Average Controllability of Random Heat Equations

by

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Abstract

In this thesis, we introduce random differential equations in an abstract framework and study their well-posedness. We study average controllability properties of a random heat equation when the diffusivity is a random variable. We show that the solutions of such random heat equations are both null and approximately controllable in average from an arbitrary open set of the domain and in an arbitrarily small time, recovering the known result when the random diffusivity is uniformly or exponentially distributed.

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Chapter 1

Introduction

A control system is a dynamical system, modelled by ordinary or partial differential equations (ODEs or PDEs), that one can act on using controls. The ability to steer the system from any given state to any other state is a desirable property of control systems. If this property holds, the system is called *controllable*. This type of problem is known as the *controllability problem*. When studying such control problems, it is useful to distinguish between finite-dimensional systems, modelled by ODEs, and infinite-dimensional systems, modelled by PDEs. The importance behind this distinction lies in the differences between the behaviour of finite and infinite-dimensional systems. For example, by the well-known Kalman rank condition, if a finite-dimensional system is controllable during some time, then it is controllable for all time (see e.g. [2]). This, however, is not true for an infinite-dimensional system modelled by the transport equation. In this case, the system is not controllable unless enough time is provided to allow for the effect of the control to propagate throughout the entire domain (see e.g. [2]). These drastic differences have lead to a separation in the studies of ODEs and PDEs. The focus of this thesis will be on control systems modelled by PDEs. This focus is motivated by the various physical phenomena that

PDEs are used to model. Examples include heat conduction, wave propagation, fluid dynamics, electromagnetism, and quantum mechanics [5].

In particular, this thesis will focus on what are known as parameter dependent control systems. A parameter dependent control system is a system whose dynamics are governed by parameter dependent operators. Each unique parameter value corresponds to a specific realization of the system. The usefulness of such systems becomes clear when considering the problem of modelling physical processes. Due to the uncertainties and complexities involved, it is difficult to perfectly model physical processes; thus it becomes natural to model them using parameter dependent coefficients. In particular, equations whose parameters are random can be used to model many uncertain physical processes [15]. An example of such a process is heat diffusion through an inhomogeneous material. Generally speaking, in order to control such systems one must use controls dependent on the parameter (see e.g. [12], [11], [6] and references therein). However, in the cases where the value of the parameter is unknown, it is not always possible to control every realization of the system using a control independent of the parameter (c.f. Remark 2.3.4); one can instead make a robust compromise to controlling every realization of the system by controlling instead the average of the state with respect to the unknown parameter. This problem was first introduced in [16]. There, the problem was formulated and solved in the setting of finite-dimensional systems. In [10] the problem of averaged controllability was studied in the context of PDEs. There, the authors focused on heat and Schrödinger equations with random parameters. Due to the recency of the averaged control problem, there are many open problems. The aim of this thesis is not to address all of these problems, but to explore how we can apply existing results on control of PDEs to the averaged control problem. Our focus will be on averaged controllability of the heat equation where the diffusivity coefficient is unknown, i.e., the diffusivity is a random variable where only its probability density function is known. The treatment of this problem will follow the treatment presented in [10].

1.1 Contribution of Thesis

The contribution of this thesis is twofold: first, we extend the result of [10] to show both null and approximate controllability in average for a random heat equation when the diffusivity is a random variable with a general probability distribution. Secondly, we characterize the necessity of a non-zero diffusivity for achieving average control properties.

1.2 Organization of Thesis

The rest of this thesis is organized as follows: in Chapter 2 we will introduce some notation. We introduce abstract parameter dependent control systems and discuss their well-posedness. We also define what it means for such systems to be exactly, null, and approximately controllable/observable in average. In Chapter 3 we state the main problem addressed in this thesis. In Chapter 4 we present a proof for the main result. In Chapter 5 we summarize the main result, and indicate interesting directions for future work. Finally, in the Appendix, Chapter 6, we include several classical results that are used throughout the thesis.

Chapter 2

Mathematical Preliminaries

2.1 Basic Notation

Throughout this thesis, we denote the set of real numbers by $\mathbb{R}$, non-negative real numbers by $\mathbb{R}_{\geq 0}$, positive real numbers by $\mathbb{R}_{>0}$, non-negative integers by $\mathbb{Z}_{\geq 0}$, positive integers by $\mathbb{Z}_{>0}$, and complex numbers by $\mathbb{C}$. Given numbers $a, b \in \mathbb{R}$, we denote by $[a, b)$ an interval inclusive of a and non-inclusive of b. We denote the real part of a number $a \in \mathbb{C}$ by $\mathrm{Re}(a)$. We will denote a norm on a vector space V by $\| \cdot \|_V$, and an inner product on a vector space H by $\langle \cdot, \cdot \rangle_H$. We will denote the space of linear operators from a vector space U to a vector space V by $\mathcal{L}(U, V)$ and from V to itself by $\mathcal{L}(V)$. We will denote by $C(U; V)$ the space of continuous functions from topological space U to topological space V and $C^k(U; V)$ the space of k-times continuously differentiable functions from U to V. We denote by $\chi_E(\cdot)$ the indicator function on E. We will often denote the partial derivative of multi-variable function $y\colon (x, t) \mapsto y(x, t)$ with respect to its second variable by $y_t(x, t)$.

2.2 Well-Posedness of Abstract Cauchy Problem

In this section we introduce an abstract Cauchy problem that can be used to model PDEs. After studying the well-posedness of this equation, we will further generalize to abstract parameter dependent control systems and discuss the averaged control problem.

Let $T > 0$, let V and U be separable Hilbert spaces. Let $A\colon D(A) \subset V \to V$, where $D(A)$ is the domain of A. Let $B \in \mathcal{L}(U, V)$, and consider the following abstract Cauchy problem:

$$\begin{cases} y_t(t) = Ay(t) + Bu(t), & t \in (0, T], \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $y_0 \in V$ and $u(t) \in U$. Our goal in this section is to provide conditions under which a solution to (2.1) exists and is unique. Before defining what it means to be a solution, we first present a formal example that will motivate the need for so-called semigroup theory.

Example 2.2.1. Consider a one-dimensional bar of length 1 that is heated along its length according to the following equation:

$$\begin{cases} y_t(x,t) = y_{xx}(x,t) + u(x,t), & (x,t) \in (0,1) \times (0,T], \\ y(x,0) = y_0(x), & x \in (0,1), \\ y_x(0,t) = y_x(1,t) = 0, & t \in [0,T], \end{cases} \tag{2.2}$$

where $y(x, t)$ is the temperature of the bar at position x and time t, y_0 is the initial temperature profile, and $u(x, t)$ is the addition of heat along the bar. We can repose this problem in the sense of an abstract Cauchy problem. To do this, we choose state space $V = L^2(0,1)$ where the state at time t is $y(\cdot,t)$. Choose $U = L^2(0,1)$, $B = I$ and

$$\begin{cases} Ah = h_{xx} \quad \text{with} \\ D(A) = \{h \in L^2(0,1) : h, h_x \text{ are absolutely continuous}, \\ \qquad\qquad h_{xx} \in L^2(0,1), h_x(0) = h_x(1) = 0\}. \end{cases}$$

It is well-known (see e.g. [3]) that for sufficiently smooth functions y_0 and u, the solution to (2.2) is given by

$$y(x,t) = \int_0^t g(t,x,y)y_0(y)dy + \int_0^t \int_0^1 g(t-\tau,x,y)u(y,s)dyd\tau,$$

where $g(t,x,y) = 1 + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\cos(n\pi x)\cos(n\pi y)$. To formulate this solution in an abstract sense, define for each $t \in [0,T]$ the operator $S(t) \in \mathcal{L}(L^2(0,1))$ by

$$S(t)y_0(x) = \int_0^1 g(t,x,y)y_0(y)dy.$$

Then, the solution becomes

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)u(\tau)d\tau.$$

This is analogous to the variation of constants formula for finite-dimensional systems.

•

This example motivates the following definition:

Definition 2.2.2. A *strongly continuous semigroup (C^0-semigroup)* is an operator-valued function, $S\colon \mathbb{R}_{\geq 0} \to \mathcal{L}(V)$, satisfying

$$S(t+s) = S(t)S(s) \quad \text{for} \quad s,t \geq 0,$$

$$S(0) = I,$$

$$\|S(t)y_0 - y_0\|_V \to 0 \quad \text{as} \quad t \to 0^+, \quad \forall\, y_0 \in V.$$

Note that $\|S(t)\|_{\mathcal{L}(V)}$ is bounded on every finite subinterval of $[0,\infty)$ (see e.g. [3, Theorem 2.1.6]). One can associate with each C^0-semigroup, a linear operator.

Definition 2.2.3. The *infinitesimal generator* A of a C^0-semigroup on a Hilbert space V is defined by

$$Ay = \lim_{t\to 0^+} \frac{1}{t}(S(t) - I)y,$$

whenever the limit exists. The domain of A, $D(A)$, is the subset of V for which the limit exists.

Next we introduce the adjoint operator of A.

Definition 2.2.4. Let A be a linear operator on a Hilbert space V that is densely defined, i.e., $D(A)$ is dense in V. The *adjoint operator* of A is the linear operator $A^*\colon D(A^*) \subset V \to V$, where $D(A^*)$ consists of all $y \in V$ such that there exists $y^* \in V$ satisfying

$$\langle Ax, y\rangle_H = \langle x, y^*\rangle_H, \quad \forall\, x \in D(A).$$

For each $y \in D(A^*)$, we define A^* as

$$A^* y = y^*.$$

Definition 2.2.5. We say that a densely defined linear operator A is *symmetric* if for all $x, y \in D(A)$

$$\langle Ax, y\rangle_H = \langle x, Ay\rangle_H.$$

A symmetric operator is *self-adjoint* if $D(A^*) = D(A)$.

We have the following semigroup result concerning the adjoint operator A^* of A:

Theorem 2.2.6. ([3, Theorem 2.2.6]): *If $S(t)$ is a C^0-semigroup with infinitesimal generator A on a Hilbert space V, then $S^*(t)$ is a C^0-semigroup with infinitesimal generator A^* on V.*

Conversely, A is not always the generator of a C^0-semigroup. We will give a sufficient condition for when A is the infinitesimal generator of a C^0-semigroup.

Theorem 2.2.7. ([3, Corollary 2.2.3]): *Let A be a densely defined closed operator, i.e., $D(A)$ is dense in V and the graph of A is a closed subset of $V \times V$. If there exists $\beta \in \mathbb{R}$ such that*

$$\mathrm{Re}\left(\langle Av, v\rangle_V\right) \leq \beta\|v\|_V, \quad \forall\, v \in D(A)$$
$$\mathrm{Re}\left(\langle A^*v, v\rangle_V\right) \leq \beta\|v\|_V, \quad \forall\, v \in D(A^*)$$

then A is the infinitesimal generator of a C^0-semigroup on V with $\|S(t)\|_{\mathcal{L}(V)} \leq e^{\beta t}$.

Throughout the rest of this section, we will assume that $A\colon D(A) \subset V \to V$ is the infinitesimal generator of a C^0-semigroup $S(t)$. We are now in position to define what it means to be a solution to (2.1).

Definition 2.2.8. The function $y(t)$ is a *classical solution* to (2.1) on $[0, T]$ if $y \in C^1([0, T]; V)$, $y(t) \in D(A)$, for all $t \in [0, T]$ and $y(t)$ satisfies (2.1) for all $t \in [0, T]$.

Lemma 2.2.9. ([3, Lemma 3.1.2]): *Assume that $Bu(\cdot) \in C([0,T];V)$ and that $y(t)$ is a classical solution of* (2.1) *on $[0,T]$. Then $Ay(\cdot) \in C([0,T];V)$ and*

$$y(t) = S(t)y_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau. \tag{2.3}$$

It is important to note that (2.3) does not always give a classical solution to (2.1). However, under certain conditions a classical solution can be obtained.

Theorem 2.2.10. ([3, Theorem 3.1.3]): *If A is the infinitesimal generator of a C^0-semigroup $S(t)$ on a Hilbert space V, $Bu(\cdot) \in C^1([0,T];V)$ and $y_0 \in D(A)$, then* (2.3) *is continuously differentiable and it is the unique classical solution of* (2.1)*.*

The condition that $Bu(\cdot) \in C^1([0,T];V)$ is generally too strong for control purposes. Instead, we wish to choose u such that $Bu(\cdot) \in L^p([0,T];V)$ for some p. This relaxation allows us to choose controls from a larger class of functions, thereby increasing the likelihood of finding a control that steers the system to the desired end state. Thus, we must introduce a weaker notion of a solution to (2.1).

Definition 2.2.11. If $Bu(\cdot) \in L^p([0,T];V)$ for some $p \geq 1$ and $y_0 \in V$ then we call (2.3) a *mild solution* of (2.1).

Note that in this context, the integral in (2.3) is well-defined since $\|S(t)\|_{\mathcal{L}(V)}$ is bounded on $[0,t]$ and if $Bu(\cdot) \in L^p([0,T];V)$ then necessarily $Bu(\cdot) \in L^1([0,T];V)$. It turns out that the mild solution coincides with the notion of weak solution used in the study of PDEs.

Definition 2.2.12. Let $Bu(\cdot) \in L^p([0,T];V)$ for some $p \geq 1$. We call $y(t)$ a *weak* *solution* of (2.1) on $[0,T]$ if $y \in C([0,T];V)$ and for all $g \in C([0,T];V)$

$$\int_0^T \langle y(t), g(t)\rangle_V \, dt - \int_0^T \left\langle Bu(t), \int_t^T S^*(\tau - t) g(\tau) d\tau \right\rangle_V dt$$
$$- \left\langle y_0, \int_0^T S^*(\tau) g(\tau) d\tau \right\rangle_V = 0.$$

Theorem 2.2.13. ([3, Theorem 3.1.7]): *Let $y_0 \in V$ and $Bu(\cdot) \in L^p([0,T];V)$ for some $p \geq 1$. Then there exists a unique weak solution of* (2.1) *that is the mild solution given by* (2.3)*.*

2.3 Parameter Dependent Control Systems

In this section we introduce a more general class of Cauchy problem with parameter dependence and describe the notions of averaged controllability and observability.

Let $T > 0$, $E \subset [0,T]$ a Lebesgue measurable set with positive Lebesgue measure. Let H and U be separable Hilbert Spaces. Let $V \subset H$ be a Hilbert space dense in H. Let V' denote the dual space of V with respect to the pivot space H, i.e.,

$$V \subset H \subset V'.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{A(\omega)\}_{\omega \in \Omega}$ be a family of operators under the following three assumptions:

(A1) $A(\omega) \in \mathcal{L}(D(A(\omega)), H)$, a.e. $\omega \in \Omega$;

(A2) $A(\omega)\colon D(A(\omega)) \to H$ generates a C^0-semigroup $\{S(t,\omega)\}_{t\geq 0}$ on H and V, a.e. $\omega \in \Omega$;

(A3) $S(t,\cdot)y \in L^1(\Omega;V)$, for all $y \in V$ and $t \in [0,T]$.

Let $B \in L^2(\Omega; \mathcal{L}(U,V))$, and consider the following control system:

$$\begin{cases} y_t(t) = A(\omega)y(t) + \chi_E(t)B(\omega)u(t) \quad \text{in } (0,T], \\ y(0) = y_0, \end{cases} \tag{2.4}$$

where $y_0 \in V$ and $u \in L^2(E;U)$ is the control. Note that the state of the system at time t, denoted by $y(t,\omega;y_0)$, depends on ω nonlinearly. According to Theorem 2.2.13, for a.e. $\omega \in \Omega$, there exists a *weak solution* $y(\cdot,\omega;y_0) \in C([0,T];V)$. Moreover, the averaged state $\int_\Omega y(\cdot,\omega;y_0)d\mathbb{P}(\omega)$ is in $C([0,T];V)$. Indeed, this follows from the fact that for a.e. $\omega \in \Omega$, $y(\cdot,\omega;y_0) \in C([0,T];V)$ and

$$\int_\Omega y(t,\omega;y_0)d\mathbb{P}(\omega) = \int_\Omega S(t,\omega)y_0 d\mathbb{P}(\omega) + \int_\Omega \int_0^t S(t-\tau,\omega)B(\omega)u(\tau)d\tau d\mathbb{P}(\omega).$$

Note that the integrals above make sense by assumption (A3).

2.3.1 Average Controllability and Observability

We now introduce the following three notions of average controllability:

Definition 2.3.1. System (2.4) is *exactly controllable in average in E with cost C* if for all $y_0, y_1 \in V$, there exists $u \in L^2(E;U)$ such that

$$\|u\|_{L^2(E;U)} \leq C(\|y_0\|_V + \|y_1\|_V)$$

and the average of the solution to (2.4) satisfies

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = y_1.$$

Definition 2.3.2. System (2.4) is *null controllable in average in E with cost C* if for all $y_0 \in V$, there exists $u \in L^2(E;U)$ such that

$$\|u\|_{L^2(E;U)} \leq C\|y_0\|_V$$

and the average of the solution to (2.4) satisfies

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = 0.$$

Definition 2.3.3. System (2.4) is *approximately controllable in average in E* if for all $y_0, y_1 \in V$, for all $\epsilon > 0$, there exists $u_\epsilon \in L^2(E;U)$ such that the average of the solution to (2.4) satisfies

$$\left\| \int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) - y_1 \right\|_V < \epsilon.$$

Remark 2.3.4. A different notion of controllability is simultaneous controllability. For an introduction to this notion see [7]. For simultaneous controllability, we are concerned with choosing a controller, independent of the random parameter, that makes every realization of the system controllable. Clearly this notion is much stronger than controllability in average. However, as one can imagine, there are many systems where simultaneous controllability is impossible to achieve but that are controllable in average; a simple example is provided in [10] which we present here. Consider the following linear, finite-dimensional system:

$$\begin{cases} y_t(t) = Ay(t) + B(\omega)u(t) \quad \text{in } (0,T], \\ y(0) = y_0 \in \mathbb{R}^2, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B(\omega) \in \{B, 2B\} \quad \text{for} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By Theorem 1 in [16], we have that the above system is null controllable in average. However, this system is not null simultaneous controllable. Otherwise, there would exist $u \in L^2(0,T)$ such that

$$e^{AT}y_0 + \int_0^T e^{A(T-t)}Bu(t)dt = e^{AT}y_0 + 2\int_0^T e^{A(T-t)}Bu(t)dt = 0.$$

But this implies that $y_0 = 0$. •

Next, we introduce the adjoint system to (2.4):

$$\begin{cases} -z_t(t) = A^*(\omega)z(t) \quad \text{in } [0,T), \\ z(T) = z_0, \end{cases} \tag{2.5}$$

where $z_0 \in V'$. We will denote the solution at time t to (2.5) by $z(t,\omega;z_0)$.

Similar to average controllability, we introduce three notions of average observability:

Definition 2.3.5. System (2.5) is *exactly observable in average in E* if there exists

$C > 0$ such that for all $z_0 \in V'$,

$$\|z_0\|_{V'}^2 \leq C \int_0^T \chi_E(t) \left\| \int_\Omega B^*(\omega) z(t, \omega; z_0) d\mathbb{P}(\omega) \right\|_U^2 dt.$$

Definition 2.3.6. System (2.5) is *null observable in average in* E if there exists $C > 0$ such that for all $z_0 \in V'$,

$$\left\| \int_\Omega z(0, \omega; z_0) d\mathbb{P}(\omega) \right\|_{V'}^2 \leq C \int_0^T \chi_E(t) \left\| \int_\Omega B^*(\omega) z(t, \omega; z_0) d\mathbb{P}(\omega) \right\|_U^2 dt. \tag{2.6}$$

Definition 2.3.7. System (2.5) is said to satisfy the *averaged unique continuation property in* E if

$$\chi_E(t) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) = 0 \quad \in L^2([0, T]; U)$$

implies $z_0 = 0$.

Note that there is no natural evolution equation to describe the dynamics of $\int_\Omega y(\cdot, \omega; y_0) d\mathbb{P}(\omega)$, the average of the solution to (2.4). Similarly, there is no natural evolution equation to describe the behaviour of $\int_\Omega z(\cdot, \omega; z_0) d\mathbb{P}(\omega)$, the average of the solution to (2.5). Because of this, we cannot directly employ existing results or techniques to prove average controllability results. However, one can still prove the classical duality between controllability and observability in the averaged sense.

Theorem 2.3.8. ([10, Theorem A.1]): *System* (2.4) *is* exactly controllable in average in E *if and only if system* (2.5) *is* exactly observable in average in E*.*

Theorem 2.3.9. ([10, Theorem A.2]): *System* (2.4) *is* null controllable in average in E *if and only if system* (2.5) *is* null observable in average in E*.*

Theorem 2.3.10. ([10, Theorem A.3]): *System* (2.4) *is* approximately controllable in average in E *if and only if system* (2.5) *satisfies the* averaged unique continuation property in E.

We prove Theorem 2.3.9 and Theorem 2.3.10. The proof for Theorem 2.3.8 is very similar to that of Theorem 2.3.9.

Proof of Theorem 2.3.9. ($\Leftarrow$): Define a linear subspace $\mathcal{X} \subset L^2(E;U)$ as

$$\mathcal{X} = \left\{ \chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) : z_0 \in V' \right\},$$

and a linear functional F on $\mathcal{X}$ as

$$F\left(\chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) \right) = -\left\langle y_0, \int_\Omega z(0, \omega; z_0) d\mathbb{P}(\omega) \right\rangle_{V,V'}.$$

We have by (2.6),

$$\begin{aligned}
&F\left(\chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) \right) \\
&\leq \|y_0\|_V \left\| \int_\Omega z(0, \omega; z_0) d\mathbb{P}(\omega) \right\|_{V'} \\
&\leq \sqrt{C} \|y_0\|_V \left\| \chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) \right\|_{L^2(E;U)}.
\end{aligned}$$

Hence, F is a bounded linear functional on $\mathcal{X}$ with norm,

$$\|F\|_{\mathcal{L}(\mathcal{X},\mathbb{R})} \leq \sqrt{C} \|y_0\|_V.$$

By Hahn–Banach theorem, F can be extended to a bounded linear functional on $L^2(E;U)$ with the same norm. With abuse of notation, we will denote this extension also by F. By Riesz representation theorem, there exists $u \in L^2(E;U)$ such that for all $v \in L^2(E;U)$,

$$F(v) = \langle v, u \rangle_{L^2(E;U)},$$

and

$$\|u\|_{L^2(E;U)} = \|F\|_{\mathcal{L}(\mathcal{X},\mathbb{R})} \leq \sqrt{C}\|y_0\|_V.$$

By definition of F,

$$\begin{aligned} -\left\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\right\rangle_{V,V'} &= \left\langle u, \chi_E(\cdot)\int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)\right\rangle_{L^2(E;U)} \\ &= \int_E \left\langle u, \int_\Omega B^*(\omega)z(\cdot,\omega;z_0)d\mathbb{P}(\omega)\right\rangle_U dt. \end{aligned}$$

Note that for a strongly measurable function $f\colon \Omega \to U$, we can write

$$\begin{aligned} \left\langle u, \int_\Omega f(\omega)d\mathbb{P}(\omega)\right\rangle_U &= \left\langle u, \sum_{j=1}^{\infty} f_j\mu(A_j)\right\rangle_U \\ &= \left\langle u, \lim_{N\to\infty}\sum_{j=1}^{n} f_j\mu(A_j)\right\rangle_U \\ &= \lim_{N\to\infty}\left\langle u, \sum_{j=1}^{n} f_j\mu(A_j)\right\rangle_U \\ &= \sum_{j=1}^{\infty} \langle u, f_j\rangle_U\, \mu(A_j) \\ &= \int_\Omega \langle u, f(\omega)\rangle_U d\mathbb{P}(\omega), \end{aligned}$$

where $f_j \in U$ with support on A_j, for all $j \in \mathbb{Z}_{>0}$. Hence,

$$-\left\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\right\rangle_{V,V'} = \int_E \int_\Omega \langle u, B^*(\omega)z(t,\omega;z_0)\rangle_U\, d\mathbb{P}(\omega)dt. \tag{2.7}$$

Also,

$$
\begin{aligned}
&\int_E \int_\Omega \langle u, B^*(\omega) z(\cdot,\omega;z_0)\rangle_U \, d\mathbb{P}(\omega) dt \\
&= \int_0^T \int_\Omega \langle \chi_E(t) B(\omega) u, z(\cdot,\omega;z_0)\rangle_{V,V'} \, d\mathbb{P}(\omega) dt \\
&= \int_0^T \int_\Omega \langle y_t(t,\omega;y_0), z(\cdot,\omega;z_0)\rangle_{V,V'} \, d\mathbb{P}(\omega) dt \\
&- \int_0^T \int_\Omega \langle A(\omega) y(t,\omega;y_0), z(\cdot,\omega;z_0)\rangle_{V,V'} \, d\mathbb{P}(\omega) dt \\
&= \int_0^T \int_\Omega \langle y_t(t,\omega;y_0), z(\cdot,\omega;z_0)\rangle_{V,V'} \, d\mathbb{P}(\omega) dt \\
&- \int_0^T \int_\Omega \langle y(t,\omega;y_0), A^*(\omega) z(\cdot,\omega;z_0)\rangle_{V,V'} \, d\mathbb{P}(\omega) dt.
\end{aligned}
$$

By Fubini–Tonelli theorem and integration by parts,

$$
\begin{aligned}
&\int_E \int_\Omega \langle u, B^*(\omega) z(t,\omega;z_0)\rangle_U \, d\mathbb{P}(\omega) dt \\
&= \int_\Omega \Big(\langle y(T,\omega;y_0), z(T,\omega;z_0)\rangle_{V,V'} - \langle y(0,\omega;y_0), z(0,\omega;z_0)\rangle_{V,V'} \\
&- \int_0^T \langle y(t,\omega;y_0), z_t(t,\omega;z_0)\rangle_{V,V'} \, dt \\
&- \int_0^T \langle y(t,\omega;y_0), A^*(\omega) z(t,\omega;z_0)\rangle_{V,V'} \, dt \Big) \, d\mathbb{P}(\omega) \\
&= \int_\Omega \Big(\langle y(T,\omega;y_0), z_0\rangle_{V,V'} - \langle y_0, z(0,\omega;z_0)\rangle_{V,V'} \Big) \, d\mathbb{P} \\
&= \left\langle \int_\Omega y(T,\omega;y_0) d\mathbb{P}, z_0 \right\rangle_{V,V'} - \left\langle y_0, \int_\Omega z(0,\omega;z_0) d\mathbb{P} \right\rangle_{V,V'}.
\end{aligned} \tag{2.8}
$$

Hence from the above and (2.7),

$$\left\langle \int_\Omega y(T,\omega;y_0)d\mathbb{P}, z_0 \right\rangle_{V,V'} = 0, \quad \forall\ z_0 \in V',$$

which shows that $\int_\Omega y(T,\omega;y_0)d\mathbb{P} = 0$.

($\Rightarrow$): Let $z_0 \in V'$. Choose $y_0 \in V$ such that

$$-\left\langle y_0, \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega) \right\rangle_{V,V'} = \left\| \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega) \right\|_{V'}. \tag{2.9}$$

Since the system is null averaged controllable in E, there exists $u \in L^2(E;U)$ such that

$$\|u\|_{L^2(E;U)} \leq C\|y_0\|_V \leq \tilde{C},$$

and

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = 0.$$

From (2.7) and (2.9), we know that

$$\int_E \left\langle u(t), \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega) \right\rangle_U dt = \left\| \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega) \right\|_{V'}.$$

By Cauchy-Schwarz,

$$\begin{aligned}\left\| \int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega) \right\|_{V'} &\leq \|u\|_{L^2(E;U)} \left\| \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega) \right\|_{L^2(E;U)} \\ &\leq \tilde{C} \left\| \int_\Omega B^*(\omega)z(t,\omega;z_0)d\mathbb{P}(\omega) \right\|_{L^2(E;U)}.\end{aligned}$$

□

Proof of Theorem 2.3.10. By linearity of (2.4), we can assume $y_0 = 0$. Indeed, if $y_0 \neq 0$ then redefine the state to be $\tilde{y} = y - y_0$.

($\Leftarrow$): Assume that (2.5) satisfies the averaged unique continuation property in E. We must prove that the set

$$\mathcal{A}_T = \left\{ \int_\Omega y(T,\omega;0)d\mathbb{P}(\omega) : y \text{ solves (2.4) with some control } u \right\}$$

is dense in V. Assume by contradiction that $\mathcal{A}_T$ is not dense in V. Then, we can find $\phi \in V'$ with $\|\phi\|_{V'} = 1$ such that

$$\langle \psi, \phi \rangle_{V,V'} = 0, \quad \forall\, \psi \in \mathcal{A}_T.$$

On the other hand, from (2.8), we have

$$\int_0^T \chi_E(t) \left\langle u(t), \int_\Omega B^*(\omega) z(t,\omega;z_0) d\mathbb{P}(\omega) \right\rangle_U dt = \left\langle \int_\Omega y(T,\omega;0) d\mathbb{P}(\omega), z_0 \right\rangle_{V,V'}.$$

Let $z_0 = \phi$ in the above. Then,

$$\int_0^T \chi_E(t) \left\langle u(t), \int_\Omega B^*(\omega) z(t,\omega;\phi) d\mathbb{P}(\omega) \right\rangle_U dt = 0, \quad \forall\, u \in L^2([0,T];U).$$

Hence,

$$\chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot,\omega;\phi) d\mathbb{P}(\omega) = 0 \quad \in L^2([0,T];U),$$

which implies by the averaged unique continuation property that $\phi = 0$. Thus we have a contradiction.

($\Rightarrow$): Assume that (2.4) is approximately controllable in average in E. Assume by contradiction that (2.5) does not satisfy the unique continuation property in E, i.e., that there is a $z_0 \in V'$ with $\|z_0\|_{V'} = 1$ such that

$$\chi_E(\cdot) \int_\Omega B^*(\omega) z(\cdot, \omega; z_0) d\mathbb{P}(\omega) = 0 \quad \in L^2([0,T]; U).$$

With (2.8), this implies

$$\left\langle \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega), z_0 \right\rangle_{V,V'} = 0, \quad \forall\, u \in L^2(E; U). \tag{2.10}$$

Now choose $y_1 \in V$ such that $\langle y_1, z_0 \rangle_{V,V'} = 1$. On the other hand,

$$\left\langle \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega), z_0 \right\rangle_{V,V'} = \left\langle \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega) - y_1, z_0 \right\rangle_{V,V'} + \langle y_1, z_0 \rangle_{V,V'}.$$

By approximate controllability, there exists $u \in L^2(E; U)$ such that

$$\left\| \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega) - y_1 \right\|_V^2 < \frac{1}{2}.$$

By Cauchy-Schwarz,

$$\left| \left\langle \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega) - y_1, z_0 \right\rangle_{V,V'} \right| < \frac{1}{2} \|z_0\|_{V'} = \frac{1}{2}.$$

Thus,

$$\left\langle \int_\Omega y(T, \omega; 0) d\mathbb{P}(\omega), z_0 \right\rangle_{V,V'} > \frac{1}{2},$$

which contradicts (2.10). $\square$

Chapter 3

Problem Statement

In this section, we will introduce the controllability problem that will be the focus for the rest of the thesis.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{Z}_{>0}$) be an open bounded and connected domain with C^2 boundary, ∂G. Consider the following random heat equation:

$$\begin{cases} y_t(x,t) - \alpha(\omega)\Delta y(x,t) = \chi_{G_0 \times E}(x,t)u(x,t) & \text{in } G \times (0,T), \\ y(x,t) = 0 & \text{on } \partial G \times (0,T), \\ y(x,0) = y_0(x) & \text{in } G, \end{cases} \tag{3.1}$$

where $y_0 \in L^2(G)$, $G_0 \subset G$ is a non-empty open subset, $E \subset [0,T]$ is a Lebesgue measurable set with positive measure, and $u \in L^2(E; L^2(G_0))$ is the control. The diffusivity $\alpha\colon \Omega \to \mathbb{R}_{>0}$ is assumed to be a random variable depending on an unknown parameter $\omega \in \Omega$. The goal is to steer the average of the solution to the system from its initial state, y_0, to any other state $y_1 \in L^2(G)$ using a control that is independent of the unknown parameter $\omega \in \Omega$.

3.1 Well-Posedness

System (3.1) can be viewed as an abstract parameter dependent control system with $V = H = L^2(G)$, $U = L^2(G_0)$, $B(\omega) = I$, for all $\omega \in \Omega$ and

$$\begin{cases} A(\omega) = \alpha(\omega)\Delta, \\ D(A(\omega)) = H^2(G) \cap H_0^1(G), \end{cases}$$

for all $\omega \in \Omega$. We now wish to verify the assumptions from Section 2.3. It is easy to see that Assumption (A1) is satisfied. To show that assumption (A2) is satisfied, we first note that $D(A(\omega))$ is dense in $L^2(G)$ for all $\omega \in \Omega$. This follows from the fact that $C_c^\infty(G)$ is dense in $L^2(G)$ and the inclusion $C_c^\infty(G) \subset (H^2(G) \cap H_0^1(G))$. By integration by parts twice, for all $y, \tilde{y} \in D(A(\omega))$

$$\langle A(\omega)y, \tilde{y}\rangle_{L^2(G)} = \int_G \alpha(\omega)\tilde{y}\Delta y dx = \int_G \alpha(\omega) y \Delta\tilde{y} dx = \langle y, A(\omega)\tilde{y}\rangle_{L^2(G)}.$$

Note also that $D(A^*(\omega)) = D(A(\omega))$ for all $\omega \in \Omega$. Hence, $A(\omega)$ is a self-adjoint operator for all $\omega \in \Omega$. Thus, $A(\omega)$ is closed for all $\omega \in \Omega$ since the adjoint of an operator is always closed. Additionally, for all $y \in D(A(\omega))$

$$\langle A(\omega)y, y\rangle_{L^2(G)} = \int_G \alpha(\omega) y \Delta y dx = -\alpha(\omega) \int_G \|\nabla y\|^2 dx \leq 0.$$

Hence from Theorem 2.2.7, $A(\omega)$ generates a C^0-semigroup on $L^2(G)$ satisfying $\|S(t,\omega)\|_{\mathcal{L}(L^2(G))} \leq 1$, for all $\omega \in \Omega$. This verifies assumption (A2). Finally, assumption (A3) follows from the fact that for all $y \in L^2(G)$

$$\int_\Omega \|S(t,\omega)y\|_{L^2(G)} d\mathbb{P}(\omega) \leq \mathbb{P}(\Omega)\|y\|_{L^2(G)} \leq \|y\|_{L^2(G)} < \infty.$$

Thus, from Theorem 2.2.13, we know that (3.1) admits a unique weak solution $y(\cdot,\omega;y_0)$ in $C([0,T];L^2(G))$ for a.e. $\omega \in \Omega$ and $\int_\Omega y(\cdot,\omega;y_0)d\mathbb{P}(\omega) \in C([0,T];L^2(G))$. In fact, the solution to the heat equation enjoys higher regularity. Given a sufficiently smooth initial condition, control function and control domain G_0, we obtain $y \in C^\infty((0,T] \times G)$ (see [5, Chapter 7, Theorem 7]). Due to this smoothing effect of the heat equation, it is not always possible to steer the average of the solution to any state $y_1 \in L^2(G)$. Thus, we focus on the problems of null and approximate controllability, as defined in Definitions 2.3.2 and 2.3.3. For later use, let us state these problems.

Problem 3.1.1. (Null Controllability in Average): Given $T > 0$, $y_0 \in L^2(G)$, and any distribution of $\alpha\colon \Omega \to \mathbb{R}_{>0}$, does there exist $u \in L^2(E;L^2(G_0))$ such that

$$\|u\|_{L^2(E;L^2(G_0))} \leq C\|y_0\|_{L^2(G)}$$

and the average of the solution to system (3.1) satisfies

$$\int_\Omega y(T,\omega;y_0)d\mathbb{P}(\omega) = 0 \ ?$$

Problem 3.1.2. (Approximate Controllability in Average): Given $T > 0$,

$\epsilon > 0$, $y_0, y_1 \in L^2(G)$, and any distribution of $\alpha\colon \Omega \to \mathbb{R}_{>0}$, does there exist $u \in L^2(E; L^2(G_0))$ such that the average of the solution to system (3.1) satisfies

$$\left\| \int_\Omega y(T, \omega; y_0) d\mathbb{P}(\omega) - y_1 \right\|_{L^2(G)} < \epsilon \ ?$$

Chapter 4

Averaged Control

In this chapter, we present our main results regarding average controllability of the heat equation. Section 4.1 contains a novel result which provides a technical inequality, c.f. Theorem 4.1.2, that will be used to build the desired average observability inequality for our random heat equation. Section 4.2 contains the proofs of the main results on average null and approximate controllability of the heat equation.

4.1 Technical Inequality

Consider system (3.1). Let $N \in (\mathbb{Z}_{>0} \cup \{+\infty\})$. Let $\{a_i\}_{i=1}^{N+1} \subset \mathbb{R}_{>0}$ and $\{b_i\}_{i=1}^{N} \subset \mathbb{R}_{\geq 0}$ be chosen such that $a_1 < a_2 < \cdots$, $b_i \neq 0$ for at least one $i \in \{1, \ldots, N\}$, and the function $\rho_N \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined as

$$\rho_N(\alpha) = \sum_{i=1}^{N} b_i \chi_{A_i}(\alpha) \tag{4.1}$$

is in $L^1(\mathbb{R})$, where $A_i \in \{(a_i, a_{i+1}), [a_i, a_{i+1}], [a_i, a_{i+1}), (a_i, a_{i+1}]\}$. It is important to note that the above sum could be infinite.

Remark 4.1.1. If we require further that $\sum_{i=1}^{N} b_i(a_{i+1} - a_i) = 1$ then $\int_{\mathbb{R}} \rho_N(\alpha) d\alpha = 1$.

Hence, ρ_N may serve as a valid probability density function for some random variable. In this case, ρ_N is just a perturbation of the probability density function corresponding to a uniform distribution. •

The adjoint equation to (3.1) is

$$\begin{cases} z_t(x,t) + \alpha(\omega)\Delta z(x,t) = 0 & \text{in } G \times (0,T), \\ z(x,t) = 0 & \text{on } \partial G \times (0,T), \\ z(x,T) = z_0(x) & \text{in } G, \end{cases} \tag{4.2}$$

where $z_0 \in L^2(G)$. We will write the solution to (4.2) as a Fourier series in terms of the basis of eigenfunctions of the Dirichlet Laplacian. To this end, consider the linear operator A_Δ given by

$$\begin{cases} D(A_\Delta) = H^2(G) \cap H_0^1(G), \\ A_\Delta f = -\Delta f, \quad \forall f \in D(A_\Delta). \end{cases}$$

We denote by $\{\lambda_j\}_{j=1}^\infty$ the eigenvalues of A_Δ and $\{e_j\}_{j=1}^\infty$ the corresponding sequence of orthonormal eigenfunctions in $L^2(G)$. Note that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ (the existence of such eigenvalues and eigenfunctions satisfying this is classical and can be found in, e.g., [5, Chapter 6.5, Theorem 1]). Assume that the final data to (4.2) is written as $z_0 = \sum_{j=1}^\infty z_{0,j} e_j$, where $z_{0,j} = \langle z_0, e_j \rangle_{L^2(G)}$. Proceeding with separation of variables, we assume that the solution to (4.2) is written as

$$z(x,t,\omega;z_0) = \sum_{j=1}^\infty f_j(t,\omega) e_j(x),$$

for some sequence $\{f_j\}_{j=1}^{\infty}$. Substituting this into (4.2) gives

$$\begin{aligned} 0 &= \sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_j(t,\omega)e_j(x) + \alpha(\omega) \sum_{j=1}^{\infty} f_j(t,\omega)\Delta e_j(x) \\ &= \sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_j(t,\omega)e_j(x) - \sum_{j=1}^{\infty} \alpha(\omega)\lambda_j f_j(t,\omega)e_j(x). \end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty} \frac{\partial}{\partial t} f_j(t,\omega)e_j(x) = \sum_{j=1}^{\infty} \alpha(\omega)\lambda_j f_j(t,\omega)e_j(x).$$

Since $\{e_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(G)$,

$$\frac{\partial}{\partial t} f_j(t,\omega) = \alpha(\omega)\lambda_j f_j(t,\omega), \quad \forall\, j \in \mathbb{Z}_{>0}.$$

Thus,

$$f_j(t) = C_j e^{\alpha(\omega)\lambda_j t}.$$

Using the final data for (4.2), we conclude that $C_j = z_{0,j} e^{-\alpha(\omega)\lambda_j T}$ and hence,

$$f_j(t,\omega) = z_{0,j} e^{-\alpha(\omega)\lambda_j (T-t)}.$$

Thus, the solution to (4.2) is

$$z(x,t,\omega;z_0) = \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha(\omega)\lambda_j (T-t)} e_j(x).$$

We can now state the technical inequality needed to show both null and approximate controllability in average for the heat equation.

Theorem 4.1.2. *There exists $C > 0$ such that*

$$\left\| \int_{\mathbb{R}} \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j T} e_j d\alpha \right\|_{L^2(G)} \leq C \int_E \left\| \int_{\mathbb{R}} \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j (T-t)} e_j d\alpha \right\|_{L^2(G_0)} dt. \tag{4.3}$$

Note that if ρ_N is the probability density function associated to the random variable α, the integrals above are expectations of the solution to (4.2).

In order to prove Theorem 4.1.2, we will need the following technical results from [13] and [9]:

Lemma 4.1.3. ([13, Proposition 2.1]): *Let $E \subset [0, T]$ be a Lebesgue measurable set of positive measure, $\mu(E)$. Let ℓ be a density point of E, i.e., $\lim_{\epsilon\to 0} \frac{\mu(E\cap(\ell-\epsilon,\ell+\epsilon))}{\mu((\ell-\epsilon,\ell+\epsilon))} = 1$. Then for each $d > 1$, there exists $\ell_1 \in (\ell, T)$ such that the sequence $\{\ell_k\}_{k=1}^{\infty}$ given by*

$$\ell_{k+1} = \ell + \frac{\ell_1 - \ell}{d^k} \tag{4.4}$$

satisfies

$$\mu(E \cap (\ell_{k+1}, \ell_k)) \geq \frac{\ell_k - \ell_{k+1}}{3}. \tag{4.5}$$

Lemma 4.1.4. ([9, Theorem 1.2]): *There exists a constant $C_1 > 0$ such that for all $r > 0$ and $\{c_j\}_{\lambda_j \leq r} \subset \mathbb{C}$,*

$$\left(\sum_{\lambda_j \leq r} |c_j|^2 \right)^{\frac{1}{2}} \leq C_1 e^{C_1\sqrt{r}} \left\| \sum_{\lambda_j \leq r} c_j e_j \right\|_{L^2(G_0)}. \tag{4.6}$$

We are now ready to prove Theorem 4.1.2, which is an adaptation of the result of [10] to our setting.

Proof of Theorem 4.1.2. Let $\tilde{z}(x,t) = \int_{\mathbb{R}} \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j t} e_j(x) d\alpha$. We have

$$\tilde{z}(\cdot,t) = \int_{\mathbb{R}} \lim_{n\to\infty} \rho_N(\alpha) \sum_{j=1}^{n} z_{0,j} e^{-\alpha\lambda_j t} e_j d\alpha.$$

In order to bring the limit outside of the integral, we will use dominated convergence theorem for Banach space-valued functions. Since $L^2(G)$ is separable, by Lemma 6.2.5, strong measurability is equivalent to weak measurability. Thus it is sufficient to check weak measurability of $\rho_N(\alpha) \sum_{j=1}^{n} z_{0,j} e^{-\alpha\lambda_j t} e_j$ as a function of α, for all $t \in [0,T]$. Indeed, let $f \in L^2(G)$ and write $f = \sum_{j=1}^{\infty} f_j e_j$. Then,

$$\left\langle \rho_N(\alpha) \sum_{j=1}^{n} z_{0,j} e^{-\alpha\lambda_j t} e_j, \sum_{j=1}^{\infty} f_j e_j \right\rangle_{L^2(G)} = \rho_N(\alpha) \sum_{j=1}^{n} z_{0,j} f_j e^{-\alpha\lambda_j t} e_j,$$

which is clearly measurable. Also,

$$\left\| \rho_N(\alpha) \sum_{j=1}^{n} z_{0,j} e^{-\alpha\lambda_j t} e_j \right\|_{L^2(G)}^2 = \rho_N(\alpha)^2 \sum_{j=1}^{n} z_{0,j}^2 e^{-2\alpha\lambda_j t} e_j \le \rho_N(\alpha)^2 \|z_0\|_{L^2(G)}^2,$$

which is integrable. Hence, by dominated convergence theorem for Banach space-valued functions (Theorem 6.2.8 in the Appendix),

$$\begin{aligned} \tilde{z}(\cdot,t) &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \rho_N(\alpha) z_{0,j} e^{-\alpha\lambda_j t} e_j d\alpha \\ &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j t} z_{0,j} \sum_{i=1}^{N} b_i (e^{-a_i\lambda_j t} - e^{-a_{i+1}\lambda_j t}) e_j. \end{aligned}$$

It is easy to verify that $\tilde{z}(\cdot,t) \in L^2(G)$ for all $t \in [0,T]$. Indeed,

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} z_{0,j} \left(\int_{\mathbb{R}} \rho_N(\alpha) e^{-\alpha\lambda_j t} d\alpha \right) e_j \right\|^2_{L^2(G)} &= \sum_{j=1}^{\infty} z_{0,j}^2 \left(\int_{\mathbb{R}} \rho_N(\alpha) e^{-\alpha\lambda_j t} d\alpha \right)^2 \\
&\leq \sum_{j=1}^{\infty} z_{0,j}^2 \left(\int_{\mathbb{R}} \rho_N(\alpha) d\alpha \right)^2 \\
&= \|z_0\|^2_{L^2(G)} \left(\int_{\mathbb{R}} \rho_N(\alpha) d\alpha \right)^2 \\
&< \infty,
\end{aligned}$$

since $\rho_N \in L^1(\mathbb{R})$.

We wish to prove (4.3), which is equivalent to proving

$$\|\tilde{z}(\cdot,T)\|_{L^2(G)} \leq C \int_{\tilde{E}} \|\tilde{z}(\cdot,t)\|_{L^2(G_0)} dt, \tag{4.7}$$

where

$$\tilde{E} = \{t : T - t \in E\}. \tag{4.8}$$

Note that $\tilde{E}$ is also a Lebesgue measurable set with positive measure. For all $\xi \in L^2(G)$, let

$$S(t,\xi) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j t} \xi_j \sum_{i=1}^{N} b_i (e^{-a_i\lambda_j t} - e^{-a_{i+1}\lambda_j t}) e_j.$$

Note that

$$\|S(t,\xi)\|^2_{L^2(G)} = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j t} \xi_j \sum_{i=1}^{N} b_i (e^{-a_i\lambda_j t} - e^{-a_{i+1}\lambda_j t}) \right)^2.$$

We claim that the following inequality holds:

$$\|S(t,\xi)\|_{L^2(G)} \leq \|S(s,\xi)\|_{L^2(G)}, \quad \text{for} \quad 0 \leq s \leq t \leq T. \tag{4.9}$$

Indeed, let $0 \leq s \leq t \leq T$. It can be easily checked that $\frac{1}{t}(e^{-a\lambda_j t} - e^{-b\lambda_j t})$ is a monotonically decreasing function in t, for all $a, b \in \mathbb{R}$ with $0 \leq a < b$. Hence,

$$\begin{aligned}
\|S(t,\xi)\|^2_{L^2(G)} &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}\xi_j^2 \left(\sum_{i=1}^{N} b_i \frac{1}{t}(e^{-a_i\lambda_j t} - e^{-a_{i+1}\lambda_j t}) \right)^2 \\
&\leq \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}\xi_j^2 \left(\sum_{i=1}^{N} b_i \frac{1}{s}(e^{-a_i\lambda_j s} - e^{-a_{i+1}\lambda_j s}) \right)^2 \\
&= \|S(s,\xi)\|^2_{L^2(G)},
\end{aligned}$$

giving (4.9).

Let $X_r = \text{span}\{e_j\}_{\lambda_j \leq r}$ for $r > 0$. Then, we claim that

$$\|S(t,\xi)\|_{L^2(G)} \leq e^{-rc(t-s)}\|S(s,\xi)\|_{L^2(G)}, \quad \forall\, \xi \in X_r^{\perp} \quad \text{and} \quad 0 \leq s \leq t \leq T, \tag{4.10}$$

where $0 < c \leq a_1$. Indeed, let $k \in \mathbb{Z}_{>0}$ be such that $\lambda_k \leq r$ and $\lambda_{k+1} > r$. Set

$$g_j(a_1, a_2, s, t) = \frac{s}{t}\frac{(e^{-a_1\lambda_j t} - e^{-a_2\lambda_j t})}{(e^{-a_1\lambda_j s} - e^{-a_2\lambda_j s})}, \quad \text{for all} \quad a_1, a_2, s, t \in \mathbb{R}.$$

We have

$$\begin{aligned}
&\|S(t,\xi)\|^2_{L^2(G)} \\
&= \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j^2}\xi_j^2 \left(\sum_{i=1}^{N} b_i \frac{1}{s}(e^{-a_i\lambda_j s} - e^{-a_{i+1}\lambda_j s}) g_j(a_i, a_{i+1}, s, t) \right)^2 \\
&= \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j^2}\xi_j^2 \left(e^{-c\lambda_j(t-s)} \sum_{i=1}^{N} b_i \frac{1}{s}(e^{-a_i\lambda_j s} - e^{-a_{i+1}\lambda_j s}) g_j(a_i - c, a_{i+1} - c, s, t) \right)^2 .
\end{aligned}$$

Since $\lambda_{k+1} < \lambda_{k+2} < \lambda_{k+3} < \dots$ we obtain

$$\begin{aligned}
&\|S(t,\xi)\|^2_{L^2(G)} \\
&\leq e^{-2cr(t-s)} \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j^2}\xi_j^2 \left(\sum_{i=1}^{N} b_i \frac{1}{s}(e^{-a_i\lambda_j s} - e^{-a_{i+1}\lambda_j s}) g_j(a_i - c, a_{i+1} - c, s, t) \right)^2 .
\end{aligned}$$

Since $0 < c \leq a_1$, we have $g_j(a_i - c, a_{i+1} - c, s, t) \leq 1$, for all $i \in \{1, \dots, N\}$, for all $j \in \mathbb{Z}_{>0}$, and for all $0 \leq s \leq t \leq T$. Hence, (4.10) holds.

Let ℓ be a density point for $\tilde{E}$, where $\tilde{E}$ is as in (4.8). By Lemma 4.1.3, for a given $d > 1$, there exists a sequence $\{\ell_k\}_{k=1}^{\infty}$ satisfying (4.4) and (4.5). Define a sequence of subsets $\{\tilde{E}_k\}_{k=1}^{\infty}$ of $(0, T)$ as follows:

$$\tilde{E}_k := \left\{ t - \frac{\ell_k - \ell_{k+1}}{6} : t \in \tilde{E} \cap \left(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k \right) \right\}, \text{ for } k \in \mathbb{Z}_{>0}.$$

Note that $\tilde{E}_k \subset (\ell_{k+1}, \ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1}))$. From (4.5) we have that

$$\begin{aligned}
\mu(\tilde{E}_k) &= \mu \left(\tilde{E} \cap \left(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k \right) \right) \\
&= \mu \left(\tilde{E} \cap \left[(\ell_{k+1}, \ell_k) \setminus \left(\ell_{k+1}, \ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6} \right) \right] \right) \\
&\geq \mu(\tilde{E} \cap (\ell_{k+1}, \ell_k)) - \frac{\ell_k - \ell_{k+1}}{6} \\
&\geq \frac{\ell_k - \ell_{k+1}}{6}. \qquad (4.11)
\end{aligned}$$

For all $k \in \mathbb{Z}_{>0}$ set $r_k = m^{2k}$, where $m \in \mathbb{R}_{>0}$ will be chosen later. From (4.9) we have for all $\xi \in X_{r_k}$,

$$\int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{\tilde{E}_k}(t) \left\| S\left(\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1}),\xi\right)\right\|_{L^2(G)} dt$$
$$\leq \int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{\tilde{E}_k}(t) \|S(t,\xi)\|_{L^2(G)} dt.$$

Hence, using (4.11), for all $\xi \in X_{r_k}$

$$\frac{\ell_k-\ell_{k+1}}{6}\left\| S\left(\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1}),\xi\right)\right\|_{L^2(G)}$$
$$\leq \mu(\tilde{E}_k)\left\| S\left(\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1}),\xi\right)\right\|_{L^2(G)}$$
$$\leq \int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{\tilde{E}_k}(t) \|S(t,\xi)\|_{L^2(G)} dt.$$

Using the above and (4.6), for all $\xi \in X_{r_k}$

$$\frac{\ell_k-\ell_{k+1}}{6}\left\| S\left(\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1}),\xi\right)\right\|_{L^2(G)}$$
$$\leq \int_{\tilde{E}_k} \left(\sum_{\lambda_j\leq r_k}\left(\frac{1}{\lambda_j t}\xi_j \sum_{i=1}^N b_i(e^{-a_i\lambda_j t}-e^{-a_{i+1}\lambda_j t})\right)^2\right)^{\frac{1}{2}} dt$$
$$\leq C_1 e^{C_1\sqrt{r_k}} \int_{\tilde{E}_k} \left\| \sum_{\lambda_j\leq r_k} \frac{1}{\lambda_j t}\xi_j \sum_{i=1}^N b_i(e^{-a_i\lambda_j t}-e^{-a_{i+1}\lambda_j t})e_j\right\|_{L^2(G_0)} dt$$
$$= C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_{k+1}+\frac{5}{6}(\ell_k-\ell_{k+1})} \chi_{\tilde{E}_k}(t) \|S(t,\xi)\|_{L^2(G_0)} dt.$$

Now write $z_0 = z_0^1 + z_0^2$, where $z_0^1 \in X_{r_k}$ and $z_0^2 \in X_{r_k}^{\perp}$. Similarly, since for all $t \in \tilde{E}_k$, we have $t + \frac{\ell_k - \ell_{k+1}}{6} \le \ell_k$ then,

$$\begin{aligned}
&\frac{\ell_k - \ell_{k+1}}{6} \|S(\ell_k, z_0^1)\|_{L^2(G)} \\
&\le \int_{\ell_{k+1}}^{\ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1})} \chi_{\tilde{E}_k}(t) \left\| S\left(t + \frac{\ell_k - \ell_{k+1}}{6}, z_0^1\right) \right\|_{L^2(G)} dt \\
&\le C_1 e^{C_1 \sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_{k+1} + \frac{5}{6}(\ell_k - \ell_{k+1})} \chi_{\tilde{E}_k}(t) \left\| S\left(t + \frac{\ell_k - \ell_{k+1}}{6}, z_0^1\right) \right\|_{L^2(G_0)} dt \\
&= C_1 e^{C_1 \sqrt{r_k}} \int_{\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}}^{\ell_k} \chi_{\tilde{E}_k}\left(t - \frac{\ell_k - \ell_{k+1}}{6}\right) \|S(t, z_0^1)\|_{L^2(G_0)} dt.
\end{aligned}$$

By definition of $\tilde{E}_k$, we have that

$$\chi_{\tilde{E}_k}\left(t - \frac{\ell_k - \ell_{k+1}}{6}\right) = \chi_{\tilde{E}}(t), \quad \forall\, t \in \left(\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}, \ell_k\right).$$

Hence,

$$\frac{\ell_k - \ell_{k+1}}{6} \|S(\ell_k, z_0^1)\|_{L^2(G)} \le C_1 e^{C_1 \sqrt{r_k}} \int_{\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}}^{\ell_k} \chi_{\tilde{E}}(t) \|S(t, z_0^1)\|_{L^2(G_0)} dt.$$

Note that by linearity in its second argument, $S(t, z_0^1) = S(t, z_0) - S(t, z_0^2)$. Using this fact and triangle inequality, we obtain

$$\begin{aligned}
&\frac{\ell_k - \ell_{k+1}}{6} \|S(\ell_k, z_0^1)\|_{L^2(G)} \\
&\le C_1 e^{C_1 \sqrt{r_k}} \int_{\ell_{k+1} + \frac{\ell_k - \ell_{k+1}}{6}}^{\ell_k} \chi_{\tilde{E}}(t) (\|S(t, z_0)\|_{L^2(G_0)} + \|S(t, z_0^2)\|_{L^2(G)}) dt.
\end{aligned}$$

By (4.9) and (4.10),

$$\begin{aligned}
&C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k} \chi_{\tilde{E}}(t)(\|S(t,z_0)\|_{L^2(G_0)} + \|S(t,z_0^2)\|_{L^2(G)})dt \\
&\leq C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}}^{\ell_k} \chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt \\
&+ C_1 e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1}) \left\| S\left(\ell_{k+1}+\frac{\ell_k-\ell_{k+1}}{6}, z_0^2\right)\right\|_{L^2(G)} \\
&\leq C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_k} \chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt \\
&+ C_1 e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1}) e^{-\frac{\ell_k-\ell_{k+1}}{6} r_k c} \|S\left(\ell_{k+1}, z_0^2\right)\|_{L^2(G)}.
\end{aligned}$$

Combining the above two inequalities yields

$$\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}\|S(\ell_k, z_0^1)\|_{L^2(G)} \\
&\leq C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_k} \chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt \\
&+ C_1 e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1}) e^{-\frac{\ell_k-\ell_{k+1}}{6} r_k c} \|S\left(\ell_{k+1}, z_0^2\right)\|_{L^2(G)}.
\end{aligned}$$

Hence, by triangle inequality

$$\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}\|S(\ell_k, z_0)\|_{L^2(G)} \\
&\leq C_1 e^{C_1\sqrt{r_k}} \int_{\ell_{k+1}}^{\ell_k} \chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt \\
&+ C_1 e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1}) e^{-\frac{\ell_k-\ell_{k+1}}{6} r_k c} \|S\left(\ell_{k+1}, z_0^2\right)\|_{L^2(G)} \\
&+ \frac{\ell_k-\ell_{k+1}}{6}\|S(\ell_k, z_0^2)\|_{L^2(G)}.
\end{aligned}$$

From (4.10),

$$\begin{aligned}
&\frac{\ell_k-\ell_{k+1}}{6}\|S(\ell_k,z_0)\|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt\\
&+C_1e^{C_1\sqrt{r_k}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_kc}\|S\left(\ell_{k+1},z_0^2\right)\|_{L^2(G)}\\
&+\frac{\ell_k-\ell_{k+1}}{6}e^{-r_kc(\ell_k-\ell_{k+1})}\|S(\ell_{k+1},z_0^2)\|_{L^2(G)}\\
&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt\\
&+(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_kc}(C_1e^{C_1\sqrt{r_k}}+1)\|S\left(\ell_{k+1},z_0^2\right)\|_{L^2(G)}.
\end{aligned}$$

We claim that $\|S(t,z_0)\|^2_{L^2(G)}=\|S(t,z_0^1)\|^2_{L^2(G)}+\|S(t,z_0^2)\|^2_{L^2(G)}$. Indeed,

$$\begin{aligned}
\|S(t,z_0)\|^2_{L^2(G)}&=\left\|\sum_{j=1}^{\infty}\frac{1}{\lambda_jt}\langle z_0^1+z_0^2,e_j\rangle_{L^2(G)}\sum_{i=1}^{N}b_i(e^{-a_i\lambda_jt}-e^{-a_{i+1}\lambda_jt})e_j\right\|^2_{L^2(G)}\\
&=\|S(t,z_0^1)\|^2_{L^2(G)}+\|S(t,z_0^2)\|^2_{L^2(G)}\\
&+2\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2t^2}\langle z_0^1,e_j\rangle_{L^2(G)}\langle z_0^2,e_j\rangle_{L^2(G)}\left(\sum_{i=1}^{N}b_i(e^{-a_i\lambda_jt}-e^{-a_{i+1}\lambda_jt})\right)^2.
\end{aligned}$$

However, $\langle z_0^1,e_j\rangle_{L^2(G)}\langle z_0^2,e_j\rangle_{L^2(G)}=0$, for all $j\in\mathbb{Z}_{>0}$, proving the claim. Thus, we clearly have $\|S(t,z_0^2)\|_{L^2(G)}\le\|S(t,z_0)\|_{L^2(G)}$. Hence,

$$\begin{aligned}
\frac{\ell_k-\ell_{k+1}}{6}\|S(\ell_k,z_0)\|_{L^2(G)}&\le C_1e^{C_1\sqrt{r_k}}\int_{\ell_{k+1}}^{\ell_k}\chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt\\
&+(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_kc}(C_1e^{C_1\sqrt{r_k}}+1)\|S(\ell_{k+1},z_0)\|_{L^2(G)}.
\end{aligned}$$

Rearranging the above yields

$$\frac{\ell_k-\ell_{k+1}}{6C_1e^{C_1\sqrt{r_k}}}\|S(\ell_k,z_0)\|_{L^2(G)}-\frac{C_1e^{C_1\sqrt{r_k}}+1}{C_1e^{C_1\sqrt{r_k}}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_kc}\|S(\ell_{k+1},z_0)\|_{L^2(G)}$$
$$\leq\int_{\ell_{k+1}}^{\ell_k}\chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt.$$

Summing the above from $k=2$ to $k=\infty$ gives

$$\frac{\ell_2-\ell_3}{6C_1e^{C_1\sqrt{r_2}}}\|S(\ell_2,z_0)\|_{L^2(G)}+\sum_{k=2}^{\infty}f_k\|S(\ell_{k+1},z_0)\|_{L^2(G)}\leq\int_0^T\chi_{\tilde{E}}(t)\|S(t,z_0)\|_{L^2(G_0)}dt,$$

where

$$f_k=\frac{\ell_{k+1}-\ell_{k+2}}{6C_1e^{C_1\sqrt{r_{k+1}}}}-\frac{C_1e^{C_1\sqrt{r_k}}+1}{C_1e^{C_1\sqrt{r_k}}}(\ell_k-\ell_{k+1})e^{-\frac{\ell_k-\ell_{k+1}}{6}r_kc},\ \forall\ k\in\{2,3,\dots\}.$$

Note that

$$\ell_k-\ell_{k+1}=\frac{(\ell_1-\ell)(d-1)}{d^k}.$$

Since $r_k=m^{2k}$,

$$f_k=\frac{(\ell_1-\ell)(d-1)}{6C_1d^{k+1}e^{C_1m^{k+1}}}-\frac{C_1e^{C_1m^k}+1}{C_1e^{C_1m^k}}\frac{(\ell_1-\ell)(d-1)}{d^k}e^{-\frac{(\ell_1-\ell)(d-1)}{6d^k}cm^{2k}}.$$

By picking m large enough, we obtain

$$f_k\geq 0,$$

for all $k \in \{2, 3, \dots\}$. Indeed,

$$\begin{aligned}
& f_k \geq 0 \\
\Leftrightarrow\ & \frac{(\ell_1 - \ell)(d-1)}{6C_1 d^{k+1} e^{C_1 m^{k+1}}} \geq \frac{C_1 e^{C_1 m^k} + 1}{C_1 e^{C_1 m^k}} \frac{(\ell_1 - \ell)(d-1)}{d^k} e^{-\frac{(\ell_1-\ell)(d-1)}{6d^k} cm^{2k}} \\
\Leftrightarrow\ & \frac{1}{6de^{C_1 m^{k+1}}} \geq \frac{C_1 e^{C_1 m^k} + 1}{e^{C_1 m^k}} e^{-\frac{(\ell_1-\ell)(d-1)}{6d^k} cm^{2k}} \\
\Leftrightarrow\ & e^{-C_1 m^{k+1}} \geq 6d \left(C_1 + e^{-C_1 m^k} \right) e^{-\frac{(\ell_1-\ell)(d-1)}{6d^k} cm^{2k}} \\
\Leftrightarrow\ & 1 \geq 6d \left(C_1 + e^{-C_1 m^k} \right) e^{-\frac{(\ell_1-\ell)(d-1)}{6d^k} cm^{2k} + C_1 m^{k+1}} \\
\Leftrightarrow\ & 1 \geq 6d \left(C_1 e^{-m^k \left(\frac{(\ell_1-\ell)(d-1)}{6} c \frac{m^k}{d^k} - C_1 m \right)} + e^{-m^k \left(\frac{(\ell_1-\ell)(d-1)}{6} c \frac{m^k}{d^k} - C_1 m + C_1 \right)} \right). \qquad (4.12)
\end{aligned}$$

Since $c > 0$, we can pick m large enough to ensure that the above holds. Hence,

$$\|S(\ell_2, z_0)\|_{L^2(G)} \leq \frac{6C_1 e^{C_1 m^2}}{\ell_2 - \ell_3} \int_{\tilde{E}} \|S(t, z_0)\|_{L^2(G_0)} dt. \qquad (4.13)$$

Recall that we wish to prove (4.7) which is equivalent to showing that there exists $C > 0$ such that

$$\|S(T, z_0)\|_{L^2(G)} \leq C \int_{\tilde{E}} \|S(t, z_0)\|_{L^2(G_0)} dt.$$

Since by (4.9) we have $\|S(T, z_0)\|_{L^2(G)} \leq \|S(\ell_2, z_0)\|_{L^2(G)}$, the statement of the theorem follows. □

Remark 4.1.5. Note that in (4.13), the constant depends on a_1, which is the left most endpoint of $\mathrm{supp}(\rho_N)$. This can be seen because the m we choose to satisfy (4.12) increases as c decreases. Recall that $0 < c \leq a_1$. Hence, as $a_1 \to 0^+$, $m \to \infty$ and thus, the constant on the right side of (4.13) goes unbounded. Hence, it is crucial to the proof technique that the left most endpoint of $\mathrm{supp}(\rho_N)$ be bounded away from 0.

This can be thought of as assigning no probability to the diffusivity being 0. This is important in showing controllability since in the case that system (3.1) has diffusivity equal to 0, the control has no effect on $G \setminus G_0$. •

4.2 Average Null and Approximate Controllability for General Density Function

We now wish to prove average controllability results for system (3.1) when the random diffusivity, $\alpha \colon \Omega \to \mathbb{R}_{>0}$ has any probability density function. This recovers the null and approximate controllability results proved in [10] when α is uniformly or exponentially distributed. Our technique will be to use simple functions to approximate the density function and invoke the result of Theorem 4.1.2.

Theorem 4.2.1. *Let system* (3.1) *be such that* α *is a random variable with Riemann integrable probability density function* ρ*. Assume* $\operatorname{supp}(\rho) \subset \mathbb{R}_{>0}$*. Then* (3.1) *is* null controllable in average in E with cost C.

Proof. Since ρ is a measurable function, there exists a sequence of functions $\{\rho_N\}_{N=1}^{\infty}$ such that

$$\rho(\alpha) = \lim_{N \to \infty} \rho_N(\alpha),$$

where

$$\rho_N(\alpha) = \sum_{i=1}^{N2^N} i2^{-N} \chi_{A_i^N}(\alpha),$$

and

$$A_i^N = \begin{cases} \{i2^{-N} \leq \rho(\alpha) < (i+1)2^{-N}\} & \text{for } i \neq N2^N, \\ \{\rho(\alpha) \geq N\} & \text{for } i = N2^N. \end{cases}$$

Note that the A_i^N are possibly countably infinite unions of disjoint intervals, for all $i \in \{1, \ldots, N2^N\}$. By definition of ρ_N, we clearly have $\rho_N \in L^1(\mathbb{R})$. Since ρ is Riemann integrable, ρ_N can be written in the same form as (4.1) in Section 4.1. Let

$$\tilde{z}_N(\cdot, t) = \int_{\mathbb{R}} \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j t} e_j d\alpha.$$

From Theorem 4.1.2, we have that

$$\|\tilde{z}_N(\cdot, T)\|_{L^2(G)} \leq C \int_E \|\tilde{z}_N(\cdot, t)\|_{L^2(G_0)} dt,$$

where C depends on A_i^N for some i (see Remark 4.1.5). Since we assume that $\text{supp}(\rho) \subset \mathbb{R}_{>0}$, then there exists $\epsilon > 0$ such that $\text{supp}(\rho_N) \subset (\epsilon, \infty)$, for all $N \in \mathbb{Z}_{>0}$. Hence, $C = C(\epsilon)$ is independent of N. Taking the limit to both sides yields

$$\| \lim_{N\to\infty} \tilde{z}_N(\cdot, T)\|_{L^2(G)} \leq C \int_E \| \lim_{N\to\infty} \tilde{z}_N(\cdot, t)\|_{L^2(G_0)} dt. \tag{4.14}$$

We claim that $\lim_{N\to\infty} \tilde{z}_N(\cdot, t) = \int_\Omega z(T-t, \omega; z_0) d\mathbb{P}(\omega) \in L^2(G)$, for all $t \in [0, T]$, where $z(t, \omega; z_0)$ is the state at time t of (4.2). To prove this claim, we use dominated convergence theorem for Banach space-valued functions. Since $L^2(G)$ is separable, we check that $\rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j t} e_j$ is weakly measurable in α. Indeed, let $f \in L^2(G)$ and write $f = \sum_{j=1}^{\infty} f_j e_j$. Then,

$$\left\langle \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j t} e_j, f \right\rangle_{L^2(G)} = \rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} f_j e^{-\alpha\lambda_j t},$$

which is clearly measurable. Thus, by Lemma 6.2.5, $\rho_N(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j t} e_j$ is a strongly measurable function of α. Now let $t \in [0,T]$. Then

$$\begin{aligned}\left\|\rho_N(\alpha)\sum_{j=1}^{\infty} z_{0,j}e^{-\alpha\lambda_j t}e_j\right\|^2_{L^2(G)} &= \rho_N(\alpha)^2\sum_{j=1}^{\infty} z_{0,j}^2 e^{-2\alpha\lambda_j t} \\ &\le \rho(\alpha)^2\sum_{j=1}^{\infty} z_{0,j}^2.\end{aligned}$$

Since $z_0 \in L^2(G)$ we have,

$$\|z_0\|^2_{L^2(G)} = \sum_{j=1}^{\infty} z_{0,j}^2 < \infty.$$

Hence,

$$\left\|\rho_N(\alpha)\sum_{j=1}^{\infty} z_{0,j}e^{-\alpha\lambda_j t}e_j\right\|_{L^2(G)} \le \rho(\alpha)\|z_0\|_{L^2(G)},$$

which is clearly integrable since $\int_0^\infty \rho(\alpha)d\alpha = 1$. Thus by Theorem 6.2.8 in the appendix,

$$\begin{aligned}\lim_{N\to\infty} \tilde{z}_N(\cdot,t) &= \int_{\mathbb{R}} \rho(\alpha)\sum_{j=1}^{\infty} z_{0,j}e^{-\alpha\lambda_j t}e_j d\alpha \\ &= \int_\Omega z(T-t,\omega;z_0)d\mathbb{P}(\omega) \quad \in L^2(G),\end{aligned}$$

where $z(t,\omega;z_0)$ is the state at time t of (4.2) when $\alpha\colon \Omega \to \mathbb{R}_{>0}$ is a random variable with probability density function ρ. Hence from the above, Hölder's inequality and (4.14),

$$\begin{aligned}\left\|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\right\|_{L^2(G)} &\leq C\int_E\left\|\int_\Omega z(t,\omega;z_0)d\mathbb{P}(\omega)\right\|_{L^2(G_0)}dt\\ &= \sqrt{\mu(E)}\left(\int_E\left\|\int_\Omega z(t,\omega;z_0)d\mathbb{P}(\omega)\right\|^2_{L^2(G_0)}dt\right)^{\frac{1}{2}},\end{aligned}$$

which is the desired observability inequality. □

Theorem 4.2.2. *Let system* (3.1) *be such that* α *is a random variable with Riemann integrable probability density function* ρ*. Assume* $\operatorname{supp}(\rho)\subset\mathbb{R}_{>0}$*. Then* (3.1) *is* approximately controllable in average in E*.*

Proof. From Theorem 2.3.10, all we need to prove is the averaged unique continuation property in E. Assume that

$$\chi_E(\cdot)\int_\Omega z(\cdot,\omega;z_0)d\mathbb{P}(\omega)=0 \quad \text{in } L^2([0,T];L^2(G_0)).$$

From Theorem 4.1.2 and Hölder's inequality, we obtain

$$\begin{aligned}\left\|\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)\right\|_{L^2(G)} &\leq C\int_E\left\|\int_\Omega z(t,\omega;z_0)d\mathbb{P}(\omega)\right\|_{L^2(G_0)}dt,\\ &= \sqrt{\mu(E)}\left\|\int_\Omega z(t,\omega;z_0)d\mathbb{P}(\omega)\right\|_{L^2(E;L^2(G_0))}.\end{aligned}$$

Hence,

$$\int_\Omega z(0,\omega;z_0)d\mathbb{P}(\omega)=0 \quad \text{in } L^2(G).$$

On the other hand,

$$\int_{\Omega} z(0,\omega;z_0)d\mathbb{P}(\omega) = \int_{\mathbb{R}} \rho(\alpha) \sum_{j=1}^{\infty} z_{0,j} e^{-\alpha\lambda_j T} e_j d\alpha.$$

Note that $\rho(\alpha)\sum_{j=1}^{n} z_{0,j}e^{-\alpha\lambda_j T}e_j$ is weakly measurable as a function of α, for all $n \in \mathbb{Z}_{>0}$. Since $L^2(G)$ is separable then $\rho(\alpha)\sum_{j=1}^{n} z_{0,j}e^{-\alpha\lambda_j T}e_j$ is strongly measurable. Also,

$$\left\| \rho(\alpha) \sum_{j=1}^{n} z_{0,j} e^{-\alpha\lambda_j T} e_j \right\|_{L^2(G)} \leq \rho(\alpha)\|z_0\|_{L^2(G)}.$$

Thus, by dominated convergence theorem for Banach space-valued functions

$$\int_{\Omega} z(0,\omega;z_0)d\mathbb{P}(\omega) = \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}} \rho(\alpha) e^{-\alpha\lambda_j T} d\alpha \right) z_{0,j} e_j.$$

Since $\{e_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(G)$ then it follows that

$$z_{0,j} = 0, \quad \forall\, j \in \mathbb{Z}_{>0}.$$

Hence, $z_0 = 0$. □

Chapter 5

Conclusions and future work

5.1 Summary

In this thesis, we have studied the average controllability properties of random heat equations whose positive diffusivity is a random variable. We have shown that such systems are both null and approximately controllable in average from arbitrary open sets of the domain and in an arbitrarily small time. The proof technique relies on the spectral decomposition of the average of the solutions. The desired averaged null observability inequality was built iteratively by decomposing the time set where the control acts into an infinite union of subsets. On these sets, we constructed an inequality observing an increasing number of Fourier components of the average of the solution. As a consequence of the averaged null observability equality, approximate controllability in average is shown.

5.2 Future Research Directions

The problem of averaged controllability has only been recently introduced in [16]. Consequently, there are many interesting open problems in this field; especially when the systems of interest are modelled by PDEs. Listed below are some open problems of interest:

- **Connection between average controllability and simultaneous controllability.**

  We have shown that under certain conditions, random evolution equations can be controlled in an averaged sense to a desired state. However, this does not indicate that any realization of the system will be steered to the desired state using this controller. The connection between average and simultaneous controllability for finite-dimensional systems is studied in [8]. This connection for general infinite-dimensional systems is still open.

- **Average controllability of random heat equations with general operators.**

  We have studied a small subclass of random heat equations whose randomness is introduced in a multiplicative manner to the Laplacian operator. In this case we were able to obtain eigenfunctions of the operator that were independent of the random parameter. This was crucial in constructing the desired observability inequality. It would be interesting to study a more general class of random operators.

- **Averaged controllability for nonlinear random evolution equations.**

  We have restricted ourselves to the study of a specific class of linear heat equations. One could pose the same average controllability problems for nonlinear equations.

Chapter 6

Appendix

6.1 Sobolev Spaces

Let $n \in \mathbb{Z}_{>0}$. Let $U \subset \mathbb{R}^n$ open. Denote $x = (x_1, \dots, x_n) \in U$. We denote the space of infinitely differentiable functions $\phi \colon U \to \mathbb{R}$, with compact support in U by $C_c^\infty(U)$. Let $k \in \mathbb{Z}_{\geq 0}$. Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$ be a multi-index such that $|\beta| = \beta_1 + \cdots + \beta_n = k$. We now define a weak derivative.

Definition 6.1.1. Suppose $f, g \in L^1_{\text{loc}}(U)$. We say that g is the *weak β-th partial derivative of f*, written

$$\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f = g$$

provided that

$$\int_U f(x) \left(\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} \phi \right)(x) dx = (-1)^k \int_U g(x)\phi(x) dx, \quad \forall\, \phi \in C_c^\infty(U).$$

Next we present a result on the uniqueness of weak derivatives.

Lemma 6.1.2. ([5, Chapter 5.2, Lemma 1]): *A weak β-th partial derivative of f is uniquely defined up to a set of measure zero, if it exists.*

We are now ready to define a special class of functions whose members belong to $L^2(U)$ and have weak derivatives.

Definition 6.1.3. The *Sobolev space* $H^k(U)$ consists of all $f \in L^1_{\text{loc}}(U)$ such that for each multi-index β with $|\beta| \leq k$,

$$\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f$$

exists in the weak sense and belongs in $L^2(U)$.

Definition 6.1.4. Let $f \in H^k(U)$. We define its *norm* to be

$$\|f\|_{H^k(U)} = \left(\sum_{|\beta| \leq k} \int_U \left| \left(\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f \right)(x) \right|^2 dx \right)^{\frac{1}{2}}.$$

We denote the closure of $C_c^\infty(U)$ in $H^k(U)$ by $H_0^k(U)$. Roughly speaking, a function $f \in H_0^k(U)$ "vanishes on ∂U". Since, ∂U has Lebesgue measure zero in U, we cannot assign meaning to the statement "f restricted to ∂U". Hence, we make this more precise using the following theorems:

Theorem 6.1.5. ([5, Chapter 5.5, Theorem 1]): *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator $T\colon H^1(U) \to L^2(\partial U)$ such that*

(i) $Tf = f|_{\partial U}$ if $f \in H^1(U) \cap C(\bar{U})$,

and

(ii) $\|Tf\|_{L^2(\partial U)} \leq C\|f\|_{H^1(U)}$ for each $f \in H^1(U)$.

Theorem 6.1.6. ([5, Chapter 5.5, Theorem 2]): *Assume U is bounded and ∂U is C^1. Suppose $f \in H^1(U)$. Then $f \in H_0^1(U)$ if and only if $Tf = 0$ on ∂U.*

6.2 Classical Analysis Results

We have the following classical theorems from analysis:

Theorem 6.2.1. ([14, Theorem 5.16]): *If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $\|F\|_{\mathcal{L}(X;\mathbb{R})} = \|f\|_{\mathcal{L}(M;\mathbb{R})}$.*

Theorem 6.2.2. ([5, Chapter D.3, Theorem 2]): *Let H be a Hilbert space with dual space H'. For every $F \in H'$, $\exists$ a unique $u \in H$, such that*

$$F(v) = \langle v, u \rangle_H, \quad \forall\, v \in H.$$

Definition 6.2.3. Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a real or complex Banach space. A function $f \colon X \to E$ is called *simple* if there exists $f_1, \ldots, f_n \in E$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $f = \sum_{j=1}^{n} f_j \chi_{A_j}$.

Definition 6.2.4. A function $f \colon X \to E$ is called *strongly measurable* if there exists a sequence of simple functions, $\{f_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} f_j(x) = f(x), \quad \text{a.e. } x \in X.$$

Lemma 6.2.5. ([3, Lemma A.5.2]): *A function $f \colon X \to V$ where V is a separable Hilbert space is strongly measurable if and only if $\langle f, v \rangle_V$ is measurable $\forall\, v \in V$ (i.e., f is weakly measurable).*

Definition 6.2.6. A strongly measurable function $f \colon X \to E$ is *Bochner integrable* if there exists a sequence of integrable simple functions $\{f_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty}\int_X \|f_j - f\|_E d\mu = 0.$$

In this case, $\int_X f d\mu$ is defined by

$$\int_X f d\mu = \lim_{j\to\infty}\int_X f_j d\mu.$$

Theorem 6.2.7. ([4, Chapter 2.2, Theorem 2]): *A strongly measurable function $f\colon X \to E$ is Bochner integrable if and only if*

$$\int_X \|f\|_E d\mu < \infty.$$

Theorem 6.2.8. ([1, Theorem E.6]): *Let $(X, \mathscr{A}, \mu)$ be a measure space, let E be a real or complex Banach space, and let $g\colon X \to \mathbb{R}_{\geq 0}$ be an integrable function. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of strongly measurable E-valued functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x), \quad a.e.\ x \in X,$$

where f is a strongly measurable E-valued function on X. Assume also that

$$\|f_n(x)\|_E \leq g(x), \quad \forall\, n \in \mathbb{Z}_{>0},\ a.e.\ x \in X.$$

Then f is Bochner integrable, and

$$\int_X f d\mu = \lim_{n\to\infty}\int_X f_n d\mu.$$

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