Experimenting with low threshold, high ceiling mathematics tasks in a Grade 10 mathematics class: a mixed method study on student engagement

by

Kariane Ouellet

A project submitted to the Department of Mathematics

In conformity with the requirements for

the degree of Masters of Science

Queen’s University

Kingston, Ontario, Canada

(September 2017)

Copyright ©Kariane Ouellet, 2017
Abstract

The experiment. In collaboration with the Mathematical Knowledge Network and the Fields institute, Professor Peter Taylor and his team created Math9-12, a new approach to curricular instruction following inquiry-based learning model, discovery learning and backwards design. A unit from this project was implemented in a Grade 10 Academic mathematics class in a high school in Southeastern Ontario, in May 2017. Purpose. The purpose of this study is to describe student engagement throughout the experiment. This study focuses on two components of student engagement: behavioural engagement and emotional engagement. Rationale. The phenomenon of interest is student engagement because it is a predictor of learning, performance, improvement, long-term achievement, expectations of abilities, and quality of social interactions. There is a need to explore students’ perspective on inquiry-based learning activities. Methods. To explore students’ perspective, the study relies on data from focus-group interviews and observational data. The data analysis is composed of qualitative and quantitative methods. The frameworks used throughout the analysis are modeled by Feldman Barrett Russell (1998), Pekrun & Linnenbrink-Garcia (2012), and Fredericks, Blumenfeld, & Paris (2004). Results. The statistical analysis revealed that the students felt generally negatively toward mathematics. Their favorite part of the experiment was the mentoring component, and their least favorite was the content of the unit. I found that individual engagement scores of the students were predictors of their achievement, based on the test they wrote. The qualitative findings for emotional engagement uncovered the students’ achievement emotions, epistemic emotions (linked to cognition), topic emotions and social emotions. Moreover, behavioural engagement was described in terms of academic participation, concentration, and social interactions benefiting and damaging learning.
Acknowledgements

First, I want to thank my parents, Pierre and Josée, for their love, their moral and financial support, constant encouragement, for bearing my ups and downs during these last challenging four years of university (and my whole life, really). This project could never have happened without knowing you were behind me.

I want to thank my supervisor, Professor Peter Taylor, for the wonderful opportunities he offered throughout the year. It was very inspiring, and I now know that I want a future in research.

Thanks to Professor Lynda Colgan, for the excellent qualitative research training and for being such a great inspiration. Moreover, I am grateful for the time you and Professor Glen Takahara took to review my project and assist to my defense.

Thank you to my friends and family members. Erik Drysdale, your help and support are beyond anything I could have hoped for. Anja Haltner, Audrey Roy, grand-maman Mariette, Graeme Baker, grand-papa Claude, Guillaume Ouellet, Kateri Gagnon, Lisa Häfele, Marc-Olivier Ouellet, Miriam Vallée, Myrka Fréchette, Reyana Hadeif, Reza Davari, Rohail Tariq, Sarah Burtch, Vidushi Nagarajan, Zahra Khambatý: you all supported me in some way and reminded me what I am capable off when I couldn’t see clearly. You rock.
Table of Contents

Abstract ...................................................................................................................................................................... ii
Acknowledgements ................................................................................................................................................iii
Table of Contents ................................................................................................................................................ iv
List of Figures ...................................................................................................................................................... viii
List of Tables........................................................................................................................................................ ix
Chapter 1 ............................................................................................................................................................... 1
  Purpose ............................................................................................................................................................... 2
  Context: The Experiment ............................................................................................................................. 3
  Rationale .......................................................................................................................................................... 11
  Overview of Project ..................................................................................................................................... 13
Chapter 2 ............................................................................................................................................................... 14
  Emotional Engagement .............................................................................................................................. 14
  Behavioral Engagement ............................................................................................................................. 16
  Cognitive Engagement ............................................................................................................................. 18
  Summary of Chapter ................................................................................................................................ 19
Chapter 3 ............................................................................................................................................................... 20
  Research Approach: Qualitative methods ................................................................................................. 21
  Defining the Case ..................................................................................................................................... 22
  Participants ................................................................................................................................................... 22
  Data Collection .......................................................................................................................................... 25
  Data Analysis: Quantitative and Qualitative methods .......................................................................... 27
  Ethics .............................................................................................................................................................. 35
  Summary of Chapter .................................................................................................................................. 36
Chapter 4 .................................................................................................................. 37
Preliminary Analysis .............................................................................................. 37

Statistical Analysis .................................................................................................. 43

Qualitative Interview Results ............................................................................... 49

Qualitative Observations Results .......................................................................... 57

Summary of the Chapter ....................................................................................... 60

Chapter 5 ................................................................................................................ 61
Description of Student Engagement during the Math9-12 May Experiment .......... 61

Limitations of the Study ....................................................................................... 66

Direction for Future Research .............................................................................. 67

Implications ........................................................................................................... 69

Conclusion ............................................................................................................ 70

What Does It Mean for Me..................................................................................... 71

References ............................................................................................................. 76
Appendix A ............................................................................................................ 81
Letter of Information for Grade 10 students and Parents .................................... 81

Appendix B ............................................................................................................ 84
Consent Form For Grade 10 Students’ Parents ...................................................... 84

Appendix C ............................................................................................................ 85
Consent Form For Grade 10 Students ................................................................ 85

Appendix D ............................................................................................................ 86
Letter of Information for Grade 12 Mentors and Parents .................................... 86

Appendix E ............................................................................................................ 89
Appendix R .......................... Assessment Rubric (Ontario Ministry of Education, 2005)................. 107

Appendix S ................................ Students test marks, and Individual Cognitive-behavioural, social-behavioural, and emotional engagement scores, and engagement scores................................................................. 109

Appendix T ................................ Student test – Solution (level 4)................................................................. 110

Appendix U ................................ Student tests samples................................................................. 115

Appendix V ................................ Student Workbooks: Introduction (Taylor, 2017)................................. 119

Appendix W ................................ Student Workbooks: First 2 examples (Taylor, 2017)................................. 128

Appendix X ................................ Student Workbooks: The algebra................................................................. 138
List of Figures

Figure 1: Sample of workbook question (Taylor, 2017) ................................................................. 8

Figure 2: Feldman Barrett and Russell’s Affective Circplex (1998), adapted by Pekrun and
and Linnenbrink-Gracia (2011) ........................................................................................................ 15

Figure 3: Classroom setting for the experiment ............................................................................. 24

Figure 4: Data analysis and deductive coding frameworks .............................................................. 28

Figure 5: Process and example of the coding for quantitative purpose .......................................... 31

Figure 6: Coding of raw data sample. .............................................................................................. 34

Figure 7: Number of students grouped by level of achievement. .................................................. 38

Figure 8: Percentage of students in each level for every question ................................................. 39

Figure 9: Sample solution level 1 .................................................................................................. 39

Figure 10: Sample solution level 2 .................................................................................................. 40

Figure 11: Sample solution level 3 ................................................................................................. 41

Figure 12: Sample solution level 4 .................................................................................................. 41

Figure 13: Daily percentage of positive cognitive and social behaviours ....................................... 42

Figure 14: Behavioural-cognitive vs. Behavioural-social engagement scores for ......................... 46

Figure 15: Linear model for achievement depending on engagement score .................................. 48

Figure 16: Curriculum approach development cycle ....................................................................... 73
List of Tables

Table 1: Experiment Agenda

Table 2: Number of participants every day of the experiment, and the days I observed.

Table 3: Classifying the codes

Table 4: Levels and respective percentage intervals.

Table 5: Number of positive and negative instances for Overall, Usual.

Table 6: Emotions toward mathematics estimators, standard deviation and p-value.

Table 7: Number of positive, negative and neutral instances for the various.

Table 8: Emotions toward the components – the statistics and p-values.

Table 9: Coefficient estimates and p-value for the linear model.

Table 10: Coefficients and p-value for the ordinal logistic model.
Chapter 1
Introduction

I took part in a working group called “Deep Understanding of School Mathematics” at the Canadian Mathematics Education Study Group (CMESG) in June 2017. At the beginning of the meeting, we considered the question of the difference, if any, between high school mathematics and mathematics. This group was composed of high school and CÉGEP teachers, university students and university professors. This discussion lead to the realization of an alarming reality: the mathematics taught in high school is not what mathematicians consider mathematics to be. Civil (2002) says that school mathematics is traditionally algorithmic and focused on “symbolic manipulation deprived of meaning” (p.40). While some classrooms may still be like this, others have changed. In non-traditional classrooms, students work in groups on richer mathematics problems (Civil, 2002). Since mathematician’s mathematics “deals with ill-defined problems; it requires time, persistence, and flexibility; mathematicians often refer to a certain element of playfulness in their work, of ‘messing around’ with ideas in their search for justifications, counterexamples, and so on” (Civil, 2002, p.42), it is posited that classroom practices should reflect these realities if we are to better serve students’ learning and bridge the worlds of school and mathematicians’ mathematics.

Connecting school mathematics to authentic mathematics is a big concern of mine, and another concern is that universities have been noticing that the students coming out of high school have gaps in their understanding of mathematics (Adamuti-Trache, Bluman, & Tiedje, 2013; Gueudet, 2008; Kajander & Lovric, 2005). Hence, bridging the gap between the nature of the educational experiences at the secondary and post-secondary levels is recurrent theme in recent research (Adamuti-Trache, Bluman, & Tiedje, 2013).
Professor Peter Taylor, from Queen’s University (Department of Mathematics and Statistics), thought that perhaps, if high school student would have the opportunity to experience authentic and meaningful mathematics experiences as mathematicians perceive them, they would understand and appreciate mathematics more deeply. Hence, the Math9-12 project was borne. Professor Taylor’s work was implemented in a grade 10 mathematics classroom, academic stream, in May 2017 in a school in Southeast Ontario. The experiment is detailed in the Context section of this chapter. Throughout the experiment, data was collected using observations, focus group interviews, and achievement marks.

In this chapter, I will explain the purpose of the study, which is to describe engagement in the context of the implementation of the Math9-12 unit. My focus on engagement is motivated by two factors: first, as discussed in the rationale, engagement is a predictor of achievement (Harbour, Lauren, Sweigart, & Hughes, 2015), which makes it desirable in the classroom. The second reason is rather personal: my own passion for mathematics. I want students to see the beauty and power of mathematics. As a future teacher, it is important for me that my students are engaged with this wonderful science/art. In this section, I will also describe the relevance of the study.

**Purpose**

The purpose of this study is to describe student engagement when a non-traditional approach to the curriculum is implemented in the classroom. Engagement is defined as having three components: behavioural, emotional and cognitive; however, the present study will focus on behavioural and emotional engagement only.

Behavioural engagement generally refers to observable behaviours in the classroom, such as participation, attention, and students discussing topics. Emotional engagement refers to
emotions in the academic context such as happiness or sadness, interest or boredom, and anxiety (Harbour, Lauren, Sweigart, & Hughes, 2015; Fredericks, Blumenfeld, & Paris, 2004).

The participants of this study included the classroom students and the teacher. The intent was to get the participants’ perspectives on the experiment. For this study, the data was analyzed with the purpose of describing student engagement using qualitative and quantitative methods.

The experiment, which is detailed in the context section, is a classroom implementation of the introduction of triangle trigonometry and analytic geometry at the grade 10 academic level. It is one component of a SSHRC-funded project led by Professor Peter Taylor in collaboration with the Fields Institute for Research in the Mathematical Sciences and a national research team. Observations were collected and focus group interviews were conducted in order to assess student response to the non-traditional module, and apply it to student engagement measures.

For the quantitative part of the study, the data was coded deductively and analyzed for emotional engagement using the framework Feldman Barrett and Russell’s (1998) Affective Circumflex Model, and for behavioural engagement using the Pekrun and Linnenbrink-Garcia (2012) framework. For the qualitative phase of the study, emotional and behavioural engagement were described following a deductive and inductive coding procedure, the first one based on the model by Pekrun and Linnenbrink-Garcia (2012), and the second one based on the model by Fredericks, Blumenfeld, and Paris, (2004).

**Context: The Experiment**

The study focused on student engagement in the context of the implementation of a non-traditional approach to instruction and curricular expectations. This is referred to as the experiment throughout the study. This section is centred on the description of the experiment.
Rationale for the experiment

The Math9-12 project is part of the Critical Transition Community of Practice, from the Mathematical Knowledge Network (Mathematics Knowledge Networks, 2016). The MKN is a KNAER Phase II Project hosted by the Fields Institute for Research in Mathematical Sciences, and is funded by/funded in part by the Ontario Ministry of Education. Math9-12 focuses on the transitions that have proven difficult for students, and in this case, the secondary to post-secondary transition. It is intended to bring new life and sophistication to the current classroom curriculum (Taylor, 2017) by giving high school students the opportunity to discover and play with mathematical ideas and processes that are typically introduced at university level. The units of Math9-12 are engineered to meet a set of curriculum expectations with respect to each high school year and the level of technical ability of the students. The units are built around what the Math9-12 team calls “Powerful Stories.” These are rich and sophisticated mathematical topics which embody many of the specific content expectations mandated in current curriculum policy documents, but are presented as a large cumulative task. They are “Low Floor High Ceiling” tasks (or Low Threshold High Ceiling), which is defined as: “a mathematical activity where everyone in the group can begin and then work at their own level, yet the task also offers lots of possibilities for learners to do much more challenging mathematics too” (McClure, Woodham, & Borthwick, 2011). The following subsections will describe the curriculum expectations that the unit was designed to meet, the instructional models, the content, and the mentoring component of Math9-12.
Curriculum Expectations

The content of the units is based on the existing Ontario Curriculum. Currently, the grade 10 Ontario curriculum involves three areas of study: quadratic relations, triangle trigonometry and analytic geometry. In the Academic stream, the students are expected to:

- Determine properties of quadratics, relate the transformation of the basic relation $y = x^2$ to the complete form $y = a(x - h)^2 + k$, solve and interpret quadratic equations, and solve problems involving quadratics. (Ontario Ministry of Education., 2005, p.47)

- Solve problems about similarities and similar triangles using their knowledge of ratios and proportions, solve problems involving right triangle by applying the Pythagorean theorem and trigonometric ratios, and solve problems using the sine and cosine laws. (Ontario Ministry of Education., 2005, p.49)

- Model and solve problems involving the intersection of straight lines, the properties of line and line segments, and verifying the geometric properties of triangles and quadrilaterals. (Ontario Ministry of Education., 2005, p.51)

In the transformation unit of Math9-12, the greater task is for students to learn how to perform transformations of shapes in a plane using geometry and matrices. The unit addresses curriculum expectations from both the triangle trigonometry and analytical geometry strands. The specific expectations that were used to design the unit were:

Analytic Geometry: Using Linear Systems to Solve Problems

- solve problems that arise from realistic situations described in words or represented by linear systems of two equations involving two variables, by choosing an appropriate algebraic or graphical method (Ontario Ministry of Education., 2005, p.49)
determine, through investigation (e.g., using dynamic geometry software, by paper folding), some characteristics and properties of geometric figures. (Ontario Ministry of Education., 2005, p.49)

verify, using algebraic techniques and analytic geometry, some characteristics of geometric figures. (Ontario Ministry of Education., 2005, p.50)

plan and implement a multi-step strategy that uses analytic geometry and algebraic techniques to verify a geometric property. (Ontario Ministry of Education., 2005, p.50)

Trigonometry: Solving Problems Involving the Trigonometry of Right Triangles

determine the measures of the sides and angles in right triangles, using the primary trigonometric ratios and the Pythagorean theorem. (Ontario Ministry of Education., 2005, p.51)

Additionally, the unit was designed to support the development of students’ spatial reasoning and computational thinking in response to the inquiry-based nature of the multi-step problems that comprised the unit.

**Instructional Models**

All Math9-12 units are based on the following models: inquiry-based learning, discovery learning, and backward-design.

**Inquiry-based learning**

Inquiry-based learning focuses on having students work in the same way professionals work (Doirer & Maass, 2014). For mathematics, this means that students are working like mathematicians and scientists in the classroom. That is, given a problem, they are to use their creativity and knowledge, and scientific processes to come up with solutions. This model
encourages in-class collaboration, and discussions (Doirer & Maass, 2014), which is aimed at increasing student engagement in the classroom, specifically with large tasks presented to them.

Moreover, Ontario’s Ministry of Education (2013) encourages teachers to adopt this model, as it is consistent with the Universal Design for Learning (UDL). The UDL generally aims at accommodating every style of learning and every exceptionality in an equitable manner.

**Discovery learning**

Discovery learning is a model which focuses on the student’s curiosity. The students are encouraged to use their curiosity and problem-solving skills to solve tasks instead of being shown the process of solving the problem and have them replicate the process (Wallace, 2015). Discovery learning is a model in which the students are engaged with the material in order to gain understanding of it (Wallace, 2015). In this model, the teacher acts as a facilitator rather than an instructor.

**Backwards-design model**

The backwards-design model is a strategy for creating classroom curriculum based on the required expectations. The key idea is while designing lesson plans, the focus should be on a larger goal and the students should be aware of what they are expected to know or realize throughout the unit. To write a unit plan, this model is composed of three main steps: consider the goals, find out what assessments are needed in order to verify that the goals are met, and decide on the most appropriate learning activities (Wiggins & McTighe, 2005). This model contrast the other intuitive way to create lesson plans which is to create learning activities and then design the assignments based on the expectations.
Content

The transformations unit began by introducing three basic transformations: horizontal and vertical dilation, horizontal shear, and rotation. Once the students were given a short lesson about how the transformations work, they were given a question. As demonstrated in Figure 1, a worksheet had four graphs: the first one had a square, and the three other images resembled the original square, but transformations had been applied. The students’ task was to determine the transformations used at each step, and the values of the parameters (e.g., how much was it dilated).

The students found the parameters using geometric reasoning at first. It involved, among other things, the Pythagorean theorem, trigonometric ratios, and spatial reasoning.

Figure 1: Sample of workbook question (Taylor, 2017)
The students worked on a few problems similar to this one, after which we introduced another method: algebraic solution. The algebraic method involved having matrices to represent each transformation:

\[
[D(a, b)] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad [R(\theta)] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad [S(h)] = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\]

The students were first introduced to matrices with basic examples, and then they practiced matrix multiplication. Through inquiry, the students found the matrix representation of the three basic transformations, and they were taught how these can help them to solve for the parameters. That is, if \( T = R(\theta) \circ S(h) \circ D(a, b) \), the matrix representation of \( T \) is: \([T] = [R(\theta)] \cdot [S(h)] \cdot [D(a, b)]\). The matrix for \( T \) is obtained by looking at the image (i.e., on the fully transformed square) of the lattice points (1,0) and (0,1) from the original square. The students were left with four equations and four unknowns (the parameters that they ultimately want to know), which they can solve. (Taylor, 2017)

For the purpose of referring to specific elements of the content later in the text, Table 1 provides a brief timeline of the experiment. Note that students were progressing at their own pace, so they were not working on the same examples necessarily. Parts of the workbook can be found in Appendices V to X.

\[\text{Table 1: Experiment Agenda}\]

<table>
<thead>
<tr>
<th>Date</th>
<th>Agenda</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>Monday, May 1, 2017</td>
</tr>
<tr>
<td></td>
<td>▪ Introduction (Researchers, mentors, students).</td>
</tr>
<tr>
<td></td>
<td>▪ Linear transformations of points and lines</td>
</tr>
<tr>
<td>Day 2</td>
<td>Tuesday, May 2, 2017</td>
</tr>
<tr>
<td></td>
<td>▪ The basic transformations: Dilatation, Rotation. Professor Taylor teaches them.</td>
</tr>
<tr>
<td></td>
<td>▪ The students practice.</td>
</tr>
<tr>
<td>Day 3</td>
<td>Wednesday, May 3, 2017</td>
</tr>
<tr>
<td></td>
<td>▪ Students keep working on basic transformation.</td>
</tr>
<tr>
<td></td>
<td>▪ Teach them the shear</td>
</tr>
<tr>
<td>Day 4</td>
<td>Thursday, May 4, 2017</td>
</tr>
<tr>
<td></td>
<td>▪ Students work with the application on iPads. The application allows them to input values for dilation, rotation and shear, and see what the result is.</td>
</tr>
<tr>
<td>Day</td>
<td>Date</td>
</tr>
<tr>
<td>-------</td>
<td>---------------------</td>
</tr>
<tr>
<td>5</td>
<td>Friday, May 5, 2017</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Monday, May 8, 2017</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Tuesday, May 9, 2017</td>
</tr>
<tr>
<td>8</td>
<td>Wednesday, May 10, 2017</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Thursday, May 11, 2017</td>
</tr>
<tr>
<td>10</td>
<td>Friday, May 12, 2017</td>
</tr>
<tr>
<td>11</td>
<td>Monday, May 15, 2017</td>
</tr>
</tbody>
</table>

**Mentoring**

Math9-12 includes a mentoring component. In the context of this particular implementation, the 10th-grade students were mentored by 12th-grade students. The potential benefit of such a structure is to offer more support to the younger students in the learning process, and provide the mentors with the opportunity to consolidate their understanding through teaching. The widespread implementation of such a structure in schools has yet to be established.

In the experiment, the mentoring component was included. However, given the early stages of the implementation of Math9-12, three master’s students who were already familiar with Math9-12 and also wished to study it also entered as mentors (and researchers). The teacher
and Professor Taylor also took on mentoring roles in addition to the grade 12 students who volunteered to be mentors in the classroom.

The role of a mentor is that of facilitator. The mentors are to be accessible to the students as they are working on the problems — providing direction, and answering their questions. The mentors are not assigned to a particular student, but rather, rotate, keep track of progress and answer questions.

**Rationale**

In order to evaluate the success of the experiment described in the previous section, one indicator is the level of engagement of students. As suggested by Lee (2014), the research community should focus on enhancing student engagement, since his findings suggest that this is a strong predictor of academic performance. In fact, engagement is one of the most well-established factor of student achievement (Harbour, Lauren, Sweigart, & Hughes, 2015).

Reeve (2012) enumerated three functions of engagement, and one of them is that it allows learning. Appleton (2008), after reviewing several studies on engagement of the previous 15 years, concluded that engagement is a “richer-get-richer” phenomenon. Students that are engaged perceive more support from their teacher and peers, which contributes to increase their level of engagement. Finn (1993) found that engagement (behaviour, participation, attendance) was strongly correlated to achievement across all genders, ethnicities, and socio-economic status. Engagement is a predictor of high school dropout, and can be used as a model to capture the gradual process which leads to dropout (Finn J. D., 1989). Overall, engagement is a predictor for learning, performance, improvement, long-term achievement, expectations of abilities, and quality of social interactions (Furrer & Skinner, 2003). In fact, emotional engagement itself is “recognized as being of critical importance for students’ academic learning, achievement,
personality development, and health…” (Pekrun & Linnenbrink-Garcia, 2012, p.260). When studied on its own, behavioural engagement itself is a strong predictor of achievement (Gregory, Allen, Mikami, Hafen, & Pianta, 2014).

Data from the National Research Council and Institute of Medicine (2004) supported decline in motivation and engagement across middle and high school. “Adolescents are too old and too independent to follow teachers’ demands mindlessly, and many are too young, inexperienced, or uninformed to fully appreciate the value of succeeding in school.” (National Research Council and Institute of Medicine, 2004, p.211) The same report mentions that schools that succeed in engaging their students provide challenging instruction and convey high expectations for the students’ success. The evidence and suggestions from this report support the need for an approach that challenges students and may serve to reverse the trend of the declining engagement.

As for student engagement in the context of inquiry-based learning, there is a need for “more in-depth research on the student learning experience from the students’ perspective.” (Buchanan, Harlan, Bruce, & Edwards, 2016, p.34) Fielding-Wells et al. (2017) mentioned that little research has been done on the impact of inquiry-based learning on student engagement and motivation. Given the need for understanding the students’ perspectives, qualitative research is the most appropriate approach to fill the gap (Buchanan, Harlan, Bruce, & Edwards, 2016).

In order to better understand students’ perspective, I conducted focus group interviews. This type of interview is appropriate when the participants share a common experience (Yin, 2011; Patton, 2002), which is the case here since they all participated to the Math9-12 experiment. I decided to conduct this type of interview because Yin (2011) suggests that youngsters and children more readily express themselves when they are part of a group as opposed to one-on-one with an interviewer.
For this research, I used classroom observations to study students’ behaviour. Patton (2002) mentioned that observational data has for purpose to describe the setting, the activities that took place and the people who participated. Observations allow the researcher to understand the interactions between people with respect to the context. Because I want to describe behavioural-engagement; the setting, the activities and the people are all important to draw an accurate and complete picture of student engagement. For this reason, according to Patton (2002), observations are an appropriate form of data in the context of this study.

Therefore, given that all the previously mentioned studies support the importance of engagement, it is beneficial to describe students’ engagement and include their perspective during the experiment under study to get insight on the success of the unit. Moreover, we may find ways to improve Math9-12 by identifying specific factors that seemed to enhance or diminish student engagement during the experiment.

**Overview of Project**

The purpose of the project is to describe student engagement throughout the experiment. In terms of engagement, I focused on behavioural and emotional engagement. The experiment is the implementation of a Math9-12 unit in a grade 10 mathematics classroom, academic stream, for a duration of 2 weeks. Professor Taylor, from Queen’s University, designed the unit. It is relevant to evaluate engagement, because it is an important predictor of achievement. There is a need for new strategies to ease the transition from high school to university, and Math9-12 is a possible solution. Therefore, I wished to describe student engagement in the context of this approach. There is a gap in research when it comes to describing student learning experience from their perspective, so this research focuses on students’ perspectives.

In the following chapter, we proceed to review the literature about student engagement.
Chapter 2

Literature Review

In this chapter, I focused on establishing the theoretical frameworks for the study. The focus of the study is student engagement, so I establish sub-focuses based on the literature to guide the analyses with the aim of having an extensive image of student engagement in the experiment.

Engagement is a concept that has been defined with several perspectives by researchers throughout the past thirty years.

Fredricks (2004) suggests that engagement is a metaconstruct that includes multiple components. Several authors pointed the different definitions of engagement, and the multiplicity of models (Appleton, Christenson, & Furlong, 2008; Fredericks, et al., 2011; Lee, 2014). Many researchers rely on the three-component model: behavioural, emotional, and cognitive engagement. Behavioural engagement generally refers to student participation (academically, socially and in extra-curricular activities) Emotional engagement refers to students’ reactions to the work, teacher, classmates, and environment. Cognitive engagement is linked to students’ motivation and investment in their own learning (self-regulation) (Fredericks, Blumenfeld, & Paris, 2004, p.60). In addition, other researchers contributed to extrapolating those definitions.

Emotional Engagement

The emotional component refers to the affective reactions of students in the academic context. Fredericks (2004) enumerates emotions such as happiness or sadness, interest or boredom, and anxiety.

The horizontal axis is called the *valence*, which means negative/positive, or unpleasant/pleasant emotions. The vertical axis represents activation and deactivation. Activation refers to physiologically activating emotions (e.g. increasing heart rate). Those two dimensions create a two-dimensional subspace which allow us to represent emotions in a cartesian way.

In addition to this model, emotions can be grouped with respect to the object of focus, which is student engagement, in this case. *Achievement emotions* is a group that includes emotions when students may have been working on a task (enjoyment, boredom, anxiety, hopelessness) and related to the outcome (pride, hope, shame). *Epistemic emotions*, another group of emotions in the context of engagement, is composed of emotions such as surprise and curiosity. It is linked to cognition, and those emotions can be felt during problem solving.

“A typical sequence of epistemic emotions induced by a cognitive problem may involve (1) surprise, (2) curiosity and situational interest if the surprise is not dissolved, (3) anxiety in case of severe incongruity and information that deeply disturbs existing cognitive schemas, (4) enjoyment and delight experienced when recombining information
such that the problem gets solved, or (5) frustration when this seems not to be possible (also see Craig, D’Mello, Witherspoon, & Graesser, 2008 ).” (Pekrun & Linnenbrink-Garcia, 2012, p.263)

Topic emotions are the emotions triggered by the content of lessons, and are linked to interest and motivation. Finally, social emotions are the group of emotions linked to other people in the social context. For example, admiration, envy, empathy, love/hate toward other classmates or the teacher, etc.

The importance of the emotional component in engagement is not negligible, as emotions might have an effect on efforts, motivation, persistence, and learning strategies. Studies found that affect (positive vs negative emotions) “influences a broad variety of cognitive processes that contribute to learning, such as perception, attention, social judgment, cognitive problem-solving, decision-making, and memory processes…” (Pekrun & Linnenbrink-Garcia, 2012, p.263).

Behavioral Engagement

Behavioural engagement is based on participation. For example, it includes participations in academic, social, and extracurricular activities. It is essential to achieve positive academic outcome, and to prevent school dropout (Finn 1989).

Fredericks (2004) enumerates three definitions of behavioural engagement:

- Positive conduct and negative conduct (following or disobeying rules, adhering to classroom conduct or skipping school, etc.) (Finn, 1993)
- Behaviour related to academic tasks: participation, efforts, concentration, attention, etc.
- Participation in extra-curricular activities.
Pekrun & Linnenbrink-Garcia (2012) note that when it comes to behavioural engagement, there is an emphasis on quantity, rather than quality. For example, someone that puts in a lot of effort would be considered someone who is behaviourally engaged. Pekrun & Linnenbrink-Garcia (2012) suggests that students’ behaviour is not independent of the emotional component. In fact, the mood and feelings have a great influence on one’s behaviour. Positive activating emotions are correlated to positive effort, and negative deactivating emotions are correlated with negative efforts. Pekrun & Linnenbrink-Garcia (2012) also define sub-categories of behavioural engagement, and two of them are called cognitive-behavioural engagement and social-behavioural engagement. The former focuses, among other things, on student’s behaviour when it comes to problem-solving.

“Experimental evidence suggests that positive mood promotes flexible, creative, and holistic ways of solving problems and a reliance on generalized, heuristic knowledge structures (Fredrickson, 2001). Conversely, negative mood has been found to promote focused, detail-oriented, and analytical ways of thinking.” (Pekrun & Linnenbrink-Garcia, 2012, p.267)

In addition, possible biological theories have been suggested, linking mood with dopamine levels which impact the students’ flexibility of problem solving strategies.

Social-behavioural engagement, on the other hand, refers to social behaviours in the academic context. For example, participation with peers and higher social interactions (Pekrun & Linnenbrink-Garcia, 2012; Linnenbrink-Garcia & Pekrun, 2011) are indicators of positive social-behavioural engagement. Therefore, it focuses on students’ behaviours when engaging with their peers, such as discussing, listening to others’ ideas, and supporting peers’ learning in a respectful and cohesive manner. In terms of emotions, a study suggested that emotions such as happiness and calmness are associated with those behaviours (Linnenbrink-Garcia & Pekrun, 2011).
Negative emotions are associated with the opposite behaviours, and also, in small-groups settings, with a detachment from the group and the task: students tend to let their peers do all the work.

**Cognitive Engagement**

Cognitive engagement is the level of student’s investment in the learning process. Fredericks (2011) explains that there are two ways to be cognitively engaged: by being purposeful and thoughtful when doing a task, and by being willing to put the necessary effort to understand complex ideas and develop mastery skills.

Appleton (2008) and Fredericks (2004) add that self-regulation is a key skill in cognitive engagement. The Ontario Ministry of Education (2010) has self-regulation as one of the learning skills that students need to develop from grade 1 to 12, and they define it as:

The student:

- sets own individual goals and monitors progress towards achieving them;
- seeks clarification or assistance when needed;
- assesses and reflects critically on own strengths, needs, and interests;
- identifies learning opportunities, choices, and strategies to meet personal needs and achieve goals;
- perseveres and makes an effort when responding to challenges

Fredericks (2004) mentioned that cognitively engaged students prefer more challenging tasks. They are flexible when it comes to problem solving, and they cope with failure in a positive way (Fredericks, 2004). She also mentions that some authors in the learning literature link this component of engagement to the student’s capacity at metacognition. Metacognition is
simply the process of assessing its own understanding. They also manage to control their effort by persisting and by supressing distractions in their environment.

Summary of Chapter

In this chapter, engagement was defined has having three components: emotional, behavioural, and cognitive. Feldman Barrett and Russell’s (1998) Affective Circumplex model can be used to understand emotional engagement in terms of valence (positive/negative) and activation. Moreover, Pekrun and Linnenbrink-Garcia (2012) grouped engagement emotions as academic, epistemic, topic and social. In terms of behavioural engagement, students can have positive and negative contact. It can be describe by student’s participation, attention, concentration and attendance. Pekrun and Linnenbrink-Garcia (2012) also identified two components of behavioural engagement: cognitive-behavioural, which is the students’ behaviour when they are solving problems, and social-behavioural, which is about students’ social interactions.

The following chapter focuses on the methodology for the study.
This study is a qualitative dominant mixed method, which “relies on a qualitative, constructivist-poststructuralist-critical view of the research process, while concurrently recognizing that the addition of quantitative data and approaches are likely to benefit most research projects.” (Johnson, Onwuegbuzie, & Turner, 2007, p.124) The constructivist view implies that the researcher recognizes that their own experiences and perceptions shapes their views, and they position themselves regarding the research based on this (Creswell, 2013). The poststructuralist view implies that “emphasis is placed on identifying meanings that are context specific and that relate to the varying discursive practices operating.” (Given, 2008) This study is qualitative dominant because it “calls for an exploration of a phenomenon; relies on the views of participants” (Plano Clark & Creswell, 2015) and it relies on qualitative data. The analysis, on the other hand, will use quantitative and qualitative processes with the aim having a deep exploration of engagement in the present context.

This chapter’s focus is to describe the methodology implemented for the study. The study is predominantly qualitative, so the first part is focuses on this research approach. Then, the case study is defined; I describe the sampling method and provide information about the demographics of the school and the participants. I discuss the data collection, which includes focus group interviews, field observations, and summative test scores. Finally, I describe the quantitative and qualitative methods that were used to analyze the data. Trustworthiness and credibility will be addressed throughout the chapter.
Research Approach: Qualitative methods

This research was conducted as a qualitative case study which triangulated through a complementary statistical interpretation of the qualitative data. Creswell (2013) defines a case study as a bounded system which the researcher explores in an in-depth manner. It is the appropriate research design because the experiment represents a unique case and it deserves to be studied on its own right (Yin, 2011). Moreover, a case study has the ability to explain “how” and “why” an experiment was a success or not, which is not something that’s possible to find out otherwise (Yin, 2009). For this study, the goal was to describe student engagement in the context of Math9-12. To describe student behaviours and interpret student responses, a case study was determined to be the most beneficial methodology (Merriam, 1991). There was a focus on an instrumental case, which Hamilton and Corbett-Whittier (2013) described as an intentional study of one particular aspect of a case. A descriptive approach was adopted for this case study because the aim of the researcher was to capture a “moment in time” picture of the case under examination (Hamilton & Corbett-Whittier, 2013). To obtain the necessary first-hand behavioural and attitudinal data from participants, it was essential to conduct multiple observations and carry out structured focus group interviews were. Creswell (2013) identifies the circumstances for which focus groups are proven to be advantageous. Since the interviewees will already be cooperating with one another in the context of the case, and because the participants are adolescents who could possibly be reluctant to share their thoughts in a one-on-one interview, Creswell (2013) and Yin (2011) would suggest that focus groups would be appropriate in order to get valuable information.
Defining the Case

To conduct a case study, a case first needs to be identified. A case can be defined as a bounded system (Creswell, 2013), which implies that it is bounded in time and in place. For the present study, the system is the Math9-12 May 2017 experiment. It is bounded in place because the experiment is limited to one classroom in one school and it has a limited number of participants; all the students in the classroom. It is bounded in time because the experiment takes place from May 1 to May 15.

Therefore, this study is interested in describing the case of the Math9-12 experiment in terms of student engagement.

A complete case study of the implementation of the Math9-12 units, which could include, for example, data collection on student achievement, on the long-term effects on achievement, long term effects on student engagement with mathematics and comparison with the “traditional” approach could provide valuable information about the implementation of this approach. However, the study is limited in terms of participants, time, and researchers, which is why there is a focus only on the participants’ perceptions and behaviour within the framework of emotional and behavioural engagement.

Participants

Convenience sampling is the method that was chosen to select participants for the present study. Plano Clark and Creswell (2015) mention that convenience sampling is to select participants based on the fact that they are available and accessible. This method works for the study, because a class had already been chosen by the principal investigator and the grade 10 math teacher of the school. The teacher had been collaborating for many years with the creator of Math9-12, Professor Taylor, and is eager to try the new approach.
The school is located in a mid-size city in Southeastern Ontario. The grade configuration is 9 to 12, and the average number of students enrolled from 2011-2015 is 923 students per year. In 2015, the average class size was 22.07 students. The school is publicly funded. The school will celebrate its 225\textsuperscript{th} anniversary in October 2017, it is considered the oldest public school in Ontario, and the second oldest in Canada. In 2012, the Fraser Institute ranked in the top 10\% of public school in Ontario in terms of performance. It is the top performing school in its schoolboard.

The student participants to the experiment are from the math teacher’s class and the mentors are grade 12 students from the school with spare time during the grade 10 math class. These mentors volunteered to participate in the Math9-12 unit. The Table 2 is a record of how many participants were present each day that the study was conducted.

Note that the mentors in this table include the grade 12 students present, the graduate students that were not collecting observational data, the classroom teacher and Professor Taylor.

\textit{Table 2: Number of participants every day of the experiment, and the days I observed.}

<table>
<thead>
<tr>
<th>Day (Day of the week)</th>
<th>Number of students</th>
<th>Number of mentors</th>
<th>Myself as observer</th>
<th>Day (Day of the week)</th>
<th>Number of students</th>
<th>Number of mentors</th>
<th>Myself as observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (M) N/D</td>
<td>6</td>
<td>√</td>
<td></td>
<td>6 (M) N/D</td>
<td>5</td>
<td>N/D</td>
<td></td>
</tr>
<tr>
<td>2 (T) 19</td>
<td>6</td>
<td>6</td>
<td>√</td>
<td>7 (T) 18</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3 (W) 18</td>
<td>4</td>
<td></td>
<td></td>
<td>8 (W) 17</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (Th) 21</td>
<td>5</td>
<td></td>
<td></td>
<td>9 (Th) 16</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 (F) 19</td>
<td>5</td>
<td>5</td>
<td></td>
<td>10 (M) N/D</td>
<td>N/D</td>
<td>N/D</td>
<td></td>
</tr>
</tbody>
</table>

On the day that the biggest number of students was recorded (Day 3, 21 students), the ratio of female:male students was 8:13. The three grade 12 mentors were male, and the three graduate student mentors were female. The regular classroom teacher is a male.
On the 5th day, the researchers could not be present, so the teacher carried on the experiment without our support and no data was collected. On the 10th day, the students wrote a test.

The class was at 8h30 in the morning, every weekday. The classroom was equipped with desks and chairs for the students, two boards on two perpendicular walls, a white screen and projector carried on a chariot. See Figure 3 for the configuration. The students’ desks were grouped by 4 or 6 to favour collaboration. The desks were placed as shown in Figure 3 for the purpose of the experiment. Prior to it, the desks were facing the board, and grouped in pair or trio.

For the purpose of the experiment, the students were provided with workbooks which contained a series of problems to work on during the class time. The student workbook in presented in Appendices V to X. On one occasion (Day 4), iPads were provided to them in order to use an application that was developed by the Math9-12 team as a complementary tool to help with the problems. Otherwise, no technology was used.
Data Collection

The data for this study is in the form of observations, interview transcripts, and written tests. They were collected with the intention to provide data for multiple studies, rather than this one only. The data was used in distinct separate ways for unique purposes by each researcher.

Focus Group Interviews

The interview data was gathered through in-person, semi-structured interviews. The semi-structured format is beneficial because it allowed the interviewers to pursue leads that arose throughout the interview that could potentially provide data for one or more of the several studies the data to which the data could potentially apply. In this study, the semi-structured format allowed me to ask for more details about the students’ emotions, which would not be possible in a structured format. The interview was introduced with an opening activity. The participants were given white adhesive labels, and asked to write words that represented mathematics for them, and then post their label on a Bristol board to create a word cloud. Throughout the interviews, the participants answered the questions and probes of the interviewers, and also confirmed and added insights to other participants’ responses. They were asked to answer with respect to their past experiences in mathematics classes and with respect to the Math9-12 experiment.

The participants of the focus group interviews came on a volunteer basis. I was aiming for a maximum of 4-6 participants for each focus groups, so two interviews were held on two different days. The two interviews happened when the experiment was completed, and most students had written the final test. The first session (I1) had 2 interviewers (female graduate students) and 4 participants (2 females, 2 males), and it lasted 40 min. The second interview (I2) had 2 interviewers (female master’s students) and 6 participants (1 female, 5 males), and lasted 36 min.
Observations

The participants recorded in Table 2 earlier in this chapter were also the subjects of the observations. Four observation foci were used during the observation periods: large group (all the participants), small group (4-6 students), mentors, and instructor. The data provided by the mentors and instructor observation were not used for this study. When the focus was the large group, the other researchers and I recorded observations about students’ attention, focus on task and what the mentors were doing. When we would look at one specific small group, we would observe:

- the interactions among the students,
- the nature and duration of collaboration (if any),
- if they were distracted and by what,
- on what they were working, and
- how they would interact with mentors.

A total of three graduate students observed throughout the experiment, rotating each day so that only one would be observing at the time, meanwhile the other two acted as mentors. Table 2 indicates when I was personally observing. The focus was chosen by the active observer. The observers were nonparticipant, which is defined to be an observer that is an outsider of the group that is being studied, who takes notes from distance (Creswell, 2013). The researcher would record descriptive and reflective notes as observed, and the notes would be transcribed into thick and rich narrative form as soon as possible after class, following Creswell’s (2013) protocol. The transcribed observations would be shared among researchers. The observations were analyzed to find incidents of behavioural engagement.
Test marks

On the last day of the experiment, the students wrote a test composed of 4 questions that reflected what they were taught throughout the experiment. The test was designed by Professor Taylor (see Appendix S). The students had the full period to write the test (70 min). It was open-book, which meant that they had access to their workbooks and the examples on which they had previously worked. No mentors were present, and only Professor Taylor and the teacher were there to supervise the examination. The data is in form of test marks.

Data Analysis: Quantitative and Qualitative methods

The study is mixed methods because it is a case study that utilized data and data analysis methods that were qualitative and quantitative. Researchers have used mixed methods for a variety of reasons, among them triangulation (which serves to validate results using different methods) (Johnson, Onwuegbuzie, & Turner, 2007), to explore or discover the main topics emerging from interviews, and to measure similarities between interviews (Sbalchiero & Tuzzi, 2016). In addition, quantitative methods for data analysis can be employed if the research question calls for hypothesis testing (Sbalchiero & Tuzzi, 2016).

In this study, mixed methods were used for means of triangulation and hypothesis testing. Hypothesis testing was used to determine if the students’ emotions towards mathematics were significantly negative and what component of the experiment generally stood out in terms of being least and most positive relative to students’ emotional engagement. Triangulation is a validation process that “involves corroborating evidence from different sources to shed light on a theme or perspective” (Creswell, 2013, p.200). In this case, triangulation was used because two types of data analysis methods were employed in order to validate the results and allow for a broader picture of student engagement in the context of the experiment.
The first step in the process of analyzing the interviews and the observations, for both qualitative and quantitative purposes, was coding. The coding process “involves aggregating the text or visual data into small categories of information, seeking evidence for the code from different databases being used in a study, and then assigning a label to the code.” (Creswell, 2013, p.153) Several iterations of coding were done in order to identify themes, and increase the accuracy of coding and the thoroughness of data analysis. This was for both the interviews and the observation, for the quantitative and qualitative purposes.

*Figure 4: Data analysis and deductive coding frameworks*

**Interviews**

The audio-recording from the focus group interviews with students was transcribed verbatim by the researchers upon completion. This resulted in two transcripts, the first having 20 pages, and the second 15 pages.
Quantitative purpose.

The coding is a qualitative procedure, but I used it for the quantitative methods as well in order to compile and quantify the data. The process was deductive because the codes were categorized following Feldman Barrett and Russell’s (1998) Affective Circumplex model adapted by Pekrun and Linnenbrink-Garcia (2012). Because the amount of data is relatively small, I used a simpler version of the model which simply takes into account positive and negative emotions. I was interested in knowing the students’ feelings with respect to mathematics, the way they usually feel in mathematics class, and their responses to the experiment. Thus, I used these factors deductively to organize the data. I was also interested to know what emotions students experienced with respect to novel and different components of the experiment (e.g. topics, mentors). These informed the final round of coding. Once the coding was complete, the number of instance in each theme was calculated, and used to do hypothesis testing. The questions that were answered through the mean of hypothesis testing were:

- Are the emotions of students towards mathematics significantly positive or negative?
- Are the emotions of students with respect to how they are usually taught significantly positive or negative?
- What component of the experiment do the student feel the most positive and least positive about?

Qualitative purpose

For the qualitative analysis, the coding process followed Creswell (2013) and is both deductive and inductive. The deductive coding strategy consist of using Pekrun & Linnenbrink-Garcia’s (2012) groups of engagement emotions in four categories: Achievement, Epistemic,
Topic and Social emotions. It is deductive because the themes are constantly being checked against the data (Creswell, 2013). Each category had the following number of codes: Academic (21), Epistemic (36), Topic (40), and Social (55), for a total of 152 codes. Inductive coding implies starting with raw data from the multiple sources and broadens to several themes. Therefore, once grouped with respect to the categories, the data was coded inductively to identify specific emotions (e.g., confusion, interest) and to link the context to emotions. Following this coding, themes emerged and are later described in Chapter 4.

**Observations**

The observations were analyzed to provide both quantitative and qualitative data. There was a total of 19 pages of single-spaced field notes, or an average of 2 pages per day of observations.

**Quantitative Purpose**

No hypothesis testing was done on the observation data. However, I sought to determine if engagement was a predictor of achievement in the context of this experiment. To develop the individual engagement scores (IES) of each student, I needed to have quantitative data about behavioural engagement. The first step was to code the data deductively according to cognitive-behavioural and social-behavioural engagement as defined by Pekrun & Linnenbrink-Garcia (2012). That is, I first coded the students’ behaviour in the context of problem-solving, and relative to social interactions. The next iteration coded the data as positive or negative, which ultimately resulted in 4 major themes:

- Positive Cognitive-Behavioural Engagement
- Negative Cognitive-Behavioural Engagement
- Positive Social-Behavioural Engagement
Negative Social-Behavioural Engagement

The number of instances in each theme was used to calculate the cognitive-behavioural and the social-behavioural scores of each student. From the interview data, I used the number of positive emotional engagement instances and negative emotional engagement instances in the “experiment” theme to calculate the emotional engagement score. From these three score, I took the average and obtained the individual engagement scores (IES) of each student.

Figure 4 is an example of the process used to obtain the individual engagement scores.

Figure 5: Process and example of the coding for quantitative purpose.
In this example, although the student got lost at when he was working on the problem, according to the framework, it’s considered positive behaviour because he recognized that he didn’t understand something, and he reached out to get help.

Later in this chapter (Example of the Process to determine the IES), there is an example of this process using a larger section of the observational data, which I coded and got the behavioural engagement scores.

Note that, in terms of credibility, this method has weaknesses. The number of instances used to calculate the scores varied considerably among students because not all students were observed for the same amount of time, and in the interviews, they didn’t share their thoughts equal amount of times. Given that the scores were established based on existing data, which was limited and mostly qualitative, there was no existing instrument (e.g. the National Survey for Student Engagement) I could use to attribute the scores. Typically, such scores are established based on quantitative data, such as surveys, or very strict observation protocols (Fredericks, et al., 2011). Conversely, the observation foci were decided at random, and each group had the full attention of an observer for approximately the same amount of time. Hence, this gives some reliability to the evidence about student engagement, which is sufficient to answer the purpose of the study.

**Qualitative Purpose**

The coding was done using deductive and inductive approaches. With respect to the definition of behavioural engagement by Fredericks, Blumenfeld, and Paris (2004), four major themes arose during the coding process: academic participation, social participation, concentration/attention, and negative social conducts (in terms of benefit to learning). Academic participation is linked to cognition. The instances in this theme are students demonstrating their
thinking by asking questions, discussing, or observation of their strategies. The theme arose as
the researcher focused on both the whole class and the small groups. Social participation is a
theme which regroups all instances where students were in interaction that would benefit their
learning, such as collaborating with peers and with the mentors. Concentration and attention is a
theme that arose as the researchers focused on the work habits of the students, recording when
they were on task or not. Finally, the negative social conducts is a theme that regroups the
instances where the students were chatting with their peers, on their phones, or are distracted.
Such actions don’t benefit learning because the students are not on task, on discussing about the
task.

The coding process led to the identification of 140 codes, which were categorized as:
academic participation (21), social participation (49), concentration and attention (51), negative
social conduct (19).

The following iteration of coding was done inductively, generally identifying contexts
within the themes. For example, within the social participation theme, there were trends linked to
the mentors, the group work and students helping each other. For the academic participation
theme, I found trends of students asking questions, students helping each other, and students
falling behind with the material. In terms of concentration and attention, students were generally
concentrated and attentive, or not. The themes were analyze to describe student behavioural
engagement throughout the experiment.

**Student Tests**

The summative test, also known as assessment of learning (Ontario Ministry of
Education, 2010) was given to the students on the last day of the experiment. Professor Taylor
designed the test and supervised the administration of the assessment along with the teacher. The
test was open book, and was composed of 4 problems which can be found in the Appendix T. The students’ tests were graded using the Ministry of Education achievement chart (Appendix R). Once those results were tabulated, a statistical analysis was performed in relation to student’s individual engagement scores, in order to verify the conclusion from previous studies that student engagement is a predictor of student achievement.

The students’ individual engagement scores (IES) were based on three categories of engagement, all weighted equally: students’ cognitive-behavioural, social-behavioural, and emotional engagement.

**Example of the Process to determine the IES**

![Figure 6: Coding of raw data sample.](Image)

This is an example of how an individual engagement score (behavioural) was calculated starting from the raw data. The goal is to clarify the process for the reader.

The blue highlighter in Figure 5 represents cognitive-behaviours and the pink underline represents social-behaviours. They are further identified as positive or negative. For example,
“quiet, working on their sheet” indicates that the students are on task, so it’s a positive cognitive behavioural instance. On the other hand, “watching videos on their phones” shows that the students are distracted and not working, so it’s interpreted as negative cognitive-behavioural engagement. The next step is to record the codes relative to the students. Table 3 is the results from the coding for id14 and id16. The text in italic is coded as positive, and the rest is negative.

Table 3: Classifying the codes

<table>
<thead>
<tr>
<th>Id14</th>
<th>Social-Behav</th>
<th>Id16</th>
<th>Social-Behav</th>
</tr>
</thead>
<tbody>
<tr>
<td>Are quiet, working on their sheet</td>
<td>Are chatting (not about math)</td>
<td>Are quiet, working on their sheet</td>
<td>Are chatting (not about math)</td>
</tr>
<tr>
<td>Are now watching the video as well, and laughing</td>
<td>Are now watching the video as well, and laughing</td>
<td>Are now watching the video as well, and laughing</td>
<td>Are now watching the video as well, and laughing</td>
</tr>
<tr>
<td>Looking at the board</td>
<td>Looking at the board</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eyes unfocused</td>
<td>Looking at the board</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taking notes</td>
<td>Back on her phone</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Based on these numbers, we find the cognitive-behavioural engagement score (CB) and the social-behavioural engagement score (SB). For both id14 and id16: $\text{CB} = \frac{2}{8} \times 100 = 25\%$, and $\text{SB} = \frac{0}{2} \times 100 = 0\%$. The next step is to use a similar process for the interview data and get the emotional engagement score, and take the average of the three to get the individual engagement score.

Ethics

In order to ensure the safety of the participants of the experiment of the study, steps were taken to obtain ethical clearance. Following Canada’s Tri-Council policy, the experiment received ethics approval from the Queen’s University General Ethics Review Board (GREB).

Moreover, I completed the *Tri-Council Policy Statement: Ethical Conduct for Research Involving*
Humans Course on Research Ethics (TCPS 2: CORE) course before beginning my research. The approval from the Limestone District School Board was obtained. I proceeded to obtain my Police Criminal Record Check with vulnerable populations.

Before the beginning of the experiment, students received letters of consent to be signed by themselves and their parents. They were made aware verbally and written form that they could withdraw from the research at any point during the experiment or after, however, they had to participate in the experiment regardless. This was reinforcing as I proceeded to the focus-group interviews at the end of the experiment.

See Appendix A to K for consent letters and ethics clearance.

Summary of Chapter

In Chapter 3, I outlined the qualitative and quantitative approaches that were used to achieve the study purpose, which was to describe students’ engagement in the context of a non-traditional approach to curriculum expectations. The qualitative data that was collected was in the form of transcribed focus group interviews and observations. The quantitative data were the students’ tests scores. The qualitative data was coded for the purpose of qualitative and quantitative analyses. For quantitative analyses, the coding process was done deductively, after which the number instances in each theme were calculated and used to test hypotheses. For the qualitative analyses, the coding was done deductively and inductively in order to describe student engagement. Moreover, individual engagement scores were calculated for each student, and an analysis was performed to evaluate if engagement is a predictor of achievement, with respect to the test marks. The findings of this study will be discussed in Chapter 4.
Chapter 4

Results

This chapter is a compilation of the results from each type of analysis. First, the preliminary analysis is the student’s scores for the test. It also includes the results from the first round of coding of the observation data, which indicates the daily behavioural engagement data. The second part of chapter is the statistical analysis results. The aim of the analysis was answer the following questions:

- Are the emotions of students towards mathematics significantly positive or negative?
- Are the emotions of students with respect to how they are usually taught significantly positive or negative?
- What component of the experiment do the student feel the most positive and less positive about?

Finally, I present the process and results of the qualitative analysis of the observational data and focus-group interviews. The results from this analysis will allow me to describe students’ engagement in the context of the experiment.

Preliminary Analysis

Student tests

The tests were marked using the Ministry of Education Achievement Chart of Mathematics for grade9-12, which can be found in Appendix S. Note that the assessment method places each student on a level, which is associated to percentages as follows:

<table>
<thead>
<tr>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>50-59%</td>
<td>60-69%</td>
<td>70-79%</td>
<td>80-100%</td>
</tr>
</tbody>
</table>
Each level represents the student’s efficacy in four categories of learning: knowledge, thinking, communication, and application. Figure 6 represents the number of students on each level, and there was a total of 20 students that wrote the test. The average of the class was a level 2. In terms of performance, a level 2 means that the students:

1. demonstrates some knowledge of the content and some understanding of the concepts
2. has some understanding of the problem, makes some sort of plan for solving it, and carries the plan with some effectiveness
3. expresses and organizes mathematical thinking with some effectiveness
4. uses conventions, vocabulary, and terminology of the discipline with some effectiveness
5. applies knowledge and skills in familiar contexts with some effectiveness
6. transfers knowledge and skills to new contexts with some effectiveness

The quantifying term in each statement is “some”. In this case, it was considered to be knowledge/skills sufficient to get a passing mark, but still required work before attaining what I would consider a reasonable level of mastery.
The test was designed so that questions 2 and 4 were, in fact, the same as questions 1 and 3 but requiring a different method to solve the problem. 1 and 4 required what we called the geometric method, and 2 and 3 the algebraic method. The level of achievement of students for each question is represented in Figure 7.

Figures 8 to 11 represent a sample of the students answer for the Question 1 on the test. There is one representing each level. The solution to the question, and the rest of the tests, is in Appendices T and U.

**Question 1.** Using powerpoint recording, find the parameters a, b, and C. Briefly describe your reasoning. Part b, explains where your answer came from is that a fellow student who hasn’t seen this type of problem before could understand the answer. Keep in mind that the only information you can be sure of is that in graphs 1 and 4 as they are correct answers to the grid points.

![Sample solution level 1](image-url)
Figure 8 was marked as a level 1. The value for \( a \) is wrong, the student most likely wrote this based on an example we worked in class, but it doesn’t make sense for this problem. In fact, because she copied blindly like she did, I conclude that she doesn’t understand what \( a \) is (horizontal dilation), which we covered on Day 2 of the experiment. Therefore, the student got a level 1 the categories Understanding and Knowledge, Thinking, Communication and Application.

![Figure 8](image)

**Figure 10: Sample solution level 2.**

Figure 9 is the solutions of a student that obtained a low level two. The values for \( a \) and \( b \) on the left of the page are right, but lower on the page, the calculations don’t make sense. They were done previously in a problem solved in class, but don’t apply in this question. The value for the shear is wrong because it should have been the displacement at \( y=1 \) rather than \( y=2 \), however, it shows some understanding of the concept. Therefore, the student obtained level 1 in the Thinking and Communication categories, and a level 2 in Understanding and Knowledge, and Application.
Figure 11: Sample solution level 3

Figure 10 is the solution of a student that got a level 3. The student’s values for all parameters are wrong. However, she shows her thinking and has reasonable explanations. Overall, she obtained level 4 in Communication, level 3 in the Thinking and Understanding and Knowledge categories, and level 2 and in Application.

Figure 12: Sample solution level 4
Finally, Figure 11 is the solution of a student that obtained a level 4. All the parameters are right, and we see evidence of thinking. Level 4 was obtained for all four categories.

The students had generally a higher level of achievement for questions 1 and 2, and a much lower level for question 3 and 4. The level of difficulty of the questions would be proportional to that trend. To the trained eye, questions 1 and 2 could be solved with few steps, and were similar to questions the students had worked on previously in the experiment. Questions 3 and 4, however, required a rigorous solving procedure.

**Daily behavioural and social engagement**

![Daily % of positive cognitive and social behaviours](image)

*Figure 13: Daily percentage of positive cognitive and social behaviours*

The observations were coded in terms of behavioural-cognitive engagement, as well as behavioural-social engagement. In the first case, instances that indicated that students were positively engaged were when they were on task, asking questions, participating, etc. Negative would be if something distracted them, generally not on-task, or had their head on their desk. In the case of behavioural-social engagement, positive engagement meant that students were
discussing, collaborating on problems, or helping each other. Negative would be if they were
isolated, or if they would interact with each other off-topic.

Figure 12 shows a correlation between the two types of engagement. Because there is
only a small amount of data (8 days), we can’t use statistical methods to confirm the correlation.

**Statistical Analysis**

For the statistical analysis, the interviews and observations were coded and based on the
themes. We seek to understand the most important factors and the trends in student engagement.
The software R was used to do the hypothesis testing and fit the models. The statistical processes
and the R code can be found in Appendices P and Q.

**Emotions toward mathematics**

The deductive coding of the interviews was done iteratively. The first round of iterations
differentiated students’ emotions about mathematics overall (*overall*), mathematics as they learn
in the class throughout the year (*usual*), and about the central experiment of this study
(*experiment*).

<table>
<thead>
<tr>
<th></th>
<th>Number of negative instances</th>
<th>Number of positive instances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>29</td>
<td>7</td>
</tr>
<tr>
<td>Usual</td>
<td>23</td>
<td>1</td>
</tr>
<tr>
<td>Experiment</td>
<td>64</td>
<td>67</td>
</tr>
</tbody>
</table>

The participants already had a view of mathematics, which is an important factor to take
into consideration. Therefore, we seek to determine if their emotions toward mathematics are
significantly negative. We use the Z-test to test H: \( p_+ = 0.5 \) vs. A: \( p_+ < 0.5 \), where \( p_+ \):
probability that students have positive emotions toward mathematics (*overall*). Using the same
test, we also test a similar hypothesis for the students’ feelings toward the mathematics they were
RUNNING HEAD: Low threshold, high ceiling tasks and engagement

previously learning in their grade 10 class. Assuming \( p \) follows a binomial distribution, we obtain:

\[
\begin{array}{cccc}
\text{Estimator} & \hat{p}_+ & \text{Standard deviation} & \text{p-value} \\
\hline
\text{Overall} & 0.1944 & 0.0659 & 1.4 \times 10^{-7} \\
\text{Usual} & 0.0416 & 0.0408 & 4.1 \times 10^{-32} \\
\end{array}
\]

Note that \( \hat{p}_+ < 0.5 \) in both cases, so we reject the null hypothesis at the \( \alpha \)-level 0.001 in both cases. In other words, students’ feelings towards both math and the way they studied it previously in the course are significantly negative (81% and 96%).

**Emotions toward the various components of the experiment**

The following iteration of the coding allows us to establish different components of the experiment and how the students feel about them.

\[
\begin{array}{cccc}
\text{Negative} & \text{Neutral} & \text{Positive} \\
\hline
\text{General} & 4 & 3 & 9 \\
\text{Content} & 37 & 2 & 12 \\
\text{Mentors} & 12 & 1 & 24 \\
\text{Group work} & 7 & 1 & 15 \\
\text{Application} & 3 & 2 & 5 \\
\text{Speed} & 0 & 0 & 1 \\
\end{array}
\]

We are interested in knowing what categories the students feel the most and least positive about. We proceed with two Z-tests. The first one tests \( H: p_c = p_o \) vs. \( A: p_c < p_o \), where \( p_c \) is the probability that the least positive feelings are toward the content, and \( p_o \) toward any of the others. The second tests, where \( p_m \) is the probability that the most positive feelings are toward the mentors, and \( p_o \) is defined as above. Based on the assumption that all parameters follow a binomial distribution.

44
Table 8: Emotions toward the components – the statistics and p-values

<table>
<thead>
<tr>
<th></th>
<th>z-statistic</th>
<th>p-value (least positive)</th>
<th>p-value (most positive)</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>0.5559</td>
<td>0.711</td>
<td>0.289</td>
</tr>
<tr>
<td>Content</td>
<td>-2.3865</td>
<td>0.009</td>
<td>0.991</td>
</tr>
<tr>
<td>Mentors</td>
<td>1.8595</td>
<td>0.969</td>
<td>0.031</td>
</tr>
<tr>
<td>Group work</td>
<td>1.4529</td>
<td>0.927</td>
<td>0.073</td>
</tr>
<tr>
<td>Application</td>
<td>0.1186</td>
<td>0.547</td>
<td>0.453</td>
</tr>
</tbody>
</table>

In the least positive case, we reject the null hypothesis with confidence $\alpha$-level 0.01, and in the most positive case we reject the null with a $\alpha$-level 0.05. We conclude that the students have significantly more positive feelings about the mentorship component compared to the other components, and significantly less positive feelings about the content of the program.

Note, however, that for the most positive component, the group work has a ratio slightly greater than the mentors (gw: 0.652, m: 0.648). However, the p-value associated to the group work component makes it less significant compared to the mentors’ due to fewer observations.

**Engagement and Achievement**

Using the data that we have about the students’ individual engagement and their test marks, we want to see if the results from the experiment are consistent with the literature, in other words, that achievement and engagement are related.

**Individual engagement scores**

In the last iteration of coding, each instance that showed evidence about the behavioural-cognitive (BC), behavioural-social (BS) and emotional (E) engagement of a student was individually recorded. The scores were calculated using the percentage of positive engagement in each category, and the individual engagement score (IES) is simply the average of the three subtypes.

$$IES = \frac{\%BC + \%BS + \%E}{3}$$
However, some participants did not have emotional engagement scores because they were not present in the interviews, so we could not collect data about their emotions. The interviews had allowed us to collect substantial amount of evidence about the emotional engagement of the participants. Moreover, literature supports that emotional engagement and achievement are heavily correlated, so I didn’t want to exclude the emotional factor.

To obtain emotional engagement scores for the students that didn’t have one, I used a k-Nearest Neighbour (k-NN) regression. This algorithm is appropriate for this case because the values for emotional engagement were missing at random. It is a supervised method, since the classification of the missing data points is based on known classification of other points. For each missing emotional engagement score, the algorithm calculates the Euclidean distances to the k nearest neighbours in terms of behavioural engagement.

![Figure 14: Behavioural-cognitive vs. behavioural-social engagement scores for each student.](image)

The distances are ordered increasingly, and based on the root-mean-square-deviation (RMSD), we find an optimal number k of neighbours. In this case, the number of data points is small, so k=2. The missing emotional scores are calculated by taking the inverse distance weighted average of the emotional scores of the nearest k neighbours.
For example, the data point (60%, 0%) belongs to id09, doesn’t have an emotional engagement score. The nearest data point $k_1$, with coordinates (40%, 0%), and Euclidean distance $d = 60 - 40 = 20\%$, has an emotional engagement score of 13%. The second nearest data point $k_2$, with coordinates (73%, 17%) and Euclidean distance $d = \sqrt{(73 - 60)^2 + (17 - 0)^2} \approx 21.40\%$, has an emotional engagement score of 75%. Hence, $E = \frac{\frac{1}{0.2140} \times 0.75 + \frac{1}{0.20} \times 0.13}{\frac{1}{0.2140} + \frac{1}{0.20}} \approx 0.44$. So, the emotional engagement score of id09 is now 44%.

**Regressions**

To determine if engagement ($X$) is a good predictor of achievement ($Y$) we first fit a linear regression. In statistics, regressions are used to find if there is a relation between two data sets. The model is $E(Y) = \beta_0 + \beta_1 X$, and we use R in order to obtain the coefficients.

| Table 9: Coefficient estimates and p-value for the linear model |
|-------------------|----------------|--------|---------|
|                   | Coefficient estimates | Standard Error | t-value | p-value |
| Intercept         | 0.7610            | 0.3685            | 2.062 | 0.0539 |
| %Engagement       | 0.0309            | 0.0089            | 3.470 | 0.0027 |

The p-value for $\beta_1$ is 0.0027, which means we reject the null hypothesis (absence of relationship between engagement an achievement) at $\alpha = 0.01$ level of significance.

From this fit, we conclude that a 1% increase in IES leads to an increase of test score by 0.03. In other words, if a student is 33% more engaged, we predict that she will reach the next achievement level.
Although the result is significant, the assumptions of a linear model may not be appropriate because of the nature of the achievement scores. The levels of achievement are not continuous, but rather ordinal variables (ordered and categorical). Moreover, the sample size is quite small, which again suggest that the simple linear model is not the best choice. The Ordinal Logistic Regression is an extension of the linear model that models ordinal variables. The model has a logit function response: \( \ln(\theta_j) = \alpha_j - \beta X \), where \( \theta_j \) is the odds of getting level \( j \) or less, and \( \alpha_j \) and \( \beta \) are coefficients. Note that \( \alpha_j \), called the threshold value, is different for every level. However, \( \beta \) is the same, which means that the effect of the independent variable is the same for the different for the logit function.

Applying the model gives the following values:

<table>
<thead>
<tr>
<th>Value</th>
<th>Standard Error</th>
<th>z value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>%Engagement</strong></td>
<td>0.1052</td>
<td>0.03883</td>
<td>2.71</td>
</tr>
</tbody>
</table>

The p-value is 0.00672, which means we reject the null hypothesis (absence of relationship between engagement and achievement) at \( \alpha = 0.01 \) level of significance.
From this model, we interpret that as engagement increases by 1%, the odds of moving from one level to the other goes up by roughly 10%.

**Qualitative Interview Results**

The deductive coding of the interviews was based on the engagement emotions categories produced Pekrun and Linnenbrink-Garcia (2012). The first category is achievement emotions, which can be enjoyment, boredom, anxiety, hopelessness, pride, hope, and shame. Epistemic emotions are feelings linked to cognition, often in the context of problem solving. For example, such emotions can be surprise, curiosity, frustration. The last category is social emotions, such as admiration, envy, empathy, love or hate.

The analysis process went as follows:

1. To get familiar with the data, the interviews were read multiple times.
2. Two careful readings of each interview led to the identifications of 152 codes.
3. The codes were categorized according to Pekrun and Linnenbrink-Garcia (2012) as previously described. Each category had the following number of codes: Academic (21), Epistemic (36), Topic (40), and Social (55).
4. I read the codes in each category to identify trends. For achievement emotions, the trends were: “pride”, “anxiety”, “hopelessness”, “confusion”, and “enjoyement”. For epistemic emotions, the trends were: “confusion”, “lack of understanding”, “like and dislike of the process”, “difficulty”, “teaching others”. For topic emotions, the trends were: “applicability”, “interest”, “dislike”, “math content in general”, “appreciation of mathematics”. For social emotions, the trends were: .
5. Finally, the trends were then analyzed and grouped into themes. For example, in social, the themes that emerged are “Asking for help”, “Mentors”, and “Working in
groups”. They respectively had 6, 15, and 23 codes. The themes for each type of emotion are enumerated and described in this chapter.

Each theme that emerged in the last step of the analysis is explained and supported by the data throughout this section of the chapter.

The analysis includes several direct quotations from the interview transcripts. The references are of the form (I#, P#, line #), where I is the interview (either I1 or I2), P refers to the participant, and line is the line number from the corresponding transcript.

Achievement emotions

Achievement emotions can be enjoyment, boredom, anxiety, hopelessness, pride, hope, and shame. For this theme, I looked for instances of those emotions in the students’ speech, but I kept a specific focus on the way in which emotions were linked to achievement, self-imposed expectations or expectations attributable to others.

Anxiety. During the interviews, 4 participants mentioned the anxiety that they feel when it comes to mathematics. The words “stress,” “tears,” “confusing,”, “sad face” came out when they were asked what their responses were to think about mathematics. One student mentioned being stressed specifically when it came to exams (I1, P2, 25). In the interview process, the participants were not questioned about anxiety in the context of the experiment, but one student mentioned “I don’t do the same work […] when someone is hovering when I’m on my own, cause I get very anxious.” (I2, P7, 354) There were many people walking around, such as researchers and mentors, and this would make him anxious.

Pride. The content of the experiment and the methods were new for the students, and having this novel experience made some of them feel proud. When they were asked what they gained from the unit, a student said: “Personally, I’ve found myself talking to my parents and be
like… ‘I know something you guys don’t’, you know, so I guess, bragging rights!” (I1, P2, 182)

In other words, this student bragged to her parents that she knew something they don’t, which, according to my interpretation, reflects pride. When we asked what she gained from the unit, a student said: “I would actually come home and tell my mother like: "hey, I actually learned something. I understood." (I1, P3, 195)

Enjoyment. In the case of achievement emotions, I looked at students’ enjoyment of the whole experiment, rather than specific components (such as mentors, or specific topics).

The students mentioned liking being able to work at their own pace, and the fact that we (researchers, instructors) didn’t expect them to do them all (I1, P1, 143). They enjoyed the fact that they could just do the examples until they understood: “He doesn't expect you to do them all, but it's just like, sort of like you do some of them until you understand, and I felt like that really helped me like, understand stuff better.” (I1, P1, 145) The absence of textbooks and homework was also pointed out as something they appreciated (I1, P1, 95). A student said he thought the unit was “pretty sweet” (I2, P5, 680).

Hopelessness. “You have to work for like, a long time, to get three simple, like variables.” (I2, P7, 396) More students mentioned finding it “a lot” to do the big problems, and have to do so many steps to get an answer. The length of the problems and the perception of the complexity was generally linked to this emotion: “…long list of all the operations and stuff, and if you, like, mess one thing, it ruins everything else.” (I1, P1, 209) Some of them struggle to decide when to apply things they previously learn, such as the Pythagorean theorem.

Hopelessness can also be seen an outcome, as a student explains it:

Sometimes I got really confused and that just made me, like, uninterested with everything. It's just like, I don't get this... whatever. Like, it's fine. It gets like, a part of it is just me losing confidence in myself. Just like... you know, not
understanding something, and then... well, I can't do it... I can't do it. (I1, P2, 108)

**Epistemic Emotions**

Epistemic emotions are linked to the cognitive processes, and problem-solving. Such emotions can be, for example, surprise, curiosity, and shame. I focused on how they felt about their understanding, and their emotions about the process of working on the problems.

**Feelings towards math and school.** Before I focus on the epistemic emotions related to the unit, I analyzed how they feel generally about mathematics and school. The students are in the school system and had emotions towards mathematics before the experiment. They reported that they were not passionate about what they are taught in schools (I1, P4, 349). Resentment is an emotion that came out with relation to math: “I don't do too well, and I guess it just kind of, causes me to be resentful towards it.” (I1, P2, 42) As they were asked what they thought math was, a student said “I don’t think math is real. I think it’s been invented. Is that a wrong answer?” (I2, P7, 66) Although it’s not easy to perceive the emotion through the words, the tone suggested frustration. On the other hand, some students like it. “I like math, but I think it’s because I know that I have a good grasp on it.” (I2, P8, 111).

**Confusion.** Students were experiencing confusion throughout the experiment, at different moment throughout the problem-solving process. For example, students mentioned being confused by matrix multiplication: “when we were doing the matrices and then, like, I guess that row by that column... I was getting really confused by that.” (I1, P2, 113). Another one said, “I didn’t know how he got from 1 to root 2.” (I1, P1, 117) A student told us he found the process of the geometric method confusing: “when we did example 4, and then you had to find how it got
Perception of their understanding. When the researchers asked the students how they felt about their own understanding of the unit, the answers fell across a full spectrum. Some said that they could do most questions or had generally a good understanding of the content: “I think I had a pretty good understanding of like all of it.” (I2, P8, 528). On the other hand, some said that even now that it’s over, they still really don’t understand what was going on: “I don’t understand it. I still don’t. I don’t get it all.” (I2, P7, 496) “As soon as something was like, a little bit different from the example we did in class, I was kind of stuck” (I2, P8, 527). As for the understanding throughout the problem-solving situation, a student mentioned that even if a mentor was coming to help, sometimes, they would still not understand (I1, P1, 257).

Challenge and difficulty. The students commented multiple times how they found the unit challenging or difficult. “It’s very tedious.” (I2, P6, 388) A student mentioned: “I thought at the start I was really just flabbergasted, because it was completely new, and hum, but later on I think I got a better a better understanding of how to, how it was.” (I2, P5, 534) The word “flabbergasted” means surprised. Hence, one can interpret the student’s reaction in how he faced this challenged: he was surprised, but he did manage, given the challenge, to get some understanding. Another participant thought that memorizing the formulas was a big challenge for him. However, a student pointed out that even if the tasks in the unit were strenuous, and that it was a change of pace compared to what they were used to, and that in the end, it wasn’t that difficult.

Doing the algebra you get to see the work that they’ve done along the way, and, but with the geometry one you kind of have to explain it afterwards what you did cause they can’t exactly see all the work you’ve done to lead up to what you got,
which is kind of annoying cause you don’t really like explaining. Once you’re done with a question you just want to be done. (I2, P6, 399)

**Surprise and empowerment.** A student thought that the experiment was generally an upgrade compared to the usual classroom routine. The students were surprised and enthused by the Tic Tac Toe game (see Table 1 in Chapter 1) that we did with them one day, mentioning how they like to see “this is how it works, and we actually got to do it instead of just like, telling us.” (I1, P3, 89). They also felt that the group setting allowed students to explain to others, which made them feel like “you can help yourself understand something.” (I1, P1, 277)

**Topic Emotions**

Topic emotions are directly linked to mathematics and the content of the unit. It might be the transformation topic, or any part of it (e.g. geometry, algebra, matrices), and also the Tic Tac Toe game we did with them. The emotions might represent their interest and their motivation with relation to the topic or sub-topics.

**Interest of mathematics.** Similar to epistemic emotions, the interest of the student in terms of topic emotions varied. On one hand, several students qualified it as “the worst subject” (I1, P2, 41; I1, P4, 365), but on the other, they admit that it’s useful and necessary (I2, P5, 95; I2, P6, 211). Two participants admitted having an interest for the subject.

**Interest of the unit.** Students mentioned that they thought the unit was different than what they typically do (I1, P5, 261). In terms of interest, two students mentioned finding the transformation interesting. More precisely, students mentioned being interested by the matrices, by the tic tac toe game, and by the introduction to linear transformations of lines and points. More generally, one participant said it was interesting to see how she could answer the problem using two perspectives, the geometric and the algebraic one. “I thought it was definitely
interesting. There was more than one way to conduct or like to do the question, and it wasn’t just black and white to the extent that you could do it a little bit in your own way.” (I2, P5, 275). However, students mentioned disliking the matrix multiplication, and the algebraic processes. In fact, when asked for suggestions, a student replied “Um, maybe just a better topic.” (I2, P6, 646)

**Applicability.** The students mentioned on several occasions that applicability is very important for them to be interested in the topics. “Right now, it feels like we are going into it blindly like, yeah, learn this, you don't know if you are going to need it but... just learn it!” (I1, P2, 395). Some students mentioned that they wished they could be taught mathematics that is more relevant to them in terms of the career they want, and “real-world applications.” Regarding the unit, when asked if they could see it being applied, a student said “it was like, good to know when we'd use that. Like if we were in animation and things like that, then you have to use that sort of thing.” (I1, P4, 191) However, most students thought that it wasn’t applicable what so ever. “I really don’t see the benefit of me learning how to, you know, for example, transform shapes when I, like algebraically, when I’m pretty confident I’m not gonna.” (I2, P7, 164). More students mentioned that they didn’t gain any real-world experience from the experiment. “It’s too specific.” (I2, P9, 488)

**Social Emotions**

Social emotions are the emotions either directed toward other people, or triggered by the social environment. Some examples are admiration, envy, empathy, and love/hate. In the context of social emotions, we saw themes emerging based type of social interaction: teacher-student (pre-experiment), mentor-student (during the experiment), and student-student (before and during the experiment).
**Asking for help (pre-experiment).** Many emotions can be linked to this action. A student mentioned that it makes her sad because she usually has to ask for help from several people before she understands in math. “Whenever I raise my hand to ask [the teacher], there is part of me that is like: ‘is he gonna think I am stupid?’” (I1, P2, 244). Students find it quite hard to access help in the regular classroom, to the point where they describe it as being “kind of awful” (I1, P4, 305).

**Mentors in the classroom.** The students thought it was helpful, useful and nice to have the mentors around in the classroom, available to help. “[The mentors] did really well in helping us. You didn’t baby us. You didn’t make us feel like: ‘you don’t know that,’ or coddle us in any way.” (I1, P2, 262). They liked having different people to give one-on-one help, and also the fact that the mentors were coming around to provide support, even if students didn’t ask for it explicitly. They find it easier to learn when there are people in the class to help them. Moreover, when a mentor came to help a student, the whole group could benefit from the instruction. On the other hand, a participant mentioned sometimes being stressed of the mentors: “I find when somebody is looking over my shoulder... I get stressed out. They are just watching me… what do I do? Sometimes it can be stressful.” (I1, P2, 328)

**Working in groups.** Working in groups means the students could help each other. “In a group, we are better cause we can help create an understanding.” (I2, P6, 511) Students can discuss (I1, P4, 294) and use different ways to explain things (I2, P8, 577). Being in a group setting provides comfort and moral support for the students (I1, P2, 223). In the case they are confused, they can ask on of their group mates and don’t feel embarrassed (I1, P2, 243). Sometimes, their classmates might not be able to help, but they’ll feel better knowing that they are not the only one confused (I1, P1, 225). On the other hand, participants mentioned that some of their classmates may not be willing to help: “But you know if it's someone that has to be there
and they feel like they *have* to help you, it's just annoying.” (I1, P1, 296) The student asking for help might feel envious too: “But again, there is also that... why do they understand and not me? So...” (I1, P3, 288) Also, the group work setting creates a lot of distractions (I2, P6, 549) and perhaps individual work is important and valuable as well (I1, P2, 235).

**Qualitative Observations Results**

The coding for the observations was done deductively and inductively based on the framework for behavioural engagement by Fredericks, Blumenfeld, and Paris, (2004). The themes that emerged following this process serve to describe students’ behavioural engagement throughout the experiment.

The coding process lead to the identification of 140 codes, which were categorized as: academic participation (21), social participation (49), concentration and attention (51), negative social conduct (19). The next iteration of coding was done inductively to identify trends. For academic participation, the trends were “asking and answering questions”, “peer mentoring”, “algorithmic thinking”. For social participation, the trends were “mentors help”, “peers help”, and “no social participation”. For concentration and attention, the trends were “on task” and “unfocused” and “distracted”. Finally, for negative social conduct, the trends were “social distractions” and “chatting”. The last step of the analysis was to group the trends into themes, and those themes are listed and described bellow.

In order to substantialize the analysis, I referenced the field notes that were taken during the observation. To reference to the transcript when I used direct quotations, I have (Day #, line), where day # refers to the day of the observation as indicated in Chapter 3, and the line is the line from the field notes on that day.
Problem solving and academic behaviours. The first theme seeks to describe students' behaviour when they were participating academically (e.g. asking questions, helping each other, solving problems).

Throughout the unit, students had the habit to reach out to the mentors and instructors for help. Sometimes, they had questions about technicalities, and other times they had questions which could be interpreted as being more complex questions. For instance, Professor Taylor once mentioned “that they have a good question” (Day 2, 87), and once, a student asked, “what the little circle means” (Day 7, 17), which led to a conversation about composition\(^1\), a topic that is not covered before grade 11.

Sometimes, students would take advantage of the small group setting to help each other. “[Student A] is explaining positive and negative rotations, using the clockwise and counter clockwise explanation.” (Day 2, 121) “Student B is helping other students with multiplication.” (Day 6, 35)

However, on other occasions, I observed students falling behind (Day 8, 8). As I reviewed their workbooks, I would “notice they follow the steps of the other examples a bit too closely without really thinking.” (Day 7, 66)

Concentration and Attention. Concentration and attention when the students are supposed to work on a certain task is an indicator of their behavioural engagement.

Throughout the experiment, students were not consistent in terms of concentration and attention. Sometimes, students would be working independently (e.g. Day 2, 60; Day 2, 127; Day 3, 31; Day 5, 81, Day 6, 28; Day 7, 55, Day 9, 51). We would observe them writing and drawing in their workbooks, and calculating.

\(^1\) Composition refers to composition of functions (transformations) in the context of the newly discovered isomorphism between the transformations and the matrices. See Context section in Chapter 1 for precisions.
Other times, they would be distracted. “…has his head on the table or is distracting another student” (Day 4, 25), “…are looking at their phones. The eyes of [the student] are unfocused.” (Day 5, 93) The day where the students could use the iPad, the researcher observed that “students are playing with the glitches of the program” (Day 4, 33), “groups a and b are not working on example, but talking loudly or on phone.” (Day 9, 21)

**Social interactions benefiting learning.** Given the group setting and the mentoring component of the unit, many students would be interacting with other people. The nature of the interactions was believed to positively impact the behavioural engagement of the students.

The mentoring component of the experiment increased the amount of interactions that benefit student learning. It allowed the mentors and instructors to help students one-on-one, in group settings, and it would also allow students to discuss about the problems among themselves. The group setting allowed for several students to benefit from mentors’ explanations. For example, we observed this when “one student talks [to the instructor], and the other two are listening to the exchange.” (Day 2, 80) Moreover, students would question answers among themselves, such as when they were “discussing whether the angle of rotation is 45°” (Day 3, 35)

**Social interactions damaging learning.** Group setting came with its downside. Certain groups of students would constantly be chatting off topic, unless a mentor was present to work with them. We observed students talking, but also being distracted by other things. “[Two students] are watching a video on their phones” and “[3 more students] are now watching the video as well, and laughing” (Day 5, 82-91). “[A group] is playing with a coin” (Day 9, 25). Some students that were working would sometimes join in conversations with their peers and lose focus.
Summary of the Chapter

In this chapter, I demonstrated the level of achievement of students on the test. I represented visually the level of behavioural engagement of students on a daily basis. The statistical analysis allowed to conclude that students feel negatively about mathematics at 81%, they feel the most positive about the mentoring component of the experiment and most negatively about the content. Moreover, we found that when we calculate the individual engagement score of each student based on the codes from the interview and observational data, engagement is a predictor of achievement.

The next chapter is a discussion of those results. The main purpose is to describe student engagement in the context of the experiment based on the results of the present chapter. I will also discuss whether the results from the quantitative and qualitative methods are consistent with one another. I will consider the limitations of this study, give direction for further studies, discuss the implications for the community and for myself.
Chapter 5

Discussion

In this chapter, the principal goal is to merge the results from the qualitative analysis and the quantitative analysis and situate the findings in the existent literature. First, I will summarize the results, compare and contrast the findings from both methods to determine consistencies or inconsistencies. I will also consider the limitations of this study, suggest possible directions for future research, discuss the implications of the study and close with some concluding thoughts.

Description of Student Engagement during the Math9-12 May Experiment

Before the experiment, the students already had emotions and opinions of mathematics based upon their past experiences in other grades and in the class where the experiment happened. Throughout the qualitative analysis, epistemic and topic emotions suggested that the students were not passionate about mathematics, and some considered it to be their worst subject in school. Half of the students also admitted to being very anxious when it came to mathematics. Pekrun and Linnenbrink-Garcia (2012) mentioned that anxiety is a multiface phenomenon: it can be linked to nervous and uneasy feelings, worries, impulses to escape a situation. It also reduces performance in difficult tasks (Pekrun & Linnenbrink-Garcia, 2012). However, about a third of students mentioned liking it and admitted that they thought it was an important subject. One student was very specific, adding that she liked mathematics mostly because she had a good grasp on it.

According to the 2015 results for Grade 8 students who participated in the Trends in International Mathematics and Science Study (TIMSS), 51% strongly value mathematics, 42% value mathematics, and 7% don’t value mathematics. The qualitative results of the present study showed that a third thought mathematics is important for the adult life - a finding consistent with
the TIMSS findings. TIMSS results also suggest that 20% of Canadian Grade 8 students very much like learning mathematics, while 40% like learning mathematics and 39% dislike learning mathematics. In terms of appreciation, the results from the statistical analysis of the present study revealed that, for the most part, the students were negative towards mathematics, with only 19% of instances being positive. This finding is not consistent with the TIMSS findings. While this could mean that the participants’ appreciation of mathematics is lower than the average in the country, it must be acknowledged that the methods to obtain the scores in the two studies are different, the scales are different, and the age groups were different (Grade 10 for the present study, Grade 8 for the TIMSS report). As noted by the National Research Council and Institute of Medicine (2004), engagement (including emotional engagement, appreciation being a factor) declines throughout high school, so the age difference between the participants could explain the difference. Moreover, during the interview, the students mentioned that they were used to working from their textbooks in a traditional setting prior to the study. Findings from Swan (2006) suggest that traditional instruction tends to demotivate students and undermine confidence. This could be a possible explanation for the students’ emotional engagement prior to the experiment.

Pekrun and Linnenbrink-Garcia (2012) found that emotional engagement is of critical importance for students’ learning, achievement, personality development, and health. Therefore, finding out the negative factors to student emotional engagement were one of the purposes of this study. Looking more specifically at the findings in the context of the experiment, a statistical test revealed that the students felt most negatively about the content of the unit. The qualitative analysis revealed that the students wished for more “applicable” mathematics, or “real-life” applications, and that the approach of the experiment did not provide them this type of experience. One student mentioned that he imagined the content of the unit being applicable in
animation, however, most students said that they did not recognize ways in which the content could apply to their own lives or future. This fact is worrying, because the Tough Choices report, (2016) found that when students perceive mathematics or science to be inapplicable, their belief that STEM subjects lead to dead-end careers is reinforced and as a result, they accord little importance to studying it in high school. On the other hand, the majority of students mentioned that the topic was a nice change compared to what they were used to, and two students thought it was interesting. Overall, when asked what they liked in the content, students mentioned they liked that they were introduced to new methods and strategies to solve a problem, (e.g., the idea of transforming shapes, matrices, and the tic tac toe game). They enjoyed being able to work at their own pace and liked the fact that there was no homework. However, close to half the students did not like matrix multiplication or the algebraic method introduced to solve the problems in the unit.

The element of the experiment that had the most positive emotional response from the students was the mentoring component. The mentors were in the classroom to facilitate students’ learning, answer their questions, and guide them. This is consistent with the results from the qualitative analysis. In general, the students appreciated the mentors, noting that they were an excellent resource for the class. The students contrasted the availability of mentors to their experience of having one teacher for 20 students - a context in which they had to wait a long time before having their questions answered, or feeling intimidated about asking for help. Since the mentors circulated around the classroom, students did not need to raise their hands to ask for help, or draw attention to their need for support. The mentors could provide one-on-one help very quickly because there were many of them (usually 5 mentors for 20 students), and the group setting allowed several students to profit from the help simultaneously. These findings were also supported by the results of the observation analysis. The Grade 12 and post-secondary student
mentors played a positive role in terms of encouraging the students to be engaged in academic tasks. In general, the idea of having mentors in the class is to increase student support. Other similar models have been tested, such as team teaching (i.e., multiple teachers in the classroom), an approach that allows for more interaction among teachers and students with similar results. In fact, such model gives more opportunities for students to connect with and understand the material, which maximizes their individual learning potential (Morin, n.d.).

From a social perspective, students found that the group setting was good for collaboration and confidence-building. For the experiment, we changed the class setting into pods, allowing for students to sit in groups of 4 or 6. Before, the desks were placed in rows to face the blackboard at the front of the classroom, grouped by two or three tables. In the new configuration, which permitted group work, students felt more comfortable to ask questions of their peers first—a situation in which the peers could either help or confirm that something is confusing to them as well. In the first case, the students who asked the question can get immediate help, while the student providing the explanation has the opportunity to consolidate his or her understanding. This is something we observed during the unit: an indicator of behavioural engagement. In the second scenario, students do not feel embarrassed to be confused because they are not alone. The small group setting in the class supports collaborative learning, a major goal of Mat9-12. One student mentioned, during the interview, that he found that “in a group we’re better. Cause we can help create an understanding.” (I2, P6, 513). Gellert (2014) found that this interactive negotiation of meaning, is crucial for the learning of mathematics. When collaboration and communication occurs, mathematical meaning, the product of social interaction is constructed. On the other hand, some students mentioned that they get off topic and distracted when working in groups, a finding consistent with the data collected during the observations. Students would chat off topic very often, sometimes watching videos on their
phones or texting, and in one case, were seen to be distracted by playing with coins. Some groups were consistently off task unless a mentor was present. One explanation for these observations may be attributable to the fact that prior to the experiment the students were not familiar with collaborative practices in their mathematics classroom. To help teachers prepare their students for collaborative and inquiry-based learning, Lucas et al. (2005) and the Ontario Ministry of Education (2013) offered guidelines and helpful tips to ease students in such models. For instance, it is essential to set expectations of students’ behaviours and contributions. On way is to model the different type of student contribution (e.g. proposing theories, building on theories and ideas, agreeing or disagreeing) when setting the groud of the inquiry-based model (Ontario Ministry of Education, 2013). Lucas et al. (2005) offer a methodology to help the teacher train their students to ask good questions that will support the learning process and is essential to the inquiry process.

In terms of behavioural engagement, the data showed that students’ attention and concentration varied greatly. The students who were working alone (generally 4-5 of them) were mostly on task, but often, students (sometimes up to the majority) were distracted or sleepy. The sleepiness could be explained by the fact that class started at 8h30. Pekrun and Linnenbrink-Garcia (2012) identified indicators of positive cognitive behaviours, and the observational data suggested that a few students were exhibiting such behaviours. For instance, students reached out for help when needed, and were participating when mentors were around. However, when they were on their own, some of them relied on worked examples which often led to mistakes. This is the type of behaviour that is induced by “traditional” teaching methods, defined as learning in a traditional context in which “individual activity based on watching, listening and imitating until fluency is attained.” (Swan, 2006, p.162)
Finally, I found that the mean of the test marks was Level 2 on a scale with four achievement levels, based on the Ontario assessment rubric. As a percent score, Level 2 is between 60% and 69%; therefore, one can conclude that most students demonstrated a minimal level of understanding. By analyzing this with respect to the individual engagement score, calculated to consider emotional and behavioural engagement, I found that engagement was a predictor of achievement. This is consistent with the findings of Harbour et al. (2015) and Lee (2014), who emphasize that student engagement should be a focus of mathematics educators. Moreover, the preliminary analysis allowed me to observe a correlation between cognitive-behavioural and social-behavioural engagement, which is consistent with the findings of Pekrun and Linnenbrink-Garcia (2012).

Limitations of the Study

Performing a statistical analysis helped to reinforce the qualitative findings of this report and provide numerical support for the researcher’s intuition. However, as the data was mainly qualitative, there was a limit to the degree to which it could be quantified. To overcome this issue, I produced values for the engagement score using the ratio of positive indicators over the total number of indicators. Given that the amount of data was different for each type of engagement, for each student, and was dependent on context, the approach used represents one researcher’s estimate for a range of possible values. A study designed with quantitative analysis in mind would reduce the error margins. I assumed the instances were following a binomial distribution for the quantitative analysis, which requires that the instances are independent. However, all the instances were coming from the same 10 students, which means that they are most likely not independent. Moreover, the framework for emotional engagement is the Feldman Barrett and Russell (1998) Affective Circumplex adapted by Pekrun and Linnenbrink-Garcia.
(2012), but we used a simplified version only considering the valence (positive/negative) aspect. This decision was made because of the limited amount of data, but Pekrun and Linnenbrink-Garcia’s (2012) research on affect and engagement found that it is valuable to consider the activation dimension.

Another limitation of the study was that the data was not collected exclusively for the purpose of describing student engagement, but was intended to provide data for multiple researchers studying different purposes. There would have been more data and a more robust representation of student engagement if the interview protocols and observations had been conducted for this specific purpose. Furthermore, the small sample size limits the generalizability of this report’s findings to other contexts.

The focus-group interviews were insightful, but adolescents are a difficult group to interview because they are often sensitive and cautious around strangers. Without a larger sample to allow for saturation of data, it is impossible to know if the students were being forthright or were stating answers that they thought the researchers wanted to hear. For the observations, three researchers rotated through responsibilities for recording the data. Even if I was aware of my biases, there is no guarantee that the other researchers were. On the other hand, the presence of many researchers may have served to mediate and reduce the effects of bias.

**Direction for Future Research**

Future research that is interested in student engagement in the context of a different approach to curriculum instruction should collect data that is quantitative in nature, in addition to qualitative data. To do so, there are several instruments, for example the National Survey for Student Engagement, which uses extensive surveys from which an individual engagement score is calculated. Normed instruments may serve to reduce bias, because each individual score is
calculated from approximately the same amount of data. Fredericks, et al. (2011) reviewed 21 different instruments used to measure a range of factors related to student engagement. These instruments (qualitative, quantitative and mixed) should be utilized for future studies. Since most of these instruments employ pre- and post- intervention scores, future quantitative research could focus on studying student engagement by measuring engagement scores prior to the experiment, and repeating the process after the experiment. Such studies could more reliably measure the real effect of a different curriculum approach on student engagement.

This research found that the students, in general, had negative feelings and attitudes toward mathematics, so future research could focus on how the influence of these emotions impact the students’ response to an unfamiliar approach to teaching and learning.

Moreover, there are other phenomena that could be explored in the context of new curricular approaches. Achievement is an important one, given the gap between high school marks and university grades in terms of students’ understanding of mathematics, i.e., there should be a focus on evaluating deep understanding, since a problem identified by McMaster is that many students come to post-secondary mathematics with only a surface understanding of most concepts (Kajander & Lovric, 2005). Research could be done to evaluate the impact of mentoring on the deep understanding and achievement of the grade 11 and grade 12 students who assume that role.

The success of new approaches to curriculum and instruction should be guided by the principles of formal program evaluation, and data should be collected in multiple sites over time. The classroom culture is a contextual factor in student participation that must be considered, as are student experience, student beliefs and student expectations. For example, implementing this unit in a class that is already familiar with inquiry-based learning may not have posed the same challenges as in a class that has always been taught in traditional way.
Implications

The findings of this study do not suggest that the students were positively engaged during the experiment.

The statistical analysis of the emotional engagement indicators revealed that the content was the most negative factor. This could imply that the students were simply not engaged with the content, and one possible explanation is their desire to see more applicable mathematics and they couldn’t see the application of the content that was presented to them. However, when they were asked to articulate what they meant by that, the only example they gave was to do taxes. Tax calculations involve simple multidigit computations (including percent) that are found in the Junior Division (Grades 4 to 6) mathematics curriculum in Ontario. The process of completing a tax form is taught in Applied, not Academic level mathematics courses, the level at which the students in this research were enrolled. Although mathematics may be applied to solve every day life problems in countless situations and a variety of contexts, the students do not recognize this explicitly and are not able to articulate examples beyond the simplest instance. Regardless of this example, however, the students all mentioned that they wish they could work on real-life problems.

One implication for curriculum developers is to reconsider the question: should we teach mathematics for its beauty and power, or its applicability, or both? It’s also curious that the students in the Academic stream wished for applicable mathematics, because they had the option to choose the Applied stream in the past. The data that was collected was limited, and perhaps large-scale survey could provide more ground for questioning the streaming structure.

So far, the explanation that was offered for the low emotional engagement relative to the content is the students’ desire for applicability. I offer another explanation: one student mentioned in the interview (I1, P4, 364) how there is stigma attached to mathematics that makes
it “the worst subject”. This implies that, perhaps, in addition to changing curricular approach, active measures should be taken to change students’ perception of mathematics in general.

Therefore, one big implication of this study is, in my opinion, that if the perception of something was reinforced negatively for years, a new approach for a short period of time is not sufficient to have a real impact on student engagement.

On the positive side, mentors in the classroom lead to positive emotional engagement results, both from the statistical and qualitative analysis. The implication from this finding is that such a program could be further tested and, if the outcomes are also positive, implemented in Ontario schools.

Conclusion

This study came from my desire to make the high school experience in mathematics better for students. Professor Taylor’s ideas are innovative, refreshing, and ingenious. There are other researchers that have wonderful ideas on how to give the curriculum a new life, but the implementation, in this case, was a bit challenging.

Student engagement has been proven to decrease throughout high school (National Research Council and Institute of Medicine, 2004), engagement is one of the most well established predictor of achievement (Harbour, Lauren, Sweigart, & Hughes, 2015), and universities realized their entering students don’t have proper mathematical knowledge and skills (Kajander & Lovric, 2005). Hence, getting students engaged in the mathematics class should be a priority (Lee, 2014). Moreover, as a future teacher, engaging my students is a true concern. I want my future students to learn in my class, but I also want them to enjoy their time and, hopefully, like what they are doing.
I think there is potential for the Math9-12 curriculum to have a positive impact on teaching and learning at the secondary school level. For future implementations, however, the classroom culture should be taken into consideration. For instance, replicating the experiment in a class where the students are already learning through the inquiry-based model, i.e., they can work in groups and learn through inquiry, makes sense. In such a case, the only “new” things that are brought to the classroom is the curriculum content, and the availability of mentors. If a class was more accustomed to traditional instructional practices and curriculum approaches, perhaps students should be eased into the inquiry model before the content of the unit is introduced. They should learn good practices, and once this is done, challenging them with the problems from Math9-12 would, perhaps, have greater success. The Galileo Educational Network (2008) created a rubric to optimize the integration of inquiry-based education. For instance, they suggest good practices such as scaffolding activities, formative feedback loops, and questioning strategies to better guide students in their learning process (Friesien & Scott, 2013).

Finally, I hope to see studies similar to this one replicated in the future. As a future teacher who is becoming familiar with the present Ontario curriculum, and a mathematics lover, I can only wish to see new life breathed into mathematics classrooms. I would really like to know how curriculum approach impacts student engagement.

What Does It Mean for Me…

I studied mathematics post-secondary mathematics for six years, I studied teaching and learning during my B.Sc. as a minor, and I am about to enter a B.Ed. program to become certified as a secondary school mathematics and physics teacher. Hence, this study has implications for me as a mathematician and researcher, and as a future educator.
…As a Mathematician and Researcher

The first thing I take from this research project is that the idea of having a more sophisticated curriculum approach is worth in the investment. The results about engagement were rather disappointing to me, as one who has been engaged with and enjoyed mathematics over time, but contrary to what the students said, I do not believe that content was the problem. The experiment and the study made me realize that my passion for mathematics creates a bias, and that I need to remember, for my future research, to put this passion in a societal context. I realize now that loving mathematics is equivalent to be an alien, in a school. Mathematics is neither universally understood nor appreciated, and for the most part, people and students do not seem to know how such passion for a subject can come to be in an individual. Although I see mathematics as an art, I do not think anyone that who has not taken courses in post-secondary mathematics can see it through this lens (at least not to the same extent someone can understand a peer’s love for music or art.) There is stigma, prejudice, fear, and anxiety to take into consideration – these things will influence the students’ perceptions of what I see as a beautiful and powerful domain. Including more sophisticated activities in the high school classroom may allow students to see mathematics through the same lens, and may serve to eradicate existing prejudices. Given that the research suggests that it might take years for students to change their beliefs about mathematics, I think that it would be worth investigating intentional, short-term, and high-impact methods that could be used to break the stigma in parallel to changing the curriculum approach.

Another thing that I learned over the past few months is that crossing the bridge from theory to practice is not an easy task. As I was observing the class from the perspective of a researcher, I could identify what seemed to work (in terms of teaching and leaning) and what did not and I had the opportunity to reflect of ways to improve instruction, assessment, and content
organization. By putting this into perspective and putting myself in the role of a curriculum developer, which is one of my long-term goals, I realized that the process of developing a new curriculum, or a new curriculum approach, is a cyclic process.

![Curriculum approach development cycle](image)

To evaluate the needs, this research allowed me to realize that engagement is a very meaningful indicator of learning. Moreover, interviewing students in focus groups provides rich feedback and many constructive suggestions. Most of the participants in the interviews were eager to share their thoughts, and I was impressed by their insights and recommendations. I think that in the future, I will want to repeat this experience formally to get a better understanding of what adolescents think and feel about mathematics.

**…As a Future Educator**

The experiment gave me a new perspective about what I am learning at The Faculty of Education. In my courses, I am learning about many educational theories and models as well as instructional/assessment strategies and methods for differentiation. At times, the volume of information has seemed overwhelming to me, and I wondered about the veracity of the various theories. I found myself asking whether I would actually apply any of this in my future daily practice.
When Math9-12 was implemented, we mostly focused on getting the students through the content and implementing the mentoring component. We did not differentiate learning. We did not take into account Individual Educational Plans for students. We did not use lesson plans, nor did we vary teaching methods on a daily basis. The omission of these important pedagogical principles was antithetical to what the literature says about effective teaching methods and how to best engage students from this age group. By attending to pedagogy and content, I think that the experiment may have resulted in more explicit instances of positive student engagement. Therefore, as a future educator, I recognize that teaching practices are as important as content.

This research project gave me the opportunity to gain a nascent understanding of adolescents’ behaviours and attitudes before starting my teaching internship. Having first-hand experience with Grade 10 students allowed me to appreciate the fact that students want to learn, but they want to be taught about things that matter to them, and to which they can relate. As a future educator, I want to be responsive to my students, and provide meaningful learning experiences for them. I hope that by having my own classes, I’ll be able to get to know my students well enough to base my teaching and build my activities around their expressed interests.

The component that was the most successful in the experiment, in terms of student engagement, was the mentoring component. Although this is not something that is common in Ontario, I think that if I am placed in a supportive and open-minded school, I would try to implement the mentorship model. The positive outcomes of the model were very powerful and meaningful, and is worth replicating in other contexts.

This study allowed me to review literature and research student engagement extensively, and come to appreciate the complexity and multifacettness of this construct. I feel that I now know the indicators of student engagement, and I know the impact of negative emotions on
learning mathematics. I hope to use this knowledge and these skills to stay constantly informed of my class’ engagement level, and use this information to continually improve my practice.

Finally, this experience allowed me to realize that I want to be a researcher in the future, and that my idea to get experience in the field will be profitable. I will get to know students from various backgrounds, and I will keep my passion for mathematics close to me and my students. I hope the experience from this study and those from my professional opportunities to learn and grow as a teacher will empower me to one day make a significant, large-scale contribution to the mathematics education field.
References


RUNNING HEAD: Low threshold, high ceiling tasks and engagement


RUNNING HEAD: Low threshold, high ceiling tasks and engagement


Appendix A

Letter of Information for Grade 10 students and Parents

1. **Invitation to Grade 10 Students.** Students in [teacher’s name] Grade 10 Math class at [school] in the winter semester 2017 are being invited to participate in a 3-week research study to be conducted in May 2017. This study is a component of a multilevel project (Math9-12) affiliated with the new Mathematics Knowledge Network of the Ontario Ministry of Education. The math9-12 project is in response to numerous calls in public media recently from academia, industry, and non-profit and government organizations for students across all levels of education to study and develop mathematical and technological skills at a higher level, with particular focus on the transition from high school to university or college.

   The purpose of this Grade 10 study is to investigate the effectiveness of a unit on transformation geometry that has been designed to fit the Ontario Academic Grade 10 Math curriculum. All students in the classroom will work with and learn the project material as it will be part of the grade 10 course. Those students for whom permission is obtained will interact with the researchers and will be asked questions about the strategies they employ, about the difficulties they encounter or the insights they develop.

2. **Purpose of the Letter.** The purpose of this letter is to provide students and their parents with the information they require to make an informed decision regarding participation in this research.

3. **Study Procedures.** The study will focus on activities co-planned with the teacher. The teacher will have the final say on design and implementation, so that they meet the teaching and learning goals of the course. The three members of the research team will observe normal classroom student engagement during the 3-week period, typically working with a small group of students. During their interaction with a student or with a small group of students, they will record the students’ comments and explanations with hand-written notes. There is no time commitment expected from students, other than their regular classroom participation.

   Students who participate in the study and who consent to be interviewed at the end of the project will be asked, in small groups, to talk for 5-10 minutes about their experience in learning the material and solving the problems. These interviews will be audio-taped but only for the purpose of transcribing the discussion to written form. The tapes themselves will never be played to any third party.

   In addition, photographs of students interacting in the class will be taken. Care will be taken to ensure that we have consent for this from all those who appear in any way in any photo that is used. Photos that do not meet this condition will be destroyed.
4. **Dissemination/publication.** Results of the study along with photos taken in class will be presented at research conferences or professional development meetings and will be displayed on the project website http://www.mast.queensu.ca/~math9-12/.

5. **Possible Risks and Harms.** There are no known or anticipated risks or discomforts associated with participating in this study.

6. **Possible Benefits.** All students in the grade 10 class will obtain the full benefit of the curriculum material whether or not they participate in the research project. Students who participate in the project and consent to be interviewed at end of the unit will have the experience of being asked to think about, evaluate and articulate the effect on them of a novel learning experience in a small-group setting in which they will also be reacting to views of their peers.

7. **Compensation.** Students will not be compensated for their participation in this research.

8. **Voluntary Participation.** Participation in this study is voluntary. Students may decline to participate or withdraw from the study at any time with no effect on their learning experience or their assessment in the course. If they withdraw from the study they also have the right to request that any data obtained from them up to that point be not used. To withdraw from the study they should inform the Principal Investigator Peter Taylor at peter.taylor@queensu.ca (613-533-2434) or the Head of the Math Department, [Name and phone number] or the School Principal [Name and phone number].

9. **Confidentiality.** Notes taken in the classroom and transcriptions of audio recordings will use pseudonyms to maintain confidentiality. Student work products that we use in the study will be confidential. The researchers will know only students’ first names and when these written notes are transcribed to text form, these names will be removed and possibly replaced by pseudonyms. The original tapes will never be made public. All data collected will remain confidential and accessible only to the investigators of this study. We will ensure that photographs taken of the students working in class involve only those students from whom consent to be photographed has been obtained. Representatives of the General Research Ethics Board of Queen’s University may require access to study-related records to monitor the conduct of the research. While we do our best to protect all the information there is no guarantee that we will be able to do so. If data is collected during the project which we may be required to report by law we have a duty to report. All original data will be destroyed after 5 years.

10. **Contacts for Further Information.** Further information regarding this research project or participation in the study can be obtained from Peter Taylor at Queen’s University (613) 533-2434 email: peter.taylor@queensu.ca. Any ethical questions about students’ rights as a research participant or the conduct of this study may be put to the Chair of the General Research Ethics Board at chair.GREB@queensu.ca or 1-844-535-2988 (Toll free in North America.)

11. **Publication.** If the results of the study are published, student names will not be used.
This letter is for the student to keep for future reference.

Peter Taylor  
Professor, Department of Mathematics and Stats  
Queen’s University  
Kingston, ON K7L 3N6  
(613) 533 2434  
peter.taylor@queensu.ca
Appendix B

Consent Form For Grade 10 Students’ Parents

I have read the Letter of Information, have had the nature of the study explained to me and I agree for my son or daughter to participate in the research study. All questions have been answered to my satisfaction.

- YES • NO

If the answer above is YES, I further agree for my son or daughter to participate in a 5-10 minute small-group interview at the end of the project about his or her experiences in working with the classroom material. These interviews will be audio-taped but only to allow the researchers to accurately transcribe student comments to text form. Excerpts from this text might be used as part of a research paper or presentation but all student names or identifying features will have been removed. The audio tapes will remain confidential.

- YES • NO

If the first answer above is YES, I further agree that my son or daughter can appear in photographs that might be taken of the class at work during the 3 weeks of the project. Such photos might be presented at research conferences or professional development meetings and be displayed on the project website.

- YES • NO

Student’s Name (please print): ____________________________________________

Parent / Legal Guardian Name (please print): _____________________________

Parent / Legal Guardian Signature: ________________________________________

Date: __________________________

Person Obtaining Consent (please print): __________________________________

Person Obtaining Consent Signature: ____________________________

Date: __________________________
Appendix C

Consent Form For Grade 10 Students

I have read the Letter of Information, have had the nature of the study explained to me and I agree to participate in the research study. All questions have been answered to my satisfaction.

- YES • NO

If the answer above is YES, I further agree to participate in a 5-10 minute small-group interview at the end of the project about my experiences in working with the classroom material. These interviews will be audio-taped but only to allow the researchers to accurately transcribe student comments to text form. Excerpts from this text might be used as part of a research paper or presentation but all student names or identifying features will have been removed. The audio tapes will remain confidential.

- YES • NO

If the first answer above is YES, I further agree that I can appear in photographs that might be taken of the class at work during the 3 weeks of the project. Such photos might be presented at research conferences or professional development meetings and be displayed on the project website.

- YES • NO

Student’s Name (please print): ____________________________________

Student’s Signature: ________________________________________

Date: ________________________________________

Person Obtaining Consent (please print): ________________________________________

Person Obtaining Consent Signature: ________________________________________

Date: ________________________________________
Appendix D

Letter of Information for Grade 12 Mentors and Parents

1. **Invitation to Grade 12 Students.** Students in [teacher’s name] Grade 10 Math class at [school] in the winter semester 2017 are being invited to participate in a 3-week research study to be conducted in May 2017. This study is a component of a multilevel project (Math9-12) affiliated with the new Mathematics Knowledge Network of the Ontario Ministry of Education. The math9-12 project is in response to numerous calls in public media recently from academia, industry, and non-profit and government organizations for students across all levels of education to study and develop mathematical and technological skills at a higher level, with particular focus on the transition from high school to university or college.

The purpose of this Grade 10 study is to investigate the effectiveness of a unit on transformation geometry that has been designed to fit the Ontario Academic Grade 10 Math curriculum. All students in the classroom will work with and learn the project material as it will be part of the grade 10 course. Those students for whom permission is obtained will interact with the researchers and will be asked questions about the strategies they employ, about the difficulties they encounter or the insights they develop.

To enrich the learning process, five students in Grade 12 will be invited in April 2017 to be tutors to work for three weeks with the study and through peer interaction with small groups of students, to contribute to the learning experience of the Grade 10 students. These tutors will receive training for this from the research team. This experience will qualify the student for credit towards community service.

2. **Purpose of the Letter.** The purpose of this letter is to provide students and their parents with the information they require to make an informed decision regarding participation in this research.

3. **Study Procedures.** For most of the class time, the Grade 10 students will be working in small groups attempting to solve a problem. The tutors will sit with these groups, observe their work, and when appropriate intervene with a well-chosen question or suggestion. There might also be a researcher working with the group, using hand-written notes to record student behaviour and possibly also taking note of any contributions of the tutor. In addition, photographs of students and tutors interacting in the class will be taken. Care will be taken to ensure that we have consent for this from all those who appear in any way in any photo that is used. Photos that do not meet this condition will be destroyed.

4. **Tutor Responsibilities.** The following time commitments will be expected from tutors:
   - Attendance at a Saturday training session in April.
   - Some time spent in the following week gaining a level of mastery of the curriculum materials,
   - hopefully working with the other tutors.
   - Regular attendance in the Grade 10 class during the 3-week study.
RUNNING HEAD: Low threshold, high ceiling tasks and engagement

- Availability at the end of the study for a 10-minute small-group interview with the researchers.
  These interviews will be audio-taped but only for the purpose of transcribing the discussion to written form. The tapes themselves will never be played to any third party.

5. **Dissemination/publication.** Results of the study along with photos taken in class will be presented at research conferences or professional development meetings and will be displayed on the project website http://www.mast.queensu.ca/~math9-12/.

6. **Possible Risks and Harms.** There are no known or anticipated risks or discomforts associated with participating in this study.

7. **Possible Benefits.** The tutors will obtain the benefit of the instruction of the researchers, of learning the curriculum material, of working under supervision as tutors in the Grade 10 classroom, and of being asked in the final interview to think about, evaluate and articulate the effect on them of a novel teaching and learning experience.

8. **Compensation.** Students will not be compensated for their participation in this research.

9. **Voluntary Participation.** Participation in this study is voluntary. Students who have been selected as tutors may withdraw from the study at any time with no penalty or effect on their academic success. If they withdraw they have the right to request that any data obtained from them up to that point be not used. To withdraw from the study they should inform the Principal Investigator Peter Taylor at peter.taylor@queensu.ca (613-533-2434) or the Head of the Math Department, [Name and phone number] or the School Principal [Name and phone number].

10. **Confidentiality.** Notes taken in the classroom and transcriptions of audio recordings will use pseudonyms to maintain confidentiality. Student work products that we use in the study will be confidential. The researchers will know only students’ first names and when these written notes are transcribed to text form, these names will be removed and possibly replaced by pseudonyms. The original tapes will never be made public. All data collected will remain confidential and accessible only to the investigators of this study. We will ensure that photographs taken of the students working in class involve only those students from whom consent to be photographed has been obtained. Representatives of the General Research Ethics Board of Queen’s University may require access to study-related records to monitor the conduct of the research. While we do our best to protect all the information there is no guarantee that we will be able to do so. If data is collected during the project which we may be required to report by law we have a duty to report. All original data will be destroyed after 5 years.

11. **Contacts for Further Information.** Further information regarding this research project or participation in the study can be obtained from Peter Taylor at Queen’s University (613) 533-2434 email: peter.taylor@queensu.ca. Any ethical questions about students’ rights as a research participant or the conduct of this study may be put to the Chair of the General Research Ethics Board at chair.GREB@queensu.ca or 1-844-535-2988 Toll free in North America.)
Publication. If the results of the study are published, student names will not be used. This letter is for the student to keep for future reference.

This letter is for the student to keep for future reference.

Peter Taylor
Professor, Department of Mathematics and Stats
Queen’s University
Kingston, ON K7L 3N6
(613) 533 2434
peter.taylor@queensu.ca
Appendix E

Consent Form for Grade 12 Mentors’ Parents

I have read the Letter of Information, have had the nature of the study explained to me and I agree for my son or daughter to participate in the research study as a tutor. As part of this responsibility I give permission for him or her to participate in a 10-minute small-group interview with the researchers at the end of the study. Such interviews will be audio-taped but only to allow the researchers to accurately transcribe student comments to text form. Excerpts from this text might be used as part of a research paper or presentation but all student names or identifying features will have been removed. All questions have been answered to my satisfaction.

• YES  • NO

If the answer above is YES, I further agree that my son or daughter can appear in photographs that might be taken of the class at work during the 3 weeks of the project. Such photos might be presented at research conferences or professional development meetings and be displayed on the project website.

• YES  • NO

Student’s Name (please print): __________________________________________

Parent / Legal Guardian Name (please print): __________________________

Parent / Legal Guardian Signature: ______________________________________

Date: ________________________________

Person Obtaining Consent (please print): _________________________________

Person Obtaining Consent Signature: _____________________________________

Date: ________________________________
Appendix F

Consent Form for Grade 12 Mentors

I have read the Letter of Information, have had the nature of the study explained to me and I agree to participate in the research study as a tutor. In particular, I agree to take on the following commitments:

- Attendance at a Saturday training session in April.
- Some time spent in the following week gaining a level of mastery of the curriculum materials, hopefully working with the other tutors.
- Regular attendance in the Grade 10 class during the 3-week study.
- Availability at the end of the study for a 10-minute small-group interview with the researchers.

These interviews will be audio-taped but only for the purpose of transcribing the discussion to written form. The tapes themselves will never be played to any third party.

- YES • NO

If the answer above is YES, I further agree that I can appear in photographs that might be taken of the class at work during the 3 weeks of the project. Such photos might be presented at research conferences or professional development meetings and be displayed on the project website.

- YES • NO

Student’s Name (please print): ________________________________

Student's Signature: ________________________________

Date: ________________________________

Person Obtaining Consent (please print): ________________________________

Person Obtaining Consent Signature: ________________________________

Date: ________________________________
Appendix G

Letter of Information for Grade 10 Math Teacher

Dear [Teacher’s Name]:

This letter is an invitation to you to take part in our research project Transformations10 in May 2017. Our plan would be for my three MSc students to spend three weeks in your Grade 10 class doing research. During this period, you and I would teach the transformations module much as we have done before, with the students mainly interacting in small groups with one another and with tutors. As part of this process the three researchers would collect data on student work. Finally, at the end of the project, the researchers hope to be able to interview you for 10-15 minutes to get your views on the impact of the project on the students and on your teaching objectives. I append below the letter of information we will send out to your students and their parents.

1. **Invitation to Grade 10 Students.** Students in [Teacher’s name] Grade 10 Math class at [school] in the winter semester 2017 are being invited to participate in a 3-week research study to be conducted in May 2017. This study is a component of a multilevel project (Math9-12) affiliated with the new Mathematics Knowledge Network of the Ontario Ministry of Education. The math9-12 project is in response to numerous calls in public media recently from academia, industry, and non-profit and government organizations for students across all levels of education to study and develop mathematical and technological skills at a higher level, with particular focus on the transition from high school to university or college.

   The purpose of this Grade 10 study is to investigate the effectiveness of a unit on transformation geometry that has been designed to fit the Ontario Academic Grade 10 Math curriculum. All students in the classroom will work with and learn the project material as it will be part of the grade 10 course. Those students for whom permission is obtained will interact with the researchers and will be asked questions about the strategies they employ, about the difficulties they encounter or the insights they develop.

2. **Purpose of the Letter.** The purpose of this letter is to provide students and their parents with the information they require to make an informed decision regarding participation in this research.

3. **Study Procedures.** The study will focus on activities co-planned with the teacher. The teacher will have the final say on design and implementation, so that they meet the teaching and learning goals of the course. The three members of the research team will observe normal classroom student engagement during the 3-week period, typically working with a small group of students. During their interaction with a student or with a small group of students, they will record the students’ comments and explanations with hand-written notes. There is no time commitment expected from students, other than their regular classroom participation.
Students who participate in the study and who consent to be interviewed at the end of the project will be asked, in small groups, to talk for 5-10 minutes about their experience in learning the material and solving the problems. These interviews will be audio-taped but only for the purpose of transcribing the discussion to written form. The tapes themselves will never be played to any third party.

4. In addition, photographs of students interacting in the class will be taken. Care will be taken to ensure that we have consent for this from all those who appear in any photo that is used. Photos that do not meet this condition will be destroyed.

5. **Dissemination/publication.** Results of the study along with photos taken in class will be presented at research conferences or professional development meetings and will be displayed on the project website http://www.mast.queensu.ca/~math9-12/ .

6. **Possible Risks and Harms.** There are no known or anticipated risks or discomforts associated with participating in this study.

7. **Possible Benefits.** The students in the grade 10 class will obtain the full benefit of the curriculum material whether or not they participate in the research project. Students who participate in the project and consent to be interviewed at end of the unit will have the experience of being asked to think about, evaluate and articulate the effect on them of a novel learning experience in a small-group setting in which they will also be reacting to views of their peers.

8. **Compensation.** Students will not be compensated for their participation in this research.

9. **Voluntary Participation.** Participation in this study is voluntary. Students may decline to participate or withdraw from the study at any time with no effect on their learning experience or their assessment in the course. If they withdraw from the study they also have the right to request that any data obtained from them up to that point be not used. To withdraw from the study they should inform the Principal Investigator Peter Taylor at peter.taylor@queensu.ca (613-533-2434) or the Head of the Math Department, [Name and phone number] or the School Principal [Name and phone number].

10. **Confidentiality.** Notes taken in the classroom and transcriptions of audio recordings will use pseudonyms to maintain confidentiality. Student work products that we use in the study will be confidential. The researchers will know only students’ first names and when these written notes are transcribed to text form, these names will be removed and possibly replaced by pseudonyms. The original tapes will never be made public. All data collected will remain confidential and accessible only to the investigators of this study. We will ensure that photographs taken of the students working in class involve only those students from whom consent to be photographed has been obtained. Representatives of the General Research Ethics Board of Queen’s University may require access to study-related records to monitor the conduct of the research. While we do our best to protect all the information there is no guarantee that we will be able to do so. If data is collected during the project which we may
be required to report by law we have a duty to report. All original data will be destroyed after 5 years.

11. **Contacts for Further Information.** Further information regarding this research project or participation in the study can be obtained from Peter Taylor at Queen’s University (613) 533-2434 email: peter.taylor@queensu.ca. Any ethical questions about students’ rights as a research participant or the conduct of this study may be put to the Chair of the General Research Ethics Board at chair.GREB@queensu.ca or 1-844-535-2988 Toll free in North America.)

**Publication.** If the results of the study are published, student names will not be used.

Peter Taylor  
Professor, Department of Mathematics and Stats  
Queen’s University  
Kingston, ON K7L 3N6  
(613) 533 2434  
peter.taylor@queensu.ca
Appendix H

Consent Form for Teacher

I have read the Letter of Information, have had the nature of the study explained to me and I agree to allow the researchers to conduct their study in my classroom. All questions have been answered to my satisfaction.

• YES  • NO

If the answer above is YES, I further agree to be interviewed (for 10-15 minutes) by the researchers at the end of the project about my experiences in working with and teaching the curriculum materials. These interviews will be audio-taped but only to allow the researchers to accurately transcribe my comments to text form.

• YES  • NO

If the first answer is YES, I further agree that I can appear in photographs that might be taken of the class at work during the 3 weeks of the project. Such photos might be presented at research conferences or professional development meetings and be displayed on the project website.

• YES  • NO

Teacher’s Name (please print): ____________________________________

Teacher’s Signature: _________________________________

Date: _________________________________

Person Obtaining Consent (please print): ____________________________________________

Person Obtaining Consent Signature: _________________________________

Date: _________________________________
Appendix I

Ethics Clearance

March 15, 2017

Dr. Peter Taylor

Queen's
University
Math&Stats
Jeffery Hall 513
Queen's
University

GREB Ref #: GMATH-012-17; TRAQ #
6020465 Title: "GMATH-012-17 Math10 KCVI May2017"

Dear Dr. Taylor:

The General Research Ethics Board (GREB), by means of a delegated board review, has cleared your proposal entitled "GMATH-012-17 Math10 KCVI May2017" for ethical compliance with the Tri-Council Guidelines (TCPS 2 (2014)) and Queen's ethics policies. In accordance with the Tri-Council Guidelines (Article 6.14) and Standard Operating Procedures (405.001), your project has been cleared for one year. You are reminded of your obligation to submit an annual renewal form prior to the annual renewal due date (access this form at http://www.queensu.ca/traq/signon.html; click on "Events"; under "Create New Event" click on "General Research Ethics Board Annual Renewal/Closure Form for Cleared Studies"). Please note that when your research project is completed, you need to submit an Annual Renewal/Closure Form in Romeo/traq indicating that the project is 'completed' so that the file can be closed. This should be submitted at the time of completion; there is no need to wait until the annual renewal due date.

You are reminded of your obligation to advise the GREB of any adverse event(s) that occur during this one year period (access this form at http://www.queensu.ca/traq/signon.html; click on "Events"; under "Create New Event" click on "General Research Ethics Board Adverse Event Form"). An adverse event includes, but is not limited to, a complaint, a change or unexpected event that alters the level of risk for the researcher or participants or situation that requires a substantial change in approach to a participant(s). You are also advised that all adverse events must be reported to the GREB within 48 hours.

You are also reminded that all changes that might affect human participants must be cleared by the GREB. For example, you must report changes to the level of risk, applicant characteristics, and implementation of new procedures. To submit an amendment form, access the application by at http://www.queensu.ca/traq/signon.html; click on "Events"; under "Create New Event" click on "General Research Ethics Board Request for the Amendment of Approved Studies". Once submitted, these changes will automatically be sent to the Ethics Coordinator, Ms. Gail Irving, at the Office of Research Services for further review and clearance by the GREB or GREB Chair.
On behalf of the General Research Ethics Board, I wish you continued success in your research.

Sincerely,

John Freeman, Ph.D.
Chair
General Research Ethics Board

c: Ms. Stefanie Knebel, Ms. Divya Lala, and Miss Kariane Ouellet, Co-investigators
Appendix J

Tri Council Politic Statement Certificate

Certificat d’accomplissement

Ce document certifie que

Kariane Ouellet

a complété le cours : l’Énoncé de politique des trois Conseils :
Éthique de la recherche avec des êtres humains :
Formation en éthique de la recherche (EPTC 2 : FER)

26 mai, 2016
April 18, 2017

Professor Peter Taylor
Math & Stats
Queen’s University
Jeffery Hall 513
Kingston, ON
K7L 3N6

Dear Professor Taylor:

Re: “GMATH-012-17 Math10 KCVI May 2017”

I am responding to your request to conduct a research project for the study “GMATH-012-17 Math10 KCVI May 2017” from Queen’s University in the Limestone District School Board, specifically at KCVI. I have reviewed your materials, and approval is granted.

You have submitted the following components required as part of the Limestone District School Board Administrative Procedure regarding External Research:

- written commitment to ensure anonymity of Board, schools, staff and students
- abstract of the research proposal
- copies of questionnaires, schedules
- participant consent form
- official ethics review
- CPIC

May I emphasize that in accordance with our Administrative Procedure 291: External Research participation in this research is entirely voluntary by the schools involved. Also, it is understood that for all research projects, the names of schools, participants, principals, teachers and students will not be identified in your final report, and participants would have the right to opt out of this project at any time. Best of luck on your research. I would appreciate a copy of your report when it is completed.

Sincerely,

Krishna Burra
Superintendent of Education
KB/ss

cc Debra Rantz, Director of Education
Talya McKenna, Principal, KCVI

Paula Murray – Chair | Debra Rantz – Director of Education and Secretary | Paul Babin – Treasurer

Our Students, Our Future
Appendix L

Proposed Interview Questions for Grade 10 Students

Affective Factors

1. How do you currently feel about the mathematics that you’ve been taught before starting this new approach?
2. What do you specifically find interesting about these transformations?
3. What do you specifically find uninteresting about the material taught?

Cognitive Factors

1. (In relation to a problem they solved that day) How did you solve this question?
   a. What steps did you take to find your solution?
2. In what way have you used the app to help you understand the problem?
3. What topics do you find difficult to understand?

Environmental Factors

1. What impact do you feel working with your classmates in small groups helps you?
   a. What role does each person play in solving the problems?
2. How do you feel about the inclusion of the grade 12 mentors?
   a. In what way does having the mentors help you with solving the problems?
   b. In what way do you feel the mentors hinder your ability to solve the problems?
Appendix M

Proposed Interview Questions for Teacher

Affective Factors

1. What aspects of the material taught, do you feel the students engage in?
2. What aspects of the material do you feel the students find uninteresting?
3. In what way do you feel the inclusion of technology helps the students engage in the problems?

Cognitive Factors

1. How do you feel the app helps the students understand the transformations?
2. In what way do you feel this approach captures the overall expectations, specifically regarding computational thinking and spatial reasoning, from the grade 10 curriculum?
3. In what way do you feel this approach does not help the students gain these overall expectations?

Environmental Factors

1. What do you feel the students are gaining by working in the small groups with their classmates?
2. What negative aspects do you feel arrive from the students working in groups?
3. What impact do you feel the grade 12 mentors have on the students?
   a. Positive impact?
   b. Negative impact?
RUNNING HEAD: Low threshold, high ceiling tasks and engagement
Appendix N

Interview Transcript Sample

P2: Yeah.

I1: Any other parts you found interesting? (Silence) It could be any part of the last two weeks. How things were taught, or the actual transformations that we did.

P1: I liked how we didn't have any textbook. It was all in the little booklet. There weren't like, any homework in the case where you didn't finish something.

I1: What about uninteresting?

P1: I didn't like the algebra. With all the multiplication, and then you had to do like, this times this equals to this - I didn't like that part.

I1: So what... could you be a little more specific about that?

P1: In like, when we did example 4, and then you had to find how it got there, with the transformations. I really thought that was like, very confusing, and I didn't know how to do it.

I2: So did you prefer the geometry?

P1: Yes.

I1: (Silence)

T2: Any differences?

P2: I mean, I... sometimes I got really confused and that just made me, like, uninterested with everything. It's just like, I don't get this... whatever. Like, it's fine. It gets like, a part of it is just me losing confidence in myself. Just like... you know, not understanding something, and then... well, I can't do it... I can't do it. Do you know what I mean?

I2: Yeah. Do you remember what portion was the most confusing to you?

P2: Probably towards... when we were doing the matrices and then, like, I guess that row by that column... I was getting really confused by that.
Appendix O

Observation Transcript Sample

<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>All 4 students participate in the conversation. Divya leaves. (id02) keeps working on the problem.</td>
</tr>
<tr>
<td>50</td>
<td>(id11) and (id?) have nothing written in their books.</td>
</tr>
<tr>
<td>51</td>
<td>(id02) and (id06) are still working. (id06) asks (id02) a question.</td>
</tr>
<tr>
<td>52</td>
<td>The other 2 are not really focused on task.</td>
</tr>
<tr>
<td>53</td>
<td>For this example (p.21), they just follow the other examples.</td>
</tr>
<tr>
<td>54</td>
<td>Now, they are just talking.</td>
</tr>
<tr>
<td>55</td>
<td>(id02) asked me how to find the shear.</td>
</tr>
<tr>
<td>56</td>
<td>I notice they follow the steps of the other examples a bit too closely without really thinking.</td>
</tr>
<tr>
<td>57</td>
<td>The teacher is explaining things to (id06).</td>
</tr>
<tr>
<td>58</td>
<td>(id06) is calculating things.</td>
</tr>
<tr>
<td>59</td>
<td>The other 2 are in their own worlds.</td>
</tr>
<tr>
<td>60</td>
<td>(id06) asks questions about the trig functions to the teacher.</td>
</tr>
</tbody>
</table>

(id20) asks me for help, I’ll be back.
Appendix P

R code

```r
library(magrittr)
library(dplyr)
library(tidyr)
library(class)
library(FNN)
library(ggplot2)
library(ordinal)

dir <- 'C:\Users\Kariane\Google Drive'
setwd(dir)

# load data
emotional <- read.csv('emotionaleng.csv',sep=';')
# Get the second row
slice <- emotional[-1,]
colnames(slice) <-
c('id','cog.neg','cog.pos','soc.neg','soc.pos','emo.neg','emo.pos')
slice[,1] <- slice[,1] %>% apply(2,as.numeric)
# Make a data set
slice.dat <- slice %>% gather(key,val,-id) %>% separate(key,c('mood','type'),'
') %>% spread(type,val) %>% tbl_df %>% mutate(ratio=pos/(pos+neg))

dep.vars <- slice.dat %>% filter(mood=='emo') %>% tbl_df %>% dplyr::select(-
c(neg,pos))
ex.vars <- slice.dat %>% filter(mood!='emo') %>% dplyr::select(-c(neg,pos)) %>%
spread(mood,ratio)
all.vars <- dep.vars %>% left_join(exp.vars,by='id')

# K nearest neighbours
train <- all.vars %>% filter(!is.na(ratio)) %>% dplyr::select(c(cog,soc)) %>%
as.matrix

test <- all.vars %>% filter(is.na(ratio)) %>% dplyr::select(c(cog,soc)) %>% as.matrix

resp <- all.vars %>% filter(!is.na(ratio)) %>% use_series(ratio)
pred <- FNN::knn.reg(train=train,test=test,y=resp,k=2)
data.frame(pred=pred$pred) %>% write.table(file='pred.csv',sep=';')

# Run a regression

test.dat <- read.csv('testvsengage.csv',sep=';')
test.dat$engagement <- as.numeric(gsub(',','.',test.dat$engagement))

# A linear regression
lm(score ~ I(engagement*100),data=test.dat) %>% summary

library(MASS)
?MASS::polr
# ordinal logistic regression
mdl.polr <- MASS::polr(score ~ I(engagement*100),data=test.dat %>%
mutate(score=as.factor(score)))
```

# log odds

104
coef(mdl.polr)
# odds
coef(mdl.polr) %>% exp
# Prob??
exp(coef(mdl.polr)) / (1 + exp(coef(mdl.polr)))
2*pt(2.71, df=nrow(test.dat)-3, lower.tail = F)

# With the other package
mdl.polr2 <- clm(score ~ I(engagement*100), data=test.dat %>% mutate(score=as.factor(score)))

ggplot(test.dat, aes(x=engagement*100, y=score)) + geom_point() +
  stat_smooth(method = "lm", col = "red", se=F) + labs(x="Engagement score", y="Achievement level", title="Linear Model")

#########################################################################
### Hypothesis Testing
#########################################################################
# Load in data
measure <- read.csv('bin_resp.csv', sep=';')
measure <- measure %>% tbl_df %>%
  mutate(NegNeut=Negative+Neutral, phat=Positive/Total) %>%
  filter(Total >= 10)
p.all <- with(measure, sum(Positive)/sum(Total))
# Get the number of classes
nam.measure <- measure$Measure %>% as.character()
n.measure <- length(nam.measure)
z.measure <- rep(NA, n.measure)
for (i in 1:n.measure) {
  m <- nam.measure[i]
  # Get the different statistics
  p.hat1 <- measure[i]$phat
  n1 <- measure[i]$Positive
  p.hat2 <- with(measure[-i], sum(Positive)/sum(Total))
  n2 <- sum(measure[-i]$Positive)
  z.temp <- (p.hat1 - p.hat2)/sqrt(p.all*(1-p.all)*(1/n1 + 1/n2))
  z.measure[i] <- z.temp
}
# Least positive test...
neg.test <- pnorm(z.measure, lower.tail = T) %>% round(3)
# Most positive test
pos.test <- pnorm(z.measure, lower.tail = F) %>% round(3)
data.frame(measure=nam.measure, z=z.measure, `pval (Least pos)`=neg.test, `Pval (Most Pos)`=pos.test)
RUNNING HEAD: Low threshold, high ceiling tasks and engagement

Appendix Q

Statistical Information

Emotions Toward Mathematics (Hypothesis testing processes)

First, we tested the hypothesis: $H_0: p = 0.5$ vs. $H_1: p < 0.5$ using a Z-test because the data points should be independent from each other. The Z-test works under the assumption that the statistic follows a normal distribution.

I assumed that the data (in this case, instances of positive and negative emotional engagement) follows a binomial distribution, i.e., $K \sim B(n, p)$, because the instances were either positive or negative. Using the Maximum Likelihood Estimator, I estimated $p$.

$$L(p|k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Taking the log and the derivative with respect to $p$, we find that the estimator of $p$ is:

$$\hat{p} = k/n$$

Now, the MLE has for property that it is asymptotically normally distributed, so in this case, $\hat{p} \sim N(p_0, -I^{-1}(p_0))$, where $I$ is the information matrix and $p_0$ is the true $p$. Using excel, I calculated the information matrix (second derivative of the log likelihood) based on the MLE of $p$, which leads to the standard deviation. The next step was to calculate the $z$ statistic:

$$z = \frac{\hat{p} - 0.5}{SE(\hat{p})}$$

Finally, I calculated the $p$-value. Recall that the $p$-value represents the smallest level of significance $\alpha$ for which the observed data indicate that the null hypothesis should be rejected.

The following table contains all the previously mentioned calculation for the first two hypothesis testing, which sought to determine if the students' feeling toward mathematics "overall" and "usual" (see chapter 4 for definitions) were negative:

<table>
<thead>
<tr>
<th>$\hat{p}$</th>
<th>$I(\hat{p})$</th>
<th>$I^{-1}(\hat{p})$</th>
<th>SE($\hat{p}$)</th>
<th>z-test</th>
<th>critical</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04166</td>
<td>601.04348</td>
<td>0.00166</td>
<td>0.04079</td>
<td>-11.237</td>
<td>1.96</td>
<td>4.13e-32</td>
</tr>
<tr>
<td>0.19444</td>
<td>229.83251</td>
<td>0.00435</td>
<td>0.06596</td>
<td>-4.632</td>
<td>1.96</td>
<td>1.43e-07</td>
</tr>
</tbody>
</table>

Emotions toward the different components of the experiment

The process is quite similar as shown previously, except that I used R instead of excel for simplicity. Moreover, for the two hypothesis tests, I used two-proportions z-tests (in the first, it was only one-proportion). I assume the data follows binomial distributions and hence, the MLE's for $p_e, p_m$ and the two values for $p_o$ are calculated using the same formula as above. For simplicity, consider the case where we were testing $H_0: p_e = p_o$ vs. $H_1: p_e < p_o$. The z-statistic is given by:

$$z = \frac{\hat{p}_e - \hat{p}_o}{\sqrt{\hat{p}(1-\hat{p})(1/n_e + 1/n_o)}}$$

Where $\hat{p} = \frac{n_e \hat{p}_e + n_o \hat{p}_o}{n_e + n_o}$ represents the proportional "successes" combined. Once these were all calculated, R outputted the p-values and they are all shown in Table 8, chapter 4.
## Appendix R

**Assessment Rubric (Ontario Ministry of Education, 2005)**

### Achievement Chart – Mathematics, Grades 9–12

<table>
<thead>
<tr>
<th>Categories</th>
<th>50–59% (Level 1)</th>
<th>60–69% (Level 2)</th>
<th>70–79% (Level 3)</th>
<th>80–100% (Level 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Knowledge and Understanding</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Knowledge of content</td>
<td>demonstrates limited knowledge of content</td>
<td>demonstrates some knowledge of content</td>
<td>demonstrates considerable knowledge of content</td>
<td>demonstrates thorough knowledge of content</td>
</tr>
<tr>
<td>Understanding of mathematical concepts</td>
<td>demonstrates limited understanding of concepts</td>
<td>demonstrates some understanding of concepts</td>
<td>demonstrates considerable understanding of concepts</td>
<td>demonstrates thorough understanding of concepts</td>
</tr>
<tr>
<td><strong>Thinking</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use of planning skills</td>
<td>uses planning skills with limited effectiveness</td>
<td>uses planning skills with some effectiveness</td>
<td>uses planning skills with considerable effectiveness</td>
<td>uses planning skills with a high degree of effectiveness</td>
</tr>
<tr>
<td>– understanding the problem (e.g., formulating and interpreting the problem, making conjectures)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>– making a plan for solving the problem</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use of processing skills</td>
<td>uses processing skills with limited effectiveness</td>
<td>uses processing skills with some effectiveness</td>
<td>uses processing skills with considerable effectiveness</td>
<td>uses processing skills with a high degree of effectiveness</td>
</tr>
<tr>
<td>– carrying out a plan (e.g., collecting data, questioning, testing, revising, modelling, solving, inferring, forming conclusions)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>– looking back at the solution (e.g., evaluating reasonableness, making convincing arguments, reasoning, justifying, proving, reflecting)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use of critical/creative thinking processes (e.g., problem solving, inquiry)</td>
<td>uses critical/creative thinking processes with limited effectiveness</td>
<td>uses critical/creative thinking processes with some effectiveness</td>
<td>uses critical/creative thinking processes with considerable effectiveness</td>
<td>uses critical/creative thinking processes with a high degree of effectiveness</td>
</tr>
</tbody>
</table>

*The processing skills and critical/creative thinking processes in the Thinking category include some but not all aspects of the mathematical processes described on pages 12–16 of this document. Some aspects of the mathematical processes relate to the other categories of the achievement chart.*
<table>
<thead>
<tr>
<th>Categories</th>
<th>50–59% (Level 1)</th>
<th>60–69% (Level 2)</th>
<th>70–79% (Level 3)</th>
<th>80–100% (Level 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Communication</strong> The conveyance of meaning through various forms</td>
<td>- expresses and organizes mathematical thinking with limited effectiveness</td>
<td>- expresses and organizes mathematical thinking with some effectiveness</td>
<td>- expresses and organizes mathematical thinking with considerable effectiveness</td>
<td>- expresses and organizes mathematical thinking with a high degree of effectiveness</td>
</tr>
<tr>
<td>Expression and organization of ideas and mathematical thinking (e.g., clarity of expression, logical organization), using oral, visual, and written forms (e.g., pictorial, graphic, dynamic, numeric, algebraic forms; concrete materials)</td>
<td>- communicates for different audiences and purposes with limited effectiveness</td>
<td>- communicates for different audiences and purposes with some effectiveness</td>
<td>- communicates for different audiences and purposes with considerable effectiveness</td>
<td>- communicates for different audiences and purposes with a high degree of effectiveness</td>
</tr>
<tr>
<td>Communication for different audiences (e.g., peers, teachers) and purposes (e.g., to present data, justify a solution, express a mathematical argument) in oral, visual, and written forms</td>
<td>- uses conventions, vocabulary, and terminology of the discipline with limited effectiveness</td>
<td>- uses conventions, vocabulary, and terminology of the discipline with some effectiveness</td>
<td>- uses conventions, vocabulary, and terminology of the discipline with considerable effectiveness</td>
<td>- uses conventions, vocabulary, and terminology of the discipline with a high degree of effectiveness</td>
</tr>
<tr>
<td>Use of conventions, vocabulary, and terminology of the discipline (e.g., terms, symbols) in oral, visual, and written forms</td>
<td>- applies knowledge and skills in familiar contexts with limited effectiveness</td>
<td>- applies knowledge and skills in familiar contexts with some effectiveness</td>
<td>- applies knowledge and skills in familiar contexts with considerable effectiveness</td>
<td>- applies knowledge and skills in familiar contexts with a high degree of effectiveness</td>
</tr>
<tr>
<td><strong>Application</strong> The use of knowledge and skills to make connections within and between various contexts</td>
<td>- transfers knowledge and skills to new contexts with limited effectiveness</td>
<td>- transfers knowledge and skills to new contexts with some effectiveness</td>
<td>- transfers knowledge and skills to new contexts with considerable effectiveness</td>
<td>- transfers knowledge and skills to new contexts with a high degree of effectiveness</td>
</tr>
<tr>
<td>Application of knowledge and skills in familiar contexts</td>
<td>- makes connections within and between various contexts with limited effectiveness</td>
<td>- makes connections within and between various contexts with some effectiveness</td>
<td>- makes connections within and between various contexts with considerable effectiveness</td>
<td>- makes connections within and between various contexts with a high degree of effectiveness</td>
</tr>
<tr>
<td>Transfer of knowledge and skills to new contexts</td>
<td>- makes connections within and between various contexts with limited effectiveness</td>
<td>- makes connections within and between various contexts with some effectiveness</td>
<td>- makes connections within and between various contexts with considerable effectiveness</td>
<td>- makes connections within and between various contexts with a high degree of effectiveness</td>
</tr>
</tbody>
</table>
Appendix S

Students test marks, and Individual Cognitive-behavioural, social-behavioural, and emotional engagement scores, and engagement scores.

<table>
<thead>
<tr>
<th>Student ID</th>
<th>Test Level</th>
<th>Cognitive-behavioural</th>
<th>Social-behavioural</th>
<th>Emotional</th>
<th>Individual engagement Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>3</td>
<td>86%</td>
<td>75%</td>
<td>33%</td>
<td>65%</td>
</tr>
<tr>
<td>02</td>
<td>2</td>
<td>55%</td>
<td>33%</td>
<td>35%</td>
<td>41%</td>
</tr>
<tr>
<td>03</td>
<td>1</td>
<td>0%</td>
<td>0%</td>
<td>29%</td>
<td>10%</td>
</tr>
<tr>
<td>04</td>
<td>2</td>
<td>100%</td>
<td>50%</td>
<td>33%</td>
<td>61%</td>
</tr>
<tr>
<td>05</td>
<td>3</td>
<td>89%</td>
<td>0%</td>
<td>65%</td>
<td>51%</td>
</tr>
<tr>
<td>06</td>
<td>2</td>
<td>33%</td>
<td>50%</td>
<td>33%</td>
<td>39%</td>
</tr>
<tr>
<td>07</td>
<td>3</td>
<td>100%</td>
<td>0%</td>
<td>65%</td>
<td>55%</td>
</tr>
<tr>
<td>08</td>
<td>1</td>
<td>17%</td>
<td>20%</td>
<td>29%</td>
<td>22%</td>
</tr>
<tr>
<td>09</td>
<td>2</td>
<td>60%</td>
<td>0%</td>
<td>44%</td>
<td>35%</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>30%</td>
<td>17%</td>
<td>29%</td>
<td>25%</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>30%</td>
<td>25%</td>
<td>29%</td>
<td>28%</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>46%</td>
<td>44%</td>
<td>33%</td>
<td>41%</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>29%</td>
<td>25%</td>
<td>50%</td>
<td>35%</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>67%</td>
<td>33%</td>
<td>29%</td>
<td>43%</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>64%</td>
<td>40%</td>
<td>38%</td>
<td>47%</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>73%</td>
<td>17%</td>
<td>75%</td>
<td>55%</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>33%</td>
<td>0%</td>
<td>7%</td>
<td>13%</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>40%</td>
<td>0%</td>
<td>13%</td>
<td>18%</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>50%</td>
<td>43%</td>
<td>32%</td>
<td>42%</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>86%</td>
<td>0%</td>
<td>56%</td>
<td>47%</td>
</tr>
</tbody>
</table>
Appendix T

Student test – Solution (level 4)

Name: Kariane Queiller

Question 1. Using geometric reasoning, find the parameters \(a, b, h\) and \(\theta\). Briefly describe your reasoning, that is, explain where your answer came from so that a fellow student who hasn’t seen this type of problem before could understand the answer. Keep in mind that the only information you can be sure of is that in graphs 1 and 4 as there the corners are tied to the grid points.

Find \(a\)
One horizontal unit is still one unit after being dilated.
So \(a = 1\).

Find \(b\)
One vertical unit is 2 units long in the second graph.
So \(b = 2\).

Find \(h\)
The point on \(y = 2\) are moved right (positive) of one unit.
\(h = 1\).

Find \(\theta\)
The horizontal line becomes vertical.
So \(\theta = 90^\circ\).
Question 2. Same problem as Question 1, but now use algebraic reasoning (matrices and equations) to find the parameters $a$, $b$, $h$ and $\theta$. Write clearly and neatly—make your symbols big enough that they are easily read. Find $\theta$, $a$, $b$ and $h$ such that:

$$T = [R(\theta)] \cdot [S(h)] \cdot [D(a, b)]$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Matrix multiplication

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$$

System of equation

1) $0 = ca$
2) $1 = sa$
3) $-2 = chb - sb$
4) $2 = shb + cb$

Solve

$$0 = ca \quad \rightarrow \quad a \neq 0 \text{ by looking at the picture.}$$

$$0 = \cos \theta$$

$$\theta = \cos^{-1}(0)$$

$$\theta = 90^\circ$$

$$1 = a \sin(90^\circ)$$

$$1 = a$$

3) $-2 = \cos 90 \cdot h \cdot b - \sin 90 \cdot b$

$$-2 = -b \quad \Rightarrow \quad b = 2$$

4) $2 = \sin 90 \cdot h \cdot b + \cos 90 \cdot b$

$$2 = h \cdot 2$$

$$h = 2$$
Question 3. This is the same transformation $T$ as Question 1, but the rotation has been put first. Use algebraic reasoning (matrices and equations) to find the parameters $a$, $b$, and $h$.

That is, you need to find $\theta$, $a$, $b$, and $h$ such that:

\[
T = [S(h)] \cdot [D(a,b)] \cdot [R(\theta)]
\]

\[
\begin{bmatrix}
0 & -2 \\
1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & h \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
a & 0 \\
b & 0
\end{bmatrix} \cdot \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

Matrix multiplication:

\[
\begin{bmatrix}
a & ab \\
b & b
\end{bmatrix} \cdot \begin{bmatrix}
c & -s \\
s & c
\end{bmatrix} = \begin{bmatrix}
a \cdot c + sb & -sa + hbc \\
sc & cb
\end{bmatrix}
\]

System of equations:

1) $0 = ac + sb$
2) $1 = sb$
3) $-2 = -sa + hbc$
4) $2 = cb$

Solve:

\[
\frac{1}{2} = \frac{sb}{cb} = \tan \theta
\]

\[
\theta = \tan^{-1} \left( \frac{1}{2} \right) \approx 26.56^\circ
\]

\[
x \text{ this, }
\]

\[
1 = \sin \left( \tan^{-1} \left( \frac{1}{2} \right) \right) \cdot b
\]

\[
1 \approx 0.44 \cdot b \\
(b \approx 2.24)
\]

3) $-2 = -\sin(26.56^\circ)a + (0.899)2.24$

\[
-2 = -0.44a - 1.78a \\
-2 = -2.22a
\]

\[
a = 0.899
\]

\[
h = -0.899 \cdot 0.899
\]

\[
h = -0.899
\]
Question 4. Same problem as Question 3, but now use geometric reasoning to find the parameters $a$, $b$, $h$ and $\theta$. This problem is more difficult than the others. Try finding the dilation multiplier $a$ first. For that you will need a pair of horizontal lines in graphs 2 and 3. The best strategy is to begin by finding a good horizontal line in graph 4—as you know it will still be horizontal in graphs 3 and 2.

If you manage to find $a$, next go after $b$.

After that, $b$ and then $h$ are a bit harder. But very satisfying. There’s a larger diagram on the next page.

---

**Find $a$**

A good horizontal line in (4) is pink (length 2). It goes from the yellow corner to the middle of the grey side, so we can trace it back to graphs 3, then 2, then 1:

\[ a = \frac{2}{\sqrt{5}} \approx 0.90 \]

---

**Find $\theta$**

The pink line in graph 2 is horizontal. Its angle in graph 1 is:

\[ \tan \theta = \frac{1}{2} \]

\[ \theta = \tan^{-1} \left( \frac{1}{2} \right) \approx 26.56^\circ \]
Find b, pink line:

\[ b = \frac{-1000}{900} = -\frac{2}{9} \times 2,2 \text{A} \]

Find c, blue line:

\[ c = \frac{0.8g}{1} \]

Find g, red line:

\[ g = 26.6 \text{C} \]

Find h, green line:

\[ h = 0.9 \text{A} \]

Find a, purple line:

\[ a = \frac{-2}{-1.2} \]

Find n, orange line:

\[ n = 0.8 \times a \]

Find m, yellow line:

\[ m = 1.34 \times 0.90 = 1.21 \]

Find τ, black line:

\[ \tau = 45 - 245 \]

Find o, white line:

\[ o = 18 - 245 \]

Find p, cyan line:

\[ p = 45 - 245 \]

Find q, magenta line:

\[ q = 18 - 245 \]

Find r, pink line:

\[ r = 26.6 \text{C} \]

Find s, blue line:

\[ s = 0.8g \]

Find t, green line:

\[ t = 1.34 \times 0.90 = 1.21 \]

Find u, red line:

\[ u = 26.6 \text{C} \]

Find v, purple line:

\[ v = 0.8g \]

Find w, orange line:

\[ w = 1.34 \times 0.90 = 1.21 \]

Find x, yellow line:

\[ x = 18 - 245 \]

Find y, cyan line:

\[ y = 45 - 245 \]

Find z, magenta line:

\[ z = 18 - 245 \]
Appendix U

Student tests samples

The questions left untouched are not shown.

Student 08: Level 1 (Q1: Level 3; Q2, Q3, Q4: Level 1)
**Question 4.** Same problem as Question 3, but now use geometric reasoning to find the parameters $a, b, h$ and $\theta$. This problem is more difficult than the others. Try finding the dilation multiplier $a$ first. For that you will need a pair of horizontal lines in graphs 2 and 3. The best strategy is to begin by finding a good horizontal line in graph 4—as you know it will still be horizontal in graphs 3 and 2.

If you manage to find $a$, next go after $\theta$.

After that, $b$ and then $h$ are a bit harder. But very satisfying. There’s a larger diagram on the next page.

---

**Question 1.** Using geometric reasoning, find the parameters $a, b, h$ and $\theta$. Briefly describe your reasoning, that is, explain where your answer came from so that a fellow student who hasn’t seen this type of problem before could understand the answer. Keep in mind that the only information you can be sure of is that in graphs 1 and 4 as there the corners are tied to the grid points.

---

Student 10: Level 2 (Q2: Level 4; Q1, Q3: Level 3; Q4: Level 1)
Question 1. Using geometric reasoning, find the parameters \( a, b, h \) and \( \theta \). Briefly describe your reasoning, that is, explain where your answer came from so that a fellow student who hasn’t seen this type of problem before could understand the answer. Keep in mind that the only information you can be sure of is that in graphs 1 and 4 as there the corners are tied to the grid points.

Student 07: Level 3 (Q1, Q2: Level 4, Q3: Level 2, Q4: Level 1)
No student reach level 4. To see what a level 4 would look like, see Appendix.
Appendix V

Student Workbooks: Introduction (Taylor, 2017)

Introduction to Transformations

Here we will be working with transformations $T$ of a particular kind. Essentially, they move points around on the plane. In this way they are functions—they associate to each point $P$ in the plane an image point $T(P)$. Here’s a diagram of an example.

The diagram uses a coordinate grid with the grid lines 1 unit apart. On the left we have the “domain” of the function where the points $P$ live. I have drawn an iconic 2×2 square centred at the origin. On the right this square has been morphed into a parallelogram centred at the origin. It’s purpose is to tell you what $T(P)$ is.

In this unit, we will use these diagrams to define or specify functions, and one thing you need to know is that the information the diagram seems to give you is “reliable.” For example, look at the $T$-image of the point $(1, 1)$. It seems to lie on the grid point $(-1, 2)$. Well it really does, that is, $T(1, 1) = (-1, 2)$. And similarly, $T(-1, 1) = (-2, 0)$, etc.

The second thing you need to know is that $T$ is what is called “linear.” You’ll have to wait for a precise definition of that, but what it means is that $T$ preserves “lines” and by this I mean that it maps lines into lines and even more, it preserves relative distances along these lines.

For example $(-1, 0)$ is halfway between $(−1, 1)$ and $(−1, −1)$, so we can deduce that $T(−1, 0)$ is halfway between $T(−1, 1)$ and $T(−1, −1)$. Now $T(−1, 1) = (−2, 0)$ and $T(−1, −1) = (1, −2)$ so $T(−1, 0)$ is halfway between these two and that’s the point $(-1/2, −1)$.
[Can you see why \((-1/2, -1)\) is halfway between \((-2, 0)\) and \((1, -2)\)? You do that coordinate-wise. The \(x\)-coordinate \(-1/2\) is halfway between the two \(x\)-coordinates \(-2\) and \(1\), and the \(y\)-coordinate \(-1\) is halfway between the two \(y\)-coordinates \(0\) and \(-2\).]
The Basic Transformations

The basic transformations are the building blocks that we will put together to make more complex transformations. They come in three distinct flavours, dilations, rotations, and shears. Here are a few examples. Describe the effect of each of these transformations in simple words—as if you were talking about them to a friend.

1. The dilations $D(a, b)$

2. The rotations $R(\theta)$

2. The (horizontal) shears $S(h)$
Some dilations
In each case, provide the diagram in the right-hand grid.

\[ D(2,3) \]

\[ D(\frac{1}{2}, \frac{5}{2}) \]

\[ D(2, 2) \]

\[ D(\frac{1}{2}, \frac{1}{2}) \]
In each case calculate $a$ and $b$
A few rotations

In each case, provide the diagram in the right-hand grid.

\[ R(180) \]

\[ R(135) \]

\[ R(-90) \]

\[ R(90) \]
In each case calculate the rotation angle $\theta$.

[When a line seems to go through a grid point assume that’s exactly the case.]
The shear

The best way to understand the shear is to imagine the square made up of horizontal lines. When we apply the shear $S(h)$, each such line is moved sideways (horizontally) but lines at different heights are moved different amounts. In fact the amount the line is moved is proportional to its height above the x-axis (its y-coordinate) and $h$ is the constant of proportionality. In the example below I have taken $h = 3/2$.

Points with $y = 1$ are moved a distance $h$.

Points with $y = 1/2$ are moved a distance $h/2$.

Points with $y = 2$ are moved a distance $2h$.

Points with $y = -1$ are moved a distance $-h$.

Points with $y = y$ are moved a distance $hy$.

For example, under $S(3/2)$ a line at height $1/2$ will move a distance $(3/2)(1/2) = 3/4$ and a line at height $2$ will move a distance $(3/2)(2) = 3$.

Also note that the line at height $0$ (on the x-axis) doesn’t move at all!

In each case, provide the diagram in the right-hand grid.
Appendix W

Student Workbooks: First 2 examples (Taylor, 2017)

A two-step transformation.

Provide the missing diagrams

\[ D(1/2, 2) \quad S(1/2) \]
Show that you can get to the same endpoint with a shear first followed by a dilation.

[Hint: Based on the properties of a shear, you can figure out what the dilation would have to be. Then work the dilation backwards to see what the intermediate diagram has to look like.]
Factoring transformations

First an important note: the transformations $T$ you are given to factor will always be assumed to be linear, and the corner points will always be mapped to lattice points. Below I give you the diagram of a transformation $T$. It is not one of the basic transformations, but you can probably see ways in which it can be constructed by putting basic transformations together. For example, can you see how to construct it as a sequence of two basic transformations? Identify the two component transformations and draw the intermediate shape. Just to emphasize, the intermediate steps can only be dilations, rotations (about the origin) or (horizontal) shears.
At the right, is the solution to the previous problem—start with a 90º rotation and follow that with a shear.

Would we get the same result if we interchanged the two steps—the shear first and the rotation second? See what happens—track the image through the two steps given below.
You do not get the same answer. In fact it’s not hard to show that you cannot obtain the original target transformation in two basic steps if the rotation has to come second, no matter what parameters are used.
But it can be done in three steps with the rotation last. The diagram below prescribes a dilation followed by a shear, followed by a rotation. Find the parameters $a$, $b$, $h$ and $\theta$ and draw the intermediate diagrams.

Given only the diagram above, this is a hard problem—at least I find it hard and a group of teachers I gave this problem to a few years ago also found it hard. The reason for that is that we rely on diagrams to help us think and it’s hard to figure out what the intermediate diagrams look like. Indeed I find that a diagram of the intermediate configurations helps the students a lot in developing their geometric thinking skills, and so I routinely give them the complete diagram. Turn the page.
Example 1.

Given the diagram, find the parameters $a$, $b$, $h$ and $\theta$. 

\[ D(a,b) \]

\[ S(h) \]

\[ R(\theta) \]

\[ T \]
Example 2.

Here is another transformation and again I have asked you to “factor” it as a composition of basic transformations. Indeed, I have used the same components as in Example 1, and in the same order—an dilation, followed by a shear, followed by a rotation. Again I have given you the intermediate diagrams to help you in your thinking.

Actually, this problem turns out to be quite similar to Example 1 so that it can serve as a good review of what you have learned. Find the parameters $a, b, h$ and $\theta$. 

![Transformation Diagrams](Image)
Example 1&2 Answers

\[ D(a, b) \]
\[ a = \sqrt{2} \]
\[ b = 1/\sqrt{2} \]

\[ h = -1 \]

\[ T \]

\[ R(\theta) \]
\[ \theta = 45 \]
Appendix X

Student Workbooks: The algebra

Geometry and Algebra

Now we are going to look at a remarkable complementarity between geometry and algebra. We have been using geometric thinking to solve a couple of transformation problems. It turns out that such problems can also be solved with a purely algebraic approach. Not only that, but the objects and operations of the geometric approach correspond faithfully to objects and operations in the algebraic approach. In an important sense, the geometric and algebraic realizations have the same underlying structure and for that reason, such a correspondence is called an isomorphism (*isoc* same, *morph* structure).

The formalization of this idea goes back to the French philosopher and mathematician René Descartes (1596 – 1650) after whom the “Cartesian” coordinate system is named. Indeed as we shall see, it is Descartes’ coordinate system that implements the isomorphism.
So, we have these two worlds, geometry and algebra. They each have their own objects and operations.

*Objects*: Geometry works with transformations—the corresponding algebraic entities are called matrices.

*Operations*: Geometry “composes” the transformations, applying them one after the other. Algebra “multiplies” its matrices, one after the other.

We will discover that the two approaches are doing exactly the same thing, but with very different modes of thought.

<table>
<thead>
<tr>
<th>A remarkable isomorphism</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Objects</strong></td>
</tr>
<tr>
<td>Geometry</td>
</tr>
<tr>
<td>transformations</td>
</tr>
<tr>
<td>composition</td>
</tr>
</tbody>
</table>
The correspondence between Transformations and Matrices

First, we take the transformation $T$ of Example 1. We associate to this a $2\times2$ matrix $[T]$ as follows.

Take the two points $E_1 = (1, 0)$ and $E_2 = (0, 1)$ on the right side and top of the starting square (they are the endpoints of that diagonal line). Find their images $T(E_1)$ and $T(E_2)$ in the transformed square. These are

$T(E_1) = (1,1)$

$T(E_2) = (-1,0)$.

We take these as the columns of the matrix of $T$:

$$[T] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Second, we take the transformation $T$ of Example 2, and using the same system, we construct its matrix $[T]$. We locate the $T$-images of the two points $E_1 = (1, 0)$ and $E_2 = (0, 1)$. These are

$T(E_1) = (1,-1)$

$T(E_2) = (0,2)$

and again these are the columns of the matrix of $T$: 
\[ [r] = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}. \]

This correspondence between linear transformations and matrices is a “bijection”—every linear transformation has a matrix and every 2×2 matrix corresponds to exactly one linear transformation.

**The Matrices of the Basic Transformations**

Now, using the system above, we construct the matrices of the basic transformations. Recall that the columns of the matrix are the \( T \)-images of \( E_1 \) and \( E_2 \). In each case fill in the entries of the matrix.

**The matrix of the dilation** \( D(a, b). \)

\[ [D(2, 3)] = \begin{bmatrix} \phantom{-}1 & \phantom{-}0 \\ -1 & \phantom{-}2 \end{bmatrix} \]

\[ [D(3/2, 3/4)] = \begin{bmatrix} \phantom{-}1 & \phantom{-}0 \\ -1 & \phantom{-}2 \end{bmatrix} \]
\[ [D(a, b)] = \begin{bmatrix} \end{bmatrix} \]
The matrix of the rotation $R(\theta)$.

$[R(-90)] = \begin{bmatrix} \cos(-90) & \sin(-90) \\ \sin(-90) & -\cos(-90) \end{bmatrix}$

$[R(45)] = \begin{bmatrix} \cos(45) & \sin(45) \\ -\sin(45) & \cos(45) \end{bmatrix}$

$[R(\theta)] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

The matrix of the rotation $R(\theta)$. The coordinate transformation $T(E_1)$ is given by:

$x = \cos(\theta)$

$y = \sin(\theta)$

$T(E_1) = (\cos(\theta), \sin(\theta))$

The coordinate transformation $T(E_2)$ is given by:

$x = -\sin(\theta)$

$y = \cos(\theta)$

$T(E_2) = (-\sin(\theta), \cos(\theta))$
The matrix of the shear $S(h)$.

$[S(2)] = \begin{bmatrix} \end{bmatrix}$

$[S(-1/2)] = \begin{bmatrix} \end{bmatrix}$

$[S(h)] = \begin{bmatrix} \end{bmatrix}$
General formulae for the matrices of the basic transformations

Dilation: \[ D(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \]

Rotation: \[ R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]

Shear: \[ S(h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \]
Matrix Multiplication

We start with the product of a 2×2 matrix and a 2×1 matrix.

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
a + 2b \\
3a + 4b
\end{pmatrix}
\]

See the pattern? We use the first row of the matrix to get the first entry and the second row to get the second entry. In each case we do a particular type of multiplication called a “dot product.”

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
a + 2b \\
3a + 4b
\end{pmatrix}
\]

Thus:

\[
\begin{pmatrix}
2 & -1 \\
3 & 5
\end{pmatrix}
\begin{pmatrix}
2 \\
4
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
26
\end{pmatrix}
\]

Calculate:

\[
\begin{pmatrix}
2 & 1 \\
4 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
5 \\
7
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
3 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
4
\end{pmatrix}
= 
\begin{pmatrix}
10 \\
10
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -2 \\
1 & 5
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
6
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
-3
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-3
\end{pmatrix}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\] is called the identity matrix. Can you see why?

Now we look at the product of two \(2 \times 2\) matrices. It turns out to be the obvious extension of what we did above:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} =
\begin{bmatrix}
a + 2c & b + 2d \\
3a + 4c & 3b + 4d \\
\end{bmatrix}
\]

See the pattern? You take a row of the first matrix and a column of the second matrix, “multiply” them in a particular way (again this is a dot product), and you get the corresponding row and column entry in the product matrix:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} =
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} =
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} =
\begin{bmatrix}
a + 2c & b + 2d \\
3a + 4c & 3b + 4d \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 \\
4 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
5 & 5 \\
7 & 9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
4 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
10 & 3 \\
10 & 4 \\
\end{bmatrix}
\]

[This shows you that matrix multiplication may not be commutative—order matters!]

\[
\begin{bmatrix}
1 & -2 \\
1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
-3 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
7 & 0 \\
-14 & 7 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
5 & 4 \\
3 & 2 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
5 & 4 \\
3 & 2 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] is called the *identity matrix*. Can you see why?

**Composition of transformations—multiplication of matrices**

When we put two or more transformations together in sequence we are forming their “composition.” For example we factored the transformation of Example 1 as a dilation \( D \), followed by a shear \( S \), followed by a rotation \( R \). To formulate this, we use the standard composition of functions symbol \( \circ \) and we write:

\[ T = R \circ S \circ D. \]

Note the unexpected “reverse” order. The transformation \( D \) is the first one we apply, but in the formula it is written last. The reason for this comes from our convention for function notation. We write \( f(x) \) for the effect of the function \( f \) on the number \( x \). In the same way we write \( T(P) \) for the effect of the transformation \( T \) on the point \( P \). Given that, the reverse order for the composition makes sense:

\[ T(P) = (R \circ S \circ D)(P) = R(S(D(P))). \]

Why does this work? How does the matrix do the job of the transformation and why does matrix multiplication correspond to composition? This is a nice question to track down, but for now I give you the key idea.

Consider the transformation of Example 1 pictured above. The matrix of \( T \) is \[ T = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}. \] Now take any point in the plane, say \((-1, 1)\), the top left corner of the square. Its \( T \)-image is \((-2, -1)\). Now calculate:

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix}^{-1} = \begin{bmatrix} 1 \\
1
\end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 \\
-1
\end{bmatrix}.
\]
This tells us that to get the image of \( P \) under \( T \), we apply \( D \) first to \( P \), and then we apply \( S \) to \( D(P) \) and then finally we apply \( R \) to \( S(D(P)) \).

It helps our thinking if we read the composition symbol \( \circ \) as “after.” Thus the equation \( T = R \circ S \circ D \) tells us that we can construct the transformation \( T \) as “\( R \) after \( S \) after \( D \).”

Okay. We have established a correspondence between transformations and matrices and are now ready for the critical second half of the isomorphism—*the composition of transformations corresponds to the multiplication of matrices.* For example, the composition

\[
T = R \circ S \circ D
\]

corresponds to the matrix product:

\[
[T] = [R] \cdot [S] \cdot [D].
\]

In the diagram above, if we take the matrices of the component transformations and multiply them together (in the correct order) we will get the matrix of \( T \). What that will
give us is an algebraic way to solve our geometric factorization problems.
Example 1--algebra

Let’s see how this works for Example 1. We need to find $\theta$, $a$, $b$ and $h$ such that:

$$T = R(\theta) \circ S(h) \circ D(a,b)$$

The first step is to form the corresponding matrix equation:

$$[T] = [R(\theta)] \cdot [S(h)] \cdot [D(a,b)]$$

Note that to move from the first transformation equation to the second matrix equation, the composition symbol $\circ$ is simply replaced by the multiplication sign. Now write out the matrices:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$ 

To simplify notation I set $s = \sin \theta$ and $c = \cos \theta$:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$ 

Multiply the matrices out. Let’s start with the last two:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix}$$

and then do the remaining multiplication:

I believe you have worked with two equations in two unknowns but few of you will ever have had to grapple with a system of four equations in four unknowns. There are good reasons for that—the calculations can be substantial.

But these equations have a simple structure and the math is a piece of cake.
Comparing each of the four entries, we get 4 equations in 4 unknowns:

(1) \[ 1 = ca \]

(2) \[ 1 = sa \]

(3) \[ -1 = chb - sb \]

(4) \[ 0 = shb + cb \]

The problem now has an algebraic formulation—solve this set of four equations for the four unknowns \( \theta, a, b \) and \( h \).
Solving the equations

(1) \[ 1 = ca \]
(2) \[ 1 = sa \]
(3) \[ -1 = chb - sb \]
(4) \[ 0 = shb + cb \]

Divide (2) by (1) to get

\[ \frac{sa}{ca} = \frac{1}{1} = 1 \]

\[ \frac{s}{c} = \frac{\sin \theta}{\cos \theta} = 1 \]  (cancel the \( a \) in top and bottom)

\[ \tan \theta = 1 \]

\[ \theta = 45. \]

Check that this fits with the diagram.

Hence \( s = \sin \theta = \sin 45 = 1/\sqrt{2} \)

\[ c = \cos \theta = \cos 45 = 1/\sqrt{2} . \]

From (2) we then get  \( a = \frac{1}{s} = \frac{1}{\sin \theta} = \sqrt{2} . \)

Since \( \sqrt{2} \approx 1.4 \), this makes sense from the diagram.
From (4) we then get \( shb = -cb \)

\[
\begin{align*}
h &= \frac{-cb}{sb} = -\frac{c}{s} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1
\end{align*}
\]

And this also makes sense from the diagram.

Finally, from (3):

\[
\begin{align*}
shb &= 1 \\
b(s - ch) &= 1 \\
\left(b \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(-1)\right)\right) &= 1 \\
\left(b \left(\frac{2}{\sqrt{2}}\right)\right) &= 1 & \Rightarrow & b &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.
\end{align*}
\]
Example 2--algebra

Now try your hand at Example 2. The sequence of basic transformations is the same as in Example 1, so we’ll get the same matrix product, but with a new left-hand side.

\[ T = R(\theta) \circ S(h) \circ D(a,b) \]

\[
[T] = [R(\theta)] \cdot [S(h)] \cdot [D(a,b)]
\]

\[
\begin{bmatrix}
1 & 0 \\
-1 & 2
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \cdot \begin{bmatrix}
1 & h \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\]

To simplify set \( s = \sin \theta \) and \( c = \cos \theta \):

\[
\begin{bmatrix}
1 & 0 \\
-1 & 2
\end{bmatrix} = \begin{bmatrix}
c & -s \\
s & c
\end{bmatrix} \cdot \begin{bmatrix}
1 & h \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\]

Now set up and solve the four equations.