THE CHOWLA PROBLEM AND
ITS GENERALIZATIONS

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Abstract

In this thesis, we study the vanishing of certain $L$-series attached to periodic arithmetical functions. Throughout the writeup, we let $f$ be an algebraic-valued (at times even rational-valued) arithmetical function, periodic with period $q$. Define

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We discuss a conjecture of Sarvadaman Chowla, made in the early 1960’s of the non-vanishing of $L(1, f)$ for rational-valued functions $f$. We further discuss the theorem of Baker-Birch-Wirsing that classifies all odd algebraic-valued, periodic arithmetical functions $f$ with $L(1, f) = 0$. We then apply a beautiful result of Bass to give a necessary condition for even algebraic-valued, periodic arithmetical functions $f$, to satisfy $L(1, f) = 0$. The sufficiency condition obtained via this method is unfortunately not very clean. This, in view of a theorem of Ram Murty and Tapas Chatterjee, completes the characterization of all algebraic-valued periodic arithmetical functions $f$ with $L(1, f) = 0$.

In the next part of the thesis, we discuss the Lerch-zeta function and its functional equation. We also deduce the transcendence of its certain special values. Keeping this in mind, we define a new $L$-series attached to periodic functions and deduce a necessary condition for the vanishing of this $L$-series at $s = 1$.

Finally, we mention some research topics that were stumbled upon during the course of this study and a few partial results regarding these questions.
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Introduction

Non-vanishing of $L$-functions has been very thoroughly studied in number theory since a long time. One of the most remarkable theorems of this kind in the past 150 years is due to Dirichlet, namely, $L(1, \chi) \neq 0$ for a non-principal Dirichlet character $\chi$. As a general principle in mathematics, many concepts become clearer on generalization. To unravel the mystery surrounding Dirichlet’s theorem, Sarvadaman Chowla [9] made the following conjecture in the early 1960s.

Conjecture. Let $f$ be a rational-valued arithmetical function, periodic with prime period $p$. Further assume that $f(p) = 0$ and

$$\sum_{n=1}^{p} f(a) = 0.$$ 

Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

unless $f$ is identically zero.

Chowla proved this conjecture in the case of odd functions i.e, $f(p-n) = -f(n)$ based on an outline of the proof by Siegel [9].

The complete resolution of Chowla’s conjecture in a wider setting was given by Baker, Birch and Wirsing in 1972 [5]. They proved the following general theorem:

Theorem. If $f$ is a non-vanishing function defined on the integers with algebraic values and period $q$ such that (i) $f(n) = 0$ whenever $1 < (n, q) < q$ and (ii) the $q^{th}$ cyclotomic polynomial $\Phi_q$ is irreducible over $\mathbb{Q}(f(1), f(2), \cdots, f(q))$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$ 

Let us observe that in the case of Chowla’s conjecture, condition (i) is vacuous as $q$ is prime and $f(q) = 0$. The condition (ii) is also satisfied as $f$ is rational-valued and $\Phi_q$ is irreducible over $\mathbb{Q}$. Thus, the Baker-Birch-Wirsing theorem implies Chowla’s conjecture.

In their paper [5], the authors mention that Chowla communicated to them that he had also solved the problem “to the extent mentioned above”. It is although unclear as to what exactly Chowla had in mind.
Using Baker’s theory of linear forms in logarithms in a minimal way and the methods known to Chowla, one can prove Chowla’s conjecture for even functions. This was done by Ram Murty [17] recently. One can therefore guess that this was the proof that Chowla had in mind when he wrote to Baker, Birch and Wirsing. We mention the proofs of Chowla [9] and Ram Murty [17] in the first chapter. This gives us an idea of the intricacies of the topic at hand.

Chowla’s question can be asked in more generality as follows: Fix a positive integer \( q \). Let \( f \) be an algebraic-valued arithmetical function, periodic with period \( q \). It is useful to define an \( L \)-function associated to the function \( f \), namely,

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.
\]

Using the analytic continuation of the Hurwitz zeta function, we can deduce that \( L(1, f) \) exists if and only if \( \sum_{a=1}^{q} f(a) = 0 \). Thus, we can ask the following question: if \( f \) is not identically zero, then is it true that

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0?
\]

The answer to this in such a generality turns out to be negative [5]. As an example, consider the function \( f \) defined such that

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (1 - p^{(1-s)})^2 \zeta(s).
\]

In particular,

\[
f(n) = \begin{cases} 
1 & \text{if } (n, p) = 1, \\
1 - 2p & \text{if } p|n, p^2 \nmid n, \\
(p - 1)^2 & \text{otherwise}.
\end{cases}
\]

Note that the function \( f \) is periodic with period \( p^2 \). Taking limit of the right hand side of (0.1) as \( s \to 1 \), we get

\[
L(1, f) = 0,
\]

because \( \zeta(s) \) has a simple pole at \( s = 1 \).

In their paper, [5], Baker, Birch and Wirsing also gave a characterization of all odd algebraic-valued periodic arithmetical functions \( f \) that satisfy \( L(1, f) = 0 \). Since their argument is short and elegant, we describe it in the first part of the second chapter. Their approach suggests a change in perspective. Instead of trying to prove the non-vanishing of an expression, we will try to characterize the functions \( f \) for which \( L(1, f) = 0 \).

Let \( f \) be any function. Then it can be written as the sum of an even function and an odd function as follows. Define \( f_o(a) := [f(a) - f(-a)]/2 \) and \( f_e(a) := [f(a) + f(-a)]/2 \). Then clearly \( f = f_o + f_e \),
where $f_o$ is the odd part of $f$ and $f_e$ is the even part of $f$. In 2014, Ram Murty and Tapas Chatterjee [8] made an important observation. They proved,

**THEOREM.** For a periodic, algebraic-valued arithmetical function $f$,

$$L(1, f) = 0 \iff L(1, f_o) = 0 \text{ and } L(1, f_e) = 0.$$

In the view of this theorem, it is enough to consider even algebraic-valued periodic arithmetical functions to understand the vanishing of $L(1, f)$. In the next part of the second chapter, we apply a beautiful result of Bass [6] to obtain a set of functions that act as building blocks for even algebraic-valued periodic arithmetical functions $f$ with $L(1, f) = 0$. This completes to an extent the characterization we were after.

In the history of number theory, various zeta functions have played a crucial role. A couple of the most extensively studied ones are probably the Riemann zeta and the Hurwitz zeta functions. As a generalization of these, Lerch (in 1887) and Lipschitz (in 1889) independently introduced another zeta function, which is more commonly known as the Lerch zeta function. The Hurwitz zeta function has one real parameter whereas the Lerch zeta function has two real parameters. We introduce this zeta function in the third chapter and include a very sleek proof of its functional equation given by B. Berndt [7]. We also mention a result of Apostol [2] wherein he explicitly computes the special values of the Lerch zeta function at negative integers and zero. As a consequence of this and the functional equation, we derive the transcendence of values of the Lerch zeta function at strictly positive integers when both the real parameters involved are rational.

Motivated by the definition of the Lerch zeta function, we define a new $L$-series attached to periodic arithmetical functions in the fourth chapter. Given such a function $f$, we define $\mathcal{L}(s, f, \alpha)$ for $0 < \alpha < 1$. We obtain a condition for the convergence of this series at $s = 1$. Using a simple observation, we express $\mathcal{L}(1, f, \alpha)$ as a linear form in logarithms of algebraic numbers. Finally, using Baker’s theory of linear forms in logarithms, we obtain a necessary condition for $\mathcal{L}(1, f, \alpha) = 0$.

In the course of this study, we stumbled upon a few unsolved and interesting questions. We were able to partially answer a few. We mention these questions and the partial answers as topics for further research in the last chapter.
CHAPTER 1

Preliminaries

The aim of this chapter is to introduce notation and some fundamental results that will be used in the later part of the thesis.

1.1. $L$-series attached to a periodic function

Let us first recall a few facts about the Hurwitz zeta function. For $0 < x \leq 1$, define the Hurwitz zeta function, $\zeta(s, x)$ as

$$
\sum_{n=0}^{\infty} \frac{1}{(n + x)^s}.
$$

This series converges absolutely for $\Re(s) > 1$. In 1882, Hurwitz [12] obtained the analytic continuation and functional equation of $\zeta(s, x)$. He proved:

**Theorem 1.1.** The Hurwitz zeta function, $\zeta(s, x)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$ with residue $1$. In particular,

$$
\zeta(s, x) = \frac{1}{s - 1} - \Psi(x) + O(s - 1),
$$

where $\Psi$ is the Digamma function.

Let $q$ be a fixed positive integer. Consider $f: \mathbb{Z} \to \mathbb{Q}$, periodic with period $q$. It is useful to define an $L$-function attached to $f$, namely

$$
L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.
$$

Let us observe that $L(s, f)$ converges absolutely for $\Re(s) > 1$. This is true because $f(n)$ is bounded and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $\Re(s) > 1$.

Since $f$ is periodic,

$$
L(s, f) = \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a + kq)^s} = \frac{1}{q^s} \sum_{a=1}^{q} f(a) \zeta\left(s, \frac{a}{q}\right),
$$

(1.1)
where $\zeta(s, x)$ is the Hurwitz zeta function. Using Theorem 1.1, we can conclude that $L(s, f)$ can be extended analytically to the entire complex plane except for a simple pole at $s = 1$ with residue $\frac{1}{q} \sum_{a=1}^{q} f(a)$.

Thus, $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ exists whenever $\sum_{a=1}^{q} f(a) = 0$, which we will assume henceforth. Hence, $L(s, f)$ is an entire function. Moreover, $L(1, f)$ can be expressed as a combination of values of the Digamma function. Theorem 1.1 implies

$$L(1, f) = -\frac{1}{q} \sum_{a=1}^{q} f(a) \Psi\left(\frac{a}{q}\right).$$

1.2. Expressing $L(1, f)$ as a linear form in logarithms of algebraic numbers

Let us first define the Fourier transform of $f$. Given a function $f$ which is periodic with period $q$,

$$\hat{f}(x) := \frac{1}{q} \sum_{a=1}^{q} f(a) \zeta_q^{-ax},$$

where $\zeta_q = e^{2\pi i / q}$. This can be inverted using the identity

$$f(n) = \sum_{x=1}^{q} \hat{f}(x) \zeta_q^{xn}.$$  (1.2)

Thus, the condition for convergence of $L(1, f)$ derived in the earlier section, i.e. $\sum_{a=1}^{q} f(a) = 0$ can be interpreted as $\hat{f}(q) = 0$.

Substituting (1.2) in the expression for $L(s, f)$ we have,

$$L(s, f) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{x=1}^{q} \hat{f}(x) \zeta_q^{xn}$$

$$= \sum_{x=1}^{q} \hat{f}(x) \sum_{n=1}^{\infty} \frac{\zeta_q^{xn}}{n^s}.$$  

Recall the partial summation or the Abel summation formula that says:

**Theorem.** Let $a_n$ be a sequence of complex numbers and $f$ be $C^1$ function on $\mathbb{R}_{>0}$. For $x > 0$, if $A(x) := \sum_{n \leq x} a_n$, then

$$\sum_{1 \leq n \leq x} a_n f(n) = A(x) f(x) - \int_{1}^{x} A(t) f'(t) dt.$$
The Abel summation formula, along with the assumption that \( \hat{f}(q) = 0 \), helps us to conclude that

\[
L(1, f) = - \sum_{x=1}^{q-1} \hat{f}(x) \log(1 - \zeta_q^x),
\]

where \( \log \) is the principal branch.

### 1.3. Baker's Theorem

Regarded as one of the most remarkable theorems of the 20th century number theory, Baker’s theorem on linear forms in logarithms is a very useful tool in proving the transcendence of many complex numbers. We will use it to prove that a certain linear form in logarithms of algebraic numbers is transcendental and hence non-zero. The statement of the theorem is

**Theorem 1.2.** If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are non-zero algebraic numbers, such that \( \log \alpha_1, \log \alpha_2, \ldots, \log \alpha_n \) are \( \mathbb{Q} \)-linearly independent, then \( 1, \log \alpha_1, \ldots, \log \alpha_n \) are \( \overline{\mathbb{Q}} \)-linearly independent.

A re-statement of Baker’s theorem is

**Theorem 1.3.** Any linear form in logarithms of algebraic numbers is either zero or transcendental. In other words, for any non-zero algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and any algebraic numbers \( \beta_0, \beta_1, \ldots, \beta_n \) with \( \beta_0 \neq 0 \) we have,

\[
\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0.
\]

A useful corollary of this statement is the following.

**Corollary 1.4.** If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are algebraic numbers different from 0 and 1, and \( \beta_1, \beta_2, \ldots, \beta_n \) are \( \mathbb{Q} \)-linearly independent algebraic numbers then,

\[
\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n
\]

is non-zero and hence, transcendental.

**Proof.** We will prove the above statement by induction on \( n \). Since \( \alpha_1 \) is algebraic, not equal to 1 and \( \beta_1 \neq 0 \), the statement is true for \( n = 1 \).

Suppose that the statement is true for all positive integers \( m \), \( m < n \). If \( \log \alpha_1, \ldots, \log \alpha_n \) are linearly independent over \( \mathbb{Q} \), then the statement is a consequence of Theorem 1.2. Hence, suppose that the numbers \( \log \alpha_1, \ldots, \log \alpha_n \) satisfy a relation over the rationals. In particular, there exist rational numbers \( b_1, \ldots, b_n \) with say \( b_r \neq 0 \), such that

\[
b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n = 0.
\]
Thus,

\[ b_r(\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n) = \beta'_1 \log \alpha_1 + \beta'_2 \log \alpha_2 + \cdots + \beta'_n \log \alpha_n, \quad (1.5) \]

where

\[ \beta'_j = b_r \beta_j - \beta_r b_j, \]

for \( 1 \leq j \leq n \). The right hand side of (1.5) is a linear form in logarithms of at most \( n - 1 \) algebraic numbers as \( \beta'_r = 0 \). Since \( \beta_1, \cdots, \beta_n \) are \( \mathbb{Q} \)-linearly independent, the set \( \{\beta'_j | 1 \leq j \leq n, j \neq r\} \) is also \( \mathbb{Q} \)-linearly independent. Thus, by induction, the right hand side of (1.5) is not zero. \( \Box \)

1.4. Linear forms in logarithms of positive algebraic numbers

The following lemma is an immediate consequence of Baker’s theorem and will be used many times.

**Lemma 1.5.** Let \( \alpha_1, \cdots, \alpha_n \) be positive algebraic numbers. If \( c_0, \cdots, c_n \) are algebraic numbers and \( c_0 \neq 0 \), then

\[ c_0 \pi + \sum_{j=1}^{n} c_j \log \alpha_j \]

is non-zero and hence, a transcendental number.

**Proof.** Suppose that \( T := \{\log \alpha_1, \cdots, \log \alpha_r\} \) is a maximal \( \mathbb{Q} \)-linearly independent set of \( \{\log \alpha_1, \log \alpha_2, \cdots, \log \alpha_n\} \) after re-labeling if necessary. Let us chose a branch of log such that

\[ i\pi = \log(-1). \quad (1.6) \]

On multiplying by \( i \), the given linear form can be re-written as

\[ c_0 \log(-1) + \sum_{j=1}^{r} d_j \log \alpha_j, \quad (1.7) \]

for some algebraic numbers \( d_j \).

Suppose \( b_0, b_1, \cdots, b_r \) are rational numbers such that \( b_0 \neq 0 \) and the linear form

\[ b_0 i\pi + \sum_{j=1}^{r} b_j \log \alpha_j = 0. \]

Thus,

\[ b_0 i\pi = -\sum_{j=1}^{r} b_j \log \alpha_j. \quad (1.8) \]

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Observe that the right hand side of the above equation is purely real whereas the left hand side is purely imaginary. This is possible only if both the sides are zero. In this case,

$$\sum_{j=1}^{r} b_j \log \alpha_j = 0. \quad (1.9)$$

This contradicts the $\mathbb{Q}$-linear independence of the set $T$. Therefore, the set $\{\log(-1), \log \alpha_1, \ldots, \log \alpha_r\}$ is $\mathbb{Q}$-linearly independent. By Baker’s Theorem 1.2, $\{1, \log(-1), \log \alpha_1, \ldots, \log \alpha_r\}$ is $\mathbb{Q}$-linearly independent. Thus, (1.7) is non-zero and hence transcendental. 

Remark. This proof also goes through when the branch of logarithm chosen in (1.6) is the principal branch. In that case, we replace $i\pi$ by $2\log i$ and proceed as before.
CHAPTER 2

The Chowla-Siegel theorem

In the early 1960s, Sarvadaman Chowla [9] considered the following problem: Let $p$ be a prime number and $f$ be a rational-valued function on the natural numbers that is periodic with period $p$ and not identically zero. Assume further that $f(p) = 0$ and that

$$\sum_{n=1}^{p} f(n) = 0. \quad (2.1)$$

Then is it true that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0 ?$$

In this context, let us define

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$ 

Chowla’s question can then be reformulated as determining the non-vanishing of $L(1, f)$ under the conditions $f(p) = 0$ and (2.1). As seen earlier, (2.1) implies the convergence of the series $\sum_{n=1}^{\infty} \frac{f(n)}{n}$.

Following an argument outlined by Siegel, Chowla proved this conjecture in the case that $f$ is an odd function, i.e, when $f(p-n) = -f(n)$. We present his proof in this chapter. We will also observe that his argument can be generalized with minor variations. The main idea here is to use the well-known cotangent expansion

$$\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{n + z}, \quad (2.2)$$

where $z \notin \mathbb{Z}$. We interpret the right hand side as

$$\frac{1}{z} + \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{z + n} + \frac{1}{z - n}.$$ 

Chowla proved:

**Theorem 2.1.** Let $f$ be a rational-valued arithmetical function, periodic with prime period $p$. Suppose $f$ is not identically zero and
$f(p) = 0$. Further assume that $f$ is odd. Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$ 

It was observed by Ram Murty [17] that the argument for the proof of the above theorem can be applied to a wider context. It can be used to prove:

**Theorem 2.2.** Let $q$ be a positive integer. Let $K$ be an algebraic number field which is disjoint from the $q^{th}$ cyclotomic field. Let $f$ be a $K$-valued arithmetical function, periodic with period $q$. Further suppose that $f$ is odd and that $f(n) = 0$ whenever $(n, q) > 1$. Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

unless $f$ is identically zero.

In a paper in 2011, Ram Murty [17] answered Chowla’s question when $f$ is an even function. In this paper, he tried to give a short argument towards the resolution of Chowla’s problem which may approximate the proof Chowla himself had in mind when he had written to Baker, Birch and Wirsing that he had solved the problem “to the extent stated above”. The ideas used in the proof were familiar to him and draw upon his earlier work. We include Ram Murty’s proof of Chowla’s conjecture when $f$ is an even function.

These theorems put together settle Chowla’s problem completely due to the lemma below. Before proceeding, let us fix the following notation: Let $f$ be any given function. The even part of $f$ is defined as

$$f_e(x) = \frac{f(x) + f(-x)}{2}.$$  

Similarly, define the odd part of $f$ as

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$  

Observe that $f_e$ is even, $f_o$ is odd and that $f = f_e + f_o$.

**Lemma 2.3.** Suppose $f$ is a rational-valued arithmetical function that is periodic with period $p$. Suppose $f(p) = 0$ and $\sum_{n=1}^{p} f(n) = 0$. Let $f_o$ denote the odd part of $f$ and $f_e$ denote the even part of $f$. Then,

$$L(1, f) = 0 \iff L(1, f_e) = 0 \text{ and } L(1, f_o) = 0.$$  

**Remark.** We will prove a similar result in the next chapter for functions that are periodic with period $q$, where $q$ is not necessarily prime and $f(q)$ is not necessarily zero.
2.1. Proof of Chowla’s theorem

Let us observe that for \( q = 2 \), the conditions \( f(2) = 0 \) and \( f(1) + f(2) = 0 \) imply that \( f \) is identically zero. Thus, we assume \( q > 2 \).

**Proof.** Since \( f \) is odd and \( f(q) = 0 \),

\[
2 \sum_{n=1}^{q} f(n) = \sum_{n=1}^{q} [f(n) + f(q-n)] = 0.
\]

Thus \( L(1, f) \) exists.

Now, suppose \( L(1, f) = 0 \). For a positive integer \( k \) such that \( (k, q) = 1 \), define

\[
S_k = \sum_{n \in \mathbb{Z}} \frac{f(kn)}{n},
\]

where the sum is to be interpreted as

\[
\lim_{N \to \infty} \sum_{n=1}^{N} \frac{f(kn)}{n} + \frac{f(-kn)}{(-n)}.
\]

Thus,

\[
S_1 = 2 \sum_{n=1}^{\infty} \frac{f(n)}{n}. \tag{2.3}
\]

We will show that

\[
S_1 = 0 \Rightarrow f \text{ is identically zero.}
\]

Since \( f(q) = 0 \),

\[
S_k = \sum_{a=1}^{q-1} f(ka) \sum_{n \equiv a \pmod{q}} \frac{1}{n}.
\]

We view the inner sum as

\[
\sum_{n \equiv a \pmod{q}} \frac{1}{n} = \frac{1}{a} + \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{a + nq} + \frac{1}{a - nq}
= \frac{1}{a} + \frac{1}{q} \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{a/q + n} + \frac{1}{a/q - n}
= \frac{\pi}{q} \cot \left( \frac{\pi a}{q} \right).
\]

Now,

\[
\frac{\pi}{q} \cot \left( \frac{\pi a}{q} \right) = \frac{i\pi}{q} \left( \frac{\zeta_q^a + 1}{\zeta_q^a - 1} \right) = -\frac{2\pi}{iq} \left( \frac{1}{2} + \frac{1}{\zeta_q^a - 1} \right),
\]

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where $\zeta_q^a = e^{2\pi ia/q}$. Thus,

$$-iS_k = \sum_{a=1}^{q} f(ka) \frac{2\pi}{q} \left( \frac{1}{2} + \frac{1}{\zeta_q^a - 1} \right).$$

Since $f$ is supported only on the co-prime residue classes mod $q$,

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(a) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(ka) = 0,$$

for $(k,q) = 1$. Hence,

$$\frac{-iq}{2\pi} S_k = \sum_{a=1}^{q} \frac{f(ka)}{\zeta_q^a - 1}.$$

In particular, for $k = 1$,

$$\frac{-iq}{2\pi} S_1 = \sum_{a=1}^{q} \frac{f(a)}{\zeta_q^a - 1}. \quad (2.4)$$

Let $\sigma_k$ be a Galois automorphism that sends $\zeta_q$ to $\zeta_q^{k'}$ for $kk' \equiv 1 \pmod{q}$. Observe that the right hand side of (2.4) is algebraic. Thus, applying $\sigma_k$ to the expression in (2.4), we have

$$\sigma_k \left( \frac{-iq}{2\pi} S_1 \right) = \sum_{a=1}^{q} \frac{f(a)}{\zeta_q^{k'a} - 1}$$

$$= \sum_{a=1}^{q} \frac{f(k'ka)}{\zeta_q^{k'a} - 1}$$

$$= \sum_{a=1}^{q} f(k'a) \zeta_q^a - 1$$

$$= -\frac{iq}{2\pi} S_k.$$

Therefore, $S_1 = 0$ implies that $S_k = 0$ for all positive integers $k$ that are coprime to $q$. Note that we use the hypothesis of $K$ being disjoint from the $q^{\text{th}}$ cyclotomic field. Also observe that this computation can also be used to calculate $L(1,\chi)$ when $\chi$ is an odd Dirichlet character mod $q$. Thus,

$$-\frac{iq}{2\pi} L(1,\chi) = \sum_{m=1}^{q} \frac{\chi(m)}{\zeta_q^m - 1}. \quad (2.5)$$

Now, $S_k = 0$ for all positive integers $k$ such that, $(k,q) = 1$. This implies that

$$0 = \sum_{k=1}^{q} \tilde{\chi}(k) S_k,$$
as $\chi$ is supported on coprime residue classes mod $q$. Thus,

$$0 = \sum_{k=1}^{q} \bar{\chi}(k)S_k$$

$$= \sum_{k=1}^{q} \bar{\chi}(k)\sum_{a=1}^{q} \frac{f(ka)}{\zeta^a - 1}$$

$$= \sum_{k=1}^{q} \sum_{a=1}^{q} f(ka) \frac{\bar{\chi}(ka)\chi(a)}{\zeta^a - 1}.$$

Now put $ka \equiv b \pmod{q}$ to get

$$0 = \sum_{b=1}^{q} f(b)\bar{\chi}(b)\sum_{k=1}^{q} \frac{\chi(k'b)}{\zeta^{k'b} - 1}.$$

As $k$ runs over all coprime residue classes mod $q$, so does $k'b$. Thus, by (2.5),

$$0 = L(1,\chi)\left(\sum_{b=1}^{q} f(b)\bar{\chi}(b)\right).$$

The non-vanishing of $L(1,\chi)$ for non-principal Dirichlet characters leads us to conclude that

$$\sum_{b=1}^{q} f(b)\bar{\chi}(b) = 0,$$

for odd characters $\chi \mod q$. Let us observe that (2.6) holds for even Dirichlet characters too. This is because

$$\sum_{b=1}^{q} f(b)\bar{\chi}(b) = \sum_{b=1}^{q} f(-b)\bar{\chi}(-b)$$

$$= -\sum_{b=1}^{q} f(b)\bar{\chi}(b).$$

Since a Dirichlet character is either odd or even, for all Dirichlet characters $\chi \mod q$,

$$\sum_{b=1}^{q} f(b)\bar{\chi}(b) = 0.$$

Hence,

$$0 = \sum_{\chi \mod q} \chi(a)\left(\sum_{b=1}^{q} f(b)\bar{\chi}(b)\right)$$

$$= \sum_{b=1}^{q} f(b)\left(\sum_{\chi \mod q} \chi(a)\bar{\chi}(b)\right).$$
By orthogonality relations, the inner sum is $\phi(q)$ whenever $b = a$ and 0 otherwise. Thus,

$$0 = \phi(q)f(a), \text{ for all } a \in \mathbb{Z}.$$Therefore, $f$ is identically 0.

\[\square\]

2.2. Non-vanishing for even functions

In this section, we prove the following theorem about the non-vanishing of $L(1, f)$ for even functions, $f$.

**Theorem 2.4.** Let $f$ be an even rational-valued arithmetical function, periodic with prime period $p$. Further assume that $f(p) = 0$. Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Let us note as before that we only need to consider the case when $p > 2$.

**Proof.** By (1.3), we know that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\sum_{x=1}^{p-1} \hat{f}(x) \log(1 - \zeta_p^x).$$

Since $f$ is even, so is $\hat{f}$, i.e., $\hat{f}(p - x) = \hat{f}(x)$. Thus,

$$L(1, f) = -2 \sum_{x=1}^{(p-1)/2} \hat{f}(x) \log |1 - \zeta_p^x|.$$Now, $0 = f(p) = \sum_{x=1}^{p} \hat{f}(x)$ implies that $\sum_{x=1}^{(p-1)/2} \hat{f}(x) = 0$ as $f$ is even. Thus,

$$\sum_{x=1}^{(p-1)/2} \hat{f}(x) \log |1 - \zeta_p| = 0.$$Hence,

$$L(1, f) = -2 \sum_{x=1}^{(p-1)/2} \hat{f}(x) \log \left| \frac{1 - \zeta_p^x}{1 - \zeta_p} \right|.$$Now observe that

$$\left| \frac{1 - \zeta_p^x}{1 - \zeta_p} \right| = \left| \frac{e^{-i\pi x/p} - e^{i\pi x/p}}{e^{-i\pi} - e^{i\pi}} \right| = \left| \frac{\sin \pi x/p}{\sin \pi/p} \right|.$$Therefore,

$$L(1, f) = -2 \sum_{x=1}^{(p-1)/2} \hat{f}(x) \log \left( \frac{\sin \pi x/p}{\sin \pi/p} \right). \quad (2.7)$$
REMARK. This expression for $L(1, f)$ also holds in the case when the period of the function is not necessarily prime. Suppose $f$ is an arithmetical function, periodic with period $q$ and $f(q) = 0$. Then,

$$L(1, f) = -2 \sum_{x=1}^{\lfloor \frac{q-1}{2} \rfloor} \hat{f}(x) \log \left( \frac{\sin \pi x/q}{\sin \pi/q} \right).$$

By Lemma 8.1 from [22], the algebraic numbers

$$\left\{ \frac{\sin \pi x/p}{\sin \pi/p} : 1 \leq x \leq \frac{p-1}{2} \right\}$$

are $\mathbb{Q}$-linearly independent. By Baker’s Theorem 1.2, they are $\overline{\mathbb{Q}}$-linearly independent. Thus, the right hand side of (2.7) is not zero. Hence, $L(1, f) = 0 \iff \hat{f}(x) = 0$ for all $1 \leq x \leq p$. This proves the theorem.

2.3. Resolution of Chowla’s problem

We have proved the non-vanishing of $L(1, f)$ when $f$ is either an odd or an even function. We can thus resolve Chowla’s question with the help of the following lemma:

**Lemma 2.5.** Suppose $f$ is a rational-valued function that is periodic mod $p$. Suppose $f(p) = 0$ and $\sum_{n=1}^{p} f(n) = 0$. Let $f_o$ denote the odd part of $f$ and $f_e$ denote the even part of $f$. Then,

$$L(1, f) = 0 \iff L(1, f_e) = 0$$

and

$$L(1, f_o) = 0.$$

**Proof.** Given the function $f$, we can write it as a sum of an even function and an odd function as seen before. Let $f = f_e + f_o$. Thus,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{\infty} \frac{f_e(n)}{n} + \sum_{n=1}^{\infty} \frac{f_o(n)}{n}.$$

If $L(1, f_e) = 0$ and $L(1, f_o) = 0$, then clearly $L(1, f) = 0$.

Now, suppose $L(1, f) = 0$. By the Chowla-Siegel theorem, the first sum is an algebraic multiple of $\pi$. By (2.7), the second sum is a linear form in logarithms of positive algebraic numbers. If $L(1, f_o) \neq 0$, then by Lemma 1.5, $L(1, f) \neq 0$. Thus, $L(1, f_o) = 0$ and in turn, $L(1, f_e) = 0$.

This completes the solution to Chowla’s problem when the function $f$ is periodic with prime period $p$ and $f(p) = 0$. The case when $f(p) \neq 0$ can also be dealt with as follows.

Define a function $g$ that is periodic with period $p$ such that

$$g(1) = g(2) = \cdots = g(p-1) = 1$$
and $g(p) = -(p - 1)$. Thus, for $1 \leq a \leq p - 1$,

$$
\hat{g}(a) = \frac{1}{p} \sum_{b=1}^{p-1} \zeta_p^{-ab} - (p - 1) = -1.
$$

Thus by (1.3),

$$
L(1, g) = \sum_{a=1}^{p-1} \log(1 - \zeta_p^a).
$$

Since,

$$
\prod_{a=1}^{p-1} (1 - \zeta_p^a) = p,
$$

we conclude that $L(1, g) = \log p$.

Suppose $f$ is an arithmetical function that is periodic mod $p$ with $\sum_{n=1}^{p} f(n) = 0$ and $f(p) \neq 0$. Assume further that $L(1, f) = 0$. Now define

$$
\tilde{f} = (p - 1)f + f(p)g.
$$

Note that $\tilde{f}(p) = (p - 1)f(p) - f(p)(p - 1) = 0$. Also observe that

$$
L(1, \tilde{f}) = f(p)L(1, g) = f(p)\log p \neq 0.
$$

On the other hand, $L(1, \tilde{f}) = L(1, \tilde{f}_e) + L(1, \tilde{f}_o)$. By the Chowla-Siegel theorem, $L(1, \tilde{f}_o)$ is an algebraic multiple of $\pi$. Also, $L(1, \tilde{f}_e)$ is a linear form in logarithms of multiplicatively independent units in the $p^{th}$ cyclotomic field. By Baker’s Theorem 1.4, $L(1, \tilde{f}) \neq 0$ implies that $L(1, \tilde{f})$ is transcendental and hence, the logarithms of multiplicatively independent units, $\log(-1)$ and $\log(p)$ are $\mathbb{Q}$-linearly dependent. Thus, $p$ can be written as a product of units. This is a contradiction. Hence, if $L(1, f) = 0$ then $f(p) = 0$. We are now reduced to the cases proved earlier. This completes the discussion of Chowla’s problem.
CHAPTER 3

A vanishing criterion for $L(1, f)$

As seen in the last chapter, Chowla had resolved the non-vanishing of $f$ when $f$ is odd and $f(p) = 0$, where $p$ is the period of $f$ and $p$ is prime. Chowla’s problem was resolved in more generality by Baker, Birch and Wirsing in 1972. In the same paper, they proved another remarkable theorem. They derived a necessary and sufficient condition for odd, algebraic-valued periodic functions $f$ that satisfy $L(1, f) = 0$. This suggests a change in perspective. Instead of proving the non-vanishing, another approach to Chowla’s problem is to characterize all functions $f$ for which $L(1, f) = 0$. Pursuing this approach, we present the characterization of odd functions as given by Baker, Birch and Wirsing in the first section. In the next, we apply a theorem of Bass towards the characterization of even functions. In the last section, we prove a theorem similar to the one in the first chapter which says that it is enough to consider only the even and the odd functions in order to obtain a complete characterization.

Let us first observe that for $q = 2$,

$$L(1, f) = f(1) \log(2),$$

which is non-zero if and only if $f(1) \neq 0$. Since, $f(1) + f(2) = 0$, this means that $L(1, f) = 0$ if and only if $f$ is identically zero. Hence, we will assume $q > 2$.

3.1. Odd functions

In this section, we reproduce a simple argument of Baker, Birch and Wirsing [5] that gives us a necessary and sufficient condition on the function $f$ such that $L(1, f) = 0$.

**Theorem 3.1.** Let $f$ be an odd algebraic-valued arithmetical function, periodic with period $q$. Then

$$L(1, f) = 0 \iff \sum_{x=1}^{q-1} x \hat{f}(x) = 0.$$

Here $\hat{f}$ denotes the Fourier transform of $f$ as defined in chapter one.

**Proof.** Let us note that

$$1 - \zeta_q^x = -(\zeta_q^{x/2} - \zeta_q^{-x/2})\zeta_q^{x/2} = -2i \left( \sin \left( \frac{x\pi}{q} \right) \right) e^{x\pi i/q},$$

(3.1)
and so the principal value of the logarithm is
\[
\log(1 - \zeta_q^x) = \log \left( 2 \sin \frac{x\pi}{q} \right) + \left( \frac{x}{q} - \frac{1}{2} \right) \pi i, \tag{3.2}
\]
for \(1 \leq x < q\). Substituting (3.2) in the expression for \(L(1, f)\) as a linear form in logarithms (1.3), we get
\[
L(1, f) = -\sum_{x=1}^{q-1} \hat{f}(x) \left[ \log \left( 2 \sin \frac{x\pi}{q} \right) + \left( \frac{x}{q} - \frac{1}{2} \right) \pi i \right]
\]
\[
= -\sum_{x=1}^{q-1} \hat{f}(x) \log \left( 2 \sin \frac{x\pi}{q} \right) - \frac{i\pi}{q} \sum_{x=1}^{q-1} x \hat{f}(x) + \frac{i\pi}{2} \sum_{x=1}^{q-1} \hat{f}(x). \tag{3.3}
\]
Since \(f\) is an odd function, \(\hat{f}\) is also an odd function. Hence,
\[
2 \sum_{x=1}^{q-1} \hat{f}(x) = \sum_{x=1}^{q-1} \left[ \hat{f}(x) + \hat{f}(q-x) \right] = 0.
\]
Therefore, the last term of (3.3) is zero. Now, note that \(\sin(\pi - \theta) = \sin(\theta)\). Thus, \(\sin(x\pi/q)\) is an even function. Hence, \(\log(2 \sin \frac{x\pi}{q})\) is even which implies that \(\hat{f}(x) \log(2 \sin \frac{x\pi}{q})\) is an odd function.
Therefore, the first term of (3.3),
\[
\sum_{x=1}^{q-1} \hat{f}(x) \log \left( 2 \sin \frac{x\pi}{q} \right) = 0.
\]
The result is immediate from here. \(\square\)

**Remark 1.** The condition obtained above is on the Fourier transform of the function and not the function itself. Using the Fourier inversion formula, we can deduce a condition on the function. The condition obtained in Theorem 3.1 can be simplified to
\[
\sum_{x=1}^{q-1} x \hat{f}(x) = \frac{1}{q} \sum_{x=1}^{q-1} x \sum_{n=1}^{q} f(n) \zeta_q^{-nx}
\]
\[
= \frac{1}{q} \sum_{n=1}^{q} f(n) \sum_{x=1}^{q-1} x \zeta_q^{-nx}
\]
\[
= \frac{1}{q} \sum_{n=1}^{q-1} f(n) \sum_{x=1}^{q-1} x \zeta_q^{-nx},
\]
as \(f(q) = f(0) = f(-q) = -f(q) = 0\) since \(f\) is odd. The innermost sum can be evaluated as follows. Let \(T\) be an indeterminate. Observe that
\[
\sum_{x=0}^{q-1} T^x = \frac{T^q - 1}{T - 1}. \tag{3.4}
\]
Differentiating (3.4) with respect to $T$, we have

$$\sum_{x=1}^{q-1} x T^{x-1} = qT^{q-1} - \frac{T^q - 1}{T - 1}.$$  

Multiplying the above equation by $T$ and substituting $T = \zeta_q^{-n}$, we get

$$\sum_{x=1}^{q-1} x \zeta_q^{-nx} = \frac{q}{\zeta_q^n - 1}.$$  

This observation along with Theorem 3.1 gives: for $f$ odd,

$$L(1, f) = 0 \iff \sum_{n=1}^{q-1} \frac{f(n)}{1 - \zeta_q^n} = 0.$$  

(3.5)

Let us note that

$$\frac{1}{1 - \zeta_q^n} = \frac{i}{2} \cot \left( \frac{n\pi}{q} \right) + \frac{1}{2}$$

and that since $f$ is odd,

$$2 \sum_{n=1}^{q-1} f(n) = \sum_{n=1}^{q-1} [f(n) + f(q - n)] = 0.$$  

Hence, the condition (3.5) can be translated as: For odd algebraic-valued periodic functions $f$,

$$L(1, f) = 0 \iff \sum_{n=1}^{q-1} f(n) \cot \left( \frac{n\pi}{q} \right) = 0.$$

We would like to mention that Baker, Birch and Wirsing also obtained a basis for the $\bar{\mathbb{Q}}$-vector space of odd algebraic-valued arithmetical functions $f$, periodic with period $q$ and $L(1, f) = 0$ in their paper [5]. Thus, the characterization of odd functions is complete.

### 3.2. Even functions

As seen earlier, the non-vanishing of $L(1, f)$ when $f$ is odd and algebraic valued has been settled. It still remains whether the same is true for even periodic functions. In this section, we will use a theorem of Bass that characterizes all the multiplicative relations among cyclotomic numbers modulo torsion ([6], [11]). We will give a necessary condition for even algebraic-valued periodic functions $f$ to satisfy $L(1, f) = 0$.  

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We define the following functions that serve as building blocks for even algebraic-valued functions, periodic with period $q$ whose $L$-series vanish at $s = 1$. Fix a divisor $d$ of $q$. For $1 \leq c \leq d - 1$, we define $F_{d,c} := F_{d,c}^{(1)} - F_{d,c}^{(2)}$ where the two functions, $F_{d,c}^{(1)}$ and $F_{d,c}^{(2)}$ are arithmetical functions, periodic with period $q$. They are defined as follows:

$$F_{d,c}^{(1)}(x) = \begin{cases} 1/2 & \text{if } x \equiv c \mod d, \\ 0 & \text{otherwise}. \end{cases}$$

$$F_{d,c}^{(2)}(x) = \begin{cases} 1/2 & \text{if } x \equiv (\frac{q}{d})c \mod q, \\ 0 & \text{otherwise}. \end{cases}$$

We will prove the following result in this section:

**Theorem 3.2.** Let $f$ be an algebraic valued, even function which is periodic with period $q$. If $L(1,f) = 0$, then $f$ is an algebraic linear combination of the functions $\{\hat{F}_{d,c}\}$ any divisor $d$ of $q$, $1 < d < q$, $1 \leq c \leq d - 1$.

Here, $\hat{F}_{d,c}$ denotes the Fourier transform of $F_{d,c}$ which can be computed as follows: For $1 \leq y \leq q$,

$$\hat{F}_{d,c}^{(1)}(y) = \frac{1}{q} \sum_{a=1}^{q} F_{d,c}^{(1)}(a) \zeta_{q}^{-ay}$$

$$= \frac{1}{2q} \sum_{j=0}^{\frac{q}{d}-1} \zeta_{q}^{-(c+dj)y}$$

$$= \frac{\zeta_{q}^{-y}}{2q} \sum_{j=0}^{\frac{q}{d}-1} \zeta_{q}^{-dyj}$$

$$= \frac{\zeta_{q}^{-y}}{2q} \sum_{j=0}^{\frac{q}{d}-1} \zeta_{\frac{q}{d}}^{-jy}.$$  

Note that the sum

$$\sum_{j=0}^{\frac{q}{d}-1} \zeta_{q/d}^{-jy} = \begin{cases} \frac{q}{d} & \text{if } y \equiv 0 \mod \frac{q}{d}, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, for $1 \leq y \leq q$,

$$\hat{F}_{d,c}^{(2)}(y) = \frac{1}{q} \sum_{a=1}^{q} F_{d,c}^{(2)}(a) \zeta_{q}^{-ay}$$

$$= \frac{1}{2q} \zeta_{q}^{-\frac{2cy}{d}} = \frac{1}{2q} \zeta_{d}^{-cy}.  $$
More precisely,
\[ \hat{F}_{d,c}(y) = \begin{cases} 
\frac{\zeta^{cy}}{2d} - \frac{\zeta^{cy}}{2q} & \text{if } y \equiv 0 \mod \frac{q}{d}, \\
-\frac{1}{2q^{cy}} & \text{otherwise.}
\end{cases} \]

Thus, we note that \( \hat{F}_{d,c} \)'s are in fact, simple functions.

**Proof.** Let \( f \) be an even algebraic-valued arithmetical, periodic function with period \( q \), not identically zero. Let \( M := \mathbb{Q}(f(1), \cdots, f(q), \zeta_q) \). Let \( \{\omega_1, \omega_2, \cdots, \omega_r\} \) be a basis for \( M \) over \( \mathbb{Q} \). Then there exists \( d_j(x) \in \mathbb{Q} \) such that,

\[ \hat{f}(x) = \sum_{j=1}^{r} d_j(x)\omega_j. \]

We can choose an integer \( N \) such that \( \forall 1 \leq x \leq q, \forall 1 \leq j \leq r, \)
\( c_j(x) := Nd_j(x) \in \mathbb{Z} \). Hence, \( N\hat{f}(x) = \sum_{j=1}^{r} c_j(x)\omega_j \). Thus,

\[ NL(1, f) = -\sum_{x=1}^{q-1} \sum_{j=1}^{r} \omega_j c_j(x) \log (1 - \zeta_q^x) \]

\[ = -\sum_{j=1}^{r} \omega_j \sum_{x=1}^{q-1} c_j(x) \log (1 - \zeta_q^x) \]

Let
\[ R_j := \sum_{x=1}^{q-1} c_j(x) \log (1 - \zeta_q^x). \]

Therefore,
\[ -NL(1, f) = \sum_{j=1}^{r} \omega_j R_j. \]

As \( \hat{f} \) is even and \( \omega_1, \cdots, \omega_r \) is a basis,
\[ c_j(x) = c_j(q - x). \]

Therefore,
\[ R_j = \sum_{x=1}^{\left\lfloor \frac{q-1}{2} \right\rfloor} c_j(x) [\log(1 - \zeta_q^x) + \log(1 - \zeta_q^{-x})]. \]

Note that \( \log \) denotes the principal branch of logarithm. Thus, \( \arg(1 - \zeta_q^x) = -\arg(1 - \zeta_q^{-x}). \) For \( 1 \leq x \leq q - 1 \), define \( a_x := \log(|1 - \zeta_q^x|). \) Thus,
\[ R_j = \sum_{x=1}^{\left\lfloor \frac{q-1}{2} \right\rfloor} 2c_j(x)a_x. \]

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Since \( c_j(x) = c_j(q - x) \), \( R_j \) can be written as
\[
R_j = \sum_{x=1}^{q-1} c_j(x) a_x = \log \left( \prod_{x=1}^{q-1} (|1 - \zeta_q^x|)^{c_j(x)} \right).
\]
Let
\[
\alpha_j := \prod_{x=1}^{q-1} (|1 - \zeta_q^x|)^{c_j(x)}.
\]
Therefore,
\[
(-N)L(1, f) = \sum_{j=1}^{r} \omega_j \log \alpha_j. \tag{3.6}
\]

Let us note that \( \alpha_j \) is non-zero, algebraic. Thus, by Corollary 1.4, if \( \alpha_j \neq 1 \) for some \( 1 \leq j \leq r \), then \( L(1, f) \) will be transcendental and hence non-zero. But \( L(1, f) = 0 \) by assumption.

Hence, \( \alpha_j = 1 \) and in turn \( R_j = 0 \), \( \forall \ 1 \leq j \leq r \). Thus, we are led to consider relations among logarithms of the cyclotomic numbers, \( 1 - \zeta_q^x \).

In an unpublished paper, Milnor conjectured the complete set of multiplicative relations among cyclotomic numbers in the set \( \{1 - \zeta_q^x : 1 \leq x \leq q - 1\} \) for a fixed positive integer \( q \). Here \( \zeta_q^x \) denotes a primitive \( q^{th} \) root of unity. This conjecture was proved by Hyman Bass [6] in 1965. In 1972, Veikko Ennola [11] realized that the conjecture was true up to a factor of 2 and not in general. He gave a different proof of the conjecture in his paper. Since his formulation of the theorem is easier to apply in our setting, we will state it here. Let, as before, \( a_x := \log(|1 - \zeta_q^x|) \). The following theorem characterizes all additive relations among the numbers \( \{a_x|1 \leq x \leq q - 1\} \).

**Theorem 3.3.** Consider the following two relations: For \( 1 \leq x \leq [(q-1)/2] \),
\[
R_1 : a_x - a_{q-x} = 0 \tag{3.7}
\]
and for any divisor \( d \) of \( q \) and \( 1 < d < q \) and \( 1 \leq c \leq d - 1 \),
\[
R_2 : a_{\frac{q}{d} c} - \sum_{j=0}^{\frac{q}{d} - 1} a_{c+dj} = 0. \tag{3.8}
\]
Let \( R \) be an additive relation among the \( a_x \)'s over the integers. Then, \( 2R \) is a \( \mathbb{Z} \)-linear combination of relations of the form \( R_1 \) and \( R_2 \).

Let \( \mathcal{R} \) denote the relation \( 2R_j = 0 \). By Theorem 3.3, \( \mathcal{R} \) belongs to the \( \mathbb{Z} \)-module generated by relations of the form (3.7) and (3.8). Since \( c_j(x) = c_j(q - x) \), \( \mathcal{R} \) belong to the \( \mathbb{Z} \)-module generated by (3.8). Indeed all relations \( R := \sum_{x=1}^{q-1} C_x a_x = 0 \) in the \( \mathbb{Z} \)-module generated by (3.7) satisfy \( C_x = -C_{q-x} \), which along with the fact that \( C_j(x) = C_j(q - x) \) (which stems from the evenness of \( f \)) imply that \( c_j(x) = 0, \ \forall \ 1 \leq x \leq q - 1 \). Thus, \( \mathcal{R} \) is a \( \mathbb{Z} \)-linear combination of (3.8). Observe that the
functions $F_{d,c}$ are precisely those that represent the relation (3.8). This implies that the functions $c_j$ are integer linear combinations of $F_{d,c}$, say

$$c_j(x) = \sum_{d \mid q} \sum_{1 < d < q} \sum_{c=1}^{d-1} m_{j,d,c} F_{d,c}(x),$$

where $m_{j,d,c} \in \mathbb{Z}$. Then,

$$N\hat{f}(x) = \sum_{j=1}^{r} c_j(x) \omega_j = \sum_{j=1}^{r} \omega_j \sum_{d \mid q} \sum_{1 < d < q} \sum_{c=1}^{d-1} m_{j,d,c} F_{d,c}(x).$$

Taking the Fourier transform of both sides and using the Fourier inversion formula, the result follows. □

Remark. On examining the above proof, we observe that we also obtain a sufficiency condition for $L(1, f) = 0$. More precisely, suppose $f$ is a given algebraic linear combination of the functions

$$\{ \hat{F}_{d,c} | \text{any divisor } d \text{ of } q, 1 < d < q, 1 \leq c \leq d - 1 \},$$

such that $f$ is an even function, then $L(1, f) = 0$. But it does not seem to be easy to determine whether a given function is an algebraic linear combination of these particular functions.

As a corollary, we have:

**Corollary 3.4.** When $q$ is prime, there are no algebraic-valued even functions $f$ such that $L(1, f) = 0$.

**Proof.** If $q$ if prime, the set

$$\{ \hat{F}_{d,c} | \text{any divisor } d \text{ of } q, 1 < d < q, 1 \leq c \leq d - 1 \}$$

is an empty set. The conclusion follows from Theorem 3.2. □

### 3.3. Combining the even and the odd

The previous two sections give a characterization of algebraic-valued periodic arithmetic functions, odd and even respectively whose $L$-series vanish at $s = 1$. In this section, we mention a theorem of Ram Murty and Tapas Chatterjee [8] which ties the non-vanishing of the odd and the even functions to give us the characterization required. This is similar to Lemma 2.5. The proof in their paper is incorrect as care was not taken about the branch of logarithm. The proof given below
follows a different argument.

As before, given any function $f$, we write it as $f = f_o + f_e$, where $f_o$ is the odd part of $f$ and $f_e$ is the even part of $f$.

**Theorem 3.5.** If $f$ is an algebraic-valued arithmetical function, periodic with period $q$, then $L(1, f) = 0 \iff L(1, f_o) = 0$ and $L(1, f_e) = 0$.

**Proof.** Let us note that

$$
\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{\infty} \frac{f_e(n)}{n} + \sum_{n=1}^{\infty} \frac{f_o(n)}{n}.
$$

Thus, it is clear that if $L(1, f_o)$ and $L(1, f_e)$ are both zero, then so is $L(1, f)$.

Now, suppose that $L(1, f) = 0$. The arguments in the first section of this chapter imply

$$
L(1, f_o) = -\frac{i\pi}{q} \sum_{x=1}^{q-1} \hat{f}_o(x).
$$

We claim that,

$$
L(1, f_e) = -2 \sum_{x=1}^{\left\lceil \frac{(q-1)}{2} \right\rceil} \hat{f}_e(x) \log \left( 2 \sin \left( \frac{x\pi}{q} \right) \right). \quad (3.9)
$$

Indeed, by (1.3), we know that

$$
\sum_{n=1}^{\infty} \frac{f_e(n)}{n} = -\sum_{x=1}^{q-1} \hat{f}_e(x) \log(1 - \zeta_q^x).
$$

Since $f_e$ is even, so is $\hat{f}_e$, i.e., $\hat{f}_e(q-x) = \hat{f}_e(x)$. Thus,

$$
L(1, f_e) = -2 \sum_{x=1}^{\left\lceil \frac{(q-1)}{2} \right\rceil} \hat{f}_e(x) \log |1 - \zeta_q^x|.
$$

By (3.1),

$$
|1 - \zeta_q^x| = \left| 2 \sin \left( \frac{x\pi}{q} \right) \right|.
$$

Hence,

$$
L(1, f_e) = -2 \sum_{x=1}^{\left\lceil \frac{(q-1)}{2} \right\rceil} \hat{f}_e(x) \log \left| 2 \sin \left( \frac{x\pi}{q} \right) \right|.
$$

Note that, for $1 \leq x \leq \left\lfloor \frac{(q-1)}{2} \right\rfloor$, $\sin(x\pi/q) > 0$. 

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Thus (3.3) and (3.9) imply
\[
L(1, f_o) = -i \left( \sum_{x=1}^{q-1} x \hat{f}_o(x) \right) \pi - 2 \sum_{x=1}^{\left\lfloor \frac{(q-1)}{2} \right\rfloor} \hat{f}_e(x) \log \left( 2 \sin \frac{x \pi}{q} \right). \tag{3.10}
\]

Suppose \( L(1, f_o) \neq 0 \). By Lemma 1.5, \( L(1, f) \neq 0 \), which contradicts our assumption. Thus, \( L(1, f_o) = 0 \) and hence, \( L(1, f_e) = 0 \). □

This gives a characterization of algebraic-valued periodic arithmetical functions \( f \) with \( L(1, f) = 0 \). But the conditions obtained in the previous characterizations do not seem to be very easy to verify.
CHAPTER 4

The Lerch zeta function

The Riemann zeta function and the Hurwitz zeta function are a couple of very well-studied functions in number theory. The Riemann zeta function was introduced by Euler (as a function of a real variable) in 1737 but was studied extensively by Riemann [20] in his 1859 paper, where he obtained the analytic continuation and functional equation for it. Riemann zeta function is given by the series
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \]
which converges absolutely for \( \Re(s) > 1 \). This function played a remarkable role in the study of the distribution of primes.

The Hurwitz zeta function was introduced by Hurwitz [12] in his 1882 paper. It depends on a real parameter, namely, \( x \). For \( 0 < x \leq 1 \), it is defined by the series
\[ \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}, \]
which converges absolutely for \( \Re(s) > 1 \). The Hurwitz zeta function is important in the study of Dirichlet \( L \)-functions. A major contribution of Riemann and Hurwitz was obtaining the analytic continuation of the respective zeta functions to the entire complex plane except for a simple pole at \( s = 1 \).

The Lerch zeta function was introduced independently by Lerch [14] in 1887 and Lipschitz [15] in 1889. It is a generalization of both the Riemann zeta and the Hurwitz zeta functions. It depends on two real parameters, \( \lambda \) and \( \alpha \). For \( \lambda \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \), the function is defined as
\[ L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}, \quad (4.1) \]
which converges for \( \Re(s) > 1 \) if \( \lambda \in \mathbb{Z} \) and for \( \Re(s) > 0 \) if \( \lambda \notin \mathbb{Z} \). In his 1887 paper, Lerch [14] obtained the analytic continuation and functional equation for \( L(\lambda, \alpha, s) \). In contrast to the Riemann and the Hurwitz zeta functions, the Lerch zeta function was not extensively studied, perhaps due to the lack of its evident applications at that time. But it has caught the attention of mathematicians in the last two decades and new results have started coming up in its theory.

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The following elementary observations can be made from the definition of $L(\lambda, \alpha, s)$:
1. For $\lambda \in \mathbb{Z}$, $L(\lambda, \alpha, s) = \zeta(s, \alpha)$, the Hurwitz zeta function.
2. For $\lambda \in \mathbb{Z}$ and $\alpha = 1$, $L(\lambda, 1, s) = \zeta(s)$, the Riemann zeta function.
3. For $\lambda = \frac{1}{2}$ and $\alpha = 1$, $L(\frac{1}{2}, 1, s) = \zeta(s)(1 - 2^{1-s})$.
4. For $\lambda \in \mathbb{Z}$ and $\alpha = \frac{1}{2}$, $L(\lambda, \frac{1}{2}, s) = \zeta(s)(2^s - 1)$.
Thus we see that values of the Lerch zeta function are intricately related to the values of the Riemann and the Hurwitz zeta functions.

A notable fact of the Lerch-zeta function as mentioned before is the following:

**Theorem 4.1.** For $\lambda \notin \mathbb{Z}$, the series (4.1) converges for $\Re(s) > 0$.

**Proof.** Without loss of generality, we can assume that $0 < \lambda < 1$. Fix such a $\lambda$. For $u > 0$, define the partial sums,

$$S(u) = \sum_{m \leq u} e^{2\pi i \lambda m}.$$

Since this is a geometric sum and $\lambda \notin \mathbb{Z}$,

$$|S(u)| = |S([u])| = \left| \frac{1 - e^{2\pi i \lambda [u]+1}}{1 - e^{2\pi i \lambda}} \right| \leq \frac{2}{|1 - e^{2\pi i \lambda}|}. \tag{4.2}$$

Recall the partial summation or the Abel summation formula that says:

**Theorem.** Let $a_n$ be a sequence of complex numbers and $f$ be $C^1$ function on $\mathbb{R}$. For $x > 0$, if $A(x) := \sum_{n \leq x} a_n$, then

$$\sum_{0 \leq m \leq x} a_m f(m) = A(x)f(x) - \int_0^x A(t)f'(t) dt.$$

This implies

$$\sum_{0 \leq m \leq x} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s} = S(x)(x + \alpha)^{-s} + s \int_0^x \frac{S(u)}{(u + \alpha)^{s+1}} du. \tag{4.3}$$

Suppose $\Re(s) > 1$. Let $x \to \infty$. Then (4.2) and (4.3) imply,

$$L(\lambda, \alpha, s) = s \int_0^\infty \frac{S(u)}{(u + \alpha)^{s+1}} du. \tag{4.4}$$

The integral on the right hand side of the above equation converges uniformly on compact subsets of the half plane $\Re(s) > 0$. This, together with (4.3) shows that the series

$$\sum_{m=0}^\infty \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}$$

converges uniformly on compact subsets of $\Re(s) > 0$. \qed

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In this chapter, we will give a proof of the analytic continuation of the Lerch zeta function and its functional equation. We will also observe a few new results about the transcendence of its certain special values. Since the Hurwitz zeta function has already been well-studied, we will only consider the case \( \lambda \notin \mathbb{Z} \) in this chapter. Indeed, we can then assume \( 0 < \lambda < 1 \).

### 4.1. Analytic continuation

**Theorem 4.2.** For \( 0 < \lambda < 1 \) and \( 0 < \alpha \leq 1 \), the Lerch zeta function, \( L(\lambda, \alpha, s) \) defined by the series (4.1) for \( \Re(s) > 0 \), can be analytically continued to an entire function of \( s \).

**Proof.** The proof given here is a reproduction of the proof by B. C. Berndt published in 1972 [7]. Let \( s > 1 \) be real. Put \( \lambda = b + \frac{1}{2} \). Hence, \( |b| < 1/2 \). Define the function
\[
F_s(z) = \frac{\pi e^{2\pi i b z}}{\sin(\pi z)(z + \alpha)^s}. \tag{4.5}
\]
Observe that \( F_s \) is analytic in the entire complex plane except for simple poles at \(-\alpha\) and the integers. Let \( k \in \mathbb{Z} \). The residue of \( F_s \) at \( k \),
\[
\text{Res}_{z=k} F_s(z) = \lim_{z \to k} (z - k) F_s(z) = \lim_{z \to k} \frac{(z - k) \pi e^{2\pi i b z}}{\sin(\pi z)(z + \alpha)^s} = \frac{\pi e^{2\pi i b k}}{(k + \alpha)^s \cos(\pi k)}.
\]
Since \( \cos(\pi k) = \pm 1 \), \( \frac{1}{\cos(\pi k)} = e^{2\pi i (k/2)} \),
\[
\text{Res}_{z=k} F_s(z) = e^{2\pi i \lambda k} \tag{4.6}
\]
Choose a real number \( c \) such that \(-\alpha < c < 0\).

For a positive integer \( m \), define the contour \( C_m \) as the positively oriented contour consisting of the right half circle with center \((c,0)\) and radius \((m + \frac{1}{2} - c)\) together with the vertical diameter through \((c,0)\). Let \( \Gamma_m \) denote the curved part of the contour.

By Cauchy’s Residue theorem,
\[
\frac{1}{2\pi i} \int_{C_m} F_s(z)dz = \sum_{k=0}^{m} \text{Res}_{z=k} F_s(z).
\]
Figure 1. The contour $C$
By (4.6),
\[
\frac{1}{2\pi i} \int_{C_m} F_s(z)dz = \sum_{k=0}^{m} e^{2\pi i \lambda k} \frac{c^{2\pi i \lambda k}}{(k + \alpha)^s}.
\] (4.7)

Since we would like to take limit as \( m \) tends to \( \infty \), we need to study the integral along \( \Gamma_m \). Since \( b \leq 1/2 \), \( \left| e^{2\pi ibz}/\sin \pi z \right| \) is bounded independent of \( m \) on \( \Gamma_m \). To see this, put \( y := \Im(z) \) and observe that:
\[
\left| \frac{e^{2\pi ibz}}{\sin \pi z} \right| = 2 \left| \frac{e^{2\pi ibz}}{e^{\pi z} - e^{-\pi z}} \right| \leq 2 \left| \frac{e^{-2\pi by}}{e^{-\pi y} - e^{\pi y}} \right| \leq c' \left| \frac{e^{\pi y}}{e^{-\pi y} - e^{\pi y}} \right|
\]
for some constant \( c' \). The above function is of the form \( t^2/|1-t^2| \) which is bounded by 1 for \( t \) sufficiently large.

Thus, there exists \( M \) such that for \( z \in \Gamma_m \),
\[
\left| \frac{e^{2\pi ibz}}{\sin \pi z} \right| \leq M.
\]
Thus,
\[
\left| \int_{\Gamma_m} F_s(z)dz \right| \leq \pi M \int_{\Gamma_m} \frac{1}{|z + \alpha|^s} dz.
\]
Recall that \( s \in \mathbb{R} \). Note that, on \( \Gamma_m \),
\[
|z + \alpha|^2 \geq (c + \alpha)^2 + \left( m + \frac{1}{2} - c \right)^2 \geq \left( m + \frac{1}{2} \right)^2.
\]
Hence,
\[
\left| \int_{\Gamma_m} F_s(z)dz \right| \leq \frac{\pi^2 M}{(m + 1/2)^s} \left( m + \frac{1}{2} - c \right).
\]
The right hand side of the above expression tends to zero as \( m \) tends to \( \infty \) since \( s > 1 \). Thus, taking limit as \( m \) tends to infinity in (4.7) gives:
\[
-\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_s(z)dz = L(\lambda, \alpha, s).
\]
Using the identity \( \sin \pi z = (e^{i\pi z} - e^{-i\pi z})/2i \), the above expression can be re-written as:

\[
L(\lambda, \alpha, s) = \int_{c-i\infty}^{c+i\infty} \frac{e^{2\pi ibz + i\pi z}}{e^{2\pi iz} - 1} (z + \alpha)^{-s} dz + \\
\int_{c-i\infty}^{c+i\infty} \frac{e^{-2\pi ibz - i\pi z}}{e^{2\pi iz} - 1} (z + \alpha)^{-s} dz. \quad (4.8)
\]

Observe that following an earlier argument, \( |b| < 1/2 \) implies that both the integrals in (4.8) converge uniformly on compact subsets of \( \mathbb{C} \) and thus define an analytic function. In this way, \( L(\lambda, \alpha, s) \) can be extended to an entire function. \( \square \)

### 4.2. Functional Equation

The Lerch zeta function satisfies a rather curious functional equation. It was first proved by Lerch \[14\] in 1887. The proof given below uses elementary contour integration. This was also given by B. Berndt in 1972 \[7\]. Other interesting proofs of the functional equation are given by:

(i) T. Apostol (1951) \[2\] - This proof follows the argument used by Riemann to prove the functional equation for the Riemann Zeta function. It uses theta-functions.

(ii) Oberhettinger (1959) \[19\] - This proof uses the Poisson summation formula.

(iii) M. Mikolás (1971) \[16\] - This proof uses the Fourier series expansion of

\[
L(\lambda, \alpha, 1-s)e^{2\pi i\lambda}, \quad 0 < \sigma < 1,
\]

as a function of \( \alpha \) over \((0, 1)\).

**Theorem 4.3.** For \( 0 < \lambda < 1 \), \( 0 < \alpha < 1 \) and for all \( s \in \mathbb{C} \), we have

\[
L(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{2\pi^s} \left\{ e^{i\pi s/2 - 2\pi i\alpha \lambda} L(-\alpha, \lambda, s) + \\
e^{-i\pi s/2 + 2\pi i\alpha(1-\lambda)} L(\alpha, 1-\lambda, s) \right\}. \quad (4.9)
\]

Before proceeding with the proof, let us note the following interesting lemma.

**Lemma 4.4.** For all \( \Re(s) > 1 \),

\[
\Gamma(s)L(\lambda, \alpha, s) = \int_0^\infty \frac{e^{(1-\alpha)x-2\pi i\lambda}x^s-1}{e^x-2\pi i\lambda - 1} dx. \quad (4.10)
\]
Proof. For $\Re(s) > 1$,
\[
\int_{0}^{\infty} e^{-(m+\alpha)x + 2\pi i \lambda m} x^{-s-1} dx = \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s} \Gamma(s),
\]
and hence, summing over $m$, we find
\[
\Gamma(s) L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-(m+\alpha)x + 2\pi i \lambda m} x^{-s-1} dx
\]
\[
= \int_{0}^{\infty} \frac{e^{(1-\alpha)x - 2\pi i \lambda} x^{-s-1}}{e^{x - 2\pi i \lambda} - 1} dx.
\]
The exchange of summation is justified by the absolute convergence of the series
\[
\sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-(m+\alpha)x + 2\pi i \lambda m} x^{-s-1} dx = \Gamma(s) L(\lambda, \alpha, s),
\]
for $\Re(s) > 1$. This proves the lemma.

We are now set to prove the functional equation of $L(\lambda, \alpha, s)$.

Proof. Suppose $-1 < s < 0$. We want to let $c$ tend to $-\alpha$ in (4.8).

Since $\sin(\pi \alpha) \neq 0$, by continuity, there exists a neighborhood around $-\alpha$ such that for $z$ in that neighborhood,
\[
\frac{1}{|\sin(\pi \alpha)|} \leq \frac{1}{|z + \alpha|}.
\]
Thus, for $z$ in that neighborhood,
\[
\left| \frac{e^{2\pi ibz \pm iz}}{(e^{2\pi iz} - 1)|z + \alpha|^s} \right| \leq A|z + \alpha|^{-s-1},
\]
for some positive constant $A$.

Since $-1 < s < 0$, $\int_{0}^{\pm 1} |y|^{-s-1} dy < \infty$.

Hence, the integrals in (4.8) converge uniformly for $-\alpha \leq c \leq -\alpha + \epsilon$, for all $\epsilon > 0$. Letting $c \to -\alpha$ in (4.8), we have
\[
L(\lambda, \alpha, s) = \int_{-\alpha}^{-\alpha + i\infty} \frac{e^{2\pi ibz - iz}}{(z + \alpha)^s(e^{2\pi iz} - 1)} dz + \int_{\alpha}^{\alpha - i\infty} \frac{e^{2\pi ibz + iz}}{(z + \alpha)^s(e^{2\pi iz} - 1)} dz.
\]
Put \( y = z + \alpha \) in the above equation to get

\[
L(\lambda, \alpha, s) = \int_0^\infty \frac{e^{2\pi ib(y-\alpha) - i\pi(y-\alpha)}}{y^s(e^{-2\pi i(y-\alpha)} - 1)} dy + \int_0^{-i\infty} \frac{e^{2\pi ib(y-\alpha) + i\pi(y-\alpha)}}{y^s(e^{2\pi i(y-\alpha)} - 1)} dy.
\]

Now, substitute \( x = -iy \) in the first integral above and \( x = iy \) in the second integral. This gives,

\[
L(\lambda, \alpha, s) = (-i) e^{(2\pi ib\alpha + i\pi\alpha)\frac{1}{2}} \int_0^\infty \frac{e^{-2\pi bx + \pi x}}{x^s(e^{2\pi x + 2\pi i\alpha} - 1)} dx + i e^{(-2\pi ib\beta - i\pi\alpha)\frac{1}{2}} \int_0^\infty \frac{e^{-2\beta x + \pi x}}{x^s(e^{2\pi x - 2\pi i\alpha} - 1)} dx.
\]

Substituting \( u = 2\pi x \), \( b = \lambda - 1/2 \) and replace \( s \) by \( 1 - s \) to get

\[
L(\lambda, \alpha, 1 - s) = 2\pi^{-s} e^{\left(\frac{\pi}{2} - 2\pi i\alpha(\lambda-1)\right)} \int_0^\infty \frac{e^{-u(\lambda-1)u^s - 1}}{e^{u+2\pi i\alpha} - 1} du + 2\pi^{-s} e^{\left(-\frac{\pi}{2} - 2\pi i\alpha\lambda\right)} \int_0^\infty \frac{e^{u\lambda} u^s - 1}{e^{u-2\pi i\alpha} - 1} du. \tag{4.11}
\]

Applying (4.10) to the integrals in (4.11), we have

\[
e^{2\pi i\alpha} \int_0^\infty \frac{e^{-u(\lambda-1)u^s - 1}}{e^{u+2\pi i\alpha} - 1} du = \Gamma(s)L(-\alpha, \lambda, s) \tag{4.12}
\]

and

\[
e^{-2\pi i\alpha} \int_0^\infty \frac{e^{u\lambda} u^s - 1}{e^{u-2\pi i\alpha} - 1} du = \Gamma(s)L(\alpha, 1 - \lambda, s). \tag{4.13}
\]

Thus, (4.11), (4.12) and (4.13) imply that the following identity holds for \( s > 1 \).

\[
L(\lambda, \alpha, 1 - s) = \frac{\Gamma(s)}{2\pi^s} \left\{ e^{i\pi\frac{s}{2} - 2\pi i\alpha\lambda} L(-\alpha, \lambda, s) + e^{-i\pi\frac{s}{2} + 2\pi i\alpha(1-\lambda)} L(\alpha, 1 - \lambda, s) \right\}.
\]

By analytic continuation, we can conclude that the above identity is true for all \( s \in \mathbb{C} \).

\[
\square
\]

### 4.3. Special values

In 1951, Apostol [2] computed the values of the Lerch zeta function at 0 and negative integers. For \( \lambda \notin \mathbb{Z}, 0 < \alpha < 1 \),

\[
L(\lambda, \alpha, 0) = \frac{1}{1 - e^{2\pi i\lambda}} = \frac{i}{2} \cot(\pi \lambda) + \frac{1}{2} \tag{4.14}
\]
It is interesting to note that the value at 0 is independent of $\alpha$.
For $k \in \mathbb{Z}_{>0}$,
\[
L(\lambda, \alpha, -k) = -\frac{\beta_{k+1}(\alpha, e^{2\pi i \lambda})}{k+1},
\] (4.15)
where $\beta_n(t, u)$ is defined by the generating function
\[
z e^{tz} e^{u(z-1)} = \sum_{n=0}^{\infty} \frac{\beta_n(t, u)}{n!} z^n.
\]
In particular, let us observe that for $\alpha \in \mathbb{Q} \cap (0, 1)$, $L(\lambda, \alpha, m) \in \bar{\mathbb{Q}}$ for $m \in \mathbb{Z}_{<0}$. This information together with the functional equation helps us to determine the transcendental nature of certain values of the Lerch zeta function.

Let $k$ be a nonzero positive integer. Substitute $s = k$ in (4.9) to get
\[
L(\alpha, \lambda, 1-k) = \frac{(k-1)!}{(2\pi)^k} \left\{ e^{i\pi k/2-2\pi i\alpha \lambda} L(-\alpha, \lambda, k) + e^{-i\pi k/2+2\pi i\alpha(1-\lambda)} L(\alpha, 1-\lambda, k) \right\}. \tag{4.16}
\]
Suppose that $\lambda, \alpha \in \mathbb{Q} \cap (0, 1)$. Let $\delta_1 := ie^{-2\pi i \lambda \alpha}$ and $\delta_2 := -ie^{2\pi i \alpha(1-\lambda)}$. Note that $\delta_1$ and $\delta_2$ are algebraic. Thus,
\[
L(\lambda, \alpha, 1-k) \frac{(2\pi)^k}{(k-1)!} = \delta_1 L(-\alpha, \lambda, k) + \delta_2 L(\alpha, 1-\lambda, k).
\]
Observe that since $\alpha$ and $\lambda$ are rational, (4.14) and (4.15) imply that $L(\lambda, \alpha, 1-k)$ is algebraic for all nonzero positive integers $k$. If $L(\lambda, \alpha, 1-k) \neq 0$ and both $L(-\alpha, \lambda, k)$ and $L(\alpha, 1-\lambda, k)$ are algebraic, then $\pi^k$ will be algebraic and we are led to a contradiction. Therefore, we have the following theorem:

**THEOREM 4.5.** If $\lambda, \alpha \in \mathbb{Q} \cap (0, 1)$ and $L(\lambda, \alpha, 1-k) \neq 0$, then at least one of $L(-\alpha, \lambda, k)$ and $L(\alpha, 1-\lambda, k)$ is transcendental.

On specializing further, we conclude that

**THEOREM 4.6.** For $\alpha \in \mathbb{Q} \cap (0, 1)$, if $L(1/2, \alpha, 1-k) \neq 0$, then $L(\alpha, 1/2, k)$ is transcendental, for all $k \in \mathbb{Z}_{>0}$.

**PROOF.** Substitute $\lambda = 1/2$ in (4.16) and observe that
\[
L\left(\alpha, \frac{1}{2}, k\right) = L\left(-\alpha, \frac{1}{2}, k\right), \quad \forall \ k \in \mathbb{Z}_{>0}.
\]
The result now follows. \qed

**REMARK.** For $\lambda = 1/2$ and $k \in \mathbb{Z}_{>0}$, the value of $L(1/2, \alpha, -k)$ as given by (4.15) can be determined by the coefficient of $z^{k+1}$ in
\[
-z e^{\alpha z} \frac{e^{az}}{e^z + 1}.
\]
We realize that the above expression is the generating function of Euler polynomials, $E_n(x)$. The real zeros of Euler polynomials in $(0, 1)$ have been completely determined. By [10], $E_{2m}(x) \neq 0$ for $0 < x < 1$ and $E_{2m+1}(x) \neq 0$ for $0 < x < 1$ except for $x = 1/2$. Hence, Theorem 4.6 indeed gives us that $L(1/2, \alpha, k)$ is transcendental for all $0 < \alpha < 1$ when $k$ is an odd positive integer and for $0 < \alpha < 1$, $\alpha \neq 1/2$ when $k$ is an even positive integer.

Thus, we see that many interesting properties of the Lerch zeta function emerge out of the functional equation. There is scope for further study of the nature of values of the Lerch Zeta function at complex arguments and other properties of its functional equation. In particular, it will be interesting to find out when $L(\lambda, \alpha, 1 - k) = 0$ for $\lambda \neq 1/2$. We believe it can be done using $\omega$-Euler polynomials and relegate this to further research.
CHAPTER 5

Generalization of the Chowla problem

Let \( q \) be a fixed positive integer and \( f \) be an arithmetical function which is periodic with period \( q \). We will define

\[
\mathcal{L}(s, f, \alpha) := \sum_{m=0}^{\infty} \frac{f(m)}{(m + \alpha)^s},
\]

for \( \Re(s) > 1 \) and \( 0 < \alpha \leq 1 \).

As in the case of the Chowla question, we can ask whether \( \mathcal{L}(1, f, \alpha) \) is non-zero for a periodic function \( f \) when \( f \) is not identically zero. We will study this question in this chapter.

5.1. Convergence of \( \mathcal{L}(1, f, \alpha) \)

Recall the Fourier inversion formula, namely,

\[
f(m) = \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{am}.\]

Substituting this in the expression for \( \mathcal{L}(s, f, \alpha) \), we get

\[
\mathcal{L}(s, f, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s} \left( \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{am} \right) \\
= \sum_{a=1}^{q} \hat{f}(a) \sum_{m=0}^{\infty} \frac{e^{2\pi iam/q}}{(m + \alpha)^s} \\
= \hat{f}(q) \zeta(s, \alpha) + \sum_{a=1}^{q-1} \hat{f}(a) \sum_{m=0}^{\infty} \frac{e^{2\pi iam/q}}{(m + \alpha)^s} \\
= \hat{f}(q) \zeta(s, \alpha) + \sum_{a=1}^{q-1} \hat{f}(a) L\left( \frac{a}{q}, \alpha, s \right),
\]

where \( \zeta(s, x) \) is the Hurwitz zeta function and \( L(\lambda, \alpha, s) \) is the Lerch zeta function.

Note that the Hurwitz zeta function is analytic in the entire complex plane except for a simple pole at \( s = 1 \) with residue 1, and the Lerch zeta function is entire for \( 0 < \lambda < 1 \) and \( 0 < \alpha \leq 1 \). Thus,
**Theorem 5.1.** For a periodic arithmetical function $f$ and $0 < \alpha \leq 1$, $\mathcal{L}(s, f, \alpha)$ is entire $\iff \hat{f}(q) = 0$.

Thus, we will assume henceforth that $\hat{f}(q) = 0$.

**5.2. Non-vanishing of $\mathcal{L}(1, f, \alpha)$**

We will now investigate the non-vanishing of $\mathcal{L}(1, f, \alpha)$. Define

$$g_\alpha(z) := \sum_{m=0}^\infty \frac{z^m}{m + \alpha}.$$  

Using summation by parts, we can see that the above series converges for $|z| \leq 1, z \neq 1$.

Recall that for $0 < \alpha \leq 1$ and $\lambda \notin \mathbb{Z}$, the series $\sum_{m=0}^\infty \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$ converges for $\Re(s) > 0$. Thus, $\mathcal{L}(1, f, \alpha) = \sum_{m=0}^\infty \frac{e^{2\pi i \lambda m}}{(m+\alpha)}$, which can be written as

$$\mathcal{L}(1, f, \alpha) = \sum_{a=1}^{q-1} \hat{f}(a) g_\alpha(\zeta_d^a).$$

**Lemma 5.2.** For a rational number $\alpha = c/d$,

$$g_\alpha(z) = -z^{-c/d} \sum_{t=0}^{d-1} \zeta_d^{ct} \log(1 - \zeta_d^{t+1/d}).$$

**Proof.** On expanding the definition of $g_\alpha$ we have,

$$g_\alpha(z) = \sum_{m=0}^\infty \frac{z^m}{m + c/d}$$

$$= d \sum_{m=0}^\infty \frac{z^m}{dm + c}$$

$$= dz^{-c/d} \sum_{m=0}^\infty \frac{z^{(dm+c)/d}}{dm + c}$$

$$= dz^{-c/d} \sum_{n=1}^\infty \frac{z^{n/d}}{n}$$

Let $\zeta_d = e^{2\pi i/d}$. Observe that

$$\sum_{t=0}^{d-1} \zeta_d^{(n-c)t} = \begin{cases} d & \text{if } n \equiv c \mod d, \\ 0 & \text{otherwise}. \end{cases}$$
Hence,
\[ g_\alpha(z) = z^{-c/d} \sum_{n=1}^{\infty} \frac{z^{n/d}}{n} \left( \sum_{t=0}^{d-1} \zeta_d^{(n-c)t} \right) \]
\[ = z^{-c/d} \sum_{t=0}^{d-1} \sum_{n=1}^{\infty} \frac{z^{n/d} \zeta_d^{nt} \zeta_{ct}}{n} \]
\[ = z^{-c/d} \sum_{t=0}^{d-1} \zeta_d^{-ct} \sum_{n=1}^{\infty} \left( \frac{z^{1/d} \zeta_d^t}{n} \right) \]
\[ = -z^{-c/d} \sum_{t=0}^{d-1} \zeta_d^{-ct} \log(1 - z^{1/d} \zeta_d^t). \]

This proves the lemma. \(\square\)

Thus, for \(\alpha \in \mathbb{Q} \cap (0, 1]\), we can express \(L(1, f, \alpha)\) as a linear form in logarithms of algebraic numbers.

Let \(\alpha = c/d\), for \(d \neq 0\) and \((c, d) = 1\). Therefore,
\[ L(1, f, \alpha) = -\sum_{a=1}^{q-1} \hat{f}(a) \zeta_q^{-ac/d} \sum_{t=0}^{d-1} \zeta_d^{-ct} \log(1 - \zeta_d^t \zeta_q^{a/d}) \]
\[ = -\sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_q^{-ac/d} \zeta_d^{-ct} \log(1 - \zeta_d^t \zeta_q^{a/d}). \]

Hence, Baker’s theorem 1.3 implies that

**Theorem 5.3.** Let \(f\) be an algebraic-valued arithmetical function which is periodic with period \(q\) and \(\alpha \in \mathbb{Q} \cap (0, 1]\). Then \(L(1, f, \alpha)\) is either zero or transcendental.

We now investigate \(L(1, f, \alpha)\) in more detail. By a computation similar to section 3.1, we can write the logarithms as a combination of their real and imaginary parts, ie,
\[ \log(1 - \zeta_d^t \zeta_q^{a/d}) = \log \left( 2 \sin \left( \frac{(qt + a) \pi}{qd} \right) \right) + i\pi \left( \frac{qt + a}{qd} - \frac{1}{2} \right). \] (5.2)

Substituting (5.2) in the expression for \(L(1, f, \alpha)\) as a linear form in logarithms, we have
\[ L(1, f, \alpha) = \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_q^{-ac/d} \zeta_d^{-ct} \log \left( 2 \sin \left( \frac{(qt + a) \pi}{qd} \right) \right) + \]
\[ i\pi \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_q^{-ac/d} \zeta_d^{-ct} \left( \frac{qt + a}{qd} - \frac{1}{2} \right). \] (5.3)

Let us recall Lemma 1.5 from the first chapter.
Lemma. Let \( \alpha_1, \cdots, \alpha_n \) be positive algebraic numbers. If \( c_0, c_1, \cdots, c_n \) are algebraic numbers and \( c_0 \neq 0 \), then
\[
c_0 \pi + \sum_{j=1}^{n} c_j \log \alpha_j
\]
is a transcendental number and hence non-zero.

Observe that (5.3) expresses \( \mathcal{L}(1, f, \alpha) \) in the setup of the above lemma. Thus, we are lead to conclude that \( \mathcal{L}(1, f, \alpha) \neq 0 \) if the coefficient of \( \pi \) in (5.3) is non-zero. Therefore, we have that if \( \mathcal{L}(1, f, \alpha) = 0 \), then
\[
0 = q^{-1} \sum_{a=1}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} \left( \frac{qt + a}{qd} - \frac{1}{2} \right).
\]

On simplifying the right hand side, we have
\[
0 = \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} \left( \frac{t}{d} - \frac{1}{2} \right) + \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} \frac{a}{qd}. \tag{5.4}
\]

The second sum in the above equation can be further simplified to get
\[
\frac{1}{qd} \sum_{a=1}^{q-1} a \hat{f}(a) \zeta_{q}^{ac/d} \sum_{t=0}^{d-1} \zeta_{d}^{-ct}.
\]

Let us note that the computations till now hold valid even for \( \alpha = 1 \).

We now assume \( 0 < \alpha < 1 \). Thus, \( c \not\equiv 0 \mod d \) and \( \sum_{t=0}^{d-1} \zeta_{d}^{-ct} = 0 \).

Therefore, the second sum in (5.4) is zero.

Consider the first sum in (5.4). It can be further split as
\[
\sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} \left( \frac{t}{d} - \frac{1}{2} \right) = \frac{1}{d} \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} t \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} - \frac{1}{2} \sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct}.
\]

The second sum on the right hand side can be re-written as
\[
\sum_{a=1}^{q-1} \sum_{t=0}^{d-1} \hat{f}(a) \zeta_{q}^{ac/d} \zeta_{d}^{-ct} = \sum_{a=1}^{q-1} \hat{f}(a) \zeta_{q}^{ac/d} \sum_{t=0}^{d-1} \zeta_{d}^{-ct},
\]

which is zero as \( \sum_{t=0}^{d-1} \zeta_{d}^{-ct} = 0 \) when \( c \not\equiv 0 \mod d \). Thus, the equation (5.4) is simplified to
\[
\left( \sum_{t=0}^{d-1} t \zeta_{d}^{-ct} \right) \left( \sum_{a=1}^{q-1} \hat{f}(a) \zeta_{q}^{ac/d} \right) = 0.
\]
A computation similar to the one in Remark 1 from Chapter 3 helps us to evaluate the first sum as

$$\sum_{t=0}^{d-1} t \zeta_d^{-ct} = \frac{d}{\zeta_d^c - 1} \neq 0.$$ 

Therefore, we have the following:

**Theorem 5.4.** Let $f$ be an algebraic-valued arithmetical function which is periodic with period $q$. Let $\alpha = c/d \in \mathbb{Q} \cap (0,1)$, $(c,d) = 1$. If $L(1, f, \alpha) = 0$, then

$$\sum_{a=1}^{q-1} \hat{f}(a) \zeta_q^{-ac/d} = 0.$$ (5.5)

**Remark.** We can “interpolate” the Fourier inversion formula to get a “continuation” of the given function $f$ to the entire complex plane by setting

$$f(z) = \sum_{a=1}^{q} \hat{f}(a) \zeta_q^{az}.$$ 

Then condition (5.5) can be interpreted as $f(-\alpha) = 0$.

This gives a clean necessary condition for $L(1, f, \alpha) = 0$. One can further study the problem and ask for sufficiency conditions.
CHAPTER 6

Topics for further research

In the course of studying the Chowla problem, many questions arose, to which I was unable to find satisfactory answers. I will state those questions and the partial progress made. These will help in organizing together, a set of problems for further research.

6.1. Transcendence of values of Dirichlet $L$-functions at $1/2$.

Let us first observe the following:

Lemma 6.1.

$$\sum_{a=1}^{q-1} \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right) = -\frac{1}{\sqrt{2}},$$

(6.1)

for $1 \leq b \leq q - 1$.

Proof. Let $\theta_a = 2\pi ab/q$.

Using the identity

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

for $\theta \in \mathbb{R}$, the sum in (6.1) becomes

$$\sum_{a=1}^{q-1} \cos \left( \frac{\pi}{4} - \theta_a \right) = \sum_{a=1}^{q} \cos \left( \frac{\pi}{4} - \theta_a \right) - \frac{1}{\sqrt{2}}$$

$$= \frac{e^{i\pi/4}}{2} \left( \sum_{a=1}^{q} e^{i\theta_a} + \sum_{a=1}^{q} e^{-i\theta_a} \right) - \frac{1}{\sqrt{2}}$$

$$= \frac{e^{i\pi/4}}{2} \left( e^{i\theta_1} \frac{e^{i\theta_q} - 1}{e^{i\theta_1} - 1} + e^{-i\theta_1} \frac{e^{-i\theta_q} - 1}{e^{-i\theta_1} - 1} \right) - \frac{1}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}},$$

since $e^{\pm i\theta_q} = 1$.

This lemma will be useful in our computation.

Let $\zeta(s, x)$ denote the Hurwitz zeta function with $0 < x \leq 1$. Consider the following functional equation of the Hurwitz zeta function [1].

$$\zeta \left( 1 - s, \frac{b}{q} \right) = \frac{2\Gamma(s)}{(2\pi q)^s} \sum_{a=1}^{q} \cos \left( \frac{\pi s}{2} - \frac{2\pi ab}{q} \right) \zeta \left( s, \frac{a}{q} \right).$$
where $1 \leq b \leq q$. Specializing at $s = 1/2$, we get,

$$\zeta \left( \frac{1}{2}, \frac{b}{q} \right) = \sqrt{\frac{2}{q}} \sum_{a=1}^{q} \zeta \left( \frac{1}{2}, \frac{a}{q} \right) \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right).$$

On separating the term with $a = q$, we have

$$\zeta \left( \frac{1}{2}, \frac{b}{q} \right) = \frac{1}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) + \sqrt{\frac{2}{q}} \sum_{a=1}^{q-1} \zeta \left( \frac{1}{2}, \frac{a}{q} \right) \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right). \quad (6.2)$$

Thus, we have the following theorem.

**Theorem 6.2.** Let $M_q$ be the $(q-1) \times (q-1)$ matrix whose $(a,b)$-th entry is $\sqrt{2/q} \cos(\pi/4 - 2\pi ab/q)$ for $1 \leq a, b \leq q - 1$. If $\bar{v} = \left[ \zeta \left( \frac{1}{2}, \frac{b}{q} \right) - \zeta \left( \frac{1}{2} \right) \right]_{b=1}^{q-1}$, then $\bar{v}$ is an eigenvector of $M_q$ with eigenvalue 1.

**Proof.** Let $\lambda \in \mathbb{R}$ such that

$$\zeta \left( \frac{1}{2}, \frac{b}{q} \right) - \frac{\lambda}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) = \sqrt{\frac{2}{q}} \sum_{a=1}^{q-1} \left[ \zeta \left( \frac{1}{2}, \frac{a}{q} \right) - \frac{\lambda}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) \right] \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right). \quad (6.3)$$

By (6.2),

$$\zeta \left( \frac{1}{2}, \frac{b}{q} \right) - \frac{\lambda}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) = \frac{1}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) + \sqrt{\frac{2}{q}} \sum_{a=1}^{q-1} \zeta \left( \frac{1}{2}, \frac{a}{q} \right) \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right).$$

On equating (6.3) to the expression obtained above, we get

$$\sqrt{\frac{2}{q}} \sum_{a=1}^{q-1} \left[ \zeta \left( \frac{1}{2}, \frac{a}{q} \right) - \frac{\lambda}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) \right] \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right) = \frac{1}{\sqrt{q}} \zeta \left( \frac{1}{2} \right) + \sqrt{\frac{2}{q}} \sum_{a=1}^{q-1} \zeta \left( \frac{1}{2}, \frac{a}{q} \right) \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right).$$

Thus,

$$-\frac{\sqrt{2}\lambda}{q} \zeta \left( \frac{1}{2} \right) \sum_{a=1}^{q-1} \cos \left( \frac{\pi}{4} - \frac{2\pi ab}{q} \right) = \frac{1}{\sqrt{q}} \zeta \left( \frac{1}{2} \right).$$

The sum on the left hand side is as calculated in Lemma 6.1. Hence,

$$\frac{\lambda}{q} = \frac{1}{\sqrt{q}};$$

as $\zeta(1/2) \neq 0$. Thus, $\lambda = \sqrt{q}$. \qed

We would like to make the following conjecture on the basis of SAGE [21] computations:
Conjecture 6.3. \( M_q \) has eigenvalue 1 with multiplicity \((q-1)/2\) and eigenvalue \(-1\) with multiplicity \((q-2)/2\). Moreover, 1, \(-1\) and \(1/\sqrt{q}\) are the only eigenvalues of \( M_q \).

Under this conjecture, we can obtain a bound on the transcendence degree of \( \mathbb{Q}(\{L(1/2, \chi)|\chi \text{ is a Dirichlet character mod } q\}) \) over \( \mathbb{Q} \) as follows.

Let us recall the following basic theorem from linear algebra [3]:

**Theorem 6.4.** Let \( K \) be a field and \( M \) be an \( n \times n \) matrix with entries in \( K \). Suppose \( M \) is diagonalizable over \( K \). Then \( M \) has \( n \) eigenvectors with entries in \( K \).

Let \( \bar{v} \) be as in Theorem 6.2. Then, Theorem 6.2 implies that \( \bar{v} \) belongs to the eigenspace corresponding to eigenvalue 1. Conjecture 6.3 along with Theorem 6.4 lead us to conclude that there exist vectors \( v_1, \ldots, v_{q-1/2} \) that form a basis for the eigenspace corresponding to eigenvalue 1 and have entries in \( \bar{Q} \).

Therefore, \( \bar{v} = \left[ \zeta\left( \frac{1}{2}, \frac{b}{q} \right) - \zeta\left( \frac{1}{2} \right) \right]_{b=1}^{q-1} \) is a \( \mathbb{C} \)-linear combination of \( v_1, \ldots, v_{q-1/2} \). In particular, there exist complex numbers \( \omega_1, \ldots, \omega_{(q-1)/2} \) such that,

\[
\zeta\left( \frac{1}{2}, \frac{b}{q} \right) - \zeta\left( \frac{1}{2} \right) = \sum_{j=1}^{q-1} \eta_{b,j} \omega_j, \tag{6.4}
\]

where \( \eta_{b,j} \in \mathbb{Q} \) for all \( 1 \leq b \leq q - 1 \).

Let \( \chi \) be a Dirichlet character mod \( q \). Consider \( \chi \) as a function on the integers in the canonical way. Let \( L(s, \chi) \) denote the Dirichlet \( L \)-function attached to \( \chi \). Since, \( \chi \) is a function that is periodic with period \( q \), by 1.1,

\[
L(s, \chi) = \sum_{b=1}^{q-1} \chi(b)\zeta\left( s, \frac{b}{q} \right).
\]

Thus,

\[
L\left( \frac{1}{2}, \chi \right) = \sum_{b=1}^{q-1} \chi(b)\zeta\left( \frac{1}{2}, \frac{b}{q} \right)
= \sum_{b=1}^{q-1} \chi(b) \left[ \zeta\left( \frac{1}{2}, \frac{b}{q} \right) - \zeta\left( \frac{1}{2} \right) \right] + \zeta\left( \frac{1}{2} \right) \sum_{b=1}^{q-1} \chi(b)
= \sum_{b=1}^{q-1} \chi(b) \left[ \zeta\left( \frac{1}{2}, \frac{b}{q} \right) - \zeta\left( \frac{1}{2} \right) \right],
\]

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since, \( \sum_{b=1}^{q} \chi(b) = 0 \) for \( \chi \neq \chi_0 \) and \( \chi(q) = 0 \). Here, \( \chi_0 \) denotes the trivial character mod \( q \), i.e,

\[
\chi_0(n) = \begin{cases} 
1 & \text{if } (n,q) = 1, \\
0 & \text{otherwise}
\end{cases}
\]

Hence, we have the following important lemma.

**Lemma 6.5.** For a Dirichlet character \( \chi \mod q \) and \( \chi \neq \chi_0 \), \( L(1/2, \chi) \) is an algebraic linear combination of \( \{ \zeta(1/2, b/q) - \zeta(1/2) : 1 \leq b < q \} \).

The above lemma along with (6.4) lead us to conclude that: \( L(1/2, \chi) \) is a \( \mathbb{Q} \)-linear combination of the complex numbers \( \omega_1, \ldots, \omega_{(q-1)/2} \). Therefore, we have the following result.

**Theorem 6.6.** Under Conjecture 6.3,

\[
t_q := \text{trdeg}_\mathbb{Q} \left( \mathbb{Q}(\{ L(1/2, \chi) : \chi - \text{ Dirichlet character mod } q \} ) \right) \leq \frac{q+1}{2}.
\]

**Remark.** The number of distinct Dirichlet characters mod \( q \) is \( \phi(q) \). So the trivial bound for \( t_q \) is \( \phi(q) \). The bound obtained above is better than the trivial bound when \( \phi(q) > (q+1)/2 \), for example, when \( q \) is prime. But in certain cases, the trivial bound is more effective.

### 6.2. Analogue of the Dedekind determinant

Let \( n \) be a fixed positive integer and \( G \) be a group of order \( n \), \( \{ x_1, \ldots, x_n \} \). Let \( f : G \to \mathbb{C} \) be any function on \( G \). Dedekind studied \( n \times n \) matrices of the form \( D = (d_{ij}) \) with

\[
d_{ij} = f(x_i^{-1}x_j).
\]

We will call such matrices as Dedekind matrices. Define

\[
S_{\chi} = \sum_{g \in G} f(g)\chi(g),
\]

for each character \( \chi \) of \( G \). Dedekind computed explicitly the determinant of such matrices to be \( \prod_{\chi} S_{\chi} \) where the product is over all characters \( \chi \) of the group \( G \). Ram Murty and Kaneenika Sinha \cite{18} also computed the eigenvalues and eigenvectors of Dedekind matrices. As an analogue of the Dedekind matrix, we can also study matrices of the form \( E = (e_{ij}) \) with

\[
e_{ij} = f(x_i x_j). \quad (6.5)
\]

A motivation to study these kind of matrices is the matrix \( \mathcal{M}_q \) from the previous section. A computation of the eigenvalues and eigenvectors of the matrices of the type mentioned in (6.5) would help to resolve the conjecture made in the previous section.

As a partial solution to the question raised, we would like to like to...
compute the determinant of such matrices in the case when the group $G$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\ast$.

Let $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$. Let $E$ be the $(p-1) \times (p-1)$ matrix whose $(i,j)$-th entry is $f(ij)$, for $1 \leq i, j \leq p-1$.

Recall the Fourier inversion formula that says

$$f(n) = \sum_{k=1}^{p} \hat{f}(k) \zeta_p^{nk}.$$  

Thus,

$$f(ij) = \sum_{k=1}^{p} \hat{f}(k) \zeta_p^{ijk}$$

$$= \hat{f}(p) + \sum_{k=1}^{p-1} \hat{f}(k) \zeta_p^{ijk}$$

$$= \hat{f}(p) + \sum_{t=1}^{p-1} \hat{f}(i^{-1}t) \zeta_p^{ij}.$$  

Hence, for all $1 \leq i, j \leq p-1$,

$$f(ij) - \hat{f}(p) = \sum_{k=1}^{p-1} \hat{f}(i^{-1}t) \zeta_p^{ij}. \quad (6.6)$$

Define the following $p-1 \times p-1$ matrices: $\mathcal{E}_0 = (\epsilon_{ij})$, with $\epsilon_{ij} = f(ij) - \hat{f}(p)$, $\mathcal{D}_0 = (\delta_{ij})$, with $\delta_{ij} = \hat{f}(i^{-1}j)$ and $\mathcal{V}_0 = (\vartheta_{ij})$, with $\vartheta_{ij} = \zeta_p^{ij}$ for all $1 \leq i, j \leq p-1$.

Thus, equation (6.6) can be re-written as a matrix equation as follows:

$$\mathcal{E}_0 = \mathcal{D}_0 \times \mathcal{V}_0. \quad (6.7)$$

Observe that $\mathcal{D}_0$ is a Dedekind matrix and $\mathcal{V}_0$ is a Vandermonde matrix, both of whose determinants can be computed using known means. Thus, the determinant of $\mathcal{E}_0$ can be computed.

Define a function

$$g(n) = f(n) - \hat{f}(p).$$

Note that $g$ is also periodic with period $p$. On taking the Fourier transform, we have

$$\hat{g}(m) = \frac{1}{p} \sum_{a=1}^{p} f(a) \zeta_p^{-am} - \frac{\hat{f}(p)}{p} \sum_{a=1}^{p} \zeta_p^{-am}.$$  

Since, $\sum_{a=1}^{p} \zeta_p^{-am} = 0$ unless $m \equiv 0 \pmod{p}$, the Fourier transform of $g$ can be re-written as

$$\hat{g}(m) = \hat{f}(m) - \hat{f}(p) \delta_0(m),$$

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where,

\[ \delta_0(m) = \begin{cases} 
1 & \text{if } m \equiv 0 \pmod{p}, \\
0 & \text{otherwise.} 
\end{cases} \]

The arguments of \( g \) under consideration are between 1 and \( p - 1 \) and hence co-prime to \( p \). Thus, for those arguments \( m \), \( \hat{g}(m) = \hat{f}(m) \). Therefore, the matrix \( D_0 = G_0 = (\gamma_{i,j}) \), where \( \gamma_{i,j} = \hat{g}(i^{-1}j) \). Thus, if \( \mathcal{G} = (g(ij)_{i,j}) \), then

\[ \det \mathcal{G} = \det G_0 \times \det \mathcal{V}_0, \quad (6.8) \]

where, \( G_0 \) is a Dedekind matrix and \( \mathcal{V}_0 \) is a Vandermonde matrix, both of whose determinants can be computed. Since the dependence on \( f \) no longer exists in (6.8), it holds for all functions, periodic with period \( p \).
Bibliography