac{1 - \tilde{\mu}_1 - \tilde{\lambda}_2 + \tilde{\mu}_2 - ((\tilde{\lambda}_2 - \tilde{\mu}_2) \tilde{\sigma}_1 - (1 - \tilde{\mu}_1) \tilde{\sigma}_2) Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \end{aligned} \quad (41)$$

$$\begin{aligned} \tilde{\tau}_{zz} = & e^{-\tau} \left[\frac{\tilde{\lambda}_2 - \tilde{\sigma} - (\tilde{\mu}_0 - \tilde{\mu}_1)(\tilde{\mu}_1 \tilde{\sigma} - \tilde{\mu}_2)}{\frac{3}{2} \tilde{\mu}_0 - \tilde{\mu}_1} Wi^2 - \frac{\tilde{v}_1}{\tilde{\sigma}_1} \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \cos \sqrt{\tilde{\sigma}_1} Wi \tau \right. \\ & \left. - \frac{\tilde{v}_1}{\sqrt{\tilde{\sigma}_1}} \frac{1 + \tilde{\sigma}_2 Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi \sin \sqrt{\tilde{\sigma}_1} Wi \tau \right] \\ & + \frac{\tilde{v}_1 - \tilde{v}_2 + (\tilde{\sigma}_2 \tilde{v}_1 - \tilde{\sigma}_1 \tilde{v}_2) Wi^2}{1 + \tilde{\sigma}_1 Wi^2} Wi^2 \end{aligned} \quad (42)$$

Once the following definitions (Eqs. (10)–(12) of [64]):

$$\tilde{\tau}_{yx} \equiv -Wi \tilde{\eta}^+, \quad \tilde{N}_1 \equiv -Wi^2 \tilde{\Psi}_1^+, \quad \tilde{N}_2 \equiv -Wi^2 \tilde{\Psi}_2^+ \quad (43)$$

are applied to Eqs. (39)–(41), we find the exact solutions for material functions in startup of steady shear flow, $\tilde{\eta}^+$, $\tilde{\Psi}_1^+$ and $\tilde{\Psi}_2^+$, as provided in Eqs. (7)–(9) above. Due to $\tilde{\tau}_{zz}$ invariance (see §III.B. of [30]), Eq. (42) is of no interest.

Table I: Exact solutions for startup of steady shear flow for special cases of the Oldroyd 8-constant framework

Model [Ref. to Model]	No. Parameters	Reduce from Oldroyd 8-Constant Framework [Eq. (4)] by Assigning	Ref. to Exact Solution (Correction to)
Newtonian	1	$\lambda_1 = \lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	
Upper convected Maxwell [65]	2	$\mu_1 = \lambda_1; \lambda_2 = \mu_0 = \mu_2 = \nu_1 = \nu_2 = 0$	
Corotational Maxwell [66]	2	$\lambda_2 = \mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	
Lower convected Maxwell (Eq. (3.23) of [67])	2	$\mu_1 = -\lambda_1; \lambda_2 = \mu_0 = \mu_2 = \nu_1 = \nu_2 = 0$	
Arbitrary normal stress ratio (ANSR) (Eq. (10) of [68])	3	$\lambda_2 = \mu_0 = \mu_2 = \nu_1 = \nu_2 = 0$	
Oldroyd B (upper convected Jeffreys) (Eq. (B) 9.1-9 of [1])	3	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \mu_0 = \nu_1 = \nu_2 = 0$	
Corotational Jeffreys [69]	3	$\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$	Eq. (7-33) of [1] (This paper)
Oldroyd A (lower convected Jeffreys) (Eq. (A) 9.1-9 of [1])	3	$\mu_1 = -\lambda_1; \mu_2 = -\lambda_2; \mu_0 = \nu_1 = \nu_2 = 0$	
Second-order (see Eq. 8.4-3 of [1])	3	$\lambda_1 = \mu_0 = \mu_1 = \nu_1 = \nu_2 = 0$	
Williams 3-constant Oldroyd [70]	3	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \nu_1 = \frac{2}{3}\lambda_1; \nu_2 = \frac{2}{3}\lambda_2; \mu_0 = 0$	
Denn modified convected Maxwell model [71]	3	$\mu_1 = \lambda_1; \lambda_2 = \mu_0 = \nu_1 = \nu_2 = 0$	
Arbitrary normal stress ratio (ANSR) (Eq. (10) of [68])	4	$\mu_0 = \mu_2 = \nu_1 = \nu_2 = 0$	
Oldroyd 4-constant [1]	4	$\mu_1 = \lambda_1; \mu_2 = \lambda_2; \nu_1 = \nu_2 = 0$	
Oldroyd 5-constant (Eq. (45) of [72])	5	$\mu_0 = \frac{2}{3}\mu_1; \nu_1 = \nu_2 = 0$	
Gordon-Schowalter*,† [73]	5	$\lambda_2 = (\eta_s/\eta_0)\lambda_1; \mu_1 = (1-\xi)\lambda_1; \mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1; \mu_0 = \nu_1 = \nu_2 = 0$	
Johnson-Segalman**,‡ [74,75]	5	$\lambda_2 = (\eta_s/\eta_0)\lambda_1; \mu_1 = (1-\xi)\lambda_1; \mu_2 = (1-\xi)(\eta_s/\eta_0)\lambda_1; \mu_0 = \nu_1 = \nu_2 = 0$	Eq. (4) of [35] (This paper)
This paper [Eq. (16)]	6	$\mu_1 = \mu_2 = 0$	
Oldroyd 6-constant [1]	6	$\nu_1 = \nu_2 = 0$	
Constant-viscosity		$\sigma_1 = \sigma_2$	
Boger, $\Psi_2(\dot{\gamma}) = 0$		$\sigma_1 = \sigma_2; \lambda_1 > \lambda_2; \lambda_1 - \mu_1 = \lambda_2 - \mu_2$	
Boger, $\Psi_2(\dot{\gamma}) < 0$		$\sigma_1 = \sigma_2; \lambda_1 > \lambda_2; \lambda_1 - \mu_1 > \lambda_2 - \mu_2$	
Oldroyd 8-constant [30]	8	-	Eqs. (7)-(9)

Remarks: *Corrects eighth row in TABLE 7.3-2 of [76]; † $\eta_0 \equiv (1-\xi)\eta_p + \eta_s$

**Corrects last row in TABLE 7.3-2 of [76]; ‡ $\eta_0 \equiv \eta_p + \eta_s$

References to the exact solutions are not provided if they are not available.

Table II: Literature review of $\eta^+(t)$ and $\Psi_1^+(t)$ measurements of wormlike micellar solutions of CPyCl

First Authors (year)	Binding Salts	Concentrations [Mol/L]		Reported Material Functions; Remarks
		CpyCl	Salts	
Rehage (1988) [26]	NaSal	0.1	0.06	$\eta^+(t)$
Rehage (1991) [77]	NaSal	0.1	0.06	$\eta^+(t)$
Berret (1997) [35]	NaSal	[CpyCl]/[Salys] = 0.5		$\eta^+(t)$
Fischer (1997) [78]	NaSal	0.1	0.25	$\eta^+(t), \Psi_1^+(t)$
Britton (1999) [79]	NaSal	0.1	0.06	$\eta^+(t)$; Homemade CPyCl
Fischer (2000) [12]	NaSal	0.04	0.04	$\eta^+(t)$
Méndez-Sánchez (2003) [45]	NaSal	0.1	0.06	$\eta^+(t)$; Flow instability
Hu (2005) [11]	NaSal	0.1648	0.0874	$\eta^+(t)$; Shear banding
López-Barrón (2014) [38]	NaSal	0.15	0.075	$\eta^+(t), \Psi_1^+(t)$
Gaudino (2017) [18]	NaSal	0.1	0.0609	$\eta^+(t)$
Al-kaby (2018) [80]	NaSal	0.1676	0.0838	$\eta^+(t)$; Shear banding
This paper	NaSal +NaDiclo	0.1	0.01+0.04; 0.02+0.03	$\eta^+(t)$

CPyCl $\equiv$ cetylpyridinium chloride; NaSal $\equiv$ sodium salicylate; NaDiclo $\equiv$ sodium diclofenac;

Table III: Dimensional variables and symbols

Name	Unit	Symbol
Complex viscosity (Eq. (15); [81])	M/Lt	$\eta^* \equiv \eta' - i\eta''$
Derivative, corotational, of a dimensionless tensor $\mathbf{b}$ (Eq. (17)ff. of [82])	t^{-1}	$\frac{\mathcal{D}\mathbf{b}}{\mathcal{D}t} \equiv \frac{D\mathbf{b}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega} \cdot \mathbf{b} - \mathbf{b} \cdot \boldsymbol{\omega}\}$
Derivative, material (Eq. (3.5-2) of [83])	t^{-1}	$\frac{D\mathbf{b}}{Dt} \equiv \frac{\partial \mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla \mathbf{b})$
Hydrodynamic stress (Eq. 1.2-2 of [83])	M/Lt^2	p
Oldroyd 8-constant fluid parameter, nonlinear (Eq. (8.1-2) of [1])	t	$\mu_0, \mu_1, \mu_2, \nu_1, \nu_2$
Oldroyd 8-constant fluid parameter, lumped (Eqs. (15) and (16) of [36])	t^2	$\sigma_1 \equiv \lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2}\nu_1) - \mu_1(\mu_1 - \nu_1),$ $\sigma_2 \equiv \lambda_1\lambda_2 + \mu_0(\mu_2 - \frac{3}{2}\nu_2) - \mu_1(\mu_2 - \nu_2)$
Rate-of-strain tensor	t^{-1}	$\dot{\gamma} \equiv \nabla \mathbf{v} + (\nabla \mathbf{v})^+$
Relaxation time	t	λ_1
Retardation time	t	λ_2
Stress tensor, extra (Eq. 1.2-2 of [83])	M/Lt^2	$\boldsymbol{\tau}$
Stress tensor, total (Eq. 1.2-2 of [83])	M/Lt^2	$\boldsymbol{\pi} \equiv \boldsymbol{\tau} + p\boldsymbol{\delta}$
Stress-growth material functions, first normal stress coefficient (Eq. (11) of [64])	M/L	$\Psi_1^+ \equiv -(\tau_{xx} - \tau_{yy})/(\dot{\gamma}^0)^2$
Stress-growth material functions, second normal stress coefficient (Eq. (12) of [64])	M/L	$\Psi_2^+ \equiv -(\tau_{yy} - \tau_{zz})/(\dot{\gamma}^0)^2$
Stress-growth material functions, viscosity (Eq. (10) of [64])	M/Lt	$\eta^+ \equiv -\tau_{yx}/\dot{\gamma}^0$
Vorticity tensor	t^{-1}	$\boldsymbol{\omega} \equiv \nabla \mathbf{v} - (\nabla \mathbf{v})^+$
Zero-shear first and second normal stress differences (Eqs. (24) and (25) of [36])	M/L	$\Psi_{1,0} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_1 = 2\eta_0(\lambda_1 - \lambda_2)$ $\Psi_{2,0} \equiv \lim_{\dot{\gamma} \rightarrow 0} \Psi_2 = -\eta_0(\lambda_1 - \lambda_2 - \mu_1 + \mu_2)$
Zero-shear viscosity	M/Lt	η_0

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

*Where τ_{ij} is the force exerted in the j th direction on a unit area of fluid surface of constant x , by fluid in the region lesser x , on fluid in the region greater x , (see “Note on the Sign Convention for the Stress Tensor” on pp. 19–20 of [83], or pp. 24–25 of [84]).

Table IV: Dimensionless variables and groups

Name	Symbol
Threshold, overshoot, undershoot, and steady state criteria [Eq. (10)ff.]	$\epsilon_o, \epsilon_u, \epsilon_{ss}$
Extra stress, yx -component [Eq. (6)]	$\tilde{\tau}_{yx} \equiv \lambda_1 \tau_{yx} / \eta_0$
Extra stress, zz -component [Eq. (6)]	$\tilde{\tau}_{zz} \equiv \lambda_1 \tau_{zz} / \eta_0$
First and second normal stress differences [Eq. (6)]	$\tilde{N}_1 \equiv \lambda_1 N_1 / \eta_0, \tilde{N}_2 \equiv \lambda_1 N_2 / \eta_0$
Oldroyd 8-constant parameters	$\tilde{\mu}_0 \equiv \mu_0 / \lambda_1, \tilde{\mu}_1 \equiv \mu_1 / \lambda_1, \tilde{\mu}_2 \equiv \mu_2 / \lambda_1, \tilde{v}_1 \equiv v_1 / \lambda_1, \tilde{v}_2 \equiv v_2 / \lambda_1$
Oldroyd 8-constant parameters, lumped	$\tilde{\sigma}_1 \equiv \sigma_1 / \lambda_1^2, \tilde{\sigma}_2 \equiv \sigma_2 / \lambda_1^2$
Oldroyd 8-constant parameter, ratio	$\sigma \equiv \sigma_2 / \sigma_1 \equiv \tilde{\sigma}_2 / \tilde{\sigma}_1$
Retardation time	$\tilde{\lambda}_2 \equiv \lambda_2 / \lambda_1$
Time	$\tau \equiv t / \lambda_1$
Weissenberg number [Eq. (2)]	$Wi \equiv \lambda_1 \dot{\gamma}$

Table V: Residual sum of squares for fitted curves in Figure 7, Figure 8 and Figure 10

40D10S		20D30S	
Experiment	RSS	Experiment	RSS
SAOS ($ \eta^* / \eta_0 $)	3.54×10^{-3}	SAOS ($ \eta^* / \eta_0 $)	1.04×10^{-2}
$\dot{\gamma} = 0.500$	0.0185	$\dot{\gamma} = 0.400$	0.112
$\dot{\gamma} = 1.25$	0.0951	$\dot{\gamma} = 0.600$	0.0593
$\dot{\gamma} = 3.00$	0.408	$\dot{\gamma} = 1.00$	0.400
Steady shear	1.78×10^{-2}	Steady shear	1.65×10^{-2}

Figure 1: Structure of wormlike micelle in aqueous solution. Penetrated and adsorbed counter-ions, NaSal and NaDiclo, shown in magnified cross section. Surfactant molecules (CPyCl) have hydrophilic heads and hydrophobic tails.

Figure 2: Flow profile of the startup of the steady shear flow in cone-plate rheometer. The WLM solution, CPyCl with two binding salts (NaSal+NaDiclo), is loaded into the shear fixture, shown in (a). A strand in the entangled network is illustrated in Figure 1. A constant shear rate is then suddenly applied, as shown in (b).

Figure 3: Four classic shapes of observed stress growth viscosity functions (top to bottom): (i) linear without overshoot, (ii) nonlinear without overshoot, (iii) nonlinear with overshoot and without undershoot, and (iv) nonlinear with both overshoot and single undershoot.

Figure 4: Exact shear stress-growth viscosity [**green**, Eq. (7)] *versus* finite difference calculations from Eq. (6) (**black**), and *versus* the well-known solution [**red**, Eq. (13)] for the special case of the corotational Jeffreys fluid, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Curves are parametrized with increasing $Wi \equiv \lambda_1 \dot{\gamma} = \frac{1}{10}, 1, 10$, from top to bottom.

Figure 5: Exact shear stress-growth first normal stress coefficient [**green**, Eq. (8)] *versus* finite difference calculations from Eq. (6) (**black**), and *versus* the well-known solution [**red**, Eq. (14)] for the special case of the corotational Jeffreys fluid, $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$. Curves are parametrized with increasing $Wi \equiv \lambda_1\dot{\gamma} = \frac{1}{10}, 1, 10$, from top to bottom.

Figure 6: Predictions of (a) first overshoot and (b) undershoot from Eq. (11). Dashed curves at low Wi indicate that the viscosity function has reached its steady state.

Figure 7: SAOS measurements of (a) 40D10S and (b) 20D30S *versus* predicted values from the Oldroyd 8-constant framework [Eq. (15)], with (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$, $\lambda_2 = 1.00 \times 10^{-2} \text{ s}$ and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$, $\lambda_2 = 5.00 \times 10^{-3} \text{ s}$. The RSS are reported in Table V.

Figure 8: Evolution of measured η^+ of (a) 40D10S and (b) 20D30S *versus* predicted values from the Oldroyd 8-constant framework [Eq. (7)], for (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$, $\lambda_2 = 1.00 \times 10^{-2} \text{ s}$, $\mu_0 = 0.205 \text{ s}$, $\mu_1 = \mu_2 = 0$, $v_1 = -2.05 \text{ s}$, $v_2 = -1.03 \text{ s}$, and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$, $\lambda_2 = 5.00 \times 10^{-3} \text{ s}$, $\mu_0 = 5.00 \times 10^{-3} \text{ s}$, $v_1 = -0.305 \text{ s}$, $v_2 = -1.13 \text{ s}$, $\mu_1 = \mu_2 = 0$. Experimental conditions: (a) $\lambda_1 \dot{\gamma} = 0$ (LVE), 0.625, 1.25, 3.75, and (b) $\lambda_1 \dot{\gamma} = 0$ (LVE), 0.608, 0.912, 1.52 (both from top to bottom). The RSS are reported in Table V.

Figure 9: η^+/η_0 versus t/λ_1 from the Oldroyd 8-constant fluid with $\lambda_2 = \mu_1 = \mu_2 = 0$ [Eq. (7)] at $Wi \equiv \lambda_1 \dot{\gamma} = 1$. Three subfigures illustrate three different values of $\tilde{v}_2 \equiv v_2/\lambda_1 = -1, 1, 0.1$. Each subfigure shows four sets of three curves for $\tilde{\mu}_0 \equiv \mu_0/\lambda_1 = -0.2, 0.05, 0.2, 0.4$. Each set of curves is parametrized with $\tilde{v}_1 \equiv v_1/\lambda_1 = -1, 0.1, 1$ (from bottom to top).

Figure 10: Steady shear viscosity measurements of (a) 40D10S and (b) 20D30S, both from Figure 8, *versus* predicted values from the Oldroyd 8-constant framework [Eq. (17)], with (a) $\eta_0 = 45.0 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.25 \text{ s}$, $\sigma_1 = 2.19 \text{ s}^2$ and $\sigma_2 = 0.329 \text{ s}^2$, and (b) $\eta_0 = 59.5 \text{ Pa}\cdot\text{s}$, $\lambda_1 = 1.52 \text{ s}$, $\sigma_1 = 2.33 \text{ s}^2$ and $\sigma_2 = 0.0931 \text{ s}^2$. The RSS are reported in Table V.

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**SUPPLEMENTARY MATERIAL TO:
STARTUP STEADY SHEAR FLOW
FROM THE OLDROYD 8-CONSTANT FRAMEWORK**

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ABSTRACT

There are two sections in our Supplementary Material. §I provides the MATLAB code that we used to generate Figure 4 through Figure 10 in our main manuscript. We devote §II to our future work on generalizing our main results to the multimode Oldroyd 8-constant framework.

CONTENTS

I	MATLAB CODE	2
a	Figures 4 and 5	2
b	Figure 6.....	4
c	Figure 7	7
d	Figure 8.....	9
e	Figure 9	12
f	Figure 10	14
II	SPRIGGS RELATIONS	15

I MATLAB CODE

This section provides MATLAB codes for constructing Figure 4 through Figure 9 in our main manuscript.

a Figures 4 and 5

In order to solve our state equation numerically, we create the following MATLAB function:

```
function ydot = O8StressGrowth(tau,y)

global lambda1 lambda2 mu0 mu1 mu2 nu1 nu2 Wi
% Note y(1) = tau_yx; y(2) = N1; y(3) = N2; y(4) = tau_zz
ydot = zeros(4,1);
ydot(1) = - y(1) - (1/2)*(1-mu1+mu0)*Wi*y(2) + (mu1-mu0)*Wi*y(3) -
((3/2)*mu0-mu1)*Wi*y(4) - Wi;
ydot(2) = 2*Wi*y(1) - y(2) + 2*lambda2*Wi^2;
ydot(3) = -(1-mu1)*Wi*y(1) - y(3) - (lambda2-mu2)*Wi^2;
ydot(4) = -nu1*Wi*y(1) - y(4) - nu2*Wi^2;

end
```

We next proceed with the following to get all three responses:

```
clear; close all; clc

global lambda2 mu0 mu1 mu2 nu1 nu2 Wi

tspan = [0 4*pi]; % defining time span
taup = linspace(tspan(1),tspan(end),100); % creating time span
IC = [0 0 0 0]; % initial conditions (stress free)
lambda2 = 2/5; % dimensionless O8 parameters
mu0 = 0;
mu1 = 0;
mu2 = 0;
nu1 = 0;
nu2 = 0;
sigma1 = 1 + mu0*(mu1 - 3/2*nu1) - mu1*(mu1-nu1);
sigma2 = lambda2 + mu0*(mu2 - 3/2*nu2) - mu1*(mu2-nu2);
sigma = sigma2/sigma1;

Wip = [0.1,1,10]; % Weissenberg numbers

Wi = Wip(1);
% Numerical Solution
[taul,y1] = ode45(@O8StressGrowth,tspan,IC);
% Solution in DPL
etaplus1 = (1 + lambda2*Wi^2)/(1+Wi^2) + (1-
lambda2)/(1+Wi^2)*(Wi*sin(Wi*taup) - cos(Wi*taup)).*exp(-taup);
psilplus1 = (1-lambda2)/(1+Wi^2) * (1 - (1/Wi*sin(Wi*taup) +
cos(Wi*taup)).*exp(-taup));
% Exact Solution, etaplus
tauyxExa = Wi*exp(-taup);
tauyxExb = (1+sigma2*Wi^2)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup);
```



```

tauyxExc = (1/sqrt(sigma1))*(sigma1-
sigma2)/(1+sigma1*Wi^2)*Wi*sin(sqrt(sigma1)*Wi*taup);
tauyxExd = -(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*Wi;
tauyxEx1 = tauyxExa.*(tauyxExb - tauyxExc) + tauyxExd;
% Exact Solution, Psilplus
psila = (sigma - lambda2) + (1-
sigma)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup) +
(1/Wi/sqrt(sigma1))*(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*sin(sqrt(sigma1)*Wi*taup)
;
psilb = (1-lambda2-(lambda2*sigma1-sigma2)*Wi^2)/(1+sigma1*Wi^2);
psil1 = -exp(-taup).*(psila) + psilb;

Wi = Wip(2);
% Numerical Solution
[tau2,y2] = ode45(@O8StressGrowth,tspan,IC);
% Solution in DPL
etaplus2 = (1 + lambda2*Wi^2)/(1+Wi^2) + (1-
lambda2)/(1+Wi^2)*(Wi*sin(Wi*taup) - cos(Wi*taup)).*exp(-taup);
psilplus2 = (1-lambda2)/(1+Wi^2) * (1 - (1/Wi*sin(Wi*taup) +
cos(Wi*taup)).*exp(-taup));
% Exact Solution, etaplus
tauyxExa = Wi*exp(-taup);
tauyxExb = (1+sigma2*Wi^2)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup);
tauyxExc = (1/sqrt(sigma1))*(sigma1-
sigma2)/(1+sigma1*Wi^2)*Wi*sin(sqrt(sigma1)*Wi*taup);
tauyxExd = -(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*Wi;
tauyxEx2 = tauyxExa.*(tauyxExb - tauyxExc) + tauyxExd;
% Exact Solution, Psilplus
psila = (sigma - lambda2) + (1-
sigma)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup) +
(1/Wi/sqrt(sigma1))*(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*sin(sqrt(sigma1)*Wi*taup)
;
psilb = (1-lambda2-(lambda2*sigma1-sigma2)*Wi^2)/(1+sigma1*Wi^2);
psil2 = -exp(-taup).*(psila) + psilb;

Wi = Wip(3);
% Numerical Solution
[tau3,y3] = ode45(@O8StressGrowth,tspan,IC);
% Solution in DPL
etaplus3 = (1 + lambda2*Wi^2)/(1+Wi^2) + (1-
lambda2)/(1+Wi^2)*(Wi*sin(Wi*taup) - cos(Wi*taup)).*exp(-taup);
psilplus3 = (1-lambda2)/(1+Wi^2) * (1 - (1/Wi*sin(Wi*taup) +
cos(Wi*taup)).*exp(-taup));
% Exact Solution, etaplus
tauyxExa = Wi*exp(-taup);
tauyxExb = (1+sigma2*Wi^2)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup);
tauyxExc = (1/sqrt(sigma1))*(sigma1-
sigma2)/(1+sigma1*Wi^2)*Wi*sin(sqrt(sigma1)*Wi*taup);
tauyxExd = -(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*Wi;
tauyxEx3 = tauyxExa.*(tauyxExb - tauyxExc) + tauyxExd;
% Exact Solution, Psilplus
psila = (sigma - lambda2) + (1-
sigma)/(1+sigma1*Wi^2)*cos(sqrt(sigma1)*Wi*taup) +
(1/Wi/sqrt(sigma1))*(1+sigma2*Wi^2)/(1+sigma1*Wi^2)*sin(sqrt(sigma1)*Wi*taup)
;
psilb = (1-lambda2-(lambda2*sigma1-sigma2)*Wi^2)/(1+sigma1*Wi^2);
psil3 = -exp(-taup).*(psila) + psilb;

```

To plot Figure 4, we code the following:

```
plot(taup,etaplus1,'-r','linewidth',2); hold on
plot(taup,-tauyxEx1/Wip(1),'-g','linewidth',4); hold on
plot(taul, -y1(:,1)/Wip(1),'-k','linewidth',2); hold on

plot(taup,etaplus2,'-r','linewidth',2); hold on
plot(taup,-tauyxEx2/Wip(2),'-g','linewidth',4); hold on
plot(tau2, -y2(:,1)/Wip(2),'-k','linewidth',2); hold on

plot(taup,etaplus3,'-r','linewidth',2); hold on
plot(taup,-tauyxEx3/Wip(3),'-g','linewidth',4); hold on
plot(tau3, -y3(:,1)/Wip(3),'-k','linewidth',2); hold on

grid on
xlim([0,6])
set(gca,'fontname','Palatino','fontsize',16)
set(gcf,'paperposition',[1 1 5 5])
axis square
print('-dtiff',' Figure4','-r500')
```

To plot Figure 5, we code the following:

```
plot(taup,psilplus1,'-r','linewidth',2); hold on
plot(taup,psil1,'-g','linewidth',4); hold on
plot(taul,-y1(:,2)/2/Wip(1)^2,'-k','linewidth',2); hold on

plot(taup,psilplus2,'-r','linewidth',2); hold on
plot(taup,psil2,'-g','linewidth',4); hold on
plot(tau2, -y2(:,2)/2/Wip(2)^2,'-k','linewidth',2); hold on

plot(taup,psilplus3,'-r','linewidth',2); hold on
plot(taup,psil3,'-g','linewidth',4); hold on
plot(tau3, -y3(:,2)/2/Wip(3)^2,'-k','linewidth',2); hold on

grid on
xlim([0,10])
ylim([-0.1,1])
set(gca,'ytick',[-0.1 0:0.2:1])
set(gca,'fontname','Palatino','fontsize',16)
set(gcf,'paperposition',[1 1 5 5])
axis square
print('-dtiff','Figure5','-r500')
```

b Figure 6

To construct Figure 6, we need to first code the overshoot criteria into MATLAB:

```
clear; close all; clc

sigma1p = [0.1 1 10]; % dimensionless sigma1
sigma2p = [0 1/27 1/9 1/3]; % dimensionless sigma2
Wip      = logspace(-1,2,1000);
```

```

np      = 1;
t       = zeros(numel(Wip),numel(sigma1p),numel(sigma2p),numel(np));
ddeta   = zeros(numel(Wip),numel(sigma1p),numel(sigma2p),numel(np));
tOS     = NaN.*t;
tOS_SS  = NaN.*t;
tUS     = NaN.*t;
tUS_SS  = NaN.*t;
BREAKOS = 0;
for inp = 1:numel(np)
    for isigma1p = 1:numel(sigma1p)
        for isigma2p = 1:numel(sigma2p)
            for iWip = 1:numel(Wip)
                sigma1 = sigma1p(isigma1p);
                sigma2 = sigma2p(isigma2p);
                Wi      = Wip(iWip);
                n       = np(inp);
                t(iWip,isigma1p,isigma2p,inp) = (n*pi -
acot(sigma2*Wi/sqrt(sigma1))/sqrt(sigma1)./Wi;
                ddeta(iWip,isigma1p,isigma2p,inp) = (1-sigma2*Wi^2)
*exp(-
t(iWip,isigma1p,isigma2p,inp))*cos(sqrt(sigma1)*Wi*t(iWip,isigma1p,isigma2p,i
np)) -...

(sigma1+sigma2)*Wi/sqrt(sigma1)*exp(-
t(iWip,isigma1p,isigma2p,inp))*sin(sqrt(sigma1)*Wi*t(iWip,isigma1p,isigma2p,i
np));

                % Overshoot Data Processing
                if ddeta(iWip,isigma1p,isigma2p,inp) > -10^-5 && BREAKOS ==
0
                    tOS_SS(iWip,isigma1p,isigma2p,inp) =
t(iWip,isigma1p,isigma2p,inp);
                    elseif ddeta(iWip,isigma1p,isigma2p,inp) < -10^-5 &&
isnan(tOS_SS(iWip,isigma1p,isigma2p,inp))
                        tOS(iWip,isigma1p,isigma2p,inp) =
t(iWip,isigma1p,isigma2p,inp);
                        BREAKOS = 1; % To prevent from assigning the calculated
t to tSS
                        elseif BREAKOS == 1
                            tOS(iWip,isigma1p,isigma2p,inp) =
t(iWip,isigma1p,isigma2p,inp);
                            end
                        end
                        BREAKOS = 0;
                    end
                end
            end
        end
    end
end
save overshoot.mat

```

We then code the undershoot criteria into MATLAB:

```

clear; close all; clc

sigma1p = [0.1 1 10]; % dimensionless sigma1
sigma2p = [0 1/27 1/9 1/3]; % dimensionless sigma2
Wip      = logspace(-1,2,1000);
np       = 2;

```

```

t      = zeros(numel(Wip),numel(sigma1p),numel(sigma2p),numel(np));
ddeta  = zeros(numel(Wip),numel(sigma1p),numel(sigma2p),numel(np));
BREAKUS = 0;
for inp = 1:numel(np)
    for isigma1p = 1:numel(sigma1p)
        for isigma2p = 1:numel(sigma2p)
            for iWip = 1:numel(Wip)
                sigma1 = sigma1p(isigma1p);
                sigma2 = sigma2p(isigma2p);
                Wi      = Wip(iWip);
                n        = np(inp);
                t(iWip,isigma1p,isigma2p,inp) = (n*pi -
acot(sigma2*Wi/sqrt(sigma1))/sqrt(sigma1)./Wi;
                ddeta(iWip,isigma1p,isigma2p,inp) = (1-sigma2*Wi^2)
*exp(-
t(iWip,isigma1p,isigma2p,inp))*cos(sqrt(sigma1)*Wi*t(iWip,isigma1p,isigma2p,i
np)) -...

(sigma1+sigma2)*Wi/sqrt(sigma1)*exp(-
t(iWip,isigma1p,isigma2p,inp))*sin(sqrt(sigma1)*Wi*t(iWip,isigma1p,isigma2p,i
np));
                if ddeta(iWip,isigma1p,isigma2p,inp) > 10^-5
                    tUS(iWip,isigma1p,isigma2p,inp) =
t(iWip,isigma1p,isigma2p,inp);
                    BREAKUS = 1;
                elseif BREAKUS == 1
                    tUS(iWip,isigma1p,isigma2p,inp) =
t(iWip,isigma1p,isigma2p,inp);
            end
        end
    end
end
end
end
end
save undershoot.mat

```

From above, we should get “[overshoot.mat](#)” and “[undershoot.mat](#).” To plot Figure 6, we finally code the following:

```

clear; clc

% This is to use the saved .mat file to plot overshoot and undershoot

load overshoot.mat;
load undershoot.mat;

subplot(2,1,1); loglog(Wip,tOS(:,1,1,1),'-k','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,2,1,1),'-k','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,3,1,1),'-k','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,1,1,1),'--k','linewidth',1); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,2,1,1),'--k','linewidth',1); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,3,1,1),'--k','linewidth',1); hold on
subplot(2,1,1); loglog(Wip,tOS(:,1,2,1),'-b','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,2,2,1),'-b','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,3,2,1),'-b','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,1,2,1),'--b','linewidth',2); hold on

```

```

subplot(2,1,1); loglog(Wip,tOS_SS(:,2,2,1),'--b','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,3,2,1),'--b','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,1,3,1),'-g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,2,3,1),'-g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,3,3,1),'-g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,1,3,1),'--g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,2,3,1),'--g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,3,3,1),'--g','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,1,4,1),'-r','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,2,4,1),'-r','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS(:,3,4,1),'-r','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,1,4,1),'--r','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,2,4,1),'--r','linewidth',2); hold on
subplot(2,1,1); loglog(Wip,tOS_SS(:,3,4,1),'--r','linewidth',2); hold on

grid on
ylim([0.005,50])
set(gca,'ytick',10.^[-2:1:1])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])

subplot(2,1,2); loglog(Wip,tUS(:,1,1,1),'-k','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,2,1,1),'-k','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,3,1,1),'-k','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,1,2,1),'-b','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,2,2,1),'-b','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,3,2,1),'-b','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS_SS(:,1,2,1),'--b','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,1,3,1),'-g','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,2,3,1),'-g','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,3,3,1),'-g','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,1,4,1),'-r','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,2,4,1),'-r','linewidth',2); hold on
subplot(2,1,2); loglog(Wip,tUS(:,3,4,1),'-r','linewidth',2); hold on

grid on
ylim([0.01,10])
set(gca,'ytick',10.^[-2:1:1])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])
print('-dtiff','Figure6','-r500')

```

c Figure 7

```

clear; close all; clc

% SAOS Measurements
% 40D10S
f40D10S = xlsread('Startup.xlsx','B2:B29'); % in [rad/s]
Gp40D10S = xlsread('Startup.xlsx','C2:C29'); % in [Pa]
Gpp40D10S= xlsread('Startup.xlsx','D2:D29'); % in [Pa]
np40D10S = Gpp40D10S./f40D10S; % eta prime
npp40D10S= Gp40D10S ./f40D10S; % eta double prime
% 20D30S
f20D30S = xlsread('Startup.xlsx','F2:F31'); % in [rad/s]
Gp20D30S = xlsread('Startup.xlsx','G2:G31'); % in [Pa]

```

```

Gpp20D30S= xlsread('Startup.xlsx','H2:H31'); % in [Pa]
np20D30S = Gpp20D30S./f20D30S; % eta prime
npp20D30S= Gp20D30S ./f20D30S; % eta double prime

%% SAOS Calculation

% Calculation of 40D10S
omegap40D10S= logspace(-3,4,500); % angular frequency
np40D10ST = zeros(1,numel(omegap40D10S)); % "T" for theory
npp40D10ST = zeros(1,numel(omegap40D10S)); % "T" for theory
alpha40D10S = 20.00; % Spriggs exponent (inf for single mode). Normally it
should range from about 2 to 4.
L140D10S = 1.25; % relaxation time
L240D10S = 0.010; % relaxation time
eta040D10S = 45; % zero shear viscosity
kend = 500; % number of terms in Spriggs
for k = 1:kend
    ep = eta040D10S./zeta(alpha40D10S).*((k^alpha40D10S +
L140D10S*L240D10S*k^(-alpha40D10S).*omegap40D10S.^2)./( k^(2*alpha40D10S) +
L140D10S^2.*omegap40D10S.^2 ));
    epp = eta040D10S./zeta(alpha40D10S).*((L140D10S -
L240D10S).*omegap40D10S.^(2*alpha40D10S)./( k^(2*alpha40D10S
) + L140D10S^2.*omegap40D10S.^2 ));
    np40D10ST = np40D10ST + ep ;
    npp40D10ST = npp40D10ST + epp;
end

% Calculation of 20D30S
omegap20D30S = logspace(-3,3,500); % angular frequency
np20D30ST = zeros(1,numel(omegap20D30S)); % "T" for theory
npp20D30ST= zeros(1,numel(omegap20D30S)); % "T" for theory
alpha20D30S = 20; % Spriggs exponent (inf for single mode). Normally it
should range from about 2 to 4.
L120D30S = 1.52; % relaxation time
L220D30S = 0.005; % relaxation time
eta020D30S = 59.5; % zero shear viscosity
kend = 500; % number of terms in Spriggs
for k = 1:kend
    ep = eta020D30S./zeta(alpha20D30S).*((k^alpha20D30S +
L120D30S*L220D30S*k^(-alpha20D30S).*omegap20D30S.^2)./( k^(2*alpha20D30S) +
L120D30S^2.*omegap20D30S.^2 ));
    epp = eta020D30S./zeta(alpha20D30S).*((L120D30S -
L220D30S).*omegap20D30S.^(2*alpha20D30S)./( k^(2*alpha20D30S
) + L120D30S^2.*omegap20D30S.^2 ));
    np20D30ST = np20D30ST + ep ;
    npp20D30ST = npp20D30ST + epp;
end

%% Figure plotting, LVE
figure
subplot(2,1,1);loglog(L140D10S*f40D10S,np40D10S/eta040D10S , 'ok', 'markersize'
,8, 'linewidth',2); hold on
subplot(2,1,1);loglog(L140D10S*f40D10S,npp40D10S/eta040D10S, 'ob', 'markersize'
,8, 'linewidth',2); hold on
subplot(2,1,1);loglog(L140D10S*omegap40D10S,np40D10ST/eta040D10S , '-
k', 'linewidth',2); hold on
subplot(2,1,1);loglog(L140D10S*omegap40D10S,npp40D10ST/eta040D10S, '-
b', 'linewidth',2); hold on

```

```

grid on
xlim([10^-1,10^1])
ylim([10^-2,10^0])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])

subplot(2,1,2);loglog(L120D30S*f20D30S,np20D30S/eta020D30S , 'ok','markersize'
,8,'linewidth',2); hold on
subplot(2,1,2);loglog(L120D30S*f20D30S,npp20D30S/eta020D30S, 'ob','markersize'
,8,'linewidth',2); hold on
subplot(2,1,2);loglog(L120D30S*omegap20D30S,np20D30ST/eta020D30S , '-
k','linewidth',2); hold on
subplot(2,1,2);loglog(L120D30S*omegap20D30S,npp20D30ST/eta020D30S, '-
b','linewidth',2); hold on

grid on
xlim([10^-1,10^1])
ylim([10^-2,10^0])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])
print('-dtiff','Figure7','-r500')

```

d Figure 8

```

clear; close all; clc

% Read the experimental data
tLVE1= xlsread('Startup.xlsx','J2:J29'); % in [s]
eLVE1= xlsread('Startup.xlsx','K2:K29'); % in [Pa s]
t1 = xlsread('Startup.xlsx','L3:L239'); % in [s]
eta1 = xlsread('Startup.xlsx','N3:N239'); % in [Pa s]
t2 = xlsread('Startup.xlsx','O3:O235'); % in [s]
eta2 = xlsread('Startup.xlsx','Q3:Q235'); % in [Pa s]
t3 = xlsread('Startup.xlsx','R3:R233'); % in [s]
eta3 = xlsread('Startup.xlsx','T3:T233'); % in [Pa s]
tLVE2= xlsread('Startup.xlsx','U2:U31'); % in [s]
eLVE2= xlsread('Startup.xlsx','V2:V31'); % in [Pa s]
t8 = xlsread('Startup.xlsx','W3:W249'); % in [s]
eta8 = xlsread('Startup.xlsx','Y3:Y249'); % in [Pa s]
t9 = xlsread('Startup.xlsx','Z3:Z241'); % in [s]
eta9 = xlsread('Startup.xlsx','AB3:AB241'); % in [Pa s]
t10 = xlsread('Startup.xlsx','AC3:AC231'); % in [s]
eta10= xlsread('Startup.xlsx','AE3:AE231'); % in [Pa s]

%% Calculated Values
nospectra = 100; % number of spectra

% 40D 10S
% The longest mode of fluid
l1Mat1 = +1.25; % relaxation time
l2Mat1 = +0.01; % retardation time
m0Mat1 = +0.20; % mu0
m1Mat1 = +0.00; % mu1
m2Mat1 = +0.00; % mu2
n1Mat1 = -2.05; % nu1
n2Mat1 = -1.03; % nu2
e0Mat1 = 45; % zeroshear viscosity

```

```

% ith spectrum
alphaMat1 = 15; % the Spriggs exponent
l1i = l1Mat1*[1:nospectra].^-alphaMat1; % relaxation time
l2i = l2Mat1*[1:nospectra].^-alphaMat1; % retardation time
m0i = m0Mat1*[1:nospectra].^-alphaMat1; % mu0
m1i = m1Mat1*[1:nospectra].^-alphaMat1; % mu1
m2i = m2Mat1*[1:nospectra].^-alphaMat1; % mu2
n1i = n1Mat1*[1:nospectra].^-alphaMat1; % nu1
n2i = n2Mat1*[1:nospectra].^-alphaMat1; % nu2
ei = e0Mat1/sum(l1i)*l1i; % zeroshear viscosity

taup = logspace(-2,2,200);
gdot0 = [0 0.5 1 3 5 7 10 20];
etaplus1 = zeros(numel(gdot0),numel(taup)); % kth eta plus
etaplusmulti = zeros(numel(gdot0),numel(taup)); % eta plus of the system
for is = 1:nospectra
    sigmalMat1 = (l1i(is)*l1i(is) + m0i(is)*(m1i(is) - (3/2)*n1i(is)) -
    m1i(is)*(m1i(is) - n1i(is)))/l1i(is)^2; % Dimensionless sigma1
    sigma2Mat1 = (l1i(is)*l2i(is) + m0i(is)*(m2i(is) - (3/2)*n2i(is)) -
    m1i(is)*(m2i(is) - n2i(is)))/l1i(is)^2; % Dimensionless sigma2
    for ig0 = 1:numel(gdot0)
        Wi = l1i(is)*gdot0(ig0); %kth Wiessenber number
        etaplusa =
        (1+sigma2Mat1*Wi^2)/(1+sigmalMat1*Wi^2)*cos(sqrt(sigmalMat1)*Wi*taup);
        etaplusb = (1/sqrt(sigmalMat1))*(sigmalMat1-
        sigma2Mat1)/(1+sigmalMat1*Wi^2)*Wi*sin(sqrt(sigmalMat1)*Wi*taup);
        etaplusc = (1+sigma2Mat1*Wi^2)/(1+sigmalMat1*Wi^2);
        etaplus1(ig0,:) = ei(is)*( -exp(-taup).*(etaplusa - etaplusb) +
        etaplusc); % Dimensional eta plus
    end
    etaplusmulti = etaplusmulti + etaplus1; % Sum all of the kth
end

% 20D 30S
l1Mat2 = +1.52; % relaxation time
l2Mat2 = +0.005; % retardation time
m0Mat2 = 0.005; % mu0
m1Mat2 = +0.00; % mu1
m2Mat2 = +0.00; % mu2
n1Mat2 = -0.305; % nu1
n2Mat2 = -1.13; % nu2
e0Mat2 = 59.5; % zeroshear viscosity

sigmalMat2 = (l1Mat2*l1Mat2 + m0Mat2*(m1Mat2 - (3/2)*n1Mat2) -
m1Mat2*(m1Mat2 - n1Mat2))/l1Mat2^2; % Dimensionless sigma1
sigma2Mat2 = (l1Mat2*l2Mat2 + m0Mat2*(m2Mat2 - (3/2)*n2Mat2) -
m1Mat2*(m2Mat2 - n2Mat2))/l1Mat2^2; % Dimensionless sigma2
Wip = l1Mat2*[0 0.4 0.6 1 3 5 10 30];
etaplus2 = zeros(numel(Wip),numel(taup));
for iWip = 1:numel(Wip)
    Wi = Wip(iWip);
    etaplusa =
    (1+sigma2Mat2*Wi^2)/(1+sigmalMat2*Wi^2)*cos(sqrt(sigmalMat2)*Wi*taup);
    etaplusb = (1/sqrt(sigmalMat2))*(sigmalMat2-
    sigma2Mat2)/(1+sigmalMat2*Wi^2)*Wi*sin(sqrt(sigmalMat2)*Wi*taup);
    etaplusc = -(1+sigma2Mat2*Wi^2)/(1+sigmalMat2*Wi^2);
    etaplus2(iWip,:) = -(exp(-taup).*(etaplusa - etaplusb) + etaplusc);
end

```



```

end

%% Figure Plotting
% 40D10S, measured values
subplot(2,1,1);
semilogx(tLVE1/l1Mat1, eLVE1/e0Mat1, 'sr', 'linewidth', 2, 'markersize', 10); hold on % LVE
subplot(2,1,1);
semilogx(t1(1:4:end)/l1Mat1, eta1(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersize', 10); hold on % gdot = 0.5 s^-1
subplot(2,1,1);
semilogx(t2(1:4:end)/l1Mat1, eta2(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersize', 10); hold on % gdot = 1 s^-1
subplot(2,1,1);
semilogx(t3(1:4:end)/l1Mat1, eta3(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersize', 10); hold on % gdot = 3 s^-1
% 40D 10S, model predictions
subplot(2,1,1); semilogx(taup, etaplusmulti(1,:)/e0Mat1, '-r', 'linewidth', 4);
hold on % LVE
subplot(2,1,1); semilogx(taup, etaplusmulti(2,:)/e0Mat1, '-r', 'linewidth', 2);
hold on % gdot = 0.5 s^-1
subplot(2,1,1); semilogx(taup, etaplusmulti(3,:)/e0Mat1, '-r', 'linewidth', 2);
hold on % gdot = 1 s^-1
subplot(2,1,1); semilogx(taup, etaplusmulti(4,:)/e0Mat1, '-r', 'linewidth', 2);
hold on % gdot = 3 s^-1
grid on
xlim([10^-1, 10^2])
ylim([0, 1])
set(gca, 'ytick', [-0.1 0:0.2:1])
set(gca, 'fontname', 'Palatino', 'fontsize', 12)
set(gcf, 'paperposition', [1 1 5 5])

% 20D 30S, measured values
subplot(2,1,2);
semilogx(tLVE2(1:end)/l1Mat2, eLVE2(1:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on % LVE
subplot(2,1,2); semilogx(t8(1:4:end)/l1Mat2, eta8(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on % gdot = 0.4 s^-1
subplot(2,1,2); semilogx(t9(1:4:end)/l1Mat2, eta9(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on % gdot = 0.6 s^-1
subplot(2,1,2); semilogx(t10(1:4:end)/l1Mat2, eta10(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on % gdot = 1 s^-1
% 20D 30S, model predictions
subplot(2,1,2); semilogx(taup, etaplus2(1,:), '-k', 'linewidth', 2); hold on % LVE
subplot(2,1,2); semilogx(taup, etaplus2(2,:), '-k', 'linewidth', 2); hold on % gdot = 0.4 s^-1
subplot(2,1,2); semilogx(taup, etaplus2(3,:), '-k', 'linewidth', 2); hold on % gdot = 0.6 s^-1
subplot(2,1,2); semilogx(taup, etaplus2(4,:), '-k', 'linewidth', 2); hold on % gdot = 1 s^-1
grid on
xlim([10^-1, 10^2])
ylim([0, 1])
set(gca, 'ytick', [-0.1 0:0.2:1])

```

```

set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])
print('-dtiff','Figure8','-r500')

```

e Figure 9

```
clear; close all; clc
```

```

% Read the experimental data
tLVE1= xlsread('Startup.xlsx','J2:J29'); % in [s]
eLVE1= xlsread('Startup.xlsx','K2:K29'); % in [Pa s]
t1 = xlsread('Startup.xlsx','L3:L239'); % in [s]
eta1 = xlsread('Startup.xlsx','N3:N239'); % in [Pa s]
t2 = xlsread('Startup.xlsx','O3:O235'); % in [s]
eta2 = xlsread('Startup.xlsx','Q3:Q235'); % in [Pa s]
t3 = xlsread('Startup.xlsx','R3:R233'); % in [s]
eta3 = xlsread('Startup.xlsx','T3:T233'); % in [Pa s]
tLVE2= xlsread('Startup.xlsx','U2:U31'); % in [s]
eLVE2= xlsread('Startup.xlsx','V2:V31'); % in [Pa s]
t8 = xlsread('Startup.xlsx','W3:W249'); % in [s]
eta8 = xlsread('Startup.xlsx','Y3:Y249'); % in [Pa s]
t9 = xlsread('Startup.xlsx','Z3:Z241'); % in [s]
eta9 = xlsread('Startup.xlsx','AB3:AB241'); % in [Pa s]
t10 = xlsread('Startup.xlsx','AC3:AC231'); % in [s]
eta10= xlsread('Startup.xlsx','AE3:AE231'); % in [Pa s]

%% Calculated Values
nospectra = 100; % number of spectra

% 40D 10S
% The longest mode of fluid
l1Mat1 = +1.25; % relaxation time
l2Mat1 = +0.01; % retardation time
m0Mat1 = +0.20; % mu0
m1Mat1 = +0.00; % mu1
m2Mat1 = +0.00; % mu2
n1Mat1 = -2.05; % nu1
n2Mat1 = -1.03; % nu2
e0Mat1 = 45; % zero-shear viscosity

taup = logspace(-2,2,200);
sigma1Mat1 = (l1Mat1*l1Mat1 + m0Mat1*(m1Mat1 - (3/2)*n1Mat1) - m1Mat1*(m1Mat1 - n1Mat1))/l1Mat1^2; % Dimensionless sigma1
sigma2Mat1 = (l1Mat1*l2Mat1 + m0Mat1*(m2Mat1 - (3/2)*n2Mat1) - m1Mat1*(m2Mat1 - n2Mat1))/l1Mat1^2; % Dimensionless sigma2
Wip = l1Mat1*[0 0.5 1 3 5 7 10 20];
etaplus1 = zeros(numel(Wip),numel(taup)); % kth eta plus
for iWip = 1:numel(Wip)
    Wi = Wip(iWip);
    etaplusa = (1+sigma2Mat1*Wi^2)/(1+sigma1Mat1*Wi^2)*cos(sqrt(sigma1Mat1)*Wi*taup);
    etaplusb = (1/sqrt(sigma1Mat1))*(sigma1Mat1 - sigma2Mat1)/(1+sigma1Mat1*Wi^2)*Wi*sin(sqrt(sigma1Mat1)*Wi*taup);
    etaplusc = -(1+sigma2Mat1*Wi^2)/(1+sigma1Mat1*Wi^2)*(taup./taup);
    etaplus1(iWip,:) = -(exp(-taup).*(etaplusa - etaplusb) + etaplusc);
end

```

```

% 20D 30S
l1Mat2 = +1.52; % relaxation time
l2Mat2 = +0.005; % retardation time
m0Mat2 = 0.005; % mu0
m1Mat2 = +0.00; % mu1
m2Mat2 = +0.00; % mu2
n1Mat2 = -0.305; % nu1
n2Mat2 = -1.13; % nu2
e0Mat2 = 59.5; % zero-shear viscosity

sigma1Mat2 = (l1Mat2*l1Mat2 + m0Mat2*(m1Mat2 - (3/2)*n1Mat2) -
m1Mat2*(m1Mat2 - n1Mat2))/l1Mat2^2; % Dimensionless sigma1
sigma2Mat2 = (l1Mat2*l2Mat2 + m0Mat2*(m2Mat2 - (3/2)*n2Mat2) -
m1Mat2*(m2Mat2 - n2Mat2))/l1Mat2^2; % Dimensionless sigma2
Wip = l1Mat2*[0 0.4 0.6 1 3 5 10 30];
etaplus2 = zeros(numel(Wip),numel(taup));
for iWip = 1:numel(Wip)
    Wi = Wip(iWip);
    etaplusa =
(1+sigma2Mat2*Wi^2)/(1+sigma1Mat2*Wi^2)*cos(sqrt(sigma1Mat2)*Wi*taup);
    etaplusb = (1/sqrt(sigma1Mat2))*(sigma1Mat2-
sigma2Mat2)/(1+sigma1Mat2*Wi^2)*Wi*sin(sqrt(sigma1Mat2)*Wi*taup);
    etaplusc = -(1+sigma2Mat2*Wi^2)/(1+sigma1Mat2*Wi^2);
    etaplus2(iWip,:) = -(exp(-taup).*(etaplusa - etaplusb) + etaplusc);
end

% %% Figure Plotting
% 40D 10S, measured values
subplot(2,1,1);
semilogx(tLVE1/l1Mat1, eLVE1/e0Mat1, 'sr', 'linewidth', 2, 'markersi
ze', 10); hold on % LVE
subplot(2,1,1);
semilogx(t1(1:4:end)/l1Mat1, eta1(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersi
ze', 10); hold on % gdot = 0.5 s^-1
subplot(2,1,1);
semilogx(t2(1:4:end)/l1Mat1, eta2(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersi
ze', 10); hold on % gdot = 1 s^-1
subplot(2,1,1);
semilogx(t3(1:4:end)/l1Mat1, eta3(1:4:end)/e0Mat1, 'sr', 'linewidth', 2, 'markersi
ze', 10); hold on % gdot = 3 s^-1
% 40D 10S, model predictions
subplot(2,1,1); semilogx(taup, etaplus1(1,:), '-r', 'linewidth', 4); hold on %
LVE
subplot(2,1,1); semilogx(taup, etaplus1(2,:), '-r', 'linewidth', 2); hold on %
gdot = 0.5 s^-1
subplot(2,1,1); semilogx(taup, etaplus1(3,:), '-r', 'linewidth', 2); hold on %
gdot = 1 s^-1
subplot(2,1,1); semilogx(taup, etaplus1(4,:), '-r', 'linewidth', 2); hold on %
gdot = 3 s^-1
grid on
xlim([10^-1, 10^2])
ylim([0, 1])
set(gca, 'ytick', [-0.1 0:0.2:1])
set(gca, 'fontname', 'Palatino', 'fontsize', 12)
set(gcf, 'paperposition', [1 1 5 5])

% 20D 30S, measured values
subplot(2,1,2);

```

```

semilogx(tLVE2(1:end)/l1Mat2, eLVE2(1:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on % LVE
subplot(2,1,2); semilogx(t8(1:4:end)/l1Mat2, eta8(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on %
gdot = 0.4 s-1
subplot(2,1,2); semilogx(t9(1:4:end)/l1Mat2, eta9(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'markersize', 8); hold on %
gdot = 0.6 s-1
subplot(2,1,2);
semilogx(t10(1:4:end)/l1Mat2, eta10(1:4:end)/e0Mat2, 'ok', 'linewidth', 2, 'marker
size', 8); hold on % gdot = 1 s-1
% 20D 30S, model predictions
subplot(2,1,2); semilogx(taup, etaplus2(1,:), '-k', 'linewidth', 2); hold on %
LVE
subplot(2,1,2); semilogx(taup, etaplus2(2,:), '-k', 'linewidth', 2); hold on %
gdot = 0.4 s-1
subplot(2,1,2); semilogx(taup, etaplus2(3,:), '-k', 'linewidth', 2); hold on %
gdot = 0.6 s-1
subplot(2,1,2); semilogx(taup, etaplus2(4,:), '-k', 'linewidth', 2); hold on %
gdot = 1 s-1
grid on
xlim([10-1, 102])
ylim([0, 1])
set(gca, 'ytick', [-0.1 0:0.2:1])
set(gca, 'fontname', 'Palatino', 'fontsize', 12)
set(gcf, 'paperposition', [1 1 5 5])
print('-dtiff', 'Nonlinear', '-r500')

```

f Figure 10

```

clear; close all; clc
% Read the experimental data
tLVE1= xlsread('Startup.xlsx', 'J2:J29'); % in [s]
eLVE1= xlsread('Startup.xlsx', 'K2:K29'); % in [Pa s]
t1 = xlsread('Startup.xlsx', 'L3:L239'); % in [s]
eta1 = xlsread('Startup.xlsx', 'N3:N239'); % in [Pa s]
t2 = xlsread('Startup.xlsx', 'O3:O235'); % in [s]
eta2 = xlsread('Startup.xlsx', 'Q3:Q235'); % in [Pa s]
t3 = xlsread('Startup.xlsx', 'R3:R233'); % in [s]
eta3 = xlsread('Startup.xlsx', 'T3:T233'); % in [Pa s]
tLVE2= xlsread('Startup.xlsx', 'U2:U31'); % in [s]
eLVE2= xlsread('Startup.xlsx', 'V2:V31'); % in [Pa s]
t8 = xlsread('Startup.xlsx', 'W3:W249'); % in [s]
eta8 = xlsread('Startup.xlsx', 'Y3:Y249'); % in [Pa s]
t9 = xlsread('Startup.xlsx', 'Z3:Z241'); % in [s]
eta9 = xlsread('Startup.xlsx', 'AB3:AB241'); % in [Pa s]
t10 = xlsread('Startup.xlsx', 'AC3:AC231'); % in [s]
eta10= xlsread('Startup.xlsx', 'AE3:AE231'); % in [Pa s]

% Screening for the last points of each curve
etaSSMat1 = [eLVE1(end) eta1(end) eta2(end) eta3(end)];
gdotMat1 = [0.05 0.500 1.25 3.00];
etaSSMat2 = [eLVE2(end) eta8(end) eta9(end) eta10(end)];
gdotMat2 = [0.05 0.400 0.60 1.00];

Wip = logspace(-2, 1, 50); % Weissenberg Number
% 40D 10S

```

```

l1Mat1 = +1.25;      % relaxation time
sigma1Mat1 = 2.19;   % sigma2 40D10S
sigma2Mat1 = 0.329;  % sigma1 20D30S
e0Mat1 = 45;         % zero-shear viscosity
% Calculated steady shear viscosity, (T for Theory)
etaSSMat1T = (1 + (sigma2Mat1/l1Mat1)*Wip.^2) ./ (1 +
(sigma1Mat1/l1Mat1)*Wip.^2);

% 20D 30S
l1Mat2 = +1.52; % relaxation time
sigma1Mat2 = 2.33;
sigma2Mat2 = 0.0931;
e0Mat2 = 59.5;      % zero-shear viscosity
% Calculated steady shear viscosity, (T for Theory)
etaSSMat2T = (1 + (sigma2Mat2/l1Mat2)*Wip.^2) ./ (1 +
(sigma1Mat2/l1Mat2)*Wip.^2);

figure
subplot(2,1,1);
semilogx(l1Mat1*gdotMat1,etaSSMat1/e0Mat1,'or','linewidth',2,'markersize',10)
; hold on % LVE
subplot(2,1,1); semilogx(Wip,etaSSMat1T,'-r','linewidth',2)
set(gca,'ytick',[-0.1 0:0.2:1])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])

% 20D 30S, measured values
subplot(2,1,2);
semilogx(l1Mat2*gdotMat2,etaSSMat2/e0Mat2,'ok','linewidth',2,'markersize',10)
; hold on % LVE
subplot(2,1,2); semilogx(Wip,etaSSMat2T,'-k','linewidth',2)
set(gca,'ytick',[-0.1 0:0.2:1])
set(gca,'fontname','Palatino','fontsize',12)
set(gcf,'paperposition',[1 1 5 5])
print('-dtiff','Figure10','-r500')

```

II SPRIGGS RELATIONS

This section provides detailed analysis of how to employ the Spriggs relations for multimode fluids. For multimode fluid, the ij component of the extra stress tensor is:

$$\tau_{ij} = \sum_{k=1}^{\infty} \tau_{ij,k} \quad (1)$$

where, for the Oldroyd 8-constant fluid, each of the k th shear stress spectrum obeys:

$$\begin{aligned} \tau_k + \lambda_1 \frac{\mathcal{D} \tau_k}{\mathcal{D} t} + \frac{1}{2} \mu_0 (\text{tr} \tau_k) \dot{\gamma}_k - \frac{1}{2} \mu_1 \{ \tau_k \cdot \dot{\gamma}_k + \dot{\gamma}_k \cdot \tau_k \} + \frac{1}{2} \nu_1 (\tau_k : \dot{\gamma}_k) \delta \\ = -\eta_0 \left(\dot{\gamma}_k + \lambda_2 \frac{\mathcal{D} \dot{\gamma}_k}{\mathcal{D} t} - \mu_2 \{ \dot{\gamma}_k \cdot \dot{\gamma}_k \} + \frac{1}{2} \nu_2 (\dot{\gamma}_k : \dot{\gamma}_k) \delta \right) \end{aligned} \quad (2)$$

Substituting the following:

$$\tau_{yx,k} = -\eta_k^+ \dot{\gamma}; \quad \tau_{xx,k} - \tau_{yy,k} = -\Psi_{1,k}^+ \dot{\gamma}^2; \quad \tau_{yy,k} - \tau_{zz,k} = -\Psi_{2,k}^+ \dot{\gamma}^2 \quad (3)$$

into Eq. (1), we find:

$$\tau_{yx} = \dot{\gamma} \sum_{k=1}^{\infty} -\eta_k^+; \quad \tau_{xx} - \tau_{yy} = -\dot{\gamma}^2 \sum_{k=1}^{\infty} \Psi_{1,k}^+; \quad \tau_{yy} - \tau_{zz} = -\dot{\gamma}^2 \sum_{k=1}^{\infty} \Psi_{2,k}^+ \quad (4)$$

or:

$$\eta^+ = \sum_{k=1}^{\infty} -\eta_k^+; \quad \Psi_1^+ = \sum_{k=1}^{\infty} \Psi_{1,k}^+; \quad \Psi_2^+ = \sum_{k=1}^{\infty} \Psi_{2,k}^+ \quad (5),(6),(7)$$

Eqs. (5)–(7) define the material functions of the multimode Oldroyd 8-constant framework in terms of their k th mode.