

## Erratum: Phase-shift cavity ring-down spectroscopy on a microsphere resonator by Rayleigh backscattering [Phys. Rev. A **87**, 053843 (2013)]

Jack A. Barnes, Gianluca Gagliardi, and Hans-Peter Loock\*

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In a recent article, we presented several theoretical models and experimental results that demonstrate phase-shift cavity ring-down experiments on a microsphere resonator. We attempted to extract the cavity ring-down time and, thereby, the optical loss from the phase shift experienced by light that is Rayleigh backscattered into counterpropagating whispering gallery modes (WGMs). At the end of the original article, we concluded that none of the three models we presented provided a fully satisfactory agreement with the experimental data and that an improved model may be required. With the present Erratum, we provide this improvement.

Phase-shift measurements have been used extensively to determine the ring-down time of optical resonators. This technique has been demonstrated for conventional mirror cavities [1,2] as well as for fiber loops [3–5] and microresonators [6]. In our previous contribution, frequency-dependent phase-shift measurements were performed on Rayleigh backscattered light generated inside a silica microsphere resonator. Large phase shifts were observed that varied approximately linearly with modulation frequency. We attempted to model the phase shift using the coupled oscillator equations given in Ref. [7], which were based on an earlier article [8]. At the time, we did not appreciate that the particular version of the equations given in these references implied that the solutions needed to be multiplied by a factor of  $\exp(i\omega t)$ , corresponding to the frequency of the source term. Our neglect of this factor led to an erroneous result. Consequently, a provisional model, based on an optical delay line coupled to an optical resonator, was proposed. This model gave a good approximation to an experimental observation, but no rigorous theoretical justification could be provided. Here, a satisfactory model, based on the coupled damped harmonic-oscillator equations, is derived. The model agrees quantitatively with our previous and with new phase-shift measurements of Rayleigh backscattered light.

In a microsphere resonator, in the absence of modal coupling, forward and backward traveling modes are degenerate due to the  $\pm m$  degeneracy of the cavity eigenfrequencies. The initially degenerate modes have frequency  $\omega_0$  and a photon cavity lifetime of  $\tau_L$ . As was described before, the lifetime captures losses due to absorption and scattering  $\tau_0$  as well as those due to coupling to a delivery waveguide  $\tau_{\text{ex}}$  where  $\tau_L^{-1} = \tau_0^{-1} + \tau_{\text{ex}}^{-1}$ .

The presence of Rayleigh scatterers, such as refractive index inhomogeneities or surface impurities, can lead to a coupling of forward and backward propagating modes. In a perturbation

approach, this can be modeled using [9]

$$\begin{aligned} \frac{da_f(t)}{dt} &= \left( j\omega_0 - \frac{1}{2\tau_L} \right) a_f(t) + \frac{j}{2\gamma} a_b(t) + \kappa s(t), \\ \frac{da_b(t)}{dt} &= \left( j\omega_0 - \frac{1}{2\tau_L} \right) a_b(t) + \frac{j}{2\gamma} a_f(t). \end{aligned} \quad (1)$$

Here  $a_f(t)$  and  $a_b(t)$  represent the time-variant field amplitudes of the forward and backward propagating modes, respectively, and  $j = \sqrt{-1}$ . The respective equation in the original paper was adopted from Refs. [7,8], but in these previous equations, the fast oscillation of the optical field was removed, and their field amplitude was written as  $a'_i = a_i \exp(-i\omega t)$ . This caused no problems in Refs. [7,8] but produced an incorrect result in the original paper.

The Rayleigh scattering lifetime, given by  $\gamma$ , describes the modal coupling. The amplitude of the input field is  $s(t)$ , and the coupling constant between the input field and the microsphere is  $\kappa$ , which is an imaginary number. Modal coupling causes a splitting of the initially degenerate modes. Using Eq. (1), the two new eigenfrequencies can be calculated as  $\omega_0 \pm 1/2\gamma$ , and the width of the eigenmodes equals  $\tau_L^{-1}$ . After some algebra, these two differential equations can be combined into a single differential equation describing the backscattered field  $a_b(t)$  in terms of the input field  $s(t)$ ,

$$\begin{aligned} \frac{d^2 a_b(t)}{dt^2} - 2 \left( j\omega_0 - \frac{1}{2\tau_L} \right) \frac{da_b(t)}{dt} \\ + \left[ \left( j\omega_0 - \frac{1}{2\tau_L} \right)^2 + \frac{1}{4\gamma^2} \right] a_b(t) = \frac{j\kappa}{2\gamma} s(t). \end{aligned} \quad (2)$$

Solutions to Eq. (2) are rather complicated algebraic functions that depend on the input function  $s(t)$ . An example is given in Fig. 1 for a Gaussian input function with a temporal width of 1 ns. The function,  $a_b(t)$ , was obtained using MAPLE software.

We wish to calculate the phase shift of  $a_b(t)$  with respect to a sinusoidally modulated input field  $s(t)$  with modulation frequency  $\omega$ . This may conveniently be performed using Laplace transform methods.

Taking the Laplace transforms of (2) gives, for the output function,  $\mathcal{L}\{a_b(t)\} = A_b(k)$ , and for the input function  $\mathcal{L}\{s(t)\} = S(k)$  where  $k = j\omega$ . With  $\mathcal{L}\{a'_b(t)\} = kA_b(k) - a_b(0)$  and  $\mathcal{L}\{a''_b(t)\} = k^2 A_b(k) - ka_b(0) - a'_b(0)$ , we rewrite Eq. (2) as

$$\begin{aligned} \left[ k^2 - k \left( 2j\omega_0 - \frac{1}{\tau_L} \right) + \left( j\omega_0 - \frac{1}{2\tau_L} \right)^2 + \frac{1}{4\gamma^2} \right] A_b(k) \\ = \frac{j\kappa}{2\gamma} S(k). \end{aligned} \quad (3)$$

\*Author to whom correspondence should be addressed: [hploock@chem.queensu.ca](mailto:hploock@chem.queensu.ca)

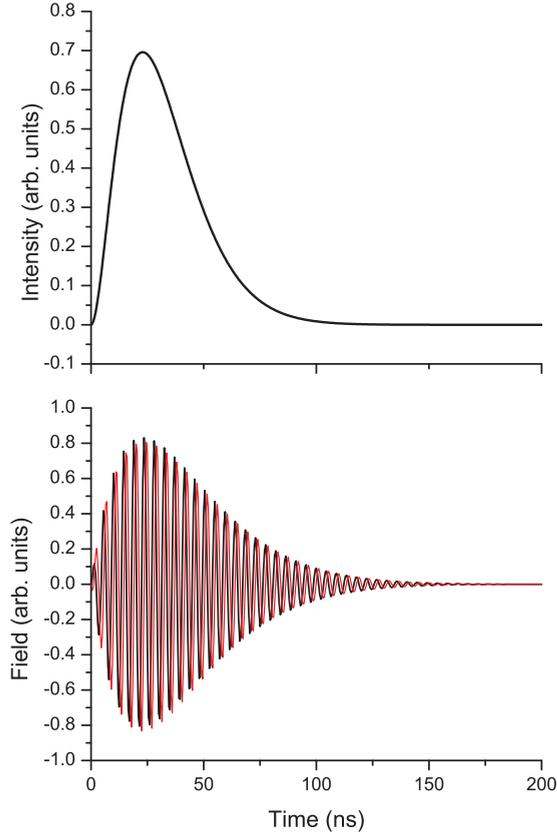


FIG. 1. (Color online) Top: Rayleigh backscattered intensity  $|a_b(t)|^2$  and bottom: field  $a_b(t)$  following a Gaussian input  $s(t) = \exp[-(t/1 \text{ ns})^2]$  with parameters  $\tau_L = 12$ ,  $\gamma = 31$  ns. We selected  $\omega = 1.4 \times 10^9$  rad/s to make the oscillations of the real part (thick black line) and imaginary part (thin red line) of the field apparent. The solution to Eq. (2) was obtained using MAPLE software with the boundary conditions  $\text{Re}\{a_b(0)\} = 0$  and  $\text{Im}\{a_b(0)\} = 0$ .

The ratio of the Laplace transform of the output and input functions defines the transfer function,

$$G(\omega) = \frac{A_b(j\omega)}{S(j\omega)} = \frac{j\kappa/2\gamma}{-(\omega - \omega_0)^2 + \frac{j}{\tau_L}(\omega - \omega_0) + \frac{1}{4}\left(\frac{1}{\tau_L^2} + \frac{1}{\gamma^2}\right)}. \quad (4)$$

The phase  $\phi$  of this transfer function equals the phase shift of the output, relative to a sinusoidally modulated input where

$$\tan(\phi) = \frac{\text{Im}[G(\omega)]}{\text{Re}[G(\omega)]} = \frac{\frac{1}{\tau_L}(\omega - \omega_0)}{(\omega - \omega_0)^2 - \frac{1}{4}\left(\frac{1}{\tau_L^2} + \frac{1}{\gamma^2}\right)}. \quad (5)$$

A similar derivation already was presented to calculate the phase shift of light that is transmitted through a

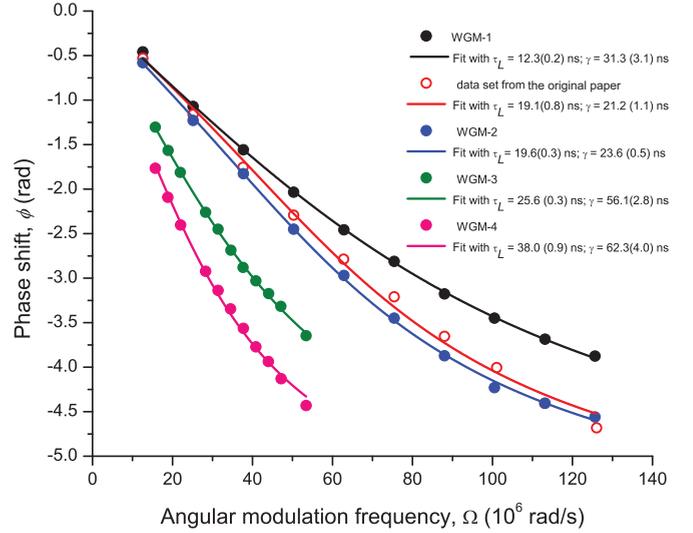


FIG. 2. (Color online) Measured phase shift of the Rayleigh backscattered light as a function of the angular modulation frequency (symbols). The solid lines are fits to  $\tau_L$  and  $\gamma$  of Eq. (6).

fiber taper which is in contact with a resonantly coupled cavity [6].

Experimentally, the microsphere is excited with an *intensity* modulated input at a frequency of  $2\Omega$ . Please see the original paper for experimental details. Modulation at megahertz frequencies gives rise to frequency sidebands in the electric field at  $\omega \pm \Omega$ . It has been shown that the phase shift in the intensity modulation envelope is equal to  $\Delta\phi = \phi_+ - \phi_-$  [6], where  $\phi_{\pm}$  is the phase shift at the sideband frequencies  $\omega_+$  and  $\omega_-$ , calculated using Eq. (5). This leads to

$$\Delta\phi = \phi_+ - \phi_- = 2 \tan^{-1} \left[ \frac{\Omega/\tau}{\Omega^2 - \frac{1}{4}\left(\frac{1}{\tau_L^2} + \frac{1}{\gamma^2}\right)} \right], \quad (6)$$

which is the desired result for excitation at the optical resonance frequency  $\omega_0$ . In Fig. 2, we show an experimentally measured Rayleigh backscattered phase shift for five whispering gallery modes along with fits of  $\tau_L$  and  $\gamma$  to Eq. (6). One of the data sets was taken from the original paper, whereas the other four data sets were obtained in the same way. The fit of Eq. (6) to the experimental data is excellent. A lack of observed splitting in the spectrum of the resonance is consistent with  $\gamma/\tau_L > 1$  obtained from the fit. The limiting value of Eq. (6) for large values of  $\Omega$  is  $\Delta\phi = -2\pi$  and is consistent with experiments.

In conclusion, we propose that Eq. (6), which was derived using a coupled oscillator model for the Rayleigh backscattering problem, accurately describes the dependence of the modulation phase shift on the modulation frequency.

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