POLYNOMIAL BEHAVIOUR OF KOSTKA NUMBERS

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ABSTRACT. Given two standard partitions $\lambda_+ = (\lambda_1^+ \geq \cdots \geq \lambda_s^+)$ and $\lambda_- = (\lambda_1^- \geq \cdots \geq \lambda_r^-)$ we write $\lambda = (\lambda_+\lambda_-)$ and set $s_\lambda(x_1, \ldots, x_t) := s_{\lambda_+}(x_1, \ldots, x_t)$ where $\lambda_t = (\lambda_1^+, \ldots, \lambda_s^+, 0, \ldots, 0, -\lambda_r^-, \ldots, -\lambda_1^-)$ has $t$ parts. We provide two proofs that the Kostka numbers $k_{\lambda\mu}(t)$ in the expansion

$$s_\lambda(x_1, \ldots, x_t) = \sum_{\mu} k_{\lambda\mu}(t)m_\mu(x_1, \ldots, x_t)$$

demonstrate polynomial behaviour, and establish the degrees and leading coefficients of these polynomials. We conclude with a discussion of two families of examples that introduce areas of possible further exploration.

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1. Introduction

The Kostka numbers, which we will denote $k_{\lambda \mu}$, were introduced by Carl Kostka in his 1882 work relating to symmetric functions. The Littlewood-Richardson coefficients $c_{\lambda \mu}^{\nu}$ were introduced by Dudley Ernest Littlewood and Archibald Read Richardson in their 1934 paper *Group Characters and Algebra*. These two combinatorial objects are ubiquitous, appearing in situations in combinatorics and representation theory, which we will address in this work, as well as algebraic geometry.

Being such natural objects, the Kostka numbers and Littlewood-Richardson coefficients have been studied extensively. There are explicit combinatorial algorithms for computing both types of coefficients, and many well-established properties. There still exist, however, open questions and areas which remain unexplored. One area of study which contains interesting open questions relating to these coefficients is that of asymptotics, which we will explore in this work.

We define the Littlewood-Richardson coefficient $c_{\lambda \mu}^{\nu}$ as the multiplicity of the Schur polynomial $s_{\nu}(x_1, \ldots, x_t)$ in the decomposition of the product of Schur polynomials

$$s_{\lambda}(x_1, \ldots, x_t)s_{\mu}(x_1, \ldots, x_t) = \sum_{\nu} c_{\lambda \mu}^{\nu} s_{\nu}(x_1, \ldots, x_t).$$

Similarly, we can define the Kostka number $k_{\lambda \mu}$ as the coefficient of $m_{\mu}(x_1, \ldots, x_t)$ in the expansion of the Schur polynomial $s_{\lambda}(x_1, \ldots, x_t)$ as

$$s_{\lambda}(x_1, \ldots, x_t) = \sum_{\mu} k_{\lambda \mu} m_{\mu}(x_1, \ldots, x_t)$$

where $m_{\mu}(x_1, \ldots, x_t)$ denotes the minimal symmetric polynomial generated by the monomial $x_{\mu}$. In the above formulas, the number of variables, $t$, may be arbitrary. However, $c_{\lambda \mu}^{\nu}$ and $k_{\lambda \mu}$ become independent of $t$ as it becomes sufficiently large.

We begin our study by extending the definition of the Schur polynomial $s_{\lambda}(x_1, \ldots, x_t)$ to arbitrary weakly decreasing sequences of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_t)$ by shifting $\lambda$ to $\lambda + e^t$, i.e.

$$s_{\lambda}(x_1, \ldots, x_t) = \frac{1}{(x_1 \cdots x_t)^e} s_{\lambda + e^t}(x_1, \ldots, x_t).$$

Given two standard partitions $\lambda_+ = (\lambda_1^+ \geq \cdots \geq \lambda_s^+)$ and $\lambda_- = (\lambda_1^- \geq \cdots \geq \lambda_r^-)$ we write $\lambda = (\lambda_+, \lambda_-)$ and set

$$s_{\lambda}(x_1, \ldots, x_t) := s_{\lambda_t}(x_1, \ldots, x_t)$$

where $\lambda_t = (\lambda_1^+, \ldots, \lambda_s^+, 0, \ldots, 0, -\lambda_r^-, \ldots, -\lambda_1^-)$ has $t$ parts.

We consider the expansions

$$s_{\lambda}(x_1, \ldots, x_t)s_{\mu}(x_1, \ldots, x_t) = \sum_{\nu} c_{\lambda \mu}^{\nu}(t) s_{\nu}(x_1, \ldots, x_t)$$

(1)
and
\[ s_\lambda(x_1, \ldots, x_t) = \sum_{\mu} k_{\lambda\mu}(t) m_\mu(x_1, \ldots, x_t) \]
where all \( \lambda, \mu, \) and \( \nu \) above are pairs \( \lambda = (\lambda_+, \lambda_-), \mu = (\mu_+, \mu_-), \) and \( \nu = (\nu_+, \nu_-). \) Both sums in (1) and (2) are finite. A result of Penkov and Styrkas, [PS], implies that the coefficients \( c_{\nu\lambda}(t) \) stabilize for sufficiently large \( t. \) The coefficients \( k_{\lambda\mu}(t), \) on the other hand, demonstrate polynomial behaviour. This result has been known for a while, see for example [BKLS] and the references therein. The methods used by Benkart and collaborators apply to representations of groups beyond \( GL_n, \) but provide little information about the Kostka polynomials themselves.

The purpose of this work is to give two proofs of the fact that \( k_{\lambda\mu}(t) \) is a polynomial in \( t. \) The proofs given are specific to \( GL_n, \) but will hopefully lead to formulas or algorithms for computing these polynomials explicitly. The first proof uses the stability of the Littlewood-Richardson coefficients, and the second uses a recurrence relation established through branching rules.

We end the document with a discussion of two families of examples which will suggest further generalizations and directions of study.

**Conventions.** Unless otherwise stated, we will be working over \( \mathbb{C}, \) representations are finite dimensional, and \( N = \mathbb{Z}_{\geq 0}. \) We denote by \([k]\) the set of integers \( \{1, \ldots, k\}. \)

# 2. Representations of \( GL_n \)

**Definition 1.** We define a polynomial representation of the group \( G = GL_n \) to be a group homomorphism
\[ \rho : G \to GL(V) \cong GL_N \]
where \( N = \dim(V) \) is the dimension of the representation, such that for every \( A = (a_{ij}) \in GL_n, \rho(A) = (\rho_{ij}(A)), \) where each entry \( \rho_{ij}(A) \) is a polynomial in the entries \( a_{kl} \) of \( A \in GL_n. \)

In this work we will refer to polynomial representations simply as representations of \( GL_n. \) For a more extensive discussion, one can refer to [GW] and [FH].

## 2.1. Representations

All finite dimensional representations of \( GL_n \) are completely reducible - i.e., if \( \rho \) is a representation of \( GL_n \) such that \( N = \dim(V) < \infty, \) then there is a decomposition of \( V \) as the direct sum of irreducible representations. It is therefore natural to want to parameterize all possible non-isomorphic irreducible representations, and when given a representation, to ask how it could be decomposed as the direct sum of irreducible components. We turn our attention to the Schur-Weyl Duality, which will allow us to build all irreducible representations of \( GL_n. \)

### 2.1.1. Schur-Weyl Duality

The Schur-Weyl Duality relates the irreducible representations of the General Linear Group, \( GL_n, \) to those of the Symmetric group \( S_k. \) We begin by considering the \( k^{th} \) tensor power of \( V, V^\otimes k, \) where \( V = \mathbb{C}^n. \) The General Linear group \( GL_n \) acts on \( \mathbb{C}^n \) by matrix multiplication, so it acts on \( V^\otimes k \) by
\[ g \cdot (v_1 \otimes \cdots \otimes v_k) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_k \]
for $g \in GL_n$. On the other hand, $S_k$ acts on $V^\otimes k$ by permuting the terms of $v_1 \otimes \cdots \otimes v_k \in V^\otimes k$. That is to say,

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$ 

These actions commute, so $V^\otimes k$ is a $GL_n \times S_k$-module. We thus have the decomposition:

$$V^\otimes k \cong \bigoplus_{\lambda} V_\lambda \otimes S^\lambda$$

where the sum runs over all $\lambda$ partitioning $k$, the $S^\lambda$ are the $S_k$-modules known as **Specht modules**, and the $V_\lambda$ are $GL_n$-modules. In particular, each $V_\lambda$ is a polynomial $GL_n$-module, and is irreducible if $\lambda$ has at most $n$ parts and is the zero module otherwise. Furthermore, every irreducible polynomial $GL_n$ module is isomorphic to a module $V_\lambda$ for $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ with $\lambda_i \in \mathbb{Z}_{\geq 0}$. Two modules $V_\lambda$ and $V_\mu$ are isomorphic if and only if $\lambda = \mu$. This gives us a concrete way of realizing the irreducible representations of $GL_n$.

We will not prove these results in this work, however proofs can be found in [GW].

**Example 1.** A simple example is the decomposition of $V^\otimes 2$:

$$V \otimes V = Sym^2 V \bigoplus Alt^2 V$$

where $Sym^2 V$ is the submodule of $V \otimes V$ generated by all elements of the form $u \otimes v + v \otimes u$ for $u, v \in V$, and $Alt^2 V$ is the submodule of $V \otimes V$ generated by the elements of the form $u \otimes v - v \otimes u$ for $u, v \in V$. While it is clear that $V \otimes V = Sym^2 V \bigoplus Alt^2 V$ as vector spaces, it is not immediately obvious that this is a decomposition into irreducible submodules. This fact can be established directly. However, it follows cleanly from the Schur-Weyl duality: $S_2$ has two irreducible representations indexed by the corresponding Young tableaux, so we have the decomposition into irreducibles

$$V \otimes V = V_{(2)} \bigoplus V_{(1,1)},$$

where $S_2$ acts on $V_{(2)}$ via the trivial representation, and on $V_{(1,1)}$ via the sign representation.

Now,

$$V_{(2)} = \{ x \in V \otimes V | \sigma \cdot x = x, \forall \sigma \in S_2 \}$$

and

$$V_{(1,1)} = \{ x \in V \otimes V | \sigma \cdot x = sgn(\sigma)x, \forall \sigma \in S_2 \},$$

but these are precisely $Sym^2 V$ and $Alt^2 V$. Thus, these modules are irreducible $GL_n$ modules by the Schur-Weyl duality.

### 2.2. Characters of $GL_n$.

**Definition 2.** We define the **character** of a representation $\rho$ of $G = GL_n$ to be the map:

$$\chi_V : G \to \mathbb{C}$$

such that $\chi_V(g) = tr(\rho(g))$ for all $g \in G$. 
Since
\[ \chi_V(g) = tr(g) = tr(ghh^{-1}) = \chi_V(hgh^{-1}), \]
the character \( \chi_V \) is invariant under conjugation and hence is constant on the conjugacy classes of \( G \).

Since the diagonalizable matrices are dense in \( GL_n \), the values of \( \chi_V \) when restricted to the diagonal matrices completely determine the character \( \chi_V \) (since all diagonalizable matrices are, by definition, conjugate to a diagonal matrix, and characters are constant on conjugacy classes). Furthermore, since the \( n \) diagonal entries completely determine a diagonal matrix, we can consider the character \( \chi_V \) to actually be a map
\[ \chi_V : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}. \]

When we consider the representations \( V_\lambda \) of \( GL_n \) as discussed in Section 2.1.1, it can be shown that \( \chi_{V_\lambda}(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n) \), i.e. the characters of the representations \( V_\lambda \) of \( GL_n \) are in fact the Schur polynomials \( s_\lambda(x_1, \ldots, x_n) \). We will discuss these polynomials in the next section.

It is important to note that two representations are isomorphic if and only if their respective characters are the same. Furthermore, we have

\[ \chi_U \oplus \chi_V = \chi_U + \chi_V \]
and
\[ \chi_U \otimes \chi_V = \chi_U \chi_V, \]
allowing us to work with the characters when examining direct sums and tensor products of modules.

3. Symmetric Polynomials

In this section we will give an overview of symmetric and Schur polynomials, following the notation of [F]. We begin our discussion of Schur polynomials by defining several combinatorial objects.

**Definition 3.** We define a partition to be a weakly decreasing sequence of non-negative integers \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t) \). We say that the number of parts of the partition is the largest integer \( t' \) such that \( \lambda_{t'} > 0 \), and set
\[ |\lambda| := \sum_{i=1}^{t'} \lambda_i. \]

**Definition 4.** For a given partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_i) \) we define the Young diagram of shape \( \lambda \) to be a left-justified array of \( |\lambda| := \sum_{i=1}^{t'} \lambda_i \) boxes with \( \lambda_i \) boxes in the \( i^{th} \) row.

We often identify the partitions \( \lambda \) and \( \lambda' \) if their corresponding Young diagrams are the same. In other words, we identify two partitions if their strictly positive parts are the same, regardless of how many parts of size zero occur.

We now consider filling these boxes, which we will also refer to as cells, with numbers. We define a filling of a Young diagram to be a numbering of the boxes using positive integers. In particular, we are interested in fillings that satisfy certain properties.
Definition 5. We say that a filling \( T \) of a Young diagram \( \lambda \) is a **Young tableau** if the numbering is weakly increasing across the rows of \( \lambda \) and strictly increasing down the columns. Given a Young tableau \( T \), we say that \( T \) has **content** \( \mu = (\mu_1, \ldots, \mu_t) \in \mathbb{N}^t \) if \( T \) contains \( \mu_i \)'s, \( 1 \leq i \leq t \).

For a Young tableau \( T \) of shape \( \lambda \) and content \( \mu = (\mu_1, \ldots, \mu_t) \), we denote by \( x^T \) the monomial:

\[
x^T := \prod_{i=1}^{t} x_i^{\mu_i}.
\]

We then define the Schur polynomial in \( t \) variables as:

\[
s_\lambda(x_1, \ldots, x_t) = \sum_T x^T
\]

where we sum over all possible tableaux \( T \) of shape \( \lambda \) filled with integers from 1 to \( t \). For ease of notation, we will denote

\[
\vec{x}_t := (x_1, \ldots, x_t).
\]

While not obvious from the definition, \( s_\lambda(\vec{x}_t) \) is a symmetric polynomial in \( t \) variables for any Young diagram of shape \( \lambda \). Furthermore, letting \( \lambda \) vary over partitions of \( k \) having at most \( t \) parts, the Schur polynomials form a basis for the symmetric polynomials in \( t \) variables of degree \( k \).

Definition 6. Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_t) \) be a partition of \( k \) having at most \( t \) strictly positive parts. We define \( m_\lambda(\vec{x}_t) \) to be the minimal symmetric polynomial generated by the monomial \( x_1^{\lambda_1} \cdots x_t^{\lambda_t} \).

3.1. Kostka Numbers.

Definition 7. We denote by \( k_{\lambda\mu}(t) \) the number of possible Young tableaux of shape \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_t) \) and content \( \mu = (\mu_1 \geq \cdots \geq \mu_t) \) (adjoining parts of size zero where necessary).

We have the relation

\[
s_\lambda(\vec{x}_t) = \sum_\mu k_{\lambda\mu}(t)m_\mu(\vec{x}_t).
\]

3.2. Littlewood-Richardson Coefficients. Given two partitions \( \lambda \) and \( \mu \) and their respective Schur polynomials \( s_\lambda(\vec{x}_t) \) and \( s_\mu(\vec{x}_t) \), we can define the **Littlewood-Richardson coefficients** \( c_\lambda^\nu(\mu)(t) \) through the expansion

\[
s_\lambda(\vec{x}_t)s_\mu(\vec{x}_t) = \sum_\nu c_\lambda^\nu(\mu)(t)s_\nu(\vec{x}_t).
\]

That is to say, when one multiplies two Schur polynomials \( s_\lambda(\vec{x}_t) \) and \( s_\mu(\vec{x}_t) \), the product, being the product of two symmetric polynomials, is again a symmetric polynomial. Since the Schur polynomials of degree \( |\lambda| + |\mu| \) in \( t \) variables form a basis of the vector space of symmetric polynomials of corresponding degree, we can rewrite the product as a sum of Schur polynomials, with \( \nu \) running over partitions of \( |\lambda| + |\mu| \). The **Littlewood-Richardson coefficient**, \( c_\lambda^\nu(\mu)(t) \) is then the coefficient of the Schur polynomial \( s_\nu(\vec{x}_t) \) in the expansion.
3.3. **Representations and Characters.** The correspondence between characters of representations of $GL_n$ and symmetric polynomials allows us to translate statements from one setting to the other. For example,

$$s_{\lambda}(\vec{x}_t)s_{\mu}(\vec{x}_t) = \sum_{\nu} c^\nu_{\lambda\mu}(t)s_{\nu}(\vec{x}_t)$$

gives us

$$V_\lambda \otimes V_\mu = \bigoplus c^\nu_{\lambda\mu}(t)V_\nu.$$ 

Denoting

$$h_s(x_1, \ldots, x_t) := \sum_{1 \leq i_1 \leq \cdots \leq i_s \leq t} x_{i_1} \cdots x_{i_s}$$

for a given $s \in \mathbb{N}$, we define

$$h_\lambda := \prod_{j=1}^t h_{\lambda_j}(x_1, \ldots, x_t)$$

for a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_t)$.

It can be shown that

$$h_s(\vec{x}_t) = \chi_{V(s)} = \chi_{\text{Sym}^s V}.$$ 

Our definition of $h_\lambda$ then allows us to directly obtain

$$h_\lambda = \chi_{\text{Sym}^{\lambda_1} V} \cdots \chi_{\text{Sym}^{\lambda_t} V}.$$ 

The identity

$$h_\mu(\vec{x}_t) = \sum_{\lambda} k_{\lambda\mu}s_{\lambda}(\vec{x}_t)$$

(see [F]) then gives us the result

$$\text{Sym}^{\mu_1} V \otimes \cdots \otimes \text{Sym}^{\mu_t} V = \bigoplus_{\lambda} k_{\lambda\mu}V_{\lambda}.$$ 

4. **Restrictions**

4.1. **Interlacing Partitions.**

**Definition 8.** For given partitions $\lambda$ and $\mu$, where $\lambda$ has $t$ parts and $\mu$ has $t-1$ parts, we say that $\mu$ **interlaces** $\lambda$, denoted $\mu \prec \lambda$, if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{t-1} \geq \mu_{t-1} \geq \lambda_t.$$ 

This concept will allow us to obtain an important relation of Schur polynomials.

**Theorem 1.**

$$s_{\lambda}(\vec{x}_t) = \sum_{\mu \prec \lambda} s_{\mu}(\vec{x}_{t-1})x_t^{[\lambda]-[\mu]}$$
Proof. For a given Young tableaux $T$ of shape $\lambda$, we let $T_t$ denote the object obtained by removing all cells with the entry $t$, and $m_T$ the number of cells removed (that is to say, the number of cells of $T$ which contained the entry $t$). We claim that $T_t$ is a Young tableau of shape $\mu$ for some $\mu$ having $r \leq t - 1$ parts and is filled with the entries from $[t - 1]$. In fact, $\mu$ interlaces $\lambda$.

First, note that $T_t$ is a Young diagram because removing cells with entry $t$ can only remove cells on the ends of rows. Furthermore, a cell with the entry $t$ cannot lie above an entry of lower value, because that would contradict the strict decrease down columns of $T$. Thus, $T_t$ is a Young diagram.

Given that we have a Young diagram with entries strictly increasing down columns and weakly increasing across rows, it is clear that $T_t$ is a Young tableau.

To see that $\mu$ interlaces $\lambda$, we first note that since $\mu$ was obtained by removing cells from $\lambda$, $\mu_i \leq \lambda_i \forall i$. Furthermore, $\mu_i \geq \lambda_{i+1}$. This follows because $\mu_i < \lambda_{i+1}$ would imply that in the initial Young tableau the $i^{th}$ row contained a $t$ which lay above an entry less than $t$ in the row below, a contradiction. Thus, $\mu$ interlaces $\lambda$.

For any Young tableau $T'$ of shape $\mu$ interlacing $\lambda$, we obtain a unique tableau $T$ of shape $\lambda$ by filling in $\lambda \setminus \mu$ with $t$'s. The result is again a tableau, as

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_t \geq \mu_t$$

ensures that the $t$'s are placed below and to the right of all lower entries.

The theorem then follows immediately from the construction of the Schur polynomials as

$$s_\lambda(\vec{x}_t) = \sum_T x^T.$$  

\[\square\]

4.2. Branching Rules. We now consider how a $GL_t$-module $V_\lambda$ restricts to $GL_{t-1} \times GL_1$. It is natural to consider the embedding of $GL_{t-1} \times GL_1 \subset GL_t$ as matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & c \end{bmatrix}$$

where $A \in GL_{t-1}$ and $c \in \mathbb{C}^x = GL_1$. It can be shown that for any irreducible polynomial representation $\rho'$ of $GL_{t-1} \times GL_1$, $\rho'$ can be written in the form $V_\mu \times \mathbb{C}^m$ for $\mu$ a partition having $s \leq t - 1$ parts, and for $m \in \mathbb{N}$, $\mathbb{C}^m$ the $\mathbb{C}^x$-module with $c$ acting by multiplication by $c^m$. Note that the module $\mathbb{C}^m \cong \mathbb{C}^\otimes m$, where $\mathbb{C}$ denotes the natural representation of $GL_1$ and $\mathbb{C}^{-1}$ denotes the dual. We will continue using the notation $\mathbb{C}^m$. We obtain the decomposition of $V_\lambda$ as a $GL_{t-1} \times GL_1$-module into

$$V_\lambda \cong \bigoplus_{(\mu,m)} r_{\mu m} V_\mu \otimes \mathbb{C}^m.$$ 

To determine the coefficients $r_{\mu m}$ we consider the characters of the representations. Since the left hand side is simply the representation $V_\lambda$, we know that the corresponding character is the Schur polynomial $s_\lambda(\vec{x}_t)$. On the right hand side, given our additive and
multiplicative properties of characters, the corresponding character should be
\[ \sum_{(\mu, m)} r_{\mu m} s_{\mu}(x_1, \ldots, x_{t-1}) x_t^m, \]
giving us the equation
\[ s_\lambda(x_1, \ldots, x_t) = \sum_{(\mu, m)} r_{\mu m} s_{\mu}(x_1, \ldots, x_{t-1}) x_t^m. \]

On the other hand, as previously discussed,
\[ s_\lambda(x_1, \ldots, x_t) = \sum_{\mu < \lambda} s_{\mu}(x_1, \ldots, x_{t-1}) x_t^{\vert \lambda \vert - \vert \mu \vert}. \]

Since these two equations must agree, we have
\[ \sum_{(\mu, m)} r_{\mu m} s_{\mu}(x_1, \ldots, x_{t-1}) x_t^m = \sum_{\mu < \lambda} s_{\mu}(x_1, \ldots, x_{t-1}) x_t^{\vert \lambda \vert - \vert \mu \vert}. \]

Identifying coefficients and setting \( m = \vert \lambda \vert - \vert \mu \vert \) gives
\[ r_{\mu m} = \begin{cases} 1 & \text{if } \mu \text{ interlaces } \lambda \\ 0 & \text{otherwise} \end{cases} \]

Moving back from characters to representations, we obtain the result that as \( GL_{t-1} \times GL_1 \)-modules
\[ V_\lambda = \bigoplus_{\mu < \lambda} V_\mu \otimes \mathbb{C}^{\vert \lambda \vert - \vert \mu \vert}. \]

5. Beyond Polynomial Representations

For a partition \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \), we can consider \( V_\lambda \) to be a representation of \( GL_t \) for \( t \geq n \) by adding \( t - n \) parts of size zero to \( \lambda \).

For \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_t) \) we now define
\[ \lambda + c^t := (\lambda_1 + c, \ldots, \lambda_t + c). \]

The corresponding Schur polynomial is
\[ s_{\lambda + c^t}(\vec{x}_t) = (x_1 \cdots x_t)^c s_\lambda(\vec{x}_t). \]

This easily follows given the construction of \( s_\lambda(\vec{x}_t) \) as \( s_\lambda(\vec{x}_t) = \sum T x^T \), since adjoining \( c \) cells to each row is equivalent to attaching a \( t \times c \) rectangle to the left of the original tableau. We can extend any filling of the original tableau of shape \( \lambda \) to a filling of the tableau of shape \( \lambda + c^t \) by filling the \( i^{th} \) row of the \( t \times c \) rectangle with \( i's \). This is in fact the only way that this rectangle can be filled, since the tableau must be strictly increasing down columns. Thus,
\[ s_{\lambda + c^t}(\vec{x}_t) = (x_1 \cdots x_t)^c s_\lambda(\vec{x}_t). \]

We can now extend the concept of a Schur polynomial to weakly decreasing sequences of integers
\[ \lambda = (\lambda_1 \geq \cdots \geq \lambda_t) \]
with \( \lambda_i \in \mathbb{Z} \). Letting \( d = \lambda_t \) we can use this shift to extend the definition of the Schur polynomial to the case when \( \lambda \) is an arbitrary sequence of weakly decreasing integers via
\[
S_\lambda(x_t) = \frac{S_{\lambda+d}(\vec{x}_t)}{(x_1 \ldots x_t)^d}.
\]
Extending \( m_\lambda(\vec{x}_t) \) similarly, we can easily verify that \( S_\lambda(\vec{x}_t) = \sum_\mu k_\lambda \mu m_\mu(\vec{x}_t) \) continues to hold by setting \( k_{\lambda+\varepsilon, \mu+\varepsilon} = k_{\lambda \mu} \).

We now extend the definition of a representation of \( GL_n \) to include \( \rho : G \to GL(V) \cong GL_N \), where for \( A = (a_{kl}) \in GL_n \)
\[
\rho(A) = (\rho_{ij}(A))
\]
where the \( \rho_{ij} \) are polynomials in the entries of \( A \) and the inverse of its determinant, \( det^{-1}(A) \). In this setting, the relation
\[
S_\lambda(x_1, \ldots, x_n) = \sum_\mu s_\mu(x_1, \ldots, x_{n-1}) x_n^{|\lambda| - |\mu|}
\]
still holds, since \( detA = (x_1 \ldots x_n) \) allows us to use our shift of representations to use the identity for partitions and shift back. Thus, we still have
\[
V_\lambda = \bigoplus_{\mu, t} V_\mu \otimes \mathbb{C}^t
\]

**Example 2.** An important example to consider is \( GL_n \) acting on \( V = \mathbb{C}^n \). This corresponds to the partition
\[
\lambda = (1, 0, \ldots, 0).
\]
\( V \) has standard basis \( e_1, \ldots, e_n \), so we have
\[
\rho = \left\{ \begin{array}{ll}
GL_n & \to GL_n \\
A & \mapsto A
\end{array} \right.
\]

**Example 3.** The dual module \( V^* \) is also a representation. \( V^* \) has dual basis \( e_1^*, \ldots, e_n^* \) where \( e_i(e_j) = \delta_{ij} \) for \( \delta_{ij} \) the Kronecker delta. We obtain the dual representation
\[
\rho^* : \left\{ \begin{array}{ll}
GL_n & \to GL_{n^*} \\
A & \mapsto (A^{-1})^T
\end{array} \right.
\]
Now, \( A^{-1} \) can be written as
\[
\frac{1}{detA} A'
\]
where \( A' \), the adjugate of \( A \), is a matrix with entries that are polynomials in the entries of \( A \). Thus, this representation is a matrix with entries that are polynomials in the entries of \( A \) and \( det^{-1}A \). The character of the dual representation, \( \chi_{V^*}(\vec{x}_n) \), will be
\[
\sum_{i=1}^n \frac{1}{x_i} = m_{(-1)}(\vec{x}_n).
\]
Consider now the representation corresponding to the partition \( \lambda = (0, 0, \ldots, 0, -1) \). Shifting this partition, we obtain the character
\[
s_\lambda(\vec{x}_n) = \frac{1}{x_1 \ldots x_n} s_{(1,1,\ldots,1,0)}(\vec{x}_n) = m_{(-1)}(\vec{x}_n).
\]
Thus, by comparing characters, we see that the dual representation corresponds to the partition \( \lambda = (0, \ldots, 0, -1) \).

### 6. Weight Decompositions

Let \( H \subset GL_n \) denote the subgroup of diagonal matrices. We define the \( H \)-module \( \mathbb{C}^\mu \) to be the one-dimensional vector space \( \mathbb{C} \) with \( H \) acting on \( \mathbb{C} \) by
\[
\text{diag}(z_1, \ldots, z_n) \cdot v = z_1^{\mu_1} \cdots z_n^{\mu_n} v
\]
for all \( v \in \mathbb{C} \).

**Definition 9.** Let \( \rho : GL_n \to GL(V) \cong GL_N \) be a representation. We refer to \( v \in V \) as a **weight vector** with weight \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \) if
\[
z \cdot v = z_1^{\mu_1} \cdots z_n^{\mu_n} v
\]
for all \( z = \text{diag}(z_1, \ldots, z_n) \in H \).

We define a **weight space** \( V^\mu \subset V \) of a representation \( V \) to be the subspace of all vectors in \( V \) that are weight vectors with weight \( \mu \).

One can show that every representation of \( GL_n \) decomposes into the direct sum of its weight spaces
\[
V = \bigoplus_{\mu} V^\mu = \bigoplus_{\mu} (\dim V^\mu) \mathbb{C}^\mu.
\]
Moreover, \( \dim V^\mu = \dim V^{\sigma(\mu)} \) for any \( \sigma \in S_n \). Moving to the character of the irreducible representation \( V_\lambda \), we obtain the expansion
\[
s_\lambda(\vec{x}_n) = \sum_{\mu} k_{\lambda\mu} m_\mu(\vec{x}_n).
\]
Comparing these we are able to conclude that
\[
\dim V^\mu_\lambda = k_{\lambda\mu}.
\]

### 7. Some Notation

So far, we have only considered the Schur polynomials as polynomials in a finite number of variables. However, we can easily extend the concept to infinitely many variables by removing the restriction on the Young tableaux \( T \) to contain only integers from 1 to some finite \( t \). Through our previous definition of the Schur polynomials, we then obtain
\[
s_\lambda = s_\lambda(x_1, x_2, x_3, \ldots) = \sum_T x^T.
\]
For partitions \( \lambda \),
\[
s_\lambda(x_1, \ldots, x_t, 0, 0, 0, \ldots) = s_\lambda(x_1, \ldots, x_t)
\]
for all \( t \in \mathbb{N}, t > 0 \). Notationally, we will use \( m_\lambda := m_\lambda(x_1, x_2, x_3, \ldots) \) to refer to the polynomial in infinitely many variables generated by the monomial \( x^\lambda \).

For example,
\[
s_{(1)}(\vec{x}_t) = x_1 + \cdots + x_t = m_{(1)}(\vec{x}_t)
\]
becomes
\[ s(1) = \sum_{i \geq 1} x_i = m(1). \]

Similarly,
\[ s(2)(\vec{x}_t) = \sum_{1 \leq i \leq t} x_i^2 + \sum_{1 \leq i < j \leq t} x_i x_j = m(2)(\vec{x}_t) + m(1,1)(\vec{x}_t) \]
becomes
\[ s(2) = \sum_{i \geq 1} x_i^2 + \sum_{1 \leq i < j} x_i x_j = m(2) + m(1,1). \]

This summation notation quickly becomes cumbersome, so we will simply use the symmetric polynomials \( m_\gamma \) in expansions of Schur polynomials.

For example,
\[ m(1,1)(\vec{x}_t) = \sum x_i x_j \]
in three variables gives the three terms
\[ m(1,1)(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3, \]
while
\[ m(2,1)(\vec{x}_t) = \sum x_i^2 x_j \]
in three variables gives
\[ m(2,1)(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_2 x_3^2 + x_2 x_3 + x_1 x_3^2, \]
which has six terms.

8. Stability

8.1. Kostka Numbers. We begin our study of stability with the Kostka numbers. Specifically, we want to know if there a value \( t_0 \) such that for partitions \( \lambda \) and \( \mu \) having \( t > t_0 \) parts,
\[ k_{\lambda \mu}(t) = k_{\lambda \mu}(t_0). \]

Again, we look at characters
\[ s_\lambda(\vec{x}_t) = \sum m_{\lambda \mu}(t) m_\mu(\vec{x}_t) \]
corresponding to
\[ V_\lambda = \bigoplus_{\lambda \mu} k_{\lambda \mu} V_\mu^\mu. \]

The Kostka numbers demonstrate stability as we let \( t \) go to infinity. We recognize that for the partitions \( \mu = \mu_1 \geq \cdots \geq \mu_k \), we have
\[ x^\mu = x_1^{\mu_1} \cdots x_t^{\mu_t}, \]
but as \( \mu_k = 0 \) for sufficiently large \( k \), \( x_k^{\mu_k} = 1 \) for these \( k \). Thus the multiplicity of the monomials \( x_1^{\mu_1} \cdots x_t^{\mu_t} \), namely the Kostka numbers \( k_{\lambda \mu}(t) \) will stabilize for sufficiently large \( t \). This stability can also be seen by looking at the Kostka numbers \( k_{\lambda \mu} \) as the number of Young tableaux of shape \( \lambda \) and content \( \mu \): as the number of parts go to infinity, the number of possible tableaux stabilizes, as adding parts of size zero to both \( \lambda \) and \( \mu \) will eventually not create any new Young tableaux of shape \( \lambda \) and content \( \mu \).
8.2. Littlewood-Richardson Coefficients. Having established the stability of the Kostka numbers for standard partitions we move to that of the Littlewood-Richardson coefficients, maintaining the restriction of λ, μ, and ν to be partitions. As previously discussed, for λ a partition having n strictly positive parts, we can view $V_\lambda$ as a $GL_1$-module for any $t \geq n$ by adding $t - n$ parts of size zero to λ. The definition of the Littlewood-Richardson coefficients as the coefficients in the expansion of the product $s_\lambda(\vec{x}_t)s_\mu(\vec{x}_t)$ is equivalent to

$$V_\lambda \otimes V_\mu = \bigoplus c^\nu_{\lambda\mu} V_\nu.$$  

However, this decomposition of the tensor product of representations of $GL_t$ depends on $t$. It is natural to then ask for what values of $t$ the formula is valid. We begin by considering an example.

**Example 4.** In terms of the bases of symmetric polynomials given by Schur polynomials, we have

$$s_{(2)}s_{(2)} = \sum_{\nu} c^\nu_{(2)(2)} s_\nu$$

where we sum over partitions of 4. In terms of the corresponding representations, this becomes

$$V_{(2)} \otimes V_{(2)} = \bigoplus c^\nu_{(2)(2)} V_\nu.$$  

The partitions of 4 are (4), (3, 1), (2, 2), (2, 1, 1) and (1, 1, 1, 1). However, (1, 1, 1, 1), having 4 parts, does not correspond to a valid irreducible representation of $GL_3$. This can be seen by moving to the corresponding characters. Considering the Schur polynomials in an indeterminate number of variables we have:

$$s_{(2)}(x_1, \ldots, x_t) = x_1^4 + \cdots + x_t^4 + x_1x_2 + \cdots x_{t-1}x_t = m_{(2)} + m_{(1, 1)},$$

so that

$$s_{(2)}s_{(2)} = \sum x_i^4 + 2 \sum x_i^3x_j + 3 \sum x_i^2x_j^2 + 4 \sum x_i^2x_jx_k + 6 \sum x_ix_jx_kx_l = m_{(4)} + 2m_{(3, 1)} + 3m_{(2, 2)} + 4m_{(2, 1, 1)} + 6m_{(1, 1, 1, 1)}.$$  

However, if we let $n = 3$, the final sum, $\sum x_ix_jx_kx_l = m_{(1, 1, 1, 1)}$ disappears, since there are no collections of four distinct indices. We now find the corresponding Littlewood-Richardson coefficients. Corresponding to the partitions of 4, we have the Schur functions:

$$s_{(4)} = \sum x_i^4 + \sum x_i^3x_j + \sum x_i^2x_j^2 + \sum x_i^2x_jx_k + \sum x_ix_jx_kx_l = m_{(4)} + m_{(3, 1)} + m_{(2, 2)} + m_{(2, 1, 1)} + m_{(1, 1, 1, 1)};$$

$$s_{(3, 1)} = \sum x_i^3x_j + \sum x_i^2x_j^2 + 2 \sum x_i^2x_jx_k + 3 \sum x_ix_jx_kx_l = m_{(3, 1)} + m_{(2, 2)} + 2m_{(2, 1, 1)} + 3m_{(1, 1, 1, 1)};$$

$$s_{(2, 2)} = \sum x_i^2x_j^2 + \sum x_i^2x_jx_k + 2 \sum x_ix_jx_kx_l = m_{(2, 2)} + m_{(2, 1, 1)} + 2m_{(1, 1, 1, 1)};$$

$$s_{(2, 1, 1)} = \sum x_i^2x_jx_k + 3 \sum x_ix_jx_kx_l = m_{(2, 1, 1)} + 3m_{(1, 1, 1, 1)}$$

and

$$s_{(1, 1, 1, 1)} = \sum x_ix_jx_kx_l = m_{(1, 1, 1, 1)}.$$
We see that
\[
\begin{align*}
s_{(2)}s_{(2)} &= x_1^4 + 2x_1^3x_2 + 3x_1^2x_2^2 + 4x_1^2x_2x_3 + 6x_1x_2x_3x_4 \\
&= m_{(4)} + 2m_{(3,1)} + 3m_{(2,2)} + 4m_{(2,1,1)} + 6m_{(1,1,1,1)} \\
&= s_{(4)} + s_{(3,1)} + s_{(2,2)}.
\end{align*}
\]

Thus, we have the Littlewood-Richardson coefficients:
\[
c_{(2)}^{(4)} = 1, \quad c_{(2)}^{(3,1)} = 1, \quad c_{(2)}^{(2,2)} = 1, \quad c_{(2)}^{(2,1,1)} = 0, \quad c_{(2)}^{(1,1,1,1)} = 0.
\]

We now examine the general situation for partitions having only non-negative parts. For clarity, we denote by \(c_{\lambda\mu}^\nu(t)\) the Littlewood-Richardson coefficient for partitions having \(t\) parts. To facilitate an inductive argument, we will use the lexicographic order on partitions: we say that \(\lambda > \mu\) if there exists \(i\) such that \(\lambda_i > \mu_i\) where \(\lambda_j = \mu_j\) for all \(j < i\).

**Theorem 2.** For any partitions \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_t), \mu = (\mu_1 \geq \cdots \geq \mu_t),\) and \(\nu = (\nu_1 \geq \cdots \geq \nu_t)\) there exists a positive integer \(t_0\) such that for \(t \geq t_0\), \(c_{\lambda\mu}^\nu(t) = c_{\lambda\mu}^\nu(t_0)\). That is to say, when increasing the number of variables in the Schur polynomials, or equivalently, the number of parts in the corresponding partitions, the Littlewood-Richardson coefficients stabilize. We will henceforth reserve \(c_{\lambda\mu}^\nu\) for this stable asymptotic value.

**Proof.** We claim that there exists \(t_0 \in \mathbb{N}\) such that for \(t > t_0\), \(c_{\lambda\mu}^\nu(t) = c_{\lambda\mu}^\nu(t_0)\). This can be seen by looking at
\[
s_{\lambda}(\vec{x}_t)s_{\mu}(\vec{x}_t) = \sum_{\nu} c_{\lambda\mu}^\nu(t)s_{\nu}(\vec{x}_t).
\]

As \(t \to \infty\), we know that the Kostka numbers \(k_{\lambda\mu}(t)\) stabilize for standard partitions. Using the expansion of the Schur polynomials as
\[
s_{\lambda}(\vec{x}_t) = \sum_{\tau} k_{\lambda\tau}(t)m_{\tau}(\vec{x}_t)
\]
we obtain
\[
(\sum_{\tau} k_{\lambda\tau}(t)m_{\tau}(\vec{x}_t))(\sum_{\tau'} k_{\mu\tau'}(t)m_{\tau'}(\vec{x}_t)) = \sum_{\nu} c_{\lambda\mu}^\nu(t)\sum_{\tau''} k_{\nu\tau''}(t)m_{\tau''}(\vec{x}_t).
\]

Since the Kostka numbers stabilize, the left hand side of the equation stabilizes. We use an inductive argument based on the lexicographic order for the right hand side.

We begin by comparing the highest terms in the expansion
\[
s_{\lambda}(\vec{x}_t)s_{\mu}(\vec{x}_t) = \sum_{\nu} c_{\lambda\mu}^\nu(t)s_{\nu}(\vec{x}_t).
\]

Since
\[
s_{\lambda}(\vec{x}_t) = m_{\lambda}(\vec{x}_t) + \sum_{\tau < \lambda} k_{\lambda\tau}(\vec{x}_t),
\]
and similarly
\[
s_{\mu}(\vec{x}_t) = m_{\mu}(\vec{x}_t) + \sum_{\tau' < \mu} k_{\mu\tau'}(\vec{x}_t),
\]

then we obtain
\[
(m_{\lambda}(\vec{x}_t)m_{\mu}(\vec{x}_t)) = \sum_{\nu} c_{\lambda\mu}^\nu(t)s_{\nu}(\vec{x}_t).
\]
We now compare coefficients in front of the terms $x$ and $c$ itself. The coefficient on the right hand side is stable. On the left hand side, this coefficient is a combination of stable coefficients, hence is itself stable. The right hand side is $c_{\lambda\mu}(t)$. We begin the expansion of the sums as

$$
\sum_{\tau} k_{\lambda,\mu}(\tau)(\bar{x}_\tau) = \sum_{\nu} c_{\lambda\mu}(t)s_{\nu}(\bar{x}_\tau).
$$

We can rearrange these sums as

$$
\sum_{\tau} k_{\lambda,\mu}(\tau)(\bar{x}_\tau) = \sum_{\nu \succ \nu_0} c_{\lambda\mu}(t)s_{\nu}(\bar{x}_\tau) = c_{\lambda\mu}(t)s_{\nu_0}(\bar{x}_\tau) + \sum_{\nu < \nu_0} c_{\lambda\mu}(t)s_{\nu}(\bar{x}_\tau).
$$

We now compare coefficients in front of the terms $x_{\nu_0}$ on both sides, using the expansion

$$
(\sum_{\tau} k_{\lambda,\mu}(\tau)(\bar{x}_\tau))(\sum_{\tau'} k_{\mu,\tau'}(\bar{x}_\tau)) = \sum_{\nu} c_{\lambda\mu}(t)s_{\nu}(\bar{x}_\tau).
$$

proving stability for the term of highest lexicographic order in the expansion. We now assume that stability holds for any $\nu_0 \succ \nu$. We begin with the expansion of the sums as

$$
(\sum_{\tau} k_{\lambda,\mu}(\tau)(\bar{x}_\tau))(\sum_{\tau'} k_{\mu,\tau'}(\bar{x}_\tau)) = \sum_{\nu_0} c_{\lambda\mu}(t)s_{\nu_0}(\bar{x}_\tau) + \sum_{\nu < \nu_0} c_{\lambda\mu}(t)s_{\nu}(\bar{x}_\tau).
$$

On the left hand side, this coefficient is a combination of stable coefficients, hence is itself stable. The coefficient on the right hand side is $c_{\lambda\mu}(t)$, establishing the result. $\square$

9. Asymptotics

We now begin our study of sequences of the form $\lambda = (\lambda_+ , \lambda_- )$. We construct these sequences by taking two partitions $\lambda_+ = (\lambda^+_1 \geq \cdots \geq \lambda^+_r)$ and $\lambda_- = (\lambda^-_1 \geq \cdots \geq \lambda^-_s)$, joining them together after negating and reversing the order of the parts of $\lambda_-$ to obtain

$$
\lambda_t = (\lambda^+_1 \geq \cdots \lambda^+_r \geq 0 \cdots \geq 0 \geq -\lambda^-_s \geq \cdots \geq -\lambda^-_1).
$$

Throughout our discussion, we assume that the total number of parts, or terms, in $\lambda$, including parts of size 0, is $t$. When we discuss asymptotic behaviour, we will be allowing $t$ to grow large by inserting parts of size zero.

We will denote by

$$
s_{\lambda}(\bar{x}_t) := s_{\lambda}(\bar{x}_t) = s_{(\lambda^+_1 , \ldots , \lambda^+_r , 0 , \ldots , 0 , -\lambda^-_1 , \ldots , -\lambda^-_s )}(\bar{x}_t)
$$

the Schur polynomial corresponding to the partition $\lambda$. For example,

$$
s_{(1,-1)}(\bar{x}_t) = m_{(1,-1)}(\bar{x}_t) + (t - 1).
$$

We now compute another example.

**Example 5.** We note that

$$
s_{(-1)} = m_{(-1)},
$$

$$
s_{(1,-2)} = m_{(1,-2)} + m_{(1,-1,-1)} + (t - 1)m_{(-1)},
$$

and

$$
s_{(1,-1,-1)} = m_{(1,-1,-1)} + (t - 2)m_{(-1)}.
$$

Then

$$
s_{(-1)}s_{(1,-1)} = m_{(-1)}(m_{(1,-1)} + (t - 1)) = m_{(1,-2)} + m_{(-1)} + 2(t - 1)m_{(-1)} + 2m_{(1,-1,-1)} = s_{(1,-2)} + s_{(1,-1,-1)} + s_{(-1)}.$$

As a corollary to Theorem 2.3 in [PS], we obtain that for given partitions \( \lambda = (\lambda_+, 0) \) and \( \mu = (0, \mu_-) \)

\[
s(\lambda_+, 0)s(\emptyset, \mu_-) = \sum_{\nu} c_{\nu}(\lambda_+, 0)(\emptyset, \mu_-) s_{\lambda-\mu\nu}
\]

where \( \nu = (\nu_+, \nu_-) \) and for \( r = |\lambda_+| - |\nu_+| = |\mu_-| - |\nu_-| \),

\[
c_{\nu}(\lambda_+, 0)(\emptyset, \mu_-) = \sum_{|\gamma|=r} c_{\nu_+\gamma}^\lambda c_{\nu_-\gamma}^{\mu_-}.
\]

Firstly, we note that \( c_{(\lambda_+, \mu_-)}^{(\lambda_+, 0)(\emptyset, \mu_-)} = 1 \), since \( \nu = (\lambda_+, \mu_-) \) gives \( |\lambda_+| - |\nu_+| = 0 \) which implies that the only summand is the one corresponding to \( \gamma = \emptyset \), for which \( c_{\lambda_+}^{\lambda_+} = c_{\mu_-}^{\mu_-} = 1 \). Secondly, for any \( \nu = (\nu_+, \nu_-) \neq (\lambda_+, \mu_-) \), \( |\nu_+| \geq |\lambda_+| \) or \( |\nu_-| \geq |\mu_-| \) implies that \( c_{\nu}(\lambda_+, 0)(\emptyset, \mu_-) = 0 \). We then have

\[
s(\lambda_+, 0)s(\emptyset, \mu_-) = s(\lambda_+, \mu_-) + \sum_{\nu} c_{\nu}(\lambda_+, 0)(\emptyset, \mu_-) s_{\lambda-\mu\nu}.
\]

Having established that the generalized Littlewood-Richardson coefficients \( c_{\nu}(\lambda_+, 0)(\emptyset, \mu_-) \) can be written as a combination of Littlewood-Richardson coefficients for standard partitions, we are able to infer their stability. ¹

In the next section, we will use this new stability to deduce the polynomial behaviour of the Kostka numbers.

10. The Polynomial Behaviour of Kostka Numbers

In this section, we will prove that \( k_{\lambda\mu}(t) \) is a polynomial in \( t \), where \( \lambda = (\lambda_+, \lambda_-) \) and \( \mu = (\mu_+, \mu_-) \). Furthermore, as an integer valued polynomial, \( k_{\lambda\mu}(t) \) can be written as a polynomial in the binomial basis for integer valued polynomials,

\[
\left\{ \binom{t+i}{i} \right\} |i \geq 0
\]

in addition to the usual basis of polynomials \( \{ t^i | i \geq 0 \} \). Our first proof of the polynomial behaviour of Kostka numbers will rely on the results from [PS]. We begin by proving a lemma.

Lemma 1. Let \( p(\vec{x}_i) \) and \( q(\vec{x}_i) \) be symmetric Laurent polynomials in \( t \) variables with coefficients that are polynomials in \( t \). Then the product \( p(\vec{x}_i)q(\vec{x}_i) \) also has coefficients that are polynomials in \( t \).

Proof. We say that \( p(\vec{x}_i) \) involves at most \( k \) variables if, when expanded as a sum of symmetric polynomials \( m_\gamma \), i.e. \( p(\vec{x}_i) = \sum \gamma m_\gamma(\vec{x}_i) \), each \( \gamma \) has at most \( k \) parts. We say that \( q(\vec{x}_i) \) involves \( s \) variables and induct on the total \( k + s \). The base case is clear, as if \( k + s = 0 \), we simply have the product \( 1 \cdot 1 = 1 \).

¹No formula for general \( c_{\lambda\mu}^{\nu} \) can be derived directly from [PS], but these coefficients likely also stabilize. We will not, however, address this question here.
We now assume that the statement holds for all $p'(\vec{x}_t)$ involving $k'$ variables and $q'(\vec{x}_t)$ involving $s'$ variables such that $k' + s' < k + s$. On one hand, as the product of symmetric polynomials, we have
\[ p(\vec{x}_t)q(\vec{x}_t) = \sum_{\gamma} a_\gamma(t) m_\gamma(\vec{x}_t), \]
where $a_\gamma(t)$ is some function. On the other hand, by sorting by the powers of $x_t$, we write
\[ p(\vec{x}_t) = \sum_i p_i(\vec{x}_{t-1}) x^i, \quad q(\vec{x}_t) = \sum_j q_j(\vec{x}_{t-1}) x^j \]
where each $p_i(\vec{x}_{t-1})$ and each $q_j(\vec{x}_{t-1})$ is a symmetric Laurent polynomial in $x_1, \ldots, x_{t-1}$. Since the inductive assumption applies for any pair $p_i, q_j$ with $(i, j) \neq (0, 0)$, we have
\[
 p(\vec{x}_t)q(\vec{x}_t) = \left( \sum_i p_i(\vec{x}_{t-1}) x^i \right) \left( \sum_j q_j(\vec{x}_{t-1}) x^j \right) \\
 = p_0(\vec{x}_{t-1}) q_0(\vec{x}_{t-1}) + \sum_{(i,j) \neq (0,0)} p_i(\vec{x}_{t-1}) q_j(\vec{x}_{t-1}) x^{i+j} \\
 = p_0(\vec{x}_{t-1}) q_0(\vec{x}_{t-1}) + \sum_d (\sum_\beta \overline{a}_{\beta,d}(t-1) m_\beta(\vec{x}_{t-1})) x^d.
\]
So,
\[
\sum_\gamma a_\gamma(t) m_\gamma(\vec{x}_t) = p_0(\vec{x}_{t-1}) q_0(\vec{x}_{t-1}) + \sum_d (\sum_\beta \overline{a}_{\beta,d}(t-1) m_\beta(\vec{x}_{t-1})) x^d.
\]

If $\gamma = (\gamma_1, \ldots, \gamma_t) \neq \emptyset$, say $\gamma_t \neq 0$, then $\sum_d (\sum_\beta \overline{a}_{\beta,d}(t-1) m_\beta(\vec{x}_{t-1})) x^d$ contributes to $\vec{x}_t^\gamma$, so $a_\gamma(t) = \overline{a}^\gamma(t) (t-1)$ for $\gamma = (\gamma_1, \ldots, \gamma_{t-1})$, hence is a polynomial in $t$ by the inductive hypothesis. If $\gamma = \emptyset$, comparing coefficients of $x^0$, i.e. comparing constant coefficients on either side of (3) we obtain
\[
a_\emptyset(t) = a_\emptyset(t-1) + \overline{a}_\emptyset(t) \\
\Rightarrow a_\emptyset(t) - a_\emptyset(t-1) = \overline{a}_\emptyset(t),
\]
giving a difference equation whose solution is a polynomial in $t$, see Appendix. This gives us that $a_\emptyset(t)$ is also a polynomial, and hence that the entire product $p(\vec{x}_t)q(\vec{x}_t)$ has polynomial coefficients, as desired.

\[\square\]

**Theorem 3.** $k_{\lambda\mu}(t)$ is a polynomial in $t$.

**Proof.** We begin by noting that we extend the lexicographic order to an order between pairs $\mu = (\mu_+, \mu_-)$ and $\nu = (\nu_+, \nu_-)$ by saying that $\mu \succ \nu$ if $\mu_+ \succ \nu_+$ and $\mu_- \succ \nu_-$. We will establish the polynomial behaviour using the stability of the Littlewood-Richardson coefficients and induction on the above order. For the base case, $s_\emptyset(\vec{x}_t) = 1$ has coefficients that are polynomials in $t$, namely, all are the zero polynomial except for $k_{\emptyset\emptyset} = 1$. We now assume that $s_{\nu'}(\vec{x}_t)$ is polynomial - meaning it has coefficients that are polynomials in $t$ - for all $\nu' \prec \nu = (\nu_+, \nu_-)$. Consider
\[
s_{(\nu_+, \emptyset)}(\vec{x}_t) s_{(\emptyset_+, \nu_-)}(\vec{x}_t) = \sum_{\lambda} c^\gamma_{\lambda\mu} s_{\lambda}(\vec{x}_t).
\]
On the right hand side, we have that $\nu = (\nu_+, \nu_-)$ is the largest element. We can therefore rearrange the equation to obtain
\[
s_{(\nu_+, \emptyset)}(\vec{x}_t) s_{(\emptyset_+, \nu_-)}(\vec{x}_t) - \sum_{\nu' \prec \nu} c_{\lambda\mu}^\gamma s_{\nu'}(\vec{x}_t) = s_{\nu}(\vec{x}_t).
\]
By Lemma 1 and the inductive hypothesis, all Schur polynomials on the left hand side of the equation are polynomial, as is the product $s_{\nu_t}(\vec{x}_t)s_{\nu}(\vec{x}_t)$. Moreover, the Littlewood-Richardson coefficients are stable. Therefore, $s_\nu(\vec{x}_t)$ is also polynomial, establishing the result. 

\[ \square \]

Neither Theorem 3 nor its proof actually provide much information about the polynomial $k_{\lambda\mu}(t)$. Approaching the problem through branching rules allows us to determine both the degree and leading coefficient of the Kostka polynomial $k_{\lambda\mu}(t)$.

**Theorem 4.** The degree of $k_{\lambda\mu}(t)$ is $d = |\lambda_+| - |\mu_+| = |\lambda_-| - |\mu_-|$. Furthermore, the leading coefficient of $k_{\lambda\mu}(t)$ is $\frac{1}{d!} k_{\lambda, \bar{\mu}_+} k_{\lambda, \bar{\mu}_-}$, where $\bar{\mu}_+$ is obtained from $\mu_+$ by adjoining $|\lambda_+| - |\mu_+|$ single cells to the corresponding Young diagram, that is to say, $\bar{\mu}_+ = (\mu_+, 1, \ldots, 1)$, (and similarly for $\bar{\mu}_-$).

**Proof.** In order to prove this result, we must first establish a recurrence relation for $k_{\lambda\mu}(t)$. We will do this using the branching rules as discussed in Section 4.2.

Recall that we have the decomposition

\[ V_\lambda = \bigoplus_{\nu < \lambda} V_\nu \otimes \mathbb{C}^{[\lambda|-|\nu]} \]

which corresponds to the relation of characters

\[ s_{\lambda_{t+1}}(\vec{x}_{t+1}) = \sum_{\nu_t < \lambda_{t+1}} s_{\nu_t}(\vec{x}_t) x_{t+1}^{[\lambda|-|\nu]} . \]

Expanding the Schur polynomials in terms of the Kostka numbers gives

\[ \sum_{\mu} k_{\lambda\mu}(t + 1) m_{\mu}(\vec{x}_{t+1}) = \sum_{\nu_t < \lambda_{t+1}} \left( \sum_{\mu} k_{\nu\mu}(t) m_{\mu}(\vec{x}_t) \right) x_{t+1}^{[\lambda|-|\nu]} . \]

We now compare coefficients of the terms

\[ \vec{x}_{t+1}^\mu = x_1^{\mu_1} \ldots x_t^{\mu_t} \]

on either side of the above equation. To simplify our induction, we look at the terms where $\mu_{t+1} = 0$, that is to say, we are really comparing coefficients for terms of the form

\[ x_1^{\mu_1} \ldots x_t^{\mu_t} . \]

This gives us

\[ k_{\lambda\mu}(t + 1) = \sum_{\nu_t < \lambda_{t+1}, |\lambda| = |\nu|} k_{\nu\mu}(t) . \]

We note that $\lambda_t < \lambda_{t+1}$. Denoting $a_t = k_{\lambda\mu}(t)$, we have

\[ k_{\lambda\mu}(t + 1) = k_{\lambda\mu}(t) + \sum_{\nu_t < \lambda_{t+1}, |\lambda| = |\nu|, \nu \neq \lambda} k_{\nu\mu}(t) \]

so

\[ a_{t+1} = a_t + \sum_{\nu_t < \lambda_{t+1}, |\lambda| = |\nu|, \nu \neq \lambda} k_{\nu\mu}(t) \]

or

\[ \Rightarrow a_{t+1} - a_t = \sum_{\nu_t < \lambda_{t+1}, |\lambda| = |\nu|, \nu \neq \lambda} k_{\nu\mu}(t) . \]

\[ (4) \]
We will now induct on \( d = |\lambda_+| - |\mu_+| \) to prove that \( k_{\lambda\mu}(t) \) is of degree \( d \) with leading coefficient \( \frac{1}{d!} k_{\lambda+\mu+} k_{\lambda-\mu-} \).

For the base case, we assume \( |\lambda_+| = |\mu_+| \). We note that the right hand side of (4) is then zero, as for any \( \nu \) such that \( \nu_t < \lambda_{t+1} \) where \( \nu \neq \lambda \),

\[
|\nu_+| < |\lambda_+| = |\mu_+| \Rightarrow k_{\nu\mu}(t) = 0.
\]

Thus, \( a_{t+1} - a_t = 0 \), so \( a_t \) is a constant. To determine this constant, we note that \( |\lambda_+| = |\mu_+| \) and \( |\lambda_-| = |\mu_-| \), so by looking at the corresponding Young diagrams with the appropriate shifts, since the positive and negative portions of the Young diagram are forced to be filled independently, we obtain \( k_{\lambda\mu}(t) = k_{\lambda+\mu+} k_{\lambda-\mu-} \), establishing the result.

We now assume that the result holds for \( \nu = (\nu_+, \nu_-) \) such that \( |\nu_+| - |\mu_+| < d \). From (4), the right hand side is the sum of polynomials of degrees \( |\nu_+| - |\mu_+| < d \). To see that the degree of the sum is, in fact, exactly \( d - 1 \), we note that sequences of the form \( \nu = (\nu_+, \nu_-) \) obtained by removing one cell each from suitable locations of \( \lambda_+ \) and \( \lambda_- \), i.e. at the end of a row that does not conflict with the weakly decreasing lengths of the rows, will interlace \( \lambda \), and will satisfy \( |\nu_+| = |\lambda_+| - 1 \). Each of the sequences of this form will give a Kostka polynomial \( k_{\nu\mu}(t) \) of degree \( d - 1 \) and positive leading coefficient, so no cancellation will occur. Thus, the degree of the right hand side of (4) is exactly \( d - 1 \), so Theorem A implies that the degree of \( k_{\lambda\mu}(t) \) is \( d \).

The leading coefficient of \( a_t = k_{\lambda\mu}(t) \) (in the expansion in the binomial basis) will be the sum of all leading coefficients on the right hand side of (4) such that \( |\lambda_+| - |\nu_+| = 1 \). In other words, it is equal to the sum

\[
\sum_{|\lambda_+| - |\nu_+| = 1} C_{\nu}
\]

where \( C_{\nu} \) denotes the leading coefficient of the Kostka polynomial \( k_{\nu\mu}(t) \). By the inductive hypothesis, we have

\[
C_{\nu} = k_{\nu_+\mu_+} k_{\nu_-\mu_-},
\]

which combined with (5) implies that the leading coefficient of \( k_{\lambda\mu}(t) \) equals

\[
\sum_{|\lambda_+| - |\nu_+|=|\lambda_-| - |\nu_-|=1} k_{\nu_+\mu_+} k_{\nu_-\mu_-} = \left( \sum_{|\lambda_+| - |\nu_+|=1} k_{\nu_+\mu_+} \right) \left( \sum_{|\lambda_-| - |\nu_-|=1} k_{\nu_-\mu_-} \right)
\]

To compute the sums on the right hand side of (6), notice that for standard partitions \( \tau \) and \( \eta \), with the definition of the Kostka number \( k_{\tau\eta}(t) \) as the number of possible Young tableaux of shape \( \tau \) and content \( \eta = (\eta_1, \ldots, \eta_t) \),

\[
k_{\tau\eta} = \sum_{\tau'} k_{\tau'\eta}
\]

where \( \tau' \) runs over all Young diagrams obtained from \( \tau \) by removing one cell, and \( \eta = \eta - (0, \ldots, 0, 1) \) (assuming that \( \eta_t > 0 \)). This follows because, for each Young tableau of shape \( \tau \) and content \( \eta \), removing a cell containing the entry \( t \) creates a new Young tableau (following much the same argument as in Section 3.1), and given the collection of all possible Young tableaux obtained by the removal of a single cell containing the entry \( t \), we can reconstruct a unique “parent” Young tableau.
Rewriting (7) as
\[ k_{\tau\eta} = \sum_{|\tau| - |\tau'| = 1} k_{\tau'\tau} \]
and combining this with (5) and (6), we conclude that the leading coefficient of \( k_{\lambda\mu}(t) \) in the binomial basis is
\[ k_{\lambda+\tilde{\mu}} k_{\lambda-\tilde{\mu}}. \]

\[ \square \]

### 11. Further Asymptotics

After proving that the Kostka numbers do exhibit polynomial behaviour, a natural next step is to try to determine what these polynomials actually are. Given partitions \( \lambda = (\lambda_+, \lambda_-) \) and \( \mu = (\mu_+, \mu_-) \), is it possible to determine the Kostka polynomial \( k_{\lambda\mu}(t) \)? In section 10 we were able to determine the degree and leading coefficient of \( k_{\lambda\mu}(t) \), however, at this time no general formula for determining \( k_{\lambda\mu}(t) \) explicitly is known. We can, however, establish formulas for two simple cases.

**Theorem 5.**

\[ k_{(s,-s),\emptyset}(t) = \binom{t + s - 2}{s} \]

**First Proof.** This statement follows from the fact that
\[ k_{(s,-s),\emptyset}(t) = k_{(2s,s^{t-2}),st} \]
where \( k_{(2s,s^{t-2}),st} \) denotes the regular Kostka number obtained by simply counting the number of Young tableaux of shape \((2s, s^{t-2})\) and content \(s^t\).

In the first row of \((s,-s) + s^t\), we have no choice but to fill the first \(s\) spots with 1’s. We then have an arbitrary choice with repetition for \(s\) elements out of \(2, \ldots, t\) for the remaining \(s\) spots in the first row. These choices completely determine the tableau, as the \(t - 2 \times s\) block can then only be filled in one way. Thus, by the formula for choosing \(s\) elements with repetition out of \(t-1\) possible choices, we have that
\[ k_{(s,-s),\emptyset}(t) = \binom{(t-1) + (s-1)}{s} = \binom{t + s - 2}{s} \]
as desired. From this formula, we can see that \(k_{(s,-s),\emptyset}(t)\) is a polynomial in \(t\) for fixed \(s\) of degree \(s\) and leading coefficient \(\frac{1}{s!}\), as expected from Theorem 4.

\[ \square \]

**Second Proof.** We can also obtain this result from branching rules and induction. Since
\[ k_{(s,-s),\emptyset}(t + 1) = k_{(s,-s),\emptyset}(t) + \sum_{j=0}^{s-1} k_{(j,-j),\emptyset}(t), \]
and
\[ k_{(s+1,-(s+1)),\emptyset}(t + 1) = k_{(s+1,-(s+1)),\emptyset}(t) + k_{(s,-s),\emptyset}(t) + \sum_{j=0}^{s-1} k_{(j,-j),\emptyset}(t), \]
we obtain
\[ k_{(s+1,-(s+1))\emptyset}(t + 1) = k_{(s+1,-(s+1))\emptyset}(t) + k_{(s,-s)\emptyset}(t + 1). \]

Defining
\[ p(t, s) := k_{(s,-s)\emptyset}(t) \]
we thus obtain the recurrence relation with boundary conditions
\[
\begin{cases}
p(t + 1, s + 1) = p(t, s + 1) + p(t + 1, s) \\
p(t, 0) = 1 \\
p(2, s) = 1
\end{cases}
\]
as
\[ p(2, s) = k_{(s,-s)\emptyset}(2) = k_{(2s,0)(s,s)} = 1. \]

We claim that \( p(t, s) = \binom{t+s-2}{s} \). Clearly,
\[
\binom{t-2}{0} = 1 = p(t, 0),
\]
and
\[
\binom{2+s-2}{s} = \binom{s}{s} = 1 = p(2, s).
\]

We now induct on \( t + s \), assuming that the statement holds for all \( t' \) and \( s' \) such that \( t' + s' < t + s \). By the recurrence relation,
\[ p(t, s) = p(t - 1, s) + p(t, s - 1) \]
which, by the inductive hypothesis, is
\[
p(t, s) = \binom{t - 1 + s - 2}{s} + \binom{t + s - 1 - 2}{s - 1}
\]
\[
\Rightarrow p(t, s) = \binom{t + s - 3}{s} + \binom{t + s - 3}{s - 1}.
\]

Using the relation \( \binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r} \), we obtain
\[ p(t, s) = \binom{t + s - 2}{s}, \]
the desired result.

\[ \square \]

We now look at \( \lambda = (1^s,-1^s) \) and \( \mu = \emptyset \). To begin our discussion, we introduce the notion of a **Dyck path**:

**Definition 10.** A **Dyck path** is a path of steps of length 1 in the \((x,y) - \text{plane}\) from the point \((0,0)\) to \((s,s)\), where each step goes either directly up or to the right and does not cross the line \(x = y\). The number of Dyck paths from \((0,0)\) to \((s,s)\) is known to be the \(s^{th}\) **Catalan number**,
\[ C_s = \frac{1}{s+1} \binom{2s}{s} \]
(see [W]).
Theorem 6.

\[ k_{(1^s,-1^s)}(t) = \binom{t}{t-s} - \sum_{i \geq 0} C_i \binom{t-(2i+1)}{t-s-i}. \]

Proof. To determine \( k_{(1^s,-1^s)}(t) \), we will look at \( k_{(2^s,1^{t-2s})}(t) \). Column one of \((1^s,-1^s) + 1^t = (2^s,1^{t-2s})\) has \( t-s \) spots, and column two has \( s \). Choosing the elements in the first column completely determines the Young tableau, as the second column will then be forced to be filled in increasing order. In total, there are \( \binom{t}{t-s} \) ways of choosing the first column. However, not all of these choices will allow for a valid Young tableau. We must therefore subtract from the total number of possibilities the number of fillings that are not valid tableaux. Let \( a_i \) denote the number in row \( i+1 \) in the first column. Thus each filling directly corresponds to a \((t-s)\)-tuple \((a_0,\ldots,a_{t-s-1})\).

We note that \( a_i < 2(i+1) \) for \( 1 \leq i \leq s \) is a necessary and sufficient condition for the corresponding filling to be a Young tableau. Thus, from the total number of choices for the first column, we must subtract all \((t-s)\)-tuples which contradict the above condition.

It is clear that there are \( \binom{t-1}{t-s} \) ways to fill the first column with \( a_0 > 1 \), i.e. with a contradiction in the first row. Thus we begin by subtracting \( \binom{t-1}{t-s} \) from the total \( \binom{t}{t-s} \).

In general, we claim that there are \( C_i \binom{t-(2i+1)}{t-s-i} \) possible fillings that do not contradict \( a_j < 2j+2 \) for \( 0 \leq j \leq i-1 \), but contradict for the first time at \( a_i \), in the \( i+1 \)st row.

Firstly, there are \( C_i \) ways to construct a valid filling for the first \( i \) rows. We establish this by filling the \( i \times 2 \) rectangle with the numbers \( 1,\ldots,2i \) in order. For a placement in the left of the two columns, we add a step to the right to our Dyck path, and for a placement in the right column we assign a vertical step. The condition \( a_i < 2j+2 \) corresponds to the condition that the Dyck path does not cross the line \( x=y \). Thus, for every valid filling of the aforementioned rectangle, we have a corresponding Dyck path and vice versa, so the number of valid \( i \times 2 \) rectangular tableaux is \( C_i \).

To introduce a contradiction at the \( i+1 \)st row, i.e. at \( a_i \), we must fill the remaining \( t-s-i \) spots with choices from the \( t-(2i+1) \) elements which would give the desired contradiction. Subtracting all possible invalid fillings by running over the \( t-s \) rows, we obtain the result:

\[ k_{(1^s,-1^s),\emptyset}(t) = \binom{t}{t-s} - \sum_{i \geq 0} C_i \binom{t-(2i+1)}{t-s-i}. \]

□

We can also prove a simpler formula for this Kostka polynomial.

Theorem 7.

\[ k_{(1^s,-1^s),\emptyset}(t + 1) = \binom{t+1}{s} - \binom{t}{s-1}. \]

Proof. By branching rules,

\[ k_{(1^s,-1^s),\emptyset}(t+1) = k_{(1^s,-1^s),\emptyset}(t) + k_{(1^{s-1},-1^{s-1}),\emptyset}(t). \]

We define

\[ q(t,s) := k_{(1^s,-1^s),\emptyset}(t), \]
so that the branching rule and our argument above gives us the recurrence relation with boundary conditions

\[
\begin{cases}
q(t+1, s) = q(t, s) + q(t, s-1) \\
q(t, 0) = 1 \\
q(2s, s) = C_s.
\end{cases}
\]

We note that the variables are constrained by \(s \geq 0\) and \(t \geq 2s\).

Now, by another straightforward inductive argument, we show that

\[
q(t, s) = \binom{t+1}{s} - 2\binom{t}{s-1}.
\]

It is clear that the base cases are satisfied, as

\[
q(t, 0) = 1 = \binom{t+1}{0} - 2\binom{t}{-1}
\]

and

\[
q(2s, s) = \binom{2s+1}{s} - 2\binom{2s}{s-1} = \binom{2s}{s} - \binom{2s}{s-1}
\]

\[
\Rightarrow q(2s, s) = \frac{(2s)!}{s!s!} - \frac{(2s)!}{(s-1)!(s+1)!} = \frac{(s+1)(2s)! - s(2s)!}{s!(s+1)!}
\]

\[
\Rightarrow q(2s, s) = \frac{(2s)!}{s!(s+1)!} = C_s.
\]

Now, assuming that the statement holds for all \(t'\) and \(s'\) such that \(t' + s' < t + s\), we use the recurrence relation to obtain

\[
q(t, s) = q(t-1, s) + q(t-1, s-1) = \binom{t}{s} - 2\binom{t-1}{s-1} + \binom{t}{s-1} - 2\binom{t-1}{s-2}
\]

\[
\Rightarrow q(t, s) = \binom{t+1}{s} - 2\binom{t}{s-1}
\]

as desired. \(\square\)

12. Remarks

The Kostka numbers \(k_{(s,-s)}(t)\) and \(k_{(1^a,-1^b)}(t)\) discussed above are examples of asymptotic behaviour involving two variables, a more general situation than that discussed in the majority of this project, which only involved the variable \(t\).

We notice that \(k_{(s,-s)}(t)\) is a polynomial in \(t\) for fixed \(s\), and is a polynomial in \(s\) for fixed \(t\). Furthermore, it can be expressed by a single binomial coefficient.

The Kostka number \(k_{(1^a,-1^b)}(t)\) on the other hand, while a polynomial in \(t\) when \(s\) is fixed, is only a polynomial in \(s\) for fixed \(t-s\). It is, however, still a combination of only two binomial coefficients.

A very general problem to study would be to examine Kostka functions

\[
p(s_1, \ldots, s_k, t_1, \ldots, t_l) = k_{(\lambda_1^+, \ldots, \lambda_k^+; \mu_1^+, \ldots, \mu_l^+)},
\]

where \(\lambda = (\lambda_1, \ldots, \lambda_k)\) and \(\mu = (\mu_1, \ldots, \mu_l)\) are weakly decreasing sequences of integers, with the hope that \(p\) can be expressed as a finite combination of multinomial coefficients.
We record a statement used repeatedly in the text.

**Theorem A.** Let

\[ p(t) = \sum_{k=0}^{d} \alpha_k \binom{t+k}{k}, \]

and let \( a_t \) be a sequence such that \( a_{t+1} - a_t = p(t) \ \forall t \geq 0 \). Then

\[ a_t = C + \sum_{k=0}^{d+1} (\alpha_{k-1} - \alpha_k) \binom{t+k}{k} \]

where \( \alpha_{d+1} = 0 \) and \( C \) is some constant. In particular, \( a_t \) is a polynomial in \( t \) of degree \( d + 1 \) and leading coefficient \( \frac{p_d}{d+1} \) where \( d = \text{deg}(p) \) and \( p_d \) is the leading coefficient of \( p(t) \).

**Proof.** We first note that if this difference equation has a solution it is unique up to a constant. This is easily seen because if

\[ b_{t+1} - b_t = a_{t+1} - a_t, \]

then

\[ b_{t+1} - a_{t+1} = b_t - a_t \ \forall t \]

so \( b_t - a_t = C \) is constant. Consider

\[ b_t = \sum_{k=1}^{d+1} (\alpha_{k-1} - \alpha_k) \binom{t+k}{k} = \sum_{k=0}^{d} \alpha_k \left( \binom{t+k+1}{k+1} - \binom{t+k}{k} \right). \]

Then

\[ b_{t+1} - b_t = \sum_{k=0}^{d} \alpha_k \left( \binom{t+k+2}{k+1} - \binom{t+k+1}{k} \right) - \alpha_k \left( \binom{t+k+1}{k+1} - \binom{t+k}{k} \right) \]

\[ = \sum_{k=0}^{d} \alpha_k \left( \binom{t+k+1}{k} + \binom{t+k+1}{k+1} - \binom{t+k+1}{k} - \binom{t+k+1}{k+1} + \binom{t+k}{k} \right) \]

\[ = \sum_{k=0}^{d} \alpha_k \binom{t+k}{k} = p(t). \]

\[ \square \]

**References**


