Extended cross decomposition for mixed-integer linear programs with strong and weak linking constraints [☆]

Emmanuel Ogbe^a, Xiang Li^{a,*}

^a*Department of Chemical Engineering, Queen's University, 19 Division st., Kingston, ON K7L 3N6*

Abstract

Large-scale mixed-integer linear programming (MILP) problems, such as those from two-stage stochastic programming, usually have a decomposable structure that can be exploited to design efficient optimization methods. Classical Benders decomposition can solve MILPs with weak linking constraints (which are decomposable when linking variables are fixed) but not strong linking constraints (which are not decomposable even when linking variables are fixed). In this paper, we first propose a new rigorous bilevel decomposition strategy for solving MILPs with strong and weak linking constraints, then extend a recently developed cross decomposition method based on this strategy. We also show how to apply the extended cross decomposition method to two-stage stochastic programming problems with conditional-value-at-risk (CVaR) constraints. In the case studies, we demonstrate the significant computational advantage of the proposed extended cross decomposition method as well as the benefit of including CVaR constraints in stochastic programming.

Keywords: Cross decomposition; Benders decomposition; Dantzig-Wolfe decomposition; Risk-averse stochastic programming; Mixed-integer linear programming.

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*Corresponding author

Email address: xiang.li@queensu.ca (Xiang Li)

9 1. Introduction

This paper aims at developing an efficient decomposition method to solve large-scale mixed-integer linear programming (MILP) problems in the following form:

$$\begin{aligned}
 & \min_{x_0, x} c_0^T x_0 + c^T x \\
 & \text{s.t. } B_1 x_0 + A_1 x \leq b_1, \\
 & \quad B_2 x_0 + A_2 x \leq b_2, \\
 & \quad x \in X, \\
 & \quad x_0 \in X_0,
 \end{aligned} \tag{P}$$

10 where variables $x_0 \in \{(x_i, x_c) : x_i \in \mathbb{Z}^{n_i}, x_c \in \mathbb{R}^{n_c}\}$, $x \in \mathbb{R}^{n_x}$, parameters $b_1 \in \mathbb{R}^{m_1}$,
 11 $b_2 \in \mathbb{R}^{m_2}$, and other parameters have conformable dimensions. The first two groups of
 12 constraints in (P) are called *linking constraints* in the sense that without them (P) can be
 13 decomposed into a number of small problems that are much easier to solve. In other words,
 14 linking constraints hinder the decomposition of (P). Among the linking constraints, the
 15 second group of constraints are called *weak linking constraints* in the sense that they do not
 16 hinder the decomposition of (P) if x_0 has fixed values, and the first group of constraints are
 17 called *strong linking constraints* in the sense that they hinder the decomposition of (P) even
 18 if x_0 has fixed values. Variables in x_0 are called *linking variables*. Sets X and X_0 are defined
 19 by linear constraints, and they are assumed to be nonempty and bounded throughout the
 20 paper for convenience of discussion.

Many realistic engineering optimization problems can be formulated in form of (P). A typical example is optimization under uncertainty through two-stage stochastic programming, which has been adopted in many areas of process systems engineering, such as supply chain optimization [1], natural gas network design and operation [2], refinery planning [3], expansion of chemical processes [4], etc. In two-stage stochastic programming, the first-stage decisions are to be implemented before the realization of uncertainty, and the second-stage (or recourse) decisions are made to satisfy the second-stage (or recourse) problem for all scenarios addressed by the problem formulation. In the context of two-stage stochastic pro-

gramming (rigorously speaking, its deterministic equivalent program [5]), Problem (P) can be written as:

$$\begin{aligned}
& \min_{x_0, x_1, \dots, x_s} \sum_{\omega=1}^s (c_{0,\omega}^T x_0 + c_{\omega}^T x_{\omega}) \\
& \text{s.t. } B_1 x_0 + \sum_{\omega=1}^s A_{1,\omega} x_{\omega} \leq b_1, \quad \textcircled{1} \\
& B_{2,\omega} x_0 + A_{2,\omega} x_{\omega} \leq b_{2,\omega}, \quad \omega \in \{1, \dots, s\}, \quad \textcircled{2} \\
& x_0 \in X_0, \\
& x_{\omega} \in X_{\omega}, \quad \omega \in \{1, \dots, s\},
\end{aligned} \tag{SP}$$

where x_0 includes the first-stage variables, x_{ω} includes the second-stage variables for scenario ω , and there are totally s scenarios addressed. $\textcircled{1}$ represents strong linking constraints that include x_{ω} for all scenarios (and therefore hinder the decomposition even when x_0 is fixed). $\textcircled{2}$ represents weak linking constraints, each linking first-stage variables and second-stage variables for one scenario. Figure 1 illustrates the structure of Problem (SP). Traditional two-stage stochastic programming formulation (which minimizes an expected cost) includes weak linking constraints but not strong linking constraints. Recently, two variants of two-stage stochastic programming formulations have attracted more attention and both of them include strong linking constraints. One variant is risk-averse two-stochastic programming [6] [7] [8] [9], which not only minimizes an expected cost, but also minimizes or limits some risk measure of loss, such as conditional value-at-risk (CVaR) [10]. When the risk measure is bounded in the formulation, the loss in different scenarios need to appear in a same constraint, which is a strong linking constraint. The other variant is chance-constrained two-stage stochastic programming [11] [12] that integrates the notion of chance constrained programming [13] into the two-stage formulation. In this formulation, the first-stage decisions are made such that the probability of the recourse problem being feasible is larger than a predefined level. The probability requirement is modeled with a constraint that includes all scenarios, so this constraint is a strong linking constraint.

When Problem (P) does not include strong linking constraints, it can be efficiently solved by Benders decomposition (BD) [14] (or called L-shaped method [15] when applied

Figure 1: Block structure of constraint ① and ② in Problem (SP)

to Problem (SP)). BD is efficient because, rather than solving the original problem directly, it solves a sequence of upper bounding problems with fixed x_0 values and a sequence of lower bounding problems that do not include x . Both upper bounding and lower bounding problems are much easier to solve than the original problem. However, when strong linking constraints are present, the upper bounding problems are not decomposable and BD does not have computational advantage. On the one hand, Dantzig-Wolfe decomposition (DWD) [16] [17] or Lagrangian decomposition [18] [19] can be applied to exploit the structure of Problem (P) by dualizing the weak and/or strong linking constraints. However, these methods are rigorous only if Problem (P) has zero dual gap, which is very unlikely considering the problem includes integer variables. While new decomposition methods are needed for solving Problem (P) with both strong linking and weak linking constraints, the relevant research is rarely seen in the literature. The existing ideas for solving Problem (P) include the modification of Benders decomposition method that yields decomposable upper bounding problems [20] [11] and the combination of BD and Lagrangian decomposition [20].

The major contribution of this paper is the development of a new decomposition method that is able to efficiently solve Problem (P), based on a novel cross decomposition (CD) method recently developed [21]. The idea of CD was first proposed by Van Roy [22] in 1983; it combines BD with Lagrangian decomposition in order to achieve improved efficiency. Several variants of CD method has then been developed (e.g., [23] [24]), and recently, we

60 have developed a novel variant of CD that combines BD and DWD [21]. There are two
 61 major advantages of our CD method. One is that the upper bound of the problem can be
 62 updated by not only the BD upper bounding problems, but also the DWD upper bounding
 63 problems (while CD methods using Lagrangian decomposition update the upper bound only
 64 through BD upper bounding problems). The other is that problem infeasibility is handled
 65 in a systematic way. However, any CD method (including the one we developed before)
 66 requires that the problem can benefit from BD, which is not the case when strong linking
 67 constraints are present. Therefore, in this paper, we extend our CD method in order to
 68 deal with the strong linking constraints in an efficient way. The key idea is to add another
 69 DWD loop in the main CD procedure, so that subproblems coupled by the strong linking
 70 constraints can be further decomposed. Furthermore, the additional DWD loop can provide
 71 extra cuts and columns to accelerate the main CD procedure.

72 The remaining part of the paper is structured as follows. In Section 2, we propose a novel
 73 bilevel decomposition strategy for solving Problem (P) and prove its validity. The discussion
 74 of this strategy motivates the extension of CD. In Section 3, we present extended CD method,
 75 which includes a Phase I procedure that prevents infeasible DWD subproblems. In section
 76 4, we discuss in detail how to apply the proposed extended CD method to CVaR constrained
 77 two-stage stochastic programming. In section 5, the advantages of CVaR constrained two-
 78 stage stochastic programming and the proposed solution method are demonstrated via three
 79 case study problems adapted from the literature. Relevant conclusions are duly drawn in
 80 Section 6.

81 **2. A bilevel decomposition strategy for (P)**

82 The extended cross decomposition is motivated by a bilevel decomposition strategy. In
 83 this strategy, Problem (P) is solved by BD via viewing x_0 as complicating variables, and
 84 the BD upper bounding problems (or called primal problems), which include strong linking
 85 constraints and cannot be directly decomposed, are solved by DWD. Figure 2 illustrate the
 86 bilevel decomposition. Here we index the upper level iterations by k and the lower level
 87 iterations for each upper level iteration by l .

Figure 2: A bilevel decomposition strategy combing BD and DWD

88 2.1. The upper level decomposition

In the upper level, BD is used to solve Problem (P). At each iteration k , x_0 is fixed to a constant x_0^k , and the following Benders feasibility problem is solved:

$$\begin{aligned}
 obj_{BFP^k} &= \min_{x, z_1 \geq 0, z_2 \geq 0} \|z_1\| + \|z_2\| \\
 \text{s.t. } & A_1 x \leq b_1 - B_1 x_0^k + z_1, \\
 & A_2 x \leq b_2 - B_2 x_0^k + z_2, \\
 & x \in X,
 \end{aligned} \tag{BFP^k}$$

where z_1, z_2 are slack variables, and $\|\cdot\|$ denotes any norm function. If $obj_{BFP^k} > 0$, then Problem (P) is infeasible for $x_0 = x_0^k$, and a BD feasibility cut will be generated. Otherwise, the following Benders primal problem will be solved and a BD optimality cut will be generated:

$$\begin{aligned}
obj_{BPP^k} = \min_x \quad & c^T x + c_0^T x_0^k \\
\text{s.t.} \quad & A_1 x \leq b_1 - B_1 x_0^k, \\
& A_2 x \leq b_2 - B_2 x_0^k, \\
& x \in X.
\end{aligned} \tag{BPP^k}$$

Note that obj_{BPP^k} is a valid upper bound for Problem (P). Then the following lower bounding problem (called BD relaxed master problem) is solved at iteration k :

$$\begin{aligned}
\min_{x_0, \eta} \quad & \eta \\
\text{s.t.} \quad & \eta \geq obj_{BPP^i} + \left(c_0^T + (\lambda_1^i)^T B_1 \right) (x_0 - x_0^i) + (\lambda_2^i)^T B_2 (x_0 - x_0^i), \quad \forall i \in S_{opt}^k, \\
& 0 \geq obj_{BFP^i} + (\mu_1^i)^T B_1 (x_0 - x_0^i) + (\mu_2^i)^T B_2 (x_0 - x_0^i), \quad \forall i \in S_{feas}^k, \\
& x_0 \in X_0.
\end{aligned} \tag{BRMP-Std^k}$$

89 Here the first group of constraints are optimality cuts generated in previous iterations i
90 at which Problem (BPP ^{i}) is feasible, and λ_1^i, λ_2^i are Lagrange multipliers for the strong
91 and weak linking constraints in (BPP ^{i}), respectively. The second group of constraints are
92 feasibility cuts generated from previous iterations i at which Problem (BPP ^{i}) is infeasible,
93 and μ_1^i, μ_2^i are Lagrange multipliers for the strong and weak linking constraints in (BFP ^{i}),
94 respectively. Readers are referred to [21] for more discussion on (BRMP-Std ^{k}).

95 2.2. The lower level decomposition

Problem (BPP ^{k}) or (BFP ^{k}) is not naturally decomposable because of the strong linking constraints, but it can be solved by a DWD procedure for efficient optimization. Specifically,

(BPP^k) can be solved by iteratively solving the following two subproblems:

$$\begin{aligned}
obj_{DWRMP-R^{k,l}} = \min_{\theta^0, \dots, \theta^{l-1} \geq 0} & c^T \left(\sum_{i=0}^{l-1} \theta^i x^{k,i} \right) + c_0^T x_0^k \\
\text{s.t. } & A_1 \left(\sum_{i=0}^{l-1} \theta^i x^{k,i} \right) \leq b_1 - B_1 x_0^k, \\
& A_2 \left(\sum_{i=0}^{l-1} \theta^i x^{k,i} \right) \leq b_2 - B_2 x_0^k, \\
& \sum_{i=0}^{l-1} \theta^i = 1.
\end{aligned} \tag{DWRMP-R^{k,l}}$$

$$\begin{aligned}
obj_{DWPP-R^{k,l}} = \min_x & c^T x + \left(\lambda_1^{k,l} \right)^T A_1 x \\
\text{s.t. } & A_2 x \leq b_2 - B_2 x_0^k, \\
& x \in X.
\end{aligned} \tag{DWPP-R^{k,l}}$$

Problem (DWRMP-R^{k,l}) represents the Dantzig-Wolfe restricted master problem and Problem (DWPP-R^{k,l}) represents the Dantzig-Wolfe pricing problem. l indexes the DWD iteration. In (DWRMP-R^{k,l}), $x^{k,i}$ represents the solution of (DWPP-R^{k,i}) that is obtained in a previous DWD iteration $i(< l)$, and it is often called a column. When $l = 1$, an initial column $x^{k,0}$ is needed to construct (DWRMP-R^{k,l}). This initial column can be obtained at the beginning of the algorithm, via solving the following initial pricing problem:

$$\begin{aligned}
& \min_x c^T x \\
& \text{s.t. } x \in X.
\end{aligned} \tag{IPP}$$

96 Let the solution of the above problem be x^0 . In the first DWD iteration for solving a (BFP^k)
97 or (BPP^k), always set $x^{k,0} = x^0$. Lagrange multipliers of the strong linking constraints
98 of (DWRMP-R^{k,l}) are represented by $\lambda_1^{k,l}$, and they are used to construct (DWPP-R^{k,l}).
99 $obj_{DWRMP-R^{k,l}}$ is a valid upper bound of Problem (BPP^k), and $obj_{DWPP-R^{k,l}} + c_0^T x_0^k +$
100 $(\lambda_1^{k,l})^T (B_1 x_0^k - b_1)$ is a valid lower bound of Problem (BPP^k). The DWD procedure reaches
101 an optimal solution of Problem (BPP^k) when the upper and lower bounds converge. Readers

102 are referred to [21] for more explanation on DWD.

Problem (BFP^k) can also be solved by a similar DWD procedure. The DWD restricted master problem is:

$$\begin{aligned}
obj_{\text{DWFRMP-R}^{k,l}} = & \min_{\theta^0, \dots, \theta^{l-1}, z_1, z_2 \geq 0} \|z_1\| + \|z_2\| \\
\text{s.t. } & A_1 \left(\sum_{i=0}^{l-1} \theta^i x^{k,i} \right) \leq b_1 - B_1 x_0^k + z_1, \\
& A_2 \left(\sum_{i=0}^{l-1} \theta^i x^{k,i} \right) \leq b_2 - B_2 x_0^k + z_2, \\
& \sum_{i=0}^{l-1} \theta^i = 1.
\end{aligned} \tag{DWFRMP-R}^{k,l}$$

Let $\mu_1^{k,l}, \mu_2^{k,l}$ be Lagrange multipliers for the strong and weak linking constraints of $(\text{DWFRMP-R}^{k,l})$, respectively, then the DWD pricing problem can be expressed as:

$$\begin{aligned}
obj_{\text{DWFP-R}^{k,l}} = & \min_{x, z_2 \geq 0} \left(\mu_1^{k,l} \right)^T A_1 x + \|z_2\| \\
\text{s.t. } & A_2 x \leq b_2 - B_2 x_0^k + z_2, \\
& x \in X.
\end{aligned} \tag{DWFP-R}^{k,l}$$

103 The proposed DWD procedure for solving Problem (BFP^k) is similar to but slightly dif-
104 ferent from the Phase I of a standard DWD procedure, so we prove the validity of the
105 procedure via the next two propositions. The first proposition states that the solution of
106 $(\text{DWFP-R}^{k,l})$ yields a valid lower bound of Problem (BFP^k) , and the second proposition
107 states the procedure terminates in a finite number of iterations with an optimal solution to
108 Problem (BFP^k) .

109 **Proposition 1.** $obj_{\text{DWFP-R}^{k,l}} + \left(\mu_1^{k,l} \right)^T (B_1 x_0^k - b_1)$ is a valid lower bound of Problem
110 (BFP^k) .

Proof. Since $(\text{DWFRMP-R}^{k,l})$ cannot be unbounded or infeasible, it must have an optimal

solution. We can express the optimal objective value of (DWFRMP-R^{k,l}) as:

$$\begin{aligned}
& obj_{DWFRMP-R^{k,l}} \\
&= \min_{x \in X^{k,l}, z_1, z_2 \geq 0} (\mu_1^{k,l})^T (A_1 x + B_1 x_0^k - b_1 - z_1) + (\mu_2^{k,l})^T (A_2 x + B_2 x_0^k - b_2 - z_2) + \|z_1\| + \|z_2\| \\
&= \left\{ \min_{x \in X^{k,l}} (\mu_1^{k,l})^T (A_1 x + B_1 x_0^k - b_1) + (\mu_2^{k,l})^T (A_2 x + B_2 x_0^k - b_2) \right\} \\
&\quad + \left\{ \min_{z_1 \geq 0} \|z_1\| - (\mu_1^{k,l})^T z_1 \right\} + \left\{ \min_{z_2 \geq 0} \|z_2\| - (\mu_2^{k,l})^T z_2 \right\},
\end{aligned}$$

111 where set $X^{k,l} = \{x \in \mathbb{R}^{n_x} : x = \sum_{i=0}^{l-1} \theta^i x^i, \sum_{i=0}^{l-1} \theta^i = 1, \theta^i \geq 0, \forall i = 0, \dots, l-1\}$.

Suppose $\exists \hat{z}_1 \geq 0$ such that $\|\hat{z}_1\| - (\mu_1^{k,l})^T \hat{z}_1 \leq -\epsilon$ ($\epsilon > 0$), then $\forall \alpha > 0$, $\|\alpha \hat{z}_1\| - (\mu_1^{k,l})^T \alpha \hat{z}_1 \leq -\alpha \epsilon$, which implies that $\min_{z_1 \geq 0} \|z_1\| - (\mu_1^{k,l})^T z_1 = -\infty$. This contradicts the fact that $obj_{DWFRMP-R^{k,l}}$ is finite. Therefore $\|z_1\| - (\mu_1^{k,l})^T z_1 \geq 0, \forall z_1 \geq 0$, which results in

$$\min_{z_1 \geq 0} \|z_1\| - (\mu_1^{k,l})^T z_1 = 0,$$

(where the minimum value is attained at $z_1 = 0$). Similarly,

$$\min_{z_2 \geq 0} \|z_2\| - (\mu_2^{k,l})^T z_2 = 0.$$

According to weak duality of Problem (BFP^k), the following value is a lower bound of (BFP^k):

$$\begin{aligned}
& \min_{\substack{x \in X, z_1, z_2 \geq 0, \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} (\mu_1^{k,l})^T (A_1 x + B_1 x_0^k - b_1 - z_1) + \|z_1\| + \|z_2\| \\
&= \left\{ \min_{\substack{x \in X, z_2 \geq 0 \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} (\mu_1^{k,l})^T A_1 x + \|z_2\| + (\mu_1^{k,l})^T (B_1 x_0^k - b_1) \right\} \\
&\quad + \left\{ \min_{z_1 \geq 0} \|z_1\| - (\mu_1^{k,l})^T z_1 \right\}. \\
&= obj_{DWFP-R^{k,l}} + (\mu_1^{k,l})^T (B_1 x_0^k - b_1).
\end{aligned}$$

113 In order to establish the finite convergence of decomposition, we make the following
 114 assumption on the linear programming (LP) solver used to solve the LP subproblems.

115 **Assumption 1.** *The primal and dual optimal solutions of a LP problem returned by the*
 116 *LP solver are extreme points of the LP and its dual problems, respectively.*

117 **Proposition 2.** *The DWD procedure for solving Problem (BFP^k), i.e., iteratively solving*
 118 *(DWFRMP-R^{k,l}) and (DWFP-R^{k,l}), is finite.*

Proof. At the DWD iteration l , suppose that the solution of (DWFP-R^{k,l}), $x^{k,l}$, is also the solution of a previously solved pricing problem. On the one hand,

$$obj_{DWFP-R^{k,l}} = \min_{\substack{z_2 \geq 0, \\ A_2 x^{k,l} \leq b_2 - B_2 x_0^k + z_2}} \|z_2\| + (\mu_1^{k,l})^T A_1 x^{k,l}.$$

On the other hand, the optimal value of (DWFRMP-R^{k,l}) can be expressed as:

$$\begin{aligned} obj_{DWFRMP-R^{k,l}} &= \min_{\substack{x \in X^{k,l}, z_1, z_2 \geq 0 \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} \|z_1\| + \|z_2\| + (\mu_1^{k,l})^T (B_1 x_0^k + A_1 x - b_1 - z_1) \\ &= \left\{ \min_{\substack{x \in X^{k,l}, z_2 \geq 0 \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} \|z_2\| + (\mu_1^{k,l})^T A_1 x \right\} + (\mu_1^{k,l})^T (B_1 x_0^k - b_1) \\ &\quad + \left\{ \min_{z_1 \geq 0} \|z_1\| - (\mu_1^{k,l})^T z_1 \right\} \\ &= \left\{ \min_{\substack{x \in X^{k,l}, z_2 \geq 0 \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} \|z_2\| + (\mu_1^{k,l})^T A_1 x \right\} + (\mu_1^{k,l})^T (B_1 x_0^k - b_1) \end{aligned}$$

where $X^{k,l} = \{x \in \mathbb{R}^{n_x} : x = \sum_{i=0}^{l-1} \theta^i x^i, \sum_{i=0}^{l-1} \theta^i = 1, \theta^i \geq 0, \forall i = 0, \dots, l-1\}$. Since $x^{k,l}$ has been generated before, it is a point in $X^{k,l}$, so

$$\min_{\substack{x \in X^{k,l}, z_2 \geq 0 \\ A_2 x \leq b_2 - B_2 x_0^k + z_2}} \|z_2\| + (\mu_1^{k,l})^T A_1 x \leq \min_{\substack{z_2 \geq 0, \\ A_2 x^{k,l} \leq b_2 - B_2 x_0^k + z_2}} \|z_2\| + (\mu_1^{k,l})^T A_1 x^{k,l},$$

and therefore

$$obj_{DWFRMP-R^{k,l}} \leq obj_{DWFP-R^{k,l}} + (\mu_1^{k,l})^T (B_1 x_0^k - b_1).$$

119 This means that, the upper and lower bounds of Problem (BFP^k) converge once an extreme
 120 point of X is generated at the second time. Since polyhedral set X has only a finite number
 121 of extreme points, so the DWD procedure always terminates in a finite number of iterations.
 122 □

123 2.3. Integration of the two levels

At the termination of the DWD procedure for solving Problem (BPP^k),

$$obj_{BPP^k} = obj_{DWPP-R^{k,l}} + c_0^T x_0^k + (\lambda_1^{k,l})^T (B_1 x_0^k - b_1), \quad (1)$$

124 where l indexes the last DWD iteration. $(x_0^k, x^{k,l})$ is a feasible solution for the original
 125 problem (P), and it can be used to update the upper bound of (P). The feasible solution
 126 that gives the best upper bound of the current BD iteration is called the incumbent solution,
 127 denoted by (x_0^*, x^*) .

Let $\lambda_2^{k,l}$ be Lagrange multipliers for the weak linking constraints in (DWPP-R^{k,l}), according to strong duality of (DWPP-R^{k,l}),

$$obj_{DWPP-R^{k,l}} = \min_{x \in X} c^T x + (\lambda_1^{k,l})^T A_1 x + (\lambda_2^{k,l})^T (A_2 x - b_2 + B_2 x_0^k). \quad (2)$$

From equations (1) and (2),

$$obj_{BPP^k} = \min_{x \in X} c^T x + c_0^T x_0^k + (\lambda_1^{k,l})^T (A_1 x - b_1 + B_1 x_0^k) + (\lambda_2^{k,l})^T (A_2 x - b_2 + B_2 x_0^k),$$

which implies that $\lambda_1^{k,l}$ and $\lambda_2^{k,l}$ are (optimal) Lagrange multipliers of (BPP^k). Therefore, the optimality cuts in the standard BD relaxed master problem (BRMP-Std^k) can be written as (via substituting (1)):

$$\eta \geq obj_{DWPP-R^{i,j}} + c_0^T x_0 + (\lambda_1^{i,j})^T (B_1 x_0 - b_1) + (\lambda_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in T_{opt}^k,$$

where T_{opt}^k includes index pairs that index the BD iteration at which an optimality cut is generated and the last DWD iteration for this BD iteration. We can also rewrite the

feasibility cuts in a similar way. As a consequence, Problem (BRMP-Std^k) can be rewritten as:

$$\begin{aligned}
& \min_{x_0, \eta} \quad \eta \\
& \text{s.t.} \quad \eta \geq \text{obj}_{DWFP-R^{i,j}} + c_0^T x_0 + \left(\lambda_1^{i,j} \right)^T (B_1 x_0 - b_1) + \left(\lambda_2^{i,j} \right)^T B_2 (x_0 - x_0^i), \\
& \quad \forall (i, j) \in T_{opt}^k, \\
& \quad 0 \geq \text{obj}_{DWFP-R^{i,j}} + \left(\mu_1^{i,j} \right)^T (B_1 x_0 - b_1) + \left(\mu_2^{i,j} \right)^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in T_{feas}^k, \\
& \quad x_0 \in X_0.
\end{aligned}
\tag{BRMP-BLD^k}$$

where T_{feas}^k includes index pairs that index the BD iteration at which a feasibility cut is generated and the last DWD iteration for this BD iteration. If Problem (BRMP-BLD^k) is infeasible, then Problem (P) is also infeasible.

Figure 3 provides the flowchart of the bilevel decomposition algorithm. The algorithm considers two tolerances, ϵ and σ . ϵ is the tolerance for the solution of Problem (P), and it is also the tolerance for the upper level decomposition. σ is the tolerance for each lower level DWD procedure. $\epsilon \gg \sigma$ is required to ensure the final solution is ϵ -optimal. The DWD loop involving (DWFRMP-R^{k,l}) and (DWFP-R^{k,l}) is to solve the feasibility problem (BFP^k), and the DWD loop involving (DWRMP-R^{k,l}) and (DWPP-R^{k,l}) is to solve the upper bounding problem (BPP^k). With Assumption 1, both the upper level BD procedure and the lower level DWD procedures can terminate in a finite number of iterations, so the bilevel decomposition method has the following finite termination property.

Theorem 1. *The bilevel decomposition algorithm shown in Figure 3 terminates in finite time with an ϵ -optimal solution or a certification that Problem (P) is infeasible, if Assumption 1 holds and all subproblems can be solved in finite time.*

We have not seen in the literature where the proposed bilevel decomposition method was ever developed for solving Problem (P), but there exist methods using similar ideas to deal with strong linking constraints, such as the one developed by Bruno and Sagastizábal [20].

Figure 3: The flowchart of the bilevel decomposition method

146 In Bruno and Sagastizábal's bilevel decomposition method, the upper level is Lagrangian
 147 decomposition and the lower level is BD, but the method is rigorous for (P) only when no
 148 integer variables are present, because it requires strong duality of (P).

149 3. The extended cross decomposition method

150 3.1. The basic ECD framework and subproblems

151 In this section, we develop a decomposition framework motivated by ideas from bilevel
 152 decomposition proposed in section 2 and the cross decomposition method proposed in [21].
 153 In this framework, which is shown in Figure 4, an upper and a lower level decomposition
 154 strategy are integrated to efficiently solve Problem (P). At the upper level, a cross decom-
 155 position approach for (P) where the following problems; DW restricted master problem, DW
 156 pricing problem, Benders primal and feasibility problems (BPP^k and BFP^k) and Benders
 157 relaxed master problems, are iteratively solved is presented. Problems (BPP^k) and (BFP^k),
 158 that are not decomposable, are then solved via a finitely convergent DWD procedure at the
 159 lower level, as in the bilevel decomposition. We refer to this approach as the *extended cross*
 160 *decomposition* (ECD) approach. The ECD lower level subproblems are same to the lower
 161 level subproblems in the bilevel decomposition method proposed in the last section, so we
 162 only describe the upper level subproblems here.

At a particular upper level iteration k of ECD, the following DW restricted master
 problem is solved first:

$$\begin{aligned}
 obj_{DWRMP-CD^k} = \min_{x_0, \theta^i} \quad & c_0^T x_0 + \sum_{i=0}^{k-1} \theta^i x^i \\
 s.t. \quad & B_1 x_0 + A_1 \sum_{i=0}^{k-1} \theta^i x^i \leq b_1, \\
 & B_2 x_0 + A_2 \sum_{i=0}^{k-1} \theta^i x^i \leq b_2, \\
 & \sum_{i=0}^{k-1} \theta^i = 1, \\
 & \theta^i \geq 0, \\
 & x_0 \in X_0,
 \end{aligned} \tag{DWRMP-CD^k}$$

163 where x^i is a column generated at the upper level and $i = 0, 1, \dots, k-1$ are indices of the
 164 generated columns. Problem (DWRMP-CD k) provides valid upper bounds to Problem (P).

Figure 4: Extended cross decomposition method

165 Just like in the bilevel decomposition, an initial feasible column, $x^0 \in X$, is needed to solve
 166 Problem (DWRMP-CD^k). It can be obtained by solving the initial pricing problem (IPP).
 167 Here we assume that Problem (DWRMP-CD^k) is always feasible for convenience, and later
 168 we will discuss how to guarantee the feasibility via a Phase I procedure.

Let λ_1^k and λ_2^k be the Lagrange multiplier associated with strong and weak linking constraints in Problem (DWRMP-CD^k) respectively, then we can construct and solve the following DW pricing problem:

$$\begin{aligned}
 obj_{DWPP^k} = \min_x \quad & c^T x + (\lambda_1^k)^T A_1 x + (\lambda_2^k)^T A_2 x \\
 s.t. \quad & x \in X.
 \end{aligned}
 \tag{DWPP^k}$$

169 The solution of Problem (DWPP^k), denoted by x^k , provides a new column for Problem
 170 (DWRMP-CD^k).

If a full DW iteration is completed and the algorithm is to enter a BD iteration, the

following Benders relaxed master problem is solved:

$$\begin{aligned}
& \min_{x_0, \eta} \eta \\
& \text{s.t. } \eta \geq \text{obj}_{DWPP^i} + c_0^T x_0 + (\lambda_1^i)^T (B_1 x_0 - b_1) + (\lambda_2^i)^T (B_2 x_0 - b_2), \quad \forall i \in U_{opt}^k, \\
& \quad \eta \geq \text{obj}_{DWPP-R^{i,j}} + c_0^T x_0 + (\lambda_1^{i,j})^T (B_1 x_0 - b_1) + (\lambda_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in T_{opt}^k, \\
& \quad 0 \geq \text{obj}_{DWFP-R^{i,j}} + (\mu_1^{i,j})^T (B_1 x_0 - b_1) + (\mu_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in T_{feas}^k, \\
& \quad x_0 \in X_0,
\end{aligned}
\tag{BRMP-CD}^k$$

171 where U_{opt}^k includes the indices of all previous iterations in which Problem (DWPP^k) is
172 solved, while T_{opt}^k and T_{feas}^k have been defined in Section 2. The optimality and feasibility
173 cuts in (BRMP-CD^k) come from previous BD iterations and DW iterations, and readers are
174 referred to [21] for more discussions on (BRMP-CD^k).

175 Note that in ECD, DW restricted master problems and DW pricing problems are included
176 in both the upper level and the lower level. However, in the upper level, both strong and
177 weak linking constraints complicate the problem and therefore they are dualized in the DW
178 pricing problems, while in the lower level, only strong linking constraints complicate the
179 problem (because the weak linking constraints no longer complicate the problem when x_0 is
180 fixed) and therefore weak linking constraints are not dualized in the DW pricing problem.

181 3.2. Synergizing the upper and the lower level

The two levels in ECD can be synergized to yield stronger bounds and accelerate convergence, like the synergy of DWD and BD iterations within the CD method. Specifically, columns and cuts can be generated from the solutions of the DW pricing problems at the lower level, and they can be added to Problem (DWRMP-CD^k) and (BRMP-CD^k) at the upper level in order to yield better upper and lower bounds for the original problem (P).

As a result, the DW restricted master problem can be enhanced as:

$$\begin{aligned}
& \min_{x_0, \theta^i} \quad c_0^T x_0 + \sum_{i=0}^{t-1} \theta^i \hat{x}^i \\
& s.t. \quad B_1 x_0 + A_1 \sum_{i=0}^{t-1} \theta^i \hat{x}^i \leq b_1, \\
& \quad \quad B_2 x_0 + A_2 \sum_{i=0}^{t-1} \theta^i \hat{x}^i \leq b_2, \\
& \quad \quad \sum_{i=0}^{t-1} \theta^i = 1, \\
& \quad \quad \theta^i \geq 0, \\
& \quad \quad x_0 \in X_0,
\end{aligned} \tag{DWRMP-ECD^k}$$

where $\hat{x}^i$ represent a column generated from either the upper level or the lower level, and t indexes all the columns. If $\hat{x}^i$ is generated at the k th upper level iteration, then $\hat{x}^i = x^k$. If $\hat{x}^i$ is generated at the l th lower level iteration for the k th upper level iteration, then $\hat{x}^i = x^{k,l}$.

The enhanced Benders relaxed master problem can be written as:

$$\begin{aligned}
& \min_{x_0, \eta} \quad \eta \\
& s.t. \quad \eta \geq obj_{DWPP^i} + c_0^T x_0 + (\lambda_1^i)^T (B_1 x_0 - b_1) + (\lambda_2^i)^T (B_2 x_0 - b_2), \quad \forall i \in U_{opt}^k, \\
& \quad \quad \eta \geq obj_{DWPP-R^{i,j}} + c_0^T x_0 + (\lambda_1^{i,j})^T (B_1 x_0 - b_1) + (\lambda_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in \hat{T}_{opt}^k, \\
& \quad \quad 0 \geq obj_{DWFP-R^{i,j}} + (\mu_1^{i,j})^T (B_1 x_0 - b_1) + (\mu_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in \hat{T}_{feas}^k, \\
& \quad \quad x_0 \in X_0,
\end{aligned} \tag{BRMP-ECD^k}$$

where $\hat{T}_{opt}^k$ includes index pairs that index the BD iteration at which an optimality cut is generated and all DWD iterations for this BD iteration, while $\hat{T}_{feas}^k$ includes index pairs that index the BD iteration at which a feasibility cut is generated and all DWD iterations for this BD iteration.

Figure 5 shows the algorithmic flowchart of ECD. The left part of the flowchart depicts the ECD upper level, where Problems (DWRMP-ECD^k), (DWPP^k) and (BRMP-ECD^k) are solved, and the right part of the flowchart depicts the ECD lower level, where Problems (DWFRMP-R^{k,l}) and (DWFP-R^{k,l}) are solved iteratively. The finite convergence property of the lower level procedure is proved in the last section, and the finite convergence property of the upper level CD procedures is proved in [21], so we have the following finite convergence property of ECD.

Theorem 2. *If Assumption 1 holds and all subproblems can be solved in finite time, then the extended cross decomposition method in Figure 5 terminates in a finite number of steps with an ϵ -optimal solution of Problem (P) or a certification that Problem (P) is infeasible.*

3.3. Further discussions

3.3.1. Phase 1 procedure and subproblems

In this section, we present a Phase I procedure to prevent the infeasibility of Problem (DWRMP-ECD^k). The Phase I procedure essentially solves the following feasibility problem:

$$\begin{aligned}
 obj_{FP} &= \min_{x_0, x, z_1 \geq 0, z_2 \geq 0} \|z_1\| + \|z_2\| \\
 \text{s.t. } & A_1 x \leq b_1 - B_1 x_0 + z_1, \\
 & A_2 x \leq b_2 - B_2 x_0 + z_2, \\
 & x \in X, \\
 & x_0 \in X_0,
 \end{aligned} \tag{FP}$$

where z_1 and z_2 are slack variables. Problem (FP) is always feasible with the assumption that X and X_0 are nonempty. The Phase I procedure actually employs ECD to solve Problem (FP), and the subproblems need to be solved in this procedure is illustrated in Figure 6. At iteration k at the upper level, the following DW restricted master problem is solved:

Figure 5: The flowchart of the extended cross decomposition method

$$\begin{aligned}
 & \min_{x_0, \theta^0, \dots, \theta^{k-1} \geq 0, z_1 \geq 0, z_2 \geq 0} \|z_1\| + \|z_2\| \\
 & \text{s.t. } B_1 x_0 + A_1 \sum_{i=0}^{t-1} \theta^i x^i \leq b_1 + z_1, \\
 & \quad B_2 x_0 + A_2 \sum_{i=0}^{t-1} \theta^i x^i \leq b_2 + z_2, \\
 & \quad \sum_{i=0}^{t-1} \theta^i = 1, \\
 & \quad x_0 \in X_0.
 \end{aligned} \tag{DWFRMP-ECD^k}$$

Figure 6: Diagram of the Phase I procedure for extended cross decomposition method

Let μ_1^k and μ_2^k be the Lagrange multipliers for the strong and weak linking constraints of the Problem (DWFRMP-ECD^k), then the following DW pricing problem can be constructed and solved:

$$\begin{aligned}
 \min_x \quad & (\mu_1^k)^T A_1 x + (\mu_2^k)^T A_2 x \\
 \text{s.t.} \quad & x \in X.
 \end{aligned}
 \tag{DWFP^k}$$

The BD relaxed master problem solved in Phase I can be written as:

$$\begin{aligned}
& \min_{x_0, \eta} \eta \\
& \text{s.t.} \quad \eta \geq \text{obj}_{DWFP^i} + (\mu_1^i)^T (B_1 x_0 - b_1) + (\mu_2^i)^T (B_2 x_0 - b_2), \quad \forall i \in U_{feas}^k, \\
& \quad \eta \geq \text{obj}_{DWFP-R^{i,j}} + (\mu_1^{i,j})^T (B_1 x_0 - b_1) + (\mu_2^{i,j})^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in \hat{T}_{feas}^k, \\
& \quad x_0 \in X_0.
\end{aligned}
\tag{BRMPI-ECD}^k$$

207 where U_{feas}^k includes the indices of all previous iterations in which Problem (DWFP)^k is
 208 solved. Note that there are no feasibility cuts generated in Phase I, because Problem (BFP)^k
 209 is always feasible (provided X and X_0 are nonempty).

210 **Remark 1.** *The Phase I procedure illustrated in Figure 6 terminates finitely with an δ -*
 211 *optimal solution of Problem (FP). If $\text{obj}_{FP} > \delta$, then Problem (P) is infeasible; otherwise,*
 212 *Problem (P) is feasible and the optimal solution of Problem (FP) is a feasible solution of*
 213 *Problem (P).*

214 In order for the optimality of Problem (FP) to precisely imply the feasibility of Problem
 215 (P), we need to select a sufficiently small optimality tolerance δ for Phase I (as least no
 216 larger than the feasibility tolerance of Problem (P)). The algorithmic flowchart of Phase I
 217 is similar to the one given in Figure 5, so it is omitted.

The optimal solution of Problem (FP) and all the other columns generated in Phase I are added to Problem (DWRMP-ECD)^k in Phase II, so Problem (DWRMP-ECD)^k is always feasible. The cuts generated in Phase I can also be added to the BD relaxed master problem in Phase II, but these cuts will all appear as feasibility cuts, as explained in [21]. The Phase

II BD relaxed master problem can be written as:

$$\begin{aligned}
& \min_{x_0, \eta} \eta \\
\text{s.t. } & \eta \geq \text{obj}_{DWPP^j} + c_0^T x_0 + \left(\lambda_1^j\right)^T (B_1 x_0 - b_1) + \left(\lambda_2^j\right)^T (B_2 x_0 - b_2) \quad \forall j \in U_{opt}^k, \\
& \eta \geq \text{obj}_{DWPP-R^{i,j}} + c_0^T x_0 + \left(\lambda_1^{i,j}\right)^T (B_1 x_0 - b_1) + \left(\lambda_2^{i,j}\right)^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in \hat{T}_{opt}^k, \\
& 0 \geq \text{obj}_{DWFP^i} + (\mu_1^i)^T (B_1 x_0 - b_1) + (\mu_2^i)^T (B_2 x_0 - b_2), \quad \forall i \in U_{feas}^k, \\
& 0 \geq \text{obj}_{DWFP-R^{i,j}} + \left(\mu_1^{i,j}\right)^T (B_1 x_0 - b_1) + \left(\mu_2^{i,j}\right)^T B_2 (x_0 - x_0^i), \quad \forall (i, j) \in \hat{T}_{feas}^k, \\
& x_0 \in X_0.
\end{aligned}$$

(BRMPHII-ECD^k)

3.3.2. Adaptive switching between upper and lower level subproblems

In [21], it is demonstrated that the CD method can be improved by switching between DWD and BD iterations in an adaptive way. The switching rules proposed in [21] can effectively avoid ineffective iterations at which columns and cuts generated do not help to close the optimality gap, so it can be adapted here to improve the convergence rate. Specifically, the ECD switching rules determine the alternation between the lower and the upper levels, following two criteria:

1. After an upper level DWD iteration, solve Problem (BRMPI-ECD^k) (for Phase I) or (BRMPHII-ECD^k) (for Phase II) to update the lower bound. If the decrease of the upper bound in the DWD iteration is more than the increase of the lower bound, then the algorithm will go to another DWD iteration. Otherwise, the algorithm will go to the lower level in order to solve (BFP^k) (for Phase I) or (BPP^k) (for Phase II).
2. After the convergence of a lower level procedure, solve Problem (DWFRMP-ECD^k) (for Phase I) or (DWRMP-ECD^k) (for Phase II). If the optimal objective value of the problem is better than the current upper bound, then the algorithm will go to an upper level DWD iteration. Otherwise, the algorithm will go to another BD iteration, which requires solving (BFP^k) (for Phase I) or (BPP^k) (for Phase II) via another lower level procedure.

236 We tag the algorithm following the above rules as "ECD2" and the standard algorithm
 237 without the rules as "ECD1". The advantage of ECD2 will be seen through the case study.

238 4. Application of ECD: Risk-averse two-stage stochastic programming

239 4.1. Background

240 Different risk measurement metrics are adopted in the literature for risk-averse two-
 241 stage stochastic programming, such as variance, variability index, probabilistic financial
 242 risk, downside risk, value-at-risk (VaR), etc [25]. In this paper, the conditional value-at-
 243 risk, CVaR, first proposed by Rockafellar and Uryasev [10] is used to measure risks because
 244 of its ability to account for the worse case scenario and computational tractability, and its
 245 close relationship to VaR [10] [26] [27] [28].

246 CVaR can be included in two-stage stochastic programming in two ways. One is to
 247 penalize CVaR in the objective function; in this case Problem (SP) does not have strong
 248 linking constraints, and it can be solved by classical decomposition methods. For example,
 249 Ahmed [29] used a cutting plane approach to solve a risk-averse stochastic linear program;
 250 the risk measured by a *dispersion* metric. Schultz and Tiedemann [30] applied Lagrangian
 251 decomposition to solve a risk averse stochastic mixed-integer programming problem. Noyan
 252 [7] developed two decomposition techniques based on generic BD to solve a stochastic LP
 253 minimizing CVaR. Qi et al. used multicut BD to solve a risk-averse problem [31].

254 The other way is to bound CVaR in constraints. Some applications with CVaR in con-
 255 straints include portfolio optimization [10], oil and energy optimization [20], oil supply chain
 256 [32], large-scale industrial batch plants [33] [34], traveling salesman's problem [28]. Modeling
 257 CVaR as objective and as constraints have been shown by Krokhmal et al. to provide the
 258 same efficient frontier [26]. However, CVaR constrained approach is preferred because de-
 259 cision makers interpret and quantify right hand side easier than penalty parameters in the
 260 objective function, as argued by Fabian and Veszpremi [35]. The research on decomposition
 261 based strategy to solve CVaR constrained two-stage stochastic programming is very limited
 262 in the literature. Huang and Zheng [28] proposed a BD-type approach for the traveling sale-
 263 man's problem with risk constraints. In their strategy, they seperated the risk constraints

264 from scenario coupling constraints. Feasibility cuts were then developed to exclude solutions
 265 already generated. Bruno and Sagastizábal [20] developed two decomposition approaches
 266 to solve a two-stage stochastic linear program with CVaR constraints. The first approach
 267 is a Benders-like procedure, and the second approach is a bilevel decomposition procedure
 268 with Lagrangian decomposition in the upper level and BD in the lower level. The second
 269 approach is rigorous only for LP problems.

270 4.2. CVaR-constrained two-stage stochastic programming

We consider the following CVaR constraints for scenario-based risk-averse two-stage stochastic programming according to [26] [33] [34]:

$$\zeta + \frac{1}{(1-\beta)} \sum_{\omega \in S} p_{\omega} \psi_{\omega} \leq b_0, \quad (3)$$

$$\psi_{\omega} \geq f_{\omega}(x_{\omega}) - \zeta, \quad \psi_{\omega} \geq 0, \quad \forall \omega \in \{1, \dots, s\}, \quad (4)$$

where $f_{\omega}(x_{\omega})$ denotes a scenario dependent loss function of the second-stage variables at scenario ω , $p_{\omega} \in [0, 1]$ denotes the probability of scenario ω ($\sum_{\omega=1}^s p_{\omega} = 1$), $\beta \in [0, 1]$ is a user specified probability, b_0 is a user specified risk threshold for the loss function over all scenarios. When the CVaR constraints are included, the two-stage stochastic programming problem can be written as:

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_s \\ \zeta, \psi_1, \dots, \psi_s}} \sum_{\omega=1}^s (c_{0,\omega}^T x_0 + c_{\omega}^T x_{\omega}) \\ & \text{s.t. } A_{0,\omega} x_0 + A_{\omega} x_{\omega} \leq b_{0,\omega}, \quad \omega \in \{1, \dots, s\}, \quad \textcircled{A} \\ & \quad \psi_{\omega} \geq f_{\omega}(x_{\omega}) - \zeta, \quad \omega \in \{1, \dots, s\}, \quad \textcircled{B} \\ & \quad \zeta + \sum_{\omega=1}^s \hat{p}_{\omega} \psi_{\omega} \leq b_0, \quad \textcircled{C} \\ & \quad x_{\omega} \in X_{\omega}, \psi_{\omega} \geq 0, \quad \forall \omega \in \{1, \dots, s\}, \\ & \quad x_0 \in X_0, \zeta \in \mathbb{R}, \end{aligned} \quad (\text{CVaR-SP})$$

271 where we define $\hat{p}_\omega = \frac{1}{(1-\beta)}p_\omega$ for convenience. In this formulation, x_0 denotes the first-
 272 stage decisions and x_ω denotes second-stage decisions for scenario ω . Constraints $\textcircled{A}$ and
 273 $\textcircled{B}$ are weak linking constraints, because when x_0 is fixed, they are decomposable over the
 274 scenarios. Constraints $\textcircled{C}$ are strong linking constraints, because they cannot be decomposed
 275 over the scenarios no matter whether x_0 is fixed.

276 ECD is developed under the assumption that all variables are explicitly bounded, because
 277 otherwise some ECD subproblems may be unbounded. In (CVaR-SP), ψ_ω and ζ are not
 278 explicitly bounded although their boundedness is already implied by the formulation, so next
 279 we are to show how to estimate the bounds for ψ_ω and ζ in order to prevent unbounded
 280 ECD subproblems.

281 First, from equation (4), a valid lower bound of ψ_ω is $\psi_\omega^{lo} = 0, \forall \omega \in \{1, \dots, s\}$.

Second, from equations (3) and (4),

$$\zeta \leq b_0 - \frac{1}{1-\beta} \sum_{\omega=1}^s p_\omega \psi_\omega \leq b_0,$$

282 so a valid upper bound on ζ is $\zeta^{up} = b_0$.

Next, we show that a valid lower bound on ζ is

$$\zeta^{lo} = \min \left\{ -\frac{(1-\beta)}{\beta} b_0 + \frac{1}{\beta} f^{lo}, f^{lo} \right\},$$

where f^{lo} is a valid lower bound for the loss function for any scenario, i.e.,

$$f^{lo} \leq \min_{\omega \in \{1, \dots, s\}} \left\{ \min_{x_\omega \in X_\omega} f_\omega(x_\omega) \right\}.$$

We show this by showing that any ζ that is less than ζ^{lo} does not satisfy (3). This is because

$$\forall \zeta < \zeta^{lo},$$

$$\begin{aligned}
\zeta + \frac{1}{1-\beta} \sum_{\omega=1}^s p_{\omega} \psi_{\omega} &\geq \zeta + \frac{1}{1-\beta} \sum_{\omega=1}^s p_{\omega} (f_{\omega}(x_{\omega}) - \zeta) \\
&\geq \zeta + \frac{1}{1-\beta} \sum_{\omega=1}^s p_{\omega} (f^{lo} - \zeta) \\
&= \zeta + \frac{(f^{lo} - \zeta)}{1-\beta} \sum_{\omega=1}^s p_{\omega} \\
&= \zeta + \frac{(f^{lo} - \zeta)}{1-\beta} \\
&= \frac{-\beta}{1-\beta} \zeta + \frac{1}{1-\beta} f^{lo} \\
&> \frac{-\beta}{1-\beta} \left\{ -\frac{(1-\beta)}{\beta} b_0 + \frac{1}{\beta} f^{lo} \right\} + \frac{1}{1-\beta} f^{lo} \\
&= b_0.
\end{aligned}$$

Finally, we show that $\psi_{\omega}^{up} = f_{\omega}^{up} - \zeta^{lo}$ is a valid upper bound for ψ_{ω} , where f_{ω}^{up} is a valid upper bound for the loss function for scenario ω , i.e.,

$$f_{\omega}^{up} \geq \max_{x_{\omega} \in X_{\omega}} f_{\omega}(x_{\omega}).$$

283 According to equation (4), this can be shown by showing that $f_{\omega}^{up} - \zeta^{lo} \geq 0$, and $f_{\omega}^{up} - \zeta^{lo} \geq$
284 $f_{\omega}(x_{\omega}) - \zeta$, and these two relations result directly from the definitions of f_{ω}^{up} and ζ^{lo} .

In summary, the bounds of ζ and ψ_{ω} can be computed as follows:

$$\begin{aligned}
\psi_{\omega}^{lo} &= 0, \quad \forall \omega \in \{1, \dots, s\}, \\
\zeta^{up} &= b_0, \\
\zeta^{lo} &= \min \left\{ -\frac{(1-\beta)}{\beta} + \frac{1}{\beta} f^{lo}, f^{lo} \right\}, \\
\psi_{\omega}^{up} &= f_{\omega}^{up} - \zeta^{lo}, \quad \forall \omega \in \{1, \dots, s\}.
\end{aligned}$$

285 The ECD subproblems for Problem (CVaR-SP), including those for Phase I and Phase
286 II, are explained in Appendix A.

5. Case Studies

We compare four solution methods for solving risk-averse stochastic programming problems in the case studies. These methods are monolith (i.e., solving the full problem using a commercial optimization solver), BLD (i.e., the bilevel decomposition method proposed in Section 2), ECD1 (the proposed ECD method without adaptive switching between the two levels), and ECD2 (the proposed ECD method with adaptive switching between the two levels). The simulation was run on a virtual machine setup running Ubuntu 16.04 with a 2.4 GHz CPU and 4 GB of memory. The two formulations and the solution methods were programmed on GAMS 24.6.1 [36], with CPLEX 12.6.3 [37] being the LP/MILP solver. GUSS [38], a GAMS extension, was utilized in all decomposition methods (with default GUSS options), in order to achieve efficient model generation and solution for the decomposed scenario problems. For all decomposition methods the norm functions in the subproblems are the 1-norm. The relative optimality tolerance used for all solution methods was 10^{-3} . Since BLD needs initial values for the first-stage decisions to start the algorithm, we set the values be zero for all problem instances.

5.1. Case A

5.1.1. Problem Description and Advantage of Risk-Averse Formulation

The case study considers an energy and bio-product supply chain optimization (SCO) problem originally studied in [39]. The supply chain considered has four layers involving material collecting, material preprocessing, energy and bio-product production and product distribution. The goal of the SCO is to determine the optimal configuration of the supply chain network and the technologies used in the production plants, such that the total profit is maximized and the customer demands at the demand locations are satisfied. McLean and Li [40] developed a two-stage stochastic MILP formulation for the SCO problem, where the first-stage decisions are binary variables determining whether or not specific units are to be developed and specific technologies are to be adopted for energy/bio-product production, and the second-stage decisions are continuous variables determining material and product flows in the operation of the supply chain.

315 The two-stage stochastic MILP in [40] does not contain strong linking constraints, so it is
 316 not suitable for evaluating the decomposition methods developed in this paper. Therefore,
 317 we modify the problem by adding a CVaR constraint. In order to do this, we first assume
 318 that the demand uncertainty is in the minimum electricity demands rather than the maxi-
 319 mum demands. We also assume that the supply chain needs to satisfy uncertain minimum
 320 electricity demand for "most scenarios" (rather than for all scenarios), so the minimum de-
 321 mand requirement can be expressed as a CVaR constraint, where the loss function $f_\omega(x_\omega)$
 322 is the total unsatisfied minimum electricity demands (i.e., the total minimum electricity
 323 demand minus the total electricity supply). The minimum electricity demand (at each de-
 324 mand location) and the yield of corn stover are the two uncertain parameters considered in
 325 the model, which are assumed to be uniformly distributed in ranges $[1 \ 3] \cdot 10^6 \cdot MWh/year$
 326 and $[790 \ 890] t/km^2 \cdot year$, respectively. The model involves 18 binary variables and $2376s$
 327 $+ 7$ continuous variables (where s refers to the number of scenarios). Other details of the
 328 model are provided in the supplemental materials.

329 In order to show the advantage of including the CVaR constraints in the formulation, we
 330 compare in Table 1 the quality of the solutions obtained from formulations with and without
 331 the CVaR constraints. 169 scenarios are addressed in each of the formulations. It can be
 332 seen that, when no CVaR constraints are included in the formulation, all preprocessing
 333 centers $(m_1, \dots, m_6)$ except m_3 are to be developed, and main plant n_2 only includes dry-
 334 grind facilities. When CVaR constraints are included in the formulation, the preprocessing
 335 center that is not developed is m_4 instead of m_3 , and main plant n_2 also needs incineration
 336 facilities in order to generate more electricity to meet the (uncertain) minimum demands
 337 for electricity. As a result, the solution from the risk-averse formulation is more robust
 338 in terms of observing the minimum electricity demand constraint (although it leads to
 339 a lower expected profit), as indicated by the lower CVaR risk measure and the higher
 340 probability of minimum electricity demand satisfaction. Note that the CVaR constraints
 341 do not enforce 100% demand satisfaction. This is different from using "hard" demand
 342 constraints that require demand satisfaction for all uncertainty realizations (which is usually
 343 overly conservative).

Table 1: Comparison of solutions from the two stochastic programming formulations - Case A

Formulation (169 scenarios)	Without CVaR constraints	With CVaR constraints
Preprocessing centers	m_1, m_2, m_4, m_5, m_6	m_1, m_2, m_3, m_5, m_6
Facilities in main plant	n_1 : Dry-grind Incineration n_2 : Dry-grind Incineration n_3 : Dry-grind Sawing Incineration	n_1 : Dry-grind Incineration n_2 : Dry-grind Incineration n_3 : Dry-grind Sawing Incineration
Predicted expected profit (Million \$)	29.43	26.96
CVaR ($\times 10^3$ MWh/year) $\dagger$	273.76	-13.73
Probability of minimum electricity demands satisfied $\ddagger$	53.3%	97.3%

$\dagger$: CVaR of the first-stage decisions, estimated by sampling 225 realizations of uncertainty (i.e., 15 samples for each uncertain parameter).

$\ddagger$: The probabilities were also estimated from the 225 realizations of uncertainty.

5.1.2. Computational Results and Discussions

Tables 2, 3, 4 and 5 show the computational results of the four solution approaches. We can see that the monolith approach is the fastest for the problem with 9 scenarios, but its solution time grows rapidly with the number of scenarios. On the other hand, the solution times of the decomposition approaches increase relatively slowly with the number of scenarios, because the number of iterations of these approaches do not increase significantly with the number of scenarios. Therefore, when the number of scenario is larger than 49, all the decomposition approaches are faster than the monolith approach (by 100% to more than an order of magnitude). We can also see that the solution time of each decomposition method is dominated by the time for solving all pricing subproblems, which depends on the total number of upper level and lower level DWD iterations. Compared to BLD, ECD1 requires fewer upper level iterations and therefore fewer total DWD iterations, because its upper level is CD which converges faster than BD (except for the 121 scenario instance). So ECD1 is mostly faster than BLD. Furthermore, ECD2 is consistently faster than BLD and mostly faster than ECD1. This is because ECD2 can avoid ineffective upper level DWD

359 iterations and it requires fewer upper level iterations.

360 Note that “total solver time” in the tables stands for the sum of solution times of all
361 subproblems solved in a decomposition method, and “wall time” in the tables stands for
362 the actual time used by a decomposition method (including the total solver time and all
363 computing overhead). The large discrepancy between the total solver time and the wall
364 time came from the fact that the data exchange (between the subproblems) on GAMS was
365 inefficient. If the methods are implemented using C++ that allows more efficient data
366 exchange, the wall times will be more close to the total solver times.

367 The advantage of ECD2 is also demonstrated by Figures 7 and 8. Figure 7 illustrates
368 how the upper and lower bounds change with the upper level iterations (but not with lower
369 level iterations at which the bounds do not change). Figure 8 shows the bound curves after
370 iteration 24 with an adjusted scale (so that the tails of the curves are more clear). We can
371 see that, in the first several iterations, the optimality gap of ECD2 is not as small as that
372 of BLD, but as the algorithm progresses, the upper and lower bounds from ECD2 approach
373 each other more quickly than those from BLD do. So compared to BLD, ECD2 converges
374 to an optimal solution in fewer upper level iterations (for all problem instances). ECD1
375 also requires fewer iterations compared to BLD for most problem instances. It should be
376 noted that, while 10^{-3} was set as the relative optimality tolerances in the case study, the
377 figures also indicate the convergence rates with difference tolerances. For example, if the
378 relative optimality tolerance is decreased to 10^{-4} , the solution times of ECD1 and ECD2
379 will not change much but the solution time of BLD will be much longer. But if the relative
380 optimality tolerance is increased to 10^{-2} , BLD will terminate more quickly than ECD1
381 while ECD2 will still be the fastest. If the relative optimality tolerance is as large as 10^{-1} ,
382 BLD will terminate within 20 iterations for all problem instances and becomes the fastest
383 method.

Table 2: Computational results of Case A - Monolith (Unit for time: sec)

Number of scenarios	9	25	49	81	121	169
Optimal obj. (Million \$)	-26.96	-26.96	-26.96	-26.96	-26.96	-26.96
Wall time	118	2563	4971	37851	55117	- ‡

Table 3: Computational results of Case A - BLD (Unit for time: sec)

Number of scenarios	9	25	49	81	121	169
Number of upper BD iterations	56	74	76	103	55	152
Number of lower DWD iterations	245	310	364	491	236	741
Optimal obj. (Million \$)	-26.96	-26.94	-26.96	-26.96	-26.96	-26.96
Time for pricing problems	232	986	2156	5132	2689	18006
Time for DW restricted MP	3.4	8.21	18.9	43.5	37.9	154
Time for BD relaxed MP	1.8	2.41	1.45	1.83	0.63	3.4
Total solver time	238	996	2176	5177	3728	18164
Wall time	392	1320	2991	8224	5012	40184

Table 4: Computational results of Case A - ECD1 (Unit for time: sec)

Number of scenarios	9	25	49	81	121	169
Num. of upper BD iterations	53	87	67	65	89	75
Num. of upper DWD iterations	27	44	34	33	45	38
Num. of lower DWD iterations	109	186	160	140	189	154
Num. of pricing problems solved #	136	230	194	173	234	192
Optimal obj. (Million \$)	-26.94	-26.95	-26.96	-26.96	-26.96	-26.93
Time for pricing problems	123	707	1114	1738	3924	4501
Time for DW restricted MP	6.1	49.7	73.4	83.3	335.6	271
Time for BD relaxed MP	0.5	1.6	0.57	0.45	0.79	0.6
Total solver time	130	759	1188	1840	4260	4774
Wall time	205	1026	1543	2368	5790	6336

Table 5: Computational results of Case A - ECD2 (Unit for time: sec)

Number of scenarios	9	25	49	81	121	169
Num. of upper BD iterations	29	36	42	52	43	36
Num. of upper DWD iterations	4	4	4	4	4	4
Num. of lower DWD iterations	108	137	157	216	166	133
Num. of pricing problems solved #	112	141	161	220	170	137
Optimal obj. (Million \$)	-26.96	-26.94	-26.96	-26.94	-26.94	-26.96
Time for pricing problems	103	425	813.1	2103.3	2611	2669
Time for DW restricted MP	6.7	29.3	55.5	181.4	159	116
Time for BD relaxed MP	0.3	0.48	0.5	1.2	0.58	0.4
Total solver time	111	455	869	2302	2771	2786
Wall time	187	632	1171	3115	3783	3924

#: Total number of IPP, DWPP, DWFP, DWPP-R and DWFP-R solved

‡: CPLEX found an MIP solution after 18 hours but could not solve final LP within 24 hours

Figure 7: Comparison of bound evolution in different decomposition methods - Case A

Figure 8: Comparison of bound evolution in different methods - Case A (after iteration 25)

5.2. Case B

5.2.1. Problem Description and Advantage of Risk-Averse Formulation

The case study considers an industrial chemical supply chain management problem that was originally studied and formulated as a two-stage stochastic MILP in [41]. The supply chain involves 5 manufacturing plants that produce 23 grades of final products from 55 grades of raw materials, a number of warehouses and distribution centers that stores raw materials and final products, and 5 regional markets. The goal of the SCO problem is to determine the optimal capacities of the 5 manufacturing plants such that the total profits are maximized and the minimum demands at the regional markets are satisfied. The first-stage decisions are the capacities of plants (represented by integer variables), and the second-stage decisions are material or product flow rates (represented by continuous variables), which reflect how the supply chain is operated.

Again, in this paper, we revise the two-stage stochastic MILP into a risk-averse two-stage stochastic MILP. In order to do this, we introduce target demands for the regional markets, which are uncertain and larger than the minimum demands. We add a CVaR constraint that requires the target demands be satisfied for "most scenarios". The loss functions in the CVaR constraint is the largest unsatisfied target demand over all regional markets. The two uncertain parameters addressed in the formulation represent the uncertain target demands at the regional markets, as defined in [41], and the minimum demands are set to be 70% of the target demands. The MILP model involves 5 positive integer variables and $8210s + 6$ continuous variables (where s refers to the number of scenarios). More details of the model are provided in the supplemental materials.

The advantage of the risk-averse formulation is demonstrated by the results in Table 6. Without the CVaR constraints, the formulation seeks to maximize the expected profit but does not ensure that all target demands are satisfied. As a result, the total plant capacity determined by this stochastic programming formulation is smaller than that determined by the risk-averse formulation. While the risk-neutral formulation predicts a higher expected profit (because of lower plant investment costs), it leads to a low target demand satisfaction probability (25%). On the other hand, the plant capacities determined by the risk-averse

Table 6: Comparison of solutions from the two stochastic programming formulations - Case B

Formulation (64 scenarios)	Without CVaR constraints	With CVaR constraints
Plant Capacities (units/year)	Plant 1: 5400 Plant 2: 1200 Plant 3: 600 Plant 4: 1200 Plant 5: 3900	Plant 1: 5100 Plant 2: 1800 Plant 3: 900 Plant 4: 1500 Plant 5: 3600
Predicted expected profit (Billion \$)	22.28	20.23
Probability of target demand satisfaction [†]	25.8%	100%

[†]: The probabilities were estimated by sampling 225 realizations of uncertainty (i.e., 15 samples for each uncertain parameter).

413 formulation enable larger total production rates, which can almost always satisfy all tar-
 414 get demands. Note that the 100% demand satisfaction rate was estimated from the 225
 415 uncertainty realizations, which does not stand for the true probability.

416 5.2.2. Computational Results and Discussion

417 We summarize the computational results of the risk-averse formulation for Case Study
 418 B in Tables 7-10. Again, we see that the monolith approach is efficient when the number
 419 of scenarios is small, but its solution time grows rapidly with the number of scenarios.
 420 On the other hand, the solution times of the three decomposition methods grows almost
 421 linearly with the number of scenarios, because the numbers of the iterations needed by
 422 the methods remain almost the same for different numbers of scenarios. In addition, for
 423 each decomposition method, most part of the solution time is devoted to pricing problems
 424 involved in upper/lower level DWD iterations. As a result, ECD2 is better than BLD and
 425 ECD1 (as it involves fewer DWD iterations), and BLD is slightly better than ECD1 (as the
 426 two methods have similar number of iterations but ECD1 spends additional time on DW
 427 restricted master problems).

428 Figure 9 compares how the bounds of the decomposition methods change with the upper
 429 level iterations for different scenario instances. We can see that for case B, BLD does not
 430 generate good upper and lower bounds at the beginning of the solution procedures but

431 ECD1 and ECD2 do. However, ECD1 and ECD2 have different rates of convergence, so
432 ECD1 is slightly slower than BLD and ECD2 is faster. Overall, all the three methods have
433 quite fast rates of convergence and ECD2 is the fastest. In addition, ECD2 will remain the
434 fastest algorithm if the tolerance is increased or decreased by an order of magnitude.

Table 7: Computational Results of Case B - Monolith (Unit for time: sec)

Number of scenarios	9	16	25	36	49	64
Optimal obj. (Billion \$)	-20.20	-20.22	-20.22	-20.22	-20.22	-20.23
Wall time	1288	3570	6521	11121	11640	20947

Table 8: Computational Results of Case B - BLD (Unit for time: sec)

Number of scenarios	9	16	25	36	49	64
Number of upper BD iterations	57	52	56	59	67	74
Number of lower DWD iterations	142	148	152	169	184	187
Optimal obj. (Billion \$)	-20.20	-20.22	-20.22	-20.22	-20.22	-20.23
Time for pricing problems	653	1095	1694	2499	3502	4369
Time for DW restricted MP	5.86	13.3	24.9	45.36	63.4	118.6
Time for BD relaxed MP	2.37	1.7	2.92	2.77	0.93	0.97
Total solver time	661	1110	1721	2547	3566	4488
Wall time	1400	1868	2710	3903	4721	5777

Table 9: Computational Results of Case B - ECD1 (Unit for time: sec)

Number of scenarios	9	16	25	36	49	64
Num. of upper BD iterations	65	71	71	77	75	77
Num. of upper DWD iterations	32	35	35	38	37	38
Num. of lower DWD iterations	128	140	137	145	137	143
Num. of pricing problems solved [#]	160	175	172	183	174	181
Optimal obj. (Billion \$)	-20.20	-20.22	-20.22	-20.22	-20.22	-20.23
Time for pricing problems	582	1056	1545	2243	2706	4501
Time for DW restricted MP	30.8	103.7	207	440	749	1245
Time for BD relaxed MP	0.43	0.4	0.45	0.3	0.33	0.4
Total solver time	613	1160	1753	2684	3456	4703
Wall time	1335	2074	2798	4040	5078	6445

Table 10: Computational Results of Case B - ECD2 (Unit for time: sec)

Number of scenarios	9	16	25	36	49	64
Num. of upper BD iterations	37	40	40	43	42	43
Num. of upper DWD iterations	4	4	4	4	4	4
Num. of lower DWD iterations	96	104	107	109	108	110
Num. of pricing problems solved [#]	100	108	111	113	112	114
Optimal obj. (Billion \$)	-20.20	-20.22	-20.22	-20.22	-20.22	-20.23
Time for pricing problems	447	791	1165	1609	2082	2554
Time for DW restricted MP	46.2	86.5	184	370	589	1028
Time for BD relaxed MP	0.24	0.28	0.28	0.32	0.3	0.3
Total solver time	493	878	1349	1980	2671	3582
Wall time	956	1515	2096	2937	3891	4938

[#]: Total number of IPP, DWPP, DWFP, DWPP-R and DWFP-R solved

Figure 9: Comparison of bound evolution in different methods - Case B

435 5.3. Case C

436 In the last two case study problems, we see how the different solution methods perform
 437 as the number of scenarios, or the number of continuous variables and constraints, increase.
 438 In this case study, we examine how the solution methods perform as the candidate integer
 439 decisions increase. We generate 6 new problem instances from the 25-scenario problem
 440 instance in Case B. In case B, the capacities of the 5 plants are multiples of 300 (ton per
 441 unit time); in other words, the interval of two consecutive candidate plant capacities is
 442 300. In this case, the capacity intervals in the 6 new problem instances are 50, 150, 250,
 443 350, 450, 550, respectively. With the increase of capacity interval, the number of candidate
 444 plant capacities decreases, and the resulting MILP tends to be easier (in spite of the same
 445 problem size), because the number of nodes to be explored in branch-and-bound search or
 446 the number of iterations in decomposition tends to reduce.

447 The performance of the different solution methods for the 6 problem instances is shown in
 448 Tables 11-14. In Table 11 that summarizes the results of the monolith approach, we can see
 449 that the problem tends to require more solution time when the capacity interval decreases
 450 (from the rightmost column toward the left). This trend is more clear and consistent in
 451 Tables 12-14 where the decomposition methods are applied. Generally, the decomposition
 452 methods are more efficient than the monolith approach (with an exception that ECD1
 453 is slower than the monolith approach when the capacity interval is 50). Among the three
 454 decomposition methods, ECD1 is the slowest, because it needs the most number of iterations
 455 to converge. When the problem is relatively easy (i.e., the capacity interval is relatively
 456 large), BLD is better than ECD2; otherwise, ECD2 is better. This is because when the
 457 problem is easy, the number of iterations BLD needs to converge is already small (e.g. only
 458 44 BD iterations for the rightmost problem instance); in this situation, integrating the extra
 459 DWD procedure into BLD does not reduce much number of iterations but requires solving
 460 more subproblems. The performance of the decomposition methods is also compared in
 461 Figure 10 in terms of how the bounds change over the iterations. Obviously, ECD2 converges
 462 the fastest for all problem instances, and ECD1 converges the slowest. The convergence rates
 463 of all decomposition methods tend to decrease when the capacity interval increases.

Table 11: Computational Results of Case C - Monolith (Unit for time: sec)

Capacity interval (ton/time)	50	150	250	350	450	550
Optimal obj. (Billion \$)	-20.32	-20.22	-19.67	-19.56	-19.66	-19.63
Wall time	4458	5120	4287	4654	4094	3438

Table 12: Computational Results of Case C - BLD (Unit for time: sec)

Capacity interval (ton/time)	50	150	250	350	450	550
Number of upper BD iterations	103	74	71	48	54	44
Number of lower DWD iterations	244	187	185	127	139	113
Optimal obj. (Billion \$)	-20.32	-20.22	-19.68	-19.56	-19.66	-19.63
Time for pricing problems	2843	2188	2216	1482	1661	1322
Time for DW restricted MP	37.8	32.3	35.5	23.6	20.3	22.6
Time for BD relaxed MP	1.4	0.7	1.2	0.6	0.7	0.8
Total solver time	2882	2221	2253	1507	1682	1345
Wall time	3942	3139	3151	2153	2367	1916

Table 13: Computational Results of Case C - ECD1 (Unit for time: sec)

Capacity interval (ton/time)	50	150	250	350	450	550
Num. of upper BD iterations	113	97	93	71	65	69
Num. of upper DWD iterations	56	48	46	35	32	34
Num. of lower DWD iterations	209	186	179	138	131	130
Num. of pricing problems solved [#]	265	234	225	173	163	163
Optimal obj. (Billion \$)	-20.33	-20.22	-19.68	-19.56	-19.66	-19.63
Time for pricing problems	2514	2218	1925	1543	1491	1324
Time for DW restricted MP	587	402	345	204	191	177
Time for BD relaxed MP	1.3	0.4	0.4	0.6	0.4	0.4
Total solver time	3103	2621	2270	1747	1683	1501
Wall time	5175	4255	3720	2862	2775	2458

Table 14: Computational Results of Case C - ECD2 (Unit for time: sec)

Capacity interval (ton/time)	50	150	250	350	450	550
Num. of upper BD iterations	61	54	51	40	37	39
Num. of upper DWD iterations	4	4	4	4	4	4
Num. of lower DWD iterations	154	146	134	102	103	99
Num. of pricing problems solved [#]	158	150	138	106	107	103
Optimal obj. (Billion \$)	-20.32	-20.22	-19.68	-19.56	-19.66	-19.63
Time for pricing problems	1785	1699	1422	1106	1138	1148
Time for DW restricted MP	465	369	294	171	171	232
Time for BD relaxed MP	0.6	0.5	0.4	0.3	0.3	0.3
Total solver time	2250	2069	1716	1277	1308	1380
Wall time	3610	3356	2792	2028	2129	2149

[#]: Total number of IPP, DWPP, DWFP, DWPP-R and DWFP-R solved

Figure 10: Comparison of bound evolution in different methods - Case C

6. Conclusions

Exploitation of problem structure is often the key to efficient solution of problems that cannot be solved efficiently by off-the-shelf optimization solvers, but the classical decomposition methods can only deal with problems with weak linking constraints. In this paper, we develop two types of decomposition methods for MILPs with both strong and weak linking constraints. The first method (represented by BLD) follows a natural two level decomposition framework, which employs BD for the upper level decomposition and DWD for the lower level decomposition. The second method (represented by ECD) follows the same bilevel strategy, but uses CD (rather than BD) in the upper level decomposition. Both decomposition methods can terminate finitely with an ϵ -optimal solution or a certificate of infeasibility. We further develop two variants of ECD (represented by ECD1 and ECD2), depending on how to alter between BD and DWD in the upper level decomposition. We also demonstrate how to apply BLD and ECD to risk-averse two-stage stochastic programming problems including CVaR constraints, which is a typical example of MILP with both strong and weak linking constraints.

The case study results show that the solution of the risk-averse two-stage stochastic programming formulation is more robust than that of the classical risk-neutral formulation. The results also demonstrate the computational advantages of the three proposed decomposition methods (i.e., BLD, ECD1, ECD2) over the monolith approach. Among the three decomposition methods, BLD tends to give the best performance when the problem needs only a small number of iterations to converge (such as some problem instances in Case C). This is because in this situation, using CD instead of BD in the upper level decomposition does not reduce the number of iteration significantly but requires solving more subproblems. However, when the problem is more difficult (either due to more continuous variables or more candidate integer decisions), using CD in the upper level is more advantageous and therefore ECD1 and ECD2 may outperform BLD (such as in Case A). In addition, ECD2 is almost always better than ECD1, because ECD2 is able to avoid ineffective CD iterations (i.e., the CD iterations that cannot reduce the gap) while ECD1 is not.

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496

List of Abbreviations

BD	Benders decomposition
BLD	Bilevel decomposition
BMP	Benders master problem
CD	Cross decomposition
VaR	Value-at-Risk
CVaR	Conditional Value-at-Risk
DWD	Dantzig-Wolfe Decomposition
DWMP	Dantzig-Wolfe Master Problem
ECD	Extended Cross Decomposition
GUSS	Gather-Update-Solve-Scatter
LD	Lagrangian Decomposition
LP	Linear Programming
MILP	Mixed-integer Linear Program/Programming
RLD	Restricted Lagrangian Dual Problem
SCO	Supply Chain Optimization
SP	Stochastic Programming
<i>Subproblems solved</i>	
BFP^k	Benders feasibility problem at iteration k
BPP^k	Benders primal problem at iteration k
BRMP-BLD^k	Benders relaxed master problem at iteration k of BLD
BRMP-ECD^k	Benders relaxed master problem at iteration k of ECD
DWRMP-ECD^k	Dantzig-Wolfe restricted master problem at iteration k of ECD
DWFRMP-ECD^k	Dantzig-Wolfe feasibility restricted master problem at iteration k of ECD
DWPP^k	Dantzig-Wolfe pricing problem at iteration k
DWFP^k	Dantzig-Wolfe feasibility pricing at iteration k
$\text{DWRMP-R}^{k,l}$	Dantzig-Wolfe restricted master problem at the k th upper level iteration and l th lower level iteration
$\text{DWFRMP-R}^{k,l}$	Dantzig-Wolfe feasibility restricted master problem at the k th upper level iteration and l th lower level iteration
$\text{DWPP-R}^{k,l}$	Dantzig-Wolfe pricing problem at the k th upper level iteration and l th lower level iteration
$\text{DWFP-R}^{k,l}$	Dantzig-Wolfe feasibility pricing problem at the k th upper level iteration and l th lower level iteration
IPP	Initial pricing problem

497 **Appendix**

498 *A. ECD subproblems for CVaR constrained two-stage stochastic programming*

We show the ECD subproblems for solving Problem (CVaR-SP). In ECD, we treat (x_0, ζ) as linking variables. First, the following subproblem is solved to provide the initial point or column for the algorithm to commence.

$$\begin{aligned} \min_{x_\omega, \psi_\omega} \quad & c_\omega^\top x_\omega \\ & x_\omega \in X_\omega. \end{aligned} \tag{CVaR-IPP}_\omega$$

499 Let x_ω^* be the optimal solution of Problem (CVaR-IPP $_\omega$). The initial column used to start
500 the algorithm is $(x_\omega^0, \psi_\omega^0) = (x_\omega^*, 0)$.

501 *A.1 Phase I subproblems*

The DW restricted master problem to solve at the upper level for Phase I is given as:

$$\begin{aligned} \min_{\substack{x_0, \zeta \\ \theta^0, \dots, \theta^{t-1} \geq 0 \\ z_{1,1}, \dots, z_{2,s} \\ z_{2,1}, \dots, z_{2,s} \\ z_3}} \quad & \|(z_{1,1}, \dots, z_{1,s})\| + \|(z_{2,1}, \dots, z_{2,s})\| + \|z_3\| \\ \text{s.t.} \quad & A_\omega x_0 + A_\omega \sum_{i=0}^{t-1} \theta^i x_\omega^i \leq b_{0,\omega} + z_{1,\omega}, \quad \omega \in \{1, \dots, s\}, \\ & - \sum_{i=0}^{t-1} \theta^i \psi_\omega^i + f_\omega(\sum_{i=0}^{t-1} \theta^i x_\omega^i) - \zeta \leq z_{2,\omega}, \quad \omega \in \{1, \dots, s\}, \tag{CVaR-DWFRMP}^k \\ & \zeta + \sum_{\omega=1}^s \hat{p}_\omega (\sum_{i=0}^{t-1} \theta^i \psi_\omega^i) \leq b_0 + z_3, \\ & \sum_{i=0}^{t-1} \theta^i = 1, \\ & x_0 \in X_0, \zeta \in \mathbb{R}, z_{1,\omega}, z_{2,\omega} \geq 0, \forall \omega \in \{1, \dots, s\}, z_3 \geq 0. \end{aligned}$$

502 Let $\mu_{1,\omega}^k, \mu_{2,\omega}^k$ be the Lagrange multiplier associated with the weak linking constraints
503 (first and second group of constraints), and, μ_3^k be associated with the strong linking con-
504 straints (third group of constraints).

The DW pricing problem for Phase I is given below:

$$\begin{aligned} \min_{x_\omega, \psi_\omega} \quad & (\mu_{1,\omega}^k)^T A_\omega x_\omega + \mu_{2,\omega}^k (-\psi_\omega + f_\omega(x_\omega)) + (\mu_3^k) \hat{p}_\omega \psi_\omega \\ & x_\omega \in X_\omega, \psi_\omega \geq 0. \end{aligned} \quad (\text{CVaR-DWFP}_\omega^k)$$

505 Let the solution be given by $(x_\omega^k, \psi_\omega^k)$ and the optimal objective $obj_{\text{CVaR-DWFP}^k} = \sum_{\omega=1}^s obj_{\text{CVaR-DWFP}_\omega^k}$.

506 The solution to Problem (CVaR-DWFP $_\omega^k$) provides a column for Problem (CVaR-DWFRMP k).

The DW restricted master problem at the lower level can be written as:

$$\begin{aligned} \min_{\substack{\theta^0, \dots, \theta^{l-1} \geq 0 \\ z_{1,1}, \dots, z_{2,s} \\ z_{2,1}, \dots, z_{2,s} \\ z_3}} \quad & ||(z_{1,1}, \dots, z_{1,s})|| + ||(z_{2,1}, \dots, z_{2,s})|| + ||z_3|| \\ \text{s.t.} \quad & A_\omega \sum_{i=0}^{l-1} \theta^i x_\omega^{k,i} \leq b_{0,\omega} - A_\omega x_0^k + z_{1,\omega}, \quad \omega \in \{1, \dots, s\}, \\ & - \sum_{i=0}^{l-1} \theta^i \psi_\omega^{k,i} + f_\omega(\sum_{i=0}^{l-1} \theta^i x_\omega^{k,i}) \leq \zeta^k + z_{2,\omega}, \quad \omega \in \{1, \dots, s\}, \quad (\text{CVaR-DWFRMPR}^{k,l}) \\ & \sum_{\omega=1}^s \hat{p}_\omega (\sum_{i=0}^{l-1} \theta^i \psi_\omega^{k,i}) \leq -\zeta^k + b_0 + z_3, \\ & \sum_{i=0}^{l-1} \theta^i = 1, \\ & z_{1,\omega}, z_{2,\omega} \geq 0, \forall \omega \in \{1, \dots, s\}, z_3 \geq 0. \end{aligned}$$

507 Let $\mu_3^{k,l}$ be the Lagrange multiplier associated with the strong linking constraints for (CVaR-DWFRMPR k,l).

The DW pricing problem at the lower level can be written as:

$$\begin{aligned} \min_{x_\omega, \psi_\omega, z_{1,\omega}, z_{2,\omega}} \quad & (\mu_3^{k,l}) \hat{p}_\omega \psi_\omega + ||z_{1,\omega}|| + ||z_{2,\omega}|| \\ \text{s.t.} \quad & A_\omega x_\omega \leq -A_{0,\omega} x_0^k + b_{0,\omega} + z_{1,\omega}, \\ & -\psi + f_\omega(x_\omega) \leq \zeta^k + z_{2,\omega}, \quad (\text{CVaR-DWFP}_\omega^{k,l}) \\ & x_\omega \in X_\omega, \psi_\omega \geq 0, \\ & z_{1,\omega}, z_{2,\omega} \geq 0, \end{aligned}$$

508 Again, let the solution of $(\text{CVaR-DWFP}^k_\omega)$ be given by $(x_\omega^{k,l}, \psi_\omega^{k,l})$ and the optimal
 509 objective $obj_{\text{CVaR-DWFP}^k,l} = \sum_{\omega=1}^s obj_{\text{CVaR-DWFP}^k,l}_\omega$. The solution provides a column for
 510 Problem $(\text{CVaR-DWFRMP}^k,l)$. An initial pricing problem is solved to generate a starting
 511 column for the DWD procedure and was given previously.

The relaxed master problem to solve at Phase I is now given as the following:

$$\begin{aligned}
 & \min_{x_0, \zeta, \eta} \eta \\
 \text{s.t. } & \eta \geq obj_{\text{CVaR-DWFP}^i} + \sum_{\omega=1}^s (\mu_{1,\omega}^i)^T A_{0,\omega} x_0 - \sum_{\omega=1}^s (\mu_{1,\omega}^i) b_{0,\omega} - \sum_{\omega=1}^s (\mu_{2,\omega}^i) \zeta \\
 & + \mu_3^i \zeta - \mu_3^i b_0, \quad \forall i \in U_{feas}^k, \\
 & \eta \geq obj_{\text{CVaR-DWFP}^{i,j}} + \sum_{\omega=1}^s (\mu_{1,\omega}^{i,j})^T A_{0,\omega} (x_0 - x_0^i) - \sum_{\omega=1}^s (\mu_{2,\omega}^{i,j}) (\zeta - \zeta^i) \\
 & + (\mu_3^{i,j}) \zeta - (\mu_3^{i,j}) b_0, \quad \forall (i,j) \in T_{feas}^k, \\
 & x_0 \in X_0, \zeta \in \mathbb{R}, \eta \in \mathbb{R},
 \end{aligned}
 \tag{CVaR-BRMPI}^k$$

512 A.2 Phase II subproblems

In Phase II, the DW restricted master problem solved at the upper level is:

$$\begin{aligned}
 & \min_{\substack{x_0, \zeta \\ \theta^0, \dots, \theta^{t-1} \geq 0}} \sum_{\omega=1}^s \left(c_{0,\omega}^T x_0 + c_\omega^T \left(\sum_{i=0}^{t-1} \theta^i x_\omega^i \right) \right) \\
 \text{s.t. } & A_\omega x_0 + A_\omega \sum_{i=0}^{t-1} \theta^i x_\omega^i \leq b_{0,\omega}, \quad \omega \in \{1, \dots, s\}, \\
 & - \sum_{i=0}^{t-1} \theta^i \psi_\omega^i + f_\omega \left(\sum_{i=0}^{t-1} \theta^i x_\omega^i \right) - \zeta \leq 0, \quad \omega \in \{1, \dots, s\}, \\
 & \zeta + \sum_{\omega=1}^s \hat{p}_\omega \left(\sum_{i=0}^{t-1} \theta^i \psi_\omega^i \right) \leq b_0, \\
 & \sum_{i=0}^{t-1} \theta^i = 1, \\
 & x_0 \in X_0, \zeta \in \mathbb{R}.
 \end{aligned}
 \tag{CVaR-DWRMP}^k$$

Let $\lambda_{1,\omega}^k$, $\lambda_{2,\omega}^k$, and λ_3^k , be the Lagrange multiplier associated with the first, second, and third group of constraints of (CVaR-DWRMP^k). The pricing problem is the following decomposable problem:

$$\begin{aligned} \min_{x_\omega, \psi_\omega} & \left(c_\omega^T + (\lambda_{1,\omega}^k)^T A_\omega \right) x_\omega + \lambda_{2,\omega}^k (-\psi_\omega + f_\omega(x_\omega)) + (\lambda_3^k) \hat{p}_\omega \psi_\omega \\ & x_\omega \in X_\omega, \psi_\omega \geq 0. \end{aligned} \quad (\text{CVaR-DWPP}_\omega^k)$$

513 Let the solution of (CVaR-DWPP _{ω} ^k) be given by $(x_\omega^k, \psi_\omega^k)$ and the optimal objective
 514 $obj_{\text{CVaR-DWPP}^k} = \sum_{\omega=1}^s obj_{\text{CVaR-DWPP}_\omega^k}$. The solution provides a column for Problem (CVaR-DWRMP^k).
 515 The Phase II DW restricted master problem solved at the lower level is given as:

$$\begin{aligned} \min_{\theta^0, \dots, \theta^{l-1} \geq 0} & \sum_{\omega=1}^s \left(c_{0,\omega}^T x_0^k + c_\omega^T \left(\sum_{i=0}^{l-1} \theta^i x_\omega^{k,i} \right) \right) \\ \text{s.t.} & A_\omega \sum_{i=0}^{l-1} \theta^i x_\omega^{k,i} \leq b_{0,\omega} - A_\omega x_0^k, \quad \omega \in \{1, \dots, s\}, \\ & - \sum_{i=0}^{l-1} \theta^i \psi_\omega^{k,i} + f_\omega \left(\sum_{i=0}^{l-1} \theta^i x_\omega^{k,i} \right) \leq \zeta^k, \quad \omega \in \{1, \dots, s\}, \\ & \sum_{\omega=1}^s \hat{p}_\omega \left(\sum_{i=0}^{l-1} \theta^i \psi_\omega^{k,i} \right) \leq -\zeta^k + b_0, \\ & \sum_{i=0}^{l-1} \theta^i = 1, \end{aligned} \quad (\text{CVaR-DWRMPR}^{k,l})$$

516 Let $\lambda_3^{k,l}$ be the Lagrange multiplier associated with the strong linking constraints. Sim-
 517 ilarly, the Phase II DW pricing problem solved at the lower level is the following:

$$\begin{aligned} \min_{x_\omega, \psi_\omega} & \left(c_\omega^T + (\lambda_{1,\omega}^k)^T A_\omega \right) x_\omega + \lambda_3^k (\hat{p}_\omega \psi_\omega) \\ \text{s.t.} & A_\omega x_\omega = -A_{0,\omega} x_0^k + b_{0,\omega}, \quad \omega \in \{1, \dots, s\}, \\ & -\psi_\omega + f_\omega(x_\omega) \leq \zeta^k, \quad \omega \in \{1, \dots, s\}, \\ & x_\omega \in x_\omega, \psi_\omega \geq 0. \end{aligned} \quad (\text{CVaR-DWPPR}_\omega^{k,l})$$

518 Again let the solution of Problem (CVaR-DWPPR $_{\omega}^{k,l}$) be given by $(x_{\omega}^{k,l}, \psi_{\omega}^{k,l})$ with the
 519 optimal objective $obj_{CVaR-DWPPR_{\omega}^{k,l}} = \sum_{\omega=1}^s obj_{CVaR-DWPPR_{\omega}^{k,l}}$. The solution provides a col-
 520 umn for Problem (CVaR-DWRMPR k,l). The Phase II relaxed master problem, a lower
 521 bound to Problem (CVaR-SP) is the following:

$$\begin{aligned}
 & \min_{x_0, \zeta, \eta} \eta \\
 \text{s.t. } & \eta \geq obj_{CVaR-DWPPR^i} + \sum_{\omega=1}^s (c_{0,\omega}^T + (\lambda_{1,\omega}^i)^T A_{0,\omega}) x_0 - \sum_{\omega=1}^s (\lambda_{1,\omega}^i)^T b_{0,\omega} \\
 & \quad - \sum_{\omega=1}^s \lambda_{2,\omega}^i \zeta + \lambda_3^i (\zeta - b_0), \quad \forall i \in U_{opt}^k, \\
 & 0 \geq obj_{CVaR-DWFP^i} + \sum_{\omega=1}^s (\mu_{1,\omega}^i)^T A_{0,\omega} x_0 - \sum_{\omega=1}^s (\mu_{1,\omega}^i)^T b_{0,\omega} - \sum_{\omega=1}^s \mu_{2,\omega}^i \zeta \\
 & \quad + \mu_3^i (\zeta - b_0), \forall i \in U_{feas}^k, \\
 & \eta \geq obj_{CVaR-DWPPR^{i,j}} + \sum_{\omega=1}^s c_{0,\omega}^T x_0 + \sum_{\omega=1}^s (\lambda_{1,\omega}^{i,j})^T A_{0,\omega} (x_0 - x_0^i) \\
 & \quad - \sum_{\omega=1}^s \lambda_{2,\omega}^{i,j} (\zeta - \zeta^i) + \lambda_3^{i,j} (\zeta - b_0), \quad \forall (i, j) \in T_{opt}^k, \\
 & 0 \geq obj_{CVaR-DWFP^{i,j}} + \sum_{\omega=1}^s (\mu_{1,\omega}^{i,j})^T A_{0,\omega} (x_0 - x_0^i) - \sum_{\omega=1}^s \mu_{2,\omega}^{i,j} (\zeta - \zeta^i) \\
 & \quad + \mu_3^{i,j} (\zeta - b_0), \forall (i, j) \in T_{feas}^k, \\
 & x_0 \in X_0, \zeta \in \mathbb{R}, \eta \in \mathbb{R}.
 \end{aligned}$$

(CVaR-BRMPII k)

522 *B. BLD subproblems for CVaR constrained two-stage stochastic programming*

The lower level subproblems for the bilevel decomposition are same to those for ECD.

At the upper level, however, the following Benders relaxed master problem is solved:

$$\begin{aligned}
& \min_{x_0, \zeta, \eta} \eta \\
& \text{s.t.} \quad \eta \geq \text{obj}_{\text{CVaR-DWPPR}^{i,j}} + \sum_{\omega=1}^s c_{0,\omega}^T x_0 + \sum_{\omega=1}^s (\lambda_{1,\omega}^{i,j})^T A_{0,\omega} (x_0 - x_0^i) \\
& \quad - \sum_{\omega=1}^s \lambda_{2,\omega}^{i,j} (\zeta - \zeta^i) + \lambda_3^{i,j} (\zeta - b_0), \quad \forall (i, j) \in T_{opt}^k, \\
& \quad 0 \geq \text{obj}_{\text{CVaR-DWFPR}^{i,j}} + \sum_{\omega=1}^s (\mu_{1,\omega}^{i,j})^T A_{0,\omega} (x_0 - x_0^i) - \sum_{\omega=1}^s \mu_{2,\omega}^{i,j} (\zeta - \zeta^i) \\
& \quad + \mu_3^{i,j} (\zeta - b_0), \forall (i, j) \in T_{feas}^k, \\
& \quad x_0 \in X_0, \zeta \in \mathbb{R}, \eta \in \mathbb{R}.
\end{aligned}$$

(CVaR-BRMP-BLD^k)

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