THE DEVELOPMENT OF UNDERSTANDING OF THE CONCEPT OF VARIABLE
IN GRADE SEVEN BEGINNING ALGEBRA STUDENTS: THE ROLE OF
STUDENT INTERACTION.

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ABSTRACT

This thesis reports on a qualitative study of student interactions in one grade seven mathematics classroom as the students worked through a series of tasks exploring multiple uses of variables. Student tasks were planned out by me, as the teacher and the researcher, and they were executed in my classroom, where I had worked to create a constructivist classroom environment. This study posed two research questions: (1) In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of variable? and (2) In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable?

I used research methods in which I was a participant. Data was collected in the form of audio taped discussions for the participants (working in three groups of four). Audio files of class discussions were accumulated and stored for later review. Written student work and reflections were collected for all class members at the time of the study. From these data sources, the relevant data set emerged.

Analysis came in the form of thick description of eight episodes of importance in which the multiple data sources came together to highlight how student interactions in the form of negotiations may have promoted a shift in understanding of variable. The data showed the complex nature of student interactions along with the potential benefits to student learning. The data showing these benefits were outlined as three patterns of negotiations. These were: negotiations with other students, negotiations with self, and negotiations with the teacher.
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CHAPTER 1
INTRODUCTION

This introductory chapter will outline my research study with its goal of investigating the ways that student interactions in mathematical conversations promote their understanding of the concept of algebraic variable. This thesis will begin by giving a background of why student understanding of algebraic variables is important. In chapter 2, I will outline the related literature in three significant sections: (1) major difficulties that students encounter as they attempt to make the transition from arithmetic to algebra, (2) constructivism and the form of this theory adopted in my classroom, and (3) how discussion and student interaction may promote student learning. I will then discuss the methodology for this study in chapter 3, including the use of a qualitative research design, studies that have been done with similar purposes, participants, procedures, data collection and analysis, and finally, efforts to ensure the quality of the data and results. In chapter 4, I will outline the unit of study from which the results were drawn. I will outline the unit tasks and describe their intended purposes. In chapter 5, the results of the study will be outlined in terms of eight narrative episodes of significance. In chapter 6, the discussion of the results will be looked at thematically according to the three main types of interactions that emerged from the data: (1) interactions with other students; (2) interactions with self; and (3) interactions with the teacher. Finally in chapter 7, the conclusions of the thesis will be presented in terms of responses to the research questions, limitations of the study, and the implications for future research.
Why Student Understanding of Algebra is of Interest

Since I began teaching mathematics I have been struck by the difficulties that students experience with the learning of algebra. These difficulties seem only to be compounded as students progress throughout high school. This study will focus on the idea of variable because the understanding of differing uses of variables is a fundamental concept in high school mathematics and in more advanced mathematics. “Algebra is clearly the backbone of secondary school mathematics. It furnishes concepts and symbolic conventions for representation of very important information in situations that affect each of us in obvious and subtle ways every day” (Christmas & Fey, 1999, p. 21). Thus, it seems appropriate to study the development of understanding of beginning algebra students in an attempt to gain an understanding of the underlying difficulties students experience in learning algebra. This type of study may shed light on the types of classroom interactions that may facilitate students in the development of understanding with regard to the concept of variable.

Variables and their multiple uses are important to study in mathematics, as “without its algebraic symbols, large parts of mathematics simply would not exist. Indeed, the issue is a deep one having to do with human cognitive abilities. The recognition of abstract concepts and the development of appropriate language are really two sides of the same coin” (Devlin, 1994, p. 5).

Understanding of the concept of variable is particularly problematic because the term variable is used to refer to different things in different contexts. Philipp (1999) identified different uses for the term variable as labels, constants (or placeholders),
unknowns, generalized numbers, varying quantities, parameters, and abstract symbols. I started this study with the assumption that most seventh grade students have limited experience with variables but have seen them used as constants or placeholders such as in a question such as, if $x + 4 = 7$, what is $x$? Often the question would not even include the ‘$x$’ but instead there may be a box or a blank space. This type of question leaves students with an understanding of variables as missing numbers, that variables have to be equal to a specific number, that in this case is three. This limited conception of variables causes difficulties for students when they are required to view variables in the other more general ways outlined by Philipp (1999). This study will look at the development of understanding of the concept of variable from that of missing number to that of more generalized abstract symbols and the role that student interaction can play.

Now that the importance of the concept of variable to the understanding of algebra has been established, the next section will look at the related literature. Studies show that students encounter a number of conceptual difficulties along the path to understanding algebraic variables and that they need to make personal understandings at each point to provide a platform for learning. Many have suggested the use of a constructivist approach, in particular a social constructivist approach that involves student interaction in the classroom environment as a key component to student learning. This study will look at how interactions among students within a constructivist classroom may help promote a more complex understanding of the concept of variable.
Potential Significance of this Study

After a close look at the related research, it is evident that although student thought, reflective abstraction, and social negotiation of mathematical meaning among students have been studied in great detail, this has occurred most often with adolescent students in clinical settings. Whole class studies such as those by Yackel and Cobb (1996) have occurred largely with young children. As the following literature review will show, there is a need for research involving the emergence of algebraic understanding of adolescent students using a combination of group activity and whole class discussions as contexts for building mathematical knowledge of the concept of variable. This study is of value because, as acknowledged by Schwartz and Whitin (2000, p. 4), “to support children’s translation of thinking into symbolic relationships about numerical relationships, patterns, and generalizations, a more comprehensive understanding of algebra is needed.” The general focus of the proposed study is interactions of students with each other and with the teacher.
CHAPTER 2
LITERATURE REVIEW

There is a need for research involving the emergence of algebraic understanding by adolescent students while using a combination of group activity and whole class discussions as contexts for building mathematical knowledge of the concept of variable. For most students, an understanding of variable occurs within a classroom context. Keeping in mind the need to link understanding and context to the ways in which learning occurs, the following literature is organized into three main sections. The first outlines the major conceptual difficulties that students encounter when learning algebra. The second outlines constructivism, its various forms, its potential for helping students develop a greater understanding of mathematical concepts, and a description of the form of constructivism that I adopted for the purposes of this study. This will include the challenges faced by a teacher attempting to implement constructivist pedagogy into a classroom that is reflective of current curriculum documents such as those from the National Council for Teachers of Mathematics (NCTM) and the Gouvernement du Québec (2004). Finally, the third section describes the ways in which discussion and student interaction have been found to promote student learning. Specifically, I will look at the ways in which student interaction, occurring in the form of negotiation, may help overcome some of the barriers to learning algebra.
Conceptual Difficulties Involved in the Transition from Arithmetic to Algebra

The area of cognitive difficulties encountered by students attempting to make the leap from arithmetic to algebra have been heavily researched (Booth, 1999; Filloy & Rojano, 1989; Gallardo, 2002; Goodson-Epsy, 1998; Herscovics & Linchevski, 1994; Kieran, 1981, 1992, 2001; Kieran & Chalough, 1999; Lodholz, 1999; Sfard, 1991; Sfard & Linchevski, 1994). The conceptual difficulties faced by students learning algebra can be divided into four main areas. For the purposes of this study, the focus will be on variables, but these four areas of difficulty are not mutually exclusive, so they will all be outlined. They are: (1) Basic Mathematics skills, integers, and order of operations; (2) the multiple meanings and uses of variable; (3) conceptions of the equality symbol; and (4) moving from arithmetical to non-arithmetical equations.

Basic Mathematics Skills, Integers, and Order of Operations

Many of the problems that students encounter with algebra are not actually problems with algebra, but problems with basic math, integers, order of operations, or number properties, which left uncorrected, manifest themselves in major difficulties in the transition from arithmetic to algebra. Swafford and Langrall (2000) stated that students seem to be able to form generalizations of arithmetic that they know well and have much difficulty forming generalizations of arithmetic that they do not understand. Gallardo (2002) and Kieran and Chalough (1999) discuss the idea that a lack of understanding of negative numbers can translate into major obstacles in the transition to algebra. Interestingly though, Gallardo (2002) concluded “it is in the transitional process from arithmetic to algebra that the analysis of students’ construction of negative numbers
becomes meaningful. During this stage, students are faced with equations and problems having negative numbers as coefficients, constants or solutions” (p. 189). English and Warren (1999) found that problems with basic operations and arithmetic weaknesses compounded the problems students had with identifying pattern relationships.

Student conceptions of the associative and commutative properties of numbers also create difficulties in algebra. Booth (1999) explained that some students think that division, like addition, is commutative, that order does not affect the answer.

Multiple Uses and Meanings of Variables

The concept of variable is one that is fundamental to high school algebra as well as more advanced mathematics. Schoenfeld and Arcavi (1999) state that while variables are formal tools in the service of generalization, their multiple uses make the term difficult to understand and are the cause of many difficulties.

Usiskin (1999b) defines variable as “a literal number that may have one or more values during a particular discussion” (p. 8). He went on to explain that variable use is related directly to the conception of algebra. If algebra is conceived as generalized arithmetic, then variables are used as pattern generalizers to translate or generalize; if algebra is conceived as a means to solve certain problems, then variables are used as unknowns or constants to solve or simply; if algebra is conceived as the study of relationships, then variables are used as arguments and parameters to relate or graph; and if algebra is conceived as a structure, then variables are used as arbitrary marks on paper to manipulate or justify.

Wagner (1999) adds an interesting dimension to the discussion of variable by explaining ways in which variables are similar and different to numbers and words.
According to Wagner, variables are like words in that they act as placeholders. They are often chosen to abbreviate words and they can mean different things in different contexts. Variables are different from words because although they can mean different things in different contexts, they must maintain the same meaning throughout the same context, whereas words can change meanings even within the same sentence. The other main difference between variables and words is that while words are made up of letters, there are specific rules about the ways they are used together. In Mathematics, however, the “freedom of delimitation” property, which allows freedom of choice of variables, gives mathematical language much more generality (Wagner, 1999, p. 318).

Wagner also explained that variables are like numbers in that some letters such as $e$ and $\pi$ are actually symbols for specific quantities, and they are also constant in the ways they appear in mathematical statements. Variables are different from numbers in that numbers represent only one quantity, but a variable can represent different numbers simultaneously, yet individually. Variables are also different from numbers because numbers that are juxtaposed are seen as a complete unit, such as 23, but with symbols the juxtaposition means that the quantities are multiplied together. The final difference between variables and numbers is that signs given to numbers match their value, but with variables this may not be the case. “That is, $x$ may represent a negative number and $–x$ may be positive” (Wagner, 1999, p. 317).

Many beginning algebra students experience difficulty with the idea of variables. The multiple ways in which variables can be used is often seen as the main culprit in causing these difficulties. Usiskin (1999a) outlines that variables can be used as unknowns, in formulas, to generalize patterns, as placeholders, and to show relationships.
Usiskin (1999b) points to the problems caused when teachers try to teach an oversimplified conception of variable. He outlines the importance of allowing students the opportunity to explore the different uses of variables. This is to allow students to develop an understanding that the use of variables is inextricably related to the conceptions that are held of algebra and the use of algebra. The different uses of variables create a conceptual dilemma, the constant-variable struggle, which beginning algebra students must come to grips with. Philipp (1999) outlined the main problem with the two conceptions. If a variable is viewed as a constant, then variables are seen as quantities that represent a single, unchanging value. Schoenfeld and Arcavi (1999) referred to a conception of a variable as an unknown, which implies that a variable has a fixed value that is unknown. Thus, the view of variable as an unknown and a constant are closely related. Philipp (1999) stated the view of variable as a variable quantity implies that students have the conception that variables represent many values and that they may assume an unlimited number of values for a particular variable. In order to address these difficulties the researchers all suggest that students be given opportunities to explore the different uses of variables and that they be given opportunities to reflect upon their conceptions of variables. Thus it is important that, when teaching algebra, teachers are aware of the different uses and meaning of variables and that they are discussed explicitly with students. Such discussions should be related to the context in which the variables were explored. Teachers need to be aware of the different ways these symbols are used and recognize the particular characteristics they exhibit in various contexts. (Wagner, 1999)
“...the symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner” (Kieran, 1981, p. 317). Actually students tend to view the equality symbol ‘=’ as a do something symbol and this conception is not easily changed (Kieran, 1981). This is partly to do with the manner in which the equal sign is used in elementary school, in questions such as $3 + 4 = ____$ where students view the left side of the question as what to do and the right side as the answer. Kieran has studied this idea quite extensively (1981). Kieran discusses Collis’ 1974 research in which he found that elementary students, ages 6-10 years old, required that two numbers connected by an operation must be replaced by a third number, the answer, in this case seven. Students of this age can certainly not handle statements such as $4 + 5 = 3 + 6$. Students between the ages of 10 and 13 years old may be able to see the outcome as unique without replacement but only after 13 years of age are students really willing to make these sorts of generalizations. Students developing an understanding of the equal sign as a symbol of equality experience a transition period between requiring an answer after the equal sign and the acceptance of the equal sign as a symbol of equivalence. Consequently, many students continue to view the equal sign in terms of a ‘do something’ symbol, even into their college Mathematics courses (Kieran, 1981). Collis attributes this to a student’s inability to accept lack of closure in a mathematical sentence. That is “their ability to hold unevaluated operations in suspension” (Kieran, 1981, p. 319).
Moving from Arithmetical to Non-Arithmetical Equations

Filloy and Rojano (1989) outline that, in order to solve equations, students must be able to make a break between arithmetical and non-arithmetical types of equations. Arithmetical equations such as $3x + 4 = 10$ are those that involve only one unknown and can be solved using only simple arithmetic or trial and error methods. Non-arithmetical equations such as $3x + 8 = 4x + 10$ are those that cannot be solved easily through arithmetic and often involve more than one unknown or a negative value. Filloy and Rojano (1989) discuss the idea of a didactical cut to express how students’ arithmetic experiences can cause difficulties with their transition to algebra. The transition is the one that occurs when students attempt to solve equations in the form $Ax + B = Cx + D$, where $A$, $B$, $C$, and $D$ are specific numbers. Although students may have been successful with equation types such as $Ax + B = C$, their success was due to their ability to use inverse arithmetical operations. Their lack of success with the first type of equation was due to an inability to use arithmetic operations to find a solution. Another challenge that occurs in making a transfer from arithmetical to non-arithmetical equations is that “…although some modification of the arithmetical notions must take place in order that the new algebraic notions may be acquired, it is also necessary for the arithmetical knowledge to be preserved” (Filloy & Rojano, 1989, p. 19).

Goodson-Epsy (1995) states that although Filloy and Rojano make the distinction between arithmetical and non-arithmetical equations, they did not describe why the difficulties occur. She explains, that in order to solve non-arithmetical questions such as
those above, “students must construct an understanding of an equation as two quantities that are the same” (Goodson-Epsy, 1995, p. 5).

*Interest in the Study of Student Learning of Algebra*

Although this study will focus on student understanding of the concept of variable, I have briefly outlined areas of possible conceptual difficulty faced by students when attempting to make the transition from arithmetic to algebra as these potential difficulties are interrelated. The choice of focus on variable is because “variables are very basic in algebra. They provide the algebraic tool for expressing generalizations. But, the concept of variable is more sophisticated than we often recognize and frequently turns out to be the concept that blocks students’ success in algebra” (Leitzel, 1989, p. 29).

Concerns regarding student difficulties with learning algebra are not new. These concerns have been around for years and have persisted through the educational reforms of the 1960s and 1980s. Dessart (1964) outlines many different techniques that were attempted in order to improve the learning of algebra at that time. These involve accepting children as active participants in their own learning, a back and forth shift between the importance of students developing correct mathematical language and the importance of correct student action, and the development and attempt to implement alternative programs. Factors such as length of class period, and student attitude were also looked at in the 1960s (Dessart, 1964). Norton (1978) discusses a team teaching and learning experiment aimed at increasing student success in algebra by having students work in groups of four to teach each other, discuss concepts, and learn to explain ideas to each other. Weber (1978) states that student involvement in learning positively influenced student achievement.
Dessart (1964) concluded that since many of the research studies in his report were contradictory in their results, more studies were needed. Dessart listed eight areas where he thought research is needed. It is interesting that although some things have changed, many of the areas requiring future study are still relevant forty years later. For example: outlining the objectives of mathematics instruction, evaluating the long-term effects of new curriculum efforts, the discovery of how student interactions with each other, teachers, and instructional material influence learning, and the coordination of studies into a comprehensive body of knowledge related to a subject area (Dessart, 1964).

Given the historical interest in the success of algebra learning, it is not surprising that the NCTM has made special provisions for algebra in its recent Standards document (2000), stating algebraic content standards for all students from pre kindergarten to the end of high school. In the process standards the NCTM (2000) outlines problem solving, reasoning and proof, and communication as key features of successful learning. The Gouvernement du Québec (2004), in its most recent reform movement, advocates similar features for the learning of mathematics with its three main competencies for student learning being their ability to solve situational problems, use mathematical reasoning and justifications, and communicate ideas to others.

The ideas of reasoning, building of understanding, and the communication of these ideas are rooted in a view of teaching and learning called constructivism. Some of the principles of constructivism and these reform movements can be seen in my classroom where this study took place. The following section will outline constructivism in its various forms, highlight possible characteristics of constructivist pedagogy, and outline the version of constructivism that formed the basis for my study.
Constructivism

This study of the development of understanding of the concept of variable occurred during classroom activities in a specific classroom that has elements that I classify within the realm of constructivism. Given the various ways in which the term constructivism has been used, the lack of a clear definition, and the obscurity of its implications for pedagogy, it is important, at this point, to highlight some agreed upon principles of constructivism. I will also outline some different forms constructivism can take, possible characteristics of pedagogical practice, and how my study falls into the realm of what is considered constructivism.

The purpose of my study relates to interaction in a mathematics classroom and is based on the idea that individual students create their own subjective knowledge within the social context of the classroom. The general term for this idea is constructivism (Ernest, 1992). Noddings (1990) outlined some general principles that are agreed upon by most constructivists. These are: (1) all knowledge is constructed, at least in part, through reflective abstraction, (2) cognitive structures are activated during the construction process, (3) these cognitive structures are under constant development, enhanced through purposeful activity, and (4) the acknowledgement of constructivism as a cognitive position leads to an adoption of methodological constructivism in research and teaching methods.

Constructivism has been highly debated in educational research especially in the context of mathematics, particularly with regards to the objectivity of knowledge. Even among those who call themselves constructivists, there is no consensus as to what it implies for learning. There are many versions of constructivism. Prawat (1996) describes
six versions and Richardson (2003) suggests that there might even be up to 18 different versions. For the purposes of situating my research into the realm of constructivism, I will describe Prawat’s (1996) six versions of constructivism, distinguish between the two main types of constructivism, radical and social, outline some characteristics of constructivist pedagogical practice, and then outline the view of constructivism that will guide this study.

**Different Versions of Constructivism**

For the purposes of this study I will expand upon Prawat’s (1996) notion that there are six versions of constructivism that fall into two main categories: modern and postmodern. The modern versions are schema (radical) constructivism and information-processing theory that represent traditional epistemological stances towards constructivism and hold that knowledge is primarily the property of individuals. The postmodern versions are generally called social constructivism and refute the idea that knowledge is primarily the property of the individual. Social constructivism can be further delineated into sociocultural theory, symbolic interactionalism, social psychological construction, and Deweyan versions of constructivism. The critical difference between the various forms of constructivism seems to be the focus given to the individual or the collective. This distinction can be described in terms of the mind-world dilemma. Cobb (1994) explains the mind-world dilemma as a general conflict between radical constructivists, who view the mind as being located in the learner’s head as an individual account of cognition and the social constructivists who view the mind as being located in the individual as part of the social action. The mind-world dilemma poses problems for pedagogy if either extreme is adopted. This will be described later. First, I
will describe Prawat’s (1996) six versions of constructivism. Each answers the mind-world dilemma differently in terms of the focus of cognition on the individual or the collective.

*Modern Versions of Constructivism*

The modern versions of constructivism are schema (radical) constructivism and information-processing theory. They hold that knowledge is the sole property of the individual.

*Schema (radical) constructivism.* Schema (radical) constructivism has roots in the ideas of Piaget and more recently in ideas of Glasersfeld (1990). The focus is on the ways in which individuals construct their own knowledge through the assimilation and accommodation of new understandings into existing cognitive schemes. Piaget attributed most learning to cognitive maturity. Piaget’s idea of reflective abstraction is one of importance in the discussion of constructivism. Cobb, Bouffi, Mclain, and Whitenack (1997) defined reflective abstraction, as Piaget did, as a process or episode that allows individual children to reorganize their mathematical activity. It may occur first as a whole class but must ultimately occur on an individual level for true conceptual development to occur. The interpretations, within radical constructivism, of Piaget’s ideas emphasizing individual construction of knowledge, have been criticized for not giving enough emphasis on social influences on learning.

*Information-processing constructivism.* The second variety of modern constructivism is information-processing. Prawat (1996) described this variety as one where cognitive structures are built in the individual’s head and the validity of these structures is judged according to their fit with structures present in the world.
Assimilation of information into existing cognitive structures is more the focus in this variety than is accommodation in the radical variety.

Postmodern Versions of Constructivism

According to Prawat (1996), the postmodern versions of constructivism came about as solutions to three main challenges to modern versions. These are: (1) knowledge is primarily the property of the individual and there may be no objective reality, (2) science will solve the mind-world dilemma, once and for all, proving where the mind is located, and (3) knowledge must be the product of an inferential system based on facts and proof of conclusions. Social constructivism has its roots in the ideas of Vygotsky (1986) that individuals construct knowledge through social interactions. Ernest (1992) stated that, at any time, mathematics is determined by a set of artifacts such as books or papers, a set of people, and a set of linguistic rules followed by these people. Therefore, social interactions and negotiation of meaning are believed to influence the construction of individual cognitive functions. Although teachers may wish to maintain classroom debate at an intellectual level, it is important to keep in mind that the negotiation of meaning in a classroom environment, as in society at large, is influenced by power relationships, and dominant opinions (Ernest, 1992; Richardson, 2003). In a classroom, power relationships such as teacher-student and dominant opinions such as those of the teacher and of students whose peers deem them adept at mathematics, have an impact on interactions and can impact the views students express in discussions.

Vygotsky (1986) called the disparity between what students can accomplish unassisted and what can be accomplished when assisted the zone of proximal development (ZPD). Learning activities, for success, must occur in this zone to result in
conceptual change. Pedagogical problems can be seen with the type of constructivism that focuses too heavily on the social collective as Vygotsky has been criticized for doing so at the expense of the individual. This will be described later. First, I will outline the four versions of social constructivism.

Despite agreement on the challenges addressed by social constructivism, the four versions represent very different solutions. The four types of social constructivism, described below, are: sociocultural theory, symbolic interactionalism, social psychological construction, and Deweyan constructivism (Prawat, 1996).

Sociocultural theory of constructivism. Sociocultural theory rejects the notion that knowledge is the property of the individual and instead views knowledge as the property of the collective, as a social construct. Psychological tools are seen as mediators between the child and reality. Objects and events in the world exert an indirect effect on tools or artifacts. Through processes of shared meaning, cultural artifacts connect individuals to society and society to individuals. The individual constructs the social and at the same time is constructed by the social. Thus “sociocultural theory relies on socially constructed artifacts to define individuals within a culture” (Prawat, 1996, p. 221).

Symbolic interactionalism. Symbolic interactionalism is the version of social constructivism that takes into consideration both the individual and the collective in equal measure. Prawat (1996) stated that its strength is that it accounts for how a group of individuals build up collective knowledge but also how individuals within the group have their own unique thoughts about the collectively created meaning. Individuals do not automatically come up with the same personal meaning as another person in the group.
Instead, individuals create their own meanings and act towards objects according to the meanings they have created for them. Artifacts are not seen as extensions of the individual but rather as part of the object world to which the individual reacts, thus “symbolic interactionalism relies upon socially constructed artifacts to define objects and events in the world” (Prawat, 1996, p. 221).

**Social psychological constructivism.** Social psychological constructivism holds that the mind is in language and that language as a result of communal relations is the carrier of truth. The view is that there is nothing outside language to which individuals may refer in order to validate the truthfulness of the language the community has chosen to use (Prawat, 1996). Prawat outlines some of the inherent weaknesses of this version of constructivism as it’s leaving no room for reality in its construction and that it does not account for how individuals manage to operate simultaneously on the individual and collective levels, in order to create new knowledge and understanding.

**Idea based (Deweyan) constructivism.** The final version of constructivism, according to Prawat (1996) is idea based (Deweyan) constructivism. He explains that this version attempts to solve the mind-world dilemma through its focus on ideas. Ideas can, under this view, offer a solution to the dilemma as they can move back and forth between the barriers that separate mind from world. Actions are also seen at the heart of ideas and knowing is seen as doing. Prawat (1996) explained that the Deweyan version of constructivism can be seen as a solution to the problem that social constructivism offers no way for students to carry on a dialogue with the real world of objects and events.
Now that the six main versions, as seen by Prawat (1996) have been outlined, I will outline some possible characteristics of constructivist pedagogy and explain the view of constructivism that will guide this study.

*Possible Characteristics of Constructivist Pedagogy*

Cobb (1988) notes that constructivism does not provide a definite pedagogical plan. He explains that while constructivism is attractive as a theory of learning, it causes great problems when one tries to apply it to instruction, because the assumptions of constructivism are in direct opposition to those of traditional transmission type teaching. He further explains that teaching results from varying degrees of imposition (transmission view) and negotiation (constructivist view) and that the prescription of instructional recommendations from theories that emphasize structure and meaning are problematic at best, because we cannot explain how students construct concepts that are more advanced and complex than before the instruction began.

The lack of a pedagogical plan in making the transition from the belief in constructivism as a learning theory, to its use as an instructional theory leaves teachers with the challenge of figuring it out for themselves. Richardson (2003) outlines five characteristics of practice that have been seen in classrooms where teachers claim to use constructivism as a guiding theory. These are (1) attention to the individual and their personal backgrounds, (2) facilitation of group dialogue with the purpose of developing shared understandings, (3) provision of opportunities for students to justify and explain responses and possibly change or add to their existing beliefs and understandings by engaging in the planned tasks, (4) development of students’ awareness of their own understandings and learning processes, and (5) planned and unplanned use of direct
instruction, references to text or web sites. This last characteristic has been questioned because direct instruction is often seen as representing the transmission view of teaching and learning and is not usually associated with constructivist pedagogy (Richardson, 2003). It can be seen however, as a practical modification as the theory of constructivism meets the reality of classroom teaching and learning. Teachers attempting to put constructivist theory into practice must strike a balance between teaching methods. Teachers have the ultimate responsibility for student learning and sometimes in the name of moving lessons along, direct questioning or instruction is used.

Although the five characteristics can be viewed as elements of constructivist pedagogy, they are not specific practices. They are approaches to strive for and eventually become part of a teacher’s practice. They represent one way of helping students learn; however, if we accept constructivism as a learning theory, then we must also accept that students can and will learn and create meaning from various sorts of instruction, many of which can be classified into the realm of constructivism so there is not only one way to teach according to constructivist principles (Richardson, 2003).

Constructivist pedagogy can be “thought of as the creation of classroom environments, activities, and methods that are grounded in a constructivist theory of learning, with goals that focus on individual students developing deep understandings in the subject matter of interest and habits of mind that aid in future learning” (Richardson, 2003, p. 1627). Given this general definition of constructivist pedagogy and the acknowledgment that students learn in different ways, teachers may find it useful to try to find a balance between what Sfard (1998) called the acquisition and participation metaphors of learning, especially as a starting point in moving towards constructivist
pedagogy. The acquisition metaphor focuses on concept development and learning acquisition of the individual. The teacher plays a central role in the transmission of knowledge. Following the participation metaphor, knowing comes about through evolving bonds between individuals and others based on communication and discourse. The teacher plays the role of facilitator and questioner, versus dispenser of knowledge.

Links to Quebec Curriculum Reform

The Quebec educational reform program (Gouvernement du Québec, 2004) focuses on the individual child and underscores the idea of success for all. It recognizes that success is defined by the individual in relation to the student’s specific challenges, talents, and needs. The Quebec reform is built around competencies that will prepare students for our fast-paced, ever-changing world. Knowledge that students will need to be successful in the work place of tomorrow is far different than the knowledge required for success in the work place of today. Thus it is vital that students learn how to learn as knowledge needed on a specific topic is no longer enclosed in a single textbook or presented by a single teacher. It is also vital that we focus on assisting students in developing competencies that will be useful in a number of fields.

The idea of teaching using constructivist principles is one that is promoted in the Quebec Educational Program (QEP) as useful across the disciplines. The following excerpt from the introductory chapter situates my research study in the context of teaching in Quebec.

The program is based on the premise that knowledge should be constructed by students rather than transmitted by teachers, because no one can learn for another person. Although it is not based on one particular approach, it draws on several theories that share a recognition that learners are the main architects of their competencies and knowledge. The constructivist, social
constructivist and cognitivist theories of learning are particularly useful in this regard:

— constructivism, because it presents knowledge as the result of actions (originally concrete and subsequently internalized), that are taken by individuals in relation to objects, representations or abstract statements.

— social constructivism, because it stresses the social character of thought and learning, and views concepts as social tools that support the exchange of viewpoints and the negotiation of meaning.

— cognitivism, because it describes the processes enabling individuals to incorporate new knowledge into their knowledge system and use it in new contexts.

People involved in applying the QEP may find these theoretical approaches helpful for purposes of constructing tools of thought and intervention strategies.

While it is the responsibility of the Ministère de l’Éducation to establish the aims of the education system, it is up to school staff to define ways to achieve them. However, since students cannot, logically, learn to think if their activities are limited to rote exercises, even without specifying any particular approach, the program has implications for pedagogical practices. It is not so much a question of following one school of thought or another, but of creating learning situations and pedagogical contexts that promote the development of competencies. This paradigm shift presents new educational challenges, but it also offers many opportunities for rich and stimulating pedagogical experiences. (Gouvernement du Québec, 2004, p.9)

This excerpt from the QEP rationalizes the type of study I conducted. Although it clearly states expectations in terms of results, it does not give concrete ways to achieve them. In fact, it states that it is up to teachers to figure out how to enact the QEP. Thus, once again, as the research on constructivist pedagogy states: suggestions are made of the successes of constructivist principles but no concrete ideas regarding pedagogical implication are given (Richardson, 2003). In relation to the teaching of Mathematics, the QEP states that “students are active when they take part in activities involving reflection, manipulation, exploration, construction or simulation and have discussions that allow them to justify their choices, compare their results and draw conclusions” (Gouvernement
de Québec, 2004, p.195). Given both the general philosophy of the curriculum reform and its implications for the teaching of Mathematics, a study of how discussion can promote understanding was justified.

*The View of Constructivism that Guided this Study*

Considering Richardson’s (2003) five characteristics of constructivist pedagogy and the fact that Baroody and Ginsburg (1990) and Goldin (1990) classify guided discovery teaching as constructivist, it is certainly accurate for me to include my classroom pedagogy within the realm of constructivism. In my classroom, activities were designed to allow students to create their own knowledge of a particular concept. There was a focus on the development of understanding in the activities that were carried out, through the encouragement of students to work as a team, and through class discussions during which activities and possible solutions were discussed overtly. Students were encouraged to create meaning from the activities, were frequently asked why, not simply how to perform a certain mathematical calculation. Students were encouraged to use the proper mathematical language and to make connections between the topics of study. During class discussions, I encouraged the students to ask questions of other students, to share ideas, to justify solutions. The students were respectful of each other and consequently, were willing to share their ideas. Student errors or limited conceptions were used as a springboard to create growth in understanding, not viewed as shortcomings. For these reasons, I qualify the teaching in my classroom within the realm of constructivism.

As mentioned above, adopting solely either pole of radical or social constructivism, can cause pedagogical problems. Piaget has been criticized for placing
too much emphasis on the individual and Vygotsky has been criticized for placing too much emphasis on the collective, at the expense of the individual. I argue that the adoption of one approach, without the other, essentially results in missing an important aspect to understanding the development of student understanding. Therefore, the view of constructivism that will guide this study is one that essentially combines the ideas of Piaget and Vygotsky into a comprehensive view of student learning.

The main problem with using the ideas of Piaget or Vygotsky in application of a pedagogical model is that neither studied learning in modern classrooms with the mandates and restrictions that teachers need to manage. Thus, the ideas of both should be taken into account when creating a practical model. Shayer (2003) argues that the views of Vygotsky and Piaget were actually complementary. In practical terms, what this means to me for this study, is that while learning takes place on a social level through discussion and negotiation, individuals may, through reflective abstraction, assimilation and accommodation, modify existing cognitive structures. However, these processes are only possible if they occur within a specific developmental range, hence, the zone of proximal development.

This view accounts for the psychological analysis of individual activity and the analysis of classroom interactions and discussions and is the model used by Cobb, Wood, and Yackel (1990) called the social-psychological approach. It emphasizes the importance of meaning and interpretation of symbol systems such as language, in developing shared meanings. It also calls for the negotiation of meaning within the classroom environment that requires students to justify and explain their solutions and ideas to the other members of the mathematics community (Cobb, Wood, Yackel &
McNeal, 1992). In the case of my classroom study the mathematics community was represented by the teacher and the students in the classroom. As students justified their answers, they pointed to reality (real objects being manipulated) and moved beyond describing and explaining. Through the process of negotiation, intersubjectivity of knowledge came about as students came to parallel conclusions that were taken as shared within the classroom community (Cobb, Yackel, & Wood, 1992).

**Learning Through Discussion and Interaction**

The notion of learning through communication and interaction with others is at the heart of social constructivism. Learning through communication is one of the underlying principles that the NCTM has built out of the idea of constructivism, despite the fact that the term constructivism cannot be found in the *Standards* document. In its Professional Standards for teaching mathematics (1991), the NCTM focuses on discourse as they outline that worthwhile mathematical tasks are ones that promote discourse and student engagement. The NCTM uses the term discourse to refer to the different ways that teachers and students represent, think, talk, agree, or disagree while engaged in worthwhile tasks. The NCTM (1991) even outlines the specific roles of teachers and students in discourse. The role of the teacher is to pose questions, listen to students’ ideas, ask students to justify their thinking, decide when to intervene with information or modeling, and monitor students’ participation in class discussions. The role of the student is to listen to the questions posed by the teacher, to use a variety of tools to help them make conjectures about them, explore examples and counterexamples to investigate conjectures, and to convince themselves and others of the validity of certain ideas and solutions. I do not think that it is coincidental that these teacher and student roles
correspond to the types of roles that teachers and students hold in the constructivist classroom. It is evident that the principles of constructivism were in mind when the Standards were written. In fact, in reflecting on the writing of the Standards, Rhomberg, the project leader, notes “The term that we did not use in writing up the Standards (but we certainly talked about) is what might be called the social constructivist’s notion of learning” (Rhomberg, as quoted in McLeod et al., 1996, p. 38).

In order to look at the ways that discussion and interaction affect student learning, it is important to determine what is meant by mathematical discussion and then look at some studies that have focused on discussion in the learning of mathematics before stating what remains to be studied in this area.

Pirie and Schwarzenberger (1988) define mathematical discussion as purposeful talk regarding a mathematical topic with genuine student contributions and interactions. Effective discussion is not automatic. It depends upon many factors, such as the structure of the task, the age and motivation of the students, the social atmosphere of the classroom, and the social relationships within the group (Hoyles, 1985). Effective discussion can promote student ability to form a view of a mathematical idea, reflect upon it, communicate and defend their ideas to others, and to incorporate elements of another’s perspective into their own (Hoyles, 1985).

Many studies looking at student discussion and interaction have occurred within a social constructivist teaching and learning environment. Social constructivist teaching methods have also been used extensively by teachers in the primary classrooms studied by Cobb, Yackel, and Wood (1992), Ball (1993), Lampert (1990), and Ball and Lampert (1999) in their studies of their own upper elementary school students. “The classroom
community is often, as the children note, a source of mathematical insight and knowledge. The students hear one another’s ideas and have opportunities to articulate and refine or revise their own. Their confidence in themselves as mathematical knowers is often enhanced through this discourse” (Ball, 1993, p. 394). Ball does caution the reader that discourse has the potential to spark confusion when consensus cannot be reached and she notes that teachers need to carefully balance the need to allow students autonomy in their discussions, with their professional responsibility to not leave students in a constant state of confusion. This is where teachers, in an attempt to apply the ideas of constructivism in a realistic classroom environment, may turn to more direct teaching methods.

Ball (1993) discusses using classroom discourse in her third grade classroom as a window into her own teaching. She discusses classroom discourse as one of three components to mathematical practice, the other two being content and community, that are interwoven with discourse. Ball notes that students’ sense making is individual and consensual as students make arguments and try to convince others of their views. The mathematical content looked at in this study was number theory (negative/positive, odd/even numbers). Ball (1993) outlines that trying to teach through discourse while respecting her belief in constructivist learning theory created some dilemmas. The first dilemma was that of representing or covering the curriculum of the content heavy Mathematics course. This dilemma came about because teaching constructively for understanding initially takes a bit longer than traditional methods; however, Ball felt that the payoff is the potential for much deeper understanding. The second dilemma, according to Ball, is respecting students’ thoughts, while trying to figure out what they
really understand from what they say. She must then help build bridges between what they know and what they need to learn. The difficulty here is often learning to respect even the students’ misconceptions. The third dilemma is creating and using community. Creating a classroom community where opinions are valued and students feel comfortable sharing their thoughts is not simple and requires a constant effort. Ball outlines that her dilemma lies in her role as the authority for knowledge and when to intervene in group discussions. Despite these three dilemmas, Ball advocates the use of class discussion to promote mathematical thinking and doing in her classroom.

Lampert (1990) looks at classroom discourse within a fifth grade classroom as a regular sequence of lessons on exponents was taped and transcribed, in order to get students’ individual understandings of the concept and perceptions on the development of their knowledge. She acknowledges that in classrooms where discourse forms the basis of building mathematical knowledge and the teacher is not the centre of activity, students' efforts result in answers that were more than solutions and problems that were deeper than the questions asked. What Lampert did not do was to verify the development of individual students’ thinking throughout the class discussions. In some cases, students changed their verbalized ideas, but no further attempts were made with individual students to see if these changes were simply in agreement with their classmates or whether they represented a true conceptual change.

Much of this deep thinking developed from these class discussions manifested itself in disagreement amongst the students. In fact, Lampert purposely designed questions that would lead to student disagreement to encourage discussion. In these discussions it is “expected that students will consider and challenge one another’s
assertions, and even challenge the teacher, and presumably by using mathematical
evidence, convince others of the reasonability of their claims” (Lampert, Rittenhouse, &
Crumbach, 1998, p. 738). It is also important that students learn to disagree respectfully,
that they present their ideas logically, and that they learn how to change their thinking
during the course of a discussion, while saving face in front of their peers. In order to
accomplish this, the class has to develop some sociomathematical norms for discussion.
The idea is to encourage a safe environment for exploring solutions, for making
justifications, and for changing one’s thinking. If students do not feel comfortable that
their ideas are valued, then they will not contribute to class discussions (Lampert,

Clarke (1998) also studied the negotiation of meaning in authentic classroom
discussions. In his study of high school mathematics and science classes, he combined
data from videotaped lessons, and student interviews. He used a team of researchers with
diverse areas of expertise in the analysis in order to come up with a multifaceted analysis
of the processes students use to create meaning during classroom discussion. For the
purposes of analysis, the transcripts were divided into episodes involving many
negotiative events. The negotiative events were subdivided into utterances to get at
student thinking. Clarke claimed that such complementary accounts, with all their
complexity, come closer than other methods, to capturing the true complexity of
classroom practices. “Such complementary accounts have the potential to be mutually
informing and to constitute in combination a richer portrayal of classroom learning than
would be possible by the consideration of either account separately” (Clarke, 1998, p. 110).
Cobb, Yackel, and Wood (1992) outline their ongoing study of second-grade mathematics classrooms that aimed at making connections between individual learning and group development. Class lessons were video taped and transcribed for analysis. The study focused on taken as shared meanings within the classroom and the development of intersubjectivity. The ongoing study has resulted in many papers, all of them relating to the learning of mathematics through class discussions. Cobb, Wood, Yackel, and McNeal (1992) focused on the types of classroom norms that elicit effective discussions. Cobb, Yackel, and Wood (1992) looked at the taken as shared meanings in a classroom and the learning that takes place through social interaction. They state that learning is a circular, not linear process. They outline that while the goal is for students to come to a taken as shared meaning, that incompatible meanings often occur. Readers of their work are cautioned that incompatible meanings often do not create discussion because communication is impossible if nothing is agreed upon.

After considering what has been studied with regards to learning through discussion and interaction, it is clear that much research has been done on the development of sociomathematical norms of class discussions and the development of collective knowledge base within the classroom. Most of these studies claim that individual understanding was affected through class discourse and use transcripts of students changed points of view as evidence. These studies do not use methods to determine if the students’ changes of opinion represent true conceptual change or whether they represent an attempt to agree with their classmates. The other thing that these studies do not do is to look specifically at the development of understanding of algebraic variable within a grade seven context. Most of these studies are located in elementary
classrooms and look at concepts related to number theory or fractions. The Clarke study looks at a high school setting, but the content of study was not made explicit. In this study I attempt to make students’ transition from arithmetic to algebra (with regards to variables) explicit, by using transcripts of class discussions and student written reflections, to allow me to determine ways in which the group and class discussions promoted understanding of variable.

Model of Growth of Understanding

Pirie and Kieren (1994, 1992) developed a psychological model to map student understanding. The model divides student thinking into eight levels of increasing sophistication. Although their model in its entirety is not applicable to my study, one key aspect of reflecting or ‘folding back’, will be looked at in terms of how it facilitates student growth in understanding. Folding back is what happens when students reflect back to what was previously done and understood in order to move ahead. Folding back occurs naturally as students face problems they are not sure how to solve. They revert in their thinking back to a level that is understood, in order to move on. Thus, folding back can be seen as having a scaffolding type role in assisting students to build layers of knowledge.

The Pirie and Kieran model, including the folding back feature, relates on an individual psychological level to the more epistemological approach of Sfard (1991). Although Sfard does not specifically mention, in her theory of reification, the idea that prior knowledge is referred back to in order to solve more complicated problems, the notion of prerequisites implies what Pirie and Kieren (1994) called folding back. Sfard contends that lower level reification and higher-level interiorization are prerequisites for
each other. This means that as a concept has become reified (structurally conceived) it forms the prerequisite upon which the operational conception of the next concept begins. This also relates to the Pirie and Kieren model as the eight stages are built upon each other. Cobb, Bouffi, Mclain, and Whitenack (1997) quote Freudenthal’s (1973, p. 125) work that “the activity of the lower level, that is the organizing activity by means of this level, becomes an object of analysis on the higher level; the operational matter of the lower level becomes a subject matter on the next.” These ideas are similar to Mason’s (1989) ideas of how shifts in attention can help individual students make connections between observed patterns and abstract mathematical principles. He contended that a shift in attention is accomplished through a process of manipulation, getting a sense of, articulating and starting over, as the original pattern moves from the foreground of discovery, to the background of investigation. He explained that being able to see the expression as a manipulable object does not replace being able to see the expression as a symbol of generality, instead Mason argued that abstraction allows learners to see mathematical constructs in both ways.

Problem Statement and Research Questions

Now that I have outlined the importance of student learning of algebra, constructivism in its various forms, and the impact of learning through discussion, I will bring together the three main areas of related literature to form my overall research question. A review of the related literature with regards to algebra, constructivism, and learning through discussion situates my study and left me with the problem statement that guided my research: How does student interaction in a constructivist classroom
environment promote student understanding of the concept of variable? My specific research questions are:

1. In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of variable?

2. In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable?

For the purposes of this thesis, student interaction will be referred to as student verbal communication or talk about Mathematics in which the concerned parties are engaged and actively attempting to construct Mathematical knowledge.

Now that my problem statement is clearly situated in the related literature, focus must shift to the methodology that is used in the study. This methods section will commence by situating my study within a qualitative research approach. It will be preceded by a statement of the significance of the proposed research as well as a restatement of the specific questions that will be investigated. Details regarding the study’s participants, procedures, data collection instruments, methods of data analysis will be outlined, ending with a discussion of how I attempted to ensure quality in my research design.
CHAPTER 3

METHODOLOGY

Qualitative Research Approach

Given that the purpose of this study is to gain an understanding at the descriptive level of the how specific interactions promote changes in student understanding, I used a qualitative methodology. As pointed out by Bogdan and Biklen (1998) qualitative research is best used in a naturalistic setting to investigate a process through the use of descriptive data. Qualitative studies are based on a process approach to causation that attempts to directly investigate causal mechanisms through a major emphasis on context. (Maxwell, 2004). Given that this study aimed at describing the development of student understanding throughout a progression of lessons on a particular topic, in a particular environment, process study was extremely important. Furthermore, learning occurs in fits and bounds over time, not in a linear fashion, thus process study is key to the effectiveness of my research design.

Participants

The participants for my study were students from my grade seven Mathematics class in the school year 2004-2005. Most of the students simply participated in regular class activities and discussions. This study will focus on the cases of twelve students, working and studied in three groups of four. These twelve students were chosen from consenting students from the population of my complete grade seven class of 32 students, from the regional high school where I was teaching in Quebec at the time. The rural public high school where the study was conducted serves a large territory and has a population of 1000 students from grades 7-11. The nearest public English high school is
over an hour’s drive away. Therefore student ability, socioeconomic status, and motivation for school success varied between students.

Procedure

From my grade seven class described above, my subjects were selected using the following procedure. First, I sent home the information letter and the letter of consent to the parents of all 32 students (see Appendix A and B). The students consenting to be full participants returned these consent forms to my vice-principal. Once the deadline had come up for the return of the forms, the vice-principal, along with my teaching partner, created groups of students for the study. My teaching partner was the language arts teacher who taught the same group of 32 students. My teaching partner helped the vice-principal to create groups based on their consent or non-consent to be full participants in the study, so that I would have intact groups of participants. I felt that the input of my teaching partner was important as she knew the students well and I did not want students to be forced to work in a conflict situation simply because they agreed to be a part of my study. I felt that it was vital to student learning and to the success of my research to have students working in groups that could cooperate and work well together. Neither the vice-principal nor my teaching partner told me who the participants were and the procedure I just described created a situation where I was blind as to who the participants were in the classroom. Instead, I was given a list of students grouped in fours (see Appendix F). The students in the class were grouped according to this list so that the full participants of the study were sitting together. Students viewed the new groupings as normal because they always sit in groups of four and they are regrouped at least at the
beginning of every term. I made sure that the groupings for the term, that started a few weeks before the study, reflected the research groupings.

During the study all the students were treated the same and were exposed to the same activities. Student work and written reflections were collected from all students at the end of each class in a manner usually done by teachers. All eight groups audio taped their group discussions using I-book laptops. For each group two different students recorded the conversation and these audio files were sent to my teaching partner via the school’s server. Students labelled the files according to group number, date, and version. For example, Group7Amay25, would be the audio file for group 7, person A on May 25. Once she received the audio files, my teaching partner erased those of non-participants and saved the others to CD Rom. She then gave the CD Roms to the vice-principal and they were stored in her office until the end of the school year in June.

Once students had been chosen, the study proceed through a series of six lessons. According to D’Ambrosio (1998) activities used to research mathematics learning should be similar to the activities that teachers would normally engage in with their students. In this study, the activities used were part of the regular curriculum and taught in a manner to which the students were accustomed. The students not participating in the study completed class activities and participate in class discussions as usual. All students explored the same tasks whether or not they were involved in the study. Teacher monitoring of students’ work during the class period was done in equal measure for all students. During the class activities, whole-class discussions were also audio taped to look at ways the interactions within the entire class promoted changes in understanding for the participants. Once the study was completed, all student work for
the unit was brought to the vice-principal’s office and stored until June. Once the school year was over and the students’ final June marks were submitted, the vice-principal gave me the audio files on CD Rom for the participant groups, as well as the written work for these groups. Data for non-consenting students was destroyed. At this point, I was able to proceed with my analysis. I found out at this point that I had four groups of consenting students. As I got into the analysis, I realized that only three of these groups were viable. The fourth group was made up of two girls and two boys. The two boys were friends and so were the two girls. During the course of the study, one of the girls had missed three out of the six lessons. This created a marked disruption in any sort of discussion in the group. What resulted was a situation where the girl figured things out quietly and said her answers on tape while the two boys talked about the weekend. There was no real discussion of any kind, so I was unable to determine if discussion was able to promote understanding of variable. For this reason, I decided to focus my analysis on three groups, who at least made an effort at discussion.

Data Collection Instruments

My two research questions, although both related to student interactions, are distinctive, so I used differing methods for collecting data for the two questions. When combined, the different data sources may help shed light on my overall question, which is: in what ways do student interactions promote the development of individual and collective understanding of the concept of variable?

First Research Question

My first question is: In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of
variable? In order to look at possible answers to this question, I used a combination of student written work, and audio taped discussions during group work sessions.

Student written work

As I was interested in the development of student understanding as students progress through mathematical activities related to variables, I analyzed the work that groups produced as a result of their collaboration. This work took the form of diagrams, notes, calculations, and any other written data that students used in order to formulate answers and solutions to problems or tasks provided. This work was written on the student worksheets that accompanied the tasks (see Appendix D). Specifically, since as a teacher, I interacted with all my students during the class, I needed artifacts to reflect upon later when listening to the audio tape of the activity and analyzing the transcriptions of the group work activities. The student work was collected from students each class as teachers do and was looked at for feedback into the learning process. At the end of the teaching unit, all the written work was stored in the vice-principal’s office until the end of the year. At this point, she gave me back the data for the consenting groups.

Audio taped data of group work

I needed to capture the verbatim accounts of the students’ discussions during the group work activities, so these portions of the classes were audio taped. Once I received the data files from the vice-principal at the end of the school year, the audio files were transcribed for analysis. These transcripts were used as a guide for looking at group understanding during the analysis and for tracking changes in understanding throughout the lessons.
Second Research Question

My second question is: In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable?

In order to look at possible answers to this question, I used a combination of my own personal field/reflective notes and audio taped data from whole-class discussions.

Field notes and personal reflective notes

While teaching I circulated among all groups of students and recorded key features of their thinking and interactions. In addition, I attempted to write down key ideas, moments, or comments made by particular students that would be used mainly as a guide for ideas when listening to the group and whole-class audiotapes. After each lesson, I used my field notes and my memory of the lesson to write reflective notes for the lesson. In these notes, I included moments I thought were important to look at in the transcripts. They included self-reflection on my biases and my dual role in the research, my opinion on the ways I may have promoted or inhibited conversation during that class, as well as my ideas regarding student understanding and links to existing theory. By keeping track of and making my biases explicit, I have used reflexivity to increase the validity of my findings (Schwandt, 2001). Although I made every attempt to write field notes during the class, they ended up being rudimentary, as in playing the dual role of teacher and researcher, my role as teacher took priority. I did use these rudimentary notes to make reflective notes after the lesson. These, however, did not prove extremely helpful in the end as they often referred to students who were not part of the full study.
They did, in a few instances, help point to places where I felt a group had made progress, or that I was too directive or gave too much detail.

*Audiotape data of whole class discussions*

As I was interested in the ways in which students interacted with each other, I audio taped the whole-class discussions in order to get verbatim accounts of the interactions in the classroom. This was necessary as it would have been impossible for me to teach the lesson and accurately write down what students said in reaction to each other. I collected all the data but did not analyze this portion until the end of the school year when I got the group audio files and written data for the participant students from the vice-principal.

*Individual Reflection Sheets*

I used the reflection sheets (see Appendix E) as a means of asking students for their current definition of variable, as well as what promoted any change in understandings. I asked students to write what comments or ideas suggested by whom, promoted a change in their understanding. The purpose of these reflections was to help pinpoint shifts in understanding. I used them to give students the opportunity to write their own ideas on paper as they understood them. This is because students may tend to agree with their peers in a group setting, so writing individual reflections gave the students an explicit chance to write their own opinions regarding the development of their mathematical understanding. The student reflections were the one aspect of the research that was not part of the regular daily activity in my classroom. Students had not been asked to write reflections on a regular basis in my classroom, so sometimes students did not complete all the questions, or they answered them in a more contrived manner. In my
context, however; with my dual role as teacher and researcher, the individual reflection acted as a substitute for individual interviews. They served the purpose of giving students an opportunity to write down their ideas of variable and of the perceived influences that the discussion with their classmates were having on their developing view of variable. In some cases, this allowed me to report on the words that students said in the group or class discussion as well as their written comments about those words, allowing me to create a more complete data set.

Researchers such as Clarke (1998) have combined data methods by using a method called complementary accounts. Clarke (1998) studied the learning of Mathematics in legitimate classroom settings, versus clinical settings. Clarke’s study looked at the negotiation of meaning through social interaction in the classroom. Lessons were videotaped during regular classroom activities and combined with subsequent student interviews and field notes. I used a modification of this approach by combining data from audio taped discussions with student written work and individual reflections.

Data Analysis

According to Bogdan and Biklen (1998) accuracy and comprehensiveness of the data are of utmost importance so was my focus during the data collection and analysis stages. My other main concern was combining data from various sources into a comprehensive account of classroom activities and student thinking.

When analyzing the data from group and class discussions I looked for situations of negotiation of meaning. These were in the form of justifications or explanations (Cobb, Yackel & Wood, 1992; Cobb, Wood, Yackel & McNeal, 1992). I was interested in how students constructed intersubjectivity from these interactions so I relied on
Clarke’s (1998) model, looking for key episodes in the interactions where understanding seemed to have changed. Jaworski (1998) defined intersubjectivity as the balance between complete objectivity and complete subjectivity that is achieved when a group comes to a common understanding through sharing and negotiation and the combination of multiple data sources. Intersubjectivity is achieved through negotiation, so I studied key episodes for negotiative events where students are either justifying or explaining solutions and then used their words as hints to their individual levels of understanding.

The results of my analysis present my interpretations of the events that have transpired in my classroom. I viewed the research of classroom events in a manner similar to Gallas (1994). That is, I collected and analyzed data that had been directly influenced by the fact that I was a participant in the classroom I studied. I do not pretend that the accounts I assembled from my various data sources represent the objective reality of classroom events, but that they represent one interpretation, mine, of the events in my classroom. “That does not negate the power and realism of the stories, or the insights about children and classrooms they might contain, but it does require that I make no claims of generalizability or reliability for my findings” (Gallas, 1994, p. 8). As reference points for reflecting upon my work, I used, as Gallas did, a combination of my past experiences and educational theories I have learned.

Enhancing the Quality of the Data and the Results of the Study

According to Jaworski (1998) the quality or rigor of a study can be enhanced by keeping the data and analysis and the interpretation connected to the social processes and context in which they were emerged. I used verbatim accounts of observations using audio taping and written student reflections (Gay & Airasian, 2003). According to
Bogdan and Biklen (1998) research results will be altered by the observer effect. This is the idea that the presence of the researcher will change the behaviour of the participants. Although there is no way to avoid changing the behaviours of the participants I observed, I tried to reduce the observer effect by treating my students the way I normally did, apart from the presence of the audio tape. I studied the behaviours of my students in my classroom so there was much less of an observer effect than if an outsider had observed the students. In this case, during class activities, the fact that students were being audiotaped could have changed student behaviour.

Another strategy used to enhance the quality of my research findings is triangulation. Triangulation is when multiple data sources are used to corroborate research findings (Mathison, 1992). My aim in using triangulation is not to claim an objective truth regarding classroom events but to describe complementary accounts that may begin to shed light on student growth in understanding. Research studies by Busse and Boromeo Ferri (2003) with seventeen year olds and by Clarke (1998) with secondary students combined group work data and student interviewing. Although their focus was not on the learning of algebra, their work combined data sources in an attempt to create complementary accounts of classroom events. Through negotiation and through the combination of multiple data sources, intersubjectivity of data can be achieved. I used the combination of audio taped discussions, student written work, and student reflections for data analysis.
Validity

Discussions of validity are usually reserved for quantitative studies; however, in the past decade researchers such as Eisenhart and Howe (1992), Goetz and LeCompte (1984), Maxwell (1992), and Phillips (1987) have extended the idea of validity into qualitative research and have developed somewhat parallel criteria for its evaluation. Others, such as Bogdan and Biklen (1998) do not discuss validity but instead the idea of credibility as whether the work is convincing, readable, and makes a contribution. In efforts to address either view, I have adopted the following ideas that guided my qualitative study.

For my purposes, I will employ the definition of validity as presented by Maxwell (1992, p. 238) “Validity, in a broad sense, pertains to this relationship between an account and something outside of that account, whether this something is construed as objective reality, the constructions of actors, or a variety of other possible interpretations”.

Maxwell (1992) presents a classification of the types of validity that, in his view, are relevant in qualitative studies. He claims that the qualitative researcher relies on understanding and corresponding types of validity in the process of describing, interpreting, and explaining the phenomena of study. Maxwell’s approach concerns not so much the data or methods, but the accounts told from the data and the inferences and conclusions. Maxwell (1992) outlined five types of validity to consider in qualitative studies, that will be described below along with how I addressed them in the presentation and the analysis of my research findings. These are descriptive, interpretative, theoretical, generalizability, and evaluative validity.
Descriptive Validity

Descriptive validity relates to the factual accuracy of the findings. What is seen and heard must be reported as such. I used verbatim accounts as often as possible that were embed into the context from which they were derived. However, the interpretations I made of the data are just one interpretation, mine, of the data.

Interpretative Validity

Interpretative validity is how meaning is made of the description. It is important to get the perspective of those being studied so I included comments made by the students during the group and class discussions as well as those written in the reflections in order to more accurately portray student opinions. It is also clear that my data does not account for any non-verbal cues.

Theoretical Validity

Theoretical validity refers to validity as a theory of some phenomenon. I was not trying to create a theory; however, I acknowledge how my data matches the research on the difficulties that have been found to complicate the understanding of variable, the theories of learning in a constructivist mathematics classroom, and theories involving the potential influence of discussion and interaction on student learning.

Generalizability

Generalizability is the extent that my findings can extend to other people, or settings. In this study, I was not aiming at generalizing beyond my immediate context, but instead at describing the context of learning within my classroom in an attempt to gain an understanding of student learning within this context. Some of my general findings may hold true for a different group of seventh graders. However, I did not
attempt to generalize to other teachers’ classes within my school or to other classes of seventh graders in other schools.

Some authors have deemphasized the importance of generalization in qualitative research. Firestone (1993) expressed the importance of providing ‘thick’ description of the case and context so that readers can decide for themselves the fit between their particular context and that of the study. “The problem of generalizing ceases to become a problem for the author. It is the reader who has to ask, what is there in this study that I can apply to my own situation, and what clearly does not apply?” (Walker, 1980, p. 34). I will leave the idea of the generalizability of my findings to the reader, as the applicability to other contexts will be highly subjective and will relate to the perceived similarities between the reader’s context and the context described in my study. Therefore it will be up to the readers to pick and choose findings that they believe will hold true in their circumstances and then to test them out if they wish. Although the findings of this study will be somewhat limited to the particular context in which they were found, it is my hope that they will at least provide hints at the ways in which similar techniques can facilitate the growth of mathematical understandings in other contexts and provide a basis to spark further research into authentic classroom learning through social interaction.

**Evaluative Validity**

In my study I attempted to describe and interpret my findings but not to evaluate them formally. I reflected upon ways in which certain of my actions or the way I organized classroom events may have affected outcomes and student interactions, but this will not be the focus of my analysis.
CHAPTER 4

OVERVIEW OF THE UNIT OF STUDY

The purpose of this chapter is to give an overview of the algebra unit that was the context of the study, describe the nineteen tasks that the students experienced, and highlight important developments in student conceptions of variable. I will also identify eight key episodes that demonstrate how discussion enhanced the understanding of the concept variable. These eight episodes will be detailed in the following chapter of analysis.

For the purpose of this study and in agreement with the Quebec curriculum, a short teaching unit of six lessons was created to allow students to explore the concept ‘variable’. Here on referred to as the unit, this six lesson introduction to variables was presented at the end of the students’ seventh grade school year. The idea of uses of variables was chosen for my study because I noticed in my teaching that student difficulties in algebra were compounded as they moved further through secondary school. The idea of variables and the understanding of their various uses is fundamental to more advanced mathematics, thus studying the development of the understanding of variables seemed logical. Once I decided that I wanted to study how students came to understand algebra and variables, and in particular the role of student interaction, I needed to research why these concepts were so difficult for students to grasp. The research suggests four main reasons why algebra is so difficult for students. They are: (1) weaknesses in basic mathematical skills, integers, and order of operations; (2) conceptions of the equality symbol; (3) moving from arithmetical to non-arithmetical equations; and (4) the multiple meanings and uses of variables. These four difficulties
were explained in detail in chapter 2 so will not be detailed here; however, I will highlight some aspects of the main focus of this chapter. That is the study of the multiple meanings and uses of variables. Put simply, the reason that variables are so complex is that their meaning relates directly to the context in which they are studied.

It is important that when teaching algebra, teachers are aware of the different uses and meaning of variables and that they are discussed explicitly with students. Out of this rationale, my six lesson unit was developed.

Over the course of the unit various uses, contexts, and meanings of variables were introduced. I purposely designed and chose activities that introduced various uses of variables because I wanted to expand the notion held by many beginning algebra students that variables are simply placeholders for numbers. I wanted students to create a working definition for the concept of variable that was related to the context of the problem being solved. This decision was influenced by my research into problems that students have learning algebra, mainly related to students being stuck on the placeholder definition of a variable. This limited conception has been found to cause great difficulties for students as they attempt to move on in algebra into solving equations, inequalities, and equations where the answer is actually an expression containing variables and not a real number. The importance of highlighting context and for providing experiences for students to study various uses for variables became vital after I read Wagner’s (1999) article in which he explains how variables are at the same time similar to and different from both numbers and words. I used these thoughts to guide the choice of activities for the lessons in this unit. Thus my rationale for the chosen lessons relates to the variety of activities to show different meanings and uses of variables. Different tasks could also have been
chosen and could have presented the various uses of variables; however, I will adhere to a description of the activities I chose and my rationale for having done so.

The next section will provide a description of the six lessons of the unit, the tasks that the students completed, as well as the main uses and meanings of variables that they were chosen to highlight. It will also point to general areas of interest in the development of group understanding of the uses of variables. These areas of interest will be referred to as episodes of importance in the development of group understanding through discussion. These episodes will be described later on chronologically in the analysis chapter and thematically in the discussion chapter.

Unit Design

This unit consisted of nineteen tasks that were completed on fifteen worksheets over the course of the six lessons. The worksheets were made up of activities that I designed, found in books, and of examples I have used previously in my teaching. The worksheets can be found in Appendix D. The introductory example as well as Worksheets 1 and 2 were taken from Witherspoon and Woodard (1998, p.10, 11) and have been included with the permission of Walsh publishing. When referencing worksheets, I will use abbreviations. For example, Worksheet 4 will be referred to as ws4.

The six lessons occurred during six consecutive mathematics classes during the course of two and a half weeks, from Tuesday, May 17, 2005 to Wednesday, June 1, 2005. The class periods were seventy-five minutes long. The students in my mathematics class were at the end of their grade seven year of schooling and had had no formal algebra training to this point. The Quebec curriculum at the time the study was
conducted did not formally introduce algebra until grade eight; however, my rationale for choosing to conduct this type of research in a grade seven class was justified in the context in which I was working. Firstly, I was studying the idea of constructivism through my course work at Queen’s University and was attempting to put constructivist principles into effect in my classroom. I was trying to create lessons in which the students were at the centre of their own learning and where students were encouraged to construct their own meaning of mathematical concepts. Secondly, the Quebec curriculum that I was accustomed to was changing. A reform was coming into effect in September of 2005 that made the formal teaching of algebra compulsory at grade seven. The reform movement also called for more constructivist teaching methods (Gouvernement du Québec, 2004). Given both the general philosophy of the curriculum reform and its implications for the teaching of Mathematics, a study of how discussion promotes understanding was justified.

Learning about constructivist teaching methods, and the changing Quebec teaching curriculum were two influences that played a large role in my decision to conduct the study I did. There was, however, a third influence. This was my recognition, during my six years of teaching, that algebra and student understanding of the uses of variables was an area that posed serious difficulty for students. It was an area where traditional teaching methods, even those of more experienced teachers whom I considered experts, were not enabling students to develop a solid understanding of the multiple uses of variables. Thus, my study of exploring the various uses of variables, based on constructivist principles, in particular the role of student interaction, followed logically.
The following chart outlines the six lessons of this unit in terms of the dates of the lessons as well as the tasks and worksheets completed during each lesson.

Table 1: Outline of Unit Tasks

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Date of Lesson</th>
<th>Tasks Completed</th>
<th>Worksheets Completed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tuesday, May 17, 2005</td>
<td>1, 2, 3</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>2</td>
<td>Tuesday, May 24, 2005</td>
<td>4, 5, 6</td>
<td>4, 5, 6</td>
</tr>
<tr>
<td>3</td>
<td>Wednesday, May 25, 2005</td>
<td>7, 8</td>
<td>7, 8</td>
</tr>
<tr>
<td>4</td>
<td>Friday, May 27, 2005</td>
<td>9, 10</td>
<td>9, 10</td>
</tr>
<tr>
<td>5</td>
<td>Monday, May 30, 2005</td>
<td>11, 12, 13, 14, 15, 16</td>
<td>11, 12, 13</td>
</tr>
<tr>
<td>6</td>
<td>Wednesday, June 1, 2005</td>
<td>17, 18, 19</td>
<td>14, 15</td>
</tr>
</tbody>
</table>

The reason for the large span of time between Lesson 1 and 2 was that there was a long weekend in between. In the next section, the tasks will be explained in detail along with the reasons they were chosen.

Each of the six lessons was conducted using the same basic structure. The class began with groups of four students each completing a series of activities. After the group tasks were completed, we had a whole class discussion where students shared the knowledge they had constructed of variables. At the end of the class, students were given an opportunity to write a student reflection, in which they answered a series of predetermined questions regarding their understanding of variables and of group influences to their understanding. The purpose of the reflection sheet was twofold. Firstly, I needed a way of tracking changes in individual’s definitions and views on variables. Secondly, I needed a way to ask students whose comments had influenced their thinking. Thus, the reflection sheet was the same every lesson and was designed to
allow me to track student development of understanding through the course of the unit.

The reflection sheet can be found in Appendix E.

Although the main structure was as outlined above, a few modifications were made due to the realities of working in a classroom. The following table outlines the activities of the unit as they were planned and as they were executed in reality.

Table 2: Modifications Made to the Unit Tasks as Planned

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Activities As Planned</th>
<th>Activities As Executed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson 1</td>
<td>Pre-definition</td>
<td>Predefinition</td>
</tr>
<tr>
<td>May 17</td>
<td>Introductory Example</td>
<td>Introductory Example</td>
</tr>
<tr>
<td></td>
<td>Tasks 1, 2, 3</td>
<td>Tasks 1, 2, 3</td>
</tr>
<tr>
<td></td>
<td>Class Discussion #1</td>
<td>**we ended up having lesson one and two flow together and had the discussion at the end of lesson two.</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #1</td>
<td>Individual Reflection #1</td>
</tr>
<tr>
<td>Lesson 2</td>
<td>Tasks 4, 5, 6</td>
<td>Tasks 4, 5, 6</td>
</tr>
<tr>
<td>May 24</td>
<td>Class Discussion #2</td>
<td>Class Discussion #1</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #2</td>
<td>Individual Reflection #1</td>
</tr>
<tr>
<td>Lesson 3</td>
<td>Tasks 7, 8</td>
<td>Tasks 7, 8</td>
</tr>
<tr>
<td>May 25</td>
<td>Class Discussion #3</td>
<td>Class Discussion #2</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #3</td>
<td>Individual Reflection #2</td>
</tr>
<tr>
<td>Lesson 4</td>
<td>Tasks 9, 10</td>
<td>Tasks 9, 10</td>
</tr>
<tr>
<td>May 27</td>
<td>Class Discussion #4</td>
<td>Class Discussion #3</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #4</td>
<td>Individual Reflection #3</td>
</tr>
<tr>
<td>Lesson 5</td>
<td>Tasks 11, 12, 13, 14, 15, 16</td>
<td>Tasks 11, 12, 13, 14, 15,16</td>
</tr>
<tr>
<td>May 30</td>
<td>Class Discussion #5</td>
<td>Class Discussion #4</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #5</td>
<td>**we ran out of time to complete the discussion and do the individual reflection.</td>
</tr>
<tr>
<td>Lesson 6</td>
<td>Tasks 17, 18, 19</td>
<td>Completion of Class Discussion #4</td>
</tr>
<tr>
<td>June 1</td>
<td>Class Discussion #6</td>
<td>Tasks 17, 18, 19</td>
</tr>
<tr>
<td></td>
<td>Individual Reflection #6</td>
<td>Class Discussion #5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Individual Reflection #5</td>
</tr>
</tbody>
</table>

The following flow chart of the unit gives a visual impression of the unit activities. It shows that each group works independently throughout the first and second lesson and that interaction among groups commences with the first class discussion of
Lesson 2. After this point, the course of future discussions was likely influenced in some way by the comments made by others during the class discussions.

Figure 1: Flow Chart of the Unit

Lesson 1
May 17

Lesson 2
May 24

Lesson 3
May 25

Lesson 4
May 27

Lesson 5
May 30

Lesson 6
June 1

Class Discussion #1

Class Discussion #2

Class Discussion #3

Class Discussion #4

Class Discussion #4 Continued

Class Discussion #5
Lessons 1 and 2

In Lessons 1 and 2, students worked on patterning as a starting point to the transition between arithmetic and algebra. Research shows that an understanding of variables and their many uses is fundamental to student success in algebra (English & Warren, 1998; Devlin, 1994; Schoenfeld & Arcavi, 1999; Philip, 1999; Usiskin, 1999a; Usiskin, 1999b and Wagner, 1999). Thus, when creating the activities of this unit, I kept in mind the need to provide my students with opportunities to explore the use of variables in different contexts. I structured the activities of the unit to provide students the opportunity to discuss and hopefully make progress in their understanding of the multiple uses of variables. Although students are often introduced to variables in equations where they represent an unknown constant value, I chose to introduce variables to my students through pattern exploration as recommended by English and Warren (1998). They suggested that this approach would be most successful with a group of students who had a fairly strong grasp of basic number theory and who had developed flexible thinking in terms of arithmetic. My study took place as my students were at the end of their grade seven year, and as we had worked hard all year on basic number manipulation and on finding many different solutions to problems, I felt that this approach could be effective with my students. Thus, students were given a series of six pattern activities to work through. Students were not given any specific strategies on how to solve the problems; however, the strategies they employed were consistent with those outlined by English and Warren (1998).

English and Warren (1998) described three strategies used by students as they progress towards being able to determine a functional relationship between the index ‘n’
and the expression. The most basic of these strategies is the ratio strategy, which would lead students to determine, for example, that if index two had ten toothpicks that index four would have twenty toothpicks representing twice as many. I did not see evidence of the use of this strategy; however, there was much evidence of the next most basic strategy, the additive or recursive strategy. English and Warren (1998) explain that the recursive strategy involves students making observations that each subsequent term increases by the same amount. For example, students would note that the pattern goes 3, 7, 11, 15, that four is added each time. Students would attempt to formalize their responses by recording arithmetic facts such as $3 + 4 = 7$, $7 + 4 = 11$, $11 + 4 = 15$. The problem with the recursive strategy is that each term is dependent upon the previous term, so if asked to determine the 100th term, the value of the 99th term would be necessary. The most sophisticated strategy, as outlined by English and Warren (1998) is that of searching for a functional relationship, known as the closed form. Students who have the most sophisticated thought processes are able to identify the function that connects the number of the index to the number of toothpicks in the indexed pattern or the output number. Students would be able to respond for example that the number of toothpicks is the index multiplied by three plus two. They would also be able to formalize their responses by using algebraic notation such as $T(n) = 3n + 2$. By the end of lesson two, students were working towards making these connections.

In the first few lessons variables were used as placeholders to represent missing numbers, showing how variables could be like words. For example $2n + 1$ meant two times the number of the sequence plus one. As variables were used as the index number, students were able to use the variables to determine patterns. We began Lesson 1 by
completing an example as a class. The introductory example that was used was the following graphic. It can be found in Appendix D and was taken from Witherspoon and Woodard (1998, p.9).

Figure 2: Introductory Example

Students were asked the pattern number and the number of matchsticks that were involved. I recorded students’ answers on the overhead. When discussing the rule, the students verbalized it as times three plus one. I wrote times 3 plus one on the overhead and in the table where we were to write the response for ‘n’, I wrote \( n \times 3 + 1 \). We never discussed what ‘n’ meant. Once the students figured out the pattern, they were more interested in getting started on the package than spending any more time on the introductory example.

The initial lessons were used as an introduction to algebra, as students could determine the pattern through arithmetic, but they were asked to write rules for the patterns using ‘n’ as a generalizations for the number of triangles or matchsticks. Worksheets 1 and 2 were taken from a book of algebra activities by Witherspoon and Woodard (1998) and were chosen as they highlighted the idea of patterning. They were also chosen because of their similarity to each other so that students could modify strategies found while completing Worksheet 1 when working on Worksheet 2. The two
questions involved the number of toothpicks compared to the number of triangles or squares that were created.

Figure 3: Worksheet 1 Graphic

![Worksheet 1 Graphic](image1)

Figure 4: Worksheet 2 Graphic

![Worksheet 2 Graphic](image2)

Figures 3 and 4 represent extracts of Worksheets 1 and 2 which were taken from Witherspoon and Woodard (Walch Publishing, 1998, p. 10, 11). The complete set of worksheets can be found in Appendix D.

Worksheets 3 to 6 were taken from a package of worksheets that were created by a group of teachers as a result of a grant awarded by the Quebec Ministry of Education (Balyta, Calder, Del Castilho, Rosciano, & D’Souza, 1998). At the time of creation, the teachers working on this project were from three school boards. Since then, two of the boards have amalgamated into the one where I work. These worksheets were ones that I used for a few years, exactly as they were created. In the first two lessons the use of variables was subtle and in some cases ignored. Students had differing opinions on what ‘n’ was. Some students thought that you could put whatever number you wanted, and
others reported that ‘n’ was where you put the pattern. By the end of Lesson 2, the groups seemed to have agreed that the pattern necessarily involved the sequence number. The development of the idea that the sequence number must be included will represent Episode 1 and it will be described in detail in the following chapter.

Episode 2 also occurred during the Lesson 2 discussion. It revolved around the idea that more than one representation of a pattern was possible. During Lesson 2, each group found their own path to representation of this pattern. During the class discussion the understanding became clear, as the students discussed that they thought the two different ways of representing the pattern were the same. This discussion is Episode 2 and it will be set out in the following chapter.

Lesson 3

In Lesson 3, students were asked to expand their meaning of ‘x’ and ‘y’ as variables. This lesson was a way to highlight that variables are different from words as they maintain the same meaning throughout one context, whereas words can change meaning even within the same sentence. This lesson consisted of Tasks 7 and 8 (ws7-8). In this lesson, task seven was actually a series of questions detailed below. Task 8 was to be completed at the end as it involved two discussion questions relating to the activities students had just completed. Task 7 involved activities that were modified from Patterson’s (1999) lesson idea for building physical models to represent variables. I called this the ‘x-box’ lesson and it can be found in Appendix D, worksheet 7 and 8. For the purposes of this lesson each group was given two different boxes on which I pre-identified the length or width as ‘x’ or ‘y’. These were actual boxes taken from my recycling bin such as boxes for Jello, tea, and paperclips. I purposely made sure that no
two groups had exactly the same combination of boxes. With their ‘x’ and ‘y’ boxes each group had to measure a pre-measured piece of paper. This paper was measured from the rolls of paper that are used with electronic calculators. Each group was given 130 cm of paper, but were not aware of how much paper they had. Before they began, they were told that each group had the same amount of paper and they were given an opportunity to hold up their ‘x’ and ‘y’ boxes for the other groups to see. Students then had to measure the length of the paper using ‘x’ and ‘y’ boxes, recording the expression for the length of the paper. For example the length of the line could have been written as $5x + 3y = \text{line length}$. After completing and recording two different expressions, students were allowed to measure their ‘x’ and ‘y’ boxes and determine if their expressions worked. After students completed the mathematical work of Task 7, they were assigned Task 8, which was to discuss what they had just done and the reasons why they thought it worked. The questions (ws8) related to why the groups thought that different expressions could be made for their same length of paper and also why the term variable would be appropriate to describe the ‘x’ and ‘y’ lengths of the boxes. The discussion of why the ‘x’ and ‘y’ boxes were variables, represents the students’ experience with the constant-variable struggle and will be outlined in the following chapter as Episode 3.

**Lesson 4**

During Lesson 4, students completed Tasks 9 and 10 (ws9-10). The lesson consisted of two parts. Task 9 consisted of discussion questions related to the previous lesson. The first part provided an opportunity for the groups to re-discuss why they thought the lengths of the ‘x’ and ‘y’ boxes were variables in the previous lesson. They had already discussed this in Task 8, but I wanted to give the groups and the individuals
in the groups a chance to re-visit their ideas after the class discussion of May 25, 2005. I gave them a discussion question that evolved out of a Lesson 3 reflection in which one student, in her definition of a variable, stated that she was stuck between numbers and letters. I gave them this question of whether variables are numbers or letters [symbols], hoping that this would provide the students with the opportunity to work through the symbol-number issue. The symbol-number issue comes from students’ previous exposure to variables. Often students are introduced to algebra through questions such as 7 + 5 =  ? and later on given questions such as 7 + ? = 12 and 7 + x = 12. This type of exposure leads students to view variables as numbers, not symbols that may represent a value. This limited conception can create problems for students as they progress into more sophisticated study of algebra. The discussion of whether a variable is a number or a letter represents Episode 4 and it will be explained in detail in the following chapter. The student debate of whether a variable is a number or letter is also simplistic when compared to sophisticated uses of variables, for example, in calculus. However, I wanted to open up the discussion of the different uses of variables, starting with cases where letters represent a specific number or set of numbers.

The second part of the lesson started in with the idea of equations. Students completed Task 10, which was a series of equations (ws10) in which the students had to determine the value of the variable. In the previous lesson, students were exposed to an expression equal to 130 cm and now understood the idea that 2x means two times the length of x, so I decided to use the idea of x boxes and line lengths to prompt exploration into equations. In this lesson all equations had multiple terms on the left hand side of the equation, equal to one term, a number on the right hand side. All questions except the
last one had only one variable to keep it simple. The last question contained two
different variables: $4x + 3y + 3 = 67$, where the value of $x$ was known to be $8.5$ cm. This
was used to highlight the importance of one variable maintaining its value within a
context. I wanted students to understand that different variables in the same context
could represent different values. The purpose of this lesson was to determine the lengths
of boxes, given the lengths of the lines. This lesson reinforced the idea that variables
maintain the same meaning within a context. Therefore, an equation with two variables
was determined by the students to have two unknown values. During the course of this
lesson, it was noted that all groups quickly recognized the strategy of using inverse,
whatever was added to one side was subtracted from the other and vice versa. The idea
behind the inverse strategy is that by using inverse operations, the value of the variable
can be determined by using the process of elimination. For example in the equation $x + 6
= 18$, the inverse strategy can help determine that ‘$x$’ has a value of 12. This is because,
on one side of the equation, six is added to ‘$x$’, therefore by subtracting six from both
sides, you are left with $x = 12$.

$$
x + 6 = 18
x + 6 - 6 = 18 - 6
x = 12
$$

The use of the inverse strategy by Group B to determine a solution, as well as
their explanation of the use of this strategy to the teacher, represents Episode 5.

*Lesson 5*

In Lesson 5, students were exposed to different letters being used as variables.
This showed that variables are unlike words because they can be put together in any order
or combination.
In Lesson 5, students completed Tasks 11 through 16 (ws11-13). In the first, Task 11, students had to come up with at least eight different answers for $20 - 12 = \underline{\hspace{1cm}}$. The purpose of this task was to develop the idea of the equality symbol. According to Kieren (1981) students have difficulty making the transition to algebra because they cannot hold the equation incomplete. For example, they can accept that $20 - 12 = 8$ but not necessarily that $20 - 12 = 4 \times 2$. The question was essentially a practice in manipulating arithmetic identities. In the next part of the lesson, Task 12 (ws12), students were given five equations to solve. The equations can be found on Worksheet 12 in Appendix D. The first two of them were basic, having the right hand side of the equation as a number only. They were: $3x + 5 = 26$ and $6y - 2 = 48$. The next two were multi-stepped as both sides contained both numbers and variables. They were $2c + 8 = 3c + 5$ and $3a + 5 = 2a + 12$. The fifth, $2x + 6 > -2$, was an inequality. The purpose of this was to show that variables represented one value within the same context in a linear equation, and that a variable simultaneously represented many different numbers in an inequality. This inequality gave the groups the opportunity to discuss a situation where the variable held more than one possible value. Each group answered the question in their own way with varying responses. Episode 6 will outline details of how Group A folded back to previous conceptions of variable and got stuck there while with teacher intervention, Group B was able to move beyond this and make progress towards a broader view of variable.

The next task in this lesson, Task 13 (ws13), was to determine whether four different expressions were equal. All four had the same numbers, but in a different order and in a different combination of addition and subtraction. I wanted students to discover
that with the same numbers and expressions, no calculations needed to be done to answer whether or not these arithmetic identities were equal.

The following task, Task 14, (ws13, #2) was another question in which logic could have been used to avoid calculation. By looking at the two sides of the equation and using logic, the value of the missing portion of the equation could have been determined. All three groups chose to calculate the answer showing that students had not developed the ability to hold equations undone (Kieran, 1992). Interestingly, this expression was used by Collis (1975) as referenced in Kieran (1992). Despite the passing of over thirty years since its original use, students still have similar problems.

The following task, Task 15 (ws13, #3), brought discussion of the use of different variables. Students debated two different variables having the same meaning in separate equations. The purpose of including this question was simply to spark debate about different variables. The question was whether or not $7(w) + 22 = 109$ and $7(n) + 22 = 109$ were equivalent. When Wagner (1981) gave these equations to students aged twelve to seventeen, he found that half the twelve-year-olds were not able to conserve that the equations were equal without having them solved. I thought that this would be an interesting question for my students who were twelve and thirteen. Thus far in the unit, different variables had represented different values; however, context must be considered. I wanted to place students in a position where the truths that they had created about variables were challenged. Episode 7 will show how the class discussion clarified the equivalence of the two equations for members of Group B who had remained undecided during the group discussion.
The last task of Lesson 5 was Task 16, (ws13, #4). This question asked students which was larger: \( x + x \) or \( x + 4 \). This example was modified from one described by English and Warren (1998). I chose to include this example as I wanted to see if students would discuss the problem context or simply assign a value to ‘\( x \)’.

**Lesson 6**

Lesson 6 was the final lesson of the unit. It provided students the opportunity to solve some problems by attempting to use variables. Up to this point student experience with variables had been in situations where the variables were already defined. For example, in Lesson 3, the variables were the lengths of the ‘\( x \)’ and ‘\( y \)’ boxes. In lesson 4 and 5, the variables were presented in equations where the values were predetermined and needed to be figured out. Lesson 6 was the students’ first attempt at using variables in questions where they did not overtly exist.

The groups were given three word problems to solve using variables. In the first question, Task 17 (ws14, #1), the variable was a specific value in an equation that was compared to the other. It represented the number of trading cards held by Lynn. Her amount was compared to Joe’s amount, the sum of the two added to a specific number of trading cards. The different groups got to various levels of sophistication with the use of variables. Episode 8 will outline the experience of Group B and the struggles they had trying to use variables, after solving the problem using recursive methods. This episode will highlight how teacher intervention helped the group move forward in the use of variables in a future problem (Task 19).

The second question, Task 18 (ws14, #2), involved two linear equations, comparing the entrance fees to a local fair. This example was modified from an example
given by Heddens and Speer (1997). It was chosen to allow students the opportunity to compare two situations algebraically and graphically. The students had to determine which plan was the best deal and how this could best be determined.

The third question, Task 19 (wk15, #3), was a question from an article written by Kieran (1992) but was modified from Bell, Malone and Taylor (1987). Students had to determine the number of rocks in three different rock piles, given certain criteria. Episode 8 will refer to the progress made by Group B and show that they were able to create and solve an equation using variables.

At the end of the sixth lesson, I felt that I could have added at least another lesson in order to allow students to continue to construct their ideas of variable and allow me to get a better picture of the students’ understanding of the use of variables, particularly with word problems. However, as time was running short at the end of the school year and I needed to prepare my students for their final exam, I decided that it was in the best interest of my students to stick to the initial plan of six lessons. The purpose of the unit was to study how student interaction facilitated student’s development of the idea of variable. In particular with regards to the potential influence on facilitating students in moving beyond the placeholder, missing number view of a variable, to a more ‘variable’ view on depending on the problem context where variables could simultaneously represent more than one value.
CHAPTER 5

RESULTS

The story of the development of understanding of the concept of variable and the student interactions related to this took place over the course of the algebra unit and will be recounted in narratives of eight episodes of significance. Throughout the lessons, students negotiated meaning and in the process developed their conceptions of variable. In this chapter, I will attempt to weave the stories of student negotiations in my classroom with current research on student learning through discussion and student learning of algebra. I will recount the stories of eight episodes that occurred during the course of the unit in terms of student negotiation and knowledge building. These episodes are but moments, or snapshots in time during the course of the unit and are not mutually exclusive from each other or any other events that occurred in the classroom over the course of the six lessons.

My class consisted of 32 students divided into groups of 4. It is important to note that all 32 of my students were exposed to the same experiences. The only difference is that three of the groups were being studied in depth. Thus, as can be seen by the flow chart of interaction, after the first class discussion during lesson two, it is difficult to attribute exact sources of influence for certain conceptions. This is due to the idea that each student builds their own understanding or truth from their experiences, including the class discussions and these shifted conceptions are then taken with them back to subsequent group activities. The flow chart situates the eight episodes into the unit. Relevant portions of this flow chart will be repeating throughout the chapter in order to situate the episodes in the context of the larger unit.
Figure 5: Flow Chart of the unit showing Episode Locations

Lesson 1
May 17

A  B  C  +  5 other Groups

Class Discussion #1

Lesson 2
May 24

A  B  C  +  5 other Groups

Class Discussion #2

Lesson 3
May 25

A  B  C  +  5 other Groups

Class Discussion #3

Lesson 4
May 27

A  B  C  +  5 other Groups

Class Discussion #4

Lesson 5
May 30

A  B  C  +  5 other Groups

Class Discussion #4 Continued

Lesson 6
June 1

A  B  C  +  5 other Groups

Class Discussion #5
The following is the story of knowledge building in one classroom with a group of 32 students and their teacher. I am this teacher. However, for the purposes of the analysis and discussion of the data, I will attempt to step back a bit in order to see the results more objectively. Thus, I will refer to myself as the teacher, in order to facilitate the critical reflection of the interactions that occurred amongst the students in my class and with myself, as they went through the experiences that I set out for them.

The eight episodes will be recounted as they help me to develop answers to my two research questions:

1) In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of variable?

2) In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable?

Throughout Episodes 1 and 2, the students used what English and Warren (1998) refer to as recursive strategies to determine the value of the indexes. By the end of Lesson 2, students were working towards being able to determine a functional relationship between the index ‘n’ of a sequence and the expression of a general term. Episode 1 relates specifically to the index ‘n’ while Episode 2 relates to the functional relationship used to create an expression of the general term.
Episode 1: The Meaning of ‘n’

Figure 6: Location of Episode 1 in the Unit

The first episode of importance occurred around the idea of the meaning of ‘n’ and will describe the experiences of Group C. As the students had never been formally introduced to algebra, the idea of ‘n’ was somewhat foreign to most. Each group struggled with the purpose of ‘n’ appearing in the last row of the table to be completed by the students.

Figure 7: Recreation of the table from Worksheet 1

<table>
<thead>
<tr>
<th>Number of Triangles</th>
<th>Number of Matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>(n)</td>
<td></td>
</tr>
</tbody>
</table>

As can be seen from the table, ‘n’ is written as a possible next index for the sequence. Students were not sure how to interpret the new symbol but through
discussions within their groups and the class at large, most seemed to develop at least a rudimentary idea of why the index ‘n’ was included in the table.

The most important shift in thinking that occurred during this episode was the shift in the idea that the ‘n’ meant to put whatever you want (GD17, C, 1-9), a comment that suggested that ‘n’ was not seen as part of the pattern, to ‘n’ being seen as an integral part of the pattern and the emerging ability to incorporate ‘n’ in a general expression. Below is the journey of Group C through the exploration of the meaning of ‘n’.

The following excerpt from the discussion of Group C clearly shows that while working on Task 1 at the beginning of Lesson 1, the group members were conflicted as to the meaning of ‘n’ and that they did not see the ‘n’ as a necessary part of the expression. They had determined the term values through recursive strategies and then tried to determine why ‘n’ was included in the table.

Lara: What does ‘n’ mean? I don’t get the ‘n’ at the bottom of the page, just the ‘n’.
Ross: That’s number. Its just whatever number you want.
Lara: ok ‘n’ how do you figure out ‘n’ just put any number or what?
Anne: I think you uh
Lara: Lets skip it.
Anne: just put any number…Isn’t ‘n’ like you have to put the pattern?
Ross: I have a question for this one…times two plus one…how would this work?
Lara: 10 times 2 is 20 plus 1 equals 21.
Anne: I think ‘n’ is the pattern. We have to fill in ‘n’ so do we just put anything in? (GD17, C, 1-9).

This excerpt shows that as early as the first task, Anne appears to have the start of an idea that ‘n’ should be incorporated in an expression as a pattern generalizer but is not yet in a position to express this in a way that helps the group. Instead, Anne led the group to filling in the chart by putting any number as Ross and Lara had suggested. Anne’s idea about using ‘n’ to write the pattern was not explored further at this time. As
interaction continues in the group, Anne becomes capable of articulating her idea and others picked up on it.

On the Student Worksheet 1, each group member crossed out the ‘n’ in the table and put a 1000 beside it with the corresponding 2001 to show the pattern of $2n + 1$ (sws1, Lara, Ross, and Anne). Allison was absent for this lesson so she did not have a response on her worksheet.

Later on the in lesson, I came over and asked the group if they had figured out an expression. The group was working on Task 3, circles (ws3) at the time. It appears as though I misunderstood Lara’s comment of plus one to mean that the first index had only one circle. This created what Mason (1989) would call a shift in thinking for Anne as Lara responded to me to clarify that each index grew by one circle. The group began to struggle with the realization that without a pattern [expression] it was impossible to figure out answers to questions where you are just given the index.

Ms. C: So have you figured out a pattern?
Lara: yes, it is plus one.
Ross: what?
Ms. C: okay, so you are saying that the first one, the first pattern has one?
Lara: no, it increases by one for each pattern.
Anne: but Lara if we’re like just given a number how would we know?
Lara: yeah, the only thing, if they just gave us a number, how would we figure it out? (GD17, C, 26-31).

This excerpt suggests that the group’s discussions on finding an expression has led them to question the use of the recursive strategy of adding one to each subsequent index but that they are not yet clear as to how to determine an expression for a given term index. This conflict served to confirm for Anne that a pattern needed to be found for the expression and that it may involve the index ‘n’. The following interaction occurred as soon as I left the group. It shows that Anne wanted to use the index as the central part of
the expression and told her group that is what she wanted to do. She then led the group through the use of the index in order to determine the expression.

Anne: ok, so you are given sequence number 1, its one you take the number of sequence it is like 1 plus two…1 plus 2 is 3, 2 plus 2 is 4.
Lara: oh, that works
Anne: 3 plus 2 is 5.
Ross: What did you do?
Anne: You take the number of sequence that’s 1, 1 plus 2 is 3, 2 plus 2 is 4, 3 plus 2 is 5.
Ross: cool. Awesome.
Anne: so number of sequence plus 2.
Anne: so the 30th term is the same as the 30th sequence right.
Lara: yeah.
Anne: so it would be 32 circles.
Ross: yeah 30 plus 2.
Lara: okay…that works. (GD17, C, 36-44).

Anne used what Cobb, Wood, Yackel, and McNeal (1992) calls ‘negotiation through explanation’ with her group members in order to use the index and the group decided that the expression was the term index plus two. Anne and Ross both wrote the expression in words on their student worksheets (sws3, Anne, Ross) but Lara wrote the expression as “n + 2 and n = the number of sequence” (sws3, Lara). This indicates a major shift for Lara to being able to write an expression in symbolic form. Lara’s ability to write a functional expression in terms of ‘n’ shows that she is moving beyond simple recursive means to more sophisticated closed form function methods for determining expressions.

When Allison returned in Lesson 2, the group took the time to go over the previous day’s examples with her. Despite the fact that Anne had led the group to using the index as the central part of the expression, the group had determined the values through recursive methods using the index. Allison seemed immediately frustrated with
the recursive methods that she called ‘trial and error’. Lara was trying to explain the group’s response to Task 3, circles, that the 100th term would have 102 circles and Allison responded that “Oh, I get it but how do you figure that out” (GD24, C, 77-78).

Allison was not satisfied with their response because although she understood that there would be 102 circles in the term, she thought it did not help her to determine the pattern.

Allison: no because if it is 100, you can’t just.
Lara: plus 2.
Allison: It would take forever. So what…you just do trial and error until you find a pattern. This could take a while.
Allison: so we know that every time it is going up by two so how else can you do it so it will be going up by 2?
Ross: plus 4, plus 4.
Allison: yeah but let’s say it is 150 and they want to know how many there will be, 150…let’s start with 8 (laughs) (GD17, C, 84-91).

Allison stated that this method or recursive expression would make it difficult to determine the value for the 150th term. This is evident by the way she stated right away the evident recursive pattern of adding two but questioned on how else it could be done. Her laugh at the end of her second statement indicates that she thinks that the recursive strategy is somewhat ridiculous.

During their work on Task 4 (ws4), the group decided that they were going to use ‘n’ or the index as an integral part of the expression and came up with an expression of index plus index minus one (n + n – 1 or 2n – 1).

Allison soon discovered that the index number plus the one before it, gave the number of pentagons. The group discussed the expression and decided that it worked. The only small difficulty was figuring out how to orally state the expression they had come up with. Through negotiation, the group agreed that the expression of the pattern was the previous index number plus the present index number. The following excerpt
from the discussion of Group C follows the conversation that led to the expression being constructed.

Allison: so it would be like $149 + 150$ would give you your answer cause $1 + 2$ is 3.
Ross: $2 + 3$ is 5 so to get 150, you would do.
Allison: $150 + 149$ to get 150
Anne: $150 + 149$. She’s right.
Allison: do you get it?
Lara: yeah, that would be a lot, like 200 and
Allison: so, how many in the $8^{th}$, well $8 + 7…15$ (with Lara).
Allison: express orally how the pattern grows—so the last number plus.
Lara: itself like.
Allison: how would you say that?
Ross: last sequence number plus.
Allison: okay wait…last sequence number plus—the present number.
Anne: yeah.
Allison: the present number.
Anne: present sequence number.
Allison: well it makes sense to us.
Lara: how many in $40^{th}$.
Allison: so $39 + 40$.
Lara: so 79. (GD17, C, 94-105)

This previous excerpt where Group C determined that the expression was index number plus previous index number will be referred to in Episode 2, as the group was able to state their expression in words but not in symbolic form.

The idea that the sequence number [index] is a necessary part of the expression was negotiated throughout Lesson 1. This occurred when the students did what Pirie and Kieren (1994) referred to as folding back. Students referred back to their solutions of the previous questions and made the connection that although they had determined the expression through recursive methods that they all involved the sequence number [index].

Allison: let’s just try to find a pattern
Allison: Does it always involve the sequence number?
Lara: I do not think necessarily.
Allison: yeah, but has it? It can’t involve anything else otherwise you would not be able to figure out 100 without knowing what 99 is.
Lara: What is the pattern for this one? (referring to a previous example)
Allison: no, it has to be the sequence number, otherwise you would not be able to figure out 100 without knowing what 99 had.
Lara: this one was times 2 plus one (referring to task #1, triangles), it wasn’t the sequence number
Ross: 1 times 2
Allison: No, that is with the sequence number.
Ross: yeah.
Allison: they are always with the sequence number
Lara: okay, now we have to figure it out (GD24, C, 139-150).

By the end of the second lesson on May 24, the worksheets completed by members of Group C showed evidence of a developing understanding of the purpose of ‘n’. This appears to be where they stopped putting random numbers in as the first coordinate of the ordered pair for the index ‘n’ in the table. Instead of crossing out the ‘n’ and replacing it with a random number, the ‘n’ is left there and its second coordinate of the ordered pair in the corresponding row of the table is an expression such as n x 3 + 1 to represent the term value. They even went back to previous worksheets and wrote in the expression for ‘n’ beside where they had put their number of choice. This indicates that students were moving from an idea of ‘n’ as a missing number to one of ‘n’ as standing for multiple different numbers or holding the place for numbers in a functional closed expression.

*Reflection Sheets*

Although not all students responded to all questions in each reflection, some student insights helped to highlight the benefits of student interaction. In her individual student reflection of May 24, Lara stated in response to the question of how her group helped her, “Anne helped me figure out patterns using the sequence number” (IR24, Lara). This reconfirms what was discussed above about Anne being a leader in Group C, when it came to using the sequence number in determining the expression for the
patterns. In response to the same question, Allison wrote “Lara helped explain the answer to my main question “how come ‘n’ is everywhere?” (IR24, Allison).

Episode 2: Diverse Rules

Figure 8: Location of Episode 2 in the Unit

Episode 2 is an extension of Episode 1, where students had been developing expressions for the first six tasks over two class periods. Some students had started to develop functional expressions and discover that an expression can be stated accurately in more than one way. This is noteworthy as algebra requires flexible thinking and often students search for what they think is the one correct answer and do not necessarily make a conscious effort to acknowledge other responses. In this episode, the varying ideas of different groups brought forward during the first class discussion put students in a position where the consideration of other responses was essential.

This episode occurred in the class discussion but roots back to the discussions held by each group, in order to come up with their expression. The students discussed two different ways of representing the expression for the pattern in Task 4 (ws4). The first three terms are shown below. The task was to determine the subsequent terms in order to answer specific questions. Students had to determine the number of pentagons in the term for the index 8, 40, and 120 and they also had to express orally how the expression grew. Finally, students were asked to determine an expression for the pattern.
Most groups chose to record their expression in one of two ways. The groups used either $2n - 1$ or $n + n - 1$. Although Group A had a discussion in which they decided that both expressions worked, each member chose one expression to record on their student worksheets. When stating the expression in words, the groups used variations of either ‘$n$’ times two minus one and/or number of sequence [index] before plus present sequence [index]. It is interesting to note that despite which of these two expressions the group members chose to use when explaining the pattern in words, the symbolic expression of $n \times 2 - 1$ was used in written form. Group B used the symbolic expression $n \times 2 - 1$ to represent the pattern and Group C came up with the following expression written in words: last index number plus present index number. Group C, however, was not able to record their expression in symbolic terms despite my intervention. During their work on Task 4, I came over to the group shortly after they determined their expression. I asked them what they had determined for an expression. When they verbalized to me that they had determined last index plus present index, I wanted to assist
them in writing the expression in the symbolic form \( n + n - 1 \). I attempted to guide them through this. It was clear that while they were able to give the expression orally and write the expression in words, they were not ready to make the jump to writing the expression in symbolic form. Just the decision to use the index as a necessary part of the pattern represented a significant growth and I think that what I was expecting them to do represented far too great a conceptual leap at this point. I ended up making the group question whether their expression actually worked but they agreed that their solution was fine and they did not alter their conceptions to adopt my ideas. This is what the members of Group C had to say when I left the group.

- Ross: then what we did for number 3 is wrong.
- Lara: no, it is still right.
- Ross: It could be another way.
- Lara: She’s (referring to me) just complicating it.
- Ross: She is complicating it.
- Allison: we’re still right, that is just another way. (GD24, C, 134-138.)

Thus Group C went into the class discussion at the end of class two with an expression that they could state in words but not symbolically.

Class Discussion

During the class discussion Cole mentioned that his group determined that the expression, multiply the index by two and minus one, satisfied the pattern (CD24, 172).

After Cole’s explanation of his group’s expression, the following interaction took place.

- Ms. C: I noticed that Mark’s group did it a different way? What did you come up with?
- Mark: sequence number of the one before it plus sequence number.
- Allison: Huh, that’s what we did
- Ms. C: That is what Allison’s group was trying to come up with too. So you came up with sequence \([\text{index}]\) number, can we say that is \(n\) plus
- Mark: sequence number of the one before it so plus \(n - 1\)
Ms. C: Allison this is what your group was coming up with. You said it in words and remember, you were having some trouble with that
Allison: yeah
Ms. C: n is the sequence number and could we say n-1.
Mark: you put that in brackets. (CD24, 177-188)

When Mark stated this group’s pattern, Allison immediately recognized that this was the pattern her group was working on. The interaction between Mark and myself served to clarify how this expression could be written symbolically.

**Reflection Sheets**

Individual reflections after this lesson showed that student comments and the class discussion did influence the ideas of their classmates. By the time the students wrote the reflections, two members of Group C had written the expression in both ways that had been discussed in class. They were not able to write the expressions symbolically while working as a group of four; however, both Lara and Anne added the symbolic expression of \( n \times 2 - 1 \) on to their student worksheets. Lara wrote this in the top left corner of the page, nowhere near the question asking for the expression. This shows that there is a great likelihood that these symbolic expressions where written in after the class discussion (sws4, Lara). Anne also appears to have added in the symbolic expressions but she did so at the bottom of the page, also not in the space provided to record the expression (sws4, Anne). Lara, in her reflection in response to the question asking students to acknowledge helpful comments during the group discussion, wrote “Mark’s group had the same pattern expressed in a different way” (IR24, Lara). Ross and Allison, the other two members of Group C did not appear to have evidence in their student worksheets or reflections that the class discussion had clarified for them how to write the expression symbolically. This lag in the ability to represent the expressions
symbolically is quite natural in beginning algebra students and was revealed in Episode 2 by the members of Group C. Many authors write of the importance of the ability to verbalize expressions before making the transition to writing them symbolically using algebra (English & Warren, 1998; Schoenfeld & Arcavi, 1999). This is an important point as the ability of the students to give a word explanation shows that they have been able to generalize the expression. They are just not able to express the generalization symbolically as it is too complicated at this point. Regardless, student ability to generalize a pattern at the end of Lesson 2, shows that the activities and the ensuing discussion had promoted growth in understanding of variable.

Mark, of Group A, also shows evidence that the ability to write expressions symbolically lags behind the ability to express it orally or in written words. Although Mark wrote the rule symbolically as $n \times 2 - 1$ as the rest of his group did, his written explanation of the rule did not correspond to the symbolic expression he provided. Mark wrote “sequence number of it before plus sequence number” (sws4, Mark). During the group work session Mark had explained to me his understanding of the two expressions. I recorded his explanation on the top right corner of his student worksheet. Mark said “so $15^{th}$ [index] is 14 plus 15 equals 29 or 15 times 2 equals 30 minus 1 equals 29 (sws4, Mark). Although this explanation makes it evident that Mark understood these two ways of representing the expression, he only recorded one of them symbolically and the other expression in a written form. Nonetheless, the recognition that ‘$n$’ is central to the expression and the recognition that more than one expression is valid for a term value shows that Mark is making progress in the ability to create closed function expressions.
that match his verbal explanations. This flexibility of thinking proves of value in Episode 3 as students begin to experience the constant-variable issue.

Episode 3: Why were “x” and “y” variables

Figure 10: Location of Episode 3 in the Unit

Episode 3 occurred during Lesson 3 in which students completed Tasks 7 and 8 (ws7, 8). These tasks were created to promote opportunities for growth related to potential struggles that students have in their conceptions of variables. The constant-variable issue is one that has been pointed out in the research by Philipp (1999), Usiskin (1999b), and Wagner (1999) and has been discussed in the literature review in chapter 2. Essentially the constant-variable issue stems from the multiple uses and meanings of variables in different contexts, sometimes having a fixed unknown value and at other times, representing several or unlimited values. Students struggle with the constant-variable issue as it requires flexibility of thinking and an understanding of the particular context of a question. Episode 3 will show that through discussions and the negotiation of meaning (Cobb, Yackel & Wood, 1992), students made progress with regard to the constant-variable issue and consequently, in their conceptions of variable. Some students
came to a solid conclusion with regard to the constant-variable struggle and others were still experiencing intrapsychological conflict with the idea.

Lesson 3 involved each group using two boxes of different lengths to measure a length of paper that I had pre cut to be 130 cm long. The boxes were actual boxes, one dimension of which was labelled ‘x’ or ‘y’. Each box was three-dimensional and the ‘x’ and ‘y’ appeared on different boxes. Therefore the students referred to the boxes as the ‘x’ box and the ‘y’ box. No group had the exact same ‘x’ and ‘y’ boxes. By using different combinations of the two boxes they were given, students recorded different expressions for the line length. I will outline and compare the discussions that occurred in the three participant groups and then describe the events of the class discussion as they pertain to the student’s work on the constant-variable issue.

*Group A: Task 8*

In Group A, the discussion of Question 10 and 11 that related to why ‘x’ and ‘y’ were variables was quite short. Question 10 was “How come different expressions work for the same length of paper using your same ‘x’ and ‘y’ boxes?” (ws8, #10). Question 11 was “why is the term variable appropriate to describe the ‘x’ and ‘y’ lengths of the boxes?” (ws8, #11). Here is what Mark had to say:

Mark: in the end…well the boxes are variables so…I think they can vary meaning that together when all put together in different orders can reach the same goal. That’s what I would say for variables (GD25, A, 171-175).

Mark: because a variable varies, it can connect—they can add up all together in like different ways—almost like a pattern but like in a lot of different ways to get the same thing (GD25, A, 176-179).

Shared understanding of Mark’s comment is implied by the fact that no other group members asked for clarification, argued about it, or even commented on it. This
implies Cobb’s (1992) notion of intersubjectivity, the idea that the group agrees with Mark’s comments. The fact that none of Mark’s group members attempted to disagree, explain, or question his conception may imply that neither Mark nor his group members realize that ‘x’ and ‘y’ are fixed values (constants) in this activity and that they see the variable as a value that varies.

*Group B: Task 8*

The discussion of Group B can be contrasted to that of Group A. Group B seems to have realized that the values of ‘x’ and ‘y’ are fixed values (constants). It is also important to note that I intervened in the discussion of Group B, providing them with a focus for their discussion as they started talking about Question 10, “How come different expressions work for the same length of paper using your same ‘x’ and ‘y’ boxes” (ws8, #10).

Ms. C.: think about that. Every team has 130 cm but each group has different boxes so why is it working? Talk about it (GD25, B, 128-130).

After I left the discussion, the group discussed the relationship between the task and their particular examples of lengths ‘x’ and ‘y’ and through this discussion agreed that different expressions worked for the same length of paper because their group used the same materials throughout the lesson. In their responses to Question 10 on their student worksheets, the students all wrote essentially the same response. Elisa’s response was representative of the group. She wrote: “because we were using the same material all the time and you are measuring the same distance all the time” (sws8, Elisa).
When it came to discussing Question 11, “why is the term variable appropriate to describe the ‘x’ and ‘y’ lengths of the boxes?” (ws8, #11), the group had the following short discussion.

Michael: because it is.
Kim: because they are unknown--because the lengths are unknown.
Linda: good answer
Elisa: good answer—like family feud (GD25, B, 163-167).

Intersubjectivity (Cobb, 1992) with Kim’s definition was implied because nobody else added a response or questioned her ideas. In response to Question 11 on the student worksheet all wrote a response reflective of Kim’s idea, that variable was an appropriate term for the ‘x’ and ‘y’ lengths because “the length of the box is unknown” (sws8, Kim). The responses of Group B indicate that this group had an image of ‘x’ and ‘y’ as unknown but fixed values, not variable.

**Group C: Task 8**

Group C made similar progress to Group B but without my intervention. They also concluded after discussion that the ‘x’ and ‘y’ values were unknown but fixed. In the discussion of Question 10, “How come different expressions work for the same length of paper using your same ‘x’ and ‘y’ boxes?” (ws8, #10), after the following discussion, Group C agreed that different patterns work because there are different ways to come up with 130 cm.

Lara: because its different patterns.
Allison: no, why does it still work?
Anne: what?
Allison: why does it still work if we go xxy or xyx, why does it still work?
Anne: because we’re still doing the same thing.
Lara: because maybe 2 y boxes equal an x box.
Allison: that doesn’t—it’s because it’s still.
Lara: why are you asking me?
Anne: because there are different ways to come up with 130 cm.
Ross: there are a lot of different ways.
Allison: its because the boxes don’t change size.
Lara: there is more than one sequence.
Anne: yeah, more patterns and the boxes don’t change.
Lara: yeah—because there is more than 1 way. (GD25, C, 209-219).

The last comments of Allison and Anne show that they have a conception of the boxes as unknowns whose values are constant. This idea is confirmed by the worksheet responses to Question 11. “I think a variable is used to replace a number. Therefore the ‘x’ and ‘y’ is replacing the actual length of the box” (sws8, Anne); “because we didn’t know the length of the boxes” (sws8, Allison); “because the length of the box was unknown and we can have different sequences” (sws8, Lara).

Class Discussion

During the class discussion I asked the class Question 11 from Worksheet 8. The resulting conversation highlights the constant-variable struggle.

Ms. C: Why do you think that the term variable is important for using for x’s and y’s in this case?
Linda: Because the length of the boxes was unknown
Bob: You keep using a different variety of box lengths
Cole: I think the same as Bob.
Ms. C: You think what?
Cole: Because the boxes vary in length different groups had different expressions
Amy: Cause it is not the same number each time, you use different numbers and different numbers of boxes, and x can be any number. (CD25, 279-288).

Linda’s statement seems to indicate that she understands that the ‘x’ and ‘y’ lengths are unknown constants. Amy and Bob’s comments imply that they see the ‘x’ and ‘y’ lengths as variable and constantly changing. Cole’s statements are contradictory, the first indicating a variable view of ‘x’ and ‘y’ and the second indicating that ‘x’ and ‘y’ are constants within the group but change only between groups.
When I asked the class what a variable was, this is what Mark had to say.

Mark: I think it is something that varies like a pattern kind of thing. Like x and y are variables because x can be anything and y can be anything and if we are given something predetermined like a certain length or whatever. I don’t know. I have it in my head the way I think it is but I can’t really say it (CD25, 298-301).

Mark’s statement above shows his personal intrapsychological conflict (Cobb, Yackel, & Wood, 1992) with the constant-variable struggle. On one hand, he stated that ‘x’ and ‘y’ could be anything, indicating a variable view of the ‘x’ and ‘y’, and on the other hand, he talked about a predetermined length, indicating that he saw that ‘x’ and ‘y’ were predetermined by the context. This is a progression from Mark’s comments from his group discussion where he stated that the term variable was appropriate for the ‘x’ and ‘y’ lengths because “because a variable varies, it can connect—they can add up all together in like different ways—almost like a pattern but like in a lot of different ways to get the same thing” (GD25, A, 177-179). Mark’s idea of predetermined values seems to indicate that he is making progress with his constant-variable struggle. Mark was starting to develop an understanding of context and the idea that although variables can change values, they must maintain the same meaning or value within the same question or equation. Mark had a hard time trying to express his thoughts and once again, understanding is more developed than can be expressed. This time, the difficulty was with verbally expressing the thoughts whereas, in the previous lesson, the difficulty was with expressing the pattern symbolically.

By the end of this class discussion students were at different places when it came to a decision with regards to the constant-variable struggle. Students started Lesson 3 seeing variables as being able to take on any value at any time. They saw variables as
being completely variable. The tasks of Lesson 3 were designed to help students see that in a particular situation, variables will have particular constant values, although we may not know what they are. A final step is the realization that in different situations, these fixed values for the variables may be different. Thus the idea of ‘varying’ has two meanings. First, varying all the time, in which case the variable can have any value. Second, that variables vary between situations but with a constant value within a particular situation.

I will highlight some of the different places the students were at the end of Lesson 3 with regard to the constant-variable issue and link the potential influence of the class discussion. I will do this by referring to Mark of Group A, Linda of Group B, and Anne of Group C.

Mark’s definition of variable at the end of Lesson 3 indicates that the class discussion has not resulted in him completely resolving his intrapsychological conflict with the constant-variable issue. On his reflection sheet, on May 25, Mark wrote that a variable is “example: x, something that is and can be any number. When predetermined, you can use it to create patterns, etc.” (IR25, Mark). This indicates that Mark still views variables as varying and equal to any number. At the same time he acknowledges what he calls predetermined values that can be used to create patterns. This shows that Mark is making progress but has still not resolved the constant-variable issue and its relationship to the problem context. Mark’s idea of predetermined values for variables will be picked up in Episode 6 where students work on solving equations with two variables, when given the value of one of the variables.
Mark was very vocal during the class discussion and his struggle with variable appears to have influenced at least one other student, Linda of Group B, away from the idea that the variable was a fixed unknown [constant].

During the discussion with her group Linda and members of Group B appeared to have reached intersubjectivity with the idea that a variable was a fixed unknown. However, Linda’s written definition of variable after the class discussion reflected the influence of her classmates, as she wrote, a variable is “a changing unknown number” (IR25, Linda). Linda attributed some of this change to comments of her classmates. Linda wrote “Mark said that a variable is a variety. Then it clicked that a variable can be varied and Elisa’s answer from our group discussion made sense” (IR25, Linda). This was likely in reference to the group discussion of Group B (GD25, B, 120-166) in which Elisa was constantly implying that the different combinations worked because they were constant so they could be used in a variety of patterns. Linda connected the ideas of Mark and Elisa even though they were representing different views of variable. This highlights that the differing views, expressed during the class discussion, created some conflict for Linda as well.

During the work of her group, Anne had clearly made a decision about what a variable was. In Anne’s response to Question 11 on student Worksheet 8 we see “I think a variable is used to replace a number. Therefore the ‘x’ and the ‘y’ is replacing the actual length of the box” (sws8, Anne). However, At the end of Lesson 3, Anne’s definition of variable was “an unknown number replaced by a letter that can vary in different situations. Yesterday I didn’t think a variable was something that could change but today I do.” (IR25, Anne). As Anne’s group never discussed Question 11 (ws8) on
tape, she must have written the above response on her own after the class discussion which likely had an influence on her thinking, leading to two distinct changes in her definition of variable. First, was the change from the variable being the letter used to replace the number, to a view that a variable is the unknown number replaced by a letter. The second change evident in her definition was that her first definition implied that the value of the variable was a fixed constant, a view consistent with her groups’ discussion while the second definition included the idea of a variable changing. Anne attributes much of this change to the comments of Amy and Cole. In response to the influence of class discussions she wrote “when Amy and Cole, etc. said that they thought a variable could change it helped me” (IR25, Anne). Cole’s comments that Anne was likely referring, to were comments such as “that’s what I was actually going to say. Like ‘x’ over here can be a completely different number than in a different place. Like it is the same thing actually, but it can vary” (CD25, 334-336). Anne’s May 25 definition of variable, above, appears to have also been influenced by Amy’s comment “x is like the number that changes its value in different situations” (CD25, 333). The subtle difference in the definitions is recognizing when a variable may vary and when it has a constant value.

Episode 4: Is a variable a number or a letter?

Figure 11: Location of Episode 4 in the Unit

Lesson 4
May 27

A  B  C  +  5 other Groups

Class Discussion #3

Episode 4
Lesson 4 started with a review of the previous lesson, in which students were asked to discuss two questions on Worksheet 9 (ws9). The first was to re-discuss why the lengths of the ‘x’ and ‘y’ boxes were variables. The second question was to discuss whether a variable is a number or a letter, allowing the students the opportunity to work on the symbol-number issue, setting the stage for the exploration of a more flexible view of variables. This issue also relates to the problems students encounter in algebra when they have a limited view of variable as a missing number or placeholder. These difficulties have been addressed by Philip (1999), Usiskin (1999b), and Wagner (1999) and are detailed in the literature review of chapter 2. The debate of whether a variable is a number or a letter is simplistic when compared to sophisticated uses of variables in calculus for example; however, the question opened up discussion of the different uses of variables, starting with cases where letters represented a specific number or set of numbers.

The acknowledgment made by some students, by the end of Lesson 4, that a variable is a symbol [letter] that represents an unknown number or numbers, represented a conceptual leap. Through the process of participation in the group and class discussions, at least two students were able to consolidate their view of variable as letter. These were Mark, during the group discussions and Cole, during the class discussions.

*Group A: Task 9, Question 1*

Back in Episode 3, I outlined that Mark was experiencing the constant-variable struggle with regards to his definition of variable. The following section of transcript outlines the debate that took place in Group A in response to why the ‘x’ and ‘y’ amounts were called variables. The transcript has been chosen as it shows that the group is
leaning towards a view of a variable as the letter and also because it highlights the progress Mark was able to make through discussion with his group.

Mark: they are variables because they can change—remember what we said last time, and they can be predetermined or undetermined.
Arial: they are all different but at the end they all end up to the same answers.
Mark: an ‘x’ is anything, like you could say you’re ‘x’ and she’s ‘y’ and x+y is.
Arial and Samantha: so like a variable is.
Arial: what is a variable, I still don’t understand.
Samantha: why are they variables?
Mark: none of us do, well starting to. I think a variable is something that can always be changing or not necessarily changing but can be anything.
Samantha: yeah, you can put it in any order and it reaches the same thing.
Mark: yeah, like the boxes.
Samantha: yeah.
Mark: if you have 3x’s and 4y’s you can have xyxyxyy or xxyyy or like that.
Mark: I think they are variables also because varying like changing because “x” can be any length. If you say the box is “x”, it doesn’t mean x=11.5cm or whatever. It means that “x” is-measure it and how much it is or maybe it is the box itself is “x”—or not an amount of centimetres, just the “x” (GD27, A, 194-212).

The comments above suggest that the students are still not confident in their understanding of variables. However, through discussion, they are starting to have an idea about what variables are. Mark’s examples indicate that he does in fact understand some fundamental ideas about variables. These are: variables can be any letter, order does not matter when variables are added or multiplied, that variables can represent specific values, and that sometimes variables remain unknown.

The Group’s debate continued with the discussion of Question 2 where students had to discuss whether a variable is a number or a letter.

Mark: I would say a variable is a letter but
Arial: a letter is a number.
Mark: the letter like represents almost a number. ‘x’ can be anything but if you make it predetermined like x = 3, then you can figure stuff out like 3x (GD27, A, 261-264).
This excerpt of the group discussion, as well as the written responses on Worksheet 9, indicate that the members of Group A seem to have decided that the variable is the letter. Mark wrote that a variable is a “letter that, when undetermined, can be anything” (sws9, Mark) which suggests that Mark thinks that the variable is the letter, representing a fixed unknown, in cases where the value is determined such as $2x = 6$, and that in cases where the value is not determined, such a $2x = y$, that the variable can be anything. This statement represents progress for Mark on both the symbol-number issue and the constant-variable issue. Mark views the variable as a symbol that in some cases represents a specific value. Mark also understands that it is the context of the problem that determines whether or not the variable is a fixed value or a varying quantity.

Class Discussion

The conversation at the end of Lesson 4, May 27, commenced with the students talking about why the lengths of the ‘x’ and ‘y’ boxes from the previous lesson were called variables and then whether the variable is a number or a letter. Although the class discussion shows that there is still some confusion with regard to the symbol-number issue, progress is being made towards the letter [symbol] view. At least one student, Cole, has consolidated his view of the variable as a letter [symbol] by the end of the lesson. The following excerpt shows that at the beginning of the discussion, Cole was leaning towards the number view of a variable but as the discussion continues, Cole’s comment about ‘x’ changing depending on the situation sparks a thought in Bob. Bob’s explanation, along with the comments of others, help Cole to consolidate his ideas about variable.
Cole: Because it’s a number that can um change but it doesn’t change the total of what it makes. Its one thing that has a lot of different numbers but it makes the same thing, it doesn’t change the total.

Linda: I think they are called variables because the length of the boxes is unknown and they can vary between each group.

Amy: I think they are called variables because they vary from problem to problem but within a problem they stay the same number at a time, but in a different problem they are a different number.

Bob: It’s kinda like any other problem.

Ms. C: In what sense?

Bob: Well, every problem is different, and they all have the same answer to the same question.

Cole: Like say if x + 4 =6 then like x would be 2 but in another problem x + 8 = say 10 no 11, then x would be 3 so it’s the same thing but it changes depending on which situation.

Bob: Cole, now I think I am wrong. I think, I said a variable is a number but now I think that it is a letter because that’s the thing that’s changing from problem to problem.

Ms. C: So what about that whole thing about being stuck between numbers and letters.

Michelle: I don’t really think it is the number or the letter, but what it stands for in a pattern or whatever.

Kim: The letter is just there cause the number is unknown so you can’t really put a number so it replaces the unknown number.

Elisa: Yeah.

Penny: I think it represents, its almost what Kim said but we talked about it as a group and we said it is a letter that represents or that stands for the unknown number

Beth: It is the number that is represented by the letter.

Allison: Its pretty much like Kim. Say your box is 10.3 but you don’t know its 10.3 so its ‘x’ and ‘x’ is the variable.

Ms. C: So is it ‘x’ all the time, or just until you know that it is 10.3

Sally: Just until you know it is 10.3

Allison: then it is x = 10.3

Elisa: I don’t think its like ‘x’. I don’t think it’s the letter. I think it’s like the actual unknown number.

Kirby: I think it is both.

Ms. C: In what sense?

Kirby: In the sense that like a variable is what’s there whether it’s a letter or a number, a variable is what is there.

Sally: You have to figure out what ‘x’ is

Cole: It’s a letter that stands for an unknown number (CD27, 350-389).

Although no real consensus comes by the end of this excerpt of class discussion, the conversation gets students thinking and it influences the ideas of many. Certainly in
Cole’s case, the discussion in which many different conceptions were presented, has allowed him to move his thinking from an uncertainty that the variable is the number to a view of certainty that the variable is the letter [symbol] that stands for an unknown value.

Some of the direct links are acknowledged in the reflection sheets although many probably are not. What is interesting is the influence on student definitions of variables at the end of this lesson on May 27. Of the twelve participating students, eleven of them were present and ten wrote definitions that included some mention of variable being a number or a letter. Six students wrote that variables were the letters that represent a missing number and four students wrote that variables are the numbers that are represented by the letters. This alone is evidence that discussion is having an influence on the development of the understanding of variables, in some cases, leading them closer to the formally accepted view of variable, and in others, leaving them with limited conception of variable as being a number. The view of variable as a missing number is a conception that many students have had since elementary school. In some cases, the class discussion served to confirm the conception that already existed. This was the case for the four students who wrote that variables are numbers. Although the idea of the variable as a number stated by Elisa and then by Beth (a non-participant), the discussion justified Allison and Anne to maintain this view of variable.

Episode 5: What is the meaning of 2x?

Figure 12: Location of Episode 5 in the Unit
Episode 5 also occurred during the class discussion of Lesson 4. Students were completing Task 10 on Worksheet 9 and 10 that consisted of five questions that needed to be solved as equations. The idea was that students would use the knowledge of the properties of variables they had developed during the x-box activity in order to solve new types of equations. This task gave students the opportunity to consolidate the following properties of variables: (1) the idea that if a variable occurs in more than one place within a context it can be assumed to stand for the same value. (2) the idea that if there are two different variables within a context, they may represent different quantities and thus cannot be combined. (3) the idea that 5x implies 5 times the value of x and consequently, (4) the idea that a single value of x can be determined from 5x by dividing by 5. Episode 5 stems from Group B’s solution to Question 6 (ws10) and the progress that is made in reference to properties two, three, and four stated above.

**Group Discussion**

The three groups held their own discussions on how to solve the equations in task 10. The discussions of Group B have been chosen as a reference because growth is evident in Linda’s understanding of the properties of variables from the beginning to the end of the lesson.

**Group B: Task 10, Question 6**

The discussion around Question 6 (ws10), determine the value of ‘y’ in the equation $4x + 3y + 3 = 67\text{cm}$ if $x = 8.5\text{cm}$, represents important developments in terms of three properties listed above. With reference to Linda’s understanding, they are: that the two variables in this equation possibly had different values, that $4x$ was four times the
value of ‘x’ and, as she was finally able to express during the class discussion, that the value of ‘y’ could be determined by dividing the value of 3y by three.

The following excerpt shows two important points. Firstly, that Linda understood the idea that in the question the two different variables had different values, one of which has a known value of 8.5cm. Secondly, this discussion shows that although Linda understood the idea four multiplied by 8.5 (the given value of x), she had confusion about how the equation was to be solved beyond this calculation.

Linda: how about we re-write the equation but add in 8.5 cm for the “x” boxes so all we have to do is figure out the “y” box.
Elisa: okay, so it will be 4…
Linda: what 4 times 8.5?
Elisa: do you need this? (**I assume she is passing a calculator).
Linda: not if you are figuring it out.
Elisa: okay, so 4x, x box is 8.5.
Linda: 4 times 8.5.
Elisa: and what 3 will be.
Kim: Its 34.
Elisa: no.
Linda: plus 3 is 67 so what’s 4 times 8.5?
Elisa: so it would be 4 times 8.5 is (**pause to calculate) 34.
Linda: 34 + 3 = 67.
Elisa: 67 subtract 34 is
Linda: why 34?
Elisa: is 33 subtract 3 equals 30 then 30 divided by 3 is 10.
Kim: wait.
Linda: you are going a little fast.

From this transcript it is evident that the students accept that the two variables may have different values and thus do not combine them. Elisa seemed to take charge and calculated the result using the inverse strategy without waiting for the others to catch up. By the end of this excerpt, Elisa’s three group members tell her to slow down. Thus the ideas expressed were Elisa’s, as the rest simply had not caught up with her yet. This
excerpt makes Elisa’s understanding clear, but it leaves me to question the understanding of the other group members. The responses on the student worksheets all reflect Elisa’s answer; however, the way in which the response was determined leads me to think they all got their work from Elisa (sws10, Linda, Kim, Michael, Elisa). In particular, Linda had expressed confusion about the subtracting of 34 (the value of 4 times 8.5) and we are left to question whether this was resolved. The discussion showed no evidence that the group had achieved a taken as shared understanding on the solution to this problem.

A few minutes later I came over to the group and asked them to justify how they determined that ‘y’ had a value of ten. Elisa provided the group’s explanation leaving the understanding of the other group members unclear.

I come over to the group. Elisa points to her y=10 and says

Elisa: that’s right, right?
Ms. C: well, explain to me what you did and we’ll see if it makes sense—you don’t need me to say that okay.
Elisa: 8.5 is “x” so 8.5 times 4 is 34.
Ms. C: why times?
Linda: because you have 4 boxes.
Elisa: then 3y we don’t know yet plus 3 equals 67 so then you’d go 67 – 34.
Ms. C: to get rid of all the “x” boxes?
Elisa: yeah, then 33 – 3 is 30 and then 3y well 30 divided by 3 is 10. (GD27, B, 236-244).

Although this is a short explanation and it does not appear that negotiation took place among the group members, Elisa’s explanation to me about how she solved the equation on her own, provided an opportunity for Linda to negotiate the solution of the question with herself and create intersubjectivity with Elisa’s solution. This is evident in the fact that during the class discussion Linda offered the following explanation of how her group had determined that the value of ‘y’ is 10 in the equation 4x + 3y + 3 = 67 cm if x = 8.5cm.
Linda: We did 4x no...yeah...4x times 8.5 because our box was that length and we got (I am writing on blackboard as she says it) 34 plus 3y plus 3 equals 67. Then we did 67 subtract 34 equals 33...subtract 3 equals 30 divided by 3 for 3 boxes so ‘y’ equals 10. (CD27, 440-442).

During the group discussion, Linda had been stuck not knowing what to do beyond 4 times 8.5. After achieving intersubjectivity with Elisa’s explanation, Linda’s ability to explain the entire solution shows a growth in understanding. The addition of the explanation of dividing 30 by 3, for 3 boxes, shows that Linda has make the solution her own and understands the four properties of variables that were discussed above.

Episode 6: Is the value of the variable predetermined?

Figure 13: Location of Episode 6 in the Unit

I chose to call this episode “Is the value of the variable predetermined?” instead of “Is there always one value determined for the variable?” in order to follow up on Mark’s ideas from the past few lessons. This is because the activities of Task 12 involved equations and inequalities in which the values for the variables were determined by the context of the question, to have either one value or a specific range of values. The growth that can be seen in this episode relates to Mark’s previous idea of predetermined values, meaning that Mark saw a variable in an unsolved equation as a predetermined number that he had not figured out yet. The tasks of this episode challenged this idea for the students. In this episode, Mark and his group make progress towards understanding
that the determined values of the variable can be more than one value simultaneously. Group B chooses to assign a particular value to the variable showing that they are not comfortable leaving the value of the variable open. Group C is able to leave the value of the variable open. This is evident in the following conversation around Question 5 of Task 12 (ws12). The question was to determine the value of ‘x’ in $2x + 6 > -2$.

**Group A: Task 12**

Group A discussed that the value of the variable could be anything and then decided it could be anything positive.

Arial: we can write any number. It can be anything.
Ray: any number.
Mark: no.
Arial: any positive number (GD30, A, 293-295).

By looking at this discussion, Mark’s comment ‘no’ may be interpreted as a misunderstanding. However, his written answer on the student worksheet clarifies what he meant and shows that Mark has a more sophisticated understanding of number that includes all positive and negative numbers. Thus he said ‘no’ to Ray’s comment that the variable could be any number. Mark wrote: “any positive number and –3, -2, -1, 0” (sws12, Mark). This means that Mark understands that the context of the equation put parameters on the possible values of the variable which includes all positive numbers as well as some negative numbers.

**Group B: Task 12**

The short discussion of Group B on determining the value of ‘x’ in the inequality: $2x + 6 > -2$ can be contrasted to Group A. Group A decided that the variable could be anything positive where Group B decided to assign a value to the variable. This discussion is interesting, as even though the conversation seems to suggest that Group B
does see that ‘x’ can have many values, folding back to their past experiences with variables tells them that variables have a single value and thus they chose to assign a single value to the variable as their answer. Although Kim had the idea that ‘x’ could be anything, she was very quick to accept $x = 4$ as an answer and to abandon her idea that ‘x’ could represent a broader range of numbers.

Kim: it could be anything.
Michael: yeah.
Elisa: yeah.
Linda: lets say its 4.
Kim: so $8 + 6$ is greater than $-2$. (GD30, B, 288-291).

Group C: Task 12

Group C also had their own way of trying to answer Question 5 from Task 12 (ws12), $2x + 6 > -2$. The conversation below is the interaction of Group C in coming to a solution to Question 5. This interaction can be contrasted to the discussion of Group B where the students folded back to a previous conception of variables having a single value and got stuck there. This discussion shows that with teacher intervention, students were able to make progress towards a more flexible view that variables can have more than one value. The beginning of this discussion shows that the group seems to be taking the same direction as Group B, folding back to their previous experience, acknowledging many values but assigning one value to the variable, two. However, Lara objects to this simplification and my intervention takes the conversation in Lara’s direction.

Anne: how can you find that out?
Ross: What is the mouth for? **this is Ross’ reference for the inequality sign
Allison: 2 times 6 is greater than negative 2.
Lara: well that could be anything because it is already bigger than negative 2.
Allison: exactly—so just make it anything. It could be 2.
Lara: because 2 is already bigger than negative 2.
Allison: Let’s make it 2, 2 times 2 is 4 plus 6 is 10 is bigger than negative 2, yeah— we’re amazing.
Lara: (to me) It could be anything because 6 is already bigger than negative 2.
Ms. C.: anything at all? Are you talking about anything positive or anything negative or any number at all?
Lara: anything positive.
Allison: if it is 2 it is 10 and 10 is bigger than negative 2.
Ross: it would have to be bigger than 2.
Ms. C.: are there any negative numbers that wouldn’t work? So find the absolute lowest number you can go because you know if it is positive it is good because you just said that plus 6 is already bigger. What if I take negative 10, or negative 8, or negative 2? How low can I go? Talk about it.
Lara: negative 4 because six plus negative 8 equals negative 2.
Allison: so anything bigger than negative four? (GD30, C, 317-336).

Lara’s comment that the variable could be anything suggested that she was wanting to develop a more sophisticated answer; however, the group had a bit of difficulty thinking about negatives until I mentioned the idea. Once I mentioned the idea, they picked up on it and were able to determine that the ‘x’ value could be anything above negative four. This discussion, and its resulting expanded view of variable, highlights that teacher intervention is sometimes necessary in order to shift student attention and assist them to move beyond their previous conceptions.

Episode 7: Do ‘w’ and ‘n’ have the same value

Figure 14: Location of Episode 7 in the Unit
Episode 7 evolved from Task 15 on Worksheet 13. This task is the third question: Are these expressions equivalent? \(7(w) + 22 = 109\) and \(7(n) + 22 = 109\). This question, and the discussions that surrounded it, are very important to the development of the idea of context with variables. The idea of context has been a theme since lesson three (the x-box activity) when the x-box was the same within a group or context but changed between groups. Through the x-box activity, students experienced a situation where two different symbols represented different values. The discussions within the group show different levels of comfort and confusion with this question. By the end of the class discussion, progress has been made towards the view that the two equations are equivalent even though they have different variables. The progress of the members of Group B will be the focus of this episode.

**Group B: Task 15**

The short discussion that Group B had regarding Task 15 suggested confusion in the thinking of the group members. The group did not have a real discussion and they did not resolve their confusion.

<table>
<thead>
<tr>
<th>Linda</th>
<th>yes, they are equivalent.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kim</td>
<td>It doesn’t mean it’s the same number.</td>
</tr>
<tr>
<td>Elisa</td>
<td>It could be</td>
</tr>
<tr>
<td>Linda</td>
<td>It could be</td>
</tr>
<tr>
<td>Kim</td>
<td>It could also not be. (GD30, B, 321-323)</td>
</tr>
</tbody>
</table>

The written responses on the student worksheet confirm the confusion I just described. Linda wrote “they would be if ‘n’ and ‘w’ are replacing the same number” (sws13, Linda). In response to the same question, Elisa wrote: “they could be if ‘w’ and ‘n’ are the same number” (sws13, Elisa). Kim wrote: “they could, if letters had the same value, but if the letters are different, they are not equivalent” (sws13, Kim). These
comments on the student worksheets show that although the general thinking within the group is that the two equations are equivalent, the pupils are unsure. They do not realize that the context of the equation necessarily makes the two variables equivalent. Group B members state that if the values for the variables are the same thing then they are equivalent, but they are still waffling in giving a definitive response.

*Class Discussion*

The class discussion for Lesson 5 ran out of time on May 30 so we started the class on June 1 by completing the discussion of the tasks of May 30, Tasks 13 to 16 on Worksheet 13. The following excerpt from the class discussion shows that many students accept that the parameters set by the two equations determines that they are equivalent.

Ms. C.: What about this one? $7w + 22 = 109$ and $7n + 22 = 109$
Cole: Yeah, they are equivalent because they have the same answer.
Ms. C.: You mean what when you say the same answer?
Cole: Well the same sum of 109.
Ms. C.: So if we think back to our x and y boxes, would it be fair to say that our w and n boxes have to be the same length?
Kim: No, yes…I don’t know.
Ms. C.: Okay. So you said no, yes, I don’t know. What were you thinking?
Kim: Because the box we used before was for distance
Ms.C: So we have 7 boxes plus 22 is 109
Elisa: Kim made a mistake because you have 7 of them is that long and 7 of that is that long so it wouldn’t match up.
Ms. C.: Because we are assuming that they both equal 109?
Allison: Because if n and w weren’t the same thing then it wouldn’t add up to 109.
Kirby: Yeah, because they are both the same equation but different letters so they have to be equal.
Sally: When it is a different letter, does it have to be a different number?
Amy: No, it is pretty much the same equation so it has to be the same number.
Ms. C.: So what about what Sally said though about a different letter equalling a different number
Kirby: Only if there was one equation with two different letters in it that they would be different. (CD1-1, 724-745).
This class discussion shows the importance of interaction on student learning. Evidence from the student worksheets of three members of Group B suggest that this discussion solidified the concept for them and that changes were made to the worksheet answers after the class discussion, to reflect solidified conceptions. For example, as previously mentioned, on the student worksheet Linda had written “they would be if ‘n’ and ‘w’ are replacing the same number” (sws13, Linda). This comment was written in blue pencil crayon and in regular pencil over it certain words were crossed out to read “they would be if ‘n’ and ‘w’ are replacing the same number” (sws13, Linda). When you read what is left after words are crossed out, it reads that ‘n’ and ‘w’ are the same number. I have to assume that the class discussion solidified her view on her response to this question and that she went back to change it. In response to the same question, Elisa had written in purple pencil crayon: “they could be if ‘w’ and ‘n’ are the same number” (sws13, Elisa). In regular pencil, words were crossed out to read “they could be if ‘w’ and ‘n’ are the same number” (sws13, Elisa). In response to the same question, Kim had written in regular pencil “They could, if the letters had the same value, but if the letters are different, they are not equivalent” (sws13, Kim). In pen underneath this response she had written in a bigger print “yes, they are equal” (sws13, Kim). Again, I draw the same conclusions, that some part of the class discussion made the students change their ideas and they went back and changed their responses in order to reflect their shifted conception. I have to conclude that the class discussion clarified the equivalence of ‘w’ and ‘n’ and that these three students went back to Worksheet 13 after the discussion to modify their responses.
Episode 8: How do we represent these trading cards?

Figure 15: Location of Episode 8 in the Unit

Episode 8 involves student experiences solving problems by creating equations using variables where they did not overtly exist. That is, the questions did not contain variables. In fact, they looked the same as the word problems that the students had been solving since elementary school. The difference was that they were expected to determine what the variable should represent, and construct and solve equations to determine the value of the variables. This was a real challenge for students. Because they were having such a hard time, my interactions with certain groups at certain points are much more elaborate than they had been previously in the unit. This is because, as students’ understandings are challenged, they fall back on previous experiences to help them cope with new situations. Since, in this case, the students had very limited experiences with variables, I do not believe that they would have made the progress they did without a challenge. The tasks themselves were the original source of the challenge as they asked the students to use variables. Nonetheless, students still used recursive methods to solve the problems and then attempted to assign variables afterwards. Thus, teacher intervention was a needed source of challenge to progress in student thought.
Episode 8 commenced with Group B’s unsuccessful attempt to use variables to solve Task 17 (Question 1, ws14) and subsequently, with my intervention with the group, in which I led them through the use of variables for this task. The episode culminates with Task 19, (Question 3, ws15) where Group B showed evidence of being able to use variables to create and solve an equation for the problem on their own.

*Group B: Task 17*

The question in Task 17 was:

Joe and Lynn collect trading cards. Joe has three times as many as Lynn does. Together they have 36 trading cards. How many does each one of them have?

All three groups used some form of logic to get the number of cards and then tried to use variables afterwards with varying degrees of success. No group was able to use variables without teacher guidance. The following transcript describes the group’s attempt to go back and use variables after they have already determined that Joe had 27 cards and Lynn had 9 cards. It shows their reluctance and lack of confidence in using variables. This is due to their lack of experience in using variables with word problems that they have solved in the past through recursive strategies and because the question was likely not complex enough to encourage students to modify existing strategies. The group even stated that they are attempting to use variables because I want them to. Their attempts to use variable end up confused and contrived.

Elisa: if she wants us to use variables we can go 9 times 3 = x (laughs)
Linda: no, it would be 9x =
Elisa: oh yeah.
Linda: 9x
Elisa: multiplied by 3—no.
Linda: wouldn’t it be 9x though
Elisa: 9x
Linda: but why would it be
Michael: I don’t really understand.
Elisa: if you have to use variables so 9x. This would be 3 so 9 times 3 is 27. Then Lynn would just have 9.
Linda: so 9x = 27?
Elisa: or use –could just go xx, x times 3 and it will be 3 times 3.
Michael: and then you would write 27 after? (GD1, B, 355-363).

By the time I came over to Group B, they were half done Question 2, Task 18. I referred to the first equation asking the students:

Ms. C: how did you do it?
Kim: well, we divided by 4.
Elisa: 36 divided by 4.
Ms. C.: why divide by 4 though?
Elisa: because you are dividing into 4 because then it would be like 3 times as much you just take the 3 like 3/4 of you …well you take your answer multiplied by 3 and then like your answer would be the one.
Ms. C.: you have to add it back?
Michael: yeah.
Ms. C.: so if you were to use an “x”, would “x” represent Lynn’s amount or Joe’s amount?
Michael: Joe’s, Lynn’s!
Ms. C.: Lynn’s amount would be x and Joe’s amount would be what?
Elisa: y (GD1, B, 398-407).

At this point in the conversation, Elisa’s comment clarifies why she was unable to connect the quarters into an equation. Because the number of cards that Joe and Lynn have was different, Elisa thought that they should be represented by two different variables, ‘x’ and ‘y’. Although her ideas would work through two simultaneous equations x + y = 36 and 3x = y, it is not the easiest approach and it does not help her in this situation. Below is the discussion I had with Group B leading them through creating an equation with variables. When Elisa says that Joe’s amount would be ‘y’, Michael says:

Michael: no
Linda: what?
Ms. C.: would it be y?
Elisa: what, oh yeah, no.
Michael: no…3x.
Ms. C.: 3x, does that make sense?
Michael: because its 2 times as much.
Elisa: what?
Ms. C.: If you use “x” for Lynn’s amount.
Elisa: okay.
Linda: so you have “x”.
Elisa: yes
Ms. C.: some amount plus Joe’s amount which would be what? Three times Lynn’s amount?
Elisa: plus
Michael: 27.
Elisa: x times 3.
Ms. C.: x times 3—3x, sure, equals what, what do they equal together?
Elisa: 36
Ms. C.: and this is how you get your four because her amount plus three times is actually four times her amount.
Ms. C.: so that’s how…you are doing a good job in terms of re-writing it as a variable but try to start out with-okay who do you need to know first. If you know Lynn you can figure out Joe right, cause you times by three so you say Lynn can be my x, she’s my missing number and Joe is my 3x, 3 times Lynn okay. Keep going—try the next one, how many rides? (GD1, B, 407-423).

In this section I suggest to the group to use ‘x’ as Lynn’s amount and then guide them to determine Joe’s amount. The previous interaction shows the complicated nature of using variables to solve a word problem and that, although with teacher guidance, the students were able to use variables, I question whether or not they would be able to use variables on their own in another context. The answer came with Group B’s response to Task 19, in which the students were able to create and solve an equation for the problem situation without teacher assistance, showing that my intervention with the group, along with their subsequent discussions, impacted their ability to use variables to create and solve equations.
Task 19 is Question 3 on Worksheet 15. The question is:

There are 14 rocks in three piles (A, B, C). Figure out how many rocks are in each pile if pile B has 2 more than pile A and pile C has 4 times as many rocks as A.

The following interaction of Group B shows significant growth from the beginning of Task 17, where they determined the number of trading cards held by Joe and Lynn, using recursive methods and then tried haphazardly to assign variables. It is clear from this interaction that their group interactions and my intervention led to a growth in understanding in using variables to solve word problems. Although the students are reluctant to use variables, they are willing to attempt it and are able to fold back to the previous examples and they are able to assign the variable in this case. As a group they are also able to take their understanding one step further and create and solve a relevant equation for the problem.

Kim: It would be easier with trial and error.
Michael: you just have to do it like this one so—x
Elisa: There are 14 rocks in a pile okay-A, B, C. Figure out how many in each pile.
Michael: so divide 14 by 6 right? because there are 6 times
Elisa: you have to put A=x, B = x+2 and then C has 4 times as many as A, so x4
Linda: 4x.
Elisa: okay, we have to find out what x is and all these equal 14. So it could be 14 minus 2 is 12.
Linda: why subtract 2
Elisa: because you are adding two here.
Elisa: so now x divided by 4, 5, divided by 6 so x = 2, so x=2 so A=2, B=4, and C=6
Michael: no 8.
Elisa: that doesn’t work—oh, yes it does (GD1, B, 451-462).
From this discussion it is evident that at least Michael and Elisa are making excellent progress with regards to their understanding of the use of variables to create equations that can be used to solve problems. Although, at the beginning of this interaction, Kim wanted to use trial and error methods to determine the number of rocks in each pile, Elisa and Michael took the lead in terms of using variables to solve the problem. The responses on the student worksheets indicate an understanding of variable usage. Elisa wrote: “A = x, B= x + 2, C = 4x. Then she wrote the expression, x+ x+ 2 + 4x = 14 and then 6x + 2 = 14 , and = 2. Then she wrote A = 2, B = 4, and C = 8” (sws15, Elisa). The only thing that Elisa did not show was how she solved the equation. The other group members wrote essentially the same thing; however, I do not know how much of it they understand or if they were just writing what Elisa and Michael had figured out.

The ability of the members of Group B to use variables to create and solve equations is likely the result of the discussions that the students had with me, in which I led them through variable use, and of the subsequent discussions they may have had as a group afterwards.

In this chapter, I have described eight episodes where student interaction promoted shifted student conceptions of variable. In the following chapter, I will further provide answers to my research questions by discussing the types of interactions that took place during the course of the unit and the possible impact that they had on student learning.
In the previous chapter eight main episodes of negotiation were described in detail and links to relevant literature were made. In this chapter, portions of the eight episodes will be readdressed thematically, in so much as they allow me to answer my two research questions.

1. In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of variable.

2. In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable.

The focus of this chapter is interaction between participants. Interactions of various types occurred throughout the unit of study. They took place in my classroom as students negotiated the meaning of variable through a series of activities that I set out for them. I will claim that the interactions were in the form of negotiations that took place, as students struggled to integrate their prior conceptions with the new experiences they were exposed to, over the course of this unit. The term negotiation has been chosen as it implies a mutual respect for the ideas and mathematical views of others, as well as a back and forth sharing of ideas, with the goal of reaching some sort of consensus. I will not claim that a consensus of ideas is always reached, or even that in cases where a consensus is reached, that this necessarily leads to the establishment of understandings of variable that are compatible with those held by the formal mathematical community; however,
this chapter will highlight cases from the context of my study where there is evidence that negotiation has resulted in a shifted conception of variable.

The negotiations engaged in by individual students during this unit can be placed into three broad categories: those that occurred with other students, those that occurred with themselves, as their prior conceptions clashed with new experiences resulting in a shifted conception, and those that occurred with the teacher. Below I will outline the different types of negotiation that occurred during the course of the unit and then provide examples of them from the data.

Student Negotiations With Other Students

The negotiations discussed in this chapter occurred in a specific classroom, mine, where both the teacher and the students had created sociomathematical norms throughout the course of the year. Negotiations with other students involve those that occurred on an interpsychological plane between students. They occurred during the process of student knowledge construction, the goal of which was for students to create what Cobb, Wood, and Yackel (1992) called the intersubjectivity of knowledge. Intersubjectivity can be achieved either through discussion with their group members or through the whole-class discussion. Students are said to have achieved intersubjectivity when they agree upon or believe that they agree upon shared meanings. These taken-as-shared meanings are then used as the basis for future knowledge development. It is important to note that intersubjectivity is a perception held by the students or in the case of this data analysis, by an onlooker, that students conceptions are similar enough to be taken-as-shared. Cobb, Yackel, and Wood (1992) discussed that intersubjectivity is achieved in one of two ways, both of which result in the students having the perception that taken-as-shared
meanings have been established. The first type is where the students have created equivalent interpretations. This is where the students have achieved interpretations that are accepted by all and where there is no evidence that other members disagree or have different perceptions. The second type is where students reach parallel interpretations. This occurs when members of the group assume that their interpretation is taken-as-shared but where there is evidence that there are differences in their individual interpretations. Given that most of my data is in the form of written work and transcripts of student conversations, it is difficult to determine which type of intersubjectivity has been achieved in all cases. I relied on verbal communication, as my data involves audio taped discussions with the corresponding written student worksheets. There are cases where students acknowledge, in the written reflections, the influence of other student’s words upon their views of variable, however, for the most part, the influence is assumed on my behalf based on the transcripts and the student definitions of variable. It is therefore clear that non-verbal cues are not accounted for in the data for this thesis.

Student growth, based on discussion, will vary for individual students as growth occurs in what Vygotsky (1986) called the zone of proximal development, as a result of student readiness. Growth will occur as a result of some sort of conflict, which may or may not be solved, through the negotiation of meaning by means of challenges, justifications, or explanations (Cobb, Wood, Yackel & McNeal; 1992). If the conflict is too great or outside the zone of proximal development, then the conflict will lead to incommensurable results (Cobb, Yackel & Wood, 1992). Essentially no discussion or growth will occur as there is nothing to talk about or reflect upon.
Evidence of growth from the interaction with other students is often clear in the dialogue of the students and can be overt as the transcripts document any change of an idea that was stated verbally, and the worksheets document any change of ideas that the students recorded in writing. The following are two examples of student negotiation with their peers. The first example shows evidence that student negotiation promoted learning towards the view of variable that is consistent with formally accepted views. The second example shows the malleable nature of an emerging view of variable. It highlights the journey of the conception of variable of one student who was influenced through discussions with her peers. At times, these influences led her towards images of variable consistent with formally accepted views and at other times, led her away from them.

First Example of Student Negotiations with other Students

The first example of student negotiation with other students highlights the importance of the class discussions. In this example, the class discussion allowed members of a group stuck on a certain conception to move forward and to expand their view. This example involves Group B and relates to Task 15 (ws13) as was described in Episode 7 of chapter 5. The question was to determine whether the expressions

\[ 7w + 22 = 109 \]
\[ 7n + 22 = 109 \]

were equivalent. The group discussion led the group to determine that the expressions could be equivalent if ‘w’ and ‘n’ represented the same number. The responses on the student worksheets state this view that was taken-as-shared for the members of Group B (sws13, Elisa, Kim, Linda, and Michael). For the same question, members of Group C had created a solid conception that the two expression were equivalent and that ‘w’ and ‘n’ necessarily had the same value. Group C had a very short group discussion that ended with each member of the group writing a simple ‘yes’
or ‘yeah’ as a response to whether or not the two expressions were equivalent (sws13, Allison, Anne, Lara, and Ross).

In the class discussion the taken-as-shared meanings created by the individual groups are brought together and had the potential to influence the understandings of their peers. This influence is only possible if the differences in the meanings are close enough to allow for a productive discussion. The difference between the conceptions of Group B and C going in to the class discussion was subtle, but important. Group B stated that the expressions were equal if ‘w’ and ‘n’ were equal to the same number. Group C had a deeper understanding that given the parameters set out by each of the equations, that ‘w’ and ‘n’ must have the same value. This difference made for a productive class discussion likely leading to the changed responses on the worksheets for three members of Group B by the end of the lesson.

The conceptions of Group C are included as I will argue that the class discussion surrounding this question, changed the conceptions for members of Group B and that the key comment in this discussion was made by Allison, a member of Group C. The following is the transcript of the class discussion surrounding this question.

Ms. C.: What about this one? 7w + 22 =109 and 7n +22 = 109
Cole: Yeah, they are equivalent because they have the same answer.
Ms. C.: You mean what when you say the same answer?
Cole: Well the same sum of 109.
Ms. C.: So if we think back to our ‘x’ and ‘y’ boxes, would it be fair to say that our ‘w’ and ‘n’ boxes have to be the same length?
Kim: No, yes…I don’t know.
Ms. C. Okay. So you said no, yes, I don’t know. What were you thinking?
Kim: Because the box we used before was for distance
Ms.C: So we have 7 boxes plus 22 is 109
Elisa: Kim made a mistake because you have 7 of them is that long and 7 of that is that long so it wouldn’t match up.
Ms. C: Because we are assuming that they both equal 109?
Allison: Because if ‘n’ and ‘w’ weren’t the same thing then it wouldn’t add up to 109.
Kirby: Yeah, because they are both the same equation but different letters so they have to be equal.
Sally: When it is a different letter, does it have to be a different number?
Amy: No, it is pretty much the same equation so it has to be the same number.
Ms. C.: So what about what Sally said though about a different letter equalling a different number
Kirby: Only if there was one equation with two different letters in it that they would be different. (CD1-1, 724-745).

At the beginning of this discussion, Kim and Elisa of Group B were stating their ideas and confusion. Then Allison made a key comment that the two letters had to be equivalent or else it would not add up to 109. This prompted Kirby and Amy to agree and Kirby’s comment about the idea that the two variables would be different amounts if they were in the same equation served to solidify that the two expressions had to be equivalent.

After this class discussion no reflection was written as the time ran out for the teaching period and the discussion was continued at the beginning of Lesson 6 on June 1. However, looking at the individual student worksheets for three of the members of Group B, evidence of changed conceptions can be seen. This is the case for Elisa, Linda, and Kim. Elisa and Linda appeared to go back and cross certain words out on their worksheets and Kim added a statement to her response. The evidence that this was done afterwards is due to the evidence from the discussion of Group B, where they had not made a decision if the expressions were equivalent and also the fact that the crossed out or added words were done with a different writing tool.

This example is a case where two groups had created closely related taken-as-shared meaning. These meanings came together during the class discussion and because
of their closeness, led to a productive discussion in which the key difference between the expressions was resolved and members of Group B decided to remove the word ‘if’ from their responses.

Second Example of Student Negotiations with other Students

The second example of negotiation between students will cite the development of understanding of Anne, a member of Group C, and will show how her negotiations with her classmates shifted her understanding of variable several times in her attempt to come to terms with both the constant-variable issue and the symbol-number issue. It will be explained that some of these shifts will be in the direction of the formally accepted views of variable, while others will be in the opposite direction.

As explained in Episode 1 of chapter five, Anne had been instrumental in her group’s decision that ‘n’ was a necessary part of the expression for the sequences that they had been given in Lessons 1 and 2. During Lesson 3 on May 25, the students were exposed to the x-box activity which posed a challenge to the idea that a variable represented “letters or symbols used to find out pattern rules. (n)” (IR24, Anne). For the first time, the variable was presented as a fixed, but unknown, constant value. At the end of the lesson, Anne determined that she thought the ‘x’ and ‘y’ lengths of the boxes were called variables because “a variable is used to replace a number therefore the ‘x’ and the ‘y’ is replacing the actual length of the box” (sws8, Anne). As her group ran out of time to discuss this question before the class discussion, I will assume that her response was based on her ideas. This response indicates that Anne has come to see a variable as a letter replacing an unknown constant value.
After the class discussion at the end of Lesson 3, Anne wrote the following definition of variable “an unknown number replaced by a letter that can vary in different situations. Yesterday I didn’t think a variable was something that could change but today I do.” (IR25, Anne). Anne’s change in definition of variable shows that the class discussion has impacted her views of variable away from that of a letter replacing a constant, towards a number that varies from situation to situation. The issue of when a variable varies and when it is constant, which is at the heart of the constant-variable issue, seems to be a challenge for Anne at this point. In her written reflection, Anne attributed these changes to comments made by Cole and Amy during the class discussion.

Amy: x is like the number that changes its value in different situations
Cole: That’s what I was actually going to say. Like x over here can be a completely different number than in a different place. Like it is the same thing actually, but it can vary. (CD25, 333-336)

The comments of Amy and Cole are key as they clarify that variables can change values depending on the context. Students are coming to terms with the main element of the constant-variable issue which is determining when a variable may vary and when it has a constant value.

During Lesson 4 on May 27 the following discussion was held in which it appears that Group C, Anne’s group, had negotiated a taken-as-shared meaning of variable as a letter. This involves a second shift in conception for Anne, this time as she works through the symbol-number issue.

Ross: a variable is a letter uh a number.
Anne: a letter.
Allison: I think a variable is a letter.
Lara: well no—a letter is just a symbol that hides the variable.
Ross: yeah, exactly.
Lara: I say that the letter is what hides the variable—the number is the variable.
Allison: I thought the letter was the variable that was hiding the number.
Anne: yeah, me too.
Lara: oh no.
Ross: I was sure of what it was and now I don’t know.
Allison: okay, let’s say you have an x box and the x is really 10.3 but you don’t know because you haven’t measured it so it is called x box so I thought the x was the variable because the x was hiding 10.3
Ross: yeah, that makes sense.
Lara: yeah, that makes more sense than mine—let’s go with that.
Allison: so the variable is the letter.
Lara: the variable is the letter.
Allison: Ross looks confused.
Ross: so a variable is a letter. (GD27, A, 235-249)

At the end of this lesson there was a class discussion following which the students wrote another individual reflection. Allison’s definition of variable was “a letter that stands for an unknown number” (IR27, Allison). This definition was representative of Lara and Ross’s definitions. However, Anne’s agreement during the group discussion that the letter was the variable hiding the number contradicts her definition that she wrote both before this discussion at the end of lesson three and also the definition she wrote immediately after the class discussion of this lesson. At the end of Lesson 4 Anne wrote “I think a variable is an unknown number represented by any letter.” (IR27, Anne). This inconsistency suggests one of two things. First, that Anne was caught up in the class discussion and going along with the group’s ideas for the sake of consensus, or second, that the inconsistency in Anne’s definitions suggest that her ideas of variable were not solid and that they shifted with each new experience. I think that the second is more likely as Anne was not simply agreeing with her group members that the variable was the letter, she actually stated it at the beginning of the discussion. This is because during the class discussion students were working on the symbol-number issue and there was debate about whether a variable is a number or a letter. Anne must have been influenced by this
discussion, outlined in detail in Episode 4, in which Cole consolidated his view of variable as letter. It was mentioned there that this discussion in which Elisa and Beth stated their ideas of variable as number, influenced Allison and Anne to change their view of variable. This is a case where discussion has not shifted a conception towards formally accepted views of variable.

Anne’s view on the symbol-number issue changed at least one more time. After Lesson 4 she had stated that the variable was a number. However, at the end of the unit in her reflection of Lesson 6, Anne wrote “I think a variable is a letter to replace an unknown (or yet to be solved) math number.” (IR1, Anne). There was no individual reflection at the end of this lesson so I cannot determine whether it was the experiences in Lesson 5, Lesson 6, or a combination of them all, that caused this final shift. Although the groups did not again explicitly discuss whether a variable was a number or letter the activities of the two lessons may have led to this shift in thinking. During Lesson 5 students worked on solving equations where the variables represented one or more fixed values. In Lesson 6 the students solved word problems assigning variables, creating an appropriate equation, then solving it. At the end of Lesson 6 there was a class discussion in which variables were mentioned. At the beginning of the discussion I asked the class:

Ms. C: A lot of us seem to be able to solve these questions. I noticed that most of you were doing it with trial and error. A few of you were trying to use variables. Lets talk about why you automatically went to trial and error instead of trying to use variables (CD1-2, 785-787)

After this comment Mark offered that trial and error was natural to them. We continued to discuss and members of groups that had used variables offered their responses. Through this discussion, it was taken-as-shared among the members of the
class that variables can be used to solve equations and that they represent unknown values. The experiences of solving the equations and having confirmation during the class discussion shifted the conception of Anne towards the formally accepted view of variable as a letter representing a value.

These two examples showed that negotiation between students have the potential to promote a shift in student understanding of variable.

Student Negotiations with Self

Students negotiated with themselves, when they referred back to previous understandings, in order to make new connections. This is what Pirie and Kieren (1994) referred to as folding back. Students also negotiated with themselves through reflective abstraction in which individuals reorganize their mathematical activity (Cobb, Bouffi, Mclain, & Whitnack, 1997). Reflective abstraction is often prompted when a delicate shift of attention (Mason, 1989) has caused students to think in new ways or to reorganize their mathematical activity. This type of development is difficult to show as it happens at an internal cognitive level, thus clues of this cognitive growth are much more subtle as there are limited external indicators. They may represent a student’s attempt to contribute to the discussion by adding in comments and then self-criticizing these comments orally. Here I will again note that the learning of individual students does not occur in a vacuum. As one student folds back or shifts the attention of the discussion, there is great potential to influence the views and thinking of the other students. In some cases, this leads to growth in conceptions that are consistent with formally accepted views of mathematics and in some cases to conceptions that are inconsistent with the formally accepted views. The irony of the conception created by a group of students is
that it is much more robust than those of an individual student. This is because students seem to be under the impression that if many others agree then they have to be on the right track. Although these situations may be examples of constructivism, the knowledge they construct is limited to the environment they constructed it in. Evidence of student negotiations with themselves is more subtle than negotiations between students, and therefore easily missed. I will however, discuss a few incidences where there is evidence that a student has shifted a conception as a result of solving some internal conflict.

The idea of internal conflict reflects the struggle that a student may experience while trying to reconcile previous understandings or truths that they have created through their experiences with current tasks in which the student’s truth is challenged in some way. The idea of internal conflict has been discussed by several authors. Cobb, Bouffi, Mclain, and Whitenack (1997) discuss Piaget’s notion of reflective abstraction as the process by which students reorganize their mathematical activity. Participation in discussions, either in their groups or in the whole class, does not determine that reflective abstraction will take place. Although in a classroom based on constructivist principles, activity occurs first on a social plane and then on an individual psychological plane, one does not automatically follow the other. This means that although reflective abstraction in a social constructivist classroom is usually the result of social interaction whereby the ideas of others cause a shift in understanding, it is possible that a student experiences reflective abstraction working on their own. It is also possible that reflective abstraction may not occur despite social interaction during discussions. It is also important to note that negotiation with self is difficult to separate from negotiations with other individuals,
as often negotiations with others will spark the negotiation with self. However, for this
section of the thesis, I will discuss the negotiations with self, regardless of whether they
were prompted by negotiations with other students or the teacher.

Cobb, Yackel and Wood (1992) also discussed the idea of internal conflict. They
used the term intrapsychological conflict to describe the conflict that a student
experiences within themselves while negotiating the coming together of previous
understandings and new experiences that challenge previously held mathematical truths.
The examples will show that students can negotiate intrapsychological conflict with
themselves as they participate in discussion with others.

*First Example of Student Negotiations with Self*

The first example that I will cite for student negotiation with themselves occurred
during the class discussion of Lesson 4 and it has been described in chapter 5, in Episode
4. During Lesson 4 students had been working on the symbol-number issue by
discussing whether a variable was a number or a letter. The following transcript shows
that at the beginning of the class discussion, Cole viewed variables as numbers. Cole is
not a participant in the full study so I am not sure whether this view of variable was
Cole’s alone or if it was a taken as shared, negotiated, definition that came about through
discussion in his group. Either way, Cole verbalized his views at the beginning of the
class discussion. Cole participated in the class discussion verbally with a few comments
but most of his participation was on a non-verbal level as he listened to the comments of
others and reflected upon them. Cole’s final comment made it evident that the class
discussion prompted a negotiation within himself and resulted in a reflective abstraction,
allowing him to modify his view of variable.
Cole: Because it’s a number that can um change but it doesn’t change the total of what it makes. Its one thing that has a lot of different numbers but it makes the same thing, it doesn’t change the total.

Linda: I think they are called variables because the length of the boxes is unknown and they can vary between each group.

Amy: I think they are called variables because they vary from problem to problem but within a problem they stay the same number at a time, but in a different problem they are a different number.

Bob: It’s kinda like any other problem.

Ms. C: In what sense?

Bob: Well, every problem is different, and they all have the same answer to the same question.

Cole: Like say if $x + 4 = 6$ then like $x$ would be 2 but in another problem $x + 8 = \text{say } 10$ no 11, then $x$ would be 3 so it’s the same thing but it changes depending on which situation.

Bob: Cole, now I think I am wrong. I think, I said a variable is a number but now I think that it is a letter because that’s the thing that’s changing from problem to problem.

Ms. C: So what about that whole thing about being stuck between numbers and letters.

Michelle: I don’t really think it is the number or the letter, but what it stands for in a pattern or whatever.

Kim: The letter is just there cause the number is unknown so you can’t really put a number so it replaces the unknown number.

Elisa: Yeah.

Penny: I think it represents, its almost what Kim said but we talked about it as a group and we said it is a letter that represents or that stands for the unknown number.

Beth: It is the number that is represented by the letter.

Allison: Its pretty much like Kim. Say your box is 10.3 but you don’t know its 10.3 so its ‘x’ and ‘x’ is the variable.

Ms. C: So is it ‘x’ all the time, or just until you know that it is 10.3

Sally: Just until you know it is 10.3

Allison: then it is $x = 10.3$

Elisa: I don’t think its like ‘x’. I don’t think it’s the letter. I think it’s like the actual unknown number.

Kirby: I think it is both.

Ms. C: In what sense?

Kirby: In the sense that like a variable is what’s there whether it’s a letter or a number, a variable is what is there.

Sally: You have to figure out what ‘x’ is

Cole: It’s a letter that stands for an unknown number (CD27, 350-389).

Thus, for Cole, the participation in the class discussion provided a background for him to experience reflective abstraction and to change his conception of variable towards
the letter [symbol] view and to resolve his intrapsychological conflict with the symbol-number issue.

**Second Example of Student Negotiations with Self**

The second example of student negotiation with self, occurred as Mark worked through his personal intrapsychological conflict with the constant-variable issue regarding variables and was able to resolve this conflict. This example shows that negotiation with self and resolution of internal conflict is facilitated by discussion in small groups and with the class.

Mark’s ideas of variable were referenced throughout chapter five in Episodes 2, 3, 4, and 6. The example of growth through negotiation with self is evident from Mark’s verbalization of his beginning thoughts of variable in Episode 3 to the more sophisticated thoughts in Episode 6. During his group’s discussion during Lesson 3, Mark made comments about variables that were undisputed by his group members implying intersubjectivity with his idea of variable as a variety.

**Mark:** because a variable varies, it can connect—they can add up all together in like different ways—almost like a pattern but like in a lot of different ways to get the same thing (GD25, A, 176-179).

This comment highlights the constant-variable struggle that Mark was experiencing. This comment states that a variable varies implying that a variable can represent anything at all.

During the class discussion of the same lesson Mark stated his definition of variable.

**Mark:** I think it is something that varies like a pattern kind of thing. Like x and y are variables because x can be anything and y can be anything and if we are given something predetermined like a certain length or whatever. I
don’t know. I have it in my head the way I think it is but I can’t really say it (CD25, 298-301).

This comment is Mark’s acknowledgement of his struggle verbalizing his views. It also shows progression from earlier in the class when Mark’s comments made it clear that he viewed a variable as something that could represent anything. This comment suggests that Mark is experiencing intrapsychological conflict, stating on one hand that ‘x’ and ‘y’ could be anything, and on the other hand, discussing predetermined values indicating that he was moving towards a constant view of variable. This indicates that Mark is making progress with his view but that this progress has created an internal conflict. When I asked Mark to clarify what he meant by a predetermined value Mark stated “I said it could be anything, but when it is predetermined” (CD25, 313). This comment indicates that Mark recognized that at least in certain cases ‘x’ was a predetermined constant value in examples such as 4x = 12.

Mark’s reflection sheet definition at the end of Lesson 3 indicates that his view is still emerging. In his reflection, Mark wrote that a variable is “example: x, something that is and can be any number. When predetermined, you can use it to create patterns, etc.” (IR25, Mark). The fact that Mark had been vocal about his views and sharing his internal struggle with the rest of his classmates had an influence on the conceptions of variable for at least one other student, Linda, as described in Episode 3 of chapter five.

Mark’s negotiation with himself through the constant-variable struggle continued through Lesson 4 as his group discussed why ‘x’ and ‘y’ had been called variables in Lesson 3 and whether or not a variable is a number or a letter. By working through the discussion of whether a variable was a number or a letter, Mark was able to resolve his
constant-variable struggle. This is evident with the following interaction of Mark with his group member Arial in reply to the question of whether a variable was a number or a letter.

Mark: I would say a variable is a letter but
Arial: a letter is a number.
Mark: the letter like represents almost a number. ‘X’ can be anything but if you make it predetermined like x=3, then you can figure stuff out like 3x (GD27, A, 261-264).

This interaction makes it clear that Mark views the variable as a letter that represents a number when the value is determined in advance by the parameters of the equation. Mark’s definition of variable, on his student Worksheet 9, in which he wrote that a variable is a “letter that, when undetermined, can be anything” (sws9, Mark) indicates that he understands that when the parameters of the equation do not define the value of the variable, then it can represent any number. Mark’s understanding is reconfirmed later on in Lesson 5, when Mark’s group is working on solving Task 12 (ws12) in which they needed to determine the value of ‘x’ in 2x + 6 > -2. Mark’s written comment that ‘x’ could be “any positive number and –3, -2, -1, 0” (sws12, Mark) shows that Mark has resolved his constant-variable struggle and that his understanding has progressed to a point where he understands that a variable can simultaneously represent more than one number, given the context of the problem.

This section showed that while progressing through the activities of the unit, students experienced internal negotiation in an effort to construct new meanings of variable.
Student Negotiations with the Teacher

Interactions with the teacher refocused a discussion, or clarified a point. In a classroom it is the professional responsibility of the teacher to create an environment where learning can take place. My classroom was no exception, and thus, my number one responsibility throughout this research was to focus on student learning, in order to prepare my students to enter the world of formal mathematics and to study algebra at a higher level. My interactions with my students during the course of the unit were like those of all teachers. These interactions were intended to focus, redirect, and move student learning forward through the experiences that were planned for them. My responsibility to set out the environment and consequently the student experience did not end once the six lessons were planned. During the course of the lessons, at times I needed to modify the flow of the lesson or the focus of the discussion in order to keep with the main learning goal of the unit.

During the course of the teaching unit I was not immune to the difficulties faced by other teachers trying to implement a constructivist pedagogy. These have been outlined by Ball (1993) and Richardson (2003). I noted the two main dilemmas that Ball (1993) outlined. These are my role as teacher as a mathematical authority in the classroom letting students struggle through the problems without allowing them to flounder. Richardson (2003) outlines that there are no real pedagogical rules for constructivist classrooms as there is no constructivist teaching theory. The implication is that teachers need to come to grips with how constructivist ideas clash with the expectations of high school Mathematics curriculum where teachers are expected to
prepare students to enter the formal Mathematics community. This is due, on one hand, to the constructivist ideas that truths are built through experience and, on the other hand, that the formal mathematics community is based on many agreed upon truths that students need to adopt in order to be successful in higher level mathematics. It could be argued that, in a truly constructivist classroom, teacher intervention would not be necessary beyond setting up tasks for students. This would be ideal if all students were able to construct mathematical knowledge that would allow them to enter the formal mathematical community, and if they all were able to do this within the prescribed amount of time allotted in the curriculum. As all teachers have a professional responsibility to teach all the assigned curriculum within a set time limit, teachers, myself included, can be found, on occasion, using techniques that are less constructivist in the name of moving things along a little bit faster. In some cases, if students were given enough time, they might have constructed the concepts and in others, I will argue, teacher intervention may be required in order to develop certain concepts.

Both Mason (1989) and Cobb, Bouffi, Mclain, and Whitenack (1997) highlight the importance of interactions with teachers and the role that the teacher needs to play despite the type of learning that the classroom supports, either traditional or more constructivist student centred. This is because students will not naturally tend to make explicit connections to previous work or to patterns. The teacher, therefore, needs to use the ideas of the students in order to overtly state patterns or to redirect the discussion. The role of the teacher in a constructivist classroom is very present and vital. The teacher needs to find the balance between allowing students to explore and justify responses and providing opportunities or overt examples that counter alternative or incomplete
conceptions. The constructivist teacher knows what types of developments students may make during the course of an exploration and once the students have made them, helps the students put mathematically appropriate names to them and apply them with mathematical conventions. In the case of school algebra, with all the time and curricular constraints, developing an understanding of the multiple uses and functions of variable may be too complex to accomplish without teacher intervention. If left long enough on their own, students may be able to construct this knowledge; however this is not practical in a school environment.

The following are two examples, from my data, that show student interaction with the teacher and a reflection of how these negotiations may have promoted learning. I will also reflect upon the necessity of these interventions and discuss whether given enough time, students would have been able to make these developments in the allotted time without teacher intervention. The first example is one where I intervened in a subtle way that promoted a shift in understanding. The second example is a case of an overt intervention resulting in a shift in understanding.

First Example of Student Negotiations with the Teacher

The first example of negotiation with the teacher has been chosen because it represents a case where a simple question posed, by the teacher, created a situation of negotiation and prompted a shift in understanding for at least one student, Linda. This is a case where the teacher intervened in the activities of Group B. The intervention was subtle but promoted a negotiation within the student and a shift in understanding. The following example refers to Episode 5 where Group B solved for the variable ‘y’ in the equation 4x + 3y + 3 = 67 where x = 8.5. Elisa had solved the equation by working
through it orally, using the inverse strategy, but the speed at which she did so left the others confused and telling her to slow down. As the teacher approached the group, Elisa asked for validation that her response of \( y = 10 \) was correct. The teacher responded by asking Elisa to explain how the answer was determined.

Elisa: that’s right, right?
Ms. C: well, explain to me what you did and we’ll see if it makes sense—you don’t need me to say that okay.
Elisa: 8.5 is “x” so 8.5 times 4 is 34.
Ms. C: why times?
Linda: because you have 4 boxes.
Elisa: then 3y we don’t know yet plus 3 equals 67 so then you’d go 67 – 34.
Ms. C: to get rid of all the “x” boxes?
Elisa: yeah, then 33 – 3 is 30 and then 3y well 30 divided by 3 is 10. (GD27, B, 236-244).

This subtle intervention of asking for an explanation prompted Elisa to explain her thought process. Linda participated, as she had when the group determined the response up to where she had gotten stuck. She knew that \( x = 8.5 \) and that \( 4 \times 8.5 \) would give the value of the ‘\( x \)’s, but did not know what to do after that. Elisa’s explanation, as well as my comment “to get rid of all the ‘x’ boxes”, (CD27, B, 242) served to clarify the solution for Linda. This allowed Linda to make the solution her own as she explained her group’s response during the class discussion.

This is a case of subtle intervention for a few reasons. First, as I approached the group, Elisa determined the course of my intervention by asking me a question which necessitated a response. Elisa asked if her response of \( y = 10 \) was correct. At this point I could have answered in many ways; however, I responded by asking Elisa to explain how her answer was determined. My response, of asking Elisa to explain, created a situation in which intersubjectivity with Elisa’s response could be achieved through explanation. This was the case for at least one other member of the group, Linda. This is shown by
the fact that Linda is the one who explained why her group viewed the variable as having a fixed value of 10, during the class discussion of May 27. Secondly, this intervention was subtle because even though the shift in Linda’s understanding came from the explanation I asked Elisa to make, I had no idea at that time that Elisa was asking me to confirm an answer that she had determined on her own. I assumed that Elisa was speaking on behalf of the group, that the response of $y = 10$ was a response that the group had agreed upon, and that Elisa was simply looking for a confirmation for her group. Only in listening to the transcripts, is it obvious that Elisa had determined the answer essentially on her own and that she was looking for a confirmation of her response. Thus my intervention seemed simple and subtle but created an important opportunity for the other group members to have an explanation of the response. Although I have no evidence that this explanation had any impact on the understanding of the other members of the group, Michael and Kim, Linda was obviously affected by this explanation. This example highlights the importance of negotiations that occur with the teacher in a constructivist classroom. Had I chosen to respond in a different way, this shift might not have occurred. For example, had I been in a rush and answered Elisa with a ‘yes’ or a ‘good work’ there would have been no opportunity for the discussion that caused a shift for Linda. It shows, that through discussion, Elisa was given an opportunity to consolidate her response and other group members were given the chance to revisit the response through an explanation of a peer.

Second Example of Student Negotiations with the Teacher

The second example of negotiation with the teacher has been chosen as it represents a situation where the teacher intervened in an overt way, that resulted in a shift
in understanding, evident in a future example of a similar type. This example refers to Episode 8 where students were working on word problems in Tasks 17 to 19 (ws 14-15). Teacher intervention, during student work on Task 17, may have resulted in the shifts in student understanding that were evident in the student responses of Task 19. This situation of negotiation with the teacher is called an overt intervention, as the teacher knew that the students had used recursive methods to determine the number of trading cards held by both Joe and Lynn (ws14, 1) and was making an overt and purposeful intervention to assist students in using variables to create and solve an equation for this problem. Thus, this intervention was more directive than those previously in the unit. Here the interventions involved more ‘telling’ rather than asking questions as before.

As explained in Episode 8, Group B had used recursive methods to determine that Lynn had 9 cards and that Joe had 27 cards. Their attempt to go back and assign variables was confused and resulted in a contrived use of the variables ‘x’. When I came over to the group the following discussion occurred.

Ms. C: how did you do it?
Kim: well, we divided by 4.
Elisa: 36 divided by 4.
Ms. C.: why divide by 4 though?
Elisa: because you are dividing into 4 because then it would be like 3 times as much you just take the 3 like 3/4 of you …well you take your answer multiplied by 3 and then like your answer would be the one.
Ms. C.: you have to add it back?
Michael: yeah.
Ms. C.: so if you were to use an “x”, would “x” represent Lynn’s amount or Joe’s amount?
Michael: Joe’s, Lynn’s!
Ms. C.: Lynn’s amount would be x and Joe’s amount would be what?
Elisa: y (GD1, B, 398-407).
Once the group started explaining what they had done, I realized that they were using ‘x’ and ‘y’ to represent the number of trading cards that Joe and Lynn had. I knew that the equation $x + y = 36$ was not one that the students would be able to solve at this point and I realized that more intervention was needed. This is because the experience that the students had had to date led to the development of the idea that each different variable represented a different number. This rule of sorts met all the examples that they had come across to date and thus, because Joe and Lynn had differing numbers of cards, the students logically decided to assign different variables ‘x’ and ‘y’ to the amounts.

Because, in my discussion with Group B, I realized that they were stuck on this idea of representing different amounts with different variables, I decided to intervene in order to help them move past this idea. In response to Elisa’s idea that ‘y’ represented Joe’s amount of cards, Michael responded and the following discussion took place.

Michael: no
Linda: what?
Ms. C.: would it be y?
Elisa: what, oh yeah, no.
Michael: no…3x.
Ms. C.: 3x, does that make sense?
Michael: because its 2 times as much.
Elisa: what?
Ms. C.: If you use “x” for Lynn’s amount.
Elisa: okay.
Linda: so you have “x”.
Elisa: yes
Ms. C.: some amount plus Joe’s amount which would be what? Three times Lynn’s amount?
Elisa: plus
Michael: 27.
Elisa: x times 3.
Ms. C.: x times 3—3x, sure, equals what, what do they equal together?
Elisa: 36
Ms. C.: and this is how you get your four because her amount plus three times is actually four times her amount.
Ms. C.: so that’s how…you are doing a good job in terms of re-writing it as a variable but try to start out with-okay who do you need to know first. If you know Lynn you can figure out Joe right, cause you times by three so you say Lynn can be my x, she’s my missing number and Joe is my 3x, 3 times Lynn okay. Keep going—try the next one, how many rides? (GD1, B, 407-423).

My intervention was an attempt to get the students to understand that expressions could be written for the amounts of cards held by both Joe and Lynn using the same variable and by writing the expressions as they compare to each other. My goal was to get the students to understand that instead of using $x = \text{Lynn’s number of cards and } y = \text{Joe’s number of cards and an equation of } x + y = 36$, that they could use $x = \text{Lynn’s number of cards, and } 3x = \text{Joe’s number of cards}$. This would create the equation $x + 3x = 36$ and allow them to determine that $4x = 36$ so $x = 9$ meaning that Lynn had 9 cards and Joe had 27 cards ($3x$). Although the students were able to solve the problem through recursive methods and although they realized it was like dividing by four, I do not believe they would have naturally moved to the use of one variable with the use of ratio without teacher intervention. This was simply too big of a jump for the students at that time and it was the last lesson of the unit and I did not want to leave the students in a state of confusion. When students create their own rules through constructivist ideas, these become truths that will only be modified if the students perceive that the truths are no longer valid. It is possible that over the course of many other examples of word problems, that students would have decided that the truth of using different letters for different numbers did not always work when trying to create and solve equations. However, these activities occurred during the last lesson of the unit and due to the unit’s timing at the end of the school year, I could not extend this unit due to the necessity of
year end review for the final exam. Had I been able to extend the unit, I could have left the students to struggle with this idea a little longer and maybe I could have provided them with word problems that were not so easily solvable through recursive methods. This may have prompted a change of methods or it may have led to extreme frustration. However, I am not convinced that the students would have naturally moved to building equations by using the same variable and ratios without teacher intervention.

Either way, I did intervene and this intervention during Episode 8 could have had an impact on the group’s approach in solving Task 19 (ws15, 3). The students were able to assign the variable ‘x’ to the number of rocks in pile A. They were also able to use the expressions of x + 2 and 4x for the number of rocks in pile B and C as expressions comparing the number of rocks in pile A to the others. The students were able to solve the equation they created and determine the number of rocks in each pile. The possible effect of my previous intervention, in Episode 8, was evident in that the students of group B chose to employ the strategy of determining what ‘x’ represented first. They applied the strategy in a way that allowed them to determine responses without using recursive methods.

The previous section shows that negotiations with other students, with themselves, and with the teacher all led to shifted conceptions of variable at different points throughout the unit. The following section will bring the three types of negotiation back together.

Interaction Between the Types of Negotiation

In the previous section, the three types of negotiation have been looked at separately. This is convenient from a research point of view but in reality they are
difficult to separate. This is because the examples given for each type of negotiation could have been simultaneously examples of more than one type. Negotiations with other students could be the trigger for a negotiation with self. The negotiations with self take place usually as a result of some sort of negotiation with other students and/or the teacher, and negotiations with the teacher result from the negotiations that students had with each other or themselves. Therefore, the negotiations have been classified according to my view of the data and may have been classified differently by a different researcher. Each of these negotiations occurred throughout the unit both in the group discussions and the whole class discussions and each may have promoted the individual and collective understanding of the concept variable. Throughout the unit, the negotiations become increasingly complex. For example, until the end of Lesson 2, students were only having discussions with the members of their own group and possibly the teacher. At the end of Lesson 2, the first class discussion was held in which all students in the class had the opportunity to interact with each other. After this point, it is very difficult to attribute the influence of a comment or a thought. This is because a comment made in a class discussion can spark a link to something said a few classes prior in a group or class discussion. Thus knowledge is complex and circular, building with each lesson, through each different type of discussion. At a basic level of the first lesson, the three types of negotiation seen in the unit, those with other students, those with self, and those with the teacher, can be seen in the following way each with the ability to influence and be influenced by each other.
However, in reality, the negotiation that occurs in a classroom setting cannot be mapped out so neatly. As negotiations in a classroom setting do not occur in a vacuum, trying to represent them is much more complex that this most basic diagram allows. The complexity of negotiation builds as the student groups come into contact with each other during the class discussions, in which students shift their conceptions and then bring these new shifted conceptions back to their next group discussion. Thus with each lesson involving a group activity and then a class discussion, the complexity of the influence of negotiation builds. The following diagram shows how in the first lesson negotiations with other students (St), with themselves (S), and with the teacher (T) are quite basic. It shows each subsequent group and class discussion sessions as an added layer of knowledge. These are labelled as Group, for Group discussions and Class for Class discussion. The arrows, going from the middle of the diagram in Lesson 1 and extending out past the boundary of Lesson 6, indicate that each lesson has the potential to influence the subsequent one. The arrow extends out past the boundary of Lesson 6 to show that the experiences that students had during this unit have the potential to impact future learning.
Figure 17: Complexity Layers of Negotiation as the Unit Progresses
CHAPTER 7

CONCLUSIONS

This chapter provides an overview of the main conclusions of the thesis and will be divided into three parts. The first part will provide answers to my research questions. The second part will discuss the limitations of this study and the third part will discuss possible implications of the study, as well as ideas for possible future research.

Responses to the Research Questions

This thesis set out to answer two research questions.

1. In what ways do student interactions during group activities promote the development of individual and collective understanding of the concept of variable.

2. In what ways do student interactions during whole-class discussions promote the development of individual and collective understanding of the concept of variable.

In chapters five and six, I outlined student interactions that appear to have influenced student conceptions of variable. In chapter five, these episodes of negotiation were described in detail in a chronological fashion. In chapter six, the student interactions were detailed in terms of the three types of negotiation: those with other students, those with themselves, and those with the teacher. The cited examples showed that student interactions during both the group and class discussion promoted individual and collective shifts in conceptions of variable. The experiences which have been cited as those promoting these shifts, were pointed to after tracing student discourse through the group and class discussions and then looking at the student worksheets and individual
reflections related to a particular discussion topic. This tracing was done when an idea, or a version of an idea, from one segment of the conversation reappeared later on. At times the ideas were readdressed by the same student and at times by a peer. Proximity of the statements that were made offered some support that they could have influenced a shift in understanding; however, the interpretation that the ideas are related comes from the words being used by the students.

The individual and collective understandings at any time were a culmination of multiple events up to that point, including activities that happened in my classroom, in the students’ previous classes, and also experiences that students had outside of class. For the purposes of this study, individual shifts were noted in cases where student definitions changed as a result of discussion, and could often be seen as evidence on the student worksheets, in the written reflections, and in the verbal comments made by the students. Collective or group shifts in understanding were noted in cases where taken-as-shared meanings allowed students to achieve intersubjectivity of knowledge (Cobb, Wood, & Yackel, 1992).

Limitations

There are two main limitations of this research. First is the fact that the data for this research comes from a single case and represents the experiences of one teacher and her students. The second is the bias created by the fact that I played the dual role of teacher and researcher in this study and its impacts on my research methodology.
Data From a Single Case

This study focused on my classroom in which I was both the teacher and the researcher. Therefore, the findings of the study, like those of all studies, may not be generalizable to other contexts. The data analysis provided a description of the events and of the context in which they occurred. Instead of attempting generalizability, I used what Firestone (1993) called thick description of the case and the context so that the readers can decide for themselves the fit between their particular context and that of the study. Therefore “the problem of generalizing ceases to become a problem for the author. It is the reader who has to ask, what is there in this study that I can apply to my own situation, and what clearly does not apply?” (Walker, 1980, p.34). The applicability of my findings to other contexts is left to the reader to decide. This will be done at a subjective level by each individual reader as they pick and choose the findings they believe hold true in their contexts and can test them out if they wish.

Bias Created by My Dual Role of Teacher and Researcher

All studies have sources of bias and this one was no exception. The main source of bias in this study was created by the unique situation I was in as I played the dual role of teacher and researcher in this study. Firstly, my desire to create a unit that emphasized student interaction, came from my reading of related literature and from my belief that such interactions could be beneficial for student learning. Thus, I entered the study with the natural inclination to look for evidence that interaction can have a positive impact on learning. Second, the bias created by my dual role had an impact on my data collection methods. I had wanted to use the method outlined by Busse and Borromeo Ferri (2003)
where student discussions were audio taped and then students were interviewed using stimulated recall to ask students about their thinking during the discussions. I could not use this method because, as both the teacher and researcher I could not be aware of which students in my class were participants in the full study and which were not. I was unaware of the participants due to a situation created for me by my teaching partner and my vice-principal, in which I was blind as to who were the participants in the study. As explained in the methods section in chapter 3, students brought their consent forms to participate in the full study back to the vice-principal. She, with the assistance of my teaching partner, created eight groups of four for the students to work in during the research unit. I was not aware of which groups of four were participants or non-participants. At the end of the school year, when I was given the audio data of the participating groups, I created transcripts of the discussions and relied on the students’ words in the data analysis. Thus, the data in this study was limited to written data and data from the transcripts of discussions. As a researcher, I was left with the task of interpreting student thinking by the words they said. I have used verbatim accounts to back up my claims but this is, nonetheless, a limitation that could not be avoided.

Implications and Possible Future Research

This study has implications for knowledge building in a high school classroom as I used methods that have been extensively researched in elementary settings and modified them to the reality of a high school classroom. Cobb (1988); Cobb, Wood and Yackel (1990); Cobb, Wood, Yackel and McNeal (1992); and Cobb, Yackel, and Wood (1992) have studied the social negotiation of meaning in beginning elementary classrooms where the teacher used constructivist principles to guide the experiences in the classroom.
Lampert (1990) used social negotiation and transcripts of class discussions to study knowledge building in her own upper elementary level classes. The study adds a dimension to the studies conducted by Lampert, as I studied a high school mathematics class and looked at the discussion students held in small groups of four, and at the impact of the class discussion on developing conceptions. The study shows that individual teachers can promote social negotiation of meaning within their classrooms.

Above I described that my study was limited because it studied a single case. This limitation can also be as the strength of this thesis as it locates, describes, and explains learning within a given learning context. The study adds to the field of knowledge on student learning within a constructivist classroom environment through negotiation of meaning with their peers and their teacher.

The study has implications for future research and curriculum policy promoting student talk and interaction. The study was built on ideas of Mathematics teaching that are consistent with those of the Quebec government (Gouvernement du Québec, 2004), and by professional organizations such as the National Council of Teachers of Mathematics (NCTM, 1991, 2000) and the Ontario Association for Mathematics Education. Although there is strong evidence supporting the positive implications of student talk and negotiation, the positive influence is not accepted universally. Websites such as Mathematically Correct (Mathematicallycorrect.com), are designed with the intent of negating, if not slandering the efforts of curriculum reform. The site refers to new teaching methods in Mathematics as ‘New Math’, ‘Fuzzy Math’ or ‘Fuzzy Crap’. The site negates the realm of positive and exciting possible learning impacts of
constructivist teaching methods and criticizes the whole movement for a lack of supporting evidence as to its benefits for student learning.

My research is unique in that it looks at social negotiation in a real high school classroom studied by the classroom teacher, not an outside researcher. For the same reason that this study is unique, it has not been a popular type of study due to the difficulties faced by playing the dual role of teacher and researcher. More research of this nature is needed if we are to work towards a pedagogical plan of constructivism. Richardson (2003) states that one of the problems with constructivism is that it works well as a philosophy but provides no pedagogical plan for teachers. The accumulation of many studies, such as the one I conducted, may help the pedagogical community close the gap between constructivism as a philosophy of knowledge construction and constructivism as a teaching/learning plan. It may also go some distance in highlighting the benefits of this philosophy to those who are sceptical of its benefits.
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Appendix A: Letter of Information

LETTER OF INFORMATION

Dear Parent/Guardian:

Your son/daughter is one of a group of Grade 7 students selected as a potential participant for a research study. The study is entitled “The Development of the Concept of Variable in Grade Seven Beginning Algebra Students”. I will conduct the study as part of my Master’s degree from Queen’s University and it will be conducted in your child’s mathematics class during the first week of May, 2005. The research has the support of the school principal. Moreover, the research has been cleared by the Queen’s University General Research Ethics Board and has been approved by the Alexander Galt Governing Board.

This letter has two purposes. First, it will describe the purpose and method of the research study. Second, it will request that both you and your son/daughter agree, in writing, to participate in this study.

First, the purpose of the study is to investigate ways in which student interactions in mathematics class promote student understanding of the algebraic concept: variable. Specifically, the study will investigate the ways that student interactions in group work settings and in whole-class discussions promote understanding in eight students over the course of a six-lesson unit on variables. The proposed method of the study requires a few different data collection methods. First, all students in the class will be audio taped while working on tasks in groups of four. Students will audiotape their own discussions while working on mathematical tasks. Students will then send the files of their conversations to their English teacher, Mrs. Spoor, via the school’s computer server. Mrs. Spoor will then burn the files of consenting groups onto a compact disc and bring it to the Vice Principal where it will stay until the end of the school year. Files of those who are not participants will be deleted by Mrs. Spoor. Second, student written work created during class activities, along with reflections of their ideas about variables will be collected. Third, whole-class discussions will be audio taped and comments, justification and explanations of consenting students will be analyzed to describe ways in which these interactions promoted the development of understanding of the concept of variable.

The second purpose of this letter is, to request that both you and your son/daughter agree, in writing, to participate in the study. Please indicate your decision to participate in the study on the attached Letter of Consent and return it to your Vice-Principal by April 19th, 2005. The Vice-Principal will then create groups of four in which students will work for the six lessons in the algebra mini unit. The Vice-Principal will create the groups so that I will have some complete groups of four that are participants in my study but that I will not know who they are. In June, after your child’s final grades are completed, I will have access to the data collected from consenting students. Thus, please note that your child’s grades will in no way be influenced by their decision to participate or not.

The teaching of the algebra unit for this study involves students in no more risk than normal classroom activities where students are encouraged to justify and explain their responses to other members of the class. Based on the counter arguments of their classmates, students may or may not choose to change their solutions. I am interested in those changes, not in the correctness of solutions. I want to know if the changes in solutions came about because of a genuine
development in understanding or whether the change was simply to agree with the rest of the group or class.

There are no known physical, psychological, economic or social risks to your son/daughter associated with participation in this research. Agreement on your part to allow your son/daughter to become a part of the study in no way obligates your son/daughter to remain a part of the study. Participation is voluntary, and your son/daughter, or you on their behalf, may choose to withdraw from the study at any time and you may request removal of all or part of your child’s data, without negative consequences. Agreement to participate in this study involves no time commitment outside of the regular class period.

The findings of this study will be written into a thesis which is the final piece of work required for the degree of Master of Education from Queen’s University. I would be happy to share my research findings, upon completion, with any interested parties. At no time will the actual identity of the students be disclosed. Students who agree to participate will be assigned a pseudonym (code name) to protect their identity and all data collected will be destroyed after the final completion of the thesis.

Should further information be required before either you or your son/daughter can make a decision about participation, please feel free to telephone me, Ms. Jodi Coleman at Alexander Galt Regional High School (819) 563-0770 ext. 404. or my thesis supervisor, Dr. Geoffrey Roulet, (613) 533-6000 ext. 74935. For questions, concerns or complaints about the research ethics of this study, contact the Dean of the Faculty of Education, Dr. Rosa Bruno-Jofré, (613) 533-6210, or the chair of the Queen’s University General Research Ethics Board, Dr. Joan Stevenson, (613) 533-6081.

Yours sincerely,

MS. JODI COLEMAN
Appendix B: Letter of Consent

LETTER OF CONSENT TO BE A PARTICIPANT IN THE STUDY

1. I agree to participate in the study entitled “The Development of the Concept of Variable in Grade Seven Beginning Algebra Students”, conducted through the Faculty of Education at Queen's University.

2. I have read and retained a copy of the Letter of Information. The purpose of the study is explained to my satisfaction and my questions have been answered to my satisfaction.

3. I understand that, upon request, I may have a full description of the results of the study after its completion.

4. I understand that the researcher is my teacher but that she will not know whether or not I am a participant in the study. The group I will work in for the research will be made up by the Vice Principal and will last for the duration of the study (six lessons). I understand that I am not to discuss with friends or the teacher whether or not I am a participant in this study.

5. I understand that the researcher intends to publish the findings of the study into a Master’s thesis, that my identity will never be revealed, and that all the data collected during the research process will be held confidentially and destroyed after the research is completed.

6. I understand that participation is voluntary, and that I am free to withdraw from this study at any time without negative consequences.

7. I understand that I will only be considered as a participant in this study if both myself and my parents agree that I participate.

8. I am aware that I can contact the researcher, Ms. Jodi Coleman at Alexander Galt Regional High School (819) 563-0770 ext. 404 or her thesis supervisor, Dr. Geoffrey Roulet, (613) 533-6000 ext. 74935 if I have any questions about this study, and I am aware that for questions, concerns or complaints about the research ethics of this study, I can contact the Dean of the Faculty of Education, Dr. Rosa Bruno-Jofré, (613) 533-6210, or the chair of the General Research Ethics Board, Dr. Joan Stevenson, (613) 533-6000 ext. 74579, email stevensj@post.queensu.ca.
FOR STUDENT: I HAVE READ AND UNDERSTOOD THIS CONSENT FORM.

IF YOU AGREE TO PARTICIPATE IN THE STUDY, PLEASE SIGN BELOW.

I agree that my comments from class discussions, and my reflections may be included in the study as long as my identity is protected with a pseudonym (code name).

Student’s name (Please Print): ____________________________

Signature of Student: ________________________________

Date: ______________________

FOR PARENTS/GUARDIANS: I HAVE READ AND UNDERSTOOD THIS CONSENT FORM. IF YOU AGREE TO ALLOW YOUR SON/DAUGHTER TO PARTICIPATE IN THE STUDY, PLEASE SIGN BELOW.

I agree that my child’s comments from class discussions and reflections may be included in the study as long as his/her identity is protected with a pseudonym (code name).

Parent/Guardian’s name (Please Print): ____________________________

Signature: ________________________________

Date: ________________ Telephone number: ____________________

PLEASE RETURN THIS FORM TO YOUR VICE-PRINCIPAL BY APRIL 19th, 2005.
THANK YOU!
Appendix C: Coding Scheme

In order to classify the data for this study, I have developed the following coding system.

1. Worksheets: are coded as (ws). So worksheet 1 is coded as ws1. This is used to reference questions on the worksheets as they were given to students.

2. Completed Student Worksheets: are coded as (sws). These are used to reference the responses of specific students to the questions on the worksheets. So for example (sws4, Linda) means that the reference is from Linda’s response to worksheet four.

3. Individual Student Reflections: are coded as (ir). For example (IR24, Allison) means that the reference comes from the individual reflection that Allison wrote on May 24.

4. Group discussion data: is coded as (gd) for group discussion followed by a number representing the lesson date, then by the Group letter, and the line of the transcript). For example (GD24, C, 139-150) means that this excerpt is from the group discussion of May 24, Group C, lines 139-150.

5. Class discussion data: is coded as (cd) for class discussion followed by a number representing the lesson date followed by the lines from the transcript). For example (CD24, 83-131) means that this excerpt is from the class discussion of May 24, lines 83-131.
Appendix D: Student Worksheets

Introductory Example, Worksheet 1, and Worksheet 2

The following introductory example as well as Worksheets 1 and 2 have been taken from Witherspoon and Woodard’s 1998 book “50 Pre-Algebra Activities” (p. 9, 10, 11) and have been included with the permission from the publisher, J. Weston Walch.

Worksheets 3 to 6

These worksheets have been taken from a package of resources created by teachers through funding from the Québec government:


These worksheets have been included in the thesis with permission of my school board, The Eastern Townships School Board that was one of the participating school boards (amalgamated with District of Bedford), and with the permission of participating teacher Andre Del Castilho.

Worksheets 7-15

I put these worksheets together based on the goals of the unit and on problem types that I found in the research on teaching pre-algebra. In cases where questions are based on the ideas or research of others, the original author has been cited in the description of the activity in Chapter 4, Overview of the Unit of Study, specifically pages 55-66.
Introductory Example

Squares can be made using matchsticks. The squares in this lesson will be made in a special way. This is shown below.

First

Second

Third
2. Triangles can be made using matchsticks. The triangles in this lesson will be made in a special way. This is shown below.

First

Second

Third

a. Draw pictures of the next two arrays of triangles.

b. Complete the following table.

<table>
<thead>
<tr>
<th>Number of Triangles</th>
<th>Number of Matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>
Worksheet 2

Geometric Patterns I

1. Squares can be made using matchsticks. The squares in this lesson will be made in a special way. This is shown below.

   First
   
   Second
   
   Third

a. Draw pictures of the next two arrays of squares.

b. Complete the following table.

<table>
<thead>
<tr>
<th>Number of Squares</th>
<th>Number of Matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
<tr>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>
PATTERNS AND SEQUENCES

"Seeing. Saying. Writing. Using!"

2) 

166

Draw the next 3 terms of this sequence.

i) How many circles will appear in the 8th term?

ii) Express orally how the pattern grows.

iii) How many circles will there be in the 30th term?

iv) Does the expression you developed in part (iii) allow you to determine the number of circles in the 30th term? If not, can you refine your expression so that you can use it to find the number of circles in the 150th term?

v) Write the rule using words or symbols.

vi) Seeing

"Writing"
Worksheet 4

PATTERNS AND SEQUENCES
“Seeing, Saying, Writing, Using!”

3)

i) Draw the next 2 terms of this sequence.

“Seeing”

ii) How many □ will appear in the 8th term?

iii) Express orally how the pattern grows.

“Saying”

iv) How many □ will there be in the 40th term?

v) Does the expression you developed in part (iii) allow you to determine the number of □ in the 40th term? If not, can you refine your expression so that you can use it to find the number of □ in the 120th term?

vi) Write the rule using words or symbols.

“Writing”

---

Math 116 materials prepared by teachers of the South Shore, Châteauguay Valley (Protestant), District of Bedford, L’Eau-Vive and Brossard School Boards with funding provided by the Direction régionale de la Montérégie (MÉO).
Worksheet 5

PATTERNS AND SEQUENCES

"Seeing, Saying, Writing, Using!"

6)

i) Draw the next 3 terms of this sequence.

ii) How many squares will appear in the 7th term?

iii) Express orally how the pattern grows.

iv) How many squares will there be in the 25th term?

v) Does the expression you developed in part (iii) allow you to determine the number of squares in the 25th term? If not, can you refine your expression so that you can use it to find the number of squares in the 100th term?

vi) Write the rule using words or symbols.

vii) A certain term contains 56 squares. What is its rank?

"Seeing"  
"Saying"  
"Writing"  
"Using"
Worksheet 6

PATTERNS AND SEQUENCES
"Seeing, Saying, Writing, Using!"

7)

i) Draw the next 3 terms of this sequence.

"Seeing"

ii) How many squares will appear in the 8th term?

iii) Express orally how the pattern grows.

"Saying"

iv) How many squares will there be in the 40th term?

v) Does the expression you developed in part (iii) allow you to determine the number of squares in the 10th term? If not, can you refine your expression so that you can use it to find the number of squares in the 100th term?

vi) Write the rule using words or symbols.

"Writing"

vii) What is the rank of the term which contains 93 squares? 

"Using"

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Worksheet 7

Name: ________________________ Group #: ____        Date:       May. 25th

Student Activity Sheet

1. Estimate how long the sheet is and record here: _______ cm

2. Expression for the length of the paper:
   
   ______ + _______ + _______ = length of sheet

3. Flip the paper and try a new arrangement:
   
   ______ + _______ + _______ = length of sheet

4. Using a ruler, record the measurements (in cm) for the length of the

   x=___________ cm       y=___________ cm       coloured tile=  2 cm each

5. Re write your algebraic expressions from #2 and #3.

<table>
<thead>
<tr>
<th></th>
<th>Expression 1</th>
<th>Expression 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re-write by filling</td>
<td>_____ + _____ + ___ = length</td>
<td>_____ + _____ + ___ = length</td>
</tr>
<tr>
<td>in blanks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Now re-write using</td>
<td></td>
<td></td>
</tr>
<tr>
<td>the measures values</td>
<td></td>
<td></td>
</tr>
<tr>
<td>to replace “x” and</td>
<td></td>
<td></td>
</tr>
<tr>
<td>“y”</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use BEDMAS to</td>
<td></td>
<td></td>
</tr>
<tr>
<td>calculate the length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>of the sheet in cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LENGTH OF LINE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>to nearest cm</td>
<td>____ cm</td>
<td>_____ cm</td>
</tr>
</tbody>
</table>
Worksheet 8

6. Find the average of Expression 1 and Expression 2:
   
   _____ + _____ = ______÷ 2 = ______

7. Actual length of the paper = ________cm. How close are you to the actual length of the paper? Explain?

8. How close was your estimate to the actual length of the paper? Explain?

In your Group discuss the following and record your answers:

9. How come your answers were so close/far from the actual measured length of the line? What would you do next time to make them more exact?

10. How come different expressions work for the same length of paper using your same “x” and “y” boxes?

11. Why is the term variable appropriate to describe the “x” and “y” lengths of the boxes?
Worksheet 9

Name: ________________________ Group #: ____ Date: May. 27th

Line Lengths using Algebraic Expressions

1. Thinking back to our last lesson, discuss why you think the lengths of the “x” and “y” boxes were variables.

2. Food for thought…In reflections of last class, someone wrote ---- “…said something about a variable changing and now I am stuck between numbers and letters.” What do you think that this person meant? Do you think that a variable is a number or a letter?

Solve the following problems using pictures, words, equations, etc.

3. Silly me. I used an “x” box to measure a 30 cm line. Before I measured the “x” box, I accidentally put the box back with the others and now it is mixed up. The algebraic expression I created for the line was $3x + 6 = 30$ cm. Can you figure out how long the “x” box was?

4. How long was my “y” box if the length of the line was 52 cm and the algebraic expression for the length of the line was $4y + 6y + 2 = 52$ cm?

5. How long was my “x” box if the line was 22 cm long and the algebraic expression was $5x – 3 = 22$ cm?
Worksheet 10

6. Determine the length of the “y” box in the following expression if the “x” box is 8.5 cm long and the line is 67 cm long. The expression is $4x + 3y + 3 = 67$ cm. How do you know?

7. What operation/s (add, subtract, multiply, divide) do you need to use when determining the length of the line?
Worksheet 11

Write at least 8 different solutions that can replace the

20 - 12 = 

1. 20 - 12 = 
2. 20 - 12 = 
3. 20 - 12 = 
4. 20 - 12 = 
5. 20 - 12 = 
6. 20 - 12 = 
7. 20 - 12 = 
8. 20 - 12 =
Worksheet 12

Name: ____________________ Group #: ____        Date:

Find the missing values in the following equations:

1. $3x + 5 = 26$

2. $6y - 2 = 48$

3. $2c + 8 = 3c + 5$

4. $3a + 5 = 2a + 12$

5. $2x + 6 > -2$
#1) Are these expressions equivalent?

\[ 685-492+947, \ 947+492-685, \ 947-685+492, \ 947-492+685 \]

#2) Find the value of \( \square \) in the following expression:

\[(235 + \square) + (679 -122) = 235 + 679\]

#3) Are these expressions equivalent?

\[7(w) +22 = 109 \quad \text{and} \quad 7(n) +22 = 109\]

#4) Which is larger, \( x + x \) or \( x + 4 \)? Explain your answer!
Problems with Variables:

1. Joe and Lynn collect trading cards. Joe has three times as many as Lynn does. Together they have 36 trading cards. How many does each one of them have?

2. At a local fair you have two choices when you go through the entrance gate. You can either pay $5.00 at the door (Option 1) and then $1.75 for each ride, or you can just pay $17.00 at the gate to ride all day (Option 2).
   a. Which option should you chose. Defend your choice using Mathematics.
   b. Draw the graphs for both options where your “x” horizontal axis is the number of rides and the “y” vertical axis is the total cost of the day. Do the two graphs intersect? What does this tell you?
Worksheet 15

3. There are 14 rocks in three piles (pile A, B, C). Figure out how many rocks are in each pile if pile B has 2 more than pile A and pile C has 4 times as many rocks as A.
Appendix E: Student Reflection Sheet

Name: ____________ Group: #____ Date: ________

Student Reflection

Today’s task was to: _______________________________
________________________________________________

My role in today’s discussion was: ___________________
________________________________________________
________________________________________________
________________________________________________

Based on what I learned today and what I already knew, my definition of a **variable** is: _________________________
________________________________________________
________________________________________________

*When* are variables used? __________________________
________________________________________________

*What* are variable used for? ________________________
________________________________________________

*How* is my definition of variable different or changed from my definition before today’s lesson? __________________
________________________________________________
What activities or parts of the class helped you change your definition of variable? __________________________________________________
________________________________________________
________________________________________________
________________________________________________
________________________________________________

What parts of today’s discussion in your group helped you change your definition of variable? Specifically, who said what and when? How did comments of others help you?
________________________________________________
________________________________________________
________________________________________________
________________________________________________
________________________________________________
________________________________________________

What parts of today’s class discussion helped you change your definition of variable? Specifically, who said what when and how did the comments of others help you?
________________________________________________
________________________________________________
________________________________________________
________________________________________________
________________________________________________
________________________________________________
Appendix F: Student Groupings and Pseudonyms

Student Participants

This study will focus on three groups of four students who were full participants in the study. Full participation meant that students and their parents had consenting in writing for the student’s work to be used in the study. This included all audio files of the group’s work sessions as well as the worksheets and reflections they completed during the course of the unit. Participating students are listed with their groups using their pseudonyms. The pseudonyms were chosen to indicate the gender of the student even though that gender differences are not examined in this study.

Group A:

Arial
Mark
Ray
Samantha

Group B:

Elisa
Linda
Michael
Kim

Group C:

Ross
Anne
Lara
Allison

There was a fourth group (D) of consenting students that will not be used as full participants for reasons outlined in the methodology section. They are:

Eric
Sophie
James
Kirby

Other students in the class were participants only in the class discussions. For the purposes of the classroom activities there was no difference between the participation of these students and the full participants. The only difference was that for the non-participants, the audio data was destroyed and they were not included in the results of the study. A third party, the students’ Language Arts teacher, deleted the audio files for
these groups. At the end of the unit, Vice-Principal discarded the student worksheets. The other worksheets along with the audio files were kept by the Vice-Principal until the end of the school year.

Names of non-participants who are referenced in class discussions are listed below using their pseudonyms. These students are listed in groups with students that they worked with. Students listed alone or in groups of two or three students indicate that their group consisted of non-participants and that the students listed were the only ones who spoke during the class discussions.

Group E

Cole
Chanel
Penny

Group F

Sally
Bob

Group G

Michelle

Group H

Jane
Amy
Marika
Beth

Throughout the study, where I am a participant, I will be referred to as Ms. C.