Tilting objects in derived categories of equivariant sheaves

by

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Abstract

We construct classical tilting objects in derived categories of equivariant sheaves on quasi-projective varieties, which give equivalences with derived categories of modules over algebras. Our applications include a conceptual explanation of the importance of the McKay quiver associated to a representation of a finite group G and the development of a McKay correspondence for the cotangent bundle of the projective line.

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Statement of Originality

Except where precise references are provided, all results of this thesis are original.

Chapter 1 gives a selective review of the history of derived categories and so contains nothing original, except possibly my point of view. Chapter 2 is a summary of basic facts about derived categories and contains nothing original. The results of Chapter 3 are generalizations and adaptations of well-known theorems for which I provide precise references. The main results of Chapter 4, Theorems 4.2.1 and 4.2.4, I found and proved independently. It has been pointed out to me that equivalent results with different proofs can be found in a recent preprint of Bocklandt-Schedler-Wemyss, for which I provide a reference. The example in 4.4 was worked out independently, then modified following a suggestion of Roman Bezrukavnikov. The results of Chapter 5 are original.

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Chapter 1

Introduction

Derived and triangulated categories were introduced by Grothendieck and Verdier [52] in the early sixties in order to establish a relative version of Serre duality for a nice morphism $f : X \to Y$ of schemes. The Grothendieck-Verdier duality theory involves constructing *derived categories* $D(X)$ and $D(Y)$ whose objects consist of complexes of sheaves with quasi-coherent cohomology, together with a derived push-forward functor $Rf_* : D(X) \to D(Y)$ and a right adjoint $f^! : D(Y) \to D(X)$.

When X is a smooth projective variety of dimension n over $Y = \operatorname{Spec} \mathbf{k}$ for some field $\mathbf{k}$, one can show that $f^!\mathbf{k} \simeq \omega_X[n]$, the complex consisting of the canonical bundle of X sitting in degree $-n$. Furthermore, the counit $Rf_*f^!\mathbf{k} \to \mathbf{k}$ of the adjunction $Rf_* \dashv f^!$ realizes the classical trace map $H^n(X, \omega_X) \to \mathbf{k}$ used in the construction of Serre duality.

Following this foundational work of Grothendieck-Verdier, derived categories found a number of uses.

In the later sixties, Grothendieck and collaborators developed intersection theory for a Noetherian scheme X using the triangulated subcategory of $D(X)$ consisting of *perfect complexes* [8], and Deligne put derived categories to use in the study of étale cohomology [23]. In the seventies derived categories provided a convenient setting in which to develop the theory of $\mathcal{D}$-modules, an intrinsic, geometric approach to linear differential equations on manifolds and varieties [14].

Beginning in the late seventies, work on derived categories accelerated. Beilinson's paper 'Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra' [3] showed that in some appropriate sense the object

$$T := \bigoplus_{d=0}^{n} \mathcal{O}(d)$$

generates $D(\mathbb{P}^n)$, the derived category of projective space, and that this implies there is an equivalence $R\mathrm{Hom}(T, -) : D(\mathbb{P}^n) \to D(\mathrm{End}(T)^{\mathrm{op}})$. Generalizing the properties of the collection $\mathcal{O}, \ldots, \mathcal{O}(n)$ led to the theory of exceptional collections on Fano varieties as developed in seminars in Moscow [47] throughout the eighties. In a related direction, Beilinson's work, together with the *reflection functors* of Bernstein-Gelfand-Ponomarev [7], gave rise to *tilting theory* in derived categories. Highlights include Happel [28], Cline-Parshall-Scott [22], and Rickard's 'Morita theory for derived categories' [46].

One of the most surprising aspects of the theories of exceptional collections and of tilting is that they realize a fixed triangulated category, say the derived category of sheaves on a variety, as the derived category of many different abelian subcategories. In order to formalize and generalize such phenomena, Beilinson-Bernstein-Deligne [4] introduced *t-structures* in triangulated categories, in terms of which they defined perverse sheaves and explained the relation of the latter to $\mathcal{D}$-modules. In further work on perverse sheaves, Beilinson-Ginzburg-Soergel [5] developed a general theory of *Koszul duality* relating the derived category of a special algebra B to the derived category of another algebra $E(B)$ canonically associated to it.

Around the same time as Grothendieck-Verdier were applying triangulated categories to duality in algebraic geometry, Puppe introduced a slightly weaker version of triangulated category to formalize constructions in stable homotopy theory [44]. In the decades that followed, great progress was made in topology, for instance the development of Brown representability [1], which guarantees that all good functors from the stable homotopy category to another triangulated category are representable, and Quillen's invention of algebraic K-theory of abelian categories [45].

In the early nineties, Robert Thomason [50] brought together ideas from homotopy theory and algebraic geometry to give an intrinsic characterization of perfect complexes $C \in D(X)$ as *compact objects*, ones for which $\mathrm{Hom}(C, -)$ commutes with small sums. He then used the triangulated subcategory of perfect complexes to develop algebraic K-theory of schemes and derived categories. Soon after, Bökstedt-Neeman [12] introduced further ideas from homotopy theory to give a slick development of unbounded derived categories and Neeman [42] used Brown representability for nice triangulated categories to give a conceptual treatment of Grothendieck-Verdier duality. Using the same methods, Keller [36] generalized tilting theory to differential-graded (dg) algebras and gave a very simple criterion for an object T in a triangulated category $\mathcal{D}$ to give an equivalence with the derived category of the dg algebra $R\mathrm{Hom}(T, T)$.

Meanwhile, in a galaxy far, far away, string theorists were discovering mirror symmetry, a mysterious relation between an algebraic variety X and a 'mirror' symplectic manifold $\widehat{X}$ [54]. In his ICM address in 1994 [38], Kontsevich outlined a program to understand mirror symmetry as an equivalence between the bounded derived category of coherent sheaves $D^b(X)$ on an algebraic variety X and the derived Fukaya category $D^b\mathrm{Fuk}(\widehat{X})$ of the mirror manifold $\widehat{X}$ whose objects roughly consist of Lagrangian submanifolds of $\widehat{X}$.

Instances of such 'homological mirror symmetry' have been constructed, notably for elliptic curves by Polishchuk-Zaslow [43]. But at least as influential as any actual instance of such an equivalence has been the very suggestion that a category depending on an algebraic variety X could be equivalent to a category depending on a symplectic manifold $\widehat{X}$. Such an equivalence would have many consequence for the geometry of either. Seidel-Thomas [48] for instance makes the following observations. While algebraic varieties are rather rigid objects, most having only finitely many automorphisms, symplectic manifolds are very flexible, having enormous symplectomorphism groups. In particular, to every Lagrangian sphere S in a symplectic manifold $\widehat{X}$, it is possible to associate a 'symplectic Dehn twist' of the manifold $\widehat{X}$ which induces an autoequivalence T_S of the derived Fukaya category $D^b\mathrm{Fuk}(\widehat{X})$. Under an equivalence $D^b\mathrm{Fuk}(\widehat{X}) \simeq D^b(X)$, the Lagrangian sphere S, thought of as an object of $D^b\mathrm{Fuk}(\widehat{X})$, should be sent to a 'spherical object' in $D^b(X)$ and the Dehn twist T_S acting on $D^b\mathrm{Fuk}(\widehat{X})$ should become an autoequivalence of $D^b(X)$, dubbed a 'spherical twist'. Since the spherical twist acts on the derived category of the variety but not on the variety itself, we might say that the derived category detects certain hidden symmetries of the variety.

The main achievement of Seidel and Thomas was to make these heuristics precise and to work out many interesting examples of spherical objects and their associated twists. Of particular interest was their discovery of *braid group actions* generated by spherical twists, which are currently finding many applications to knot theory and representation theory. See for instance Cautis-Kamnitzer [20] which constructs knot homology theories using braid group actions on derived categories, Bezrukavnikov [9] which discusses t-structures indexed by elements of braid groups, and Huybrechts-Thomas [33] which introduces $\mathbb{P}$-objects and $\mathbb{P}$-twists, hyper-Kähler analogues of spherical objects and spherical twists.

The history of derived categories is of course much richer than my biased survey suggests. I have chosen to emphasize themes that will play an important part in this thesis, such as tilting theory, Koszul duality, compact objects, and braid group actions. For more details, I suggest Weibel's 'History of homological algebra' [53], which I found to be a helpful reference.

1.1 Overview of the thesis

The chapters of this thesis are arranged in order of increasing originality and depth. In particular, I consider Chapter 5, The projective McKay correspondence, to be the most important and interesting.

The beginning of each chapter contains a summary of its contents. Here we shall just give a brief overview of each chapter together with the statements of the major results.

Chapter 2 In this expository chapter we review standard definitions of triangulated categories, derived categories, and derived functors. The reader should take note of Section 2.5, in which we introduce the derived category of equivariant sheaves on a variety $D_G(X)$, the main object of study in this thesis. The reader should also give attention to Lemma 2.9.4, which gives the correct signs for Serre duality, a technical point which often causes confusion.

Chapter 3 We adapt work of Keller [36] to give a criterion for an object $T \in D_G(X)$ to produce an equivalence $R\mathrm{Hom}_G(T,-) : D_G(X) \to D(A)$, where $A := \mathrm{End}(T)^{\mathrm{op}}$. This criterion is the main technical tool on which all of our subsequent results depend. To illustrate the criterion, we give an exposition of an equivariant version of Beilinson's work [3] and deduce a description of the equivariant K-theory of projective space.

The major results of Chapter 3 are the following:

Theorem. 3.1.1

1. *The functor* $\Psi = R\mathrm{Hom}_{\mathrm{G}}(T,-) : D_G(X) \to D(A)$ *is an equivalence if and only if the object* $T \in D_G(X)$ *is compact, meaning* $\mathrm{Hom}_G(T,-)$ *commutes with small direct sums, and generates, meaning that* $R\mathrm{Hom}_G(T,\mathcal{F}) = 0$ *implies* $\mathcal{F} = 0$.

   If moreover $\mathrm{Hom}_G(T,T[i]) = 0$ *for* $i \neq 0$ *(*'Ext vanishing'*), then the dg algebra is just (quasi-isomorphic to) an algebra.*

2. *If* T *satisfies the hypotheses of* 1., *and if* X *is smooth and quasi-projective, then the equivalence* $\Psi = R\mathrm{Hom}_G(T,-)$ *restricts to an equivalence* $\Psi : D^b_G(X) \to \mathrm{perf}A$, *where* $\mathrm{perf}(A) \subset D(A)$ *is the full triangulated subcategory of* perfect *complexes, those quasi-isomorphic to a bounded complex of finitely generated projective modules.*

3. *If every finitely generated* A*-module has a finite resolution by finitely generated projectives, then* $\mathrm{perf}(A) \simeq D^b(A)$, *the bounded derived category of finitely generated* A*-modules. In this case, if the hypotheses of* 1. *and* 2. *are satisfied, then the equivalence* Ψ *restricts to an equivalence*

$$\Psi = R\mathrm{Hom}_{\mathrm{G}}(T,-) : D^b_G(X) \to D^b(A).$$

Theorem. 3.2.1 *Let G be a finite group acting linearly on a vector space V of dimension $n+1$, and let W_i, $i \in I$ be the irreducible representations of G. Then*

$$T := \bigoplus_{i \in I} \bigoplus_{d=0}^{n} \mathcal{O}(d) \otimes W_i$$

is a classical tilting object in $D_G(\mathbb{P}(V))$ and the algebra $A := \mathrm{End}_G(\mathcal{F})^{\mathrm{op}}$ has finite global dimension.

By Theorem 3.1.1, the usual functor $R\mathrm{Hom}_G(\mathcal{F}, -)$ gives an equivalence

$$D^b_G(\mathbb{P}(V)) \simeq D^b(A).$$

Chapter 4

The classical McKay correspondence establishes a bijection between conjugacy classes of finite subgroups $G \subset SL_2(\mathbb{C})$ and affine Dynkin diagrams Γ, where the nodes of Γ are indexed by the irreducible representations of G and the edges are determined by the values of certain characters of G. Taking into account the order in which these values are computed leads to an orientation on Γ, giving the so-called 'McKay quiver'. The path algebra of the McKay quiver modulo certain quadratic relations gives a Koszul algebra known as the preprojective algebra Π_Γ, and there is a well-known equivalence between the categories of G-equivariant coherent sheaves on $\mathbb{C}^2$ and of modules over Π_Γ. As their basic examples of spherical objects, Seidel-Thomas [48] took the skyscraper sheaves at the origin of $\mathbb{C}^2$ whose fibres are the irreducible representation of G. Seidel-Thomas showed that the dimensions of the Ext-groups between these spherical objects are encoded in the adjacencies of the graph Γ and that this implies the associated spherical twists generate the action of a braid group of type Γ.

We apply Theorem 3.1.1 to the higher dimensional McKay correspondence, for subgroups $G \subset GL(V)$ for V an arbitrary vector space, and show that algebras appearing here are Koszul, a fact which leads to a conceptual explanation of the definition of the 'McKay quiver' in this context. Finally, we consider higher dimensional examples involving the symmetric group, where we find $\mathbb{P}$-objects in the sense of [33]

In order to state our main results, let us introduce some notation. Let G be a finite group with non-modular representation V and let $W := \bigoplus_i W_i$, the sum of the distinct irreducible representations of G. Consider V as the total space of a G-equivariant vector bundle over a point, with projection π and zero-section s. We may pull-back W from the point to get an equivariant vector bundle $\pi^* W$ on V. Set $B := \mathrm{End}_G(\pi^* W)^{\mathrm{op}}$, the opposite algebra of equivariant endomorphisms of $\pi^* W$.

Theorem. 4.2.1 *There is an equivalence*

$$\Psi = \mathrm{Hom}_G(\pi^* W, -) : \mathrm{Coh}_G(V) \longrightarrow B\text{-mod}$$

from G-equivariant coherent sheaves to left B-modules, where the action on an object $\Psi(\mathcal{F})$ is given by precomposition with elements of $B^{\mathrm{op}} = \mathrm{End}_G(\pi^* W)$.

Theorem. 4.2.4 *The algebra $B = \mathrm{End}_G(\pi^* W)^{\mathrm{op}}$ is Koszul.*

Proposition. 4.6.2 *In the notation of Theorem 4.2.1, let $G = S_n$, let V be the sum of two copies of the standard representation, let W_λ be an irreducible representation of G indexed by partitions in the usual way, and let $s_* W_\lambda := W_\lambda \otimes \mathcal{O}_0$ (a skyscraper sheaf at the origin of V with fibre W_λ). When W_λ is the trivial or the sign representation, then $s_* W_\lambda \in D_G(V)$ is a $\mathbb{P}$-object.*

Chapter 5

Kirillov [37] has developed a McKay correspondence for subgroups $\tilde{G} \subset PSL_2(\mathbb{C})$, relating the geometry of $\tilde{G}$ acting on $\mathbb{P}^1$ and the combinatorics of the affine Dynkin diagram Γ associated to the double cover $G \subset SL_2(\mathbb{C})$ of $\tilde{G}$. There are two main aspects of this correspondence.

First, for every 'height function' h on the set of vertices of Γ, there is an associated quiver Q_h with underlying graph Γ and a collection of $\tilde{G}$-equivariant vector bundles F_i^h on $\mathbb{P}^1$. Letting $T := \bigoplus_i F_i^h$, there is an isomorphism $\mathrm{End}_G(T)^{\mathrm{op}} \simeq \mathbb{C}Q_h$ between the opposite algebra of the equivariant endomorphisms of T and the path algebra of the quiver Q_h and also a derived equivalence

$$R\Phi_h := R\mathrm{Hom}_G(T, -) : D^b_{\tilde{G}}(\mathbb{P}^1) \to D^b(Q_h)$$

from equivariant sheaves on $\mathbb{P}^1$ to representations of the quiver Q_h. Set $E_i^h := R\Phi_h^{-1}(S_i^h)$, where S_i^h is the simple representation of Q_h corresponding to the ith vertex.

Second, the equivalences for different height functions h and $\tilde{h}$ are related by the reflection functors of Bernstein-Gelfand-Ponomarev:

$$\begin{array}{ccc} & D^b_{\tilde{G}}(\mathbb{P}^1) & \\ {\scriptstyle R\Phi_h} \swarrow & & \searrow {\scriptstyle R\Phi_{\tilde{h}}} \\ D^b(Q_h) & \xrightarrow{BGP} & D^b(Q_{\tilde{h}}). \end{array}$$

Our main results are analogues for the cotangent bundle $T^*\mathbb{P}^1$ of the two aspects of the McKay correspondence for $\mathbb{P}^1$.

Let π be the projection of $T^*\mathbb{P}^1$ onto $\mathbb{P}^1$ and s the zero-section. Let $\mathcal{F}_i^h := \pi^* F_i^h$ and set $B_h := \mathrm{End}_G(\bigoplus_i \mathcal{F}_i^h)^{\mathrm{op}}$.

Our first result gives the analogue of the equivalences $R\Phi_h$.

Theorem. 5.3.1 *For each height function h, there is an equivalence*

$$R\Psi_h := R\mathrm{Hom}_G(\bigoplus_i \mathcal{F}_i^h, -) : D^b_{\tilde{G}}(T^*\mathbb{P}^1) \to D^b(B_h).$$

The following two results explain the sense in which the different equivalences $R\Psi_h$ differ by spherical twists, which take the place of reflection functors.

Proposition. 5.3.4 *Let h be a height function on Γ and set $\mathcal{E}_i^h = s_* E_i^h$. The objects $\mathcal{E}_i^h$ are spherical objects in $D^b_{\tilde{G}}(T^*\mathbb{P}^1)$ and the associated spherical twists generate an action of a braid group of type Γ on $D^b_{\tilde{G}}(T^*\mathbb{P}^1)$.*

For a given height function h, we can use the inverse equivalence $R\Psi_h^{-1}$ to produce a heart $\mathcal{B}_h$ of a non-standard t-structure on (a subcategory of) $D^b_{\tilde{G}}(T^*\mathbb{P}^1)$. The objects $\mathcal{E}_i^h$ are the simple objects of the abelian category $\mathcal{B}_h$. Our final result says that the hearts $\mathcal{B}_h$ differ by the action of spherical twists.

Theorem. 5.4.8 *If $i \in Q_h$ is a source, then $T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^- h}$. Likewise, if i is a sink, then $T_{\mathcal{E}_i^h}^{-1}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^+ h}$. In particular, since the hearts are finite length and hence determined by their simples, $T_{\mathcal{E}_i^h}(\mathcal{B}_h) = \mathcal{B}_{\sigma_i^- h}$ for i a source and $T_{\mathcal{E}_i^h}^{-1}(\mathcal{B}_h) = \mathcal{B}_{\sigma_i^+ h}$ for i a sink.*

Chapter 2

Derived and triangulated categories

We provide some background on and establish notation for the main objects of interest in this thesis, derived categories of equivariant sheaves and derived categories of modules over algebras. For a first introduction to derived categories, I recommend the reviews of Caldararu [19] and Thomas [49]. For more details, I would look at the beautiful books of Huybrechts [32] and Gelfand-Manin [26]. Finally, to see how the whole theory works in greater generality, in particular avoiding boundedness hypotheses, consult Lipman [39].

2.1 Complexes

Let $\mathcal{A}$ be an abelian category and denote by $\mathrm{Kom}(\mathcal{A})$ the category whose objects are complexes of objects in $\mathcal{A}$ and whose morphisms are maps of complexes. $\mathrm{Kom}(\mathcal{A})$ is again abelian. In particular, the kernel of a morphism of complexes consists of the kernels of the components of the morphism. The most important example of an abelian category is Ab, the category of abelian groups.

Denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, which has the same objects as $\mathrm{Kom}(\mathcal{A})$, but whose morphisms are homotopy classes of maps of complexes. $\mathrm{K}(\mathcal{A})$ is not abelian in general (the component-wise kernel in $\mathrm{Kom}(\mathcal{A})$ is not well-defined up to homotopy). There is a natural functor $\mathrm{Kom}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ identifying the objects of the two categories and sending a morphism to its homotopy class.

Given a complex $A^\bullet = \cdots \to A^{k-1} \to A^k \to A^{k+1} \to \cdots$ with differential d, define a new complex $A^\bullet[1]$ whose ith term is A^{i+1} and whose differential is $-d$. For a morphism f of complexes, define $f[1]$ by letting the ith component be f^{i+1}. The operation [1] then gives an autoequivalence of $\mathrm{Kom}(\mathcal{A})$, called the **shift** or *translation* functor. For any integer n, denote the nth power of the shift [1] by $[n]$, which acts on a complex by dragging the nth term into degree zero.

We denote the cohomology of the complex $A^\bullet$ at the ith spot by $H^i(A^\bullet)$. Note the relation $H^i(A^\bullet) = H^0(A^\bullet[i])$. A map of complexes $f : A^\bullet \to B^\bullet$ induces a map of cohomology groups $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ which depends only on the homotopy class of f. The map f is called a **quasi-isomorphism** when the induced maps $H^i(f)$ are isomorphisms for all i.

Using the snake lemma, one can show that a short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ in $\mathrm{Kom}(\mathcal{A})$ gives rise to a long exact sequence of cohomology groups

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Given two complexes $A^\bullet, B^\bullet$, define a new complex $\mathrm{Hom}(A^\bullet, B^\bullet)$ with ith term $\bigoplus_p \mathrm{Hom}(A^p, B^{p+i})$ and differential $d(f) := d_B \circ f - (-1)^i f \circ d_A$. Note that when $A := A^\bullet$ and $B := B^\bullet$ are complexes concentrated in degree zero, the Hom complex has $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in degree zero and is trivial elsewhere. Given $A^\bullet \in \mathrm{Kom}(\mathcal{A})$, we have functors $\mathrm{Hom}(A^\bullet, -) : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathrm{Ab})$ and $\mathrm{Hom}(-, A^\bullet) : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathrm{Ab})$.

Suppose $\mathcal{A}$ has a tensor product $\otimes$. Given two complexes $A^\bullet$ and $B^\bullet$, we define a new complex $A^\bullet \otimes B^\bullet$ with ith term $\bigoplus_{p+q=i} A^p \otimes B^q$ and with differential $d := d_{B^\bullet} \otimes 1 + (-1)^i \otimes d_{A^\bullet}$. Given $A^\bullet \in \mathrm{Kom}(\mathcal{A})$, we have a functor $A^\bullet \otimes - : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$.

Note that any additive functor $\Phi : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{B})$ descends to a natural functor $\Phi : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ since additivity preserves homotopies of morphisms.

Given a morphism of complexes $f : A^\bullet \to B^\bullet$, we define a new complex $\mathrm{C}(f)$, the **mapping cone** of f, with

(**2.1.1**)
$$\mathrm{C}(f)^i := A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{\mathrm{C}(f)} := \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

The natural injection $B^i \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$ induce complex morphisms $\pi : B^\bullet \to \mathrm{C}(f)$ and $\delta : \mathrm{C}(f) \to A^\bullet[1]$ respectively. Altogether we have the important sequence of morphisms

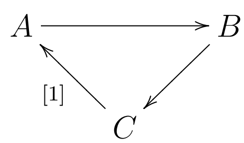
(**2.1.2**)
$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\pi} \mathrm{C}(f) \xrightarrow{\delta} A^\bullet[1]$$

Remark 2.1.1. Similarly, one can consider categories $\mathrm{Kom}^*(\mathcal{A})$ and $\mathrm{K}^*(\mathcal{A})$, where $* = +, -,$ or b restricts to complexes $A^\bullet$ such that $A^i = 0$ for $i \ll 0, i \gg 0,$ and $|i| \gg 0$ respectively.

2.2 Triangulated categories

While the homotopy category of an abelian category is not abelian, it is **triangulated**, a structure which allows us to talk of *exactness* in some looser sense.

Let $\mathcal{T}$ be a category with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ which we call the *shift* or *translation*. A **triangle** in $\mathcal{T}$ is a collection of objects and morphisms $A \to B \to C \to A[1]$. Triangles are often represented by diagrams of the form



or by a triple $A \to B \to C$ with the final morphism $C \to A[1]$ understood.

A *morphism* of triangles is a diagram

$$\begin{CD} A @>>> B @>>> C @>>> A[1] \\ @VfVV @VgVV @VhVV @VVf[1]V \\ A' @>>> B' @>>> C' @>>> A'[1] \end{CD}$$

with commutative squares. The morphism is an *isomorphism* when f, g, and h are isomorphisms.

Definition 2.2.1. A **triangulated category** is an additive category $\mathcal{T}$ endowed with the structure of a **shift** functor [1] and a specified set of triangles called **exact** satisfying the following axioms:

TR1 i) Any triangle of the form

$$A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow A[1]$$

is exact.

ii) Any triangle isomorphic to an exact triangle is exact.

iii) Any morphism $f : A \to B$ can be completed to an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} \mathrm{C}(f) \xrightarrow{h} A[1].$$

TR2 A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].$$

is exact if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} C[1].$$

is also exact.

TR3 Given a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

whose rows are exact triangles, there exists $h : C \to C'$ completing the diagram to a morphism of triangles.

TR4 ('Octahedron') For each pair of morphisms $f : A \to B$, $g : B \to C$, there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & \mathrm{C}(f) & \longrightarrow & A[1] \\ \| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\ A & \xrightarrow{g\circ f} & C & \longrightarrow & \mathrm{C}(g\circ f) & \longrightarrow & A[1] \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ & & \mathrm{C}(g) & = & \mathrm{C}(g) & \longrightarrow & B[1] \\ & & \downarrow & & \downarrow & & \\ & & B[1] & \longrightarrow & \mathrm{C}(f)[1] & & \end{array}$$

in which the first two rows are exact triangles and the two central columns are exact triangles with the horizontal arrows forming a morphism between them.

Remark 2.2.2. One can show that for an object X in a triangulated category $\mathcal{T}$, the functor $\mathrm{Hom}(X,-) : \mathcal{T} \to \mathrm{Ab}$ sends an exact triangle $A \to B \to C$ to an exact sequence $\mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C)$ of abelian groups, and similarly for the contravariant functor $\mathrm{Hom}(-,Y)$. More generally, a functor $\Phi : \mathcal{T} \to \mathrm{Ab}$ sending exact triangles to exact sequences of abelian groups is called **cohomological**.

The first algebraic example that one encounters is the homotopy category $\mathrm{K}(\mathcal{A})$, in which the translation functor is taken to be the shift [1] of complexes and the exact triangles are taken to be anything isomorphic to a triangle of the form

$$A^\bullet \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1],$$

as introduced in (2.1.2). That this choice of shift functor and exact triangles satisfies the above axioms and so endows $\mathrm{K}(\mathcal{A})$ with the structure of triangulated category is verified in [26, IV.1.9].

Remark 2.2.3. Let us comment on the axioms for a triangulated category. One can show that the completion of $f : A \to B$ to a triangle $A \to B \to \mathrm{C}(f) \to A[1]$ from axiom TR1, iii) is unique up to a non-unique isomorphism. The object $\mathrm{C}(f)$ is often called the 'cone of f', or sometimes the 'cofibre of f' [30].

The sign in $-f[1]$ from TR2 is necessary to make the standard choices of shift functor and exact triangles on $\mathrm{K}(\mathcal{A})$ satisfy the axioms for a triangulated category. This sign must *not* be ignored since it has non-trivial consequences for the theory of triangulated categories. See for instance Remark 2.2.5 and the discussion in Section 2.9 of signs in Serre duality.

The octahedron axiom TR4 has a fearsome reputation, more for typographical than for conceptual complexity. Indeed, the octahedron has the following intuitive interpretation. Thinking of the exact triangles $A \to B \to \mathrm{C}(f)$ and $A \to C \to \mathrm{C}(g)$ as analogous to short exact sequences in an abelian category, we roughly have '$\mathrm{C}(f) \simeq B/A$' and '$\mathrm{C}(g) \simeq C/A$'. The second vertical triangle is then analogous to the standard short exact sequence $B/A \to C/A \to C/B$.

When considering functors between triangulated categories, we should like for them to respect the triangulated structures.

Definition 2.2.4. A **triangle functor** between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ is a pair (Φ, η) with $\Phi : \mathcal{T} \to \mathcal{T}'$ a functor and $\eta : \Phi \circ [1] \Rightarrow [1] \circ \Phi$ a natural isomorphism of functors such that for any exact triangle

$$A \to B \to C \to A[1]$$

in $\mathcal{T}$, the triangle

$$\Phi(A) \to \Phi(B) \to \Phi(C) \to \Phi(A[1]) \simeq \Phi(A)[1]$$

is exact in $\mathcal{T}'$, where the isomorphism on the right-hand side is η_A.

Remark 2.2.5. Because of the signs in TR2, the pair $([1], \mathrm{id})$ is *not* a triangulated functor, while the pair $([1], -\mathrm{id})$ is.

One has to take similar care to make the additive functors $\mathrm{Hom}(A^\bullet, -) : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathrm{Ab})$ and $A^\bullet \otimes - : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ into triangulated functors. See [39, 1.5.3,1.5.4] for the appropriate signs.

Definition 2.2.6. A strict subcategory $i : \mathcal{T}' \hookrightarrow \mathcal{T}$ of a triangulated category is called a **triangulated subcategory** if the pair (i, id) is a triangulated functor. Equivalently, a subcategory is triangulated if it is stable under the shift functor and if $A \to B \to C$ in $\mathcal{T}$ is any exact triangle with $A, B \in \mathcal{T}'$, then $C \in \mathcal{T}'$.

2.3 Derived categories

For the purposes of homological algebra it is convenient to consider two complexes more-or-less the same when they are quasi-isomorphic. This leads to the following definition.

Theorem 2.3.1. *Given an abelian category $\mathcal{A}$, there exists a category $D(\mathcal{A})$, called the* **derived category** *of $\mathcal{A}$, such that*

1. *There is a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(A)$ which sends quasi-isomorphisms to isomorphisms.*

2. *The functor Q is universal with respect to property* 1.*: given any category $\mathcal{D}$ and any functor $\mathcal{F} : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ which sends quasi-isomorphisms to isomorphisms, there is a unique functor $\mathcal{G} : D(A) \to \mathcal{D}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(A) \\ & {}_{\mathcal{F}}\searrow \quad \swarrow_{\mathcal{G}} & \\ & \mathcal{D} & \end{array}$$

Remark 2.3.2. $D(\mathcal{A})$ is usually constructed as a 'triangle quotient' $\mathrm{K}(\mathcal{A}) \to D(\mathcal{A})$ of the homotopy category by the subcategory of complexes quasi-isomorphic to 0. The functor Q factors as $\mathrm{Kom}(\mathcal{A}) \to \mathrm{K}(\mathcal{A}) \to D(\mathcal{A})$ and identifies the objects of $D(\mathcal{A})$ with those of $\mathrm{Kom}(\mathcal{A})$ or $\mathrm{K}(\mathcal{A})$. The derived category $D(\mathcal{A})$ inherits the structure of triangulated category from $\mathrm{K}(\mathcal{A})$. The shift functor [1] is the same as for $\mathrm{K}(\mathcal{A})$ and an exact triangle in $D(\mathcal{A})$ is any triangle isomorphic to one of the form

$$A^\bullet \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1].$$

Since quasi-isomorphisms become isomorphisms in $D(\mathcal{A})$ and since triangles that are isomorphic to exact triangles must be exact, there are more exact triangles in $D(\mathcal{A})$ than in $\mathrm{K}(\mathcal{A})$. In particular, one can show ([26, IV.2.8]) that every short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ can be completed to an exact triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D(\mathcal{A})$ and that every exact triangle is isomorphic to one of this form. In contrast, a morphism $C^\bullet \to A^\bullet[1]$ extending $A^\bullet \to B^\bullet \to C^\bullet$ to an exact triangle need not exist in $\mathrm{K}(\mathcal{A})$.

Remark 2.3.3. Similarly, for $* = +, -,$ and b, one can construct a derived category $D^*(\mathcal{A})$ as a quotient of $\mathrm{K}^*(\mathcal{A})$ and show that the natural functor $D^*(\mathcal{A}) \to D(A)$ identifies $D^*(\mathcal{A})$ with the full subcategory of $D(\mathcal{A})$ consisting of complexes $A^\bullet$ with $H^i(A^\bullet) = 0$ for $i \ll 0, i \gg 0,$ and $|i| \gg 0$ as $* = +, -,$ and b respectively.

While the objects of $D(\mathcal{A})$ are the same as those of $\mathrm{Kom}(\mathcal{A})$, the morphisms in $D(\mathcal{A})$ between complexes $A^\bullet$ and $B^\bullet$ are in general rather difficult to compute.

Note that they need not be the same as the morphisms in $\mathrm{Kom}(\mathcal{A})$. For instance, if $A^\bullet = 0 \to A \to A \to 0$, where the central arrow is the identity morphism, then endomorphisms of $A^\bullet$ in the category of complexes are in bijection with endomorphisms of A in $\mathcal{A}$. On the other hand $A^\bullet \simeq 0$ in $D(\mathcal{A})$, so $A^\bullet$ has no non-zero endomorphisms in the derived category.

When $\mathcal{A}$ has enough injectives, one can show that for $A^\bullet, B^\bullet, I^\bullet \in D^+(\mathcal{A})$, where $I^\bullet$ is a complex of injectives and $B^\bullet \simeq I^\bullet$, there is a natural isomorphism

$$\mathrm{Hom}_{D(A)}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet).$$

Likewise, when $\mathcal{A}$ has enough projectives, one can show that for $A^\bullet, B^\bullet, P^\bullet \in D^-(\mathcal{A})$, where $P^\bullet$ is a complex of projectives and $P^\bullet \simeq A^\bullet$, there is a natural isomorphism

$$\mathrm{Hom}_{D(A)}(A^\bullet, B^\bullet) \simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P^\bullet, B^\bullet).$$

Since injective resolutions are usually too big to handle and projective resolutions do not exist in many situations that we are interested in, the above isomorphisms do not provide us a practical way to compute morphisms in the derived category.

In special cases, however, we can compute morphisms by other means. The natural functor $\mathcal{A} \hookrightarrow \mathrm{K}(\mathcal{A}) \to D(\mathcal{A})$ that sends an object $A \in \mathcal{A}$ to the complex with A in degree zero and zeroes elsewhere, is known to be an equivalence with the full subcategory of $D(\mathcal{A})$ consisting of complexes with cohomology concentrated in degree zero and we therefore already know how to compute morphisms between such complexes. If $\mathcal{A}$ has enough injectives, then given objects $A, B \in \mathcal{A}$, there is a natural isomorphism

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[p]) \simeq \mathrm{Ext}^p_{\mathcal{A}}(A, B),$$

so we can hope to compute morphisms between complexes concentrated in a single degree. Finally, if $\mathcal{A}$ has enough injectives, then there are some spectral sequences which sometimes allow us to compute morphisms between more general complexes from knowledge of morphisms between simpler complexes: when $B^\bullet$ is bounded below,

$$E_2^{p,q} = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, H^q(B^\bullet)[p]) \Rightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) \tag{2.3.1}$$

and when $A^\bullet$ is bounded above,

$$E_2^{p,q} = \mathrm{Hom}_{D(\mathcal{A})}(H^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[p+q]) \tag{2.3.2}$$

2.4 Derived functors

Given a left exact functor $\Phi : \mathcal{A} \to \mathcal{B}$ between abelian categories, we would like to promote Φ to a functor $R\Phi : D(\mathcal{A}) \to D(\mathcal{B})$ so that we have a natural isomorphism of functors $Q_{\mathcal{B}} \circ \Phi \simeq R\Phi \circ Q_{\mathcal{A}}$, where the Qs are the natural functors from the homotopy categories to the derived categories. Similarly, for a right exact functor $\Psi : \mathcal{A} \to \mathcal{B}$, we would like to construct a functor $L\Psi : D(\mathcal{A}) \to D(\mathcal{B})$ so that we have a natural isomorphism $Q_{\mathcal{B}} \circ \Psi \simeq L\Psi \circ Q_{B}$.

If Φ and Ψ are exact, then we can just apply them to each term of a complex to produce the desired functors, but without exactness this naive application of Φ and Ψ does not work. In particular, applying Φ and Ψ to complexes component-wise need not always be well-defined since quasi-isomorphisms might not be sent to quasi-isomorphisms.

Instead, we seek $R\Phi$ and $L\Psi$ that satisfy a weaker compatability with functors between homotopy categories.

Definition 2.4.1. The **right derived functor** of a left exact functor $\Phi : \mathcal{A} \to \mathcal{B}$ is a triangle functor $R\Phi : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation of functors $\rho_\Phi : Q_\mathcal{B} \circ \Phi \Rightarrow R\Phi \circ Q_\mathcal{A}$ satisfying the following universal property: for any triangle functor $\Theta : D(\mathcal{A}) \to D(\mathcal{B})$ with natural transformation $\rho_\Theta : Q_\mathcal{B} \circ \Phi \Rightarrow \Theta \circ Q_\mathcal{A}$, there is a unique natural transformation $\rho : R\Phi \Rightarrow \Theta$ such that $(\rho \circ Q_\mathcal{A}) \circ \rho_\Phi = \rho_\Theta$.

Likewise, the **left derived functor** of a right exact functor $\Psi : \mathcal{A} \to \mathcal{B}$ is a triangle functor $L\Psi : D(\mathcal{A}) \to D(\mathcal{B})$ together with a natural transformation of functors $\lambda_\Psi : L\Psi \circ Q_\mathcal{A} \Rightarrow Q_\mathcal{B} \circ \Psi$ satisfying the following universal property: for any triangle functor $\Upsilon : D(\mathcal{A}) \to D(\mathcal{B})$ with natural transformation $\lambda_\Upsilon : \Upsilon \circ Q_\mathcal{A} \Rightarrow Q_\mathcal{B} \circ \Psi$, there is a unique natural transformation $\lambda : \Upsilon \Rightarrow L\Psi$ such that $\lambda_\Psi \circ (\lambda \circ Q_\mathcal{A}) = \lambda_\Upsilon$.

Remark 2.4.2. More generally, given a triangle functor $\Phi : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$, we can ask for it to have a right derived functor $R\Phi : D(\mathcal{A}) \to D(\mathcal{B})$ in the sense that there should be a natural transformation of functors $\rho_\Phi : Q_\mathcal{B} \circ \Phi \Rightarrow R\Phi \circ Q_B$ satisfying the same universal property as above. Similarly for left derived functors.

One can show that, when they exist, derived functors are unique up to unique isomorphism.

For a left exact functor $\Phi : \mathcal{A} \to \mathcal{B}$, the cohomology of the right derived functor computes the classical 'higher right derived functors' of Φ:

$$H^i(R\Phi(A)) \simeq R^i\Phi(A) \text{ for } A \in \mathcal{A} .$$

Likewise, for a right exact functor $\Psi : \mathcal{A} \to \mathcal{B}$, the cohomology of the left derived functor computes the classical 'higher left derived functors' of Ψ:

$$H^i(L\Psi(A)) \simeq L^i\Psi(A) \text{ for } A \in \mathcal{A} .$$

In fact, Gelfand-Manin [26, III.13] *define* the classical derived functors in this way.

One of the wonderful things about derived functors is that, under weak hypotheses [39, Corollary 2.2.7] that are satisfied in all of the examples that we consider in Section 2.6, the composition of two derived functors $R\Phi : D(\mathcal{A}) \to D(\mathcal{B})$ and $R\Phi : D(\mathcal{B}) \to D(\mathcal{C})$ is the derived functor of their composition:

$$R(\Psi \circ \Phi) \simeq R\Psi \circ R\Phi.$$

Similarly, $L(\Psi \circ \Phi) \simeq L\Psi \circ L\Phi$.

When $\mathcal{A}$ has enough injectives, this compatibility of derived functors with composition is reflected in the Grothendieck spectral sequence

$$\textbf{(2.4.1)} \qquad E_2^{p,q} = R^p\Psi \circ R^q\Phi(A^\bullet) \Rightarrow R^{p+q}(\Psi \circ \Phi)(A^\bullet) \text{ for } A^\bullet \in D^+(\mathcal{A})$$

When $\Phi = \mathrm{Id}$, the spectral sequence reads $E_2^{p,q} = R^p\Psi(H^q(A^\bullet)) \Rightarrow R^{p+q}\Psi(A^\bullet)$, which is particularly interesting since it gives a means of computing the higher derived functors of Ψ applied to a complex $A^\bullet$ from the higher derived functors applied to the cohomology of $A^\bullet$.

Remark 2.4.3. A similar spectral sequence exists for left derived functors, but we shall not need it.

2.5 Examples of derived categories

Let X be a scheme over a field $\mathbf{k}$, G a finite group such that the order of $\mathbf{k}$ does not divide the order of the group (the 'non-modular' case), and let G act by automorphisms on X. A G-**equivariant quasi-coherent sheaf** on X is a quasi-coherent sheaf $\mathcal{F}$ together with an isomorphism $\lambda_g : \mathcal{F} \longrightarrow g^*\mathcal{F}$ for each $g \in G$ such that $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$.

In fact, we shall mostly be interested in G-equivariant *coherent* sheaves, but certain constructions will sometimes lead to more general quasi-coherent sheaves. We shall often say 'G-sheaf' for 'G-equivariant coherent sheaf' and when we wish to emphasize that the sheaf is not necessarily coherent, we shall say 'quasi-coherent G-sheaf'.

Given two quasi-coherent G-sheaves $\mathcal{F}$ and $\mathcal{G}$ with equivariant structures λ and μ, we get a representation of G on $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$: given $g \in G$, $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$, $g \cdot \phi = \mu_g^{-1} \circ \phi \circ \lambda_g$. The homomorphisms commuting with the equivariant structures are therefore precisely those invariant under this action and we define $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}(\mathcal{F}, \mathcal{G})^G \subset \mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

Denote by $\mathrm{QCoh}_G(X)$ the category whose objects are quasi-coherent G-sheaves and whose morphisms are elements of $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G})$. The category $\mathrm{QCoh}_G(X)$ is abelian and has enough injectives. Denote by $\mathrm{Coh}_G(X)$ the full (abelian) subcategory of $\mathrm{QCoh}_G(X)$ whose objects are coherent G-sheaves.

When necessary and particularly in applications, we shall assume that X is smooth and quasi-projective over the field $\mathbf{k}$, so that every object of the category $\mathrm{Coh}_G(X)$ has a finite locally free resolution.

Note that the abelian category $\mathrm{QCoh}_G(X)$ is $\mathbf{k}$-linear, meaning the Hom groups are $\mathbf{k}$-vector spaces and composition is bilinear. Similarly A-Mod, the category of modules over a $\mathbf{k}$-algebra A, is $\mathbf{k}$-linear. We denote their unbounded derived categories respectively by

$$D_G(X) \quad \text{and} \quad D(A).$$

Further, we denote by $D^b_G(X)$ the full triangulated subcategory of $D_G(X)$ with bounded, coherent cohomology, and by $D^b(A)$ the full triangulated subcategory of $D(A)$ with bounded, finitely generated cohomology.

2.6 Examples of derived functors

We assume that $\mathcal{A}$ is an abelian category, **k**-linear (for simplicity), has enough injectives, admits infinite direct sums, and that direct sums of exact sequences are exact. These conditions ensure that we can work with unbounded complexes. Examples of abelian categories satisfying these conditions include the category of G-equivariant quasi-coherent sheaves on a scheme and the category of modules over an algebra.

We consider the examples of derived functors important for our applications.

Given an object $A^\bullet \in \mathrm{K}(\mathcal{A})$, we have a functor $\mathrm{Hom}(A^\bullet, -) : \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathrm{Vect}(\mathrm{k}))$, the latter being the homotopy category of vector spaces over **k**. One can show that since $\mathcal{A}$ is assumed to have enough injectives, $\mathrm{Hom}(A^\bullet, -)$ has a right derived functor $R\mathrm{Hom}(A^\bullet, -) : D(\mathcal{A}) \to D(\mathbf{k})$. Furthermore, the cohomology of this functor computes Homs between shifts of objects in $D(\mathcal{A})$:

$$H^p(R\mathrm{Hom}(A^\bullet, B^\bullet)) = \mathrm{Hom}(A^\bullet, B^\bullet[p]).$$

Remark 2.6.1. As a matter of notation, when $A^\bullet \in \mathrm{K}(\mathrm{QCoh}_G(X))$, we let $R\mathrm{Hom}(A^\bullet, -)$ denote the derived functor of $\mathrm{Hom}(A^\bullet, -)$, which computes all morphisms, not just those commuting with the G-action. Thus $R\mathrm{Hom}(A^\bullet, -)$ actually takes values in $D(\mathrm{Rep}\ G)$, the derived category of representations of G. Under the non-modularity assumptions we have made on G, taking G-invariants is an exact functor, so we may compute the derived functor $R\mathrm{Hom}_G(A^\bullet, -) : D(\mathrm{QCoh}_G(X)) \to D(\mathbf{k})$ by taking invariants of $R\mathrm{Hom}(A^\bullet, -)$ term-by-term.

Given a complex $\mathcal{F}^\bullet \in \mathrm{K}(\mathrm{QCoh}_G(X))$, we have functors $\mathcal{H}om(\mathcal{F}^\bullet, -) : \mathrm{K}(\mathrm{QCoh}_G(X)) \to \mathrm{K}(\mathrm{QCoh}_G(X))$ and $\mathcal{F}^\bullet \otimes - : \mathrm{K}(\mathrm{QCoh}_G(X)) \to \mathrm{K}(\mathrm{QCoh}_G(X))$ and one can show that they have right and left derived functors respectively:

$$R\mathcal{H}om(\mathcal{F}^\bullet, -) : D_G(X) \to D_G(X)$$
$$\mathcal{F}^\bullet \otimes^L - : D_G(X) \to D_G(X)$$

Similarly, for $\mathcal{G} \in D_G(X)$, we have a contravariant functor $R\mathcal{H}om(-, \mathcal{G}) : D_G(X) \to D_G(X)$. In particular, $R\mathcal{H}om(-, \mathcal{O}_X)$ is called the (derived) dual. Its action on an object $\mathcal{F}$ is denoted $\mathcal{F}^\vee := R\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Remark 2.6.2. On a smooth, quasi-projective variety X we can resolve $\mathcal{F}^\bullet \in D^b_G(X)$ by a finite complex of vector bundles $\mathcal{E}^\bullet$. In this case, there are isomorphisms of functors

$$\mathcal{F}^\bullet \otimes^L - \simeq \mathcal{E}^\bullet \otimes - \quad \text{and} \quad R\mathcal{H}om(\mathcal{F}^\bullet, -) \simeq \mathcal{H}om(\mathcal{E}^\bullet, -) \simeq \mathcal{E}^{\bullet\vee} \otimes -.$$

From now on we shall drop the L in $\otimes^L$ and just write $\otimes$. This should not cause confusion.

Restricted to objects $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}_G(X)$, the cohomology of $R\mathcal{H}om$ computes local Ext while the cohomology of $\otimes$ computes local Tor, both carrying natural G-equivariant structures:

$$H^i(R\mathcal{H}om(\mathcal{F}, \mathcal{G})) = \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$$
$$H^i(\mathcal{F} \otimes \mathcal{G}) = \mathcal{T}or_i(\mathcal{F}, \mathcal{G})$$

Given a morphism $f : X \to Z$ of G-schemes, we have functors $f^* : \mathrm{K}(\mathrm{QCoh}_G(Z)) \to \mathrm{K}(\mathrm{QCoh}_G(X))$ and $f_* : \mathrm{K}(\mathrm{QCoh}_G(X)) \to \mathrm{K}(\mathrm{QCoh}_G(Z))$ and derived functors

$$Lf^* : D_G(Z) \to D_G(X)$$
$$Rf_* : D_G(X) \to D_G(Z)$$

When f is flat, f^* is exact and we may drop the L in Lf^*. Similarly when f is a closed immersion, f_* is flat and may drop the R in Rf_*.

Restricted to objects $\mathcal{G} \in \mathrm{QCoh_G(Z)}$ and $\mathcal{F} \in \mathrm{QCoh_G(X)}$, the cohomology of these functors computes the classical higher derived pull-back and push-forward:

$$H^i(Lf^*\mathcal{G}) = L^i f^*(\mathcal{G})$$
$$H^i(Rf_*\mathcal{F}) = R^i f_*(\mathcal{F})$$

In particular, when $Z = \mathrm{Spec}\,\mathbf{k}$ with the trivial G-action, then $Rf_* = R\mathrm{Hom}(\mathcal{O}_X, -) = R\Gamma$, the derived functor of the global sections functor, and $R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$, the ith sheaf-cohomology group as a G-representation.

Note that we have the following compatibilities with composition:

$R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq R\Gamma \circ R\mathcal{H}om(\mathcal{F}, \mathcal{G})$, $R(g \circ f)_* \simeq Rg_* \circ Rf_*$, and $L(g \circ f)^* \simeq Lf^* \circ Lg^*$.

2.7 Projection formula and flat base-change

We need three very important relations between the derived pull-back and pushforward:

Adjunction Lf^* is left adjoint to Rf_*. We shall often speak of 'the adjunction $Lf^* \dashv Rf_*$'.

Projection formula

For any two objects $\mathcal{E}^\bullet \in D_G(X), \mathcal{F}^\bullet \in D_G(Z)$, there is a canonical isomorphism

$$Rf_*(\mathcal{E}^\bullet \otimes Lf^*\mathcal{F}^\bullet) \simeq Rf_*\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$$

in $D_G(Z)$

Flat base-change (see [39, Proposition 3.9.5])

Given a fibre square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

with f flat and g quasi-compact and quasi-separated, we have a natural isomorphism

$$f^* Rg_* \mathcal{F}^\bullet \simeq Rp_* q^* \mathcal{F}^\bullet.$$

Note that we dropped the Ls in Lf^* and Lq^* since f and hence q are flat and so have no higher derived pull-backs.

In particular, when $Z = \operatorname{Spec} \mathbf{k}$ so that push-forward along f is the functor of global sections, flat base-change gives

$$f^* R\Gamma(\mathcal{F}^\bullet) \simeq Rp_* q^* \mathcal{F}^\bullet.$$

2.8 Integral transforms

Given an object $\mathcal{P} \in D(X \times Y)$, define the **integral transform** with **kernel** $\mathcal{P}$ to be the functor

$$\Phi^{\rightarrow}_{\mathcal{P}} := R\pi_{Y*}(\mathcal{P} \otimes L\pi^*_X(-))$$

from $D(X)$ to $D(Y)$. Similarly, in the other direction, we define the functor $\Phi^{\leftarrow}_{\mathcal{P}} := R\pi_{X*}(\mathcal{P} \otimes L\pi^*_Y(-))$ from $D(Y)$ to $D(X)$.

An important special case is when $X = Y$ and $\mathcal{P} = \mathcal{O}_\Delta = \Delta_* \mathcal{O}_X$. Given an object $\mathcal{F} \in D(X)$, we have

$$
\begin{aligned}
\Phi^{\rightarrow}_{\mathcal{O}_\Delta}(\mathcal{F}) &= Rq_*(\Delta_*\mathcal{O}_X \otimes Lp^*\mathcal{F}) \\
&\simeq Rq_*(\Delta_*(\mathcal{O}_X \otimes L\Delta^* Lp^*\mathcal{F})) && \text{using the projection formula for } \Delta \\
&\simeq Rq_*(\Delta_*\mathcal{F}) && \text{since } L\Delta^* Lp^* = \mathrm{Id}_X \\
&\simeq \mathcal{F} && \text{since } Rq_* \circ \Delta_* = \mathrm{Id_X}
\end{aligned}
$$

Thus we see that $\mathcal{O}_\Delta$ provides a kernel for the identity functor Id_X.

Remark 2.8.1. Our whole discussion of derived functors, the projection formula, flat base-change, and integral transforms restricts to the level of bounded categories with coherent or finitely generated cohomology, provided that whenever Rf_* appears, we require f to be proper so that coherent cohomology is sent to coherent cohomology.

2.9 Serre functors

One of the most useful theorems in algebraic geometry is Serre duality, which is formalized in the following notion.

Definition 2.9.1. Let $\mathcal{T}$ be a Hom-finite $\mathbf{k}$-linear category. A **Serre functor** is a $\mathbf{k}$-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ together with isomorphisms

$$
\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{G}, S\mathcal{F})^* \tag{2.9.1}
$$

natural in $\mathcal{F}, \mathcal{G} \in \mathcal{A}$.

Letting $\mathcal{G} = \mathcal{F}$, the image of $1 \in \mathrm{Hom}(\mathcal{F}, \mathcal{F})$ under the isomorphism 2.9.1 gives a canonical **trace** element

$$
Tr_{\mathcal{F}} : \mathrm{Hom}(\mathcal{F}, S\mathcal{F}) \to \mathbf{k}.
$$

The composition of morphisms followed by the trace,

$$\mathrm{Hom}(\mathcal{G}, S\mathcal{F}) \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}(\mathcal{F}, S\mathcal{F}) \xrightarrow{\mathrm{Tr}_{\mathcal{F}}} \mathbf{k},$$

is then a non-degenerate pairing and realizes the duality in 2.9.1.

Definition 2.9.2. A triangulated category $\mathcal{T}$ with Serre functor S is called n-**Calabi-Yau** (n-**CY**) if there is a natural isomorphism of functors $S \simeq [n]$.

Remark 2.9.3. When Serre functors exist, they are known to be unique up to natural isomorphism.

The example to keep in mind is the derived category $D^b(X)$ of a smooth projective variety of dimension n, where the Serre functor is $S \simeq (\omega_X \otimes -)[n]$, with ω_X the canonical bundle of X. For sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, the isomorphism $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{G}, S\mathcal{F})^*$ is the more familiar $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^n(\mathcal{G}, \mathcal{F} \otimes \omega_X)$.

When X is smooth quasi-projective, we shall also use Serre duality for a full triangulated subcategory $\mathcal{D} \subset D^b(X)$ consisting of objects with compact support.

When $\omega_X \simeq \mathcal{O}_X$ (so that X is Calabi-Yau or holomorphic symplectic), the Serre functor in $D^b(X)$ is isomorphic to the shift $[n]$. This is the origin of 'Calabi-Yau' in 'n-Calabi-Yau category'.

The following is Lemma A.5.2 of Michel Van den Bergh's appendix to [10].

Lemma 2.9.4. *If $f : A \to B[i]$ and $g : B \to A[n-i]$ are morphisms in an n-CY triangulated category $\mathcal{T}$, then*

$$\mathrm{Tr}_A(g[i] \circ f) = (-1)^{i(n-i)} \mathrm{Tr}_B(f[n-i] \circ g).$$

The proof involves a careful analysis of the signs arising in triangulated categories. The lemma is useful for understanding multiplication in an Ext algebra. Say for instance that $\mathcal{D}$ is 2-CY and $\mathcal{E} \in \mathcal{D}$. Then the natural pairing

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \otimes \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \longrightarrow \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{Tr}_{\mathcal{E}}} \mathbf{k} \tag{2.9.2}$$

is antisymmetric.This will be important in Section **??**.

2.10 t-structures

In this section we follow [26, IV.4].

A general triangulated category $\mathcal{D}$ need not be equivalent to the derived category of an abelian category, and when it is, it might be equivalent to the derived categories of many non-equivalent abelian categories. This leads to the notion of a **t-structure** on a triangulated category $\mathcal{D}$, which is, roughly speaking, a way of looking at $\mathcal{D}$ as if it were the derived category of some specific abelian category called the **heart** of the t-structure.

The motivating example is the **standard t-structure** on the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$, which consists of the two strictly full subcategories $\mathcal{D}^{\leq 0} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for } i > 0\}$ and $\mathcal{D}^{\geq 0} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for } i < 0\}$. The heart of the standard t-structure is the intersection of these two subcategories $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, which consists of complexes with cohomology only in degree zero so is identified with the original abelian category $\mathcal{A}$ under the usual inclusion $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$.

Abstracting the properties of the standard t-structure on $D(\mathcal{A})$, one arrives at the following definition.

Definition 2.10.1. Let $\mathcal{D}$ be a triangulated category with a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and set $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$ if it satisfies the following conditions.

1. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.

2. $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for $\mathcal{F} \in \mathcal{D}^{\leq 0}, \mathcal{G} \in \mathcal{D}^{\geq 1}$.

3. For any $\mathcal{F} \in \mathcal{D}$ there is a triangle $A \to \mathcal{F} \to B$ with $A \in \mathcal{D}^{\leq 0}, B \in \mathcal{D}^{\geq 1}$.

The **heart** of a t-structure is the full subcategory $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \subset \mathcal{D}$.

Remark 2.10.2. One can show that the standard t-structure on $D(\mathcal{A})$ is indeed a t-structure.

In the case of the derived category, the triangle in 3. can be constructed using the **truncation functors** $\tau_{\leq 0}$, $\tau_{\geq 1}$. Given a complex $A^\bullet$, we have a short exact sequence of complexes

$$\begin{array}{ccccccccccc}
\cdots & \to & A^{-2} & \to & A^{-1} & \to & \ker d^0 & \to & 0 & \to & 0 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & A^{-2} & \to & A^{-1} & \to & A^0 & \to & A^1 & \to & A^2 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & 0 & \to & 0 & \to & \mathrm{im}\, d^0 & \to & A^1 & \to & A^2 & \to & \cdots
\end{array}$$

where the first row is by definition $\tau_{\leq 0} A^\bullet$ and has cohomology only in degrees less than or equal to zero, and where the last row is by definition $\tau_{\geq 1} A^\bullet$ and can have cohomology only in degrees greater than or equal to one. This short exact sequence thus gives rise to the desired exact triangle $\tau_{\leq 0} A^\bullet \to A^\bullet \to \tau_{\geq 1} A^\bullet$ with $\tau_{\leq 0} A^\bullet \in \mathcal{D}^{\leq 0}$ and $\tau_{\geq 1} A^\bullet \in \mathcal{D}^{\geq 1}$. For any integers n, m, the functors $\tau_{\leq n}, \tau_{\geq m}$ are defined similarly by translation.

More intrinsically, the functor $\tau_{\leq n}$ can be characterized as the right adjoint of the inclusion $i_{\leq n} : \mathcal{D}^{\leq n} \hookrightarrow D^b(\mathcal{A})$ and the functor $\tau_{\geq m}$ as the left adjoint of the inclusion $i_{\geq m} : \mathcal{D}^{\geq n} \hookrightarrow D^b(\mathcal{A})$, with the obvious unit $\mathrm{Id}_{\leq n} \Rightarrow \tau_{\leq n} i_{\leq n}$ and counit $\tau_{\geq m} i_{\geq m} \Rightarrow \mathrm{Id}_{\geq m}$.

Finally, note that for $m \leq n$, there is a natural isomorphism of functors $\tau_{\geq m}\tau_{\leq n} \simeq \tau_{\leq n}\tau_{\geq m}$.

A simple but important feature of a t-structure is the existence of truncation functors with analogous properties.

Lemma 2.10.3. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

1. The inclusion $i_{\leq n} : \mathcal{D}^{\leq n} \hookrightarrow \mathcal{D}$ has a right adjoint $\tau_{\leq n}\mathcal{D} \to \mathcal{D}^{\leq n}$. Similarly $i_{\geq m} : \mathcal{D}^{\geq m} \hookrightarrow \mathcal{D}$ has a left adjoint $\tau_{\geq m}\mathcal{D} \to \mathcal{D}^{\geq m}$.

2. For any $\mathcal{F} \in \mathcal{D}$, there is a triangle

$$\tau_{\leq 0}\mathcal{F} \to \mathcal{F} \to \tau_{\geq 1}\mathcal{F}$$

and any two triangles $A \to \mathcal{F} \to B$ with $A \in \mathcal{D}^{\leq 0}, B \in \mathcal{D}^{\geq 1}$ are canonically isomorphic.

3. For $m \leq n$ there are natural isomorphisms of functors $\tau_{\geq m}\tau_{\leq n} \simeq \tau_{\leq n}\tau_{\geq m}$. Let $\tau_{[m,n]}$ be either functor, well-defined up to a natural isomorphism.

To show the existence of $\tau_{\leq 0}$ and $\tau_{\geq 1}$, one invokes condition 3. of the definition of t-structure and checks functoriality. The functors $\tau_{\leq n}, \tau_{\geq m}$ are then constructed by translation.

Theorem 2.10.4. *The heart $\mathcal{A}$ of a t-structure is abelian. Given a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathcal{A}$, let $\mathcal{H} \in \mathcal{D}$ complete the morphism to a triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$. Then* *the composition* $\tau_{\leq -1}\mathcal{H}[-1] \to \mathcal{H}[-1] \to \mathcal{F}$ *is the kernel of* f *and the composition* $\mathcal{G} \to \mathcal{H} \to \tau_{\geq 0}\mathcal{H}$ *is the cokernel of* f*.*

The next theorem justifies our description of a t-structure as a way of looking at a triangulated category $\mathcal{D}$ as if it were the derived category of its heart.

Theorem 2.10.5. *Let* $\mathcal{A}$ *be the heart of a* t*-structure* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *on a triangulated category* $\mathcal{D}$*. Define a functor*

$$H^0 := \tau_{[0,0]} : \mathcal{D} \to \mathcal{A}.$$

Similarly, by translation, define $H^i(\mathcal{F}) := H^0(\mathcal{F}[i])$ *for* $\mathcal{F} \in \mathcal{D}$*.*

1. H^0 *is a cohomological functor, so for each exact triangle* $A \to B \to C \to A[1]$ *there is a long exact sequence*

$$(\mathbf{2.10.1}) \qquad \cdots \to H^{-1}(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to \cdots$$

If in addition $\cap_n \mathcal{D}^{\leq n} = \cap_n \mathcal{D}^{\geq n} = \{0\}$*, then*

2. A morphism $f : \mathcal{F} \to \mathcal{G}$ *is an isomorphism in* $\mathcal{D}$ *if and only if* $H^i(f)$ *is an isomorphism for all* i*.*

3. $\mathcal{D}^{\leq n} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for all } i > n\}$ *and* $\mathcal{D}^{\geq n} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for all } i < n\}$*.*

The functors H^i are called the cohomology functors on $\mathcal{D}$ with respect to the given t-structure.

In order for t-structures and cohomology functors to behave as expected, one usually imposes the following condition satisfied by the standard t-structure on $D^b(\mathcal{A})$.

Definition 2.10.6. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is **bounded** if $\cap_n \mathcal{D}^{\leq n} = \cap_n \mathcal{D}^{\geq n} = \{0\}$ and $H^i(\mathcal{F})$ is non-zero for only finitely many i.

The following useful lemma is proved using the octahedron axiom.

Lemma 2.10.7. *A bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is determined by its heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. $\mathcal{D}^{\leq 0}$ is the extension-closed subcategory of $\mathcal{D}$ generated by $\mathcal{A}[i]$ for $i \geq 0$. If $\mathcal{D}$ has two bounded t-structures with nested hearts $\mathcal{A}' \subseteq \mathcal{A}$, then in fact $\mathcal{A}' = \mathcal{A}$ and so the t-structures are the same.*

2.11 Grothendieck groups

Definition 2.11.1. The **Grothendieck group** $K_0(\mathcal{D})$ of a triangulated category is the free abelian group on the objects of $\mathcal{D}$ modulo the subgroup generated by elements $A - B + C$ such that there is an exact triangle $A \to B \to C$.

Remark 2.11.2. If $\mathcal{D}$ is a triangulated category carrying a bounded t-structure with heart $\mathcal{A}$, then there is an isomorphism

$$K_0(\mathcal{D}) \simeq K_0(\mathcal{A}),$$

where the group on the right is the usual Grothendieck group of an abelian category. In one direction, there is a natural homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{A})$ which on the class of an object $A \in \mathcal{D}$ is $[A] \mapsto \sum_i (-1)^i [H^i(A)]$. The sum is finite by boundedness of the t-structure, and we see that the map indeed vanishes on the relations $A - B + C$ by considering the long exact sequence in cohomology (2.10.1) for the exact triangle $A \to B \to C$. In the other direction, the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$ induces a homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{D})$ since short exact sequences in $\mathcal{A}$ are sent to exact triangles in $\mathcal{D}$.

These two natural homomorphisms are inverse to each other and so we have the identification $K_0(\mathcal{D}) \simeq K_0(\mathcal{A})$.

Since a triangle functor between triangulated categories $\Phi : \mathcal{D} \to \mathcal{D}'$ sends exact triangles to exact triangles, it induces a group homomorphism $K_0(\mathcal{D}) \to K_0(\mathcal{D}')$. For instance, if $\mathcal{D} = D^b(\mathcal{A}), \mathcal{D}' = D^b(\mathcal{B})$ and $\Phi : \mathcal{A} \to \mathcal{B}$ is left exact with total derived functor $R\Phi : D^b(\mathcal{A}) \to D^b(\mathcal{B})$, then under the isomorphisms $K_0(\mathcal{D}) \simeq K_0(\mathcal{A})$ and $K_0() \simeq K_0(\mathcal{B})$, the induced map between the Grothendieck groups is given by

$$A \mapsto \sum_i (-1)^i R^i \Phi(A).$$

for $A \in \mathcal{D} = D^b(\mathcal{A})$.

Chapter 3

Tilting objects and the resolution of the diagonal

The central theme of this thesis is the construction of derived equivalences between sheaves on varieties and modules over non-commutative algebras. Theorem 3.1.1 formulates some conditions for constructing such an equivalence from a 'tilting object' and provides the foundation for all subsequent results.

In Section 3.2 we construct 'Beilinson's resolution of the diagonal' for projective space [3] which we use in Theorem 3.2.1 to construct a tilting object in the equivariant derived category. As a consequence of this construction we are able to describe a good basis for the equivariant K-theory of projective space.

3.1 Tilting objects

Given an object $T \in D_G(X)$, the unbounded derived category of G-equivariant quasi-coherent sheaves, we have a differential graded (dg) algebra $A := R\mathrm{Hom}_{\mathrm{G}}(T, T)^{\mathrm{op}}$ and a natural functor

$$\Psi := R\mathrm{Hom}_{\mathrm{G}}(T, -) : D_G(X) \to D(A),$$

where the action of A on $R\mathrm{Hom}_{\mathrm{G}}(T, \mathcal{F})$ is given by precomposition with elements of $R\mathrm{Hom}_{\mathrm{G}}(T, T)$.

Conditions for this functor to be an equivalence and to restrict to an equivalence between bounded derived categories of finitely generated objects are known under very general hypotheses by theorems of Keller and Ben-Zvi-Francis-Nadler. The following statement will be sufficient for our purposes.

Theorem 3.1.1. *1. The functor* $\Psi = R\mathrm{Hom}_{\mathrm{G}}(T, -) : D_G(X) \to D(A)$ *is an equivalence if and only if the object* $T \in D_G(X)$ *is* **compact**, *meaning* $\mathrm{Hom}_G(T, -)$ *commutes with small direct sums, and* **generates**, *meaning that* $R\mathrm{Hom}_G(T, \mathcal{F}) = 0$ *implies* $\mathcal{F} = 0$.

If moreover $\mathrm{Hom}_G(T, T[i]) = 0$ *for* $i \neq 0$ *(*'Ext vanishing'*), then the dg algebra is just (quasi-isomorphic to) an algebra.*

2. *If* T *satisfies the hypotheses of* 1. *and if* X *is smooth and quasi-projective, then the equivalence* $\Psi = R\mathrm{Hom}_G(T, -)$ *restricts to an equivalence* $\Psi : D^b_G(X) \to \mathrm{perf}A$, *where* $\mathrm{perf}(A) \subset D(A)$ *is the full triangulated subcategory of* perfect *complexes, those quasi-isomorphic to a bounded complex of finitely generated projective modules.*

3. *If every finitely generated A-module has a finite resolution by finitely generated projectives, then* $\mathrm{perf}(A) \simeq D^b(A)$, *the bounded derived category of finitely generated A-modules. In this case, if the hypotheses of* 1. *and* 2. *are satisfied, then*

the equivalence Ψ *restricts to an equivalence*

$$\Psi = R\mathrm{Hom}_{\mathrm{G}}(T, -) : D^b_G(X) \to D^b(A).$$

A compact generator T *such that* $\mathrm{Hom}_G(T, T[i]) = 0$ *for* $i \neq 0$ *is known as a* **classical tilting object**.

Proof. 1. This is just a theorem of Keller [36, Theorem 8.5, part b)], which works more generally upon replacing $D_G(X)$ with an 'algebraic' triangulated category (more or less, one admitting a notion of $R\mathrm{Hom}$) having arbitrary set-indexed direct sums.

2. First note that the equivalence Ψ restricts to an equivalence between the subcategories of compact objects. Indeed, given a compact object $C \in D_G(X)$, we can use the adjunction $\Psi \dashv \Psi^{-1}$ and the fact that equivalences preserve sums to see that $\mathrm{Hom}_A(\Psi(C), -)$ commutes with sums:

$$\mathrm{Hom}_A(\Psi(C), \bigoplus_i \mathcal{F}_i) \simeq \mathrm{Hom}_G(C, \bigoplus_i \Psi^{-1}(\mathcal{F}_i))$$
$$\simeq \bigoplus_i \mathrm{Hom}_G(C, \Psi^{-1}(\mathcal{F}_i)) \simeq \bigoplus_i \mathrm{Hom}_A(\Psi(C), \mathcal{F}_i).$$

For X quasi-projective, the compact objects of $D_G(X)$ are known to be the bounded complexes of equivariant vector bundles [6, Corollary 4.15], and when X is smooth, every G-sheaf has a finite equivariant resolution by vector bundles, so the compact objects are all of $D^b_G(X)$.

On the algebraic side, the compact objects in $D(A)$ are precisely the perfect complexes [36, pg. 8.2]. Thus the promised equivalence $\Psi : D^b_G(X) \to \mathrm{perf}A$ is simply that between categories of compact objects.

3. This is clear.

□

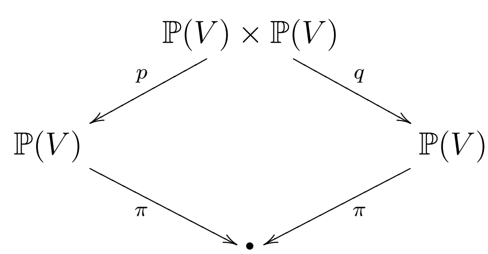
3.2 Beilinson's resolution of the diagonal

Recall from Section 2.8 that there is a natural isomorphism of functors $\Phi_{\vec{\mathcal{O}}_\Delta} \simeq \mathrm{Id}_X$, where $\Phi_{\vec{\mathcal{O}}_\Delta}$ is the integral transform with kernel $\mathcal{O}_\Delta \in D(X \times X)$.

For $X = \mathbb{P}(V)$, Beilinson [3] gave a canonical construction of a bounded complex of vector bundles $\mathcal{E}^\bullet$ on $X \times X$ together with an isomorphism $\mathcal{E}^\bullet \simeq \mathcal{O}_\Delta$ as objects of $D(\mathbb{P}(V) \times \mathbb{P}(V))$. This construction, known as 'Beilinson's resolution of the diagonal', is very useful for understanding quasi-coherent sheaves on $\mathbb{P}(V)$. The strategy is to take a sheaf $\mathcal{F}$ and feed it through the two integral transforms $\Phi_{\vec{\mathcal{E}}^\bullet}$ and $\Phi_{\vec{\mathcal{O}}_\Delta}$. The isomorphism $\mathcal{E}^\bullet \simeq \mathcal{O}_\Delta$ of kernels then produces an isomorphism $\Phi_{\vec{\mathcal{E}}^\bullet}(\mathcal{F}) \simeq \Phi_{\vec{\mathcal{O}}_\Delta}(\mathcal{F}) \simeq \mathcal{F}$ in $D(\mathbb{P}(V))$. Like the Fourier transform of a function, the object $\Phi_{\vec{\mathcal{E}}^\bullet}(\mathcal{F})$ is sometimes easier to understand than the original sheaf $\mathcal{F}$.

Let V be an $n+1$-dimensional vector space and consider the diagram of the product over a point:

(3.2.1)



Letting V denote the trivial vector bundle with fibre V (on whichever space is understood from the context) and T the tangent bundle of $\mathbb{P}(V)$, the Euler sequence on $\mathbb{P}(V)$ is

$$0 \to \mathcal{O}(-1) \to V \to T(-1) \to 0. \tag{3.2.2}$$

Pulling-back along p and along q, we obtain two exact sequences of sheaves on $\mathbb{P}(V) \times \mathbb{P}(V)$:

$$0 \to p^*\mathcal{O}(-1) \to V \to p^*T(-1) \to 0$$
$$0 \to q^*\mathcal{O}(-1) \to V \to q^*T(-1) \to 0$$

The composition $p^*\mathcal{O}(-1) \to V \to q^*T(-1)$ can be thought of as a section σ of $\mathcal{O}(1) \boxtimes T(-1) := p^*\mathcal{O}(1) \otimes q^*T(-1)$.

I claim that the vanishing locus of σ is precisely the diagonal $\Delta \subset \mathbb{P}(V) \times \mathbb{P}(V)$. To see this, consider the composition $p^*\mathcal{O}(-1) \to V \to q^*T(-1)$ restricted to a point (L, L'), where L and L' are lines in V or points in $\mathbb{P}(V)$. By definition, the fibre of $p^*\mathcal{O}(-1)$ at (L, L') is just the line $L \subset V$, and the kernel of $V \to q^*T(-1)$ is just $L' \subset V$. Thus the composition $p^*\mathcal{O}(-1) \to V \to q^*T(-1)$ is zero over the point (L, L') if and only if $L = L'$. This shows that set-theoretically the zero-locus of σ is the diagonal $\Delta \subset \mathbb{P}(V) \times \mathbb{P}(V)$. Moreover, by looking at local coordinates, one can check that in fact the zero-locus is reduced and hence is equal to Δ scheme-theoretically.

Since σ vanishes precisely along Δ, the dual of the Koszul complex for $\sigma : \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V)} \to \mathcal{O}(1) \boxtimes T(-1)$ gives the desired locally free resolution of $\mathcal{O}_\Delta$:

$$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0. \tag{3.2.3}$$

Because the resolution was constructed canonically from the Euler sequence on $\mathbb{P}(V)$, which is naturally equivariant for any linear action, the resolution is automatically equivariant.

It will be useful to know what an individual term $\mathcal{P} := \mathcal{O}(-d) \boxtimes \Omega^d(d)$ of the resolution does when used as the kernel of an integral transform. For the left-to-right integral transform, we see that

$$\begin{aligned} \Phi_{\mathcal{P}}^{\rightarrow}(\mathcal{F}) &\simeq Rq_*(Lp^*\mathcal{F}(-d) \otimes Lq^*(\Omega^d(d))) \\ &\simeq (Rq_*Lp^*\mathcal{F}(-d)) \otimes \Omega^d(d) \\ &\simeq R\Gamma(\mathcal{F}(-d)) \otimes \Omega^d(d) \end{aligned} \tag{3.2.4}$$

where the first isomorphism is the definition of the integral transform $\Phi_{\mathcal{P}}^{\rightarrow}$, the second is by the projection formula, and the last is by flat base change for the diagram (3.2.1). Similarly, going right-to-left, we have

$$\Phi_{\mathcal{P}}^{\leftarrow}(\mathcal{F}) \simeq R\Gamma(\mathcal{F} \otimes \Omega^d(d)) \otimes \mathcal{O}(-d).$$

Our main application of the resolution of the diagonal is the following.

Theorem 3.2.1. *Let G be a finite group acting linearly on a vector space V of dimension $n+1$, and let W_i, $i \in I$ be the irreducible representations of G. Then*

$$T := \bigoplus_{i \in I} \bigoplus_{d=0}^{n} \mathcal{O}(d) \otimes W_i$$

is a classical tilting object in $D_G(\mathbb{P}(V))$ and the algebra $A := \operatorname{End}_G(T)^{\mathrm{op}}$ has finite global dimension.

By Theorem 3.1.1, the usual functor $R\operatorname{Hom}_G(T, -)$ gives an equivalence

$$D^b_G(\mathbb{P}(V)) \simeq D^b(A)$$

where $A := \operatorname{End}_G(T)^{\mathrm{op}}$.

Proof. We divide the proof into a few parts. First we show that T is a classical tilting object in $D_G(X)$. By Theorem 3.1.1, this will establish an equivalence Ψ :

$D_G(\mathbb{P}(V)) \to D(A)$ that restricts to an equivalence $\Psi : D^b_G(\mathbb{P}(V)) \to \operatorname{perf} A$. Next, to show that A is of finite global dimension, we use these derived equivalences to reduce the problem to understanding how the simple A-modules behave under the inverse equivalence Ψ^{-1}. Finally, this behaviour is described in Lemmas 3.2.2 and 3.2.3, which will complete the proof of the present theorem.

To show that T is a classical tilting object, we must check generation and Ext-vanishing. Note that since T is locally free, it is immediately compact.

To begin with, consider the case $G = 1$.

It is enough to check Ext-vanishing between the summands $\mathcal{O}, \mathcal{O}(1) \ldots, \mathcal{O}(n)$ of T'. But for this, note that since $|l - k| \leq n$, we have $\operatorname{Ext}^i(\mathcal{O}(k), \mathcal{O}(l)) \simeq H^i(\mathbb{P}(V), \mathcal{O}(l-k)) = 0$ for all $i > 0$.

Since $\operatorname{Ext}^k(T, \mathcal{F}) \simeq \bigoplus_{i=0}^n \operatorname{Ext}^k(\mathcal{O}(d), \mathcal{F})$, to show that T generates $D(\mathbb{P}(V))$ it is enough to see that $\operatorname{Ext}^k(\mathcal{O}(d), \mathcal{F}) \simeq R^k\Gamma(\mathcal{F}(-d)) = 0$ for all k and for all $d = 0, \ldots, n$ implies $\mathcal{F} \simeq 0$.

So assume $R^k\Gamma(\mathcal{F}(-d)) = 0$ for all k and for all $d = 0, \ldots, n$. To establish the implication $\mathcal{F} \simeq 0$, break up the resolution (3.2.3) into a collection of short exact sequences

$$\begin{gathered} 0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \mathcal{O}(-n+1) \boxtimes \Omega^{n-1}(n-1) \to \mathcal{P}_{n-1} \to 0 \\ 0 \to \mathcal{P}_{n-1} \to \mathcal{O}(-n+2) \boxtimes \Omega^{n-2}(n-2) \to \mathcal{P}_{n-2} \to 0 \\ \vdots \\ 0 \to \mathcal{P}_1 \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0 \end{gathered} \tag{3.2.5}$$

Taking the terms of the first short exact sequence as kernels, apply the left-to-right integral transforms to $\mathcal{F}$ and use (3.2.4) to get an exact triangle

$$R\Gamma(\mathcal{F} \otimes \mathcal{O}(-n)) \otimes \Omega^n(n) \to R\Gamma(\mathcal{F} \otimes \mathcal{O}(-n+1)) \otimes \Omega^{n-1}(n-1) \to \Phi^{\rightarrow}_{\mathcal{P}_{n-1}}(\mathcal{F}).$$

The first two terms vanish by assumption, so $\Phi_{\overrightarrow{\mathcal{P}_{n-1}}}(\mathcal{F}) = 0$. Repeating this process for each exact sequence $0 \to \mathcal{P}_k \to \mathcal{O}(-k+1) \boxtimes \Omega^{k-1}(k-1) \to \mathcal{P}_{k-1} \to 0$, we end up at the last step with an exact triangle

$$R\Gamma(\mathcal{F} \otimes \mathcal{O}(-1)) \otimes \Omega(1) \to R\Gamma(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}$$

in which the first two terms and hence the last term $\mathcal{F}$ are zero. Hence T generates $D(\mathbb{P}(V))$, as claimed.

Now let G be any finite group acting linearly on V. We check Ext-vanishing and generation for $T \in D_G(\mathbb{P}(V))$.

Again, we check Ext-vanishing between summands of T. Note that for every $k \in \mathbb{Z}$ we have canonical isomorphisms $\mathrm{Ext}^k(\mathcal{O}(d) \otimes W_i, \mathcal{O}(e) \otimes W_j) \simeq \mathrm{Hom}(W_i, W_j) \otimes \mathrm{Ext}^k(\mathcal{O}(d), \mathcal{O}(e))$. By Ext-vanishing for the case $G = 1$, the second factors in these tensor products are zero for $k \neq 0$. Thus after taking G-invariants

$$\mathrm{Ext}^k_G(\mathcal{O}(d) \otimes W_i, \mathcal{O}(e) \otimes W_j) = 0$$

for $k \neq 0$, every $d, e = 0, \ldots, n$, and every $i, j \in I$, which is the Ext-vanishing that we need.

To see that T generates $D_G(\mathbb{P}(V))$, suppose that $\mathrm{Ext}^k_G(\mathcal{O}(d) \otimes W_i, \mathcal{F}) \simeq (W_i^* \otimes \mathrm{Ext}^k(\mathcal{O}(d), \mathcal{F}))^G = 0$ for every $d = 0, \ldots, n$ and $i \in I$. Then $\mathrm{Ext}^k(\mathcal{O}(d), \mathcal{F})$ contains no copy of any irreducible representation W_i and so must be zero. By generation for the case $G = 1$, we must have $\mathcal{F} = 0$, and so T generates $D_G(\mathbb{P}(V))$.

Having established that T is a classical tilting object, we know by Theorem 3.2.1 that the functor $\Psi = R\mathrm{Hom}_{\mathrm{G}}(T, -) : D_G(\mathbb{P}(V)) \to D(A)$ is an equivalence and restricts to an equivalence $\Psi : D^b_G(\mathbb{P}(V)) \to \mathrm{perf}\ A$.

It thus remains to see that A has finite global dimension, which in particular will imply that $D^b(A) \simeq \operatorname{perf} A$. Since A is finite dimensional, it is enough to see that each simple module S has a finite length projective resolution [2, Proposition 5.1] or, equivalently, that S is (isomorphic to) an object of perf A.

To see that $S \in \operatorname{perf} A$, consider the object $\Psi^{-1}(S) \in D_G(\mathbb{P}(V))$. The following two lemmas describe the object $\Psi^{-1}(S)$ precisely, showing in particular that it lies in the subcategory $D^b_G(\mathbb{P}(V)) \subset D_G(\mathbb{P}(V))$. Thus under the restricted equivalence $\Psi : D^b_G(\mathbb{P}(V)) \to \operatorname{perf} A$ we have $\Psi(\Psi^{-1}(S)) \simeq S \in \operatorname{perf} A$, and so A has finite global dimension.

The following two lemmas describe precisely the simple modules S of the algebra A and the objects $\Psi^{-1}(S) \in D^b_G(\mathbb{P}(V))$. □

Lemma 3.2.2. *Let $e_{i,d}$ for $i \in I$, $d = 0, \dots, n$ be the projection of A onto the summand* $\operatorname{Hom_G}(\mathcal{O}(d) \otimes W_i, \mathcal{O}(d) \otimes W_i)$.

1. *The $e_{i,d}$ form a complete set of primitive, orthogonal idempotents in A.*

2. *The modules $P_{i,d} := A \cdot e_{i,d}$ form a complete list of indecomposable projectives. The modules $S_{i,d} := e_{i,d} \cdot A \cdot e_{i,d}$ form a complete list of simples.*

3. *Under the equivalence Ψ, there are isomorphisms $\Psi(\mathcal{O}(d) \otimes W_i) \simeq P_{i,d}$ and $\Psi(\Omega^d(d) \otimes W_i[d]) \simeq S_{i,d}$.*

Proof. 1. The $e_{i,d}$ are a complete set of orthogonal idempotents by definition. To see that they are primitive, we use the equivalence Ψ. Note that there are isomorphisms

$$\begin{aligned}\Psi(\mathcal{O}(d)\otimes W_i) &\simeq R\mathrm{Hom}_G(\bigoplus_{p,j}\mathcal{O}(p)\otimes W_j, \mathcal{O}(d)\otimes W_i) \\ &\simeq \mathrm{Hom}_G(\bigoplus_{p,j}\mathcal{O}(p)\otimes W_j, \mathcal{O}(d)\otimes W_i) \qquad \text{since } |p-d|\leq n \\ &\simeq A\cdot e_{i,d} \qquad \text{since } e_{i,d} \text{ is projection}\end{aligned}$$

Thus $\mathrm{Hom}_A(A\cdot e_{i,d}, A\cdot e_{i,d}) \simeq \mathrm{Hom}_G(\mathcal{O}(d)\otimes W_i, \mathcal{O}(d)\otimes W_i) \simeq \mathrm{Hom}_G(W_i, W_i)$. By Schur's lemma, the latter is a finite dimensional division algebra over the ground field k, so the idempotent $e_{i,d}$ must be primitive [2, Chapter 1, Proposition 4.7].

2. See [2, Proposition 4.8, c] for the standard relation between primitive idempotents and the indecomposable projectives $P_{i,d}$. See [2, Propositon 4.9] for the simplicity of the modules $S_{i,d}$.

3. We have already noted that there is an isomorphism $\Psi(\mathcal{O}(d)\otimes W_i) \simeq P_{i,d}$ in the proof of 1. To establish the isomorphism $\Psi(\Omega^d(d)\otimes W_i[d]) \simeq S_{i,d}$, note that

$$\begin{aligned}\Psi(\Omega^d(d)\otimes W_i[d]) &\simeq R\mathrm{Hom}_G(\bigoplus_{p,j}\mathcal{O}(p)\otimes W_j, \Omega^d(d)\otimes W_i[d]) \\ &\simeq (\bigoplus_{p,j} R\mathrm{Hom}(\mathcal{O}(p), \Omega^d(d)[d])\otimes \mathrm{Hom}(W_j, W_i))^G.\end{aligned}$$

I claim that $R\mathrm{Hom}(\mathcal{O}(p), \Omega^d(d)[d]) = 0$ if $p\neq d$ and $R\mathrm{Hom}(\mathcal{O}(d), \Omega^d(d)[d]) \simeq \mathrm{k}$ (concentrated in degree zero), so that

$$\Psi(\Omega^d(d)\otimes W_i[d]) \simeq \mathrm{Hom}_{\mathrm{G}}(W_i, W_i) \simeq S_{i,d},$$

as desired. To sustain the claim, note that since $\mathcal{O}(p)$ is locally free, $R\mathrm{Hom}(\mathcal{O}(p), \Omega^d(d)[d]) \simeq R\Gamma(\Omega^d(d-p)[d])$, and the claim follows from the following lemma.

□

Lemma 3.2.3. *1. If $-n+k \leq d < 0$ or $0 < d \leq k$, then $H^i(\Omega^k(d)) = 0$ for all i.*

2. For $0 \leq k \leq n$,

$$H^i(\Omega^k) = \begin{cases} \mathrm{k} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Proof. The first claim is 'Bott vanishing' from [15, Proposition 14.4].

For the second claim, we induct on k. If $k = 0$, we are looking at the structure sheaf $\mathcal{O}$, and the statement is immediate. Suppose the statement is true for $k-1$ and consider the short exact sequence

$$0 \to \Omega^k \to \bigwedge^k V^*(-k) \to \Omega^{k-1} \to 0$$

obtained by taking the kth wedge power of the dual of the Euler sequence 3.2.2 and twisting by $-k$. We have already treated the case $k = 0$. If $0 < k \leq n$, then all cohomology groups of the middle term vanish and the long exact sequence in cohomology gives $H^i(\Omega^{k-1}) \simeq H^{i+1}(\Omega^k)$ for all i. But by induction, the former group is isomorphic to k when $i = k-1$ or $i+1 = k$, and zero otherwise, which establishes the claim. □

Remark 3.2.4. Note that since tensoring with $\mathcal{O}(-n)$ is an autoequivalence of $D_G(V)$, the object

$$T := \bigoplus_{i \in I} \bigoplus_{d=0}^{n} \mathcal{O}(-d) \otimes W_i \tag{3.2.6}$$

is another compact generator for $D_G(\mathbb{P}(V))$ with the same endomorphism algebra as the generator from Theorem 3.2.1.

3.3 Equivariant Grothendieck group

As we saw in Section 3.2, Beilinson's resolution of the diagonal $\mathcal{E}^\bullet \simeq \mathcal{O}_\Delta$ gives an isomorphism $\Phi_{\vec{\mathcal{E}^\bullet}}(\mathcal{F}) \simeq \mathcal{F}$ of objects for each $\mathcal{F} \in D_G(\mathbb{P}(V))$. Like the Fourier transform of a function, the object $\Phi_{\vec{\mathcal{E}^\bullet}}(\mathcal{F})$ is sometimes easier to understand than the original object $\mathcal{F}$. In particular, we shall see how Theorem 3.2.1 provides us with a good basis for $K_0^G(\mathbb{P}(V))$, the Grothendieck group of G-equivariant coherent sheaves on $\mathbb{P}(V)$, and how the integral transform $\Phi_{\vec{\mathcal{E}^\bullet}}$ will tell us precisely how to express the class of an object $\mathcal{F} \in D_G^b(\mathbb{P}(V))$ in this basis.

Using the compact generator $T := \bigoplus \mathcal{O}(-d) \otimes W_i$ from Remark 3.2.6, we have an equivalence $\Psi := R\mathrm{Hom}_{\mathrm{G}}(T, -) : D_G^b(\mathbb{P}(V)) \to D^b(A)$, where $A := \mathrm{End}_G(T)^{\mathrm{op}}$. Applying the inverse equivalence Ψ^{-1} to the standard t-structure on $D^b(A)$ endows $D_G^b(\mathbb{P}(V))$ with a non-standard t-structure with heart

$$\Psi^{-1}(A\text{-mod}) = \{\mathcal{G} \in D_G^b(\mathbb{P}(V)) \mid \mathrm{Hom}_G(\mathcal{F}, \mathcal{G}[i]) = 0 \text{ for } i \neq 0\}.$$

This is very useful and in some ways surprising since $\mathrm{Coh}_G(\mathbb{P}(V))$, the heart of the standard t-structure on $D_G^b(\mathbb{P}(V))$, lacks both injectives and projectives, while A-mod has enough of both and is furthermore of finite length, meaning that every module has a filtration with simple quotients which, by the Jordan-Holder theorem, depend only on the module, not on the filtration.

As one application of this we get a useful basis for $K_0^G(\mathbb{P}(V))$. Under the equivalence Ψ the object $\mathcal{O}(-d)\otimes W_j \in D_G^b(\mathbb{P}(V))$ is sent to the A-module $\mathrm{Hom}_{\mathrm{G}}(\oplus_{p,i}\mathcal{O}(-p)\otimes W_i, \mathcal{O}(-d) \otimes W_j) \simeq A \cdot e_{d,j}$, where as in Lemma 3.2.2 $e_{d,j}$ is the projection onto the summand $\mathrm{Hom}_{\mathrm{G}}(\mathcal{O}(-d) \otimes W_j, \mathcal{O}(-d) \otimes W_j)$ and the $A \cdot e_{d,j}$ are a complete set of indecomposable projectives of the algebra A. Their classes therefore form a basis for the Grothendieck group $K_0(A)$ of finitely generated A-modules and so, under the isomorphism $K_0(A) \simeq K_0^G(\mathbb{P}(V))$, the classes of the $\mathcal{O}(-d) \otimes W_j$ form a basis for $K_0^G(\mathbb{P}(V))$.

Among the many bases we could find for $K_0^G(\mathbb{P}(V))$, this one is particularly convenient for multiplication. Since

$$[\mathcal{O}(-d) \otimes W_i] \cdot [\mathcal{O}(-e) \otimes W_j] = [\mathcal{O}(-d-e) \otimes W_i \otimes W_j],$$

in order to express any product in our basis we need to be able to decompose $W_i \otimes W_j$ into irreducible representations, which can be done using character theory, and we need to know how to express $\mathcal{O}(-d-e)$ in our basis. In fact, once we know how to express $\mathcal{O}(-n-1)$, we can work out the expression for $\mathcal{O}(-d-e)$ iteratively.

Using the terms of the short exact sequences 3.2.5 as kernels for integral transforms, we have exact triangles

$$\Phi_{\mathcal{P}_{k+1}}^{\rightarrow}(\mathcal{O}(-n-1)) \to R\Gamma(\Omega^k(k-n-1)) \otimes \mathcal{O}(-k) \to \Phi_{\mathcal{P}_k}^{\rightarrow}(\mathcal{O}(-n-1)).$$

Working from the last exact sequence up to the first, we see then that in $K_0^G(\mathbb{P}(V))$ we have an equality

$$\mathcal{O}(-n-1) = \sum_{k=0}^{n} (-1)^k R\Gamma(\Omega^k(k-n-1)) \otimes \mathcal{O}(-k).$$

The following lemma determines $R\Gamma(\Omega^k(k-n-1))$.

Lemma 3.3.1. *For $k = 0, \ldots, n$,*

$$H^i(\Omega^k(k-n-1)) = \begin{cases} \bigwedge^{n+1-k} V & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We use induction on k.

If $k=0$, then $H^i(\mathcal{O}(-n-1))$ is non-zero only for $i=n$. In this case, $H^n(\mathcal{O}(-n-1)) \simeq H^0(\omega(n+1))^* \simeq \bigwedge^{n+1} V$, where the first isomorphism is by Serre duality and the second is induced by the isomorphism of vector bundles $\omega(n+1) \simeq \bigwedge^{n+1} V^*$ coming from the $n+1$st wedge power of the dual Euler sequence.

Now assume the statement holds for $k-1$ and consider the sequence $0 \to \Omega^k(k-n-1) \to \bigwedge^k V^*(-n-1) \to \Omega^{k-1}(k-n-1) \to 0$. All cohomology of the last term vanishes by Lemma 3.2.3, so for each i we have an isomorphism $H^i(\Omega^k(k-n-1)) \simeq H^i(\bigwedge^k V^*(-n-1))$. The latter is non-zero only when $i=n$, in which case $H^n(\Omega^k(k-n-1)) \simeq H^n(\bigwedge^k V^*(-n-1)) \simeq \bigwedge^k V^* \otimes \bigwedge^{n+1} V \simeq \bigwedge^{n+1-k} V$, where the second to last isomorphism uses $H^n(\mathcal{O}(-n-1)) \simeq \bigwedge^{n+1} V$ (case $k=0$ of the present lemma) and the last isomorphism is induced by the non-degenerate pairing $\bigwedge^{n+1-k} V \otimes \bigwedge^k V \to \bigwedge^{n+1} V$. $\square$

Proposition 3.3.2. *The classes of $\mathcal{O}(-d) \otimes W_i$ where $d=0,\ldots,n$ and $i \in I$, form a basis for $K_0^G(\mathbb{P}(V))$. We have the equality*

$$\mathcal{O}(-n-1) = \sum_k (-1)^{k+n} \bigwedge^{n+1-k} V \otimes \mathcal{O}(-k).$$

Proof. As we noted above, $\mathcal{O}(-n-1) = \sum_k (-1)^k R\Gamma(\Omega^k(k-n-1)) \otimes \mathcal{O}(-k)$, and by part 2. of Lemma 3.3.1, $R\Gamma(\Omega^k(k-n-1)) = (-1)^n H^n(\Omega^k(k-n-1)) = \bigwedge^{n+1-k} V$. $\square$

Chapter 4

Equivariant sheaves and Koszul duality

4.1 Equivariant sheaves and Koszul duality

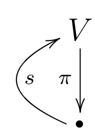
We construct an exact equivalence between the categories of G-equivariant sheaves on affine space V and modules over a certain algebra B. We show that this algebra is Koszul and use this to describe the algebra B as the quotient of the McKay quiver by quadratic relations.

When $G \subset SL_2(\mathbb{C})$ and Γ the affine Dynkin diagram associated to G by McKay's observation, we note that the Koszul dual $E(B)$ is the Ext-algebra of the Γ-configuration of spherical objects indexed by irreducible representations of G introducted by Seidel-Thomas [48] and studied in [16]. Using Serre duality we compute a presentation for $E(B)$ and the dual presentation for B, which turns out to be the preprojective algebra of the diagram Γ. This example has been much studied in the literature. We include it to illustrate technique and to provide a reference for Chapter 5.

We conclude with another example, the symmetric group S_n acting on $\mathfrak{h}^* \oplus \mathfrak{h}$, where $\mathfrak{h}^*$ is the complexified root system of type A_{n-1}. We describe the McKay quiver for this example. Looking for an analogue of the spherical objects from the case $G \subset SL_2(\mathbb{C})$, we compute the Poincaré polynomials of some natural objects indexed by irreducible representations of S_n. For the trivial and sign representation, we see that we get $\mathbb{P}$-objects in the sense of Huybrechts-Thomas [33].

4.2 The general case

Let G be a finite group and V a finite dimensional representation over $\mathbb{C}$. Let I be an index set for the irreducible representations of G, and for $i \in I$, let W_i be the corresponding irreducible representation. We think of V as the total space of a G-equivariant vector bundle over a point, with projection π and zero-section s:



Taking the pull-back π^*W of the equivariant vector bundle $W = \oplus_i W_i$ on the point, we set

$$\textbf{(4.2.1)} \qquad B := \mathrm{End}_G(\pi^*W)^{\mathrm{op}} \simeq (\pi^*\mathrm{End}(W)^{\mathrm{op}})^G \simeq (S^\bullet V^* \otimes \mathrm{End}(W)^{\mathrm{op}})^G.$$

The following theorem shows that the algebra B encodes everything there is to know about G-equivariant coherent sheaves on the affine space V.

Theorem 4.2.1. *There is an equivalence*

$$\Psi = \mathrm{Hom}_G(\pi^*W, -) : \mathrm{Coh}_G(V) \longrightarrow B\text{-mod}$$

from G-equivariant coherent sheaves to left B-modules, where the action on an object $\Psi(\mathcal{F})$ is given by precomposition with elements of $B^{\mathrm{op}} = \mathrm{End}_G(\pi^* W)$.

Proof. Let us give a plan of the proof.

As in the proof of Theorem 3.2.1, we first establish an equivalence $R\Psi : D^b_G(V) \to \mathrm{perf}\ B$ between the bounded derived category of G-sheaves on V and the subcategory $\mathrm{perf}\ B \subset D^b(B)$ of bounded complexes of finitely generated projective B-modules. We shall see later (Theorem 4.2.4) that the algebra B is Koszul, and so by Theorem 4.2.3, B has finite global dimension. It follows that there is an equivalence $\mathrm{perf}\ B \simeq D^b(B)$ and hence an equivalence $R\Psi : D^b_G(V) \to D^b(B)$. To conclude, we show that the derived equivalence $R\Psi : D^b_G(V) \to D^b(B)$ implies that the functor $\Psi : \mathrm{Coh}_G(V) \to B\text{-mod}$ is an equivalence.

To establish the equivalence $R\Psi : D^b_G(V) \to \mathrm{perf}\ B$, we must check that $\pi^* W$ is a classical tilting object in the sense of Theorem 3.1.1, that is, we should have $\mathrm{Ext}^k_G(\pi^* W, \pi^* W) = 0$ for $k > 0$ and $\pi^* W$ should generate $D_G(V)$ in the sense that $\mathrm{Hom}_G(\pi^* W, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$. For the Ext vanishing, note that in fact for any $\mathcal{F}$, $\mathrm{Ext}^k_G(\pi^* W_i, \mathcal{F}) = 0$ for all $k > 0$ since $\pi^* W_i$ is projective (and in fact free). For generation, suppose $\mathrm{Hom}_G(\pi^* W_i, \mathcal{F}) \simeq \mathrm{Hom}_G(W_i, \pi_* \mathcal{F}) = 0$ for all i. This means that $\pi_* \mathcal{F}$ (the global sections of $\mathcal{F}$) contains no irreducible, so $\mathcal{F} = 0$ since V is affine. Hence $\pi^* W$ is a classical tilting object, as claimed.

Since we shall show in Theorem 4.2.4 that the algebra B is Koszul and hence of finite global dimension, we in fact have an equivalence $D^b_G(V) \simeq D^b(B)$. Now we want to see that this derived equivalence restricts to an equivalence $\mathrm{Coh}_G(V) \simeq B\text{-mod}$. By the above Ext vanishing, the functor $R\Psi = \Psi$ is exact and sends coherent sheaves to actual modules, not just complexes. By the derived equivalence, we know that Ψ

is full and faithful. It therefore remains to see that it is essentially surjective.

For this, given a B-module M, the derived equivalence implies that there is a complex $\mathcal{F}^\bullet$ of G-equivariant sheaves such that $R\Psi(\mathcal{F}^\bullet) = M$. We want to see that the cohomology of $\mathcal{F}^\bullet$ is concentrated in degree zero so that $\mathcal{F}^\bullet$ is quasi-isomorphic to an actual G-sheaf and thus Ψ will be essentially surjective. Recall the spectral sequence 2.3.1

$$E_2^{p,q} = \mathrm{Hom}_G(\pi^*W, H^q(\mathcal{F}^\bullet)[p]) \Rightarrow \mathrm{Hom}_G(\pi^*W, \mathcal{F}^\bullet[p+q]),$$

where the Homs are understood to be in the category $D^b_G(V)$.

By the Ext vanishing, only the first column ($p = 0$) of the E_2 term is non-zero so the sequence immediately degenerates and we have isomorphisms

$$\mathrm{Hom}_G(\pi^*W, H^q(\mathcal{F}^\bullet)) \simeq \mathrm{Ext}^q_G(\pi^*W, \mathcal{F}^\bullet) = R^q\Psi(\mathcal{F}^\bullet).$$

But the right hand side is zero for $q \neq 0$ since we assumed that $R\Psi(\mathcal{F}^\bullet) = M$, a module. Since π^*W generates, $H^q(\mathcal{F}^\bullet) = 0$ for $q \neq 0$, so $\mathcal{F}^\bullet$ is concentrated in degree zero, as desired. □

To understand the algebra B, we use some basic facts about Koszul algebras. A good reference for this material is [5].

Let B be a graded algebra with semisimple degree zero part B_0, which we also consider as a B-module via $B/B_{\geq 1} \simeq B_0$.

Definition 4.2.2. B is called **Koszul** if the algebra

$$E(B) := \mathrm{Ext}^\bullet_B(B_0, B_0)$$

is generated in degree 1.

We assume further that B is **finite**, meaning that each B_i is finitely generated as a left and a right B_0-module, and that B is Noetherian. We summarize the facts that we need in the following theorem, the proofs of which can be found in [5].

Theorem 4.2.3. *1. If B is Koszul, then B is* **quadratic***, meaning that the natural ring homomorphism $T^{\bullet}_{B_0}B_1 \to B$ is a surjection with kernel generated by a space of relations R in degree 2. Define the quadratic dual $B^!$ to be the algebra with dual generators B_1^* and dual relations $R^{\perp}$.*

Note that the tensor algebra is taken over B_0.

2. If B is Koszul, then so are $B^!$ and $E(B)$. There are canonical isomorphisms $E(B) \simeq B^{!\mathrm{op}}$ and $E(E(B)) \simeq (B^{!\mathrm{op}})^{!\mathrm{op}} \simeq B$.

3. If B is Koszul, it has finite global dimension.

4. (Numerical criterion) Assume B is an algebra over a field F and that there is a finite set of orthogonal idempotents $e_i \in B_0$ such that $B_0 = \oplus_i F e_i$. We can thus form a matrix of Poincaré series

$$P(B,t)_{i,j} = \sum_d t^d \dim_F\, e_i B_d e_j.$$

Since $E(B)_0 = \mathrm{Hom}_{B_0}(B_0, B_0) \simeq B_0^{\mathrm{op}} \simeq B_0$, we can also form the matrix $P(E(B),t)$.

Then B is Koszul if and only if

$$P(B,t) \cdot P(E(B),-t) = \mathrm{Id}.$$

Let us return to the algebra $B = \mathrm{End}_G(\pi^* W) \simeq (S^{\bullet} V^* \otimes \mathrm{End}(W)^{\mathrm{op}})^G$.

First, I claim that B is Noetherian. To see this, first note that the endomorphism algebra $\mathrm{End}_X(\mathcal{F})$ of any coherent sheaf $\mathcal{F}$ on a Noetherian scheme X must be Noetherian, since any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ of $\mathrm{End}_X(\mathcal{F})$ is in particular an ascending chain of submodules of $\mathrm{End}_X(\mathcal{F})$ thought of as a finitely generated module over the Noetherian algebra $A = \mathcal{O}_X(X)$ and so must eventually terminate. If X carries the action of a finite group, then by an ancient theorem of Noether A is finite as a module over the ring of invariants A^G, and so the invariant endomorphisms $\mathrm{End}_G(\mathcal{F})$ are a finitely generated module over A^G. Then the same argument on the ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ of $\mathrm{End}_X(\mathcal{F})$ works to show that the algebra $\mathrm{End}_G(\mathcal{F})$ is Noetherian.

Next, note that $B_0 = \oplus_i \mathrm{Hom}_{\mathrm{G}}(W_i, W_i) \simeq \oplus_i \mathbb{C} \cdot 1_{W_i}$, a commutative semi-simple algebra. Given this property of B_0 and the fact that B is Noetherian, we may try to apply the numerical criterion to check that our algebra B is Koszul.

To apply the numerical criterion, we need to understand $E(B)$. By the adjunction $\pi^* \dashv \pi_*$, we see that the image of $s_* W$ under the equivalence $\Psi : D^b_G(V) \simeq \mathrm{perf}\, B$ is

$$\mathrm{Hom}_G(\pi^* W, s_* W) \simeq \mathrm{Hom}_G(W, \pi_* s_* W) \simeq \oplus_i \mathrm{Hom}_G(W_i, W_i) \simeq B_0$$

and thus we have an isomorphism $E(B) \simeq \mathrm{Ext}^\bullet_B(B_0, B_0)\mathrm{Ext}^\bullet_G(s_* W, s_* W)$. The Koszul resolution $0 \to \pi^* \bigwedge^n V^* \otimes W \to \cdots \to \pi^* V^* \otimes W \to \pi^* W \to s_* W \to 0$ and the adjunction $Ls^* \vdash s_*$ then give isomorphisms

$$E(B) \simeq \mathrm{Ext}^\bullet_G(s_* W, s_* W) \simeq (\bigwedge^\bullet V \otimes \mathrm{End}(W))^G. \tag{4.2.2}$$

In order to apply the numerical criterion to our algebra B we need expressions for the Poincaré series of the graded vector spaces $(S^\bullet V^* \otimes U)^G$ and $(\bigwedge^\bullet V \otimes U)^G$ where U is some G-representation. Letting $S_U(t)$ be Poincaré series of the first and $E_U(t)$ the series of the second, Molien's formulae are

(4.2.3)

$$S_U(t) = \frac{1}{|G|}\sum_{g\in G} \frac{\chi_U(g)}{\det_V(1-g^{-1}\cdot t)}$$

$$E_U(t) = \frac{1}{|G|}\sum_{g\in G} \chi_U(g)\det_V(1+g\cdot t)$$

A proof of the first formula can be found in [24, Theorem 3.2.2], and the second formula follows similarly.

Theorem 4.2.4. *The algebra* $B = (S^\bullet V^* \otimes \mathrm{End}(W)^{\mathrm{op}})^G$ *is Koszul.*

Proof. We use the numerical criterion and the isomorphism $E(B) \simeq (\bigwedge^\bullet V \otimes \mathrm{End}(W))^G$. The degree zero part of B is commutative semisimple with one idempotent $e_i = 1_{W_i} \in \mathrm{Hom}_G(W_i, W_i)$ for each irreducible W_i. Thus we have matrices $S(t) := P(B,t)$ and $E(t) := P(E(B),t)$ of Poincaré series with rows and columns indexed by irreducibles.

We need to check that $S(t)\cdot E(-t) = \mathrm{Id}$. The (p,r) entry of the product takes the form $\sum_q S_{pq}(t)\cdot E_{qr}(-t)$. Letting χ_{kl} be the character of the representation $\mathrm{Hom}(W_k, W_l)$ and setting $\Delta_g = \det_V(1-g^{-1}\cdot t)$, Molien's formulae give

$$S_{pq} = \frac{1}{|G|}\sum_g \frac{\chi_{qp}(g)}{\Delta_g} \quad \text{and} \quad E_{qr}(-t) = \frac{1}{|G|}\sum_h \chi_{qr}(h^{-1})\Delta_h,$$

where in the expression for $S_{pq}(t)$ we have χ_{qp} instead of χ_{pq} because we have taken the opposite algebra and in the expression for $E_{qr}(-t)$ we take the value of the character on h^{-1} so that we have Δ_h instead of $\Delta_{h^{-1}}$.

Letting $\Delta = \prod_{g\in G}\Delta_g$, the (p,r) entry of our product is

$$\sum_q S_{pq}(t)\cdot E_{qr}(-t) = \frac{1}{|G|^2\Delta}\sum_q\sum_{g,h} \chi_q(g^{-1})\chi_p(g)\chi_q(h)\chi_r(h^{-1})\Delta_h\prod_{k\neq g}\Delta_k.$$

We claim that this is equal to zero when $p\neq r$ and equal to one when $p=r$. Thus the product of our matrices will indeed be the identity. To see this, notice that the contribution to the above expression coming from a fixed pair g, h and summing over q will be

$$\frac{1}{|G|^2\Delta}\chi_p(g)\chi_r(h^{-1})\Delta_h \prod_{k\neq g}\Delta_k \sum_q \chi_q(g^{-1})\chi_q(h).$$

The factor before the sum is constant for a fixed pair g, h and the sum itself can be determined by the second orthogonality relation for characters: it is equal to zero when g and h are not conjugate and is equal to $|C_G(h)|$ when they are conjugate.

Let $g \sim h$ denote that g and h are conjugate in G. After summing up q we are left with

$$\sum_q S_{pq}(t)\cdot E_{qr}(-t) = \frac{1}{|G|\Delta}\sum_{\substack{(g,h)\\ g\sim h}} \frac{|C_G(h)|}{|G|}\chi_p(g)\chi_r(h^{-1})\Delta_h \prod_{k\neq g}\Delta_k.$$

Note that the sum on the right is over conjugate pairs.

For each g, we have $\Delta_h \prod_{k\neq g}\Delta_k = \Delta$, so we can cancel the $1/\Delta$ in front of the sum. The summand $\chi_p(g)\chi_r(h^{-1})$ only depends on the conjugacy class of h since g and h are conjugate, and so the number of times $\chi_p(g)\chi_r(h^{-1})$ is counted in the sum is the number of elements in the conjugacy class of h. Since the factor $\frac{|C_G(h)|}{|G|}$ is precisely the reciprocal of this number, we are left with

$$\sum_q S_{pq}(t)\cdot E_{qr}(-t) = \frac{1}{|G|}\sum_h \chi_p(h)\chi_r(h^{-1}) = \delta_{pr},$$

where the last equality is from orthogonality of the irreducible characters. □

One interesting consequence of Koszulity for the algebra B is that it must be quadratic, by Theorem 4.2.3. That is, the natural homomorphism $T^{\bullet}_{B_0}B_1 \to B$ is surjective with kernel generated in degree 2. In fact, since $B_0 = \oplus_{i\in I}\mathrm{Hom}_G(W_i, W_i)$ is commutative semisimple with primitive orthogonal idempotents $e_i = 1 \in \mathrm{Hom}_G(W_i, W_i)$, we can realize $T^{\bullet}_{B_0}B$ as the path algebra of a quiver whose vertices are labeled by the e_i and whose arrows from e_i to e_j are identified with a basis for $e_iB_1e_j$. Since in fact $e_iB_1e_j = (V^* \otimes \mathrm{Hom}(W_i, W_j))^G \simeq \mathrm{Hom}_G(W_i, V \otimes W_j)$, we get precisely the McKay quiver, as usually defined. (Let us point out here that the definition of the McKay quiver is therefore not as fanciful as might first appear.) Thus B itself is realized as the quotient of the path algebra of the McKay quiver by some quadratic relations.

In general the relations can be quite difficult to write down. The best method here is to find a 'superpotential' for the algebra B whose derivatives give the relations. For more on this and for another approach to proving Koszulity for an algebra isomorphic to B, see the paper of Bocklandt-Schedler-Wemyss [11].

4.3 Spherical objects

As observed by McKay [40], when $G \subset SL_2(\mathbb{C})$ the McKay quiver is given by taking the affine Dynkin diagram Γ corresponding to G and turning each edge into a pair of arrows pointing in opposite directions.

For our purposes, having a presentation of B as the quotient of a path algebra will be less important than the characterization of B in terms of configurations of spherical objects, as introduced by Seidel and Thomas [48]. See [32, Chapter 8] for a nice exposition.

In our applications we'll be interested in a smooth, quasi-projective surface X carrying the action of a finite group G and whose canonical bundle is trivial as a G-sheaf (for instance, $G \subset SL_2(\mathbb{C})$, $X = V$ a 2-dimensional vector space). In this case, an object $\mathcal{E} \in D^b_G(X)$ is called **spherical** if there is a graded ring isomorphism $\mathrm{Ext}^{\bullet}_G(\mathcal{E}, \mathcal{E}) \simeq H^{\bullet}(S^2, \mathbb{C})$, where S^2 is the 2-sphere.

Given a graph Γ, a Γ-**configuration** of spherical objects is a collection of spherical objects $\mathcal{E}_i$ indexed by the nodes of Γ such that

$$\dim \operatorname{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) = \#\text{ edges from } i \text{ to } j$$

and all other Exts vanish.

In all the cases that we consider, the objects $\mathcal{E}_i$ lie in a full triangulated subcategory $\mathcal{D}$ of $D^b_G(X)$ consisting of objects whose cohomology is supported on a fixed compact subvariety. Compact support ensures that all Homs are finite dimensional and that the subcategory $\mathcal{D}$ has Serre duality, while triviality of ω_X means that duality takes the simple form $\operatorname{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{F}[2])^*$. A triangulated category with these properties is commonly called 2-**Calabi-Yau** (2-**CY**).

A much-studied example (see for instance [16]) is when $X = V$, a 2-dimensional vector space, $G \subset SL(V)$, and $\mathcal{D} \subset D^b_G(V)$ is the full triangulated subcategory supported at the origin. Here we have a Γ-configuration for Γ the affine Dynkin diagram associated to G. The spherical objects are $s_*W_i \simeq W_i \otimes \mathcal{O}_0$ for the irreducible G-representations W_i. To see that these objects are spherical, recall the isomorphism 4.2.2:

$$\operatorname{Ext}^\bullet_G(s_*W_i, s_*W_i) \simeq (\bigwedge^\bullet V \otimes \operatorname{Hom}(W_i, W_i))^G.$$

Since $V \otimes \operatorname{Hom}(W_i, W_i)$ has no invariants by McKay's observation and since $\bigwedge^2 V$ is trivial, the latter algebra is indeed isomorphic to $H^\bullet(S^2, \mathbb{C})$. To see that the objects s_*W_i, $i \in \Gamma$ form a Γ-configuration, note that if $i \neq j$, then $\operatorname{Ext}^\bullet_G(s_*W_i, s_*W_j) \simeq (\bigwedge^\bullet V \otimes \operatorname{Hom}(W_i, W_j))^G$ is zero in degrees 0 and 2 by Schur's lemma and

$$\dim \operatorname{Ext}^1_G(s_*W_i, s_*W_j) = \dim \operatorname{Hom}_G(W_i, V \otimes W_j) = \#\text{ edges beween } i \text{ and } j$$

by McKay's observation.

It is known that the classes of the $\mathcal{E}_i$ form a basis for $K_0(\mathcal{D})$ (compare with Proposition 5.4.5). The above discussion shows that in the basis $\mathcal{E}_i$, the natural **Euler form**

$$\langle \mathcal{E}, \mathcal{F} \rangle = \sum_k (-1)^k \dim \mathrm{Ext}^k_G(\mathcal{E}, \mathcal{F})$$

on $K_0(\mathcal{D})$ is given by the Cartan matrix of Γ, so we may identify $K_0(\mathcal{D})$ with the affine root lattice associated to Γ and the $\mathcal{E}_i$ with a base of simple roots.

Associated to each spherical object $\mathcal{E} \in \mathcal{D}$ is an autoequivalence of $\mathcal{D}$, the so-called **spherical twist** $T_{\mathcal{E}}$ whose action on an object $\mathcal{F}$ is given by

(4.3.1) $$T_{\mathcal{E}}(\mathcal{F}) \simeq \mathrm{Cone}(\mathrm{RHom_G}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}).$$

Furthermore, it is known that for a Γ-configuration $\mathcal{E}_i$, the spherical twists $T_{\mathcal{E}_i}$ generate an action of the braid group B_Γ on $\mathcal{D}$, where B_Γ has generators σ_i for each vertex $i \in \Gamma$ and relations $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if i and j are joined by an edge and $\sigma_i\sigma_j = \sigma_j\sigma_i$ otherwise. From 4.3.1 we see that the induced action of $T_{\mathcal{E}_i}$ on $K_0(\mathcal{D})$ takes the form

$$[T_{\mathcal{E}_i}(\mathcal{F})] = [\mathcal{F}] - \langle \mathcal{E}_i, \mathcal{F} \rangle [\mathcal{E}_i].$$

That is, $T_{\mathcal{E}_i}$ induces the reflection at the simple root $[\mathcal{E}_i]$ and so the braid group action on $\mathcal{D}$ induces the Weyl group action on the affine root lattice $K_0(\mathcal{D})$.

In this way we think of $\mathcal{D}$ as a categorification of the affine root lattice, with bases of simple roots replaced by Γ-configurations of spherical objects and the action of the Weyl group replaced by the action of the braid group B_Γ. For more on this point of view, see Khovanov and Huerfano [31], who use the category $\mathcal{D}$ to categorify the adjoint representation of the quantum group associated to Γ.

The following lemma shows that in dimension two this example is in some sense universal.

Lemma 4.3.1. *1. The objects s_*W_i, $i \in \Gamma$ form a Γ-configuration of spherical objects in the subcategory $\mathcal{D} \subset D^b_G(\mathbb{C}^2)$ of objects supported at the origin.*

2. The Ext*-algebra of this Γ configuration,*

$$\mathrm{Ext}^\bullet_G(\bigoplus_{i\in\Gamma}\mathcal{E}_i, \bigoplus_{i\in\Gamma}\mathcal{E}_i),$$

is Koszul with Koszul dual Π_Γ, the preprojective algebra of the diagram Γ.

3. Let Γ be an affine Dynkin diagram of type ADE*, $\mathcal{E}_i$, $i \in \Gamma$ a Γ-configuration of spherical objects in a 2-CY category $\mathcal{D}$, and $\mathcal{E}'_i$, $i \in \Gamma$ another Γ-configuration in a possibly different 2-CY category $\mathcal{D}'$. Then the* Ext*-algebras of the two Γ-configurations are isomorphic:*

$$\mathrm{Ext}^\bullet_{\mathcal{D}}(\bigoplus_{i\in\Gamma}\mathcal{E}_i, \bigoplus_{i\in\Gamma}\mathcal{E}_i) \simeq \mathrm{Ext}^\bullet_{\mathcal{D}'}(\bigoplus_{i\in\Gamma}\mathcal{E}'_i, \bigoplus_{i\in\Gamma}\mathcal{E}'_i).$$

By 2., any such Ext*-algebra is Koszul.*

Proof. 1. We have already seen this in the discussion before the lemma.

2. This is easy and well-known. See Example 4.3.3 for an argument in Type A. Types D and E are dealt with similarly.

3. Let $\{\mathcal{E}_i\}$ be a Γ-configuration in a 2-CY category. Then $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i) \simeq \mathbb{C}$ and $\mathrm{Ext}^2(\mathcal{E}_i, \mathcal{E}_i) \simeq \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i)^* \simeq \mathbb{C}$ by sphericity and Serre duality and all other Homs and Ext^2s are zero by the condition of being a Γ-configuration. Thus the composition $\mathrm{Ext}^1(\mathcal{E}_j, \mathcal{E}_k) \otimes \mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) \to \mathrm{Ext}^2(\mathcal{E}_i, \mathcal{E}_k)$ is zero unless $i = k$, in which case the composition

$$\mathrm{Ext}^1(\mathcal{E}_j, \mathcal{E}_i) \otimes \mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) \to \mathrm{Ext}^2(\mathcal{E}_i, \mathcal{E}_i) \simeq \mathbb{C}$$

is just the Serre pairing and we have $\mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) \simeq \mathrm{Ext}^1(\mathcal{E}_j, \mathcal{E}_i)^*$.

Now let $\{\mathcal{E}'_i\}$ be another Γ-configuration in a possibly different 2-Calabi-Yau category. To establish an isomorphism between the Ext-algebras of $\oplus_i \mathcal{E}_i$ and $\oplus_i \mathcal{E}'_i$, it is enough to take the natural identifications for Homs and Ext^2s and then to give an isomorphism on Ext^1s compatible with the above pairing. To achieve this, choose for every pair of adjacent vertices i and j a postive direction $i \to j$ and give an isomorphism $\mathrm{Ext}^1(\mathcal{E}_i, \mathcal{E}_j) \simeq \mathrm{Ext}^1(\mathcal{E}'_i, \mathcal{E}'_j)$. Then letting the isomorphism for the negative direction $j \to i$ be determined by duality ensures compatability with the pairing. $\square$

Remark 4.3.2. The lemma is very useful. If B is any graded algebra for which $E(B)$ is an Ext-algebra of a Γ-configuration, then by the third part of the lemma there is an isomorphism

$$E(B) \simeq \mathrm{Ext}^\bullet_G(\bigoplus_{i \in \Gamma} \mathcal{E}_i, \bigoplus_{i \in \Gamma} \mathcal{E}_i).$$

Then by Koszul duality (Theorem 4.2.3) and the second part of the lemma,

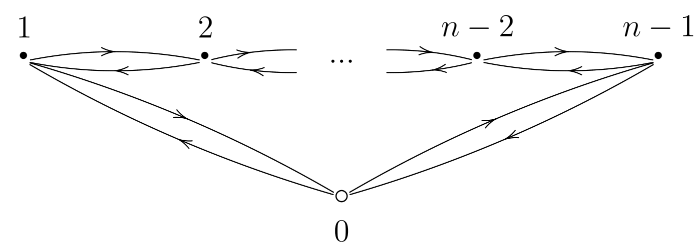
$$B \simeq E(E(B)) \simeq \Pi_\Gamma.$$

This will be important in Chapter 5, where we use Γ-configurations of spherical objects to relate equivariant sheaves on the cotangent bundle $\mathcal{T}$ to the above universal example.

Example 4.3.3. Let $G = \mathbb{Z}/n\mathbb{Z}$. Defining W_1 to be the irreducible representation of $\mathbb{Z}/n\mathbb{Z}$ where 1 acts as multiplication by $\zeta = e^{2\pi i/n}$, all of the other irreducible representations are just powers of this one, which we denote by $W_0, W_1, \ldots, W_{n-1}$. The group $\mathbb{Z}/n\mathbb{Z}$ embeds in $SL_2(\mathbb{C})$ with cyclic generator

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

so that the standard representation takes the form $V = W_1 \oplus W_{n-1}$. In this case it is straight-forward to check that the McKay quiver is



To find a presentation for $B = \mathrm{End}_G(\pi^* W)^{\mathrm{op}}$ we first find a presentation for $E(B) = \mathrm{Ext}^{\bullet}_G(s_* W, s_* W)$ and then compute its quadratic dual $E(B)^!$. By Theorem 4.2.3, there is a canonical isomorphism $B \simeq E(B)^{!\mathrm{op}}$, so this will be sufficient.

First we find a nice basis of generators for $E(B)_1$ over $E(B)_0 = \bigoplus_i \mathrm{Hom}_G(s_* W_i, s_* W_i) \simeq \bigoplus_i \mathrm{Hom}_G(W_i, W_i) \simeq B_0$. Each summand in $E(B)_1 = \bigoplus_{i \to j} \mathrm{Ext}^1_G(s_* W_i, s_* W_j)$ is one dimensional since the $s_* W_i$ form a Γ-configuration. Choose non-zero clockwise arrows $(i|i+1) \in \mathrm{Ext}^1(s_* W_i, s_* W_{i+1})$ and define the counterclockwise arrows $(i+1|i) \in \mathrm{Ext}^1(s_* W_i, s_* W_{i-1})$ by requiring that the whole collection form Darboux coordinates for the antisymmetric Serre pairing 2.9.2

$$\mathrm{Ext}^1_G(s_* W, s_* W) \otimes \mathrm{Ext}^1_G(s_* W, s_* W) \longrightarrow \mathrm{Ext}^2_G(s_* W, s_* W) \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

Denoting the product $(j|k) \cdot (i|j) = (i|j|k)$, then component by component, this means that $\mathrm{Tr}((i|i+1|i)) = 1$ and $\mathrm{Tr}((i+1|i|i+1)) = -1$.

Since $\mathrm{Ext}^2(s_* W_i, s_* W_k) = 0$ for $k \neq i$, we certainly have relations

$$(j|k) \otimes (i|j) = 0$$

whenever $k \neq i$. By the antisymmetry of the Serre pairing we also have a relation

$$(i+1|i) \otimes (i|i+1) + (i-1|i) \otimes (i|i-1) = 0$$

for each i. I claim that these two kinds of relations form a basis for the space R of relations, that is, for the kernel of $\mathrm{Ext}^1_G(s_*W, s_*W) \otimes_{B_0} \mathrm{Ext}^1_G(s_*W, s_*W) \to \mathrm{Ext}^2_G(s_*W, s_*W)$. Indeed, the space on the left has dimension equal to the number of paths of length two, while the space on the right has dimension equal to the number of nodes of the quiver. Since the map is surjective, the kernel has dimension equal to the number of length two paths minus the number of nodes. But this is precisely the number of relations that we have given. It will thus be enough to see that our relations span the kernel. So consider a relation on paths of length two: $\sum a_{ijk}(i|j|k) = 0$. Since $(i|j|k) = 0$ when $k \neq i$, we can ignore those terms, and by the splitting $\mathrm{Ext}^2_G(s_*W, s_*W) = \bigoplus_i \mathrm{Ext}^2_G(s_*W_i, W_i)$, we can consider the individual pieces $a_{ii+1i}(i|i+1|i) + a_{ii-1i}(i|i-1|i) = 0$ of the sum. Taking the trace of this latter relation, we see that $a_{ii+1i} - a_{ii-1i} = 0$, so the relation is a scalar multiple of $(i|i+1|i) + (i|i-1|i) = 0$, and so our relations are indeed sufficient.

Having found bases of generators and of relations for $E(B)$ we can compute the quadratic dual $E(B)^!$. Let $\langle i|i+1\rangle$ and $\langle i|i-1\rangle$ be dual to $(i+1|i)$ and $(i-1|i)$. I claim that a basis for the dual relations $R^\perp$ is given by the elements

$$\langle i+1|i\rangle \otimes \langle i|i+1\rangle - \langle i-1|i\rangle \otimes \langle i|i-1\rangle \tag{4.3.2}$$

for each i. The number of such elements is equal to the number of nodes, which is exactly the dimension of $R^\perp$, and they certainly kill the space R, so it is enough to see that they span $R^\perp$. Suppose that $\sum b_{ijk}\langle j|k\rangle \otimes \langle i|j\rangle \in R^\perp$. Then in particular it must kill $(m|n) \otimes (l|m)$ when $l \neq n$, and so $b_{nml} = 0$. This leaves $\sum b_{iji}\langle j|i\rangle \otimes \langle i|j\rangle$, which must kill $(k+1|k) \otimes (k|k+1) + (k-1|k) \otimes (k|k-1)$ for all k, so $b_{kk+1k} + b_{kk-1k} = 0$, and we see our relations are sufficient.

If $\alpha = \langle i|i+1\rangle$, then $\overline{\alpha} = \langle i+1|i\rangle$ and the relation in 4.3.2 is precisely the ith component of the preprojective relations.

4.4 The symmetric group

We now consider another family of examples involving the symmetric group. Our main results are the description of the McKay quiver in Section 4.4.1, and the discovery of $\mathbb{P}$-objects in the derived category analogous to the spherical objects encountered when $G \subset SL_2(\mathbb{C})$.

Let $G = S_n$, the symmetric group on n-letters, and let P be its permutation representation with basis $\varepsilon_1, \dots, \varepsilon_n$. P decomposes into one copy of the trivial repesentation, with basis $\varepsilon_1 + \cdots + \varepsilon_n$, and one copy of the so-called standard representation, with basis $\alpha_1 := \varepsilon_1 - \varepsilon_2, \dots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n$, which we identify with a base of simple roots for the the complexified root lattice $\mathfrak{h}^*$ of type A_{n-1}. The transpositions $s_1 = (1,2), \dots, s_{n-1} = (n-1, n)$ act as reflections in these simple roots and generate the action of G.

Now let $V = \mathfrak{h} \oplus \mathfrak{h}^*$, which we endow with the natural, G-invariant symplectic form

$$\omega((x,\varphi),(y,\psi)) = \psi(x) - \varphi(y)$$

for $(x,\varphi),(y,\psi) \in \mathfrak{h} \oplus \mathfrak{h}^*$.

Let us establish some notation and recall some standard facts about partitions of integers and representations of symmetric groups (see [25, Chapter 7]).

Given partitions $\lambda = (l_1, \ldots, l_r)$ and $\mu = (m_1, \ldots, m_s)$ of n, we say that μ **dominates** λ, denoted $\mu \trianglerighteq \lambda$, if

$$m_1 + \cdots + m_i \geq l_1 + \cdots l_i \text{ for all } i \ .$$

Dominance is a partial order on the set of partitions of n. To each partition $\lambda = (l_1, \ldots, l_r)$ with $l_1 \geq l_2 \geq \cdots \geq l_r$, we associate a left justified Young diagram with l_1 boxes in the top row, l_2 boxes in the second row, and so on. A filling with positive integers of the boxes of a Young diagram associated to λ is called a **tableau** if the integers are weakly increasing across the rows and strictly increasing down the columns. A tableau is said to have **content** $\mu = (m_1, \ldots, m_s)$ if it contains the number 1 with multiplicity m_1, the number 2 with multiplicity m_2, and so on. The **Kostka number** $K_{\lambda\mu}$ is the number of tableau of shape λ and content μ.

For each partition λ of n, we have an associated Young subgroup $S_\lambda \simeq S_{l_1} \times \cdots \times S_{l_k} \subset S_n$, where S_{l_1} permutes the first l_1 letters, S_{l_2} permutes the next l_2 letters, and so on. Let $U_\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C})$ be the induction of the trivial representation from the Young subgroup S_λ to the full group S_n.

Theorem 4.4.1. *1. For each λ there is a distinguished irreducible submodule $W_\lambda \subset U_\lambda$. As λ runs over all partitions, the representations W_λ give a complete list of irreducibles for S_n. In particular, if $\lambda = (n-1, 1)$, then $W_\lambda \simeq \mathfrak{h}^* \simeq \mathfrak{h}$, the standard representation of S_n, and if $\lambda = (n-k, k)$, then $W_\lambda \simeq \bigwedge^k \mathfrak{h}^* \simeq \bigwedge^k \mathfrak{h}$.*

2. (Young's rule) We have a decomposition

$$U_\lambda = \bigoplus_{\mu \trianglerighteq \lambda} W_\mu^{\oplus K_{\mu\lambda}},$$

Note that $K_{\lambda\lambda} = 1$, so the irreducible W_λ appears with multiplicity one.

3. *(Branching rule) If λ is a partition of $n-1$, then $\mathrm{Ind}_{S_{n-1}}^{S_n}(W_\lambda)$ decomposes as a sum over W_μ for partitions μ that are obtained from λ by adding a box, each occurring with multiplicity one. By Frobenius reciprocity, we can equivalently say that for a partition λ of n, $\mathrm{Res}_{S_{n-1}}^{S_n}(W_\lambda)$ decomposes as a sum over partitions obtained from λ by removing a box, each occurring with multiplicity one.*

4.4.1 Quiver for the symmetric group

Consider the algebra

$$B = \mathrm{End}_G(\bigoplus_\lambda \pi^* W_\lambda)^{\mathrm{op}} \simeq (S(V^*) \otimes \mathrm{End}(\bigoplus_\lambda W_\lambda)^{\mathrm{op}})^G,$$

where the W_λ are the irreducibles of S_n labeled by partitions of n or Young diagrams of size n, as in Theorem 4.4.1. By Theorem 4.2.4, B is Koszul and so generated in degree one with relations in degree 2 over $B_0 = \bigoplus_\lambda \mathrm{Hom}_G(W_\lambda, W_\lambda)$, a commutative semi-simple algebra with a primitive idempotent $e_\lambda = 1_\lambda \in \mathrm{Hom}_G(W_\lambda, W_\lambda)$ for each λ.

As noted before, this allows us to describe B as a quotient of the path algebra of the McKay quiver Q, which has vertices labeled by the λ and arrows from λ to μ given by a basis for $e_\mu \cdot B_1 \cdot e_\lambda = \mathrm{Hom}_G(W_\lambda, V^* \otimes W_\mu)$. To describe this quiver it is then enough to give the dimensions of these spaces.

To compute these dimensions, note that in the representation ring we have $V^* = 2 \cdot W_\nu$, where $\nu = (n-1, 1)$, the partition giving the standard representation. Using brackets to denote the usual pairing in the representation ring, we have

$$\mathrm{Hom}_G(W_\lambda, V^* \otimes W_\mu) = 2\langle W_\lambda, W_\nu \otimes W_\mu \rangle.$$

Young's rule together with a simple computation of Kostka numbers gives $U_\nu = W_\nu \oplus$

$W_{(n)}$, so in the representation ring we have the equality $W_\nu = U_\nu - W_{(n)} = U_\nu - U_{(n)}$ (note that $W_{(n)} \simeq U_{(n)} \simeq \mathbb{C}$, the trivial G-representation, again by Young's rule or because there is nowhere to induce up to). Thus we have

$$W_\nu \otimes W_\mu = U_\nu \otimes W_\mu - U_{(n)} \otimes W_\mu = \operatorname{Ind}(\operatorname{Res} W_\mu) - W_\mu,$$

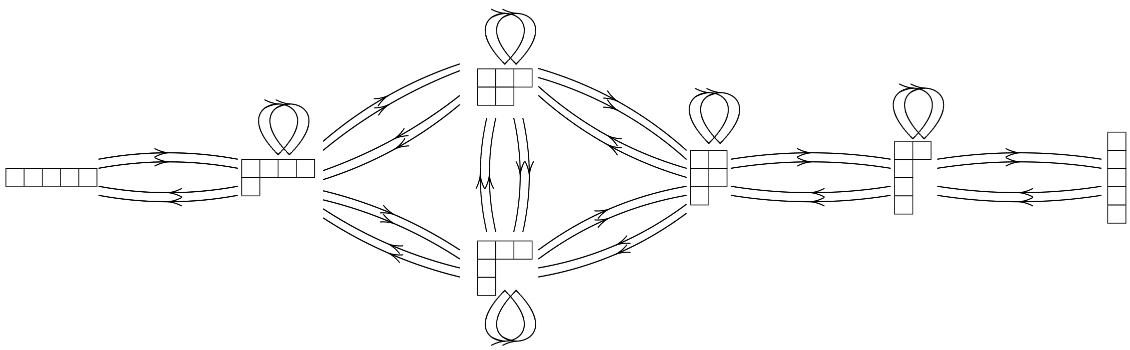
where the restriction and induction are taking place between S_{n-1} and S_n and we have used the isomorphism $\operatorname{Ind}_H^G((\operatorname{Res}_H^G U) \otimes W) \simeq U \otimes \operatorname{Ind}_H^G W$, which is straight-forward to verify and can be interpreted as the projection formula for equivariant sheaves for the morphism $BH \to BG$ of classifying spaces.

The branching rule then gives $\operatorname{Ind}(\operatorname{Res} W_\mu) = W_\mu^{\oplus d_\mu} \bigoplus_{\beta \in S_\mu} W_\beta$, where d_μ is the number of steps in the Young diagram of the partition μ (in the branching rule, we remove a box to restrict and then put it back where we found it), and S_μ is the set of partitions obtained from λ by removing a box and then putting it back somewhere else (so the Young diagram differs in two rows from that of μ).

Altogether then the number of arrows $n_{\lambda\mu}$ from λ to μ in our quiver is

$$n_{\lambda\mu} = 2\langle W_\lambda, W_\nu \otimes W_\mu \rangle = \begin{cases} 2(d_\mu - 1) & \text{if } \lambda = \mu \\ 2 & \text{if } \lambda \in S_\mu \end{cases}$$

Example 4.4.2. When $G = S_5$, the quiver is



4.4.2 Poincaré polynomials

We saw that when $G \subset SL_2(\mathbb{C})$, the special objects $s_*W_i \simeq W_i \otimes \mathcal{O}_0$ were spherical, meaning their Ext algebras were isomorphic to $H^{\bullet}(S^2, \mathbb{C})$, the cohomology ring of the 2-sphere. In this section we consider the Ext-algebras of the objects $W_\lambda \otimes \mathcal{O}_0$ in our present example when $G = S_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, giving a formula for their Poincaré polynomials.

It will be easier to begin by considering $U = U_{(1^n)} \oplus U^*_{(1^n)} \supset V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $U_{(1^n)}$ is the permutation representation and compute the Poincaré polynomials P'_λ in this case. Using the Koszul resolution of $\mathcal{O}_0$, we saw in 4.2.2 that

$$\mathrm{Ext}^{\bullet}(W_\lambda \otimes \mathcal{O}_0, W_\lambda \otimes \mathcal{O}_0) \simeq \bigwedge^{\bullet} U \otimes \mathrm{End}(W_\lambda)$$

and so by Molien's formula 4.2.3, we have

$$P'_\lambda = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^2_\lambda(\sigma) \det{}^2_P(1 + \sigma \cdot t).$$

To compute $\det_{U_{(1^n)}}(1 + \sigma \cdot t)$, consider the cycle decomposition $\sigma = \sigma_1 \cdots \sigma_m$ of cycle/conjugacy type $\kappa = (k_1, ..., k_m)$. The action of σ on $U_{(1^n)}$ splits into blocks, one for each cycle, with the block for a given cycle σ_i being diagonalizable with the k_ith roots down the diagonal. Thus choosing a primitive k_ith root of unity ζ_i, σ_i contributes $\prod_j (1+\zeta_i^j \cdot t) = 1-(-t)^{k_i}$ to $\det_{U_{(1^n)}}(1+\sigma\cdot t)$. We see then that $\det_{U_{(1^n)}}(1+\sigma\cdot t)$ depends only on the conjugacy class κ of σ and $\det_{U_{(1^n)}}(1+\sigma\cdot t) = \prod_{i=1}^m (1-(-t)^{k_i})$.

To get from $\det_{U_{(1^n)}}(1+\sigma\cdot t)$ to $\det_{\mathfrak{h}}(1+\sigma\cdot t)$ we simply divide by $1+t$. Setting

$$\Delta_\kappa := \det_{\mathfrak{h}}(1+\sigma\cdot t) = \frac{\prod_{i=1}^m (1-(-t)^{k_i})}{1+t},$$

we have the compact formula

$$P_\lambda = \sum_\kappa \frac{1}{z(\kappa)} \chi_\lambda^2(\kappa)\Delta_\kappa^2. \tag{4.4.3}$$

where $z(\kappa) = \prod_r r^{m_r} m_r!$, with m_r the number of times r appears in the partition κ and $n!/z(\kappa)$ is the number of permutations of type κ.

We include a table of the first few values of P_λ.

The Poincaré polynomials P_λ		
$n=1$	$n=2$	$n=3$
	$P_{(1^2)} = t^2+1$	$P_{((1^3)} = t^4+t^2+1$
$P_{(1)} = 1$	$P_{(2)} = t^2+1$	$P_{(2,1)} = t^4+2t^3+5t^2+2t+1$
		$P_{(3)} = t^4+t^2+1$
$n=4$		
$P_{(1^4)} = t^6+t^4+t^2+1$		
$P_{(2,1^2)} = t^6+2t^5+6t^4+6t^3+6t^2+2t+1$		
$P_{(2^2)} = t^6+2t^4+6t^3+2t^2+1$		
$P_{(3,1)} = t^6+2t^5+6t^4+6t^3+6t^2+2t+1$		
$P_{(4)} = t^6+t^4+t^2+1$		

4.4.4 $\mathbb{P}$-objects

Note that in the above table, the Poincaré polynomials of the trivial representation (n) and the sign representation (1^n) are the Poincaré polynomials of the cohomology ring of $\mathbb{P}^{2n-2}$. This is no coincidence.

Proposition 4.4.3. *When W_λ is the trivial or the sign representation, there is a canonical algebra isomorphism* $\mathrm{Ext}^\bullet_G(s_*W_\lambda, s_*W_\lambda) \simeq H^\bullet(\mathbb{P}^{2n-2}, \mathbb{C})$.

Proof. When W_λ is the trival or sign representation, we have $\mathrm{Ext}^\bullet_G(s_*W_\lambda, s_*W_\lambda) \simeq (\bigwedge^\bullet V \otimes \mathrm{Hom}(W_\lambda, W_\lambda))^G \simeq (\bigwedge^\bullet V)^G$. In a particular degree d, $\bigwedge^d V \simeq \bigoplus_{k+l=d} \bigwedge^k \mathfrak{h} \otimes \bigwedge^l \mathfrak{h}^*$, so to compute invariants of $\bigwedge^d V$, we need to see when $\bigwedge^k \mathfrak{h} \otimes \bigwedge^l \mathfrak{h}^*$ contains a trivial representation.

By [34, Theorem 2.9.18], $\langle W_\lambda \otimes W_\mu, W_\gamma \rangle > 0$ only if $|\lambda - \mu| \leq 2(n - \gamma_1)$. In particular, when $\gamma = (n)$, this shows that $W_\lambda \otimes W_\mu$ contains a copy of the trivial representation only if $\lambda = \mu$. Since $\bigwedge^k \mathfrak{h}$ is irreducible for all k, we see that the only possible invariants in $\bigwedge^\bullet V$ must come from summands of the form $\bigwedge^k \mathfrak{h} \otimes \bigwedge^k \mathfrak{h}^* \simeq \mathrm{Hom}(\bigwedge^k \mathfrak{h}, \bigwedge^k \mathfrak{h})$. By Schur's lemma, each such summand contributes a canonical invariant, the identity morphism. In fact, the canonical invariants in $\bigwedge^k \mathfrak{h} \otimes \bigwedge^k \mathfrak{h}^*$ are just wedges of the canonical invariant in $\mathfrak{h} \otimes \mathfrak{h}^*$. The latter invariant must therefore generate the whole algebra and so sending it to the class of a hyperplane gives the isomorphism $\mathrm{Ext}^\bullet_G(s_*W_\lambda, s_*W_\lambda) \simeq H^\bullet(\mathbb{P}^n, \mathbb{C})$. □

The interest in this proposition is that Huybrechts-Thomas [33] develops a theory of $\mathbb{P}^m$-objects (those whose Ext algebra is isomorphic to the cohomology of $\mathbb{P}^m$) analogous to the theory of spherical objects from Seidel-Thomas [48].

For a spherical object $\mathcal{E} \in D^b_G(X)$, the spherical twist $T_\mathcal{E}$ can be realized as an integral transform whose kernel is the cone of $\mathrm{tr} : \mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta$. Similarly, given a $\mathbb{P}^m$-object in $\mathcal{E} \in D^b_G(X)$, the twist $T_\mathcal{E}$ is given by an integral transform whose kernel we construct in a few steps.

To begin with, let $h \in \mathrm{Hom}_\mathrm{G}(\mathcal{E}, \mathcal{E}[2]) \simeq \mathrm{Hom}_\mathrm{G}(\mathcal{E}[-2], \mathcal{E})$ to be the class of the hyper-plane section and $h^\vee \in \mathrm{Hom}_\mathrm{G}(\mathcal{E}^\vee[-2], \mathcal{E}^\vee)$ the image of h under the natural isomorphism $\mathrm{Hom}_\mathrm{G}(\mathcal{E}[-2], \mathcal{E}) \simeq \mathrm{Hom}_\mathrm{G}(\mathcal{E}^\vee[-2], \mathcal{E}^\vee)$. Next consider the morphism $H := h^\vee \boxtimes 1 - 1 \boxtimes h$ on $X \times X$, which fits into an exact triangle

$$\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \xrightarrow{H} \mathcal{E}^\vee \boxtimes \mathcal{E} \longrightarrow \mathcal{H}.$$

Now define t to be the unique morphism $\mathcal{H} \to \mathcal{O}_\Delta$ factoring the trace $\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta$.

Finally, we define the kernel whose corresponding integral transform gives the desired $\mathbb{P}^m$-twist:

$$\mathcal{P}_\mathcal{E} := \mathrm{Cone}(t).$$

Remark 4.4.4.

When I originally considered this example, I worked with two copies of the permutation representation rather than with two copies of the standard representation. In this case, if λ is the trivial or the sign representation, then the Poincarè polynomial is $P_\lambda = t^{2n} + 2t^{2n-1} + \cdots + 2t + 1$. Roman Bezrukavnikov pointed out to me the obvious: if we replace the permutation representation with the standard representation, the Poincaré polynomial will be divided by $(t+1)^2$ and we get $\mathbb{P}^{2n-2}$-objects.

The hope is that we could associate autoequivalences of a derived category $D^b_G(X)$ to objects with more general Ext-algebras. David Ploog and Bernd Kreussler have made some progress in this direction. I plan to look into this in the future.

Chapter 5

The projective McKay correspondence

Kirillov [37] has described a McKay correspondence for finite subgroups of $PSL_2(\mathbb{C})$ which associates to each 'height' function an affine Dynkin quiver together with a derived equivalence between equivariant sheaves on $\mathbb{P}^1$ and representations of this quiver. The equivalences for different height functions are then related by reflection functors for quiver representations.

The main goal of this chapter is to develop an analogous story for the cotangent bundle of $\mathbb{P}^1$. We show that each height function gives rise to a derived equivalence between equivariant sheaves on the cotangent bundle $T^*\mathbb{P}^1$ and modules over the preprojective algebra of an affine Dynkin quiver. These different equivalences are related by spherical twists, which take the place of the reflection functors for $\mathbb{P}^1$.

5.1 Introduction

In [41], John McKay associated to a finite group $G \subset SL_2(\mathbb{C})$ a graph Γ in which the vertices are labeled by the irreducible representations W_i of G and the number of edges n_{ij} between two irreducibles W_i, W_j is given by $n_{ij} = \dim \mathrm{Hom}_{\mathrm{G}}(W_i, V \otimes W_j)$, where V is the standard two dimensional representation of G coming from its embedding in $SL_2(\mathbb{C})$. McKay then observed that the graph Γ is an affine Dynkin diagram of type A,D, or E.

As discussed in **??**, this relation between the representation theory of finite subgroups of $SL_2(\mathbb{C})$ and affine Dynkin diagrams has a description in terms of G-equivariant sheaves on $\mathbb{C}^2$. More precisely, there is an equivalence

$$\mathrm{Coh}_G(\mathbb{C}^2) \simeq \Pi_\Gamma\text{-mod} \tag{5.1.1}$$

between the category of G-equivariant coherent sheaves on $\mathbb{C}^2$ and the category of finitely generated modules over the preprojective algebra Π_Γ. For the purposes of this chapter we shall refer to this equivalence as the McKay correspondence for $\mathbb{C}^2$.

More geometrically, Kapranov-Vasserot [35], building on work of Gonzalez-Sprinberg-Verdier [27], construct a derived equivalence

$$D^b_G(\mathbb{C}^2) \simeq D^b(\widehat{X})$$

where $\widehat{X} \to \mathbb{C}^2/\!\!/G$ is the minimal resolution of the 'Kleinian singularity' $\mathbb{C}^2/\!\!/G$. It is this equivalence that usually goes under the name 'McKay correspondence'.

In another direction, Kirillov [37] has described a projective McKay correspondence for finite subgroups $\widetilde{G}$ of $PSL_2(\mathbb{C})$. Letting Γ be the graph associated by McKay to the double cover $G \subset SL_2(\mathbb{C})$ of $\widetilde{G}$, this projective correspondence relates equivariant sheaves on $\mathbb{P}^1$ to representations of the path algebra of a quiver with underlying graph Γ. More precisely, to each 'height' function

$$h : \Gamma \to \mathbb{Z}$$

on the vertices of Γ (defined in Section 5.2), Kirillov associates a quiver Q_h on Γ and an exact equivalence

$$D^b_{\widetilde{G}}(\mathbb{P}^1) \xrightarrow{R\Phi_h} D^b(Q_h)$$

where $D^b_{\widetilde{G}}(\mathbb{P}^1)$ is the bounded derived category of $\widetilde{G}$-equivariant coherent sheaves on $\mathbb{P}^1$ and $D^b(Q_h)$ is the bounded derived categories of representations of Q_h. Furthermore, the equivalences for different height functions h and $\tilde{h}$ differ by a sequence of the reflection functors of Bernstein-Gelfand-Ponamarev [7]:

$$\begin{array}{ccc} & D^b_{\widetilde{G}}(\mathbb{P}^1) & \\ {}^{R\Phi_h}\swarrow & & \searrow^{R\Phi_{\tilde{h}}} \\ D^b(Q_h) & \xrightarrow{BGP} & D^b(Q_{\tilde{h}}). \end{array}$$

It is well-known that the Grothendieck groups of the various quivers Q_h can be identified with the affine root lattice associated to the diagram Γ, and that under this identification, the reflection functors generate the action of the affine Weyl group.

The main goal of this chapter is to develop an analogous story for the cotangent bundle $T^*\mathbb{P}^1$. Theorem 5.3.1, together with Proposition 5.3.5, gives for each height function h an equivalence

$$D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \xrightarrow{R\Psi_h} D^b(\Pi_\Gamma),$$

where Π_Γ is the preprojective algebra of the diagram Γ.

In order to relate the various equivalences $R\Psi_h$, we consider for each h a 'Γ-configuration' of spherical objects $\mathcal{E}_i^h$, $i \in I$, together with the associated spherical twists of Seidel-Thomas [48] which act as autoequivalences on the derived category. Just as the equivalences $R\Phi_h$ in the $\mathbb{P}^1$-case differed by reflection functors, Theorem 5.4.8 explains how the equivalences $R\Psi_h$ differ by spherical twists.

To make the analogy between spherical twists and reflection functors more precise, Proposition 5.4.4 reinterprets the latter purely in terms of $D^b_{\widetilde{G}}(\mathbb{P}^1)$. Under the inverse equivalence $R\Phi_h^{-1}$, the heart of the standard t-structure on $D^b(Q_h)$ is sent to a heart $\mathcal{A}_h \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$ with simple objects E_i^h, $i \in I$ and the various hearts are related by tilting at the simple objects E_i^h in the sense of Happel-Reiten-Smalø[29].

Similarly, under the inverse equivalence $R\Psi_h^{-1}$, the standard t-structure on $D^b(\Pi_\Gamma)$ gives a non-standard t-structure on $D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$. Restricting this t-structure to the subcategory $\mathcal{D} \subset D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$ of objects supported along the zero section gives a heart $\mathcal{B}_h \subset \mathcal{D}$ whose simple objects are the spherical objects $\mathcal{E}_i^h$ that we have already encountered. Proposition 5.4.9 shows how the action of the spherical twists can be described in terms of tilting at the simple objects $\mathcal{E}_i^h$.

Note that, although the spherical twists are indeed the right analogues of the reflection functors, the situation for $T^*\mathbb{P}^1$ is richer than for $\mathbb{P}^1$, since the spherical twists actually act by *autoequivalences* on the category $\mathcal{D}$, while the reflection functors are derived equivalences between categories of *different* quivers and the effect of the reflection functors on $D^b_{\widetilde{G}}(\mathbb{P}^1)$ is merely to tilt t-structures.

Completing the analogy between $\mathbb{P}^1$ and $T^*\mathbb{P}^1$, let us note that there is an isomorphism $K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \simeq K_0(\mathcal{D})$ sending the class of E_i^h to the class of $\mathcal{E}_i^h$, that these collections form bases of simple roots for $K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \simeq K_0(\mathcal{D})$ thought of as the affine root lattice, and that the spherical twists generate an action of a braid group B_Γ on $\mathcal{D}$, which induces the action of the affine Weyl group on $K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \simeq K_0(\mathcal{D})$ agreeing with that coming from the reflection functors.

We summarize the relation between $\mathbb{P}^1$ and $T^*\mathbb{P}^1$ in the following table.

$\mathbb{P}^1$	$T^*\mathbb{P}^1$
$R\Phi_h : D^b_{\widetilde{G}}(\mathbb{P}^1) \simeq D^b(Q_h)$	$R\Psi_h : D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \simeq D^b(\Pi_\Gamma)$
Hearts $\mathcal{A}_h \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$	Hearts $\mathcal{B}_h \subset \mathcal{D} \subset D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$
Simples E_i^h	Spherical objects $\mathcal{E}_i^h$
Reflection functors	Spherical twists
Affine Weyl group	Braid group

Furthermore, together with the equivalences Π_Γ-mod $\simeq \mathrm{Coh}_G(\mathbb{C}^2)$ and $D^b_G(\mathbb{C}^2) \simeq D^b(\widehat{X})$ for the resolution $\widehat{X} \to \mathbb{C}^2/\!\!/G$, our results provide a chain of equivalences

$$D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \simeq D^b(\Pi_\Gamma) \simeq D^b_G(\mathbb{C}^2) \simeq D^b(\widehat{X}).$$

We may thus view $T^*\mathbb{P}^1$ as providing a bridge between the McKay correspondence for $\mathbb{P}^1$ of [37] and the usual McKay correspondence for $\mathbb{C}^2$.

Let us point out that the structures appearing in the above table are very similar to those in Bridgeland's paper [17], in which exceptional collections on certain Fano varieties are related to collections of spherical objects on canonical bundles. Since the combinatorics of affine ADE diagrams with varying orientation is more complicated than that of the one-way oriented A diagrams appearing in theory of exceptional collections, finding analogies for the all of the fine results in [17] requires further work.

5.2 McKay correspondence for $\mathbb{P}^1$

We review here Kirillov [37] and prove some related facts that will be useful later.

Let V be a 2-dimensional vector space, set $\mathbb{P}^1 = \mathbb{P}(V)$, and assume that our finite subgroup $G \subset SL(V)$ contains $\pm I$. We divide G-representations and G-sheaves into two types, **even** and **odd**, depending on whether $-I$ acts trivially or non-trivially.

We shall be mostly interested in coherent, even G-sheaves, which we can also think of as $\widetilde{G} = G/\pm I$-sheaves, where $\widetilde{G}$ is now a subgroup of $PSL(V)$. We denote by $\mathrm{Coh}_{\widetilde{G}}$ the category whose objects are even G-sheaves and whose morphisms lie in $\mathrm{Hom}_{\mathrm{G}}$, the invariant part of Hom in the category of coherent sheaves. $\mathrm{Coh}_{\widetilde{G}}$ is abelian and we denote its bounded derived category by $D^b_{\widetilde{G}}(\mathbb{P}^1)$.

It will be convenient to work with odd sheaves as well. For instance, the G-action on the trivial bundle V stabilizes the tautological sub-bundle $\mathcal{O}(-1)$. With this natural G-action, $\mathcal{O}(-1)$ is an odd sheaf since $-I$ acts non-trivially on the fibres. As a tensor power of $\mathcal{O}(-1)$, the line bundle $\mathcal{O}(d)$ inherits a natural G-action, and its parity as a G-sheaf agrees with the parity of the integer d. Thus $\mathcal{O}(d) \otimes W_i$ will be an even G-sheaf precisely when the integer d and the representation W_i have the same parity.

In order to keep track of even G-sheaves of the above form, we introduce a **parity function** $p : \Gamma \to \mathbb{Z}$ on the vertices of Γ, where $p(i) = 0$ if the irreducible G-representation W_i is even and $p(j) = 1$ if the irreducible W_j is odd. Notice also that if two edges in the diagram Γ are connected, then they have opposite parity and their heights differ by one.

Generalizing these properties of the parity function, we define a **height function** to be a function $h : \Gamma \to \mathbb{Z}$ on the vertices of Γ satisfying the conditions:

1. $h(i) \equiv p(i) \bmod 2$,

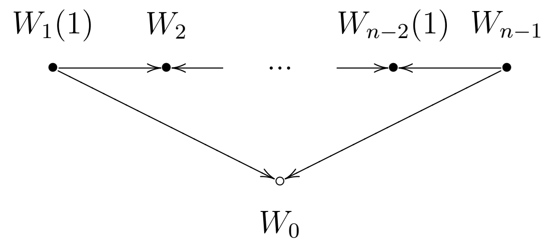
2. $|h(i) - h(j)| = 1$ if i is connected to j in Γ.

The first condition says that the parity of the height of a vertex agrees with the parity of the representation W_i, so each height function h gives rise to a collection of even G-sheaves

$$F_i^h = W_i \otimes \mathcal{O}(h(i))$$

indexed by the nodes $i \in \Gamma$. The second condition says that the height goes up or down one step between neighboring vertices of Γ. The height function then determines an orientation on the edges of Γ by letting the edges flow downhill. We denote the resulting quiver by Q_h.

Example 5.2.1. Let $G = \mathbb{Z}/n\mathbb{Z}$ and take the height h to be equal to the parity function p. The resulting quiver is pictured below, with the vertices labeled by the even G-sheaves.



We can now give a main result of [37], stated in a form convenient for us.

Theorem 5.2.2. *Let $A_h = \mathrm{End}_G(\oplus_i F_i^h)^{\mathrm{op}}$ be the opposite algebra of the endomorphism algebra of the collection F_i^h, $i \in \Gamma$. Then the natural functor $R\Phi_h := R\mathrm{Hom}_G(\oplus_i F_i^h, -)$*

$$D^b_{\widetilde{G}}(\mathbb{P}^1) \xrightarrow{R\Phi_h} D^b(A_h)$$

to the bounded derived category of finitely generated left A_h-modules is an equivalence.

Kirillov shows moreover that there is an isomorphism of algebras $A_h \simeq \mathbb{C}Q_h$, where $\mathbb{C}Q_h$ is the path algebra of the quiver Q_h, so the functor can also be thought of as taking values in the derived category of representations of Q_h. More precisely, under the isomorphism $A_h \simeq \mathbb{C}Q_h$, the space of paths in Q_h from i to j is given by $\mathrm{Hom}_G(F_j^h, F_i^h) = e_j A_h e_i$, where the identity morphisms $e_i = 1 \in \mathrm{Hom}_G(F_i^h, F_i^h)$ form a set of primitive, orthogonal idempotents in A_h.

In the rest of this section we give some definitions and comments about the category $D^b_{\widetilde{G}}(\mathbb{P}^1)$ that will be useful later. We let T denote the tangent bundle of $\mathbb{P}^1$ and ω the canonical bundle. Since $G \subset SL_2(\mathbb{C})$, any isomorphisms $T \simeq \mathcal{O}(2)$ and $\omega \simeq \mathcal{O}(-2)$ will be G-equivariant.

We call a vertex i of Q_h a **sink** if it has lower height than its neighbors, so that arrows are coming in to i, and a **source** if it has greater height than its neighbors, so that arrows are coming out of i. Given a height function h for which i is a sink or a source, define a new height function $\sigma_i^+ h$ or $\sigma_i^- h$ by

$$\sigma_i^+ h(k) = \begin{cases} h(k) & \text{if } i \neq k \\ h(k)+2 & \text{if } i = k \end{cases} \qquad \text{or} \qquad \sigma_i^- h(k) = \begin{cases} h(k) & \text{if } i \neq k \\ h(j)-2 & \text{if } i = k. \end{cases}$$

Since we have assumed $\pm I \subset G$, the Dynkin diagrams that we are considering are bipartite and one can check that one height function differs from another by a sequence of such operations, turning sinks into sources and sources into sinks.

The following observation of Kirillov is essential. If $i \in Q_h$ is a source, then

$$V \otimes F_i^h(-1) \simeq V \otimes W_i \otimes \mathcal{O}(h(i)-1) \simeq \bigoplus_{i \to j} W_j(h(j)) = \bigoplus_{i \to j} F_j^h,$$

where the first isomorphism is by definition of F_i^h, the second by McKay's observation and the step-wise nature of height functions, and the last is again by definition. Here the sum is over arrows $i \to j$ leaving the source i. Thus tensoring the Euler sequence $0 \to \mathcal{O}(-1) \to V \to T(1) \to 0$ with $F_i^h(-1)$ and using the above isomorphisms gives

$$0 \to F_i^{\sigma_i^- h} \to \bigoplus_{i \to k} F_k^h \to F_i^h \to 0. \tag{5.2.1}$$

Likewise, if $i \in Q_h$ is a sink, then tensoring the Euler sequence with $F_i^h(1)$ gives

$$0 \to F_i^h \to \bigoplus_{k \to i} F_k^h \to F_i^{\sigma_i^+ h} \to 0. \tag{5.2.2}$$

Lemma 5.2.3. $\mathrm{Ext}^1_G(F_k^h, F_l^h \otimes T^{\otimes d}) = 0$ *for all height functions* h*, all* $k, l \in I$*, and all* $d \in \mathbb{N}$*.*

Proof. First we check that the statement is true when h is the parity function. Then we show that if the statement is true for a height function h, it is also true for the modified height functions $\sigma_i^+ h$ and $\sigma_i^- h$. Since every height can be obtained from the parity function by a sequence of such modifications, this will establish the lemma.

By Serre duality, $\mathrm{Ext}^1_G(F_k^h, F_l^h \otimes T^{\otimes d}) \simeq \mathrm{Hom}_G(F_l^h \otimes T^{\otimes(d+1)}, F_k^h)^*$. If h is the parity function, the latter space is zero since $F_l^h \otimes T^{\otimes(d+1)}$ has higher degree than F_k^h, so the lemma holds for the parity functions.

Now assume the lemma is true for a height h and that i is a sink in h. We want to see that the lemma must hold for $\sigma_i^+ h$. Consider the possible values of k and l.

If $k = l = i$, then $\mathrm{Ext}^1_G(F_i^{\sigma_i^+ h}, F_i^{\sigma_i^+ h} \otimes T^{\otimes d}) \simeq \mathrm{Ext}^1_G(F_i^h, F_i^h \otimes T^{\otimes d})$, since tensoring with $\mathcal{O}(2)$ is an equivalence. The latter space is zero by assumption on h.

If $k, l \neq i$, then $\sigma_i^+ h(k) = k$ and we have $\mathrm{Ext}^1_G(F_k^{\sigma_i^+ h}, F_l^{\sigma_i^+ h} \otimes T^{\otimes d}) \simeq \mathrm{Ext}^1_G(F_k^h, F_l^h \otimes T^{\otimes d})$. The latter space is zero by assumption on h.

If $l = i$ and $k \neq i$, then $\sigma_i^+ h(k) = k$ and $F_l^{\sigma_i^+ h} = F_i^{\sigma_i^+ h} \simeq F_i^h(2)$, so $\mathrm{Ext}^1_G(F_k^{\sigma_i^+ h}, F_l^{\sigma_i^+ h} \otimes T^{\otimes d}) \simeq \mathrm{Ext}^1_G(F_k^h, F_i^h \otimes T^{\otimes(d+1)})$. The latter space is zero by assumption.

Finally, consider the case $k = i$ and $l \neq i$. If $d \geq 1$, $\mathrm{Ext}^1_G(F_i^h(2), F_l^h \otimes T^{\otimes d}) \simeq \mathrm{Ext}^1_G(F_i^h, F_l^h \otimes T^{\otimes(d-1)})$, and we are done by assumption. If $d = 0$, then by Serre duality $\mathrm{Ext}^1_G(F_i^h(2), F_l^h) \simeq \mathrm{Hom}_G(F_j^h, F_i^h)^*$. The dimension of the latter space is the number of paths from i to j in Q_h, which is zero since i is a sink.

Thus if the lemma holds for a height function h, then it also holds for $\sigma_i^+ h$. A similar argument shows that it also holds for $\sigma_i^- h$. □

For the proof of the next lemma we use Beilinson's resolution of the diagonal [3], which on $\mathbb{P}^1 \times \mathbb{P}^1$ takes the form

$$0 \to p^*\mathcal{O}(-1) \otimes q^*\omega(1) \to p^*\mathcal{O} \otimes q^*\mathcal{O} \to \mathcal{O}_\Delta \to 0,$$

for p and q the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the left and right factors respectively. The resolution is canonically constructed and so is automatically G-equivariant, and in fact each of its terms is an even G-sheaf.

Taking $\mathcal{F} \in D(\mathrm{QCoh}_{\widetilde{G}}(\mathbb{P}^1))$ and using the resolution of the diagonal as the kernel of a derived integral transform, we get an exact triangle

$$Rq_*(p^*\mathcal{F}(-1) \otimes q^*\omega(1)) \to Rq_*(p^*\mathcal{F} \otimes q^*\mathcal{O}) \to Rq_*(p^*\mathcal{F} \otimes \mathcal{O}_\Delta)$$

in $D(\mathrm{QCoh}_{\widetilde{G}}(\mathbb{P}^1))$. Applying the projection formula and flat base-change then gives the exact triangle

$$R\Gamma(\mathcal{F}(-1)) \otimes \omega(1) \to R\Gamma(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}. \tag{5.2.3}$$

Notice that since $R\Gamma(\mathcal{F}) \otimes \mathcal{O} \in D(\mathrm{QCoh}_{\widetilde{G}}(\mathbb{P}^1))$, $R\Gamma(\mathcal{F})$ must be a complex of even G-representations. Similarly, since $\omega(1)$ is an odd sheaf and $R\Gamma(\mathcal{F}(-1)) \otimes \omega(1) \in D(\mathrm{QCoh}_{\widetilde{G}}(\mathbb{P}^1))$, $R\Gamma(\mathcal{F}(-1))$ must be a complex of odd representations in order to make the tensor product with $\omega(1)$ even.

Lemma 5.2.4. *For any height function h, the collection F_i^h generates $D(\mathrm{QCoh}_{\widetilde{G}}(\mathbb{P}^1))$ in the sense that if $R\mathrm{Hom}_G(F_i^h, \mathcal{F}) = 0$ for all i, then $\mathcal{F} = 0$.*

Proof. As in the proof of Lemma 5.2.3, we first show the statement is true when h is the parity function and then show that if the statement is true for a height function h, then it is also true for $\sigma_i^+ h$ and $\sigma_i^- h$.

First let h be the parity function and assume $R\mathrm{Hom}_G(F_i^h, \mathcal{F}) = 0$ for all F_i^h. Note that $R\Gamma(\mathcal{F}) = R\mathrm{Hom}(\mathcal{O}, \mathcal{F})$ and $R\Gamma(\mathcal{F}(-1)) \simeq R\mathrm{Hom}(\mathcal{O}(1), \mathcal{F})$. We claim that the assumption $R\mathrm{Hom}_G(F_i^h, \mathcal{F}) = 0$ implies both are zero, and so $\mathcal{F} = 0$ from the exactness of triangle 5.2.3.

Indeed, if $R\Gamma(\mathcal{F}) = R\mathrm{Hom}(\mathcal{O}, \mathcal{F})$ were non-zero, then since it consists of even representations it would contain some non-zero irreducible even representation W_i and we would have $(W_i^* \otimes R\mathrm{Hom}(\mathcal{O},\mathcal{F}))^G \simeq R\mathrm{Hom}_{\mathrm{G}}(W_i,\mathcal{F}) \neq 0$. This contradicts the assumption that $R\mathrm{Hom}_G(F_i^h,\mathcal{F}) = 0$ for all i, since $W_i = F_i^h$ for h the parity function and i even. Likewise, by assumption $R\mathrm{Hom}_G(W_i(1),\mathcal{F}) \simeq (W_i^* \otimes R\mathrm{Hom}(\mathcal{O}(1),\mathcal{F}))^G = 0$ for all odd representations W_i, but $R\mathrm{Hom}(\mathcal{O}(1),\mathcal{F})$ consists of odd representations, and so must be zero.

Now assume the conclusion of the lemma holds for a height function h. We will show that this implies the lemma for $\sigma_i^- h$, where i is a source in h. Suppose that $R\mathrm{Hom}_G(F_k^{\sigma_i^- h},\mathcal{F}) = 0$ for all k. We want to see that this implies $\mathcal{F} = 0$. Since $\sigma_i^- h$ differs from h only at i, we have $R\mathrm{Hom}_G(F_k^{\sigma_i^- h},\mathcal{F}) = 0$ for all $k \neq i$ by assumption on h. We claim further that $R\mathrm{Hom}_G(F_i^h,\mathcal{F}) = 0$, and so we shall have $\mathcal{F} = 0$ by the assumption on h. To sustain the claim, recall sequence 5.2.1: $0 \to F_i^{\sigma_i^- h} \to \bigoplus_{i\to j} F_j^h \to F_i^h \to 0$. Applying $R\mathrm{Hom}_G(-,\mathcal{F})$ gives an exact triangle of complexes of vector spaces

$$R\mathrm{Hom}_G(F_i^h,\mathcal{F}) \to \bigoplus_{i\to j} R\mathrm{Hom}_G(F_j^h,\mathcal{F}) \to R\mathrm{Hom}_G(F_i^{\sigma_i^- h},\mathcal{F}).$$

The last two terms are zero by assumption on $\sigma_i^- h$, so the first term must be zero too, as claimed.

A similar argument shows that if the lemma holds for h, then it also holds for $\sigma_i^+ h$ when i is a sink. Thus the lemma holds for all height functions. □

We conclude this section by making some standard remarks about categories of modules over finite dimensional algebras (see [2]) and introducing some important t-structures on the category $D^b_{\tilde{G}}(\mathbb{P}^1)$.

Since the algebra $A_h = \mathrm{End}_G(\oplus_i F_i^h)^{\mathrm{op}}$ is finite dimensional, the category A_h-mod of finitely generated modules is of finite length, meaning that every object has a finite filtration with simple quotients, and by the Jordan-Holder theorem these simples and their multiplicities do not depend on the filtration. The simple representations of the algebra A_h are indexed by the vertices of the diagram Γ. Given a vertex i, we have the ith idempotent $e_i = 1 \in \mathrm{Hom}_{\mathrm{G}}(F_i^h, F_i^h)$ and the corresponding simple is

$$S_i^h := e_i A_h e_i = e_i \mathrm{Hom}_{\mathrm{G}}(\oplus_j F_j^h, \oplus_j F_j^h) = \mathrm{Hom}_{\mathrm{G}}(F_i^h, F_i^h).$$

In terms of the quiver Q_h, the simple consists of a one-dimensional vector space at i and zeroes elsewhere. By the Jordan-Holder theorem, the classes of the simples S_i^h form a basis for $K_0(Q_h)$.

We also have the indecomposable projectives $P_i^h = A_h e_i$, which are dual to the S_i^h under the Euler form on $K_0(Q_h)$:

$$\langle P_i^h, S_j^h \rangle = \sum_k (-1)^k \dim \mathrm{Ext}^k(P_i^h, S_j^h) = \dim \mathrm{Hom}(P_i^h, S_j^h) = \delta_{ij}.$$

Since every representation has a resolution by sums of the P_i^h, the classes of the P_i^h span $K_0(Q_h)$, and by duality with S_i^h, they are linearly independent, so the indecomposable projectives provide another basis for $K_0(Q_h)$.

Applying the inverse equivalences from Theorem 5.2.2, we get for each height function h the heart of a bounded t-structure

$$\mathcal{A}_h := R\Phi_h^{-1}(A_h\text{-mod}) \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$$

with simple objects $E_i^h = R\Phi_h^{-1}(S_i^h)$ and indecomposable projectives $R\Phi_h^{-1}(P_i^h)$, which are dual with respect to the Euler form on $K_0(\mathcal{D}^b_{\widetilde{G}}(\mathbb{P}^1))$. In fact, $R\Phi_h(F_i^h) = \mathrm{Hom}_{\mathrm{G}}(\oplus_j F_j^h, F_i^h) = A_h e_i = P_i^h$, so the indecomposable projectives of $\mathcal{A}_h$ are just the objects $F_i^h = R\Phi_h^{-1}(P_i^h)$.

Remark 5.2.5. Notice that since $R\Phi_h(E_i^h) = R\mathrm{Hom}_{\mathrm{G}}(\oplus_j F_j^h, E_i^h) = S_i^h$, which is just a one-dimensional vector space concentrated at the ith vertex of Q_h, the simple $E_i^h \in \mathcal{A}_h$ is the unique object in $D^b_{\widetilde{G}}(\mathbb{P}^1)$ for which

$$R\mathrm{Hom}_{\mathrm{G}}(F_k^h, E_i^h) = \begin{cases} \mathbb{C} & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where $\mathbb{C}$ denotes the complex with $\mathbb{C}$ in degree zero and zeroes elsewhere.

The hearts $\mathcal{A}_h$ will play an important role in our discussion. In particular, we need the following two lemmas, which describe how the simples of one heart are related to each other and to the simples of another heart.

Lemma 5.2.6. *We have*

$$\dim \mathrm{Ext}^k_G(E_i^h, E_j^h) = \begin{cases} 1 & \textit{if } i = j \textit{ and } k = 0 \\ 1 & \textit{if } i \to j \textit{ and } k = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The corresponding statement can be easily checked for the simples S_i^h (just think about when there can be morphisms and extensions between the simple representations of Q_h), and so the lemma follows upon applying the inverse equivalence $R\Phi_h^{-1}$. □

Lemma 5.2.7. *If i is a source, the simples of $\mathcal{A}_{\sigma_i^- h}$ are given by*

$$E_j^{\sigma_i^- h} = \begin{cases} E_i^h[-1] & \textit{if } i = j \\ E_j^h & \textit{if } i \neq j, i \nrightarrow j \\ \mathrm{Cone}(E_i^h[-1] \to E_j^h) & \textit{if } i \to j \end{cases}$$

If i is a sink, the simples of $\mathcal{A}_{\sigma_i^+ h}$ are

$$E_j^{\sigma_i^+ h} = \begin{cases} E_i^h[1] & \text{if } i = j \\ E_j^h & \text{if } i \neq j, j \nrightarrow i \\ \mathrm{Cone}(E_j^h[-1] \to E_i^h) & \text{if } j \to i \end{cases}$$

Here, when $i \to j$, $E_i^h[-1] \to E_j^h$ is the non-zero morphism, unique up to scalar, provided by Lemma 5.2.6, and likewise for $E_j^h[-1] \to E_i^h$ when $j \to i$.

Proof. We give the proof for $\mathcal{A}_{\sigma_i^- h}$. The argument for $\mathcal{A}_{\sigma_i^+ h}$ is similar.

Claim 1: $E_i^{\sigma_i^- h} = E_i^h[-1]$.

Up to isomorphism, $E_i^{\sigma_i^- h}$ is the unique object of $D^b_{\widetilde{G}}(\mathbb{P}^1)$ such that $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, E_i^{\sigma_i^- h}) = 0$ for $k \neq i$ and $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^{\sigma_i^- h}) \simeq \mathbb{C}$ (see Remark 5.2.5), so to establish the claim we check that $E_i^h[-1]$ satisfies these two conditions. For the first condition, note that $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, E_i^h[-1]) \simeq R\mathrm{Hom}_{\mathrm{G}}(F_k^h, E_i^h)[-1] = 0$, since $F_k^{\sigma_i^- h} \simeq F_k^h$ for $k \neq i$. For the second condition, we need $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^h[-1]) \simeq \mathbb{C}$. Applying $R\mathrm{Hom}_{\mathrm{G}}(-, E_i^h)$ to the sequence $0 \to F_i^{\sigma_i^- h} \to \bigoplus_{i \to k} F_k^h \to F_i^h \to 0$, we get an exact triangle

$$R\mathrm{Hom}_{\mathrm{G}}(F_i^h, E_i^h) \to \bigoplus_{i \to k} R\mathrm{Hom}_{\mathrm{G}}(F_k^h, E_i^h) \to R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^h).$$

Since $R\mathrm{Hom}_{\mathrm{G}}(F_i^h, E_i^h) \simeq \mathbb{C}$ and $\bigoplus_{i \to k} R\mathrm{Hom}_{\mathrm{G}}(F_k^h, E_i^h) = 0$, we have the desired isomorphism $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^h[-1]) \simeq R\mathrm{Hom}_{\mathrm{G}}(F_i^h, E_i^h) \simeq \mathbb{C}$.

Claim 2: $E_j^{\sigma_i^- h} = E_j^h$ if $i \neq j, i \nrightarrow j$.

As in Claim 1, we check that E_j^h satisfies the characteristic properties of $E_j^{\sigma_i^- h}$. Since $F_k^{\sigma_i^- h} = F_k^h$ for $k \neq i$, note that $R\mathrm{Hom}_{\mathrm{G}}(F_j^{\sigma_i^- h}, E_j^h) \simeq \mathbb{C}$ and $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, E_j^h) = 0$ for $k \neq i, j$. It remains to show that $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_j^h) = 0$. Applying $R\mathrm{Hom}_{\mathrm{G}}(-, E_j^h)$ to sequence 5.2.1 gives us an exact triangle $R\mathrm{Hom}_{\mathrm{G}}(F_i^h, E_j^h) \to \bigoplus_{i \to k} R\mathrm{Hom}_{\mathrm{G}}(F_k^h, E_j^h) \to R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_j^h)$, whose first two terms and hence last term are zero.

Claim 3: $E_j^{\sigma_i^- h} = \mathrm{Cone}(E_i^h[-1] \to E_j^h)$ if $i \to j$.

Let $X = \mathrm{Cone}(E_i^h[-1] \to E_j^h)$. We check that X satisfies the characteristic properties of $E_j^{\sigma_i^- h}$. Applying $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, -)$ to the exact triangle $E_i^h[-1] \to E_j^h \to X$ gives the exact triangle (**5.2.4**) $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, E_i^h[-1]) \to R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, E_j^h) \to R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, X)$.

If $k \neq i, j$, then $F_k^{\sigma_i^- h} = F_k^h$, the first two terms and hence the last term of the triangle are zero. if $k = j$, then $F_k^{\sigma_i^- h} = F_j^h$ and so the first term is zero, giving an isomorphism $R\mathrm{Hom}_{\mathrm{G}}(F_j^{\sigma_i^- h}, X) \simeq R\mathrm{Hom}_{\mathrm{G}}(F_j^h, E_j^h) \simeq \mathbb{C}$. Finally, if $k = i$, then applying $R\mathrm{Hom}_{\mathrm{G}}(-, E_j^h)$ and $R\mathrm{Hom}_{\mathrm{G}}(-, E_i^h)$ to the sequence 5.2.1 shows that $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^h[-1]) \simeq \mathbb{C}$ and $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_j^h) \simeq \mathbb{C}$ (both concentrated in degree 0). Thus if the first arrow in the triangle is non-zero, it must give an isomorphism $R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_i^h[-1]) \simeq R\mathrm{Hom}_{\mathrm{G}}(F_i^{\sigma_i^- h}, E_j^h)$ and so $R\mathrm{Hom}_{\mathrm{G}}(F_k^{\sigma_i^- h}, X) = 0$, and we shall have verified that $X = \mathrm{Cone}(E_i^h[-1] \to E_j^h)$ satisfies the characteristic properties of $E_j^{\sigma_i^- h}$.

That the first arrow in the triangle 5.2.4 is indeed non-zero follows from the fact that $\oplus_k F_k^{\sigma_i^- h}$ generates the derived category. Indeed, apply the functor $R\mathrm{Hom}_{\mathrm{G}}(\oplus_k F_k^{\sigma_i^- h}, -)$ to the morphism $E_i^h[-1] \to E_j^h$. We have seen in the course of our argument that the result is zero when restricted to summands with $k \neq i$. If it were also zero when $k = i$, then the morphism $E_i^h[-1] \to E_j^h$ would have to be zero by faithfulness of $R\mathrm{Hom}_{\mathrm{G}}(\oplus_k F_k^{\sigma_i^- h}, -)$, a contradiction. □

5.3 McKay correspondence for $T^*\mathbb{P}^1$

We now give the analogue for the cotangent bundle $T^*\mathbb{P}^1$ of the equivalences $R\Phi_h : D^b_{\widetilde{G}}(\mathbb{P}^1) \simeq D^b(A_h)$ from Theorem 5.2.2.

Let π be the projection and s the zero-section for $T^*\mathbb{P}^1$:

$$\begin{array}{c} T^*\mathbb{P}^1 \\ {\scriptstyle s}\uparrow\ \ \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 \end{array}$$

Define $\mathcal{F}_i^h := \pi^* F_i^h$ and $\mathcal{E}_i^h := s_* E_i^h$, analogues for $T^*\mathbb{P}^1$ of the indecomposable projectives and the simples for the heart $\mathcal{A}_h \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$. Setting $B_h := \mathrm{End}_G(\oplus_i \mathcal{F}_i^h)^{\mathrm{op}}$, consider the natural functor $R\Psi_h = R\mathrm{Hom}_{\mathrm{G}}(\oplus_i \mathcal{F}_i^h, -)$ from the derived category of $\widetilde{G}$-sheaves on $T^*\mathbb{P}^1$ to the derived category of finitely generated B_h-modules.

Theorem 5.3.1. *For each height function h, we have an equivalence*

$$D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \xrightarrow{R\Psi_h} D^b(B_h).$$

Proof. Like in the proofs of 3.2.1 and 4.2.1, we must check that there are no higher Exts between the $\mathcal{F}_i^h$ and that the $\mathcal{F}_i^h$ generate $D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$. This will establish that $R\Psi_h$ gives an equivalence $D^b_G(T^*\mathbb{P}^1) \simeq \mathrm{perf}\ B_h$. In Proposition 5.3.5, we shall see that B_h is Koszul and hence of finite global dimension, so $\mathrm{perf}\ B_h \simeq D^b(B_h)$.

To compute Exts, use the adjunction $\pi^* \dashv \pi_*$ and the projection formula:

$$\mathrm{Ext}^k_G(\mathcal{F}_i^h, \mathcal{F}_j^h) = \mathrm{Ext}^k_G(\pi^* F_i^h, \pi^* F_j^h) \simeq$$
$$\mathrm{Ext}^k_G(F_i^h, \pi_*\pi^* F_j^h) \simeq \bigoplus_{d\geq 0} \mathrm{Ext}^k_G(F_i^h, \oplus_k F_j^h \otimes T^{\otimes d}),$$

where T denotes the tangent bundle of $\mathbb{P}^1$. Each summand on the right is zero by Lemma 5.2.3, so indeed we have vanishing of the higher Exts.

Next we establish spanning. Suppose that we have $\mathcal{G} \in D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$ such that $R\mathrm{Hom}_{\mathrm{G}}(\pi^* F_i^h, \mathcal{G}) = 0$ for all i. Applying the adjunction, we have $R\mathrm{Hom}_{\mathrm{G}}(F_i^h, R\pi_*\mathcal{G}) =$

0 for all i, so by Lemma 5.2.4 above, $R\pi_*\mathcal{G} = 0$. But π is an affine map, so π_* is exact and has no kernel, hence $\mathcal{G} = 0$. □

Remark 5.3.2. Note that our algebra $B_h \simeq \bigoplus_{i,j,d} \mathrm{Hom_G}(F_i^h, F_j^h \otimes T^{\otimes d})^{\mathrm{op}}$ is graded by the difference $h(j) + 2d - h(i)$. Since $F_i^h = \mathcal{O}(h_i) \otimes W_i$ and $F_j^h \otimes T^{\otimes d} \simeq \mathcal{O}(h(j) + 2d) \otimes W_j$, there is an isomorphism $\mathrm{Hom_G}(F_i^h, F_j^h \otimes T^{\otimes d}) \simeq \mathrm{Hom_G}(W_i, \mathcal{O}(h(j) + 2d - h(j)) \otimes W_j)$, so the degree zero part of the algebra B_h is just $B_0 = \bigoplus_i \mathrm{Hom_G}(F_i^h, F_i^h)$, which is a commutative semisimple $\mathbb{C}$-algebra with one summand for each i.

As in Chapter 4, we shall apply Koszul duality to understand the graded algebra B_h. For this, we need to compute some Exts, which we shall do using the following lemma from [17, pg. 20].

Lemma 5.3.3. *For $\mathcal{F}, \mathcal{G} \in D^b_{\widetilde{G}}(\mathbb{P}^1)$ we have*

$$\mathrm{Ext}^k_G(s_*\mathcal{F}, s_*\mathcal{G}) \simeq \mathrm{Ext}^k_G(\mathcal{F}, \mathcal{G}) \oplus \mathrm{Ext}^{2-k}_G(\mathcal{G}, \mathcal{F})^*$$

In particular, the lemma allows us to compute Exts between the objects $\mathcal{E}_i^h = s_*E_i^h$.

Proposition 5.3.4. *Let h be a height function on Γ and set $\mathcal{E}_i^h = s_*E_i^h$. Then we have*

$$\begin{aligned} \mathrm{Hom_G}(\mathcal{E}_i^h, \mathcal{E}_j^h) &\simeq \mathrm{Hom_G}(E_i^h, E_j^h) \\ \mathrm{Ext}^1_G(\mathcal{E}_i^h, \mathcal{E}_j^h) &\simeq \mathrm{Ext}^1_G(E_i^h, E_j^h) \oplus \mathrm{Ext}^1_G(E_j^h, E_i^h)^* \\ \mathrm{Ext}^2_G(\mathcal{E}_i^h, \mathcal{E}_j^h) &\simeq \mathrm{Hom_G}^2(E_j^h, E_i^h)^* \end{aligned}$$

For any height function h, the $\mathcal{E}_i^h$ form a Γ-configuration of spherical objects.

Proof. The three isomorphisms are just Lemma 5.3.3.

To see the the $\mathcal{E}_i^h$ form a Γ-configuration, note that $\mathrm{Hom_G}(\mathcal{E}_i^h, \mathcal{E}_i^h) \simeq \mathrm{Ext}^2_G(\mathcal{E}_i^h, \mathcal{E}_i^h)^* \simeq \mathbb{C}$ and $\mathrm{Ext}^1_G(\mathcal{E}_i^h, \mathcal{E}_i^h) = 0$ by Lemma 5.2.6 together with the three isomorphisms. Thus $\mathcal{E}_i^h$ is indeed spherical. One can see in the same way that $\mathrm{Ext}^1_G(\mathcal{E}_i^h, \mathcal{E}_j^h) \simeq \mathbb{C}$ exactly when i and j are connected in Q and that $\mathrm{Hom_G}(\mathcal{E}_i^h, \mathcal{E}_j^h) = \mathrm{Ext}^2_G(\mathcal{E}_i^h, \mathcal{E}_j^h) = 0$ when $i \neq j$. □

Proposition 5.3.5. *B_h is Koszul with Koszul dual $E(B_h) \simeq \mathrm{Ext}^\bullet_G(\oplus_i \mathcal{E}_i^h, \oplus_i \mathcal{E}_i^h)$, the* Ext *algebra of the spherical Γ-collection $\mathcal{E}_i^h$. Thus by Lemma 4.3.1, there is an isomorphism $B_h \simeq \Pi_\Gamma$ with the preprojective algebra of Γ.*

Proof. From the adjunction $s^* \dashv Ls_*$, we see that

$$R\mathrm{Hom_G}(\pi^* F_i^h, s_* E_j^h) \simeq R\mathrm{Hom_G}(Ls^* \pi^* F_i^h, E_j^h) \simeq R\mathrm{Hom_G}(F_i^h, E_j^h).$$

By the remarks before Lemma 5.2.6, the right hand side is zero when $i \neq j$ and is 1-dimensional and concentrated in degree zero when $i = j$. Thus we see that $R\Psi_h(\mathcal{E}_i^h) \simeq e_i B_h e_i$, the ith simple of the algebra B_h. Since $R\Psi_h$ is an equivalence,

$$E(B_h) = \mathrm{Ext}^\bullet_{B_h}(B_0, B_0) \simeq \mathrm{Ext}^\bullet_G(\oplus_i \mathcal{E}_i^h, \oplus_i \mathcal{E}_i^h).$$

Thus $E(B_h)$ is the Ext-algebra of a Γ-configuration and by Lemma 4.3.1, there is an isomorphism $B_h \simeq \Pi_\Gamma$ and B_h is Koszul. □

Remark 5.3.6. Putting together the equivalences of Theorems 5.3.1 and 4.2.1 and the isomorphisms $B_h \simeq \Pi_\Gamma$, we see that for each height function h there is a chain of equivalences

$$D^b_{\tilde{G}}(T^*\mathbb{P}^1) \simeq D^b(B_h) \simeq D^b(\Pi_\Gamma) \simeq D^b_G(\mathbb{C}^2),$$

which provides a bridge between the projective McKay correspondence of [37] and the usual McKay correspondence for $\mathbb{C}^2$.

As pointed out by Khovanov-Huerfano [31], a single equivalence $D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \simeq D^b_G(\mathbb{C}^2)$ can be obtained by noting that there is an isomorphism of resolutions $\widehat{Y} \to T^*\mathbb{P}^1/\widetilde{G}$ and $\widehat{X} \to \mathbb{C}^2/G$. Applying the celebrated theorem of Bridgeland-King-Reid [18] then gives equivalences

$$D^b_{\widetilde{G}}(T^*\mathbb{P}^1) \simeq D^b(Y) \simeq D^b(Y') \simeq D^b_G(\mathbb{C}^2).$$

5.4 Reflection functors and spherical twists

One of the most interesting aspects of Kirillov's paper [37] is that the equivalences $R\Phi_h$ for different h are related by the reflection functors of Bernstein-Gelfand-Ponamarev [7]. We show that in terms of $D^b_{\widetilde{G}}(\mathbb{P}^1)$, the reflection functors amount to tilting at a simple object. On $D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$ the reflection functors are replaced by spherical twists which relate the various equivalences $R\Psi_h$. We also note that the action of the twist can be described in terms of tilting at a simple object. This completes our description of the relation betweeen the McKay correspondences for $T^*\mathbb{P}^1$ and $\mathbb{P}^1$ as outlined in the table from the introduction.

5.4.1 Reflection functors

Recall from Section 5.2 that if i is a sink in a quiver Q, we define a new quiver $\sigma_i^+ Q$ by reversing all arrows adjacent to i so that it becomes a source. Likewise, if i is a source, we define $\sigma_i^- Q$ so that i becomes a sink.

Accompanying these operations on quivers are the **reflection functors** of Bernstein-Gelfand-Ponomarev **reflection functors** [7]

$$\operatorname{Rep} Q \xrightarrow{\sigma_i^+} \operatorname{Rep} \sigma_i^+ Q \qquad \text{and} \qquad \operatorname{Rep} Q \xrightarrow{\sigma_i^-} \operatorname{Rep} \sigma_i^- Q.$$

In the first case, given a sink i in Q and representation V, define $\sigma_i^+ V$ to be the same as V away from i, and at i replace V_i with the kernel of the natural morphism $\bigoplus_{j \to i} V_j \to V_i$. The arrows from $(\sigma_i^+ V)_i$ to adjacent V_j are given by the composition $(\sigma_i^+ V)_i \hookrightarrow \bigoplus_{j \to i} V_j \to V_j$. This defines the functor on objects and its definition on morphisms is the obvious one. Likewise, if i is a source, σ_i^- does nothing away from i, and at i replace V_i with the cokernel of the morphism $V_i \to \bigoplus_{i \to j} V_j$. The arrows from adjacent V_j to $(\sigma_i^- V)_i$ are given by the composition $V_j \hookrightarrow \bigoplus_{i \to j} V_j \to (\sigma_i^- V)_i$ and the definition of the functor on morphisms is obvious.

We record some basic and well-known facts about the reflection functors.

Lemma 5.4.1.

1. *The functor σ_i^+ is left exact, while σ_i^- is right exact, and we have an adjunction $\sigma_i^- \dashv \sigma_i^+$.*

2. *The derived functors $R\sigma_i^+$ and $L\sigma_i^-$ are inverse equivalences. Identifying the Grothendieck groups of Q and $\sigma_i^{\pm} Q$ using the bases of simple representations, the automorphisms of the Grothendieck group induced by $R\sigma_i^+$ and $L\sigma_i^-$ are simply reflections at the ith simple.*

In the case of a Dynkin diagram, the functors thus generate the action of the Weyl group on the root lattice, which we identify with K_0 of the quiver.

Theorem 8.9 in [37] gives the relation between the equivalences $R\Phi_h$ for different height functions in terms of reflection functors.

Theorem 5.4.2. *We have a commutative diagram of equivalences*

$$\begin{array}{ccc} & D^b_{\widetilde{G}}(\mathbb{P}^1) & \\ {}^{R\Phi_h}\swarrow & & \searrow^{R\Phi_{\sigma_i^- h}} \\ D^b(Q_h) & \xrightarrow{\ L\sigma_i^-\ } & D^b(Q_{\sigma_i^- h}). \end{array}$$

Likewise, we have $R\sigma_i^+ \circ R\Phi_h \simeq R\Phi_{\sigma_i^+ h}$.

As discussed in the comments before Lemma 5.2.6, applying the inverse equivalence $R\Phi_h^{-1}$ to the standard heart of $D^b(Q_h)$ gives a non-standard heart $\mathcal{A}_h \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$ of finite length with simples E_i^h and indecomposable projectives F_i^h.

To relate the various hearts $\mathcal{A}_h$ we use the following proposition from [17].

Proposition 5.4.3. *Let $\mathcal{A} \subset \mathcal{D}$ be a finite length heart of a bounded t-structure for $\mathcal{D}$ and let $S \in \mathcal{A}$ be a simple object, Set $\langle S \rangle^\perp = \{\mathcal{F} \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(S, \mathcal{F}) = 0\}$ and ${}^\perp\langle S \rangle = \{\mathcal{F} \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, S) = 0\}$. Then the full subcategories*

$$L_S\mathcal{A} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for } i \neq 0, 1, H^0 \in \langle S \rangle^\perp \text{ and } H^1(\mathcal{F}) \in \langle S \rangle\}$$

and

$$R_S\mathcal{A} = \{\mathcal{F} \in \mathcal{D} \mid H^i(\mathcal{F}) = 0 \text{ for } i \neq -1, 0, H^{-1} \in \langle S \rangle \text{ and } H^0(\mathcal{F}) \in {}^\perp\langle S \rangle\}$$

are hearts of bounded t-structures on $\mathcal{D}$. $L_S\mathcal{A}$ is called the **left tilt** *at S, $R_S\mathcal{A}$ the* **right tilt**.

We can now state the relation between various hearts $\mathcal{A}_h$ in terms of tilting.

Proposition 5.4.4. *Denoting the left and right tilts at $E_i^h \in \mathcal{A}_h \subset D^b_{\widetilde{G}}(\mathbb{P}^1)$ by L_i and R_i, we have*

$$L_i\mathcal{A}_h = \mathcal{A}_{\sigma_i^{-1}h} \quad \text{and} \quad R_i\mathcal{A}_h = \mathcal{A}_{\sigma_i^+ h}.$$

Proof. This follows essentially from the well-known relation between the reflection functors and tilting (in fact tilting was invented to generalize the reflection functors). Letting $\mathcal{C}_h \subset D^b(Q_h)$ denote the standard heart, the relation is that $L\sigma_i^-(\mathcal{C}_{\sigma_i^+ h}) = R_i\mathcal{C}_h$ and $R\sigma_i^+(\mathcal{C}_{\sigma_i^- h}) = L_i\mathcal{C}_h$. We check the first and the second is similar.

Since both $L\sigma_i^-(\mathcal{C}_{\sigma_i^+ h})$ and $R_i\mathcal{C}_h$ form hearts of bounded t-structures for $D^b(Q_h)$ and nested hearts are in fact equal, it is enough to see that $L\sigma_i^-(\mathcal{C}_{\sigma_i^+ h}) \subseteq R_i\mathcal{C}_h$. In fact, $L\sigma_i^-(\mathcal{C}_{\sigma_i^+ h})$ is finite length and is the smallest extension closed subcategory containing its simples, so it is enough to check that $L\sigma_i^-(S_j^{\sigma_i^+ h}) \in R_i\mathcal{C}_h$ for every simple $S_j^{\sigma_i^+ h} \in \mathcal{C}_{\sigma_i^+ h}$.

By the definition of right tilting, we then must see that $H^0(L\sigma_i^-(S_j^{\sigma_i^+ h})) = \sigma_i^-(S_j^{\sigma_i^+ h}) \in {}^\perp\langle S_i^h\rangle$ and $H^{-1}(L\sigma_i^-(S_j^{\sigma_i^+ h})) \in \langle S_i^h\rangle$. First note that

$$\sigma_i^-(S_j^{\sigma_i^+ h}) = \begin{cases} S_j^h & \text{if } i \not\to j \\ W & \text{if } i \to j \\ 0 & \text{if } i = j \end{cases}$$

where W is the quiver representation with $\mathbb{C}$ at i and j and an isomorphism for the arrow joining them. Thus in all cases $H^0(L\sigma_i^-(S_j^{\sigma_i^+ h})) = \sigma_i^-(S_j^{\sigma_i^+ h}) \in {}^\perp\langle S_i^h\rangle$.

For H^{-1}, consider a projective resolution $0 \to P^{-1} \to P^0 \to S_j^{\sigma_i^+ h} \to 0$. After applying the functor, the map $\sigma_i^- P^{-1} \to \sigma_i^- P^0$ is still injective, except possibly at i, so $H^{-1}(L\sigma_i^-(S_j^{\sigma_i^+ h})) \in \langle S_i^h\rangle$.

Since tilting commutes with equivalences and we have $R\Phi_h^{-1} \circ L\sigma_i^- \simeq R\Phi_{\sigma_i^+ h}^{-1}$ and $R\Phi_h^{-1} \circ R\sigma_i^+ \simeq R\Phi_{\sigma_i^- h}^{-1}$, the proposition follows. □

5.4.2 Spherical twists

In the category $D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$ the role of the objects $E^h_i \in D^b_{\widetilde{G}}(\mathbb{P}^1)$ is played by the spherical objects $\mathcal{E}^h_i = s_* E^h_i$. In fact, for this section it is more convenient to think of the $\mathcal{E}^h_i$ as objects in the full triangulated subcategory

$$\mathcal{D} \subset D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$$

consisting of objects (set-theoretically) supported along the zero-section, since $\mathcal{D}$ is a 2-CY category in the sense of 2.9.

By Proposition 5.3.4, the spherical objects $\mathcal{E}^h_i$ form a Γ-configuration. The spherical twists $T_{\mathcal{E}^h_i}$ therefore generate an action of the braid group B_Γ on $\mathcal{D}$, as discussed in Section **??**.

We saw above that if we identify all of the Grothendieck groups $K_0(Q_h)$ for different h with the affine root lattice associated to Γ, then the reflection functors induce the action of the Weyl group. Now we want to see that $K_0(\mathcal{D})$ can be identified with the affine root lattice and that the action of the braid group B_Γ induces that of the Weyl group.

Proposition 5.4.5. *The classes of the $\mathcal{F}^h_i$ and $\mathcal{E}^h_j$ form dual bases with respect to the natural pairing $\langle\ ,\ \rangle : K_0(D^b_{\widetilde{G}}(T^*\mathbb{P}^1)) \otimes K_0(\mathcal{D}) \to \mathbb{Z}$, where $\langle \mathcal{E}, \mathcal{F}\rangle = \sum_k (-1)^k \dim \mathrm{Ext}^k_G(\mathcal{E}, \mathcal{F})$.*

Proof. First, recall that the pullback π^* gives an isomorphism $K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \simeq K_0(D^b_{\widetilde{G}}(T^*\mathbb{P}^1))$ (see [21, Theorem 5.4.17]). Since the F^h_i form a basis for the former, the $\mathcal{F}^h_i = \pi^* F^h_i$ form a basis for the latter. Since $R\mathrm{Hom}_{\mathrm{G}}(\pi^* F^h_i, s_* E^h_j) \simeq R\mathrm{Hom}_{\mathrm{G}}(s^*\pi^* F^h_i, E^h_j) \simeq R\mathrm{Hom}_{\mathrm{G}}(F^h_i, E^h_j)$, the duality between $\mathcal{F}^h_i$ and $\mathcal{E}^h_j$ follows from that between F^h_i and E^h_j discussed in Remark 5.2.5. Then the linear independence of the $\mathcal{E}^h_j$ follows from the duality between the $\mathcal{F}^h_i$ and the $\mathcal{E}^h_j$.

It remains to show that the $\mathcal{E}_j^h$ span $K_0(\mathcal{D})$. For this, consider an object $\mathcal{F} \in \mathcal{D}$. Its class in $K_0(\mathcal{D})$ may be written as $[\mathcal{F}] = \sum_k (-1)^k [\mathcal{H}^k(\mathcal{F})]$, where the $\mathcal{H}^k(\mathcal{F})$ are the cohomology sheaves of $\mathcal{F}$. Although the support of $\mathcal{G} = \mathcal{H}^k(\mathcal{F})$ may be non-reduced and so $\mathcal{G}$ might not be the push-forward of an object from $\mathbb{P}^1$, there is a natural filtration $0 = \mathcal{G}^m \subseteq \cdots \subseteq \mathcal{G}^1 \subseteq \mathcal{G}$ whose associated graded pieces have reduced support along Z, where $\mathcal{G}^k = \sqrt{\mathcal{I}}^k \cdot \mathcal{G}$ for $\mathcal{I} = \mathrm{Ann}(\mathcal{G})$. Thus the class of $\mathcal{G}$ and hence the class of $\mathcal{F}$ can be written as a combination of classes pushed-forward from $\mathbb{P}^1$. By the comments before Lemma 5.2.6, classes on $\mathbb{P}^1$ are combinations of the classes E_i^h, and so $K_0(\mathcal{D})$ is indeed spanned by the $\mathcal{E}_i^h$. □

Proposition 5.4.6. *In the basis $\mathcal{E}_i^h$, the Euler form on $K_0(\mathcal{D})$ is given by the Cartan matrix of Γ, so $K_0(\mathcal{D})$ is an affine root lattice with the $\mathcal{E}_i^h$ as a base of simple roots. Moreover, the twists $T_{\mathcal{E}_i^h}$ induce the corresponding simple reflections.*

Proof. For the first claim, simply note that by Lemmas 5.2.6 and 5.3.3 $\langle \mathcal{E}_i^h, \mathcal{E}_i^h \rangle = \dim \mathrm{Hom_G}^k(\mathcal{E}_i^h, \mathcal{E}_i^h) + \dim \mathrm{Ext}^2_G(\mathcal{E}_i^h, \mathcal{E}_i^h) = 2$ and for $i \neq j$ $\langle \mathcal{E}_i^h, \mathcal{E}_j^h \rangle = -\dim \mathrm{Ext}^1_G(\mathcal{E}_i^h, \mathcal{E}_j^h) - \dim \mathrm{Ext}^1_G(\mathcal{E}_j^h, \mathcal{E}_i^h)^* = -n_{ij}$. The second claim then follows from the expression $[T_{\mathcal{E}_i^h}(\mathcal{E}_j^h)] = [\mathcal{E}_j^h] - \langle \mathcal{E}_i^h, \mathcal{E}_j^h \rangle [\mathcal{E}_i^h]$.

□

Remark 5.4.7. It can be shown that while the push-forward along the zero-section gives an isomorphism $K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \simeq K_0(\mathcal{D})$, the push-forward map $\sigma_* : K_0(D^b_{\widetilde{G}}(\mathbb{P}^1)) \to K_0(D^b_{\widetilde{G}}(T^*\mathbb{P}^1))$ has kernel consisting of imaginary roots, and so the image can be thought of as a root lattice of finite type.

Under the inverse equivalence $R\Psi_h^{-1}$, the standard t-structure on $D^b(B_h)$ is sent to a non-standard t-structure on $D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$, which we may restrict to $\mathcal{D} \subset D^b_{\widetilde{G}}(T^*\mathbb{P}^1)$. The resulting heart, which we denote $\mathcal{B}_h \subset \mathcal{D}$, is of finite length.

Our final result shows that the spherical twists not only realize the action of the Weyl group on the affine root lattice but also relate the various hearts $\mathcal{B}_h \subset \mathcal{D}$.

Theorem 5.4.8. *If $i \in Q_h$ is a source, then $T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^- h}$. Likewise, if i is a sink, then $T_{\mathcal{E}_i^h}^{-1}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^+ h}$. In particular, since the hearts are finite length and hence determined by their simples, $T_{\mathcal{E}_i^h}(\mathcal{B}_h) = \mathcal{B}_{\sigma_i^- h}$ for i a source and $T_{\mathcal{E}_i^h}^{-1}(\mathcal{B}_h) = \mathcal{B}_{\sigma_i^+ h}$ for i a sink.*

Proof. We prove $T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^- h}$ for i a source. The proof of $T_{\mathcal{E}_i^h}^{-1}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^+ h}$ is similar.

Consider the defining exact triangle $R\mathrm{Hom}_{\mathrm{G}}(\mathcal{E}_i^h, \mathcal{E}_j^h) \otimes \mathcal{E}_i^h \to \mathcal{E}_j^h \to T_{\mathcal{E}_i^h}(\mathcal{E}_j^h)$.

If $i = j$, then $T_{\mathcal{E}_i^h}(\mathcal{E}_i^h) \simeq \mathcal{E}_i^h[-1] \simeq s_* E_i^h[-1] \simeq s_* E_i^{\sigma_i^- h} \simeq \mathcal{E}_i^{\sigma_i^- h}$ with the first isomorphism being a standard property of spherical twists in a 2-CY category, the second isomorphism is from exactness of s_*, the third by Lemma 5.2.7, and the last by definition.

If $i \neq j$ and $i \nrightarrow j$, then $R\mathrm{Hom}_{\mathrm{G}}(\mathcal{E}_i^h, \mathcal{E}_j^h) = 0$ because the $\mathcal{E}_k^h$ form a Γ-configuration, so $T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^h \simeq \mathcal{E}_j^{\sigma_i^- h}$, with the last isomorphism coming from Lemma 5.2.7.

If $i \to j$, then $R\mathrm{Hom}_{\mathrm{G}}(\mathcal{E}_i^h, \mathcal{E}_j^h) \simeq \mathrm{Ext}_G^1(\mathcal{E}_i^h, \mathcal{E}_j^h) \simeq \mathbb{C}$ by the properties of Γ-configurations. Thus the defining exact triangle is of the form $\mathcal{E}_i^h[-1] \to \mathcal{E}_j^h \to T_{\mathcal{E}_i^h}(\mathcal{E}_j^h)$. But by Lemma 5.2.7, $E_j^{\sigma_i^- h} \simeq \mathrm{Cone}(E_i^h[-1] \to E_j^h)$, so indeed $T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{\sigma_i^- h}$.

□

Note that the relation among the hearts $\mathcal{B}_h \subset \mathcal{D}$ by autoequivalences is stronger than the relation among the hearts $\mathcal{A}_h$ by tilting (Proposition 5.4.4). Our final result, which is well-known to experts, shows that the weaker relation of tilting is induced by the spherical twists, thus completing the analogy between the spherical twists and the reflection functors outlined in the table from the Introduction of this chapter.

Proposition 5.4.9. *Let $\mathcal{D}$ be a 2-CY triangulated category with $\mathcal{B} \subset \mathcal{D}$ the heart of a bounded t-structure that is of finite length. Then twists along simple, spherical objects realize tilting at S:*

$$T_S(\mathcal{B}) = L_S\mathcal{B} \quad \textit{and} \quad T_S^{-1}(\mathcal{B}) = R_S\mathcal{B}.$$

Proof. Since bounded t-structures with nested hearts are equal (Lemma 2.10.7), it is enough to check that $T_S(\mathcal{A}) \subseteq L_S(\mathcal{A})$, and since the $T_S(\mathcal{A})$ is finite length, it is enough to check that $T_S(S') \in L_S(\mathcal{A})$ for every simple $S' \in \mathcal{A}$.

When $S = S'$, we know that $T_S(S) = S[-1]$ so that indeed $H^0(T_S(S)) = 0 \in \langle S\rangle^\perp$ and $H^1(T_S(S)) = S \in \langle S\rangle$. Thus $T_S(S) \in L_S\mathcal{A}$.

Otherwise consider the exact triangle

$$R\mathrm{Hom}(S, S') \otimes S \to S' \to T_S(S').$$

By Schur's lemma, $\mathrm{Hom}(S, S') = \mathrm{Hom}(S', S) = 0$, and so by Serre duality $\mathrm{Ext}^2(S, S') = 0$. Then from the long exact sequence in cohomology we see that $H^i(T_S(S')) = 0$ for $i \neq 0$ so that $T_S(S') \simeq H^0(T_S(S'))$. The non-zero part of the long exact sequence is thus

$$0 \to S' \to T_S(S') \to \mathrm{Ext}^1(S, S') \otimes S \to 0.$$

Applying $\mathrm{Hom}(S, -)$ gives

$$0 \to \mathrm{Hom}(S, T_S(S')) \to \mathrm{Ext}^1(S, S') \otimes \mathrm{Hom}(S, S) \;\to \mathrm{Ext}^1(S, S') \to 0.$$

The map on the right being an isomorphism, we have $\mathrm{Hom}(S, T_S(S')) = 0$, whence $T_S(S') \in \langle S\rangle^\perp$. □

Chapter 6

Conclusions

A recurring theme in this thesis has been the construction of ‘classical tilting objects’ $T \in D_G(X)$ in derived categories of equivariant sheaves on a scheme X, which by Theorem 3.1.1 give rise to equivalences $R\mathrm{Hom}_G(T, -) : D_G(X) \to D(\mathrm{End}(T)^{\mathrm{op}})$.

If we consider more carefully the proof of Theorem 3.1.1, we notice that it depended on Keller [36, Theorem 8.5, part b)], which actually gives a much more general means of constructing derived equivalences. Recall that an object T in a triangulated category $\mathcal{D}$ is *compact* if the functor $\mathrm{Hom}(T, -)$ commutes with small sums and is a *generator* of $\mathcal{D}$ if $\mathrm{Hom}(T, \mathcal{F}) = 0$ implies $\mathcal{F} = 0$ in $\mathcal{D}$. Keller in fact proves that given any compact generator T in a nice triangulated category $\mathcal{D}$, there is a differential-graded algebra $R\mathrm{Hom}(T, T)$ and an equivalence $R\mathrm{Hom}(T, -) : \mathcal{D} \to D(R\mathrm{Hom}(T, T))$ with the derived category of $R\mathrm{Hom}(T, T)$. The ‘classical’ in classical tilting object requires that not only should T be a compact generator, but that also all of the higher Ext groups of T should vanish and so the differential-graded algebra $R\mathrm{Hom}(T, T)$ can be replaced with the classical algebra $\mathrm{Hom}(T, T)$.

While in all of the examples that we considered, our compact generators did satisfy this Ext-vanishing, Keller's theorem shows that it is quite natural to allow oneself the freedom of working with differential-graded algebras. Furthermore, Bondal and Van den Bergh [13, Theorem 3.1.1] have shown that for any quasi-compact, quasi-separated scheme X, there exists a compact generator $T \in D(X)$ and hence an equivalence $D(X) \simeq D(R\mathrm{Hom}(T,T))$. On the other hand, it is very rare indeed to have Ext-vanishing for this compact generator T. If for instance X were smooth and projective over a field, then a classical tilting object T would give an equivalence $D^b(X) \simeq \operatorname{perf} A$ with A a finite dimensional algebra, and hence an isomorphism of Grothendieck groups $K_0(X) \simeq K_0(A)$. But when A is finite dimensional, the latter group is free of finite rank, which is hardly ever true for a smooth projective variety. In particular, for a smooth projective curve C, $K_0(C)$ is finitely generated if and only if $C \simeq \mathbb{P}^1$. To apply tilting theory to a general variety it is thus not only natural but also absolutely necessary to consider differential-graded algebras.

Another important theme of this thesis has been the study of derived symmetries of a variety such as spherical twists and $\mathbb{P}$-twists, both of which are constructed as integral transforms. Recall that given an object $\mathcal{P}$ in the derived category of a product $X \times Y$, the integral transforms with kernel $\mathcal{P}$ are the functors $\Phi^{\rightarrow}_{\mathcal{P}} := R\pi_{Y*}(\mathcal{P} \otimes L\pi_X^*(-))$ from $D(X)$ to $D(Y)$ and $\Phi^{\leftarrow}_{\mathcal{P}} := R\pi_{X*}(\mathcal{P} \otimes L\pi_Y^*(-))$ from $D(Y)$ to $D(X)$.

There is a functor

$$D(X \times Y) \to \mathrm{ExFunk}(D(X), D(Y)) \tag{6.0.3}$$

from the derived category of the product $X \times Y$ to the category whose objects are triangle functors and whose morphisms are natural transformations. The image of this functor includes any geometrically relevant functor. In particular, Orlov has shown that any exact autoequivalence of $D(X)$ for X smooth projective is naturally isomorphic to an integral transform for a unique kernel. For a proof and discussion of this result, I recommend [32, Theorem 5.14].

The relation between the derived category of the product and the category of triangle functors is however imperfect. In general the functor 6.0.3 is neither full nor faithful and we cannot rule out strange triangle functors with no kernel. Even worse, the category $\mathrm{ExFunk}(D(X), D(Y))$ has no reasonable structure.

Toën [51] and Ben-Zvi-Francis-Nadler [6] have shown however that if we replace $D(X \times Y)$ with a suitable ∞-enhancement, then the analogue of the functor (6.0.3) is an equivalence of ∞-categories.

Keller's theorem together with the results of Toën [51] and Ben-Zvi-Francis-Nadler have convinced me to begin working with differential-graded algebras and ∞-enhancements of derived categories as soon as possible. In particular, I plan to study the derived symmetries of a variety X by using compact generators to construct kernels of autoequivalences in ∞-enhancements of $D(X \times X)$.

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