Riding the waves: The formation and evolution of vertical bending waves in Milky Way-like disc galaxies

by

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Abstract

In this thesis we study the dynamics of vertical (bending) waves in Milky Way-like disc galaxies. The main goals of the thesis are to form a coherent picture for the formation and evolution of bending waves using three-dimensional N-body simulations, develop tools and machinery to aid in the study of vertical waves in both simulations and astrometric surveys, and to make predictions for what we might see in the analysis of Gaia data.

The hallmark simulations presented in this thesis evolve a two-component disc embedded in smooth and ‘clumpy’ haloes, where the latter contains multiple orbiting substructure. One of our main conclusions is that vertical (bending) waves should be generic, long-lived features of massive discs that can form with and without provocation from external agents, though the predominant type of vertical wave induced by the latter is dependent on the perturber’s orbital parameters. Furthermore, we find that bending is a property almost exclusive to kinematically cooler (thin disc) stellar populations.

In our simulations bending waves manifest as variations in mean vertical displacement and bulk vertical motions, that together behave largely as simple monochromatic plane waves. At intermediate radii they appear as tightly wound, short-wave corrugations that match smoothly onto the warp near the edge of the disc. By way
of Fourier and spectral analyses, based on classical studies of in-plane density waves and further developed in this thesis for the study of vertical waves, we find that in general the waves are a superposition of modes that comprise two main branches on the radius-rotational frequency plane.

The preeminent result of this thesis is a novel sequence of events describing the life-cycle of bending waves, which involves excitation, dispersion, phase-wrapping or shearing across the disc, self-gravitating wave-like action, and disc heating. This conclusion is largely borne out of the comparison between frequency power spectra of waves in our simulations with predictions from linear perturbation theory, and suggests that the wave-like features in astrometric surveys such as *Gaia* may indicate the existence of long-lived modes of a dynamically active disc in addition to perturbations from recent disc-satellite interactions.
Statement of Co-Authorship

The research presented in this thesis was completed under the supervision of Prof. Lawrence M. Widrow at Queen’s University. All of the work presented here was done by the author (Matthew H. Chequers) except where explicitly stated otherwise.

Chapter 3 contains a version of a paper published in Monthly Notices of the Royal Astronomical Society as: Matthew H. Chequers, Kristine Spekkens, Lawrence M. Widrow and Colleen Gilhuly. Simulating a slow bar in the low surface brightness galaxy UGC 628. *Monthly Notices of the Royal Astronomical Society*, 463:1751-1758, 2016. I am the lead author of this paper. Prof. Kristine Spekkens, Prof. Lawrence M. Widrow, and I wrote the manuscript. I produced all of the figures and performed all of the calculations, except for the analysis of photometric data (Section 3.3.1) which was done by Colleen Gilhuly.

Chapter 4 contains a version of a paper published in Monthly Notices of the Royal Astronomical Society as: Matthew H. Chequers and Lawrence M. Widrow. Spontaneous generation of bending waves in isolated Milky Way-like discs. *Monthly Notices of the Royal Astronomical Society*, 472:2751-2763, 2017. I am the lead author of this paper. Prof. Lawrence M. Widrow and I wrote the manuscript. I produced all of the figures and performed all of the calculations, except for the content of the linear ring analysis (Section 4.5) which was done by Prof. Lawrence M. Widrow.

Chapter 6 contains a version of a revised paper submitted on July 17, 2018 to Monthly Notices of the Royal Astronomical Society as: Matthew H. Chequers and Lawrence M. Widrow. Bending Waves in the Milky Way’s disc from halo substructure. Manuscript ID: MN-18-1949-MJ.R1. I am the lead author of this paper. Prof. Lawrence M. Widrow and I wrote the manuscript. I produced all of the figures and performed all of the calculations. Keir Darling helped with getting the code used to generate $N$-body initial conditions to compile and produced the first version of a script to run the code.
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Last, but certainly not least, I would like to thank my dear friends: Ananthan Karunakaran, Cory Wagner, and Majd Abdelqader. I could not imagine what grad school would have been like without all of you.
So remember to look up at the stars and not down at your feet. Try to make sense of what you see and wonder about what makes the Universe exist. Be curious. And however difficult life may seem, there is always something you can do and succeed at. It matters that you don’t just give up.

-Dr. Stephen Hawking,
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<tr>
<td>AGAMA</td>
<td>Action-based Galaxy Modelling Architecture - Code package used to construct action-based initial conditions of N-body galaxies</td>
</tr>
<tr>
<td>CALIFA</td>
<td>Calar Alto Legacy Integral Field Area – An observational survey</td>
</tr>
<tr>
<td>DF</td>
<td>Distribution Function – A general function of the phase space coordinates</td>
</tr>
<tr>
<td>LAMOST</td>
<td>Large Sky Area Multi-Object Fiber Spectroscopic Telescope – An observational survey</td>
</tr>
<tr>
<td>LON</td>
<td>Line-of Nodes – A line representing the intersection of an orbital plane with the midplane of the galactic disc</td>
</tr>
<tr>
<td>LSB</td>
<td>Low surface brightness – A class of, often, low-mass galaxies that are fainter than, say, Milky Way-like galaxies</td>
</tr>
<tr>
<td>M31</td>
<td>Messier object 31 – The Andromeda galaxy</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
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<tr>
<td>MaNGA</td>
<td>Mapping Nearby Galaxies at Apache Point Observatory – An observational survey</td>
</tr>
<tr>
<td>MW</td>
<td>Milky Way – Our Galaxy</td>
</tr>
<tr>
<td>NFW</td>
<td>Navarro-Frenk-White – A ‘universal’ dark matter halo density profile</td>
</tr>
<tr>
<td>RAVE</td>
<td>Radial Velocity Experiment - An observational survey</td>
</tr>
<tr>
<td>SDSS</td>
<td>Sloan Digital Sky Survey – An observational survey</td>
</tr>
<tr>
<td>SEGUE</td>
<td>Sloan Extension for Galactic Understanding and Exploration – An observational Survey</td>
</tr>
<tr>
<td>SN</td>
<td>Solar Neighbourhood – Usually referred to as the space volume within a radius of a few kpc of the Sun in the Milky Way disc</td>
</tr>
<tr>
<td>WKB</td>
<td>Wentzel-Kramers-Brillouin – An approximation for solving linear differential equations</td>
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<th>Meaning</th>
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<tr>
<td>$X^0$</td>
<td>$X$ number of degrees</td>
</tr>
<tr>
<td>arcsec</td>
<td>Second of arc = $1/3600$ degrees</td>
</tr>
<tr>
<td>Gyr</td>
<td>$10^9$ years</td>
</tr>
<tr>
<td>$Xk$</td>
<td>$10^3$ times $X$</td>
</tr>
<tr>
<td>km</td>
<td>Kilometre</td>
</tr>
<tr>
<td>kpc</td>
<td>Kiloparsec</td>
</tr>
<tr>
<td>$XM$</td>
<td>$10^6$ times $X$</td>
</tr>
<tr>
<td>$M_\odot$</td>
<td>Solar mass, i.e., the mass of the Sun (\simeq 2 \times 10^{30}) kilograms</td>
</tr>
<tr>
<td>mag</td>
<td>magnitude, a logarithmic measure of the brightness of an object</td>
</tr>
<tr>
<td>Mpc</td>
<td>$10^6$ parsecs</td>
</tr>
<tr>
<td>Myr</td>
<td>$10^9$ years</td>
</tr>
<tr>
<td>$O(\cdot)$</td>
<td>‘Order of’ the function ((\cdot))</td>
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Parsec $\simeq 3.26$ light-years $\simeq 3.09 \times 10^{16}$ metres. For reference, the stellar disc of the Milky Way extends out to a radius of $\sim 20$ kpc, and the Sun sits at a distance of $\sim 8$ kpc from the Galactic centre.
Chapter 1

General Introduction and Literature Review
1.1 Density Wave Theory

This thesis is concerned with the dynamics of the formation and evolution of (vertical) density wave structures in disc galaxies. Bars and spiral structure are the most common examples of density wave features a disc might exhibit. The seminal work of Lin and Shu (1964, but see also Lindblad 1963, Lin and Shu 1966, and Lin et al. 1969) proposed that these structures are not material in nature, but can be thought of as ‘density waves’, i.e. a standing wave between two reflection points. In this scenario matter that makes up the disc – primarily stars, gas, and dust – is compressed as it moves through the waves (via the rotation of the disc itself), resulting in areas in the disc with greater mass density than others. More technically, this phenomenon arises via the orbits of stars near a putative density perturbation being perturbed and ‘trapped’ by the perturbation itself, which subsequently grows and traps more stars in non-linear manner.

Dynamically speaking, spiral structure and, in particular, bars are instabilities (Miller et al. 1970; Hohl 1971; Kalnajs 1972, 1977; Jalali 2007). Thus, the strength of these perturbations will increase exponentially over time as the waves are reflected and amplified (Goldreich and Lynden-Bell 1965; Julian and Toomre 1966) in the absence of a suppression mechanism. Both the random motions of the stars (Athanassoula and Sellwood 1986) and the presence of a sufficiently dense central concentration of mass (Mark 1974; Toomre 1981), such as a bulge, have been invoked to suppress the bar instability. Ostriker and Peebles (1973) further showed that a bar could be quelled if the disc is immersed in a massive rigid spheroidal component, such as a dark matter halo. The increased density from the halo acts to shorten the wavelength of disturbances in the disc. Bars that form in live $N$-body haloes, where the halo is able
to respond to the bar perturbation, also appear to be at least moderately stabilized, although the response of the halo does seem to invigorate bar growth (see Sellwood 2016).

The key idea from Ostriker and Peebles (1973) was that the growth of an instability strongly depends on the relative density fraction between the baryonic and dark matter components of a galaxy. Thus, more massive discs, such as our own Milky Way (MW), are generally more susceptible to the formation of density wave features. Indeed, we do observe a preponderance of bars and strong amplitude spiral structure in high-mass disc galaxies.

Of course, the responsiveness of a disc to an instability is more complicated than a single number, and in general depends on the structural properties of the disc as well. As such, an ultra-responsive disc can amplify even the subtlest putative density perturbations. Although rare, it is not surprising then that we also observe bars and other density wave features in low-mass, or low surface brightness, disc galaxies, which are thought to be dark matter dominated (see Chapter 3).

Bars and spiral structure are just a couple of examples of how density waves might manifest in discs, and are specific examples of waves that lie within the disc midplane. Their vertical counterparts will be introduced in Section 1.3. It cannot be stressed enough though that, once formed, density waves play a dominant role in driving the secular evolution of disc galaxies (Kormendy 2013; Sellwood 2013, 2014).
1.2 In-Plane Waves within a Cosmological Context

Modern cosmological simulations of a Lambda Cold Dark Matter Universe predict that galaxies form through the hierarchical merging of dark matter haloes, resulting in a large scale filamentary ‘cosmic web’ structure (A small excerpt from a long list of relevant publications include: Davis et al. 1985; Frenk et al. 1988; Warren et al. 1992; Cen et al. 1994; Gelb and Bertschinger 1994; Hernquist et al. 1996; Navarro et al. 1996, 1997; Jenkins et al. 2001; Doroshkevich et al. 2004; Wambsganss et al. 2004; Springel et al. 2005, 2006, 2008). Baryonic matter resides at the center of these dark matter haloes and follows the central potential throughout this merging scenario. Remnants of this active formation history in MW sized haloes survive at low redshifts in the form of smaller dark matter subhaloes that orbit within the host halo (Klypin et al. 1999; Moore et al. 1999; Gao et al. 2004). These subhaloes (halo substructure) can also harbour baryons, which we observe as satellite dwarf galaxies (Tolstoy et al. 2009; McConnachie 2012).

There are currently \( \sim 60 \) observed MW satellite galaxies located at distances out to \( \sim 400 \) kpc\(^1\) (McConnachie 2012; Bechtol et al. 2015; Drlica-Wagner et al. 2015; Koposov et al. 2015; Kim and Jerjen 2015), which is in contradiction with the hundreds of subhaloes predicted by cosmological simulations. This is known as the ‘missing satellite problem’ (Klypin et al. 1999; Moore et al. 1999). Despite attempts to remedy this discrepancy by accounting for dust obscuration and the difficulty of observing

\(^1\)For context, the MW’s stellar disc has a radius of \( \sim 20 \) kpc with the Sun residing \( \sim 8 \) kpc from the Galactic center.
through the plane of the MW at low latitudes, it is unlikely that the order of magnitude of missing satellites lay hidden from view in these manners. It was therefore proposed that the missing satellites are, in fact, present in the form of dark matter dominated subhaloes (Klypin et al. 1999; Moore et al. 1999; Gao et al. 2004). This hypothesis is not unrealistic given the recent observations of about two dozen ultra faint dwarfs in the MW (Tolstoy et al. 2009; McConnachie 2012; Bechtol et al. 2015; Drlica-Wagner et al. 2015; Koposov et al. 2015; Kim and Jerjen 2015).

Satellites and dark matter subhaloes, and globular clusters for that matter, can interact with the MW disc as they traverse their orbits. If a satellite’s orbit intersects the MW disc the satellite will impart some of its orbital energy to the disc as it passes through, effectively heating the disc stars (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998). These heating encounters can cause the disc to spread out radially and produce warps and flares in the outer disc (Quinn and Goodman 1986; Quinn et al. 1993; Walker et al. 1996; Velazquez and White 1999).

Furthermore, and as discussed in Section 1.1, stellar discs are typically unstable to the formation of non-axisymmetric features, such as bars and spiral structure. Satellite-disc interactions can excite and provoke the formation of these structures if their orbits intersect the stellar disc (Gauthier et al. 2006; Dubinski et al. 2008; Sellwood 2013). For example, consider the Gauthier et al. (2006) simulations of an M31-like disc-bulge-halo model. When simulated with a smooth halo the disc was stable against the formation of a bar for at least 10 Gyr and only formed flocculent spiral structure with moderate density contrasts. When the halo was re-initiated with \(~10\) per cent of its mass in the form of 100 orbiting subhaloes, the disc formed a strong bar and large amplitude multi-armed spiral structure (see also the series of simulations
Figure 1.1: An example of a normalized breathing mode perturbation DF in the $z-v_z$ plane for a truncated isothermal sheet (Weinberg 1991; Widrow et al. 2014). This figure is a reproduction of figure 7 in Widrow et al. (2014).

by Kazantzidis et al. 2008). Thus, although discs can form bars in isolation, the continual bombardment by a large number of subhalos over a disc’s lifetime also plays a significant role in driving the secular evolution of the disc (Dubinski et al. 2008).

1.3 Vertical Waves in Stellar Discs

Stellar discs are, however, stable in the vertical direction – i.e. the direction perpendicular to the disc midplane. Thus, any perturbation (for example, a transfer of orbital energy to the disc stars from a passing dark matter subhalo or satellite galaxy)
to the vertical motions of stars can produce vertical waves – coherent motions of stars – that oscillate throughout the disc and dissipate as they damp (Weinberg 1991; Sellwood et al. 1998). The increase in the vertical energy of disc stars results in heating and thickening of the disc as the disc settles down into a new state of equilibrium (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998).

Theoretically, satellite interactions can excite intrinsic vertical oscillation modes in the disc, such as bending, breathing, and higher order modes (Toomre 1966; Araki 1985; Mathur 1990; Weinberg 1991; Widrow and Bonner 2015). The lowest order mode, the bending mode, corresponds to a bulk displacement and vertical motion of stars either above or below the disc midplane. A breathing mode corresponds to an oscillatory pattern in phase space with two-fold symmetry – a complimentary expansion and compression of the equilibrium distribution function (DF), $f(x, v)$. An example of an isolated breathing mode DF in a truncated isothermal sheet (Spitzer 1942; Camm 1950) is shown in Fig. 1.1 (Weinberg 1991; Widrow et al. 2014). The maximal phase of the mode is shown in Fig. 1.1, which corresponds to stars above and below the disc midplane moving with velocities in opposite directions, relative to the bulk motion of the whole system. That is, for $z < 0$ there are more stars with positive than negative velocities, while for $z > 0$ the opposite is true. In this case, the ‘breathing’ pattern of the mode in phase space also manifests as a physical rarefaction and compression of the sheet. The perturbation in Fig. 1.1 rotates in a clockwise direction in the $z$-$v_z$ plane as it evolves. In a real galaxy the situation is more complex, where different modes can be superposed and there are multiple stellar populations.
1.4 Observations of Vertical Waves in the Milky Way Disc

1.4.1 The Solar Neighbourhood

Dynamical effects from a bar (Dehnen 2000; Sellwood 2013) or spiral structure (de Simone et al. 2004; Quillen and Minchev 2005; Chakrabarty 2007; Sellwood 2013), tidally stripped stars from accreted satellite galaxies (Navarro et al. 2004; Helmi et al. 2006), and vertical waves induced by interacting satellite galaxies (Minchev et al. 2009; Gómez et al. 2013) can all leave an imprint in the velocity-space of stars in the disc. Minchev et al. (2009) conjectured that the observed structure in the velocity distribution of stars in the Solar Neighbourhood (SN; Chereul et al. 1998; Dehnen 1998; Chereul et al. 1999; Nordström et al. 2004) is due to the latter – what we are observing is the ripples in the velocity-space of stars left by an interacting satellite depositing some of its orbital energy in the disc.

Recently, Widrow et al. (2012) analysed the kinematics of 11k main-sequence stars from the Sloan Extension for Galactic Understanding and Exploration (SEGUE) survey (Yanny et al. 2009) in the SN (within ∼2 kpc of the Sun). In that paper they reported a detection of a north-south asymmetry in the number counts of stars as a function of height relative to the Galactic midplane. The asymmetry shows a ∼10% (north-south)/(north+south) deficit and excess at |z| ≃ 400 pc and |z| ≃ 800 pc, respectively. Yanny and Gardner (2013) confirmed this number count asymmetry with a more rigorous analysis of uncertainties and systematic effects. Furthermore, Widrow et al. (2012) showed that the bulk vertical motions as a function of stars in the SN display characteristics of a breathing wave perturbation, with a slope of
Figure 1.2: The bulk vertical motion of stars as a function of height above and below the Galactic midplane for data from Widrow et al. (2012) (black dots) and the breathing mode perturbation from Fig. 1.1 rotated by 90° (dotted line). The solid black lines correspond to the breathing mode perturbation rotated by a further 15° and 30° relative to the dotted black line. Note, the perturbation in Fig. 1.1 was scaled to match the data. This figure is a reproduction of figure 8 in Widrow et al. (2014).

\[ \sim 3 - 5 \text{ km s}^{-1}\text{kpc}^{-1} \] in the \( z-v_z \) plane. We show their data in Fig. 1.2. The dotted line corresponds to the breathing mode perturbation in Fig. 1.1 rotated by 90°. The two solid black curves correspond to the breathing mode perturbation from Fig. 1.1 rotated another 15° and 30° relative to the dotted black line. The perturbation in Fig. 1.1 was scaled to fit the data. From Fig. 1.2 it is clear that the bulk vertical motions of stars display evidence of a breathing mode perturbation, especially at large \( |z| \) where the slope of the perturbation increases as a function of \( |z| \).
Williams et al. (2013) also found vertical bulk motions in their analysis of 72k red-clump stars from the Radial Velocity Experiment (RAVE) survey (Steinmetz et al. 2006). In that paper, they considered the average vertical motion of stars as a function of cylindrical radius and height, and found evidence of rarefaction and compression of the Galactic disc at radii extending from just interior to exterior of the Solar radius, respectively, with peak bulk vertical motions of $\sim \pm 15 \, \text{km s}^{-1}$. Furthermore, Carlin et al. (2013) found similar results in their study of 400k F-type stars using spectroscopic radial velocity measurements from the Large Sky Area Multi-Object Fiber Spectroscopic Telescope (LAMOST) survey (Cui et al. 2012; Zhao et al. 2012) and proper motions from the PPMXL catalogue (Roeser et al. 2010). (see also Sun et al. 2015, Ferguson et al. 2017, Pearl et al. 2017, Carrillo et al. 2018, and Wang et al. 2018 for more recent confirmations and developments regarding the vertical structure of the extended SN).

As noted by Carlin et al. (2013), the three surveys probe different parts of the local Galactic disc and consider different spatial projections of the data, and so it is not surprising that they disagree quantitatively. Also, it is unlikely that the qualitative agreement between these surveys is simply the result of distance measurement systematics masquerading as bulk motions (see Schönrich et al. 2012) since the surveys are independent from one another. Thus, the key idea is that the MW is not in a state of equilibrium, at least in the vertical direction, which many dynamical studies assume to be true, and indicates that interesting physical processes are currently playing an active role in shaping the dynamical evolution of the Galactic disc. This idea, coupled with the hypothesis of Widrow et al. (2012) that a recent passage of a satellite galaxy or dark matter subhalo is responsible for the perturbations seen
in the data (see Section 1.5 for an overview of possible satellite galaxy candidates), serves as the primary motivation for this thesis.

1.4.2 Large-Scale Bending Waves

Warpers are perhaps the most common and conspicuous examples of bending waves in disc galaxies. The existence of warps has been attributed to the tidal effects of satellite galaxies, interactions of the disc with its dark halo, internal excitation of bending instabilities in gaseous discs, and intergalactic winds and magnetic fields, to name just a few (Kahn and Woltjer 1959; Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988; Battaner et al. 1990; Binney 1992; Debattista and Sellwood 1999; López-Corredoira et al. 2002a; Revaz and Pfenniger 2004; Shen and Sellwood 2006). Warps are generally observed in the outer regions of both H\textsc{i} and stellar discs (see Binney 1992, Cox et al. 1996, and Sellwood 2013, and references therein), and typically exhibit an ‘integral sign’ shape, although warps with more complicated morphologies also exist.

Indeed, detailed H\textsc{i} maps of the MW reveal a complex pattern of warping and flaring (Levine et al. 2006). We also observe the MW warp in the stellar (Djorgovski and Sosin 1989) and dust (Freudenreich et al. 1994) components of the disc. It starts just inside the Solar radius and increases in amplitude toward the edge of the disc (Drimmel and Spergel 2001; López-Corredoira et al. 2002b; Momany et al. 2006; Reylé et al. 2009; Schönrich and Dehnen 2017).

However, warps are but one example of how a galactic disc might bend in and out of its midplane. A disc may also exhibit corrugations or wave-like structures in the vertical direction. Such structures are most easily observed in the MW where
we have access to the full six-dimensional phase space of nearby stars. For example, Xu et al. (2015) recently mapped the North-South number count asymmetry of stars as a function of position in the disc plane towards the Galactic anti-centre. Their observations revealed a pattern of short-wavelength bending waves that oscillate above and below the disc midplane. The amplitudes of the corrugations increase towards the disc’s edge, out to a distance of \( \sim 20 - 25 \) kpc from the Galactic centre. They also found that the waves were not azimuthally symmetric and proposed they might be associated with the MW’s spiral structure (Hou and Han 2014) or related to perturbations from a satellite galaxy.

The velocity counterpart to the Xu et al. (2015) bending waves may have been detected by Schönrich and Dehnen (2017). In that paper they carefully considered the selection function bias of the Gaia-TGAS data set when determining distances and mapped out a trend between bulk vertical motion and angular momentum of stars in the SN. More specifically, they found an oscillatory behaviour in the bulk vertical motions that generally increases with galactocentric radius. One the main conclusions was that the trend cannot be explained alone by a warp pattern, but rather is evidence for a significant wave-like pattern on top of a warp signal, which suggests a relation between short-wavelength oscillations just past the Solar radius and the Galactic warp near the edge of the disc.

1.5 Vertical Waves in Numerical Simulations

Since the recent detection of vertical waves in the SN (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013) there has been extensive progress made in attempting to determine the source of the perturbations using numerical simulations. For instance,
it has been shown that vertical perturbations can be excited by internal mechanisms such as a bar (Monari et al. 2015), spiral structure (Debattista 2014; Faure et al. 2014; Monari et al. 2016a), or some (non-linear) combination of the two (Monari et al. 2016b). Indeed, one expects that a time-dependent perturbing potential that sweeps through the disc will cause it to contract and expand in the direction perpendicular to the midplane. For the case of bending waves, a perturbation that breaks symmetry about the midplane is required, such as a buckling bar. However, perturbations of these origins likely contribute to a small degree, if any, to the waves observed in the SN since waves that arise due to a bar are localized to the bar itself, and we observe spiral structure with only moderate density contrasts in the MW.

Other authors have invoked satellites and halo substructure to explain the bending and breathing waves seen in the MW (Gómez et al. 2013; Widrow et al. 2014; Feldmann and Spolyar 2015; de la Vega et al. 2015; D’Onghia et al. 2016; Gómez et al. 2016, 2017; Laporte et al. 2017, 2018). Satellite interactions are particularly intriguing since they can produce vertical perturbations very similar to what we observe. For example, consider the simulation by Purcell et al. (2011) where a model MW disc was perturbed by a single satellite galaxy. The prototype for their perturber was the Sagittarius dwarf spheroidal galaxy (Ibata et al. 1994, 1997), which is believed to have survived several orbits about the Galaxy. The hallmark of the Purcell et al. (2011) simulation was its ability to reproduce features similar to what we observe in the MW, such as spiral structure, a bar, and the Monoceros ring (Newberg et al. 2002; Yanny et al. 2003; Li et al. 2012; Morganson et al. 2016). Gómez et al. (2013) reanalysed this simulation in the context of vertical waves and found striking parallels between both the bulk vertical motions and vertical number density profiles of disc
particles at roughly the Sun’s location in the simulation and that seen in the data (Widrow et al. 2012; Yanny and Gardner 2013). Other studies have attributed at least some of the observed wave behaviour in the MW to both the Sagittarius dwarf galaxy and the Large Magellanic cloud (Laporte et al. 2017, 2018).

Interestingly, de la Vega et al. (2015) have suggested that the evolution of perturbations in the disc can be understood as purely kinematical features. To demonstrate this, they followed test-particle representations of disc perturbations in a time-independent unperturbed potential. Their perturbations therefore phase-mix but do not self-gravitate. The resulting structures were found to be qualitatively similar to the bending and breathing wave patterns seen in the SN. This contrasts, at least for bending waves, with predictions from linear perturbation theory that, in general, an isolated disc (embedded in a flattened halo) supports a continuum of bending modes, and therefore a generic bending perturbation will disperse (Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988; Nelson and Tremaine 1995, but see also Chequers and Widrow 2017). The key ingredient to the linear theory is that the modes are self-gravitating. However, within the linear framework the halo is static and is therefore unable to respond to the time-dependent disc potential (and vice versa). Using N-body simulations, Binney et al. (1998) speculate that if the halo is allowed to respond then true modes of the disc, qualitatively similar to but quantitatively different from the Sparke-Casertano modes, will still exist. Thus, it is unclear whether the vertical waves observed in the SN, and simulations for that matter, are in fact bonafide self-gravitating modes of the MW disc, purely kinematic phenomenon, or perhaps some combination of the two.
Most simulations examining the effect of satellite collisions on the vertical structure of the MW consider only single satellite encounters. In contrast, Gómez et al. (2017, but see also Gómez et al. 2016) studied bending waves in 16 MW-like haloes found in fully cosmological magneto hydrodynamic simulations from the Auriga Project (Grand et al. 2017). They found that nearly three-quarters of their galactic discs exhibited complex corrugation patterns as well as strong ‘integral sign’ warps with amplitudes in excess of 2 kpc, and thus bending waves in MW-like discs should be very common. One of their main conclusions was that the spatial patterns of warps and vertical waves similarly manifested in maps of bulk vertical motion in all cases, albeit with an azimuthal phase offset. This anticorrelation between mean vertical displacement and velocity implies a true wave-like oscillatory nature of the bending waves (see also Gómez et al. 2013 and Gómez et al. 2016), which can be exploited to observe corrugations in nearly face-on external galaxies using line-of-sight velocity measurements. Furthermore, by combining both position and velocity phase space information, we can gain enormous insight into the nature of vertical waves in the MW disc.

1.6 Enter Gaia and this Thesis

Beginning with the Hipparcos observing mission (ESA 1997), humankind has been charting the phase space distribution of stars in the MW for just over 20 years. Hipparcos offered complete phase space information for \( \sim 100k \) stars in the vicinity of the Sun, and completely revolutionized our understanding of dynamics in the MW.

The successor to the Hipparcos mission, the Gaia space telescope, was launched in late 2013 and is currently taking data. Gaia is an all-sky survey that will measure the
complete phase space and spectroscopic information of some one billion MW stars in unprecedented detail across a substantial portion of the disc (Perryman et al. 2001; Gaia Collaboration et al. 2016b). Thus, position and velocity information will soon be available for four orders of magnitude more stars than Hipparcos gave us. The final complete Gaia catalogue is expected to be released by the end of 2022.

The magnitude of detail and spatial coverage Gaia has to offer will surely aid in the study of vertical waves in the MW. Already within the early stages of the Gaia second data release (Gaia Collaboration et al. 2018a) we are seeing evidence of vertical wave-like structure in the MW disc (Antoja et al. 2018; Gaia Collaboration et al. 2018b; Poggio et al. 2018). In particular, Antoja et al. (2018) found very intriguing and intricate spiral-like structures in the phase space distribution of stars in the extended SN at Galacocentric radii just beyond that of the Sun, indicating the disc is far from being in a state of equilibrium.

This thesis approaches the topic of vertical oscillations in the MW from a theoretical perspective by way of $N$-body simulations of MW-like galaxy models, and focuses particularly on bending waves. This undertaking is extremely open ended since there has been little recent work on the subject, especially regarding more realistic three dimensional simulations, prior to the initial detection of waves in the SN by Widrow et al. (2012). Thus, this thesis primarily aims to understand the physics of vertical oscillations at a fundamental level, and answer questions such as how do these oscillations form and evolve in MW-like discs? And, moreover, are the perturbations observed in the SN purely kinematic phenomena, or is self-gravity essential to their evolution? Also, the timing of this thesis fortuitously coincides with the Gaia observing mission. As such, one of our main intentions is to also serve as a theoretical
precursor to the catalogue and determine what impact vertical oscillations have on observables that we might see in the Gaia data.

The outline of the thesis is as follows. In Chapter 2 we give an overview of the code packages used for constructing our galaxy equilibrium initial conditions and the code used to run our simulations, as well as the main Fourier and spectral analysis code developed for this thesis. The next four chapters are based on a series of independent academic papers that have either been published or submitted. The chapters are presented in a pedagogical sense; the order does not correspond to the chronological sequence of publication. We begin in Chapter 3 with the exploration of the dynamics and formation of ‘classical’ in-plane density waves, a bar. In particular, we use $N$-body methods to examine the formation and evolution of a bar in a model of an especially peculiar low surface brightness galaxy, UGC 628. This chapter also serves as an introduction to Fourier and spectral techniques in the context of in-plane density waves, which are extended to vertical waves in later chapters. In Chapter 4 we begin our study of vertical oscillations in MW-like galaxies. There, we analyse bending waves that arise seemingly without provocation from an external agent in single-component MW-like discs. We then explore interactions between a single satellite galaxy and a model single-component MW disc, and characterize the formation of the induced vertical waves (bending and breathing) according to satellite orbital parameters in Chapter 5. Our ‘capstone’ project is presented in Chapter 6. In that chapter we analyse a simulation comprising a two-component MW-like disc embedded in a ‘clumpy’ halo with multiple orbiting subhaloes. Finally, in Chapter 7 we summarize the research presented here, state the main conclusions of this thesis, and propose a novel sequence of events describing the formation and evolution of bending
waves following satellite interactions. It is also here that we highlight recommendations for future research projects. Note that we have compiled the references from all chapters at the end of the thesis, rather than at the end of each chapter, to avoid citations common to chapters being listed multiple times.
Chapter 2

Technical Details of Code
2.1 Introduction

This chapter is dedicated to describing technical details of code packages that were used to generate equilibrium initial conditions of $N$-body galaxy models and run the simulations in this thesis. Most of the information regarding constructing initial conditions can be found within the first few sections of each subsequent chapter, as is necessary for publication. For reference and completeness, we include a more in-depth description in Section 2.2. However, it is not customary to describe, in any great detail, the code used to run simulations within a publication other than simply citing the code paper. Thus, an overview of the algorithms utilized throughout this thesis to compute the gravitational evolution of our $N$-body models is given in Section 2.3. Also, a brief overview of the Fourier and spectral decomposition code developed for this thesis is given in Section 2.4. The disinterested reader may skip this chapter without any loss of continuity.

2.2 Self-Consistent Modelling of Galaxy Initial Conditions in Dynamical Equilibrium

2.2.1 Overview of Self-Consistent Modelling

For a large number of particles, a collisionless system, such as a self-gravitating system of stars, can be described by a distribution function (DF), $f(x, v, t)$ (section 4.1 of Binney and Tremaine 2008). In essence, the DF is the density of particles within a phase space element $d^3x d^3v$. According to the Jeans theorem (Jeans 1915) a system is in approximate dynamical equilibrium only if the DF describing the system
depends on the phase space coordinates through integrals of motion, $I(x, v)$, in a given potential (section 4.2 of Binney and Tremaine 2008). The exact choice for the integrals of motion is arbitrary, as long as they are constant along a particle’s orbit. The potential is known from the Poisson equation, $\nabla^2 \Phi(x) = 4\pi G \rho(x)$, and the fact that the density is the zeroth moment of the DF, i.e. $\rho(x) = \int f(I[x, v]) \, d^3v$.

The objective of self-consistent modelling is to solve this system of coupled equations for an arbitrary number of model components – that is, to simultaneously solve for $I(\Phi)$, the Poisson equation, and the density expressed as the velocity integral of the DF. The circular dependence of these relations requires a numerical iterative procedure to solve for a unique potential-density pair. In general, convergence is met for sensible galaxy models, although, to the best of our knowledge, no strict mathematical proof exists (see section 2.6.4 of Vasiliev 2018). The general outline of the iterative procedure is as follows:

1. Assume a DF for each component of the model, $f_c(I)$.
2. Make an initial guess for the total potential, $\Phi(x)$.
3. Establish $I$ in this potential, $I(\Phi)$.
4. Compute the total density via $\rho(x) = \sum_c \rho_c(x)$, where $\rho_c(x) = \int f_c(I[x, v]) \, d^3v$ is the density of each component. This step maintains the ‘self-consistency’ of the scheme.
5. Solve the Poisson equation to obtain a new approximation for the total potential.
6. Repeat from step 3 until the potential-density pair converges.
An $N$-body representation of the model can then be realized by sampling the DF using the integrals of motion that correspond to the converged potential-density pair.

### 2.2.2 GALACTICS

We used galactics (Kuijken and Dubinski 1995; Widrow et al. 2008) to generate initial $N$-body initial conditions in Chapters 3-5. galactics allows one to construct axisymmetric disc-bulge-halo systems in approximate dynamical equilibrium, and uses the procedure outlined in Section 2.2.1 to iteratively compute the potential corresponding to desired forms of the density distribution for each component.

The DF for the disc is a function of two exact integrals of motion, the total energy and vertical angular momentum, and an approximate third integral corresponding to the relative vertical energy,

$$E_z = \frac{1}{2} v_z^2 + \Phi(R, z) - \Phi(R, 0).$$

This third integral arises when the potential is assumed to be separable in $R$ and $z$ (section 3.2.2 of Binney and Tremaine 2008) such that $\Phi(R, z) = \Phi_R(R) + \Phi_z(z)$, which is approximately valid for nearly circular orbits with small radial and vertical excursions (i.e. thin discs).

The disc is constructed to obtain a desired density distribution well-approximated by

$$\rho_d(R, z) = \frac{M_d}{4\pi R_d^2 h} e^{-R/R_d} \text{sech}^2(z/h) C \left( (R - R_t) / \delta R_t \right),$$

(2.1)

where $R$ and $z$ are cylindrical coordinates, $C$ is a truncation function that goes smoothly from unity to zero at $R \simeq R_t$ over a width of order $\delta R_t$, $M_d$ is the total disc mass, $R_d$ is the radial scale length, and $h$ is the scale height. The radial velocity
dispersion is an exponential function of radius,

\[ \sigma_R(R) = \sigma_{R0} e^{-R/R_\sigma}, \tag{2.2} \]

where \( \sigma_{R0} \) is the central dispersion and \( R_\sigma \) is the radial scale length of the squared dispersion profile. The vertical velocity dispersion is determined from the local surface density and the model assumption that the disc is isothermal in the direction perpendicular to the disc plane.

The DFs for the bulge and halo are assumed to be functions only of the total energy, and therefore are isotropic in velocity space. The bulge DF is constructed to yield a density profile that is given, approximately, by

\[ \rho_b(r) = \frac{v_b^2}{4\pi G R_e^2 c(n)} \left( \frac{r}{R_e} \right)^{-p} e^{-b(r/R_e)^{1/n}}, \tag{2.3} \]

where \( v_b \) is the characteristic velocity scale of the bulge. Equation (2.3) yields the Sérsic law for the projected surface density profile with index \( n \) so long as one sets \( p = 1 - 0.6097/n + 0.05563/n^2 \). The constant \( b \) is adjusted so that \( R_e \) encloses half the total projected mass and \( c(n) = (nb^n(p-2)) \Gamma(n(2-p)) \) \cite{Prugniel1997, Terzic2005}.

Finally, the halo DF is constructed to yield the Navarro-Frenk-White (NFW) density profile \cite{Navarro1996}

\[ \rho_h(r) = \frac{a_h v_h^2}{4\pi G r (r + a_h)^2}, \tag{2.4} \]

where \( a_h \) is the NFW scale length and \( v_h \) is the characteristic velocity scale. While the velocity distributions of the halo and bulge are isotropic, their space densities are
slightly flattened due to the disc potential.

### 2.2.3 Action-Based Modelling with AGAMA

To construct equilibrium initial conditions for our \( N \)-body models in Chapter 6 we used the code Action-based Galaxy Modelling Architecture (AGAMA; Vasiliev 2018). As the name implies, AGAMA uses actions, \( \mathbf{J} = (J_R, J_\phi, J_z) \), as the integrals of motion. The actions can conveniently be defined in terms of phase space coordinates as (section 3.5 of Binney and Tremaine 2008)

\[
J_i = \frac{1}{2\pi} \oint p_i \, dq_i ,
\]

where \( q_i \) and \( p_i \) are canonical coordinates, and the integration is over one period in the corresponding phase space. In the case of cylindrical coordinates, \((q_1, q_2, q_3) = (R, \phi, z)\) and \((p_1, p_2, p_3) = (p_R, p_\phi, p_z)\) are the conjugate momenta. There are many advantages to using actions for the integrals, namely that they are identically conserved. This contrasts with galactics (Section 2.2.2) where the vertical energy is used as a third integral of motion, which is only approximately conserved in thin, cool discs and not at all conserved in thick, warm ones. The price to pay in action-based modelling is the computational challenge of computing the actions.

Transformations from physical phase space coordinates to action space (i.e., step 3 in the iterative scheme outlined in Section 2.2.1) are exactly known only for the class of Stäckel potentials (see section 3.5.3a of Binney and Tremaine 2008). Thus, in practice, the actions for an arbitrary potential are computed using the so-called ‘Stäckel fudge’ (Binney 2012), which assumes the motions of particles is separable and the potential is locally well-described by the Stäckel potential - i.e. the actions
are computed as if the potential is of a Stäckel form. AGAMA uses an efficient and
accurate implementation of the Stäckel fudge to compute the action transformations
for all galaxy components (Vasiliev 2018, but see also Sanders and Binney 2016 for a
review of approaches). Estimating the actions using this scheme typically results in
errors on the order of only a few per cent (Binney 2012; Vasiliev 2018).

With AGAMA, the input parameters determine the functional form of the DFs in
action space and only approximately determine the physical structural properties of
the actual model. Thus, a certain amount of trial and error is required to construct a
model with the desired properties. Spheroidal galaxy components are modelled using
the double power law DF from Posti et al. (2015), who showed that DFs of this form
do well in generating the family of density profiles

$$\rho(r) = \frac{\rho_0}{(r/r_b)^\alpha(1 + r/r_b)^{\beta-\alpha}},$$  \hspace{1cm} (2.6)

where \(r_b\) is the break radius. \((\alpha, \beta) = (1,4)\) and \((1,3)\) correspond to the popular
Hernquist (Hernquist 1990) and NFW (Navarro et al. 1996) profiles, respectively.
The amount of velocity (an)isotropy in the spheroidal components is governed by a
linear function of the actions encoded in the DF, where the linear coefficients can be
varied to obtain any desired amount of (an)isotropy.

The disc components in AGAMA are modelled using the pseudo-isothermal DF
from Binney and McMillan (2011). In constructing these models one uses the fact
that although the galactocentric radius \(R\) of a star is not an integral of motion, its
angular momentum \(L_z = J_\phi\) is. The radius of a circular orbit, \(R_c\), with angular
momentum, $L_z$, is defined as to satisfy (section 3.2.1 of Binney and Tremaine 2008)

\[
\left( \frac{\partial \Phi}{\partial R} \right)_{(R_c, 0)} = \frac{L_z^2}{R_c^3},
\]

where the derivative in equation (2.7) is evaluated in the equatorial plane of the galaxy. One can therefore numerically solve equation (2.7) for $R_c$ given the potential and $L_z$, and use it as a proxy for $R$. The radial profiles of the surface density and radial velocity dispersion are assumed to be decreasing exponential functions of $R_c$, which yields profiles that are only approximately exponential functions of $R$. The radial vertical velocity dispersion profile controls the vertical structure of the disc and is also assumed to be exponentially decreasing. As such, the disc is constructed to be isothermal, but only when in isolation – the inclusion of other galaxy components, such as a bulge and/or halo, results in a disc with only approximately constant scale height.

### 2.3 Running N-body Simulations

The code we used to run our simulations was GADGET-2 (Chapters 3 and 5; Springel 2005) and GADGET-3 (Chapters 4 and 6). GADGET-2/3 is a very flexible and massively parallel code designed to efficiently follow the gravitational evolution of a collisionless fluid using $N$-body techniques as well as an ideal gas using smoothed particle hydrodynamics. The simulations employed in this thesis are purely collisionless, and thus we defer the interested reader to the code paper for details of the gas dynamics (i.e., radiative cooling, star formation, feedback processes, chemical enrichment models, etc) implemented in GADGET-2 (section 2.3 of Springel 2005).
2.3.1 Computing Gravitational Forces

Central to any $N$-body code is the means by which it computes the gravitational force between $N$-body particles. The naive approach – called direct-summation – computes the total gravitational force on each particle by summing up the contributions from every other particle. Although this method is exact it requires enormous computation times for any reasonably high resolution simulation since the number of force evaluations scales like $O(N^2)$, where $N$ is the number of particles. We therefore turn to algorithms that compute gravity in a more approximate, yet sufficiently accurate, way.

Depending on the types of simulations the user wishes to run, GADGET allows for the computation of gravity using tree (more technically referred to as a hierarchical multipole expansion) or hybrid tree-particle-mesh algorithms with and without periodic boundary conditions and/or comoving coordinates. In this thesis we only consider non-cosmological simulations of isolated disc-bulge-halo systems, and therefore employ the tree algorithm to compute gravity using physical coordinates and vacuum boundary conditions.

The tree algorithm is based on Barnes and Hut (1986). As the name implies, tree algorithms involve decomposing the particle distribution into a tree data structure (an octree for the case of Barnes and Hut 1986). Calculating the forces on particles then consists of traversing this tree. In essence, this amounts to grouping distant particles together and computing their contribution to the total force on a given particle as a single multipole force. The gravitational forces between nearby particles are calculated using direct-summation. This allows the force on a particle to be calculated with $O(\ln N)$ operations, and therefore the algorithm globally scales as
\( \mathcal{O}(N \ln N) \), which is much more computationally efficient than direct-summation.

The tree decomposition recursively divides the volume containing all particles (the root node) into eight cubes of equal volume (child nodes) until one ends up with nodes containing single particles (leaf nodes). The force computation then involves ‘walking’ the tree and summing up the contributions from tree nodes. In the classical Barnes and Hut (1986) approach the multipole expansion of a node is used in computing the force on a given particle if the ‘opening criterion’,

\[
\frac{l}{d} < \theta ,
\]

is met, where \( d \) is the distance between the particle of interest and the center-of-mass of a node, \( l \) is the side length of the node, and \( \theta \) is a tolerance parameter. If equation (2.8) is not satisfied, the node is ‘opened’ and the tree walk is continued on the opened node’s children nodes. This scheme is continued until a leaf node is reached, in which case the gravitational force is calculated using direct-summation. The force errors of the tree algorithm can be made arbitrarily small by decreasing \( \theta \), at a cost of computation time. For the case of \( \theta = 0 \) the algorithm degenerates to direct-summation.

The Barnes and Hut (1986) opening criterion (equation 2.8) can be interpreted geometrically, but technically speaking it is a comparison between the quadrupole and monopole terms of the multipole force expansion for nodes assumed to be homogeneously distributed in mass (Barnes and Hut 1986; Springel et al. 2001). To see this, consider the multipole expansion of the gravitational potential due to a mass
density (see section 2.4 of Binney and Tremaine 2008)

\[ V(d) = \frac{G}{d} \sum_{n=0}^{\infty} \int \rho(r') \left( \frac{r'}{d} \right)^n P_n(\cos \phi) dV', \]  \hspace{1cm} (2.9)

where \( \rho \) is the mass density, \( P_n \) are the Legendre Polynomials, and \( \phi \) is the angle between the vectors pointing from the origin to the mass element imparting a force and the point the potential is being evaluated at. The first two non-zero terms in equation (2.9) correspond to the monopole and quadrupole moments of the density distribution. If we assume the density is approximately homogeneously distributed within each node, the force contribution from these terms is \( \sim GM/d^2 \) and \( GMl^2/d^4 \), respectively, where \( M \) is the total mass in the node.

The tree algorithm implemented in gadget-2/3 is the same as that of Barnes and Hut (1986) except for the inclusion of a ‘relative opening criterion’ in place of equation (2.8):

\[ \frac{GM}{d^2} \left( \frac{l}{d} \right)^2 < \alpha |a_{\text{old}}|, \]  \hspace{1cm} (2.10)

where \( |a_{\text{old}}| \) is the total acceleration of the particle of interest in the last time-step and \( \alpha \) is the tolerance parameter. If equation (2.10) is satisfied the force contribution from the distant node is approximated by its monopole expansion (i.e., as if all of the mass within the node is concentrated at the center-of-mass location). Equation (2.10) is therefore representative of the comparison between a rough estimate for the multipole expansion truncation error and the total expected force on a particle, and is used in an attempt to limit the force error in each particle-node interaction to a higher degree than the standard Barnes and Hut (1986) criterion. In practice, we use a rather
conservative value of $\alpha = 0.0015$ in our simulations which, for a typical disc-bulge-halo run, results in $\sim 99$ per cent of the relative force errors being $\leq 0.3$ per cent. For interactions between nearby particles, GADGET-2/3 softens the gravitational force (see section 2.9.1 of Binney and Tremaine 2008) with user-provided softening lengths that can vary between galaxy components.

### 2.3.2 Time Integration of N-body Particles

GADGET-2/3 uses a symplectic kick-drift-kick leapfrog scheme (Springel 2005, but see section 3.4 of Binney and Tremaine 2008) to evolve particles over time. Particle time-steps are chosen in an adaptive way, according to a particle’s acceleration, since evolving all particles in a system with a constant (small) time-step is computationally wasteful. Time-steps are assigned to each particle as

\[
\Delta t = \min \left[ \Delta t_{\text{max}}, \left( \frac{2 \eta \epsilon_c}{|a|} \right)^{1/2} \right],
\]

where $\Delta t_{\text{max}}$ is the maximum time-step allowed and is a free parameter usually set to some small fraction of the dynamical time, $\epsilon_c$ is the gravitational softening length of the component the particle belongs to, $|a|$ is the particle’s acceleration, and $\eta$ is an accuracy parameter. For all simulations presented in this thesis we set $\eta$ to a rather conservative value of 0.005, which gives sub-per cent fractional energy conservation accuracies (a benchmark for integration scheme accuracy, assuming force errors are sufficiently small) over simulation times of many Gyr.

There are two ‘modes’ offered by GADGET-2/3 to synchronously integrate particles, both of which require the specification of $\Delta t_{\text{max}}$ and a minimum time-step, which we set to some vanishingly small value. The modes differ only in various
computational efficiency trade-offs. We use the default scheme where time-steps are discretized as power of 2 subdivisions of a global time-step. Particles can always move to a smaller time-step but can only move to a higher time-step every other step in their integration. Therefore, the code does not advance all particles at every step, but rather advances particles with relatively large accelerations (and hence the particles that require their orbits to be broken down into smaller time intervals) more frequently.

### 2.3.3 Switching to GADGET-3

The main differences between GADGET-2 and GADGET-3 are improved prescriptions for gas dynamics (often user defined, and not used in this thesis) and enhanced computational efficiency in the latter. Therefore, we switched to GADGET-3 in Chapters 4 and 6 to take advantage of its increased efficiency, which allows for higher resolution simulations to be run. Unlike GADGET-2, GADGET-3 is not a publicly accessible code since it often contains (proprietary) user defined recipes for gas dynamics and other physics. We obtained a version of GADGET-3 through another graduate student in the department, Jacob Bauer.

### 2.4 Fourier and Spectral Analysis Code Developed for this Thesis

The Fourier and spectral techniques developed in the pioneering work of Sellwood and Athanassoula (1986) have proved to be an indispensable tool for the dynamical study of in-plane density waves. Using such an analysis one can decompose the surface
density of a disc into its constituent Fourier components and, using a time series of Fourier modes, infer rotational frequencies as a function of position within the disc. Chequers and Widrow (2017, i.e., Chapter 4 of this thesis) extended these methods to the study of bending waves, although the framework allows for a trivial further extension to higher order vertical waves, such as breathing waves. This analysis forms the basis for essentially all major conclusions of this thesis. Since Chapters 4 and 6 both give comprehensive descriptions of the techniques, we defer the reader to those chapters for the technical details. Here, we simply wish to highlight that the code developed to implement these analyses for this thesis was written from scratch, i.e. without the aid of any numerical analysis packages.
Chapter 3

Introduction to Fourier and Spectral Analyses Applied to In-Plane Density Waves

CHAPTER 3. INTRO TO FOURIER & SPECTRAL ANALYSES

3.1 Abstract

We present a disc-halo $N$-body model of the low surface brightness galaxy UGC 628, one of the few systems that harbours a ‘slow’ bar with a ratio of corotation radius to bar length of $R \equiv R_c/a_b \sim 2$. We select our initial conditions using SDSS DR10 photometry, a physically motivated radially variable mass-to-light ratio profile, and rotation curve data from the literature. A global bar instability grows in our submaximal disc model, and the disc morphology and dynamics agree broadly with the photometry and kinematics of UGC 628 at times between peak bar strength and the onset of buckling. Prior to bar formation, the disc and halo contribute roughly equally to the potential in the galaxy’s inner region, giving the disc enough self gravity for bar modes to grow. After bar formation there is significant mass redistribution, creating a baryon dominated inner and dark matter dominated outer disc. This implies that, unlike most other low surface brightness galaxies, UGC 628 is not dark matter dominated everywhere. Our model nonetheless implies that UGC 628 falls on the same relationship between dark matter fraction and rotation velocity found for high surface brightness galaxies, and lends credence to the argument that the disc mass fraction measured at the location where its contribution to the potential peaks is not a reliable indicator of its dynamical importance at all radii.

3.2 Introduction

Bars provide a testing ground for our theories of galaxy evolution, structure and dynamics. In particular, since the susceptibility of a galactic disc to bar-like instabilities is a function of the relative contributions to the gravitational potential of the disc,
bulge, and dark halo, bars can be used to constrain mass models and break the disc-halo degeneracy that plagues rotation curve decomposition. The strength, pattern speed, and length of a bar all depend on the structure of the host galaxy, a point illustrated in numerous N-body experiments. For example, the quintessential strong, thin, and long bar can develop through an \( m=2 \) instability in galaxy models where the disc contribution to the centripetal force is comparable to that of the bulge and halo inside a few disc scale lengths (e.g see the review by Sellwood 2014). By contrast, if the disc completely dominates the centripetal force in the inner region, the bar that develops will be shorter, fatter, and more boxy (Athanassoula 2003).

The contribution of the disc to the centripetal force can be represented by the dimensionless ratio

\[
\mathcal{F}_X = \frac{V_d(XR_d)}{V(XR_d)}, \tag{3.1}
\]

where \( V(R) \) is the total circular speed at radius \( R \), \( V_d \) is the contribution to \( V \) from the disc, and \( X \) denotes a multiple of the exponential disc scale length \( R_d \) (van Albada et al. 1985; Sackett 1997). \( \mathcal{F}_X \) is commonly evaluated at \( R = 2.2R_d \), where the disc contribution to the mass distribution peaks. It is often more useful to consider disc maximality in terms of \( \mathcal{F}^2_X \), rather than the classical quantity defined in equation (3.1), since \( \mathcal{F}^2_X \) is a more direct measure of the mass contribution from the disc to the total mass budget of the galaxy enclosed within the radius \( XR_d \). By convention, discs are described as maximal if \( \mathcal{F}^2_{2,2} > 0.72 \). The first case described above corresponds to a submaximal disc \( (\mathcal{F}^2_{2,2} \approx 0.5) \) while the second, a maximal one.

In this paper, we consider the low surface brightness (LSB) galaxy UGC 628. This galaxy has a clear photometric bar (de Blok et al. 2001; de Blok and Bosma 2002;
Chemin and Hernandez (2009) and is commonly classified as Sbc–Sm (de Vaucouleurs et al. 1991). LSB galaxies are thought to be dark matter dominated in the inner disc (Bothun et al. 1997; de Blok and McGaugh 1997; de Blok et al. 2001; Combes 2002; de Blok and Bosma 2002; Kuzio de Naray et al. 2008) and indeed, the mass model for UGC 628 by de Blok and Bosma (2002) has $\mathcal{F}_{2,2}^2 \simeq 0.09–0.16$. The mere presence of a bar in a galaxy with such a small $\mathcal{F}_{2,2}^2$ value already challenges our understanding of bar formation because self-gravity in the disc is so low (Mayer and Wadsley 2004). Indeed, the bar fraction in LSB galaxies is a mere $\sim 4\%$ (Mihos et al. 1997), while that in their high surface brightness counterparts is about an order of magnitude higher (e.g. Marinova and Jogee 2007).

Our main interest in UGC 628 is due to the claim by Chemin and Hernandez (2009) that its bar is ‘slow’. A bar with pattern speed $\Omega_p$ has a corotation resonance at radius $R_c$ defined by the condition $\Omega(R_c) = V(R_c)/R_c = \Omega_p$. Corotation sets a theoretical upper bound on the length of the bar $a_B$ and therefore the dimensionless ratio $\mathcal{R} \equiv R_c/a_B$ is expected to be greater than unity. Moreover, for a given $a_B$, $\Omega_p < \Omega(a_B)$ so long as $\Omega$ is a decreasing function of $R$, which is almost always the case. Thus, bars with $\mathcal{R}$ close to unity have a pattern speed as fast as nature will allow. By convention, bars are defined respectively as ‘fast’ or ‘slow’ depending on whether $\mathcal{R}$ is less than or greater than 1.4.

It is notoriously difficult to measure the length and pattern speed of bars in real galaxies (and for that matter, simulated ones) and therefore estimates of $\mathcal{R}$ are generally plagued by large uncertainties. Nevertheless, of the tens of galaxies (Rautiainen et al. 2008; Aguerri et al. 2015) for which $\mathcal{R}$ has now been measured, the vast majority appear to be fast. Indeed, UGC 628 is one of only three galaxies that are observed to
have a slow bar, the others being the blue compact dwarf NGC 2915 (Bureau et al. 1999) and the dwarf irregular NGC 3741 (Banerjee et al. 2013).

The preponderance of fast bars especially among massive, bright galaxies is easy to understand. Once a bar-like perturbation develops, it causes circular orbits inside corotation to become elongated in the same sense as the perturbation thereby enhancing the putative bar. By contrast, circular orbits outside corotation are elongated perpendicular to the perturbation (See Contopoulos 1980). The bar rapidly develops and grows out to its corotation radius. The bar pattern speed may decrease with time, via dynamical friction (Chandrasekhar 1943; Mulder 1983; Weinberg 1985) for example, however, this only allows for more stars to participate in the bar mode as corotation is pushed to larger radii. Thus, $\Omega_p$ can decrease but the bar remains ‘fast’ (Athanassoula 2013).

These arguments may not hold for LSB galaxies. Marinova and Jogee (2007) showed that $a_B$ rarely exceeds $R_{25}$, the radius the surface brightness isophote equals 25 mag arcsec$^{-2}$. Evidently, once $R_c$ approaches $R_{25}$ discs lack the surface density out to corotation necessary to support a fast bar. Additionally, the strong shear in this region implied by flat galaxy rotation curves destabilizes the development of precessing bar orbits. Thus, it may not be surprising that an LSB galaxy (where the surface brightness is everywhere lower than in typical discs) like UGC 628 harbours one of the few known examples of a slow bar.

The previous discussion suggests that $R$ should depend on a galaxy’s morphological type. This hypothesis is supported by Rautiainen et al. (2008) who estimated pattern speeds for 38 barred galaxies by modelling near-infrared and optical images. In short, they simulated the response of a gas and stellar disc to a rigidly rotating $m=2$
potential perturbation and varied \( \Omega_p \) until the simulated disc morphologies matched observations. They found that \( \mathcal{R} \) gradually increased from early to late type galaxies; Sa through Sb galaxies tended to have fast bars while Sbc through Scd galaxies tended to have slow ones. By contrast, the bar pattern speeds measured by Aguerri et al. (2015), who applied the Tremaine-Weinberg method (Tremaine and Weinberg 1984) to stellar absorption line maps and optical images of 15 Calar Alto Legacy Integral Field Area (CALIFA) survey galaxies, all point to fast bars with no dependence on morphology. However, they did not include galaxies with Hubble type later than Sbc. Thus, whether bars are generally fast or slow remains an open question.

In this paper, we present a dynamical \( N \)-body model for UGC 628 by evolving an (initially) exponential stellar disc in a live Navarro-Frenk-White (NFW) halo (Navarro et al. 1996) for \( \sim 12 \) Gyr. Our goals are two-fold. The first is to construct a model of this LSB galaxy that forms a bar. Indeed, the exponential disc inferred by de Blok and Bosma (2002) for UGC 628 has so little mass that, when evolved as an \( N \)-body model, it develops flocculent spiral structure but no bar. We re-examine the surface photometry of UGC 628 using updated multi-colour images and a variable stellar mass-to-light ratio. We find a steeper surface density profile than previously reported, from which we construct a bar-unstable \( N \)-body model that reproduces the morphology and kinematics of UGC 628 at late times. With this model in hand, we then proceed to examine how the bar in UGC 628 redistributes mass in the disc and the implications of that re-distribution for the dark matter mass fraction in this LSB galaxy.

Our procedure for constructing the dynamical model of UGC 628 is outlined in Section 3.3. This model provides initial conditions for our \( N \)-body simulations, which
are described in Section 3.4 where we compare our model to observations of UGC 628 and discuss the quantitative properties of the bar. We discuss the implications of our results for the mass distribution in UGC 628 and summarize in Section 3.5. Throughout, we adopt a distance for UGC 628 of 71.2 Mpc and an inclination of 56° (Chemin and Hernandez 2009).

3.3 Modelling UGC 628

In this section we present our approach for constructing $N$-body models of UGC 628. We derive the galaxy’s surface density profile from extant Sloan Digital Sky Survey (SDSS) data and colour-mass-to-light ratio relations in Section 3.3.1 and use it together with rotation curve data described in Section 3.3.2 to generate initial conditions for the simulations detailed in Section 3.3.3.

3.3.1 Surface Density Profile from Photometry

Fig. 3.1 describes how we infer the surface density profile of UGC 628 from photometric data. The image processing suite XVISTA\(^1\) was used to fit isophotal contours to an SDSS DR10 (Ahn et al. 2014) $r$-band image. The break in the resulting $r$-band isophotal profile (Fig. 3.1a) at $\sim$4 kpc divides the disc into two regions, with the inner region corresponding to the bar. The $r$-band solution was imposed on $g$- and $i$-band images and surface brightness profiles were generated in each band.

Into and Portinari (2013) tabulated colour-mass-to-light relations for a variety bands, and mass-to-light profiles derived using different combinations of $g$, $i$, and $r$.

\(^1\)http://ganymede.nmsu.edu/holtz/xvista
Figure 3.1: Derivation of the stellar surface density profile of UGC 628. Panel a: SDSS DR10 $r$-band surface brightness profile. Panel b: $g-i$ colour profile derived from the $r$-band isophotal solution. Panel c: Mass-to-light profile derived from the Into and Portinari (2013) ($g-i$, $r$) colour-band relation. The piecewise linear fit is shown as a solid red line. Panel d: Surface density profile produced by multiplying the surface brightness profile in panel a and the piecewise linear fit in panel c.
for UGC 628 are comparable. The $g$-$i$ colour profile presented in Fig. 3.1b was selected to take advantage of the colour stability due to the largest difference between bands. The $g$-$i$ colour profile shows a number of features with amplitudes $\sim 0.2$ mag and length scales of order 1 kpc. The most pronounced of these occur for $R$ between 3 and 9 kpc and are likely due to colour variations in the bar and spiral arms. We note that the bar is also responsible for large, rapid variations in the ellipticity and position angle of the $r$-band isophotal contours, which were used to derive the $g$- and $i$-band surface brightness profiles. The inner portion of the resulting mass-to-light ratio profile, shown in Fig. 3.1c, has higher mass-to-light and a steeper gradient than the outer portion of the profile. Thus, we model the mass-to-light ratio profile as a continuous piecewise linear function with two segments.

Fig. 3.1d shows the inferred surface density profile that was constructed by combining the $r$-band surface brightness profile in Fig. 3.1a and the mass-to-light profile fit in Fig. 3.1c. The uncertainties were calculated by propagating uncertainty from the $r$-band surface brightness profile and the piecewise linear fit of the mass-to-light profile. This surface density profile guided the selection of initial disc parameters in the $N$-body simulations of UGC 628, and serves as a basis of comparison with simulation snapshots. In comparison, the surface density profile of the de Blok and Bosma (2002) model of UGC 628 possesses a shallower slope and much smaller central density, which results from measuring an exponential scale length in the outer part of the surface brightness profile (de Blok et al. 1995) and applying a constant mass-to-light ratio.
CHAPTER 3. INTRO TO FOURIER & SPECTRAL ANALYSES

Figure 3.2: The rotation curve and surface density profile of the equilibrium initial conditions for our UGC 628 model. Panel a: Initial disc (red), halo (blue), and total (black) circular velocity curves for the model, and derived rotation curves from de Blok and Bosma (2002) (dark orange points) and Chemin and Hernandez (2009) (magenta points). Panel b: Initial surface density profile (solid red line) overplotted on the inferred UGC 628 profile (black points; see Section 3.3.1).

3.3.2 Rotation Curve Data

Rotation curve data derived from optical Hα velocity field observations from de Blok and Bosma (2002) and Chemin and Hernandez (2009) is shown in Fig. 3.2a. We see that the two data sets are formally inconsistent with one another, particularly in the range $R \sim 3$-6 kpc. This discrepancy in the inner region is mitigated when improved Hα velocity field data with higher angular sampling, compared to Chemin and Hernandez (2009), are considered (L. Chemin, private communication). However, we note
that neither Chemin and Hernandez (2009) nor de Blok and Bosma (2002) accounted for bisymmetric flows when deriving rotation curves from the measured line-of-sight velocities. Because the bar in UGC 628 projects near the major axis, rotation curves derived ignoring the bar flows are biased low relative to the true circular velocity (Spekkens and Sellwood 2007; Dicaire et al. 2008; Randriamampandry et al. 2016). We account for this observational bias when comparing our models to the rotation curve data sets in Section 3.4.1. We do use the general shape of the rotation curve, especially in the outer regions of the galaxy where \( V(R) \) is approximately constant, to guide the selection of halo model parameters.

### 3.3.3 Initial Conditions for \( N \)-body Simulations

We generate initial conditions for our \( N \)-body simulations using the GALACTICS code (Kuijken and Dubinski 1995; Widrow et al. 2008), which allows one to construct axisymmetric, multicomponent galaxy models that are in approximate dynamical equilibrium. The models in the present study comprise a stellar disc and an NFW dark halo. Particles for the disc are drawn from a phase space distribution function (DF) that depends on the energy, angular momentum about the symmetry axis, and vertical energy. By construction, this DF yields a disc whose space density is given by

\[
\rho_d(R, z) = \frac{M_d}{4\pi R_d^2 z_d^2} e^{-R/R_d} \text{sech}^2 \left( \frac{z}{z_d} \right),
\]

where \( M_d \) is the total disc mass and \( z_d \) is the disc scale height. In addition, the radial
velocity dispersion of the disc is an exponential function of $R$:

$$
\sigma_R(R) = \sigma_0 e^{-R/2R_\sigma},
$$

where $\sigma_0$ is the central dispersion and $R_\sigma$ is the radial scale length of the squared dispersion profile. The velocity dispersion in the azimuthal direction is calculated from the epicycle approximation. Finally, the disc is constructed to be approximately isothermal in the direction perpendicular to the disc plane.

The halo DF depends only on the energy and is constructed to yield an NFW density profile (Navarro et al. 1996)

$$
\rho_h(r) = \frac{a_h \sigma_h^2}{4\pi G r (r + a_h)^2},
$$

where $a_h$ is the NFW scale length and $\sigma_h$ is the characteristic velocity scale.

The input values for $R_d$ and $M_d$ are found by fitting the surface density profile to a single exponential function, as shown in Fig. 3.2b. We then set $z_d = R_d/6$ as suggested in a study of edge-on galaxies by van der Kruit and Searle (1981). Furthermore, we assume that the exponential scale length for $\sigma_R^2$ is the same as for the surface density, i.e. $R_\sigma = R_d$ (Bottema 1993). Finally, we set the central radial velocity dispersion so that the Toomre $Q$ parameter (Toomre 1964) is equal to unity at $R = 2.5R_d$.

The halo parameters $a_h$ and $\sigma_h$ were chosen to produce a reasonable fit (chi-by-eye) to the observed outer rotation curve. A comparison of the disc-halo rotation curve decomposition of our initial conditions and the de Blok and Bosma (2002) and Chemin and Hernandez (2009) data is shown in Fig. 3.2a. If we define the total mass in dark matter by the standard $M_{200}$ (mass interior to radius $r_{200}$, which is defined as the radius inside which the mean density is 200 times the critical density), then
Table 3.1: Initial galaxy model parameters. The first block corresponds to parameters used to build the initial equilibrium $N$-body models using the GALACTICS code (Kuijken and Dubinski 1995; Widrow et al. 2008). The second block corresponds to parameters derived from values in the first block.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Our Model</th>
<th>de Blok and Bosma (2002)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_d [10^9 , M_\odot]^a$</td>
<td>9.239</td>
<td>5.882</td>
</tr>
<tr>
<td>$R_d [\text{kpc}]^b$</td>
<td>2.909</td>
<td>4.7</td>
</tr>
<tr>
<td>$z_d [\text{kpc}]^c$</td>
<td>0.485</td>
<td>0.783</td>
</tr>
<tr>
<td>$R_{\text{out}} [\text{kpc}]^d$</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>$\sigma_0 [\text{km s}^{-1}]^e$</td>
<td>31.7</td>
<td>10.4</td>
</tr>
<tr>
<td>$R_\sigma [\text{kpc}]^f$</td>
<td>2.909</td>
<td>4.7</td>
</tr>
<tr>
<td>$a_h [\text{kpc}]^g$</td>
<td>14</td>
<td>10.8</td>
</tr>
<tr>
<td>$\sigma_h [\text{km s}^{-1}]^h$</td>
<td>210.0</td>
<td>262.5</td>
</tr>
<tr>
<td>$r_{200} [\text{kpc}]^i$</td>
<td>112</td>
<td>130</td>
</tr>
<tr>
<td>$Q(2.5R_d)^j$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$M_{200} [10^{11} , M_\odot]^k$</td>
<td>1.843</td>
<td>2.807</td>
</tr>
<tr>
<td>$c^l$</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>$\mathcal{J}_{2,2}^{m}$</td>
<td>0.44</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Parameters: $a$; radial disc scale length; $b$; vertical disc scale height; $c$; outer disc truncation radius; $d$; central radial velocity dispersion in the disc; $e$; radial scale length of the radial velocity dispersion profile in the disc; $f$; NFW halo scale length; $g$; central velocity dispersion of the halo; $h$; outer radius of the halo; $i$; Toomre $Q$ parameter at $2.5R_d$; $j$; halo mass; $k$; halo concentration; $l$; disc mass contribution to the total galaxy mass within $2.2R_d$.

Our model has a disc mass fraction of $f_d = M_d/M_{200} = 0.05$ and a halo concentration $c = 8$. As noted by Mayer and Wadsley (2004), LSB galaxies typically have mass fractions $f_d \leq 0.1$ and can be bar unstable with halo concentrations as low as $c = 4$.

Parameter values used for our model are presented in Table 3.1. The disc and halo DFs were populated with $10^6$ and $2 \times 10^6$ particles, respectively. The models were simulated using GADGET-2 (Springel 2005) for $\sim 11.7$ Gyr (roughly 195 dynamical times at $R = 2.2R_d$) with a softening length of 40 pc for both disc and halo particles. Minimum and maximum time steps were set to 0.01% and 0.2% of the galactic
dynamical time defined at a radius of 20 kpc. Energy was conserved to within a maximum of 0.04% over the simulation runtime.

Additional simulations were run to test the sensitivity of our results to the model parameters; and we have verified that models with initial parameter values of $2 < R_d < 3.5 \text{ kpc}$ and $1 < Q(2.5R_d) < 1.5$ produce similar results. The same cannot be said for an $N$-body model with the shallower, less dense disc of de Blok and Bosma (2002). The parameters for our realization of their model are also given in Table 3.1. In this case, the galaxy develops flocculent spiral structure but never forms a bar.

3.4 Bar Formation and Evolution in the Simulated Models

3.4.1 Comparison to UGC 628

In Fig. 3.3 we show a sky subtracted log-scale $r$-band image of UGC 628 alongside the projected log-scale disc surface brightness from our simulation when the bar mode begins to grow ($t = 3 \text{ Gyr}$), at peak bar strength ($t = 5 \text{ Gyr}$), and after buckling ($t = 8.5 \text{ Gyr}$; see Section 3.4.2). The surface brightness images were constructed by applying the mass-to-light fit (Fig. 3.1c) to convert $N$-body particle mass to luminosity. The bar position angle in the simulations was chosen to match that in the observations. Additionally, the simulated discs were truncated at $R = 12.5 \text{ kpc}$ before they were projected onto the sky to mimic the sky subtraction used in producing the $r$-band image.

At all times, the inner disc of our model has a clear central bar. In addition, spiral arms emanating from the ends of the bar are visible, especially in the $t = 3$
Figure 3.3: The surface brightness distributions of our UGC 628 model when the bar mode begins to grow ($t = 3$ Gyr), at peak bar strength ($t = 5$ Gyr), and after buckling ($t = 8.5$ Gyr), projected on the sky plane using the same log-scale. The leftmost image is a sky-subtracted log-scale $r$-band SDSS image of UGC 628. A distance legend corresponding to both the simulated and observed images is also shown.
and $t = 5$ Gyr snapshots. The bar at 5 Gyr is far narrower than that in the $r$-band image. Conversely, the bar at 8.5 Gyr does not extend quite as far in length as the bar in UGC 628. Thus, the bar in UGC 628 may be best-fit by our model at a time between peak strength and buckling.

The surface density and rotation curve decomposition profiles at $t = 3$, 5, and 8.5 Gyr are presented in Fig. 3.4. The evolution of the surface density profile in the top row of Fig. 3.4 shows considerable mass redistribution within the disc. Disc material initially at $R \sim 5$ kpc is mainly redistributed to the central bar region, although some disc material moves to the outer disc near $R_c$. We note that, even just after the formation of the bar at $t = 3$ Gyr, the disc surface density at $R_c$ is low and the shear is high because the rotation curve is already flat.

The surface density profile at $t = 8.5$ Gyr matches the observed profile decently, and the break in the profile occurs around the correct radius. Additionally, the surface density in the outer disc captures the proper decrease in density apparent in the observational data. Of course, choosing to compare the surface density profile at $t = 8.5$ Gyr to observations is arbitrary, and we find slightly better or worse agreement depending on the simulation snapshot. However, after the bar buckles the surface density stays roughly constant while the disc is continually being heated in both the in-plane and vertical directions.

The circular velocity curves in the bottom row of Fig. 3.4 were computed by calculating the total in-plane force from all particles (black line) and from the disc (red line) and halo (blue line) potentials using direct summation on 5000 test particles uniformly distributed within the disc mid-plane. For $R > 5$ kpc, the model rotation curves are in good agreement with the observed rotation curves at all times. At later
Figure 3.4: Evolution of the mass distribution and rotation curve in our model of UGC 628. Columns correspond to various times in the simulation as indicated. Top row: Surface density profile. The observed surface density profile is shown as black points, the initial conditions of the model are shown as a red dashed line, and the surface density profile of the model at the indicated time is shown as a solid red line. Only uncertainties larger than the data points are shown. Vertical black dashed lines indicate the radius of corotation at each time. Bottom row: Circular rotation curve. The red, blue, and black lines correspond the disc, halo, and total rotation curves, respectively. The model initial conditions correspond to the dashed lines, while the solid lines show the rotation curve at the time indicated. Observational data from de Blok and Bosma (2002) and Chemin and Hernandez (2009) are shown as dark orange and magenta points, respectively. The cyan line shows the rotation curve derived from the line-of-sight velocities for the model projections shown in Fig. 3.3, under the assumption of axisymmetry, for comparison with the observational data.
times once the bar has buckled, the disc contribution in the bar region is strongly increased. For $R < 5$ kpc, the initial circular velocity is considerably larger than that measured by Chemin and Hernandez (2009), but agrees with that obtained by de Blok and Bosma (2002). The increased circular velocity of the system at later times in the simulation is also broadly consistent with the de Blok and Bosma (2002) measurements except at $R = 1$ kpc, where the simulated circular velocity exceeds the measured value. As discussed above, this discrepancy between our model and the data is not surprising because Chemin and Hernandez (2009) and de Blok and Bosma (2002) do not account for bisymmetric flows when deriving the rotation curves (see Section 3.3.2). The cyan lines in the bottom panels of Fig. 3.4 show the rotation curves that we derive from line-of-sight velocities of the model projections in Fig. 3.3, under the assumption of axisymmetry. There is good agreement between these curves and the observational data.

### 3.4.2 Bar Properties

Section 3.4.1 established good qualitative agreement between the photometry and kinematics of UGC 628 and those of our barred galaxy model around the time of buckling. To quantify the strength of the bar modes in our model we expand the surface density of the disc into a Fourier series in azimuthal angle $\phi$ of $m$-fold density symmetries and find the magnitude of the $m=2$ Fourier coefficient (Sellwood and Athanassoula 1986) for all disc particles with cylindrical radius $R < 5$ kpc, the region that encompasses the bar at late times. We assume that the bar is the only bisymmetric mode present for $R < 5$ kpc at all times, and that contamination from bisymmetric spiral modes is limited. We define the Fourier coefficient for the $m$th
mode at time $t$ as

$$A_m(t) = \frac{1}{N} \sum_{j=1}^{N} e^{im\phi_j}, \quad (3.5)$$

where $N$ is the number of particles with $R < 5$ kpc, $m=2$ for a bisymmetric bar mode, and $\phi_j$ is the azimuthal position of the $j$th particle. We define the bar strength as the magnitude of equation (3.5).

The bar strength as a function of time for our model is shown in Fig. 3.5a. After buckling, the bar settles in strength for the remainder of the simulation. In comparison, the bisymmetric mode strength in the constant mass-to-light de Blok and Bosma (2002) model remained at the initial noise level of $|A_2| \sim 10^{-2}$ over the entire simulation runtime, which is more than an order of magnitude smaller than the bar strength measured in any of the variable mass-to-light models that we simulated.

To compute the bar pattern speed $\Omega_p$, we evaluate the numerical derivative of the cumulative azimuthal phase of the $m=2$ density mode in the inner disc (Ridders 1982; Press et al. 2007). The result is presented in Fig. 3.5b and is generally consistent with measurements by Chemin and Hernandez (2009) over the lifetime of the bar, particularly after the time of peak strength at $t \sim 5$ Gyr. The pattern speed begins to decrease once the bar reaches maximum strength, presumably due to dynamical friction from the halo (Chandrasekhar 1943; Mulder 1983; Weinberg 1985). At the onset of bar buckling ($t \sim 7$ Gyr) the rate at which the pattern speed decreases changes and stays constant for the remainder of the simulation.

To quantify the relative speed of the bar we compute $R$ as a function of time. The radius at which $\Omega(R) = \Omega_p$ in Fig. 3.5b defines $R_c$. Accurate and robust determination of bar lengths in simulations is a long standing challenge (Michel-Dansac and Wozniak...
Figure 3.5: Quantitative properties of the bar and disc as a function of time for the UGC 628 model. All quantities are averaged over a galactic dynamical time defined at a radius of 20 kpc. Vertical black dashed lines indicate times of peak strength, start of buckling, and the end of buckling, as labelled. Panel a: Bar strength. Panel b: Bar pattern speed. The range of measured values allowed by the 1σ uncertainties of Chemin and Hernandez (2009) is indicated as horizontal blue dashed lines. Panel c: Ratio of corotation radius and bar semi-major axis length. The range of measured values allowed by the 1σ uncertainties of Chemin and Hernandez (2009) are indicated as horizontal blue dashed lines. The horizontal solid green line indicates the cut-off between fast and slow bars, $R = 1.4$. Panel d: Disk mass fraction. The black and red lines correspond to measurements evaluated at $0.5R_d$ and $2.2R_d$, respectively.
We estimate the bar semi-major axis using a method similar to that described in Debattista and Sellwood (2000), where $a_B$ is defined as the location of the upturn in the radial profile of $|A_2|$ near the bar edge. As noted by Debattista and Sellwood (2000), this method tends to underestimate the length of the bar, particularly in the presence of noise. We mitigate this effect by considering the weighted mean of $R$ over a dynamical time, where outlying values are down-weighted.

The resulting values of $R$ are shown in Fig. 3.5c, and fall within the confidence bounds of measurements by Chemin and Hernandez (2009) for most of the simulation once the bar has formed, and especially at later times after the bar has begun to buckle. The large variability in $R$ relative to $\Omega_p$ and $|A_2|$ (Fig. 3.5a,b) arises because it is a ratio of two noisy quantities, $R_c$ and $a_B$, particularly for $t < 5$ Gyr when the bar is still growing. It is clear that the bar in our model is slow, with $R > 1.4$ over the entire bar lifetime. We note that both $R_c$ and $a_B$ do increase over time, however, $a_B$ does so at a lesser rate and would have to approximately double in value for the bar to be considered ‘fast’.

A more visual depiction of the bar’s relative speed is shown in Fig. 3.6, where the power spectrum of bisymmetric mode frequencies is plotted as a function of radius for $5 < t < 8.5$ Gyr (Sellwood and Athanassoula 1986; Press et al. 2007, see also Roškar et al. (2012) for a succinct description). The epicyclic frequency used to compute resonant frequencies in Fig. 3.6 was calculated from

$$\kappa^2 = 4\Omega^2(R) + R \frac{d}{dR} \Omega^2(R), \quad (3.6)$$

where $\Omega(R) = V(R)/R$ is the circular frequency profile of the galaxy and $V(R)$ is the circular speed computed from the in-plane forces, as described in Section 3.4.1.
Figure 3.6: Power spectrum of $m=2$ density mode frequencies in the simulated model as a function of cylindrical radius for $5 < t < 8.5$ Gyr. The vertical black lines correspond to the average measurement of $a_B$ over the specified time frame (solid) and the associated $1\sigma$ uncertainty (dashed). The red lines show the inner Lindblad, corotation, and outer Lindblad resonances (from left to right) at $t = 5$ Gyr (solid) and 8.5 Gyr (dashed).

(Binney and Tremaine 2008). The peak in power at $(R, \Omega) \sim (3$ kpc, $10$ km s$^{-1}$ kpc$^{-1}$) corresponds to the bar. The structure at $\Omega \sim 25$ km s$^{-1}$ kpc$^{-1}$ between corotation and the outer Lindblad resonance corresponds to a bisymmetric spiral mode emanating from the edge of the bar. Also present in Fig. 3.6 are bisymmetric density modes that have accumulated along various resonances. The bar structure clearly does not extend to corotation, which is apparent from the order of magnitude difference in power between the peak power at $R \sim 3$ kpc and the structure closer to corotation along $\Omega \sim 10$ km s$^{-1}$ kpc$^{-1}$. Thus, the bar is slow since it is rotating at a frequency
far less than $\Omega(a_B)$.

To investigate how the relative disc and halo mass distributions change as a function of time we compute $F^2$, from equation (3.1), as a function of time evaluated at various multiples of the disc radial scale length of an exponential fit to the radial surface density profile for $R < 12$ kpc (Press et al. 2007). Fig. 3.5d shows $F^2_{0.5}$ and $F^2_{2.2}$ as a function of time, which serves as proxies for the mass contribution of the disc within $R = 0.5R_d$ and $R = 2.2R_d$, respectively. When the bar starts to form, $F^2_{0.5} \sim F^2_{2.2} \sim 0.4$. By the time the bar reaches peak strength there are equal mass fractions between the disc and halo in the bar region. The disc mass contribution steadily increases over time in the inner region and slightly decreases within $R = 2.2R_d$ until the bar buckles. After buckling, the relative fraction of disc and halo mass remains roughly constant in time with $F^2_{0.5} \sim 0.6$ and $F^2_{2.2} \sim 0.4$. Thus, between the times of peak bar strength and on the onset of buckling, when the model best corresponds to the photometry and kinematics of UGC 628, the model is baryon dominated in the bar region and dark matter dominated farther out.

### 3.5 Discussion and Summary

We have presented $N$-body models of the barred LSB galaxy UGC 628, one of the few systems for which a slow ($R = R_c/a_B > 1.4$) bar has been measured (Chemin and Hernandez 2009). We re-examined the surface brightness profile of UGC 628 by fitting SDSS DR10 photometry and applying the colour-mass-to-light ratio transformations of Into and Portinari (2013), finding a higher surface density than previously reported (de Blok and Bosma 2002). We used this surface density distribution together with
rotation curve data from the literature to initialize our simulations with an exponential stellar disc with a fractional mass contribution enclosed within $R = 2.2R_d$ of $\mathcal{F}_{2.2}^2 = 0.44$ embedded in an NFW halo. The disc developed a bar that begins to grow at $t = 3$ Gyr, peaks in strength at $t = 5$ Gyr, and finishes buckling by $t = 8.5$ Gyr.

The model provides a good description of the available photometry and kinematics of UGC 628 at relatively late times, between the bar’s peak strength and buckling. The bar length and pattern speed that we measure from the simulations imply $\mathcal{R} \sim 2$ after the bar reaches peak strength, in broad agreement with the measurements of Chemin and Hernandez (2009). We find that the bar redistributes mass effectively in the inner disc; our model therefore implies that UGC 628 is baryon dominated in the bar region and dark matter dominated further out.

The observations for UGC 628 by Chemin and Hernandez (2009) and the models presented in this paper imply that, in agreement with the photometric models of Rautiainen et al. (2008), at least some late-type galaxies host slow bars. The pattern speed evolution of the bar in our model is broadly consistent with theoretical considerations and $N$-body simulations of submaximal discs, where drag from dynamical friction decreases the pattern speed (see Sellwood 2014, for a review). However, the bars in most simulations are ‘born fast’ with $\mathcal{R} \sim 1$ just after they form (e.g. Debattista and Sellwood 2000), while $\mathcal{R} \sim 2$ in our model at all times. We attribute this difference to the relatively low stellar surface density and high shear in our model at $R = R_c$, which follow from the LSB classification and flat rotation curve of UGC 628, even at early times (Fig. 3.4): the disc therefore does not have sufficient self-gravity to support a bar mode out to corotation even before any braking has taken place.

Our model has a disc mass fraction within 2.2 exponential scale lengths of $\mathcal{F}_{2.2}^2 \sim$
0.4 throughout the simulation (Fig. 3.5d), and is therefore dark matter dominated according to that definition. With $V_{\text{tot}} = 110 \text{ km s}^{-1}$ and $f_{DM} = 1 - F_{2.2}^2 \sim 0.6$, our model sits on the lower $1\sigma$ envelope of the universal relation proposed by Courteau and Dutton (2015), commensurate with LSB galaxies having systematically larger discs than their high surface brightness counterparts (e.g. Zwaan et al. 1995). Contrary to the conclusion of Chemin and Hernandez (2009) given the low pattern speed of the bar, however, our model suggests that UGC 628 is not dark matter dominated at all radii but instead has $F_{0.5}^2 \sim 0.6$ at the onset of bar buckling. Thus our model of UGC 628 provides an example of the fact that, as argued by many authors in the past (Debattista and Sellwood 2000; Courteau and Dutton 2015), the disc mass fraction enclosed within the radius where its contribution peaks is not a reliable indicator of its dynamical importance at all radii. Pairing $F_{2.2}^2$ with a measure at smaller radii, such as $F_{0.5}^2$, may be more informative (Fig. 3.5d), although the latter is even harder to constrain observationally than the former due to uncertainties in stellar population models, dust obscuration corrections, and bulge contributions near galaxy centres, particularly in more massive systems than the one considered here.

We note that simulations show that the presence of gas during bar growth and evolution can influence its final properties (e.g. Bournaud et al. 2005; Villa-Vargas et al. 2010; Athanassoula et al. 2013; Athanassoula 2014, but see Sellwood and Debattista 2014). Most of these simulations distribute the gas identically to the stars at the outset, and therefore reflect the influence of molecular gas discs rather than the more extended atomic gas ones (e.g. Broeils and Rhee 1997; Frank et al. 2016). The low star formation efficiency and quiescent star formation history of UGC 628 (Young et al. 2015) suggest a low molecular-to-atomic gas ratio both now and in
the past, consistent with CO non-detections in late-type LSB galaxies (Das et al. 2006). In addition, UGC 628 itself has an unusually low atomic-to-stellar mass ratio of $M_{HI}/M_*$ $\sim$ 0.2 for an LSB galaxy (Springob et al. 2005; Kim 2007). We therefore expect the molecular gas fraction in UGC 628 to be low and its inclusion in our models to have little influence on the bar properties that we report here (Berentzen et al. 2007).

The baryon dominated central regions of UGC 628 implied by our model distinguish it from most others LSB galaxies, where kinematics and photometry suggest a dark matter dominated inner disc (Bothun et al. 1997; de Blok and McGaugh 1997; de Blok et al. 2001; Combes 2002; de Blok and Bosma 2002; Kuzio de Naray et al. 2008). The disc in UGC 628 therefore has enough self-gravity to support a bar but also a sufficiently low central surface brightness ($\mu_{B,0} = 23.1$ mag arcsec$^{-2}$, Kim 2007) for the system to be classified as an LSB. Indeed, the rarity of bars in LSB galaxies (Mihos et al. 1997) suggests that this balance is a delicate one. It may be the case that the properties of the bar and mass distribution in UGC 628 found here also apply to the few other known barred LSB galaxies, though to our knowledge pattern speeds for these systems have not been measured. Estimates of $R$ and detailed dynamical models for these LSB galaxies as well as the other known systems with slow bars (Bureau et al. 1999; Banerjee et al. 2013) may help further explore the properties of such instabilities in late-type, low-mass, and LSB discs.

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Chapter 4

Spontaneous Generation of Bending Waves in Single-Component Milky Way-like Discs

4.1 Abstract

We study the spontaneous generation and evolution of bending waves in $N$-body simulations of two isolated Milky Way-like galaxy models. The models differ by their disc-to-halo mass ratios, and hence by their susceptibility to the formation of a bar and spiral structure. Seeded from shot noise in the particle distribution, bending waves rapidly form in both models and persist for many billions of years. Waves at intermediate radii manifest as corrugated structures in vertical position and velocity that are tightly wound, morphologically leading, and dominated by the $m = 1$ azimuthal Fourier component. A spectral analysis of the waves suggests they are a superposition of modes from two continuous branches in the Galactocentric radius-rotational frequency plane. The lower-frequency branch is dominant and is responsible for the corrugated, leading, and warped structure. Over time, power in this branch migrates outward, lending credence to an inside-out formation scenario for the warp. Our power spectra qualitatively agree with results from linear perturbation theory and a WKB analysis, both of which include self-gravity. Thus, we conclude that the waves in our simulations are self-gravitating and not purely kinematic. These waves are reminiscent of the wave-like pattern recently found in Galactic star counts from the Sloan Digital Sky Survey and smoothly transition to a warp near the disc’s edge. Velocity measurements from *Gaia* data will be instrumental in testing the true wave nature of the corrugations. We also compile a list of ‘minimum requirements’ needed to observe bending waves in external galaxies.
4.2 Introduction

It is common for disc galaxies to bend in and out of their respective midplanes. The most conspicuous examples are the warps seen in the outer regions of both H\textsc{i} and stellar discs (see Binney 1992 and Sellwood 2013, and references therein). There are classic ‘integral sign’ warps as well as warps with more complicated morphologies. Indeed, detailed H\textsc{i} maps of the Milky Way (MW) reveal a complex pattern of warping and flaring beyond the Solar circle (Levine et al. 2006). More recently, Reylé et al. (2009) showed that warped distortions similar to that in the gas also exist in the stellar and dust components of the MW (see also Cox et al. 1996 for an earlier discussion of warps in the stellar and gaseous components of external discs).

Warps are but one example of how a galactic disc might deviate from planarity. For example, a disc may also exhibit corrugations or wave-like structures in the direction normal to its midplane. Such structures are most easily observed in the MW where we have access to the full six-dimensional phase space. Consider the Monoceros ring, an overdensity in stars that arcs around the Galactic centre at a Galactocentric radius of about 18 kpc (Newberg et al. 2002; Yanny et al. 2003; Li et al. 2012; Morganson et al. 2016). The Monoceros ring was originally thought to comprise tidal debris from a disrupted dwarf galaxy (Martin et al. 2004; Peñarrubia et al. 2005). The alternative is that it is a feature intrinsic to the disc. For example, several authors have pointed out that the Monoceros ring arose from warping and flaring of the disc (Momany et al. 2004, 2006; Hammersley and López-Corredoira 2011) or resulted from the disc interacting with a massive satellite (Kazantzidis et al. 2008; Younger et al. 2008; Purcell et al. 2011; Gómez et al. 2016). More recently, Xu et al. (2015) suggested that the Monoceros ring is one of several crests of an oscillatory bending
wave observed in the direction of the Galactic anti-centre.

The counterpart to these Galactic waves may have been observed in external galaxies. For example, corrugations in the line-of-sight velocity field have been detected in the Hα emission of nearly face-on spiral galaxies by Alfaro et al. (2001) and Sánchez-Gil et al. (2015). These wave-like variations in the vertical velocity field appear to coincide with large amplitude spiral arms. One explanation is that they arise from the interaction between spiral density waves and the galaxy’s gaseous disc via the hydraulic bore mechanism (Martos and Cox 1998; Martos et al. 1999). On the other hand, simulations show that corrugated velocity patterns in the vertical velocity field arise in both stellar and gaseous discs (Gómez et al. 2016, 2017), a result that suggests a gravitational rather than hydrodynamic origin.

Departures from planarity in the disc of the MW often appear as asymmetries in the phase space distribution of stars North and South of the Galactic midplane. For example, the discovery of bending waves by Xu et al. (2015) involved mapping the North-South number count asymmetry as a function of position in the disc plane. Evidence for a North-South asymmetry in the vertical profile of number counts for a 1 kpc cylinder centered on the Sun had already been found by Widrow et al. (2012) and Yanny and Gardner (2013) (see Ferguson et al. 2017 for a more recent confirmation using a larger photometric sample). Furthermore, Widrow et al. (2012), Williams et al. (2013), Carlin et al. (2013), and Sun et al. (2015) observed bulk motions in Solar Neighbourhood (SN) disc stars perpendicular to the midplane (vertical motions). These motions can be interpreted as a combination of the bending and breathing modes that are theoretically predicted for a plane-symmetric system (Mathur 1990; Weinberg 1991; Gómez et al. 2013; Widrow et al. 2014; Widrow and Bonner 2015;
The existence of warps has been attributed to the tidal effects of satellite galaxies, interactions of the disc with its dark halo, internal excitation of bending instabilities in gaseous discs, and intergalactic winds and magnetic fields, to name just a few (Kahn and Woltjer 1959; Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988; Battaner et al. 1990; Binney 1992; Debattista and Sellwood 1999; López-Corredoira et al. 2002a; Revaz and Pfenniger 2004; Shen and Sellwood 2006). Satellites and halo substructure have been invoked to explain the bending and breathing waves seen in the MW (Widrow et al. 2012; Gómez et al. 2013; Widrow et al. 2014; Feldmann and Spolyar 2015; D’Onghia et al. 2016; Gómez et al. 2016, 2017), though these features can also be generated by spiral structure (Debattista 2014; Faure et al. 2014; Monari et al. 2016a), the bar (Monari et al. 2015), or some (non-linear) combination of the two (Monari et al. 2016b).

The conclusion is that a galactic disc continually experiences perturbations that can set up waves perpendicular to its midplane. The question remains as to what happens to these waves once they are produced. Are they long-lived or do they quickly decohere, heating and thickening the disc? Can the bending waves be understood as purely kinematic structure, as has been suggested by de la Vega et al. (2015), or is self-gravity essential to their evolution? Finally, is there a connection between the corrugations of Xu et al. (2015) and the bulk motions in the Solar neighbourhood on the one hand, and the warp at the edge of the Galactic disc on the other?

In this paper, we address these and other questions through a series of idealized isolated galaxy simulations of MW-like models, as well as theoretical arguments based on both the eigenmode and WKB analysis of bending waves by Hunter and Toomre
The bending waves in our simulations arise without any provocation apart from the random noise of the particle distributions used to characterize the (smooth) halo, bulge, and disc. We emphasize that our results are specific to MW-like galaxies and may not generalize to other stellar discs. Indeed, there are very thin disc galaxies that show no sign of bending. These galaxies, which may well be low surface brightness galaxies viewed edge-on (Bizyaev et al. 2017), may not be as susceptible to the dynamical mechanisms described in this work as the galaxies considered here.

Attempts to understand the evolution of galactic bending waves in general, and warps in particular, date back over half a century. Much of the discussion parallels efforts to understand spiral structure. In particular, it was recognized early on that in the absence of self-gravity, an initial warp will shear due to differential precession (Kahn and Woltjer 1959) in a process akin to the winding problem for kinematic spiral structure. However, Lynden-Bell (1965) argued that a self-gravitating disc might support true bending modes, which makes warps long-lived. This idea was pursued in detail by Hunter and Toomre (1969) who found that in general, an isolated disc supports a continuum of modes and therefore a generic bending perturbation will disperse. For a disc embedded in a spherically symmetric halo, the only discrete ‘mode’ is the trivial zero-frequency tilt of the disc as a whole. If, on the other hand, the disc is embedded in a flattened halo, then the tilt mode is distorted but remains discrete and is therefore a candidate for explaining long-lived warps. This idea, first proposed by Dekel and Shlosman (1983) and Toomre (1983), was studied in detail by Sparke (1984) and Sparke and Casertano (1988) who treated the disc as a system of concentric rings embedded in the static potential of a flattened halo (see Revaz and
One of the main limitations in Sparke (1984) and Sparke and Casertano (1988) is that the halo is treated as a static potential, whereas live haloes can respond to the time-evolving gravitational field of the disc. In addition, the disc will set up wakes in the halo, which then act back on the disc via dynamical friction (Chandrasekhar 1943). Bertin and Mark (1980) argued that the halo-disc interaction can actually excite bending waves that are inside their co-rotation radius, though waves outside co-rotation are damped. Their analysis made a number of approximations, most notably that the halo comprised a spatially uniform Maxwellian distribution of particles. Also, they did not consider the halo contribution to the vertical restoring force. Nelson and Tremaine (1995) re-examined these claims in the context of more realistic halo models and concluded that the halo is more likely to damp rather than excite bending waves. However, this conclusion itself was called into question by Binney et al. (1998) who carried out numerical experiments that incorporated a ring model for the disc into a conventional $N$-body simulation of a dark halo. They found that when the disc is initialized to the Sparke-Casertano tilt-mode, the warp rapidly winds up while its energy remains constant or even increases. Binney et al. (1998) interpret these results in terms of time-scales: the inner halo responds to the disc on a time-scale comparable to the precession period and therefore the potential that the disc precesses in will be markedly different from the one assumed in the static-potential analysis of Sparke (1984) and Sparke and Casertano (1988). They go on to speculate that there will be true modes of the disc-live halo system, qualitatively similar to but quantitatively different from the Sparke-Casertano modes, that nevertheless avoid the rapid damping predicted in Nelson and Tremaine (1995).
In this work, we show that a Fourier and spectral analysis of bending waves yields a spectrum that is broadly consistent with both the eigenmode and WKB analysis. Moreover, there is a low-frequency branch of modes with little differential precession. Waves along this branch are therefore long-lived and thus candidates for the bending waves seen in the Galaxy.

Our work is based on \( N \)-body simulations of two isolated MW-like galaxy models, which are distinguished by the relative contributions of the disc and halo to the rotation curve, and thus by their susceptibility to the formation of a bar. The models and simulations are described in 4.3. In Section 4.4, we analyse the vertical bending waves that emerge in our simulated galaxies. In particular, we decompose the waves according to their azimuthal symmetry and present a spectral analysis that is reminiscent of the classic studies of density waves (see Sellwood and Athanassoula 1986). In Section 4.5 we show that the qualitative features seen in our spectral analysis are in good agreement with an eigenmode analysis of the linear ring model as well as the dispersion relation derived in the WKB approximation. Section 4.6 gives a brief description of how we might observe vertical bending waves in external galaxies. We discuss our results in Section 4.7 and conclude in Section 4.8.

4.3 Simulations

4.3.1 Two Milky Way-like Models

We generate the initial conditions for our \( N \)-body simulations using GALACTICS (Kuijken and Dubinski 1995; Widrow et al. 2008), which allows one to construct axisymmetric disc-bulge-halo systems that are in approximate dynamical equilibrium. The
phase space distribution function (DF) for the disc in these models, \( f_d \), is a function of two exact integrals of motion, the energy and the component of the angular momentum about the symmetry axis, and an approximate third integral that corresponds to the vertical energy. By construction, the disc density obtained by integrating \( f_d \) over all velocities is well-approximated by the function

\[
\rho_d (R, z) = \frac{M_d}{4\pi R_d^2 h} e^{-R/R_d} \text{sech}^2 (z/h) C ((R - R_t) / \delta R_t),
\]

(4.1)

where \( R \) and \( z \) are cylindrical coordinates and \( C \) is a truncation function that goes smoothly from unity to zero at \( R \approx R_t \) over a width of order \( \delta R_t \). The radial velocity dispersion is an exponentially decreasing function of \( R \):

\[
\sigma_R (R) = \sigma_{R0} \exp (-R/2R_d),
\]

(4.2)

where we assume that the radial scale length for the square of the velocity dispersion is the same as the scale length for the surface density, in accord with observations (Bottema 1993). The azimuthal velocity dispersion profile is set by the epicycle approximation (see Kuijken and Dubinski 1995 and Section 3.2.3 of Binney and Tremaine 2008) while the vertical velocity dispersion is determined from the local surface density and the model assumption that the disc has a constant scale height.

The DFs for the bulge and halo are functions solely of the energy and are therefore isotropic in velocity space. The bulge DF is constructed to yield a density profile that is given, approximately, by

\[
\rho_b (r) = \frac{v_{b0}^2}{4\pi G R_e^2 c(n)} \left( \frac{r}{R_e} \right)^{-p} e^{-b(r/R_e)^{1/n}},
\]

(4.3)
which yields the Sérsic law for the projected surface density profile with index \( n \) so long as one sets \( p = 1 - 0.6097/n + 0.05563/n^2 \). The constant \( b \) is adjusted so that \( R_e \) encloses half the total projected mass and \( c(n) = (nb^{n(p-2)}) \Gamma (n(2-p)) \) (Prugniel and Simien 1997; Terzić and Graham 2005).

The halo DF is constructed to yield the Navarro-Frenk-White (NFW) profile (Navarro et al. 1996):

\[
\rho_h(r) = \frac{a_h v_h^2}{4\pi G} \frac{1}{r (r + a_h)^2}. \tag{4.4}
\]

While the velocity distributions of the halo and bulge are isotropic, their space densities are slightly flattened due to the disc potential.

In this paper, we consider two GALACTICS models. The first (Model 1) is from Widrow et al. (2008). In that paper, Bayesian and Markov chain Monte Carlo methods were used to constrain the GALACTICS parameters so as to match kinematic and photometric observations of the MW. That analysis yielded a suite of models that varied in their susceptibility to bar formation. The model that we chose for this paper has a relatively weak bar instability. For Model 2, we reduced the disc mass by roughly a third and increased the masses of the bulge and halo components so as to obtain a similar rotation curve. The disc in Model 2 is therefore so light that it never forms a bar.

While our models yield very similar circular speed curves they differ in the relative contributions of the disc and the dynamically hot components (the bulge and halo) to the radial force, as shown in Fig. 4.1. In particular, Model 1 has a disc mass of \( M_d \approx 4 \times 10^{10} M_\odot \), which impies that at a radius of \( 2.2 R_d \), \( V_d^2/V_c^2 = 0.51 \), where \( V_c = V_c(R) \) is the circular speed at radius \( R \) and \( V_d \) is the contribution to the circular speed
CHAPTER 4. SPONTANEOUS BENDING WAVES

Figure 4.1: Circular speed curves for the initial conditions of Model 1 (top) and Model 2 (bottom). The solid black curve shows the total circular speed of the model. Also shown are the contributions to the circular speed curve from the bulge (blue, short-dashed), the disc (red, dotted), and the dark matter halo (magenta, long-dashed). Model 1 corresponds to the most stable MW-like model of Widrow et al. (2008), and possesses a disc with enough self-gravity in the inner region to trigger bar formation. Model 2 was constructed with a disc mass 60 per cent of that in Model 1, while the bulge and halo parameters were adjusted to yield a similar total circular speed curve. Model 2 only forms flocculent spiral structure due to the dominant bulge and halo relative to the disc.

from the disc. The central radial velocity dispersion is set to $90\text{ km s}^{-1}$, which yields a Toomre $Q$ parameter (Toomre 1964) at $2.2R_d$ of 1.2. For Model 2, $V_d^2/V_c^2 = 0.33$ at $2.2R_d$. In addition, the central radial velocity dispersion in the disc is decreased so that the Toomre $Q$ parameter at $2.2R_d$ is the same as in Model 1. The GALACTICS
### Table 4.1: Initial model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Model 1</th>
<th>Model 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_d [10^{10} M_\odot] )</td>
<td>3.9</td>
<td>2.3</td>
</tr>
<tr>
<td>( R_d [\text{kpc}] )</td>
<td>2.8</td>
<td>2.8</td>
</tr>
<tr>
<td>( h [\text{kpc}] )</td>
<td>0.44</td>
<td>0.44</td>
</tr>
<tr>
<td>( R_t [\text{kpc}] )</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>( \delta R_t [\text{kpc}] )</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( \sigma_{R0} [\text{km s}^{-1}] )</td>
<td>90</td>
<td>56.3</td>
</tr>
<tr>
<td>( M_h [10^{12} M_\odot] )</td>
<td>1.2</td>
<td>1.3</td>
</tr>
<tr>
<td>( v_h [\text{km s}^{-1}] )</td>
<td>481</td>
<td>505</td>
</tr>
<tr>
<td>( a_h [\text{kpc}] )</td>
<td>43</td>
<td>43</td>
</tr>
<tr>
<td>( M_b [10^9 M_\odot] )</td>
<td>9.9</td>
<td>17.9</td>
</tr>
<tr>
<td>( v_b [\text{km s}^{-1}] )</td>
<td>272</td>
<td>375</td>
</tr>
<tr>
<td>( R_e [\text{kpc}] )</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>( n )</td>
<td>1.32</td>
<td>1.32</td>
</tr>
<tr>
<td>( Q(2.2R_d) )</td>
<td>1.2</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Disc mass\(^a\); radial disc scale length\(^b\); vertical disc scale height\(^c\); truncation radius of the disc\(^d\); truncation width of the disc\(^e\); central radial velocity dispersion in the disc\(^f\); halo mass\(^g\); characteristic velocity scale of the halo\(^h\); NFW halo scale length\(^i\); bulge mass\(^j\); characteristic velocity scale of the bulge\(^k\); bulge effective radius\(^l\); Sérsic index\(^m\); Toomre \( Q \) parameter at \( 2.2R_d \).

parameters for the two models are given in Table 4.1.

#### 4.3.2 N-body Simulations

We use GALACTICS to generate N-body realizations for the two models just described. These realizations have \( 2.5M \) disc particles, \( 250k \) bulge particles, and \( 5M \) halo particles. The masses of particles within each component are identical. The initial conditions were evolved for \( \sim 10 \) Gyr using GADGET-3 (Springel 2005) with a softening length of 40 pc for all particle types. The maximum time step was set to 0.2 Myr, which is \( \sim 0.2 \) per cent of the Galactic dynamical time at a radius of 20 kpc, where the warp occurs, and \( \sim 1 \) per cent of a Galactic dynamical time at 4 kpc. Total
energy was conserved to within 0.06 per cent. We start with ‘pristine’ equilibrium initial conditions, which allow us to focus on key dynamical processes such as the secular evolution of bending waves. Our simulations ignore gas dynamics and do not capture the physics of galaxy formation.

Over time the discs in our simulations rotate and drift slightly about the global coordinate origin. To account for this when analyzing face-on maps of our discs we center the disc and rotate it into the \( x-y \) plane by way of the following iterative scheme. We compute the center of mass of disc particles within a cylinder of radius 20 kpc and height 2 kpc centered in the \( x-y \) plane and translate all disc particles to that frame. This procedure is repeated until a translation of less than 1 pc is achieved in all coordinate directions. Next, we account for any tilt of the disc relative to the \( x-y \) plane by using a two-dimensional Newton-Raphson scheme to find the Euler angles about the \( x- \) and \( y- \) axes that minimize the root mean square vertical displacement of disc particles within a cylinder of radius 20 kpc and height 2 kpc centered in the \( x-y \) plane. This method does a good job of diagonalizing the moment of inertia tensor of the disc.

In Fig. 4.2, we show the disc surface density relative to the equilibrium surface density for the two models at various times. The maps extend to \( R = 25 \text{kpc} \), or about \( 9R_d \), which is the radius at which we begin to truncate the disc. This radius also corresponds to the outermost radius at which the warp in the MW’s stellar disc has been observed (Momany et al. 2006). The decrease of \( \sim 20 \text{–} 55 \) per cent in the surface density for \( R > 22 \text{kpc} \), which develops at early times, is likely due to a relaxation in the initial conditions.

The disc in Model 1 forms a bar at \( \sim 2.5 \text{Gyr} \), which eventually extends to \( R \sim \)
Figure 4.2: Face-on disc surface density enhancement for Model 1 (top row) and Model 2 (bottom row) at five epochs throughout the simulation, as indicated at the top of each column. The colour map indicates the logarithm (base 10) of the ratio of the surface density at each epoch relative to the initial equilibrium surface density. Dotted concentric circles indicate increments of 5 kpc in radius. The rotation of the discs is counter-clockwise. The bar that forms in the higher mass disc of Model 1 is clearly visible, as well as a ring structure between 10 and 15 kpc. Model 2 only forms flocculent tightly wound spiral structure and displays only modest surface density enhancements relative to Model 1.
5 kpc. In addition, a prominent ring (again, an enhancement relative to the initial exponential profile) develops at a radius of \( \sim 10 - 15 \) kpc. This feature is likely a result of mass redistribution due to the bar. On the other hand, the disc in Model 2 forms flocculent spiral structure but no bar.

### 4.4 Analysis of Bending Waves

#### 4.4.1 Fourier Analysis

The surface density shown in Fig. 4.2 represents the zeroth moment, with respect to \( z \), of the coarse-grained stellar DF, \( \tilde{f} \),

\[
\Sigma (R, \phi, t) = \int \tilde{f}(r, \mathbf{v}, t) \, d^3v \, dz .
\]  

(The true DF for an \( N \)-body system is a sum of six-dimensional \( \delta \)-functions at the positions of the particles. In practice, coarse-graining amounts to calculating an average surface density over a small patch of the disc.) Likewise, the mean vertical displacement of the disc is given by the first moment with respect to \( z \),

\[
Z (R, \phi, t) = \Sigma (R, \phi, t)^{-1} \int \tilde{f}(r, \mathbf{v}, t) \, z \, d^3v \, dz .
\]  

In Fig. 4.3, we show mean vertical displacement and vertical bulk velocity maps for the two models at the same five epochs shown in Fig. 4.2. During the first Gyr, displacements of order 100 pc and bulk vertical motions of a few \( \text{km} \, \text{s}^{-1} \) are found across the discs in both models. We interpret these displacements and bulk motions as bending waves. By 4 Gyr, the presence of the bar is clearly evident in Model 1.
Figure 4.3: Face-on maps of the mean vertical displacement, $Z(R, \phi, t)$ (top set of rows), and mean vertical velocity, $V_z(R, \phi, t)$ (bottom set of rows), for Model 1 and Model 2, as indicated on the left side. Columns correspond to the same five epochs in Fig. 4.2, as indicated at the top of each column. Dotted concentric circles indicate increments of 5 kpc in radius. The rotation of the discs is counter-clockwise. At later times, the leading nature of the bending waves, especially in the outer disc, is very clear. In general, bending is slightly stronger and more pervasive in the less massive and barless disc of Model 2 compared to Model 1, especially at later epochs. Furthermore, the average height and vertical velocity seem to be coupled, qualitatively similar to the oscillatory anti-correlation previously reported (Gómez et al. 2013, 2016, 2017).
Interestingly enough, by this epoch, the vertical displacements are stronger in Model 2 and extend further in toward the centre of the disc, as compared with Model 1. The implication is that the bar and/or more massive nature of the disc in Model 1 has a damping effect on bending waves.

The waves have a strong $m = 1$ component, where $m$ is the azimuthal mode number. Moreover, these waves appear to be leading in the sense that the radius of the peak vertical displacement ridge increases with increasing $\phi$, that is, in the direction of rotation. The $m = 1$ waves are tightly wound at intermediate radii but transition smoothly to a warp at the edge of the disc. In general, the amplitude of the waves in the outer disc grows over time while the amplitude decreases in the inner disc. We return to this point below.

The bulk vertical motions follow the same general pattern of the displacements except that they are out of phase, as one would expect for wave-like motion (see Gómez et al. 2013, 2016, 2017). From the amplitudes of the velocity and height perturbations, we infer a characteristic pattern speed for the bending waves of $\sim 10 - 20 \text{ km s}^{-1} \text{ kpc}^{-1}$.

In the study of density waves, it has proved instructive to write the surface density as a Fourier series in $\phi$ (see, for example, the seminal work of Sellwood and Athanassoula 1986). Formally, one has

$$\Sigma (R, \phi, t) = \text{Re} \left\{ \sum_m \Sigma_m (R, t) e^{-im\phi} \right\},$$

where

$$\Sigma_m (R, t) = \frac{1}{2\pi} \int_0^{2\pi} \Sigma (R, \phi, t) e^{im\phi} d\phi.$$
In practice, the $\Sigma_m$ are calculated over a ring of finite radius (essentially, coarse graining in $R$). For a ring at radius $R_\alpha$ and width $\Delta R_\alpha$ we have

$$\Sigma_m (R, t) = A_\alpha^{-1} \sum_{j \in \alpha} m_j e^{im\phi_j} ,$$

(4.9)

where $A_\alpha = 2\pi R_\alpha \Delta R_\alpha$ is the area of the ring and the sum is over all particles in the ring. Similar to the surface density, we can construct a Fourier series for the vertical displacement:

$$Z(R, \phi, t) = \text{Re} \left\{ \sum_{m=0}^{\infty} Z_m (R, t) e^{-im\phi} \right\} ,$$

(4.10)

where we define

$$Z_m (R, t) \equiv A_\alpha^{-1} \sum_{j \in \alpha} \frac{z_j m_j}{\Sigma (R_j, \phi_j, t)} e^{im\phi_j} .$$

(4.11)

The surface density at particle $j$, $\Sigma (R_j, \phi_j, t)$, is estimated by using the Fourier series for $\Sigma$ (equation 4.7) truncated at the 5th order terms.

In Fig. 4.4 we show the $m = 0$, 1, 2, and 3 Fourier modes of $Z$ for the $t \sim 4$ Gyr snapshot of the Model 1 simulation. We see that while the $m = 1$ mode is clearly leading, the $m = 2$ term is trailing, and the $m = 3$ term is a combination of both leading and trailing. Furthermore, the $m = 1$ mode is the dominant mode beyond a radius of $\sim 15$ kpc.

We explore how this dominant mode in the outer disc forms and evolves in Fig. 4.5 by showing $|Z_1(R)|$ for both models at various snapshots. Both models show a trend of increasing $|Z_1(R)|$ from intermediate radii to the outer disc with time. This is most evident in Model 1 where there is a factor of $\sim 2 - 3$ decrease in $|Z_1|$ at $R \sim 10 - 15$ kpc.
Figure 4.4: Fourier mode decomposition of $Z(R, \phi, t \sim 4 \text{ Gyr})$ for $m = 0, 1, 2,$ and $3,$ as indicated, for Model 1. Dotted concentric circles indicate increments of 5 kpc in radius. The rotation of the disc is counter-clockwise. The colour scale is in units of kpc, and differs between the upper and lower panels in order to highlight the extremal values for each $m$. The factor of $1/2$ for the zeroth order term in the Fourier Series is reflected in the $m = 0$ panel. From the figure it is clear that the (leading) $m = 1$ term is the dominant one in the decomposition.

and an approximately equal increase in the outer disc after $t \sim 1 \text{ Gyr}$. We attribute the large suppression of $|Z_1|$ at intermediate radii to the bar, which forms at $\sim 2.5 \text{ Gyr}$. This trend, which is also present in Model 2 though not so pronounced, suggests that warps may arise from the outward migration of $m = 1$ bending waves.

4.4.2 Spectral Analysis

To connect our 3D simulations with the linear analysis of Section 4.5, we perform a spectral analysis of bending waves in our simulated discs. Our method uses the
Figure 4.5: Time evolution of $|Z_1(R)|$. The upper and lower panels show Model 1 and Model 2, respectively. Both models show a general trend of increasing strength propagating outward from intermediate radii to the outer disc over time.

techniques from Sellwood and Athanassoula (1986) (see also Roškar et al. 2012 for a succinct description). We calculate the Fourier coefficients $Z_m(R,t)$ (see Section 4.4.1, equation 4.11) for the $t_j = j \Delta t + t_0$ snapshots where $\Delta t$ is the time between snapshots, $j = 0 \ldots N - 1$, and $N$ is even. We then perform a discrete Fourier transform in time
to obtain the two-sided spectral coefficients

\[ Z_m(R, \omega_k) = \sum_{j=0}^{N-1} Z_m(R, t_j) w(j) e^{2\pi i j k/N}, \]  

(4.12)

(Press et al. 2007, Section 13.4). Here, \( w(j) \) is a Gaussian window function with a standard deviation of \( N/2^{5/2} \), which is introduced to mitigate spectral leakage from high frequencies. We note that using similarly shaped window functions, such as the Hamming window, does not significantly alter the resulting spectra or our conclusions. The discrete frequencies\(^1\) are given by

\[ \omega_k = \frac{2\pi}{m N\Delta t} k, \]  

(4.13)

with \( k = -N/2 \ldots N/2 \), and the Nyquist frequency corresponding to \( \omega_k=\pm N/2 \). The two-sided power spectrum is then computed as

\[ P_m(R, \omega_k) = \frac{1}{W} |Z_m(R, \omega_k)|^2, \]  

(4.14)

where \( W = N \sum_{j=0}^{N-1} w(j) \) is the window function normalization.

In Fig. 4.6, we explore the time evolution of the \( m = 1 \) bending wave power spectra for Model 1. To do so, we divide the simulation into five \( \sim 2 \) Gyr intervals and calculate \( P_1 \) using the method outlined above. Each interval has \( N = 200 \) snapshots with a time resolution of \( \Delta t \approx 9.8 \) Myr, which gives a frequency resolution of \( \sim 3.14 \) km s\(^{-1}\) kpc\(^{-1}\). Prominent bands of power develop at early times and persist

---

\(^1\)Since we consider negative frequencies, we note a difference between our index \( k \), and that used in the conventional spectral analysis. For example, the mapping between our \( k = -N/2 \ldots N/2 \) and the index used in Roskar et al. (2012), \( k' = 0 \ldots N - 1 \), is \( k' = k \) for \( k \geq 0 \) and \( k' = k + N \) for \( k < 0 \). Therefore, the range \( k' = 0 \ldots N/2 - 1 \) corresponds to increasing positive frequencies, and \( k' = N/2 + 1 \ldots N - 1 \) corresponds to increasing negative frequencies. The Nyquist frequency for \( k' = N/2 \) corresponds to \( k = \pm N/2 \).
Figure 4.6: Two-sided frequency power spectra of $m = 1$ bending waves for various $\sim 2$ Gyr time intervals, as indicated, for Model 1. The scale of power is logarithmic and is constant across all time intervals shown. Overlaid on all panels in red are total circular frequency (solid) and vertical resonance (dashed) curves, $\omega = \Omega \pm \nu_x$, where $\nu_x = \sqrt{\nu_b^2 + \nu_h^2}$ is the vertical forcing frequency from the bulge and halo, for the equilibrium initial conditions. The shaded red indicates the ‘forbidden’ region between the resonances according to the WKB approximation (see Section 6.6.1 of Binney and Tremaine 2008 and Section 4.5). The horizontal dotted green line references zero frequency. The horizontal dashed blue lines in the middle top panel indicate the frequencies considered in Fig. 4.7.
over the entire simulation. Much of the power lies along two main branches, which, as we describe below, follow the two lowest-order vertical resonances from linear theory. Power along the lower branch, which corresponds to slowly counter-rotating waves, is seen across the disc at all times. On the other hand, power along the upper branch (more rapidly rotating waves) diminishes with time. In general, over time power migrates to larger radii.

We note the presence of spurious bands of power at fixed radii that extend almost the entire frequency range in Fig. 4.6. More specifically, there is a band present in all time intervals at $R < 0.25 \text{kpc}$, as well as weaker bands at larger radii that come and go across the various time intervals. These bands arise when the local surface density term in the vertical displacement Fourier coefficients ($\Sigma (R_j, \phi_j)$ in equation 4.11) for one or more particles is computed to be nearly zero. This situation is an artefact of our scheme to estimate the local surface density as a truncated Fourier series.

Overlaid in all panels of Fig. 4.6 are the $m = 1$ vertical resonances (analogous to the Lindblad resonances for density waves), $\omega = \Omega \pm \nu_x$, where $\Omega$ is the circular frequency of the disc and $\nu_x = \sqrt{\nu_b^2 + \nu_h^2}$ is the vertical force oscillation frequency due to the bulge and halo potential, of the initial conditions. Power in the $m = 1$ bending waves collects along the vertical resonances and tends to lie outside of the forbidden region between them, as predicted by the WKB approximation (Binney and Tremaine 2008, Section 6.6.1, but also see Section 4.5). Furthermore, agreement between our power spectra and the vertical resonances of the initial conditions endures for many billions of years.

We can then ask the question as to why higher frequency waves subside over time, or conversely, why low-frequency waves persist? To this end, we consider a form
of the tip-Line-of-Nodes (LON) plot (see Briggs 1990) for waves of various pattern speeds. A tip-LON plot is constructed by calculating, for a series of concentric rings, the tilt, or tip away from the midplane, of the disc and the position of the LON. To do this, we construct an effective vertical displacement as a function of $R$ and $\phi$ for a narrow range in frequency, $\Delta \omega_k$, using only the $m = 1$ spectral coefficient of the $t - \omega_k$ Fourier transform,

$$Z_1(R, \phi, \omega_k) = \int_{\omega_k-\Delta\omega_k/2}^{\omega_k+\Delta\omega_k/2} \text{Re}\left\{Z_1(R, \omega_k) e^{i\phi}\right\} d\omega_k.$$  

The function $Z_1(R, \phi, \omega_k)$ is essentially our series of rings from which the tilt and position angle for the LON as a function of $R$ can be easily determined.

Tip-LON plots are shown in Fig 4.7. For illustrative purposes we only consider the time interval $2 \lesssim t \lesssim 4$ Gyr for Model 1 (i.e. Fig. 4.6, top middle panel) and show the results for $\omega_k = 28.3, 3.1, 0, \text{and} -3.1 \text{km s}^{-1}\text{kpc}^{-1}$. These four frequencies, which are indicated by horizontal dashed lines in Fig. 4.6, correspond to a typical frequency in the upper branch and three frequencies in the lower branch. Each dot of a tip-LON plot corresponds to a ring within the disc whose radius, tilt, and position angle are encoded in the colour, radial coordinate, and angular position, respectively, of the dot. Thus, dots that trace a clockwise pattern with increasing ring radius indicate a trailing wave.

Large tilt angles for $\omega_k = 28.3 \text{km s}^{-1}\text{kpc}^{-1}$ are localized to $15 \lesssim R \lesssim 20 \text{kpc}$ where there is a prominent peak in the power spectrum (see Fig. 4.6). These are clearly trailing waves. On the other hand, the tilt is stronger across a greater range in $R$ at $\omega_k = -3.1, 0, \text{and} 3.1 \text{km s}^{-1}\text{kpc}^{-1}$. These are leading waves, which experience less shear in the outer disc and therefore persist over a longer period of time.
Figure 4.7: A form of tip-LON plots (so-called ‘Briggs’ plots, see Briggs 1990) for four wave frequencies within the time interval $2 \lesssim t \lesssim 4$ Gyr (indicated in Fig. 4.6, top middle panel). For each frequency the disc is divided into concentric rings in radius. Each point in the figure corresponds to a radial ring, with the radius indicated by colour. The radial coordinate is the tilt, in degrees, of that ring, and the azimuthal coordinate is the position of maximum vertical displacement. Concentric dotted circles in the radial direction indicate tilt increments of $5^\circ$. High-frequency waves are morphologically trailing since the azimuth decreases with increasing radius. In contrast, waves near zero frequency are leading.

4.5 Linear Ring Model

In this section, we study the linear bending waves of a razor thin, self-gravitating disc embedded in the static gravitational potential of a bulge and halo. The starting point for this investigation is the second-order integro-differential equation for local vertical displacements of the disc, $Z(R, \phi, t)$, which was introduced by Hunter and
Toomre (1969). In order to relate this $Z$ to the mean vertical displacement mapped in our simulations, we first show that the equation from Hunter and Toomre (1969) can be derived from the collisionless Boltzmann equation.

### 4.5.1 Equations of Motion

We consider a system of collisionless ‘stars’ whose equilibrium distribution is that of a self-gravitating disc embedded in the external potential of an extended dark matter halo and a centrally concentrated bulge. We assume that the stellar orbits, when projected onto the disc plane, are circular. The DF for the stars can then be written

$$f(r,v,t) = \delta(v_R) \delta(v_\phi - R \Omega(R)) F(R, \phi, z, v_z, t), \quad (4.16)$$

where $\Omega(R) = v_c(R)/R$, $\delta(\cdot)$ is the Dirac delta function, and $F(R, \phi, z, v_z, t)$ is the reduced DF. Upon integration over $v_R$ and $v_\phi$, the collisionless Boltzmann equation becomes

$$\frac{\partial F}{\partial t} + \Omega(R) \frac{\partial F}{\partial \phi} + v_z \frac{\partial F}{\partial z} = \frac{\partial \psi}{\partial z} \frac{\partial F}{\partial v_z}, \quad (4.17)$$

where $\psi$ is the gravitational potential.

The vertical displacement of the disc is found by taking the $z$-moment of the DF:

$$Z(R, \phi, t) = \int z F(R, \phi, z, v_z, t) \, dz \, dv_z. \quad (4.18)$$

This quantity is essentially the analog of $Z$ in equation (4.6) for the thin, cold disc.
Likewise, the vertical motion of the disc is

\[ V(R, \phi, t) = \int v_z F(R, \phi, z, v_z, t) \, dz \, dv_z. \]  \hspace{1cm} (4.19)

Multiply equation (4.17) by either \( z \) or \( v_z \) and integrate over \( z \) and \( v_z \) and we obtain

\[ \left( \frac{\partial}{\partial t} + \Omega(R) \frac{\partial}{\partial \phi} \right) Z = V \]  \hspace{1cm} (4.20)

and

\[ \left( \frac{\partial}{\partial t} + \Omega(R) \frac{\partial}{\partial \phi} \right) V = -\frac{\partial \psi}{\partial z} = F_b + F_h + F_d, \]  \hspace{1cm} (4.21)

where in the last line, we’ve written out the separate contributions to the vertical force from the bulge, halo, and disc. Together, these equations lead to the standard second order equation for \( Z \), which was first introduced in Hunter and Toomre (1969).

For definiteness, we assume that the potential from the bulge and halo are the same as in the galactics model described in Section 4.3. Near \( z = 0 \), the contributions to the vertical restoring force due to these components are given, to a good approximation, by

\[ F_b(R, t) = -\nu_b^2 Z(R, t) \quad F_h(R, t) = -\nu_h^2 Z(R, t), \]  \hspace{1cm} (4.22)

where \( R = (R, \phi) \) in cylindrical coordinates. The vertical force due to the disc, to first order in \( Z/R \), is

\[ F_d = -G \int dR' \Sigma(R') \frac{Z(R, t) - Z(R', t)}{(R^2 + R'^2 - 2RR' \cos \varphi + z_0^2)^{3/2}}, \]  \hspace{1cm} (4.23)
where $\varphi \equiv \phi - \phi'$ and the ‘softening’ parameter $z_0$ is introduced to make the integral regular at $R = R'$ (Sparke and Casertano 1988). (See Hunter and Toomre 1969, Sparke and Casertano 1988, and Nelson and Tremaine 1995 for alternative ways of handling the $R = R'$ singularity in the integrand.) Note that the term on the right-hand side that is proportional to $Z(R, t)$ may be written $F_1 = -\nu_d^2 Z$, where

$$
\nu_d^2 = G \int_0^\infty dR' R' \Sigma(R') H(R, R') \quad (4.24)
$$

and

$$
H(R, R') = \int_0^{2\pi} \frac{d\varphi}{(R^2 + R'^2 - 2RR' \cos \varphi + z_0^2)^{3/2}}. \quad (4.25)
$$

This term describes the vertical restoring force due to the equilibrium disc that acts on an element of the disc displaced from the midplane. We can then write

$$
\nu_t^2 = \nu_d^2 + \nu_h^2 + \nu_i^2, \quad (4.26)
$$

where $\nu_t$ is the total vertical epicyclic frequency. The term in equation (4.23) that is proportional to $Z(R', t)$ describes the perturbing force on an unperturbed element in the disc due to those parts of the disc that have been displaced from the midplane.

For small displacements, we can write $Z$ and $V$ as superpositions of linear modes with $m$-fold azimuthal symmetry so that each mode has the form

$$
Z(R, \phi, t) = \text{Re} \left\{ Z(R, \omega)e^{im(\phi - \omega t)} \right\} \quad (4.27)
$$
and

\[ V(R, \phi, t) = \text{Re} \left\{ V(R, \omega) e^{i(m \phi - \omega t)} \right\} . \]  \hspace{1cm} (4.28)

We then have

\[ -i (\omega - m \Omega) Z(R, \omega) = V(R, \omega) \]  \hspace{1cm} (4.29)

and

\[ -i (\omega - m \Omega) V(R, \omega) = -\nu^2 \frac{d}{dR} Z(R, \omega) \]
\[ + G \int_0^\infty dR' R' \Sigma(R', \omega) I_m(R, R') , \]  \hspace{1cm} (4.30)

where

\[ I_m(R, R') = \int_0^{2\pi} \frac{d\varphi \cos m \varphi}{(R^2 + R'^2 - 2RR' \cos \varphi + z_0^2)^{3/2}} . \]  \hspace{1cm} (4.31)

### 4.5.2 Linear Ring Model and the WKB Approximation

Following Sparke (1984) and Sparke and Casertano (1988), we divide the disc into \( N \) concentric rings. Equations (4.29) and (4.30) together constitute an eigenvalue equation for the \( 2N \)-dimensional vector \( (Z, V)^T \). The eigenvalues \( \omega_i \) \( (i = 1, 2N) \) are all real. We find that each eigenmode departs significantly from zero over a relatively narrow range in \( R \). This point is illustrated in Fig. 4.8 for the case \( m = 1 \) where we plot a horizontal line segment across the range in \( R \) where the amplitude of the eigenmode exceeds 10 per cent of its maximum value. The parameters for the \( N \)-ring disc and the external potential of the bulge and halo are chosen to match our
CHAPTER 4. SPONTANEOUS BENDING WAVES

Figure 4.8: Frequency as a function of radius for $m = 1$ bending wave eigenmodes of the linear ring model and the WKB analysis of Model 1 (see Section 4.5.2). The horizontal solid black line references zero frequency. Horizontal solid blue lines show the range in radius where the amplitude of each frequency eigenmode exceeds 10 per cent of its maximum value. The solid red curve corresponds to the total circular frequency curve of the linear model, $\Omega$. Dashed red illustrates the $m = 1$ vertical resonances, $\omega = \Omega \pm \nu_x$. Shaded red indicates the ‘forbidden’ region between the two resonances, as predicted by the WKB approximation. Evidently, the linear eigenmodes lie outside of the forbidden region and tend to lie along the two vertical resonances.

Evidently, there are two continuous branches of eigenmodes. The patterns of modes along the upper-$\omega$ branch rotate in the same sense as the disc while those modes along the lower branch are counter-rotating or rotating at very small (nearly zero) positive frequency. In general, a disturbance localized in $R$ will involve a superposition of modes from the two branches. The part of the wave associated with modes from
the upper branch, where $\omega$ is a decreasing function of $R$, will shear into a trailing spiral while the part associated with the lower branch will shear into a leading spiral. Since the magnitude of the differential rotation rate $|d\omega/dR|$ is smaller along the lower branch, phase mixing occurs more slowly for this part of the wave. Thus, over time we expect more pronounced leading waves of vertical oscillation, which is indeed what we find in the simulations (cf. Figs. 4.3 and 4.4).

For wavelengths that are small as compared with the scale length of the disc we may use a WKB approximation. The wavefronts are assumed to be perpendicular to the radial direction and the dispersion relation becomes (Hunter and Toomre 1969; Sparke and Casertano 1988; Nelson and Tremaine 1995)

$$
(\omega - m\Omega(R))^2 - 4\pi^2 G\Sigma(R)/\lambda - \nu^2_h - \nu^2_b = 0,
$$

where $\lambda$ is the wavelength of the perturbation. Corotation occurs when $\omega = \Omega$, while vertical resonances (the analogs of Linblad resonances) occur when $\omega = \Omega \pm \nu_x$ where $\nu_x = \sqrt{\nu^2_h + \nu^2_b}$ is the vertical oscillation frequency associated with the halo and bulge (i.e., the fixed spheroidal components). From equation (4.32), we see that waves are excluded from the region in frequency space between these two resonances, the so-called ‘forbidden’ region (Nelson and Tremaine 1995; Binney and Tremaine 2008, Section 6.6.1).

In addition to the results of the eigenmode analysis (Sparke and Casertano 1988) in Fig. 4.8, we also show the corotation and vertical resonance curves as calculated in the WKB approximation. Indeed, the linear eigenmodes tend to lie along the two vertical resonances.
4.6 Corrugations in External Galaxies

If warps and corrugations in MW-like disc galaxies are easily excited and long lived then they should be observable in similar external galaxies. As shown in this work, and demonstrated in others (for example, see Gómez et al. 2017), spatial corrugations in the vertical direction are coupled to corrugations in stellar vertical velocity. Furthermore, Gómez et al. (2017) showed that vertical perturbations in simulated disc galaxies manifest in both the gaseous and stellar components. Thus, velocity corrugations should be observable in the line-of-sight kinematics of nearly face-on external disc galaxies. Integral field spectroscopy surveys such as Calar Alto Legacy Integral Field Area (CALIFA; Sánchez et al. 2012, but also see Falcón-Barroso et al. 2017), Mapping Nearby Galaxies at APO (MaNGA; Bundy et al. 2015), and DiskMass (Bershady et al. 2010) hold great promise for detecting bending waves in external galaxies since they gather spatially resolved information on both the stellar and gaseous components. It is most likely easier to detect any corrugations in the gaseous disc rather than the stellar component due to larger velocity dispersions in the latter. In this Section we outline a number of requirements needed to observe extragalactic bending waves and corrugations with respect to our simulations.

In Fig. 4.9 we show line-of-sight velocity maps for our Model 1 as viewed face-on and at inclinations $i = 4^\circ$, $8^\circ$, and $12^\circ$. Contributions to the line-of-sight velocity from both radial and azimuthal flows increase with increasing inclination. Nevertheless, hints of corrugation are still visible even at $i = 12^\circ$. In principle, one might attempt to incorporate vertical motions into a model for two-dimensional velocity maps of spiral galaxies, as is done for non-circular flows in the presence of a bar (Spekkens and Sellwood 2007; Sellwood and Sánchez 2010; Sellwood and Spekkens 2015).
We may thus ask, at what inclination do the motions in the vertical direction dominate the in-plane flows along the line-of-sight? That is, how face-on does a galaxy need to be? In Fig. 4.10 we consider the radial, azimuthal, and vertical bulk motions for Model 1. In particular, we show a time sequence of $m = 1$ and $m = 2$ Fourier mode amplitudes as a function of radius (i.e. equations 4.10 and 4.11, but for velocity rather than vertical displacement). In the inner disc, the largest amplitudes
Figure 4.10: Time evolution of the $m = 1$ and $m = 2$ (left and right columns, respectively) Fourier coefficient magnitude as function of radius for cylindrical radial, azimuthal, and vertical velocities (top, middle, and bottom rows, respectively), for Model 1. Colour indicates the epoch of each magnitude’s profile. The large in-plane $m = 2$ amplitudes in the central disc are due to the bar, while the large $m = 1$ amplitude of vertical velocity in the outer disc at later times is a manifestation of the warp and other vertical corrugation patterns. As the disc is projected onto the sky and inclined the $m = 1$, 2, and higher order terms (not shown) mix into the line-of-sight velocity. In the text we discuss at what inclinations the $m = 1$ vertical motions dominate the mixed non-circular in-plane motions in the line-of-sight velocity field.
correspond to the \( m = 1 \) and \( m = 2 \) radial and azimuthal velocities, and are due to the bar. On the other hand, large \( m = 1 \) vertical velocity amplitudes in the outer disc, especially at late times, are due to the warp and associated corrugations.

For an inclined disc, the \( m = 1 \) and \( m = 2 \) contributions, as well as higher \( m \) terms from lower amplitude waves, mix into the line-of-sight velocity field. After accounting for the systematic velocity of a galaxy, the line-of-sight velocity can be decomposed as

\[
v_{\text{los}} = (v_R \sin \theta + v_\phi \cos \theta) \sin i + v_z \cos i ,
\]

where \( v_R, v_\phi, \) and \( v_z \) are Fourier expansions of each respective velocity component, \( \theta \) is the position angle of the disc, and \( i \) is the inclination. In Fig. 4.10, non-circular radial and azimuthal flows roughly equal that of the \( m = 1 \) vertical component in the outer disc. Thus, upon averaging over the position angle term on the interval \([0, \pi]\), bulk \( m = 1 \) vertical motions in the outer disc will be comparable to the integrated \( m = 1 \) and \( m = 2 \) in-plane contributions along the line-of-sight for an inclination of \( \sim 38^\circ \). For inclinations of \( \sim 20^\circ \) and \( 10^\circ \), the average vertical contribution to the line-of-sight velocity field would be roughly double and four times that, respectively, of the non-circular in-plane flows.

In addition to an inclination constraint, a very large field of view is needed in order to resolve the velocity corrugations that we find in the outer disc. The \( m = 1 \) bending waves for Model 1 (Fig. 4.10) dominate in the region \( R \sim 20 - 25 \) kpc, which corresponds to \( \sim 6 - 8 \) disc scale lengths, or \( \sim 4 - 5 \) effective radii (assuming an exponential disc and \( 1R_e \simeq 1.7R_d \)). Furthermore, the spatial resolution required to resolve corrugations similar to those presented in this work should be at least
\[ \sim 2 - 3 \text{ kpc}. \]

The presence of strong spiral arms may complicate the interpretation of any observed ‘vertical motions’. Since stars on the side of a face-on disc furthest from an observer are obscured by dust, compression and rarefaction of the disc perpendicular to the disc plane will appear as bulk vertical motions. Spiral arms can generate such breathing wave motions in the disc, and even arms with modest density contrasts can produce bulk motions comparable to what we observe in this work (Debattista 2014; Faure et al. 2014; Monari et al. 2016a,b). It could be that the corrugations observed by Sánchez-Gil et al. (2015) using long slit spectroscopy are associated with breathing mode perturbations from the spiral structure. However, the vertical motions will trace the spiral arms and follow a distinct compression and rarefaction pattern on either side of the arms. The bending waves we observe in our simulations are tightly wound and leading, and therefore should be distinguishable from bulk motions induced by (trailing) spiral arms.

### 4.7 Discussion

A satellite or dark matter subhalo that passes through a stellar disc will set up disturbances in the disc, which include vertical bending and breathing waves. The simulations presented in this paper demonstrate that bending waves also arise in an isolated MW-like stellar disc without provocation from an outside agent such as a passing satellite.

At intermediate radii \( R \sim 8 - 15 \text{ kpc} \) for a MW-like galaxy), these waves develop into tightly wound leading spirals with an amplitude and radial wavelength that is similar to those recently discovered in the MW (Xu et al. 2015). In contrast to the
hypothesis of Xu et al. (2015) that the corrugated structures are associated with spiral arms, we find that they are independent, both in morphology and pattern speed.

The bending waves in our simulations persist for many billions of years. Over time, bending wave power migrates outward. The warp begins at $R \sim 10 - 15$ kpc with an amplitude of $\sim 200$ pc that increases to $\sim 300 - 500$ pc closer to the disc’s edge. These amplitudes are in agreement with recent measurements of the stellar warp in the MW (see for example Reylé et al. 2009, and references therein). The outward migration of power is more dramatic for Model 1 where the bar and/or more massive disc plays a dominant role in suppressing bending waves, especially at intermediate radii. Since our Galaxy has a central bar, the implication is that bending wave corrugations are efficiently damped in the inner disc, and should only be detected at Galactocentric radii greater than that of the Sun.

What then is the origin of the bending waves in our simulated galaxies? A likely culprit is the random noise of the halo and bulge particle distributions. To test this hypothesis, we re-simulated Model 1 with a fixed analytic potential for the bulge and halo and found that the vertical perturbations were negligible even though the disc developed spiral structure and a bar. We note bar formation was delayed as compared with the live halo case (see Sellwood 2016). As a further check, we evolved the live disc in a fixed potential that was generated by halo and bulge particles frozen in their initial positions. In this case, the disc developed a strong corrugation pattern of bending waves at intermediate radii and a large warp near the edge of the disc. Evidently, large-scale vertical waves can be excited by particle noise in the halo but not by structures in the disc itself, such as a bar or spiral arms.
Once formed, the evolution of bending waves is remarkably well described by linear perturbation theory, especially during the early stages of the simulations. The linear eigenmodes of a razor thin, dynamically cold disc form a continuum of bending waves, rather than discrete modes, with rotational frequencies that lie along the vertical resonance curves at $\omega = \Omega \pm \nu_x$ (Hunter and Toomre 1969; Sparke and Casertano 1988). Of course, our $N$-body discs have a non-zero vertical velocity dispersion and thickness. Also, the subsequent evolution of our live models obviously differs from that in the linear regime, where dynamical friction from the halo tends to efficiently damp bending waves on very short time-scales (Nelson and Tremaine 1995). Furthermore, as the disc evolves, in-plane and vertical (time-dependent) inhomogeneities and asymmetries emerge, driven by the bar, spiral structure, mass redistribution within the disc, and resolution effects, which the linear theory and WKB analysis does not incorporate. Nevertheless, power in the $R - \omega$ plane concentrates along the two resonances and lies, for the most part, outside the region between them (i.e., the forbidden region as predicted by WKB theory).

Since both linear theory and the WKB analysis include self-gravity of the perturbations, we conclude that the phenomena observed in our discs are true bending waves. This contrasts with kinematical phase-wrapping explanations of the observed local bending and breathing modes in the SN as suggested by de la Vega et al. (2015). However, de la Vega et al. (2015) studied bending and breathing waves that arose from satellite interactions with the disc and close flybys of a fairly massive Sagittarius dwarf model. Admittedly, we do not consider direct interactions between the disc and external perturbing agents in this paper. One could imagine a two-stage scenario in which the waves present at the early stages of a satellite encounter behave at least
somewhat kinematically. Later on, when these perturbations have phase-mixed and sheared into extended arcs they can be described by a ring model, in which self-gravity is a key ingredient.

We also conclude that our bending waves are not strongly Landau-damped, as one might have anticipated from the analysis of vertical oscillations in one-dimensional systems (Mathur 1990; Weinberg 1991; Louis 1992; Widrow and Bonner 2015). Likewise, the waves are not strongly damped by dynamical friction due to the halo. If anything, a live halo (and bulge) appear to give rise to more vigorous perturbations perpendicular to the disc as predicted by Bertin and Mark (1980) and in discord with the analysis of Nelson and Tremaine (1995).

Our simulations are consistent with the picture presented in Binney et al. (1998). Recall that the authors of that paper simulated a disc-halo system where the disc was represented by rigid concentric rings while the halo constituted standard $N$-body particles. They showed that if the disc was initialized in the configuration of a discrete warp mode (e.g., the modified-tilt mode of Sparke and Casertano 1988) the warp rapidly winds up while retaining its warp energy. They conclude that the true modes of the disc-halo system are different from those of the disc-static halo system, essentially because the halo responds to the warp and thus changes the potential in which the disc evolves. The spectral analysis of our simulations suggests that while the modes of the disc-halo system might differ quantitatively from those of the disk-static halo system, the general structure of the modes, at least those of the continuum, are qualitatively similar. The key point is that there exists a low-frequency branch where the differential precession is small at larger radii. Thus, the winding problem is alleviated and bending waves persist for many dynamical times.
One might expect, then, that waves at smaller radii would be sheared out via differential rotation. The survival of these waves in our simulations, especially those along the low-frequency branch, is likely attributed to the strong self-gravity of the inner disc. Moreover, random in-plane motions act to further stiffen the inner disc and resist bending (Debattista and Sellwood 1999). A straight LON and constant pattern speed in the inner disc indicates that the waves rotate coherently and cohesively, connecting the global structure of low-frequency waves between the inner and outer disc. The waves in the inner disc of our simulations differ from that predicted by the linear ring model since neighbouring rings cannot be considered to oscillate independently and epicyclic motions are not taken into account. This point implies at least some degree of non-linearity present in our simulations regarding the bending waves – the waves we see may very well be akin to the true modes of the disc hinted at by Binney et al. (1998).

4.8 Conclusions

Our main conclusion is that bending waves should be generic, long-lived features of MW-like disc galaxies irrespective of whether the disc has been perturbed by a passing satellite or dark matter subhalo. This conclusion is based on the results from $N$-body simulations of isolated MW-like galaxies. In particular, we focus on two models: one with a maximal bar-forming disc and the other with a submaximal disc which only forms flocculent spiral structure. In both models, bending waves develop across the entire disc within the first billion years of the simulation. At intermediate radii the waves manifest as leading, tightly wound corrugations, which extend over a large range in azimuth and match smoothly on to the warp that develops at the edge
A major goal of this paper is to develop tools for the study of bending waves in simulations and observations. Our approach is based on the spectral analysis of surface density maps for simulated disc galaxies – a tool that has proved indispensable for understanding the dynamics of bars and spiral structure. In principle, the complete dynamical state of a stellar disc is encapsulated in the DF, a complicated function of the six phase space coordinates and time. To simplify the analysis, one can derive a surface density map, essentially by integrating the DF over $v$ and $z$. The standard procedure is to then divide the disc into radial bins and write the surface density in each bin as a series of functions with $m$-fold azimuthal symmetry. The coefficients provide the relative strength of the different angular modes as a function of $R$ and $t$. By Fourier transforming in time, we can also obtain a power spectrum in $R$ and $\omega$.

For this work, we begin with the $z$ and $v_z$ moments of the DF across the disc, which provide a measure of bending waves in the disc. As with surface density maps, we can decompose maps of $\langle z \rangle$ and $\langle v_z \rangle$ as Fourier series in $\phi$. We focus on the $m = 1$ waves, which is the dominant term not only for the warp but also for the bending waves that occur at intermediate radii. Furthermore, the $m = 1$ waves provide a direct link to tilted ring models for warped galaxies and the eigenfunction calculations of Hunter and Toomre (1969) and Sparke and Casertano (1988). From Fig. 4.5, we conclude that at a given $R$, the amplitude of the $m = 1$ bending waves grows with time until reaching some maximal value, after which it remains roughly constant. The maximum amplitude is an increasing function in $R$ as is the time at which it is reached. The upshot is that over time, the strength of the bending waves near the edge of the disc increases relative to that of waves in the inner part of the
The long-lived nature of bending waves is consistent with the conclusions of Binney et al. (1998), who showed that a live halo responds to vertical displacements on a relatively short time-scale. In essence, bending waves are a phenomena of the disc-halo system. In the absence of this coupling between the disc and live halo, dynamical friction would efficiently damp the bending waves, as was suggested by Nelson and Tremaine (1995).

The connection between linear theory, in the form of either eigenfunction or WKB analyses, and the simulation results is most clearly seen in Figs. 4.6 and 4.8. The agreement is impressive, as power in the $m = 1$ waves tends to reside just outside the WKB-forbidden region between the two vertical resonance curves. At early times, the power is distributed approximately uniformly throughout the disc while at late times power is concentrated near the outer edge of the disc, lending credence to an inside-out formation scenario of the warp. We emphasize that central to the linear theory calculations is a gravitational restoring force that acts on perturbations in the disc. Hence, the agreement between simulations and linear theory bolsters our claim that we are observing (at least in the simulations) bona fide gravity waves.

It was a surprise (at least to us) that bending waves arose in a disc embedded in a smooth dark matter halo. Evidently, particle noise in the halo is enough to provoke vertical waves in a stellar disc. Of course, dark matter haloes in ΛCDM cosmologies are predicted to host a system of satellite galaxies and dark matter subhaloes (for example, see the early works of Klypin et al. 1999 and Moore et al. 1999). In a realistic galaxy, a small fraction of these halo objects will pass through the disc, an interaction that is characterized by a series of phases. In the first, a local region of the disc is
perturbed by the passing object. The response of individual stars will depend on how well their vertical epicyclic motions match the time-dependence of the gravitational field (Sellwood et al. 1998; Widrow et al. 2014). The localized perturbation is then sheared out due to differential rotation and phase mixing (Widrow et al. 2014; de la Vega et al. 2015) but soon begins to behave as a tightly wound bending wave with gravity providing the restoring force. In essence, this is the phase investigated here. Eventually, the wave energy is dissipated through further phase mixing and Landau damping. Consequently, the disc then becomes a little thicker and kinematically hotter. Using the shot noise in our simulations as a proxy for asymmetries in haloes, it is tangible to imagine bending waves in MW-like (stellar) discs to be continually excited.

One of the goals of the Gaia observing mission (Perryman et al. 2001; Gaia Collaboration et al. 2016b, but see Gaia Collaboration et al. 2016a for an overview of DR1) is to provide a map of the mean velocity field over a substantial fraction of the Galactic disc. The picture presented here suggests that this map will reveal a mix of localized flows and larger scale motions that connect waves at intermediate radii with the outer warp (see, for example, Abedi et al. 2014 regarding Gaia and the Galactic warp). In principle, the former will tell us something about perturbers roaming the halo while the latter may say more about the structure of the disc. We predict that less dispersive low-frequency \( m = 1 \) bending waves, which manifest as tightly wound and leading radially dependent vertical asymmetries in position and velocity, should be prevalent at radii just beyond the Sun. Although similar features have already been observed in the MW’s density distribution (Xu et al. 2015), the key will be testing the true wave nature of these patterns using proper motion and
radial velocity measurements from the next Gaia data release. In theory, the analysis tools based on moments in $z$ and $v_z$, and azimuthal Fourier modes, can be applied to Gaia data, though there, one must contend with selection effects, observational uncertainties, and obscuration due to dust.

One possible complication, not discussed here, is that different components of the disc may respond differently to a passing satellite. Bovy and Rix (2013) have suggested that to do an Oort type analysis (that is, to infer the surface density, vertical force, and density of dark matter in the disc) one should divide disc stars into bins based on [$\alpha$/Fe] and [Fe/H] – the idea being that the stars in each bin can be treated as a distinct isothermal tracer of the gravitational potential. On the other hand, if the disc is in a continual state of disequilibrium, and if the manifestation of disequilibrium varies from one population to the next, then population-dependent systematic errors might creep into this sort of analysis (Banik et al. 2017). Put another way, discrepancies in the inferred potential might signal that the Galaxy is in a perturbed state.

In summary, Gaia may well find complicated patterns in the bulk motions of disc stars. The challenge will then be to characterize these motions, perhaps with the spectral methods described here, and disentangle the effects of external perturbers and internal dynamics.

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Chapter 5

Vertical Waves in
Single-Component Milky Way-like Discs Induced by Single Satellite Encounters

This chapter borrows heavily from a paper titled ‘Bending and breathing modes of the Galactic disc’ published in Monthly Notices of the Royal Astronomical Society as: Lawrence M. Widrow, Jarrett Barber, Matthew H. Chequers and Edward Cheng, Monthly Notices of the Royal Astronomical Society, 440:1971-1981, 2014. In particular, this chapter highlights only the contributions Matthew H. Chequers made to that paper. We further note that keeping with the spirit of a ‘manuscript style’ thesis, the following is written as if it were a standalone manuscript at the time.
of publishing. Thus, no attempt is made to add content in hindsight of more recent published work, except for the convention of referring to the perturbations as ‘waves’, rather than ‘modes’ as was done in the 2014 paper.
5.1 Abstract

We explore the hypothesis that a passing satellite galaxy or dark matter subhalo has excited coherent oscillations of the Milky Way’s stellar disc in the direction perpendicular to the Galactic midplane. This work is motivated by recent observations of spatially dependent bulk vertical motions within a radius of 2 kpc around the Sun. A satellite can transfer a fraction of its orbital energy to the disc stars as it plunges through the Galactic midplane, thereby heating and thickening the disc. Bulk motions arise during the early stages of such an event when the disc is still in an unrelaxed state. Using a suite of three-dimensional $N$-body simulations modelling disc-satellite interactions, we show that the response of the disc depends on the satellite orbital parameters. Most interestingly, we find that when the component of the satellite’s velocity perpendicular to the disc is small compared with that of the disc particles, bending wave perturbations are predominant. Conversely, breathing waves are excited to a greater degree when the vertical velocity of the satellite is larger than that of the disc particles. We argue that the compression and rarefaction motions seen in three different surveys are in fact breathing wave perturbations of the Galactic disc.

5.2 Introduction

Recently, bulk motions in the direction perpendicular to the Milky Way’s (MW’s) midplane within the Solar Neighbourhood (SN; within a heliocentric distance of $\sim 2$ kpc) have been uncovered in the kinematic data of three separate astronomical surveys (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013). Widrow et al. (2012) analysed a sample from the Sloan Extension for Galactic Understanding and
Exploration (SEGUE) survey (Yanny et al. 2009), and found that the relation between the bulk motions and distance from the midplane is indicative of a breathing wave perturbation with an amplitude of $\sim 3.5 \text{ km s}^{-1} \text{kpc}^{-1}$. Williams et al. (2013) explored the kinematics of stars from the Radial Velocity Experiment (RAVE) survey (Steinmetz et al. 2006) and found peak bulk vertical motions of $\pm 15 \text{ km s}^{-1}$. Moreover, when projected onto a plane of galacocentric radius and height, the vertical motions indicate an expansion of the disc inside the solar radius and a compression outside. Carlin et al. (2013) came to similar conclusions in their analysis of stars with spectroscopic radial velocity information from the Large Sky Area Multi-Object Fibre Spectroscopic Telescope (LAMOST) survey (Cui et al. 2012; Zhao et al. 2012) and proper motion measurements from the PPMXL catalogue (Roeser et al. 2010). It is worth noting that these three surveys agree only qualitatively since they examine different spatial footprints of the SN.

In addition to observing bulk vertical motions, Widrow et al. (2012) detected an asymmetry in the number counts of stars above and below the Galactic midplane. More specifically, Widrow et al. (2012) showed there is an $\sim 10$ per cent (north - south)/(north + south) underdensity at $|z| \simeq 400 \text{ pc}$ and overdensity at $|z| \simeq 800 \text{ pc}$. Widrow et al. (2012) postulated that the number count asymmetry coupled with bulk motions implies a coherent density oscillation of the MW’s disc in the vertical direction that has been excited by a recent passage of an orbiting satellite galaxy or dark matter subhalo.

As a massive object, be it a satellite galaxy or dark matter subhalo, passes through a stellar disc it will transfer a portion of its orbital energy to the stars in the disc (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998) and excite
modes of the disc such as bending and breathing modes (Araki 1985; Mathur 1990; Weinberg 1991). Stellar discs are stable in the vertical direction, thus these modes oscillate as waves and act to thicken the disc as their energy dissipates due to Landau Damping and phase mixing (Sellwood et al. 1998), and leave an imprint in the phase space distribution of disc stars.

Interactions between a stellar disc and an orbiting satellite can also excite non-axisymmetric structure, such as bars and spiral structure (see Sellwood 2013 for a review of the dynamics of stellar discs). For example, consider the simulations of Purcell et al. (2011) where a MW-like disc was perturbed by several passages of a satellite galaxy. The incident satellite was modelled after the Sagittarius dwarf spheroidal galaxy (Ibata et al. 1994, 1997), which is believed to have survived several orbits about the Galaxy. The main result of Purcell et al. (2011) was that the spiral structure in their model disc, excited by the perturbing satellite, bore an uncanny resemblance to what we observe in the Galaxy, and they were able to reproduce other features in the Galactic disc such as the Monoceros ring (Newberg et al. 2002; Yanny et al. 2003; Li et al. 2012). Interestingly, Gómez et al. (2013) reanalysed this simulation in the context of vertical density waves and were able to reproduce number density profiles of disc particles within SN-like volumes ($\sim 8$ kpc from the Galactic centre) that were qualitatively similar to the wave-like perturbations observed in Widrow et al. (2012) and Yanny and Gardner (2013).

Recent cosmological simulations show that within the framework of a Lambda cold dark matter Universe the host haloes of MW-like galaxies maintain a plethora of dark matter subhaloes that orbit within the host (Klypin et al. 1999; Moore et al. 1999; Gao et al. 2004). Therefore, the stellar discs residing at the center of host haloes
should be prone to frequent bombardment from the subhalo system and density modes will be continually excited. The excitation of in-plane density modes was explored by Gauthier et al. (2006) with their \( N \)-body simulations of an M31-like galaxy. They found that if the host halo was smooth their disc remained stable against the formation of a bar for at least 10 Gyr and exhibited only flocculent spiral structure. When \( \sim 10 \) per cent of the halo mass was initiated in the form of orbiting subhaloes, the disc formed prominent spiral structure and a strong bar. Thus, the subhalo system plays a significant role in driving the secular evolution of the disc (Dubinski et al. 2008). Similar results have also been echoed in other studies (for example, see Kazantzidis et al. 2008). The effect that a system of subhaloes has on the vertical structure of the disc has yet to be investigated.

In this work we explore the hypothesis of Widrow et al. (2012) that the bulk vertical motions observed in the SN were induced by a recent disc-satellite interaction. Using \( N \)-body simulations, we model the interaction between a MW-like disc and relatively low-mass satellite galaxies. In particular, we characterize the generation of bending and breathing waves as a function of satellite orbital parameters, namely the vertical velocity of the orbiting object as it crosses the disc midplane. We also reanalyse snapshots of the Gauthier et al. (2006) simulation in the context of vertical waves.

The layout of this manuscript is as follows. Section 5.3 gives an overview of our simulations, the models we use, and the perturbations that arise in such scenarios. In Section 5.4 we present our analysis of snapshots from the Gauthier et al. (2006) simulation. Finally, we summarize and conclude in Section 5.5.
5.3 Single Satellite $N$-body Simulations

We employ a suite of $N$-body simulations where a fiducial parent galaxy is bombarded with a single satellite to study the effects such encounters have on exciting vertical perturbations. We consider ‘light’ and ‘heavy’ models for our satellite on orbits that are prograde and retrograde compared to that of the disc, as well as ‘polar’ orbits (i.e., when the satellite’s orbital angular momentum vector is perpendicular to the spin angular momentum of the disc). For each of these orbits and satellite models we further consider the vertical velocity of the satellite as it passes through the disc midplane being lesser and greater than that of the vertical velocity dispersion of the disc particles.

5.3.1 Generating Initial Conditions and Running the Simulations

The initial conditions of the parent galaxy and the satellites were constructed using GALACTICS (Kuijken and Dubinski 1995; Widrow et al. 2008). For the parent galaxy we chose the most stable of the Galactic models presented in Widrow et al. (2008), comprising a disc, bulge, and halo. The distribution function (DF) used to characterize the disc yields a space density given by

$$\rho_d(R, z) = \frac{M_d}{4\pi R_d^2 z_d} e^{-R/R_d} \text{sech}^2(z/z_d),$$  \hspace{1cm} (5.1)

where $M_d$ is the disc mass, $R_d$ is the disc scale length, and $z_d$ is the scale height. The
radial velocity dispersion of the disc is given by

\[ \sigma_R(R) = \sigma_0 e^{-R/2R_s}, \quad (5.2) \]

where \( \sigma_0 \) is the central dispersion and \( R_s \) is the scale length of the squared dispersion.

The vertical dispersion is determined from the assumption of constant scale height, and the azimuthal velocity dispersion is calculated from the epicycle approximation (see Kuijken and Dubinski 1995 and Section 3.2.3 of Binney and Tremaine 2008).

The DFs for the halo and bulge are isotropic in velocity space. The halo DF produces a density profile that is Navarro-Frenk-White (NFW; Navarro et al. 1996), while the bulge DF produces a density profile approximately given by

\[ \rho_b(r) = \frac{v_b^2}{4\pi G R_e^2 c(n) \left( \frac{r}{R_e} \right)^{-p} e^{-b(r/R_e)^{1/n}}}, \quad (5.3) \]

where the constant \( b \) is set such that \( R_e \) encloses half the total projected mass and \( c(n) = (nb^n(p-2)) \Gamma(n(2-p)) \) (Prugniel and Simien 1997; Terzić and Graham 2005).

The bulge DF is constructed such that \( p = 1 - 0.6097/n + 0.05563/n^2 \), and so equation (5.3) yields a projected surface density profile corresponding to the Sérsic law with index \( n \). The galactics parameters for our parent galaxy model are presented in Table 5.1.

Our satellites are modelled with a NFW density profile. Both of our satellite models have a scale radius of 0.22 kpc and are smoothly truncated at a radius of \( \sim 0.9 \) kpc over a distance of \( \sim 0.5 \) kpc. The masses of our light and heavy models are \( 8 \times 10^8 M_\odot \) and \( 4 \times 10^9 M_\odot \), respectively.

We sample the DFs of each galaxy component to generate N-body realizations.
Table 5.1: Initial model parameters for our parent galaxy.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_d$ [$10^{10} M_\odot$]</td>
<td>4.1</td>
</tr>
<tr>
<td>$R_d$ [kpc]</td>
<td>2.8</td>
</tr>
<tr>
<td>$h$ [kpc]</td>
<td>0.5</td>
</tr>
<tr>
<td>$R_t$ [kpc]</td>
<td>25</td>
</tr>
<tr>
<td>$\delta R_t$ [kpc]</td>
<td>3</td>
</tr>
<tr>
<td>$\sigma_R_{0}$ [km s$^{-1}$]</td>
<td>57.5</td>
</tr>
<tr>
<td>$M_h$ [$10^{12} M_\odot$]</td>
<td>1.3</td>
</tr>
<tr>
<td>$v_h$ [km s$^{-1}$]</td>
<td>481</td>
</tr>
<tr>
<td>$a_h$ [kpc]</td>
<td>43</td>
</tr>
<tr>
<td>$M_b$ [$10^{10} M_\odot$]</td>
<td>1</td>
</tr>
<tr>
<td>$v_b$ [km s$^{-1}$]</td>
<td>272</td>
</tr>
<tr>
<td>$R_e$ [kpc]</td>
<td>0.64</td>
</tr>
<tr>
<td>$n$</td>
<td>1.32</td>
</tr>
<tr>
<td>$Q(2.2 R_d)$</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Disc mass$^a$; radial disc scale length$^b$; vertical disc scale height$^c$; truncation radius of the disc$^d$; truncation width of the disc$^e$; central radial velocity dispersion in the disc$^f$; halo mass$^g$; characteristic velocity scale of the halo$^h$; NFW halo scale length$^i$; bulge mass$^j$; characteristic velocity scale of the bulge$^k$; bulge effective radius$^l$; Sérsic index$^m$; Toomre $Q$ parameter at $2.2 R_d$.$^n$

of our models. The disc and halo were sampled with 500$k$ and 300$k$ particles, respectively, while the bulge and satellites were sampled with 50$k$ particles. We used 

**GADGET-2 (Springel 2005)** to run our suite of simulations.

Throughout the evolution of our simulations the central disc drifts and tilts slightly due to the satellite collisions. To account for this when analyzing face-on maps of our discs we center the disc and rotate it into the $x$-$y$ plane by way of the following iterative scheme. We compute the center of mass of disc particles within a cylinder of radius 20 kpc and height 2 kpc centered in the $x$-$y$ plane and translate all disc particles to that frame. This procedure is repeated until a translation of less than 1 pc is achieved in all coordinate directions. Next, we account for any tilt of the
disc relative to the $x$-$y$ plane by using a two-dimensional Newton-Raphson scheme to find the Euler angles about the $x$- and $y$- axes that minimize the root mean square vertical displacement of disc particles within a cylinder of radius 20 kpc and height 2 kpc centered in the $x$-$y$ plane. This method does a good job of diagonalizing the moment of inertia tensor of the disc.

5.3.2 Single Satellite Perturbations

In all of our simulations the passage of a satellite through the disc induces both bending and breathing waves in the disc. In particular, we find that perturbations are more robust and longer-lived for the case of massive satellites on prograde orbits (i.e. spin-orbit resonance, see Toomre and Toomre 1972). Furthermore, while both bending and breathing waves are excited by a given interaction, bending seems to be stronger for the case of satellites with low vertical velocities relative to the disc stars, while breathing waves are excited to a greater degree when a satellite passes through the disc midplane more quickly\(^1\).

We show an example of perturbations that arise from a satellite collision in Fig. 5.1. For illustrative purposes, we only show the interaction for the case our heavy satellite on a prograde orbit. The satellite passes through the midplane of the disc at a Galactocentric radius of 8 kpc with a vertical speed of $\sim 60 \text{ km s}^{-1}$ (i.e. greater than that of the disc particles). The strong wake generated by the satellite is clearly visible in the upper left panel of Fig. 5.1. Over time, the disc develops prominent flocculent spiral structure.

\(^1\)Note that the Widrow et al. (2014) paper, which this chapter is based on, did not explicitly present the suite of simulations, but rather focused on the creation of bending and breathing waves relative to satellite vertical velocity. An in-depth study of the dependencies of vertical oscillations on satellite orbital parameters was presented the following year by Feldmann and Spolyar (2015).
Figure 5.1: Bending and breathing perturbations induced by a $4 \times 10^9 M_\odot$ satellite. The left-hand column shows the face-on disc just as the satellite passes through the midplane while the right-hand column shows the disc at $t = 250$ Myr. The top panels show the density map (logarithmic grey-scale shading). The middle panels show the bending strength parameter $B$ as defined in equation (5.4) in units of km s$^{-1}$. The bottom set of panels shows the breathing strength parameter $A$ in units of km s$^{-1}$ kpc$^{-1}$. This figure is a reproduction of figure 10 in Widrow et al. (2014).
To quantify the strength of bending and breathing waves we model the bulk vertical velocity across the disc plane as

\[ \bar{v}_z = (z|x, y) = A(x, y)z + B(x, y) \]  

That is, for disc particles within a two-dimensional cylinder centred on the point \((x, y)\) we fit \(\bar{v}_z\) to a linear function in \(z\). The coefficient \(B\) is a measure of bending wave strength and is shown in the two middle panels of Fig. 5.1, while \(A\) is a measure of breathing wave strength and is shown in the bottom two panels. From these plots we clearly see that both types of waves are excited as the satellite passes through the disc and are localized to the vicinity of the impact site. Over time these perturbations are sheared out into more extended arcs due to the differential rotation of the disc and continue to exist for many hundreds of Myrs.

### 5.4 System of Satellites

Gauthier et al. (2006) followed the evolution of a disc-bulge-halo galaxy in which the halo comprised a smooth NFW (Navarro et al. 1996) component as well as a cosmologically motivated system of 100 orbiting subhaloes. The masses of the subhalo population ranged from \(8.7 \times 10^7 M_\odot\) to \(1.2 \times 10^{10} M_\odot\) and followed a mass function given by \(dN/dM_s \propto M_s^{-1.9}\) (Gao et al. 2004). For the parent galaxy, they used the self-consistent equilibrium model of M31 from Widrow and Dubinski (2005), labelled M31a. This model provides a good match to the observed rotation curve, surface brightness profile, and velocity dispersion profile and is stable against the formation of a bar for 10 Gyr so long as the halo mass is assumed to be smoothly distributed. The disc has a mass of \(7.8 \times 10^{10} M_\odot\) and an exponential scale length of 5.6 kpc. The
circular speed curve reaches a peak value of $260\,\text{km}\,\text{s}^{-1}$ at a radius of $\sim 10\,\text{kpc}$. The complete list of model parameters can be found in table 2 of Widrow and Dubinski (2005) or table 1 of Gauthier et al. (2006). In principle, the model could be rescaled to make better contact with MW observations. Nevertheless, the qualitative features of the simulations should be applicable to the Galaxy.

One of the most striking results from the Gauthier et al. (2006) simulation is the formation of a strong bar at about 5 Gyr (see also Kazantzidis et al. 2008). In Fig 5.2, we show the surface density and vertical wave strength at 2.5 and 10 Gyr. The length of the bar is $\sim 20\,\text{kpc}$ as can be seen in the upper-right panel of Fig 5.2 and figure 5 of Gauthier et al. (2006). Prominent spiral structure also develops early on and appears to be a precursor to the formation of the bar. Though there is some disc heating during the first 4 Gyr, the most significant heating and thickening occurs after the bar forms (see Dubinski et al. 2008 for a further discussion). By contrast, no bar and only weak spiral structure develops in the control experiment, which assumes a smooth halo. Evidently, satellites and subhaloes provoke spiral structure and bar formation (see also Kazantzidis et al. 2008 and Purcell et al. 2011).

It is not at all surprising that a system of subhaloes also excites vertical oscillations in the stellar disc. Bending and breathing wave perturbations are found across the disc and throughout the simulation, as can be seen in the middle and lower panels of Fig 5.2. Prior to bar formation, there are strong large-scale bending waves across the disc with amplitudes of the order of $\sim 5\,\text{km}\,\text{s}^{-1}$. The breathing waves have a somewhat smaller amplitude ($\sim 1 - 2\,\text{km}\,\text{s}^{-1}\,\text{kpc}^{-1}$) and vary on smaller scales. At later times, after the bar has formed and the bulk of subhalo bombardment has occurred, the bending wave perturbations have diminished. Moreover, in the inner
Figure 5.2: Face-on maps of the surface density and the bending and breathing wave strengths for the Gauthier et al. (2006) simulation. Left-hand panels show the disc at 2.5 Gyr, which is prior to the formation of the bar, while the right-hand panels show the galaxy at 10 Gyr, well after the bar has formed. The top panels show logarithmic grey-scale maps of surface density. The middle panels show the bending strength in units of km s$^{-1}$, while the bottom panels show the breathing wave strength in units of km s$^{-1}$ kpc$^{-1}$. This figure is a reproduction of figure 12 in Widrow et al. (2014).
parts of the Galaxy, the breathing wave pattern mirrors that of the bar. Thus, while subhaloes may have triggered the formation of the bar, it is the bar that generates and maintains compression and rarefaction motions in the inner Galaxy. Of course, the Sun sits well beyond the region of the bar and it is therefore unlikely that the bar could cause the bulk motions observed in the solar neighbourhood (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013).

5.5 Summary and Conclusions

The implications of spatially dependent bulk motions perpendicular to the Galactic disc were highlighted by Oort (1932) in his seminal work on the structure of the Galactic disc. Oort’s aim was to determine the vertical potential from the local stellar density and velocity distribution. He based his analysis on the assumption that the local distribution of stars is in equilibrium. To test the assumption, he computed the mean vertical velocity for stars in four separate bins, $100 < \pm z < 200$ and $200 < \pm z < 500$ pc, but did not find evidence for systematic motions, a result that he notes ‘lends some support to the assumption . . . that in the $z$-direction the stars are thoroughly mixed’ (Oort 1932). Turning Oort’s argument around, the detection of bulk vertical motions in the SEGUE, RAVE, and LAMOST surveys (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013) suggests that the local Galactic disc is not in equilibrium in the $z$-direction.

In this paper, we considered the hypothesis that the observed bulk vertical motions were generated by a passing satellite or dark matter subhalo. The idea that dark matter, in one form or another, might be responsible for heating and thickening the disc dates back to the 1980s. Lacey and Ostriker (1985) calculated disc heating by a
dark halo of supermassive black holes while Carr and Lacey (1987) considered dark matter in the form of $10^6 M_\odot$ dark clusters. In essence, our hypothesis is that the bulk motions seen in the data represent the early stages of a disc heating event brought on by the transfer of orbital energy from a passing satellite to the disc (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998).

Our focus has been to explore the theoretical aspects of disc-satellite interactions. Our main conclusion is that a satellite collision excites both bending and breathing wave perturbations in the disc. Moreover, we found that the nature of the perturbations is controlled largely by the satellite’s vertical velocity relative to the disc. In particular, a slow moving (as measured in the local standard of rest) satellite induces a bending wave perturbation. With a higher vertical velocity, higher order waves, such as the breathing wave, are excited. Thus, if a satellite is indeed responsible for the bulk vertical motions seen in the SN, then its vertical velocity through the disc would likely have been $> 50 \text{ km s}^{-1}$. We also found that the amplitude of vertical perturbations scales with satellite mass. Since the response of the disc is governed by the local restoring force, and hence the local surface density, the surface density of a passing satellite would have to be at least comparable to that of the disc in order to produce an appreciable perturbation. The model satellites considered by Gómez et al. (2013) satisfy these conditions (vertical velocity and surface density) and so it is not surprising that they found vertical perturbations in the disc that were qualitatively similar to what was found in the data.

Our simulations show that after a localized breathing wave perturbation is produced, it is sheared by the differential rotation of the disc. After several orbital periods of the disc (many hundreds of Myr), the perturbation still exist and assumes
a spiral-like pattern. The situation is more complicated for the case of a population of satellites (c.f. Fig. 5.2) and it may be difficult to disentangle initial perturbations from the accumulated long-lived perturbations.

There are 25 known satellites of the MW. Moreover, in a Lambda cold dark matter cosmology, the dark halo of a MW-sized galaxy is expected to harbour many more non-luminous subhaloes (Klypin et al. 1999; Moore et al. 1999). Thus, it is likely that the Galactic disc has been continually perturbed over its lifetime. In principle, observations of bulk motions in the stellar disc could provide a probe of the subhalo distribution. To do so will require a suite of simulations where the slope and amplitude of the subhalo mass function are varied.

Over the next few years, Gaia will provide an unprecedented snapshot of the Galaxy by making astrometric, spectral, and photometric observations of approximately one billion MW stars (see Perryman et al. 2001 and de Bruijne 2012). This data set will yield a more accurate and complete map of bulk motions in the stellar disc. By bringing together these observations, theoretical analysis, and N-body simulations, we hope to better understand Galactic dynamics, and, in particular, interactions between the MW’s disc and its system of satellites and dark matter subhaloes.

5.6 Acknowledgements

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Chapter 6

Bending Waves in Two-Component Milky Way-like Discs from Halo Substructure

6.1 Abstract

We use $N$-body simulations to investigate the excitation of bending waves in a Milky Way-like disc-bulge-halo system. The dark matter halo consists of a smooth component and a population of subhaloes while the disc is composed of thin and thick components. Also considered is a control simulation where all of the halo mass is smoothly distributed. We find that bending waves are more vigorously excited in the thin disc than the thick one and that they are strongest in the outer regions of the disc, especially at late times. By way of a Fourier decomposition, we find that the complicated pattern of bending across the disc can be described as a superposition of waves, which concentrate along two branches in the radius-rotational frequency plane. These branches correspond to vertical resonance curves as predicted by a WKB analysis. Bending waves in the simulation with substructure have a higher amplitude than those in the smooth-halo simulation, though the frequency-radius characteristics of the waves in the two simulations are very similar. A cross correlation analysis of vertical displacement and bulk vertical velocity suggests that the waves oscillate largely as simple plane waves. We suggest that the wave-like features in astrometric surveys such as the Second Data Release from Gaia may be due to long-lived waves of a dynamically active disc rather than, or in addition to, perturbations from a recent satellite-disc encounter.

6.2 Introduction

The Milky Way’s (MW’s) disc bends in and out of its midplane. The most conspicuous example of this is the large scale warp, which has been observed in H\textsc{i} (Levine et al.
2006), dust (Freudenreich et al. 1994), and stars (Djorgovski and Sosin 1989). It starts within the Solar Circle and increases in amplitude toward the edge of the disc (Drimmel and Spergel 2001; López-Corredoira et al. 2002b; Momany et al. 2006; Reylé et al. 2009; Schönrich and Dehnen 2017). There is also evidence for short-wavelength ripples or corrugations in the disc (Xu et al. 2015; Schönrich and Dehnen 2017) as well as a mix of localized bending and breathing motions in the Solar Neighbourhood (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013, but also see Sun et al. 2015, Ferguson et al. 2017, Pearl et al. 2017, Carrillo et al. 2018, and Wang et al. 2018 for more recent developments). Further evidence for perturbations perpendicular to the disc midplane comes from North-South asymmetries in stellar number counts (Widrow et al. 2012; Yanny and Gardner 2013; Ferguson et al. 2017). Finally, wave-like features have been seen in the Gaia Data Release 2 (GDR2) (Gaia Collaboration et al. 2018b; Antoja et al. 2018; Poggio et al. 2018). In particular, Antoja et al. (2018) found spiral-like structures in the \{z, v_R, v_\phi, v_z\} phase space distribution of stars within a circular arc at a Galactocentric radius just beyond the Solar Circle.

In this paper, we focus on the MW’s stellar disc and adopt the view that the aforementioned phenomena can be understood as a superposition of bending waves. On the observational front, the \textit{Gaia} space observatory (Perryman et al. 2001; Gaia Collaboration et al. 2016b, but see Gaia Collaboration et al. 2018a for an overview of GDR2) will allow us to determine the bulk vertical velocity, \(V_z\), and midplane displacement, \(Z\), as a function of position in the disc plane across a substantial fraction of the disc (see Gaia Collaboration et al. 2018b). In principle, one might then try to decompose \(Z\) and \(V_z\) into waves that are defined by their azimuthal wave number, radial wavelength, and pattern speed. As with spiral structure, one
might further attempt to determine whether bending waves are leading or trailing and whether they are prograde or retrograde with respect to the Galactic rotation.

Our aim in this paper is to study the physics of bending waves through $N$-body simulations of an isolated disc-bulge-halo system. Our work differs from previous simulation-based studies in several important respects. First, we follow the evolution of a model galaxy when a fraction of the halo mass is replaced by a system of subhaloes. For the most part, earlier simulations followed a single satellite as it plunged through the disc with the goal of connecting observations with specific disc-subhalo (or disc-satellite galaxy) events (Gómez et al. 2013; Widrow et al. 2014; Feldmann and Spolyar 2015; de la Vega et al. 2015; D’Onghia et al. 2016; Gómez et al. 2016, 2017; Laporte et al. 2018). Indeed, some of the observed phenomena have been attributed to the Sagittarius dwarf galaxy and/or the Large Magellanic cloud (Gómez et al. 2013; Laporte et al. 2017).

We take the somewhat different viewpoint that the disc is continually perturbed by subhaloes and satellite galaxies. Local perturbations are sheared into circular arcs that then oscillate as self-gravitating waves. The key idea is that the vertical perturbations observed today may come more from long-lived waves than very recent disc-substructure interactions.

de la Vega et al. (2015) have suggested that the evolution of perturbations in the disc can be understood as purely kinematical. To make their case, they follow test-particle representations of disc perturbations in a time-independent unperturbed potential. Their perturbations therefore phase-mix but do not self-gravitate. The resulting structures are found to be qualitatively similar to the bending and breathing patterns seen in the Solar Neighbourhood. The viewpoint that perturbations
evolve kinematically forms the basis of the analysis by Antoja et al. (2018) who use phase-mixing arguments to ‘date’ the event that produced the phase space features they found in GDR2. However, the inconsistent timescales between various phase-wrapping models (for example, Minchev et al. 2009, Antoja et al. 2018, and Monari et al. 2018) suggests the situation is more complex in that other dynamical processes, such as the bar or spiral structure (Kawata et al. 2018; Quillen et al. 2018), may also play a significant role in driving the evolution of features seen in the phase space distribution of stars.

Our view is that immediately after the disc encounters a satellite the resulting vertical perturbation may well evolve kinematically, but that once sheared into an extended arc, it will behave as a self-gravitating wave. This picture is supported by the good agreement between the results from linear perturbation theory, which includes self-gravity (Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988; Nelson and Tremaine 1995), and simulations (Chequers and Widrow 2017). Eventually, the energy imparted to the disc by a passing subhalo or satellite is converted into random motions of the disc stars thereby vertically heating the disc (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998). Thus, the life-cycle of a perturbation involves excitation, phase-wrapping or shearing across the disc, self-gravitating wave-like action, and disc heating.

The challenge then is to see if one can disentangle these different processes. Adding to the complexity of the problem is the realization that vertical bending waves are excited even in simulations where, apart from the shot noise of the $N$-body distribution, the halo is smooth and axisymmetric (Chequers and Widrow 2017). In what follows, we make direct comparisons between a control simulation, where the halo is
smooth, and one run with a clumpy halo. Our set-up closely follows that of Gauthier et al. (2006) who considered an equilibrium disc-bulge-halo model for M31, which, when initialized with a smooth halo, was stable against bar formation for at least 10 Gyr. However, when $\sim 10$ per cent of the halo mass was replaced by orbiting substructure, the disc developed more vigorous spiral structure and a strong bar (see also Dubinski et al. 2008 and Kazantzidis et al. 2008). Not surprisingly, halo substructure also excites vertical oscillations of the disc. We find that the amplitude of bending waves, including the warp, increases by a factor $\sim 5$ relative to the waves that arise in the smooth halo run.

Vertical perturbations can also be excited by internal mechanisms, such as a bar (Monari et al. 2015) and spiral structure (Debattista 2014; Faure et al. 2014; Monari et al. 2016a,b). Indeed, one expects that any time-dependent perturbing potential that sweeps through the disc will cause it to contract and expand in the direction perpendicular to the midplane. Bending waves require a perturbation that breaks symmetry about the midplane, as would occur with a buckling bar. Since the focus of this paper is on the effects of substructure, we consider a relatively low mass disc that is stable to bar formation for at least 10 Gyr. We stress that our goal is not to develop the most realistic model for the MW but rather to study the physics of bending waves that are generated by halo substructure.

A second feature of our simulations is the inclusion of both thin and thick disc components. We use the Action-based Galaxy Modelling Architecture (AGAMA, Vasiliev 2018) code to generate initial conditions. Action-based methods do not rely on the epicycle approximation and are therefore particularly well-suited for models that include a thick disc component. One motivation for including two components in our
model disc is the conjecture that dynamically distinct components will respond differently to a passing subhalo (Banik et al. 2017). In particular, we expect that the transfer of energy between a subhalo and disc is enhanced when the timescale for changes in the gravitational potential match (are in resonance with) the vertical epicyclic motions of the disc stars (Sellwood et al. 1998). Moreover, slow moving subhaloes tend to excite bending motions while fast moving ones excite compression and rarefaction. Since the vertical frequency for thick disc stars is larger than for thin disc stars, one might expect a preference for bending waves in the thin disc and breathing waves in the thick disc (Widrow et al. 2014).

As mentioned above, one might hope to use Gaia data to characterize bending waves in the MW’s stellar disc according to properties such as wavelength and pattern speed. Ultimately, one might imagine decomposing disc perturbations into something akin to normal modes. The theoretical problem of determining the normal modes of a self-gravitating stellar dynamical disc is challenging even in highly idealized cases. Mathur (1990), Weinberg (1991), and Widrow and Bonner (2015), for example, worked out the vertical oscillations of a self-gravitating, isothermal sheet (Spitzer 1942; Camm 1950). Though these one-dimensional modes can be long-lived with respect to the timescale for the vertical epicyclic motions of the stars, they eventually decay due to phase mixing and Landau damping. Toomre (1966) and Araki (1985) stressed the difference between bending and density waves in a self-gravitating plane-symmetric system. With density waves, gravity causes overdense regions to collapse with velocity dispersion (i.e., kinematic pressure) providing the restoring force. By contrast, bending waves are enhanced by the centrifugal force as particles pass through a bend in the disc. In this case, gravity acts to restore a perturbed region of the disc.
A complementary approach is to ignore the velocity dispersion of the stars and focus instead on the dynamics of a rotating disc. In the pioneering work of Hunter and Toomre (1969), and subsequent analyses by Sparke (1984) and Sparke and Casertano (1988), the disc is treated as a set of concentric rings, which interact with one another gravitationally. The normal modes of the ‘$N$-ring’ system can then be found using standard eigenvalue methods. Of particular interest to Sparke and Casertano (1988) and others is the existence of the ‘modified tilt mode’, which arises when the disc is embedded in a flattened halo. This mode is the only isolated one of the system and therefore might explain the existence of long-lived warps. In the limit of large $N$, the remaining modes form a continuum. Therefore any initial perturbation, which naturally involve a superposition of these modes, will disperse or phase mix.

With simulations, such as the ones described in this paper, we have the luxury of knowing the full phase space distribution function (DF), or at least an $N$-body sample of the DF of the stars, across the entire disc at all times. As described below, this allows us to analyse the perturbations using various mathematical tools. Here, we draw on previous studies of spiral density waves and consider representations of $Z$ and $V_z$ in terms of azimuthal wave number and/or frequency. This spectral analysis demonstrates a clear connection between results from linear theory and WKB analyses, on the one hand, and fully self-consistent, non-linear simulations on the other. In addition, we compute the cross-correlation between $Z$ and $V_z$ and find a rich structure in time and Galactocentric radius.

The outline of the paper is as follows: In Section 6.3 we describe the initial conditions and $N$-body parameters for our simulations. We focus on subhaloes and
CHAPTER 6. MULTIPLE SATELLITE ENCOUNTERS

their interactions with the disc in Section 6.4. In Sections 6.5 and 6.6 we apply Fourier methods in the analysis of these simulations. We further explore the wave-like nature of bending waves by comparing midplane displacement and vertical bulk motion in Section 6.7. We discuss some implications of our results in Section 6.8 before concluding in Section 6.9.

6.3 Simulations

We simulate the evolution of a disc-bulge-halo model galaxy comprised of thin and thick disc components as well as a halo where 10 per cent of the halo mass is initially in subhaloes and the remaining 90 per cent is smoothly distributed. We also run a simulation where the halo is smooth. A comparison of the two simulations, which we refer to as the satellite and control/isolated simulations, allows us to quantify the effects of disc-subhalo interactions.

6.3.1 An action-based equilibrium model for the Galaxy

We construct an equilibrium model for a MW-like galaxy using the action-based code\textsuperscript{1} AGAMA (Vasiliev 2018). The code employs an iterative scheme to find self-consistent forms for the DF of the different components and the gravitational potential. The general idea is to begin with an initial guess for the potential and expressions for the DFs in terms of a set of three integrals of motion. One then computes the density as a function of coordinates and solves Poisson’s equation to obtain a new approximation for the potential. The procedure is repeated until the density-potential

\textsuperscript{1}https://github.com/GalacticDynamics-Oxford/Agama. The version of AGAMA we used pre-dates the official public release and was downloaded on July 27, 2017.
pair converges. In our previous work, we used the GALACTICS code where the DFs were chosen to be functions of the angular momentum about the symmetry axis of the galaxy, the total energy, and the vertical energy (Kuijken and Dubinski 1995; Widrow et al. 2008). The latter is only approximately conserved in thin discs and not at all well conserved in thick ones. An action-based code avoids this problem since the actions are identically conserved. The price one pays is the computational challenge of computing the actions. AGAMA uses an efficient and accurate implementation of the so-called ‘Stäckel fudge’ (Binney 2012) to compute the action transformations for all galaxy components (Vasiliev 2018, but see also Sanders and Binney 2016 for a review of approaches).

With AGAMA, the input parameters explicitly determine the functional form of the DFs in action space and only implicitly determine the galaxy’s structure. Thus, a certain amount of trial and error is required to construct a model with the desired properties. In this section, we outline the properties of our ‘target’ model. The most important AGAMA input file parameters are given in Table 6.1.

We assume that the density profiles of the bulge and halo are given, at least approximately, by the Hernquist (Hernquist 1990) and NFW (Navarro et al. 1996) profiles, respectively. Both of these components can be modelled within AGAMA by the double-power DF from Posti et al. (2015), which yields a system whose spherical density profile is given, to a good approximation, by

$$\rho(r) = \frac{\rho_0}{\left(\frac{r}{r_b}\right)^\alpha(1 + \frac{r}{r_b})^{3-\alpha}}. \quad (6.1)$$

The disc components in AGAMA are modelled using the pseudo-isothermal DF from Binney and McMillan (2011). In constructing these models one uses the fact that though the galactocentric radius $R$ of a star is not an integral of motion, its angular
momentum $L_z$ is. One can therefore use $R_c$, the radius of a circular orbit with angular momentum $L_z$, as a proxy for $R$. For example, the factor in the disc DF that controls the vertical structure has the form

$$f(L_z, J_z) = \frac{\nu_z}{(2\pi \sigma_z^2)^{1/2}} e^{-\nu_z J_z / \sigma_z^2}, \quad (6.2)$$

where $\nu_z$ is the vertical epicycle frequency and $J_z$ the vertical action. The quantity $\sigma_z$ is a function of $R_c$ (i.e., a function of $L_z$) and controls the radial profiles of the vertical scale height and velocity dispersion. In our models, $\sigma_z$ is assumed to be an exponentially decreasing function of $R_c$, which yields a disc with an approximately exponential vertical velocity dispersion profile.

Our choice of disc parameters is guided by the analysis of Bovy et al. (2012) and Bovy and Rix (2013) as well as our desire for a model that is stable against the formation of a bar for the duration of the simulation runtime. As discussed in Section 3.2, this model allows us to focus on the effect substructure, rather than a bar or large-amplitude spiral structure, has on vertical oscillations. The thin disc has a radial exponential scale length of $3.6 \text{kpc}$ and a mass of $2.6 \times 10^{10} M_\odot$. The corresponding quantities for the thick disc are $2.0 \text{kpc}$ and $2.2 \times 10^{10} M_\odot$. The total disc mass of $4.8 \times 10^{10} M_\odot$ is in accord with the total stellar disc mass for the MW reported in the literature (Bovy and Rix 2013, but see also Courteau et al. 2014 for a review). The vertical structure of the discs are well-described by a $\text{sech}^2(z/2 z_d)$ profile with $z_d = 260 \text{pc}$ for the thin disc and $690 \text{pc}$ for the thick (Bovy et al. 2012).

Our bulge has a target density profile given by the Hernquist profile (equation 6.1 with $(\alpha, \beta) = (1, 4)$) and a total mass of $2 \times 10^{10} M_\odot$, which is slightly higher (by about 5-10 per cent) than the current best estimates for the MW’s bulge (see Bland-Hawthorn and Gerhard 2016). The bulge scale radius is taken to be $600 \text{pc}$ as in the
Figure 6.1: Circular speed curve decomposition for our MW-like model. The solid magenta curve shows the total circular speed of the model. Also shown are the contributions to the circular speed curve from the bulge (cyan), halo (green), thin disc (blue), thick disc (red), and total disc (black).

Bovy and Rix (2013) model and the velocity distribution is approximately isotropic.

The halo target density profile is given by the NFW profile (equation 6.1 with $(\alpha, \beta) = (1, 3)$; see Navarro et al. 1996) with scale radius $r_b = 22 \text{kpc}$ and $\rho_0 = 9 \times 10^6 \text{M}_\odot \text{kpc}^{-3}$. The parameters for the DF are chosen so as to yield a total circular speed curve that is approximately flat with $V_c \simeq 220 - 250 \text{km s}^{-1}$ for $2 \text{kpc} < R < 30 \text{kpc}$ (see Fig. 6.1). Note as well that the total disc is submaximal. In particular, $V_d^2/V_c^2 \approx 0.37$ at the peak of the total disc’s circular speed curve ($R \sim 7 \text{kpc}$), where $V_d$ is the contribution to the circular speed from the total disc.

Fig. 6.2 shows the surface density profiles of the thin and thick discs. We note that, at small radii, the surface densities are supressed relative to a pure exponential.
Figure 6.2: Radial properties of our MW-like disc. The colour corresponds to the thin (red), thick (blue), and total (black) discs. Top panel: Surface density profiles. Solid lines correspond to our $N$-body model and dashed lines indicate the target profiles. Middle panel: Root-mean-square height profiles. Bottom panel: Toomre $Q$ parameter profile of the total disc.
This effect is likely due to the presence of a cuspy bulge and the fact that in action-based modelling the resulting density law is only approximately equal to the target. Our Hernquist bulge causes the epicyclic frequencies to rapidly increase as $R \rightarrow 0$ and consequently the pseudo-isothermal DF is not well behaved in the central disc (Vasiliev 2018).

The root-mean-square thicknesses of the discs in our model vary with $R$ by a factor of $\sim 2$ as seen in the middle panel of Fig. 6.2. The depression at small radii is, again, due to the inclusion of a cuspy bulge. The decrease in thickness at larger radii is presumably due to using an exponential vertical velocity dispersion profile, which would produce a constant scale height only for a thin disc in isolation.

The central radial dispersion was chosen to yield a relatively high Toomre $Q$ parameter (Toomre 1964) across the disc, as can been seen in the bottom panel of Fig. 6.2. The fact that we have a submaximal disc with $Q > 1.5$ indicates that very little structure will develop in the disc, at least in the absence of external perturbations.

In the top panel of Fig. 6.3 we show the vertical density profiles for the thin, thick, and total discs evaluated at $R = 8 \text{kpc}$. On average, the density profiles roughly correspond to our target parameters. In the bottom panel we show the vertical velocity dispersion profile, and see that the thin and thick discs are roughly isothermal. The vertical profiles are shown for the Solar circle though the same general trends are observed at all other radii.
Figure 6.3: Vertical properties of our MW-like disc evaluated at a galactocentric radius of 8 kpc. The colour corresponds to the thin (red), thick (blue), and total (black) discs. Top panel: Vertical density profiles. Solid lines correspond to our $N$-body model and dashed lines indicate the target profiles. Bottom panel: Vertical velocity dispersion as a function of distance from the midplane.

6.3.2 Halo substructure

For our satellite simulation we replace 10 per cent of the halo mass with 100 subhaloes by a scheme similar to the one presented in Gauthier et al. (2006). We randomly remove 10 per cent of the particles that characterize the smooth halo and insert an equivalent amount of mass in the form of spherical subhaloes. These subhaloes have
initial center-of-mass positions and velocities drawn from the same DF as the original halo.

The initial cumulative distribution of subhaloes as a function of galactocentric radius is shown in Fig. 6.4. Note that while only a few subhaloes begin with an initial radius inside the region of the disc (i.e., $r \lesssim 30$ kpc), 20 or so subhaloes follow orbits with a perigalacticon less than 30 kpc. Thus, most of the subhaloes that pass through the disc were initialized well outside the disc region.
We assume an initial subhalo mass function given by

\[
\frac{dN_{\text{sat}}}{dM_{\text{sat}}} = \begin{cases} 
AM_{\text{sat}}^{-2} & \text{for } M_{\text{min}} \leq M_{\text{sat}} \leq M_{\text{max}}, \\
0 & \text{otherwise},
\end{cases}
\]  

(6.3)

where \(A\) is a normalization constant. In addition, we assume that the total mass in subhaloes is one tenth of the virial mass of the halo and the maximum subhalo mass is \(M_{\text{max}} = 0.1 (M_{\text{disc}} + M_{\text{bulge}})\). These two conditions allow us to determine \(M_{\text{min}}\) and \(A\). Equation (6.3) is then sampled to determine initial virial masses for the subhaloes. We use \textsc{galactics} (Kuijken and Dubinski 1995; Widrow et al. 2008) to generate \(N\)-body realizations of the subhaloes assuming that each has a truncated NFW density profile. \textsc{galactics} requires the specification of NFW mass in addition to scale and truncation radii. We compute scale radii for each subhalo as \(c = r_{200}/a_s\), where \(c\) is the halo concentration, \(r_{200}\) is the radius in which the mean halo density is 200 times the critical density, and \(a_s\) is the NFW scale radius. We adopt a constant value of \(c = 20\) for all of the subhaloes, which is in agreement with cosmological simulations for the mass range of our subhalo population over a significant look-back time (~10 Gyr) (Macciò et al. 2007; Zhao et al. 2009). A histogram of the subhaloes as a function of their virial mass is shown in Fig. 6.5.

We smoothly truncate the \(N\)-body subhaloes at their tidal radii which are numerically computed using the Jacobi approximation (Binney and Tremaine 2008, section 8.3)

\[
\bar{\rho}_{\text{sat}}(r_t) = 3\bar{\rho}_{\text{halo}}(r_{\text{sat}}),
\]  

(6.4)

where \(\bar{\rho}_{\text{sat}}(r_t)\) is the subhalo’s mean density inside the truncation radius \(r_t\), and \(\bar{\rho}_{\text{halo}}\) is the mean density of the halo inside the initial radius, \(r_{\text{sat}}\), of the subhalo. Truncation is meant to take into account tidal stripping that may have occurred prior to the
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Figure 6.5: Number distribution of substructure virial mass. Solid black represents the distribution of the substructure population in our subhalo model. The dashed red curve corresponds to the number distribution derived from the mass spectrum from which our population was sampled (equation 6.3).

initial epoch of the simulation.

6.3.3 N-body Simulations

Multidimensional sampling routines built into AGAMA were used to generate an N-body realization of our model galaxy with $2.5M$ thin disc, $2.5M$ thick disc, $250k$ bulge, and $5M$ halo particles. When subhaloes were included, the smooth component of the halo had $4.5M$ particles while each subhalo had $25k$ particles. The initial conditions were evolved for $\sim 10$ Gyr using GADGET-3 (Springel 2005) with softening lengths of 40, 15, 100, and 80 pc for the disc, bulge, halo, and subhalo particles, respectively. The maximum time step was set to 0.18 Myr, which is $\sim 0.6$ per cent of the galactic
dynamical time defined at the radius of peak disc contribution to the rotation curve, $R \sim 7$ kpc. Total energy was conserved to within 0.06 per cent. The ratio of total kinetic to potential energy for the particle distributions comprising the thin and thick discs, bulge, and halo in the isolated model was $-0.503$, $-0.502$, $-0.505$, and $-0.503$, respectively, indicating that the initial conditions were sufficiently close to equilibrium. Indeed, in the first 250 Myr we observed a maximum increase in the vertical velocity dispersion and $Z_{RMS}$ profiles of only $\sim 1 \text{ km s}^{-1}$ and $\sim 60 \text{ pc}$, respectively.

Over time the discs in our simulations rotate and drift about the global coordinate origin, particularly in the simulation with halo substructure. To account for this when analyzing our models we center and rotate all model components by the following scheme: We first iteratively compute the center of mass of disc particles within a cylinder of radius 20 kpc and height 2 kpc. Next, we reorient the system so that the disc lies in the $x$-$y$ plane by using a two-dimensional Newton-Raphson scheme to find the orientation that minimizes the root mean square vertical displacement of disc particles within the same cylinder used to find the centre of mass.

### 6.4 Disc-Subhalo interactions

In this section we focus on the evolution of the subhalo population in our satellite simulation. We define the position and velocity of a particular subhalo as the average position and velocity of its 100 initially most energetically bound particles. This method does a good job at accurately tracking the phase space coordinates of a subhalo until it is completely disrupted. Computation of the bound mass within each subhalo has historically involved an iterative procedure concerning energy calculations.
Figure 6.6: Mass bound to subhaloes relative to the total mass contained in the halo and subhalo population as a function of time (solid black). For comparison we also show the corresponding relation for the two distinct phases of subhalo mass loss in the Gauthier et al. (2006) simulation (dashed red).

(e.g. Benson et al. 2004). Here, we employ a simpler approach where the bound mass of each subhalo at each time output is taken to be the mass contained within the Jacobi sphere as defined in equation (6.4).

In Fig. 6.6 we show the temporal evolution of the mass bound in subhaloes relative to the total mass of the halo and subhalo population. Our initial total substructure mass fraction, $\sim 0.08$, is in rough agreement with that of MW-like haloes found in cosmological simulations, such as the Aquarius Project (Springel et al. 2008, figure 12). If we consider only subhaloes initiated within a radius of 50 kpc, the substructure mass fraction decreases to $\sim 10^{-2}$. This value also agrees with that found by the Aquarius Project, although in the case of our simulation is due to the stochastic
nature of sampling the initial positions and masses of only 100 subhaloes. This is supported by Fig. 6.4, where the inner galaxy contains fewer subhaloes than expected, and the fact that those subhaloes initiated within a radius of $\sim 90$ kpc have masses from the lower end of the mass spectrum we sampled.

Our results can also be compared with those from Gauthier et al. (2006) who argued that the time-dependence of the mass in subhaloes could be described by two distinct phases. During the first 4 Gyr $f \propto \exp\left(-t/\tau_f\right)$ with $\tau_f \approx 5.1$ Gyr, while from 4 to 10 Gyr $\tau_f \approx 13.4$ Gyr. We find a similar trend where the exponential decay rate of subhalo mass decreases with time, though our initial ‘rapid decay phase’ is shorter. The difference may be a simple reflection of the fact that our distribution extends to much larger radii and therefore a greater fraction of our subhaloes do not experience tidal disruption.

The amplitude and extent of a perturbation in the disc due to a passing subhalo will depend on the position at which the subhalo’s orbit crosses the disc plane, its mass and velocity, and the orientation of the orbital angular momentum vector with respect to that of the disc’s (i.e. prograde, retrograde, or vertical orbits; Gómez et al. 2013; Widrow et al. 2014; Feldmann and Spolyar 2015). The two panels in Fig. 6.7 are meant to give a sense of which subhaloes will cause the most significant perturbations. In both panels we show the times and cylindrical radii at which a subhalo crosses the disc midplane. In the top panel we colour the points according to the ratio of subhalo surface density to the locally azimuthally averaged surface density of the disc at the time of impact. The restoring force of the disc is proportional to the disc surface density, so by comparing this with the corresponding quantity for the subhaloes we get a sense of how significantly a given interaction might perturb
Figure 6.7: Time evolution of subhalo impacts on the disc. Each point represents the cylindrical radius and the time at which a subhalo crosses the $z = 0$ plane. The star symbol indicates the specific impact event discussed in Section 6.5. Colour coding in the top panel corresponds to the ratio of projected subhalo surface density to that of the local disc at the time of impact - a proxy for the efficacy of each impact to produce vertical waves. In the bottom panel, the colour corresponds to the ratio of subhalo vertical velocity to the local vertical velocity dispersion of the disc at that time.

In general, we see that the subhaloes that cross the disc plane with $R < 10$ kpc have relatively low surface densities, except perhaps during the first few Gyr. The number of significant subhalo interactions (interactions
where the subhalo surface density is comparable to or greater than that of the disc) increases significantly at larger radii, in part because the density of the disc decreases exponentially, and in part because subhaloes tend to be disrupted if their orbits take them close to the galactic centre.

In the bottom panel of Fig. 6.7 we show the subhalo’s vertical velocity as it passes through the disc midplane relative to the local vertical velocity dispersion of the disc as a function of $R$ and simulation time. We see that satellites that pass through the outer disc have speeds 10-100 times higher than the typical vertical velocities of the disc stars. In this region, the epicyclic motion can all but be ignored during a subhalo-disc interaction. Conversely, subhaloes passing through the disc at intermediate radii ($5 \text{kpc} < R < 15 \text{kpc}$) have vertical velocities more typical of the vertical velocities in the disc. This sets up the possibility for resonant excitation of both vertical bending and breathing waves (Sellwood et al. 1998; Widrow et al. 2014).

### 6.5 Fourier analysis of density and vertical waves

In Fig. 6.8 we show the normalized surface density in the disc, $\Sigma(R, \phi, t)/\Sigma(R, t_0)$, where $\Sigma(R, t_0)$ is the initial surface density profile (see top panel Fig. 6.2). The normalization is chosen to highlight non-axisymmetric features that develop across the full range in $R$. Evidently, neither disc forms a bar. The disc in the control simulation develops tightly wrapped flocculent spiral arms with a density contrast that tends to be more pronounced at late times and toward the outer region of the disc. The spiral structure is considerably stronger in the satellite simulation. At early times the density features are more asymmetric and irregular. Note, in particular, what appears to be evidence of a single satellite interaction in the $t = 500 \text{Myr}$
Figure 6.8: Surface overdensity for the control (top row) and satellite (bottom row) simulations at five epochs. The colour map indicates the logarithm (base 10) of the ratio of the local surface density at the time indicated to the azimuthally-average surface density of the initial conditions. Dotted concentric circles indicate increments of 5 kpc in radius. The rotation of the discs is counter-clockwise.
snapshot at $R \sim 20 - 25\, \text{kpc}$ and roughly the ‘5-o’clock’ position of the disc. At later times large density contrasts survive only past $R \sim 10 - 15\, \text{kpc}$ and extend in azimuthal angle around a large fraction of the disc. By the end of the simulation the disc develops a dominant $m = 1$ spiral arm that is morphologically leading. We attribute these density enhancements and morphological differences to collisions or torques imparted from the orbiting subhalo system since they are not present in the control run (Gauthier et al. 2006; Dubinski et al. 2008).

In Fig. 6.9 we show vertical displacement, $Z(R, \phi, t)$, across the disc at the same epochs as those shown in Fig. 6.8. The trends are similar to those seen in the surface density plot. Bending waves appear in both the control and satellite simulations though they have a higher amplitude by a factor of $\sim 5$ in the satellite case. In both cases, strength in the bending waves tends to migrate toward the edge of the disc over time.

At early times, there are localized bending waves in the satellite case. In particular, we see evidence of an apparent interaction event in the 500 Myr snapshot at the same location in the disc as seen in the surface density, and have identified the subhalo responsible for this event. This satellite plunged through the disc midplane at a radius of $\sim 26.5\, \text{kpc}$ at $\sim 400\, \text{Myr}$ (indicated by the star symbol in Fig. 6.7) on a prograde orbit with cylindrical velocities $(V_R, V_\phi, V_z) \simeq (-160, 50, 210)\, \text{km}\, \text{s}^{-1}$. At the time of impact the satellite had a mass of $\sim 8 \times 10^8\, M_\odot$ within its 2.7 kpc radius Jacobi sphere. Despite crossing through the disc with such a large vertical velocity, the radial extent, surface density, and in-plane velocity of the satellite are significant, and therefore it is not surprising that the impact left such a clear and spatially extended mark in the disc, as seen in Figs. 6.8 and 6.9.
Figure 6.9: Face-on maps of mean vertical displacement $Z(R, \phi, t)$ for the isolated (top row) and satellite (bottom row) simulations at the same five epochs in Fig. 6.8. Dotted concentric circles indicate increments of 5 kpc in radius. The rotation of the discs is counter-clockwise.
We next consider the application of Fourier methods to the simulations. Formally, the DF of an \( N \)-body system can be expressed as the sum of six-dimensional \( \delta \)-functions centred on the phase space coordinates (positions and velocities) of the particles and weighted by their mass. The density is found by integrating over velocities while the surface density is found by further integrating over \( z \). Thus, the surface density is given by

\[
\Sigma(R, \phi, t) = \sum_j m_j \delta(\phi - \phi_j(t)) \frac{\delta(R - R_j(t))}{R} .
\]  

(6.5)

(For the maps in Fig. 6.8 we effectively integrate over bins in a polar grid and divide by the area of each bin. In Fig. 6.9, we compute the average \( z \) in each of these bins).

Beginning with the pioneering work of Sellwood and Athanassoula (1986) it has proved useful to consider various transforms of the surface density when studying the formation of warps, bars, and spiral structure. A Fourier series in \( \phi \) yields a decomposition of \( \Sigma \) in terms of azimuthal wave number \( m \), with warps dominated by \( m = 1 \) while bars and bisymmetric spiral structure are dominated by \( m = 2 \). A Fourier transform in \( t \) yields the surface density amplitude as a function of \( \omega, R, \phi, \omega, R, \phi, \) and \( m \). The frequency \( \omega \) can be used as a proxy for angular pattern speed. Sellwood and Athanassoula (1986) also consider a Fourier transform in \( \ln R \).

In what follows, we apply Fourier methods to bending waves. We begin by dividing the disc into centric rings each labelled by \( \alpha \) and centred on a radius \( R_\alpha \) with width \( \Delta R_\alpha \) and surface area \( S_\alpha = 2\pi R_\alpha \Delta R_\alpha \). The radially-smoothed surface density is then

\[
\Sigma(R_\alpha, \phi, t) = 2\pi S_\alpha^{-1} \sum_{j \in \alpha} m_j \delta(\phi - \phi_j(t)) ,
\]  

(6.6)

where \( R_\alpha \rightarrow R \) in the limit of small \( \Delta R_\alpha \) and the sum is over all particles in ring \( \alpha \). The vertical displacement, \( Z(R, \phi, t) \), and vertical bulk motion, \( V_z(R, \phi, t) \), can be
constructed by taking moments of the DF. In analogy with equation (6.6) we have
\[ Q(R_\alpha \phi, t) = 2\pi S_\alpha^{-1} \Sigma^{-1}(R_\alpha, \phi, t) \sum_{j \in \alpha} m_j q_j(t) \delta(\phi - \phi_j(t)) , \] (6.7)
where \( Q = \{Z, V_z, \ldots\} \) and \( q_j = \{z_j, v_{z,j}, \ldots\} \), and the \ldots refer to any non-linear function of \( z \) and \( v_z \).

The Fourier series for the surface density is given by
\[ \Sigma(R_\alpha, \phi, t) = \sum_{m=0}^{\infty} \Sigma_m(R_\alpha, t) e^{-im\phi} , \] (6.8)
where
\[ \Sigma_m(R_\alpha, t) = \frac{1}{2\pi} \int_0^{2\pi} \Sigma(R_\alpha, \phi, t) e^{im\phi} d\phi \]
\[ = S_\alpha^{-1} \sum_{j \in \alpha} m_j e^{im\phi_j} . \] (6.9)
Likewise, the Fourier coefficients of \( Q(R_\alpha, \phi, t) \) are
\[ Q_m(R_\alpha, t) = S_\alpha^{-1} \sum_{j \in \alpha} \frac{m_j q_j e^{-im\phi_j}}{\Sigma(R_j, \phi_j, t)} , \] (6.10)
where \( \Sigma(R_j, \phi_j, t) \) is the interpolated surface density at the position of the \( j \)th particle.

In Fig. 6.10 we plot \( |\Sigma_1(R, t)| / \Sigma(R, t_0), |Z_1(R, t)|, \) and \( |V_{z,1}(R, t)| \). In particular, we show the separate contributions from the thin and thick discs, and compare the isolated and satellite simulations. Generally, we see that the thick disc is much less responsive by a factor of \( 4 \sim 5 \) and the subhaloes induce perturbations with peak amplitudes \( \sim 5 \) times greater than those in the control simulation.

Bending waves are strongest in the outer disc for both control and satellite runs. This result is expected for several reasons. First, the transition from a relatively quiet disc to one with stronger vertical waves occurs at \( R \sim 15 \)kpc, which roughly coincides with region where the surface density of the thick disc falls to negligible values. Thus,
Figure 6.10: Time evolution of $m = 1$ Fourier coefficients for in-plane density and bending waves in both the thin and thick discs of the isolated and satellite simulations, as indicated. Note the difference in colour scale between the thin and thick discs.
the increased gravity from a relatively warm component may make the inner disc less responsive to bending perturbations. Moreover, the higher surface density of the thin disc itself acts to stiffen the disc. The outer disc is dominated by the cooler thin disc though much of the gravitational field is from the halo, and it is therefore more prone to non-axisymmetric structure formation and bending waves. Finally, as noted in section 6.4 the strongest subhalo encounters occur in the outer disc.

To determine the morphology of \( m = 1 \) bending waves we compute the complex phase for \( Z_1, \)

\[
\alpha(R, t) = \tan^{-1} \left[ \frac{\text{Im}\{Z_1(R, t)\}}{\text{Re}\{Z_1(R, t)\}} \right],
\]

as a function of \( R \) and \( t \). In Fig. 6.11 we show \( \alpha(R, t) \) for the thin and thick discs in both simulations. Attention is immediately drawn to ridges of constant phase that arc outward in radius over time. Since the coordinate system in our simulations has the disc rotating counter-clockwise (i.e., in the direction of increasing \( \alpha \)), we conclude that the bending waves are either prograde and trailing or retrograde and leading. A distinction between the two cases is made when considering the behaviour of \( \alpha \) over slices in \( R \) and time. The signature of leading waves is for \( \alpha \) to increase with increasing \( R \) at fixed time. On the other hand, the signature for a retrograde bending wave is for \( \alpha \) to decrease with time at fixed \( R \). Indeed, we observe these very relationships in Fig. 6.11 and conclude that the dominant long-lived bending waves in our simulations are morphologically leading with patterns that rotate in a retrograde sense, as is expected for neutrally stable bending waves (see section 8.1 of Sellwood 2013). We note that though we only show \( \alpha \) for vertical displacements the phase angle for bulk vertical motions displays the same behaviour as a function of \( R \) and \( t \), albeit with an offset.
Figure 6.11: Time evolution of Fourier phase angle $\alpha(R, t)$ for $m = 1$ vertical displacement bending waves in the thin and thick discs of both simulations, as indicated.
6.6 Spectral wave Analysis

The spectral methods developed by Sellwood and Athanassoula (1986) allow one to characterize density waves in terms of angular frequency and galactocentric radius. These methods provide valuable insight into disc dynamics and have been extended to the study of bending waves (see Chequers and Widrow 2017).

Our spectral analysis for in-plane density waves and bending waves begins with the Fourier coefficients, \( \Sigma_m(R, t_j) \) and \( Q_m(R, t_j) \), for \( N \) snapshots at times \( t_j = j \Delta_t + t_0 \), where \( \Delta_t \) is the time between snapshots, \( j = 0 \ldots N - 1 \), and \( N \) is even (see Fig. 6.10). We then perform a discrete Fourier transform on the time series to obtain the two-sided frequency coefficients (Press et al. 2007, Section 13.4) as

\[
F_m(R, \omega_k) = \begin{cases} 
\Sigma^{-1}(R, t_0) \sum_{j=0}^{N-1} \Sigma_m(R, t_j) w(j) e^{2\pi i j k/N}, \\
\sum_{j=0}^{N-1} Q_m(R, t_j) w(j) e^{2\pi i j k/N}.
\end{cases}
\]  

(6.12)

Here, \( w(j) \) is a Gaussian window function with a standard deviation of \( N/2^{5/2} \), which is introduced to diminish high-frequency spectral leakage. The discrete frequencies are given by

\[
\omega_k = \frac{2\pi}{m} \frac{k}{N \Delta_t},
\]  

(6.13)

with \( k = -N/2 \ldots N/2 \). The frequency resolution is regulated by the length of the time baseline, \( N \Delta_t \), while the Nyquist frequency, corresponding to \( \omega_k = \pm N/2 \), is derived from the time resolution, \( \Delta_t \). The frequency power spectrum is then computed as

\[
P_m(R, \omega_k) = \frac{1}{W} |F_m(R, \omega_k)|^2,
\]  

(6.14)

where \( W = N \sum_{j=0}^{N-1} w(j) \) is the window function normalization.

In Fig. 6.12 we plot the frequency power spectra for \( m = 1 \) density and bending waves over a time baseline of \( 0 \leq t \lesssim 10 \) Gyr. The general properties of relative
Figure 6.12: Frequency power spectra for $m = 1$ in-plane density and bending waves in the thin and thick discs of both simulations over a time baseline of $0 \leq t \lesssim 10$ Gyr. The layout of the figure is analogous to Fig. 6.10. The horizontal dashed red line references zero frequency while the dashed green curve corresponds to corotation. Overlaid as dashed blue and cyan curves are orbital resonances for kinematic density waves in a cold disc (see equation 6.15), as indicated, corresponding to the initial conditions.
frequency power between the isolated and satellite runs as well as the thin and thick discs is analogous to that of the time series of Fourier coefficient magnitude in Fig. 6.10 (Note the different scales used for power between the thin and thick discs).

Evidently, bending wave power follows one of a series of branches that arc across the $R$-$\omega$ plane and is concentrated in the outer disc at lower, or even counter-rotating, frequencies. These observations are in accord with our analysis in Section 6.5 where we concluded that the dominant waves in the disc were morphologically leading and propagate in a retrograde fashion compared to the rotation of the disc.

The power spectrum branches in Fig. 6.12 roughly coincide with the orbital resonance curves

$$\omega(R) = \begin{cases} 
\Omega(R) + n\kappa(R), & m = 1 \text{ in-plane density waves,} \\
\Omega(R) + n\nu(R), & m = 1 \text{ bending waves.} 
\end{cases} \tag{6.15}$$

Here, $\Omega$ is the circular frequency, $\kappa$ is the total epicyclic frequency, and $\nu$ is the total vertical forcing frequency from the disc, bulge, and halo. The integer $n$ corresponds to the number of azimuthal periods an orbit takes to close during one radial oscillation period in a frame rotating with angular frequency $\omega$. The Lindblad resonances and their vertical analogs correspond to $n = \pm 1$.

Many of the features in Fig. 6.12 can be understood by appealing to the WKB approximation, where one assumes that the waves (either density or bending) are tightly wound. In this approximation, and under the further assumption that the disc is cold and thin, the dispersion relations are (Binney and Tremaine 2008, sections 6.2.1 and 6.6.1, and section 8.1 from Sellwood 2013)

$$\begin{align*}
(\omega - \Omega)^2 &= \begin{cases} 
\kappa^2 - 4\pi^2 G\Sigma/\lambda_d, & \text{for } m = 1 \text{ density waves,} \\
\nu^2 + 4\pi^2 G\Sigma/\lambda_b, & \text{for } m = 1 \text{ bending waves,} 
\end{cases} \tag{6.16}
\end{align*}$$
where $\Sigma$ is the radial surface density and $\lambda_d$ ($\lambda_b$) is the wavelength of a density (bending) wave. From equation (6.16) we see that WKB density waves are ‘forbidden’ in the region outside the Lindblad resonances while bending waves are forbidden between the corresponding vertical resonances (equation 6.15). Indeed, we find that in the case of density waves power mainly lies between the Lindblad resonance curves whereas power lies outside the $n = \pm 1$ vertical resonances in the case of bending waves.

The Lindblad resonances and their vertical analogs emerge from equation (6.16) in the limit that the surface density terms (i.e. the terms that encode the forcing frequency due to the self-gravity of the perturbation) on the right hand side are negligible when compared to the $\kappa^2$ or $\nu^2$ terms. Thus, we can predict the degree to which power should align with the resonance curves by comparing the two terms on the right-hand side of equation (6.16). In general, we expect this alignment to be better at large radii since the surface density falls off exponentially while $\kappa$ and $\nu$ fall as as powers of $R$. Furthermore, following similar arguments to those found in Binney and Tremaine (2008, equation 3.89), one can show that

$$\frac{\nu^2}{\kappa^2} \simeq \frac{3}{2} \frac{\rho}{\bar{\rho}}$$

(6.17)

where $\bar{\rho}$ is the mean density within a given radius and $\rho$ is the density of the galaxy in the midplane of the disc at that radius. Since the bulge and halo are nearly spherically symmetric with monotonically decreasing density profiles and the submaximal disc lies in the plane, we conclude that $\nu < \kappa$. Therefore the surface density term in equation (6.16) will play a larger role for bending waves than for density waves. It is therefore not surprising that bands in power are more closely aligned with the resonance curves in the density wave case than the bending wave one.
Figure 6.13: Time evolution of $m = 1$ height/bending frequency power spectra evaluated over $\sim 2$ Gyr intervals for the thin disc in both simulations, as indicated. The dashed curves correspond to the same quantities as in Fig. 6.12.
In Fig. 6.13 we show a time sequence of $m = 1$ bending waves over $\sim 2$ Gyr intervals for the thin disc. For each of the time intervals in the control simulation, power lies along branches just outside the $n = \pm 1$ resonance curves, as expected. The situation is more complicated in the case with satellites. At early times, there are numerous regions of power across the $R - \omega$ plane that are not particularly well correlated with the resonance curves. Over time, the positive frequencies diminish in power, although localized waves still exist, and the strength of low-frequency counter-rotating waves increases in strength and migrates outward. By the end of the simulation, the branches of power roughly mirror that of the isolated simulation, though the relative weighting of the waves localized in radius is quite different. These results make intuitive sense. Bending waves in the control simulation involve the growth of linear perturbations, which are well described by a WKB analysis. The satellite-provoked waves, on the other hand, are more random in nature, though over time, the most persistent ones also appear to coincide with predictions from linear theory.

### 6.7 Correlation between $Z$ and $V_z$

Simultaneous observations of bulk vertical motions and vertical displacement provide an avenue toward understanding wave dynamics in the Galactic disc. Roughly speaking the angular frequency of a wave will be given by the amplitude of its oscillations in velocity divided by the amplitude of its displacement oscillations. Xu et al. (2015) found evidence for corrugations in the disc with an amplitude of about 120 pc and a wavelength of 7 kpc at Galactocentric radii of 10 – 16 kpc. On the other hand, Schönrich and Dehnen (2017) identified ripples in the velocity field of the Solar
Neighbourhood with an amplitude of $\sim 0.8 \, \text{km s}^{-1}$ and a wavelength of $2-2.5 \, \text{kpc}$. Of course, the wavelength and amplitude of a bending wave might change with Galactocentric radius. As noted in Schönrich and Dehnen (2017), the extent of their sample was not sufficient to detect such changes. In any case, if one assumes that the Xu et al. (2015) and Schönrich and Dehnen (2017) oscillations are related, then the inferred angular frequency is $\simeq 6.7 \, \text{km s}^{-1} \text{kpc}^{-1}$, which corresponds to a period of about 920 Myr. Obviously, the caveats mentioned above imply that this is only a ballpark value and valid only if the two observations are of the same wave-like structure.

The correlation between bulk vertical motion and midplane displacement in simulations has been discussed by Gómez et al. (2013), Gómez et al. (2016), Chequers and Widrow (2017), and Gómez et al. (2017). In both of our control and satellite simulations the bulk vertical motions follow the same general patterns as the displacements (c.f. Figs. 6.10-6.12) except for a phase offset. This behaviour is expected for wave-like motion and allows one to infer a rotational frequency from the respective amplitudes of each pattern.

To further explore the connection between velocity and displacement oscillations, we compute the azimuthally-averaged cross correlation function between $Z_1$ and $V_{z,1}$:

$$C(R, \tau) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \int_{t_0}^{t_f} V_{z,1}(R, \phi, t) Z_1(R, \phi, t+\tau) \, dt,$$

where $\tau$ is the lag time between the signals. To highlight small amplitude wave-like features we normalize to the root-mean-square of $Z$ and $V_z$ at a given radius,

$$\langle Q^2 \rangle^{1/2} = \sqrt{\frac{1}{t_f} \int_{0}^{t_f} q^2(t) \, dt},$$

where $Q = \{Z, V_z\}$ and $q = \{z, v_z\}$, respectively.
For the highly idealized case of a linear one-dimensional monochromatic plane wave, the displacement, \( z = D \sin(\omega t + \phi) \), is related to the velocity by \( v = d\,z/dt = D\,\omega \cos(\omega t + \phi) \). The displacement pattern will lag the velocity by an offset in phase of \( \pi/2 \) and a time \( \pi/2\omega \). Thus, the cross correlation between the two waveforms, \( C(\tau) \propto \sin(\omega \tau) \), will display periodic peaks at \( 2p - 1 \) multiples of \( \tau = \pi/2\omega \), where \( p \) is an integer, that decreases in amplitude with increasing \(|\tau|\).

In Fig. 6.14 we show the normalized cross correlation between the \( m = 1 \) displacement and velocity patterns in our simulations and immediately notice large correlation amplitudes that are localized in \( R \), periodic in \( \tau \), and differ between the two simulations. We also note that the radii at which these peaks in correlation amplitude occur seem to correspond to radii that feature large frequency power as seen in the spectra in Fig. 6.12.

Of course, in the case of our simulations the situation is more complex than simple one-dimensional monochromatic waves since our discs comprise waves of varying frequency localized in radius. Quite remarkably, for a given \( R \) the cross correlation patterns we see in Fig. 6.14 generally exhibit the exact behaviour that is expected for simple plane waves. In particular, we observe fluctuations in amplitude that are periodic in \( \tau \) and are able to recover the localized frequency \( \omega = \pi/2\tau \). More globally, we see intricate patterns of correlation amplitude that highlight the dominant long-lived waves themselves and the spatial transitions between them.

We note that in the case of the thin disc in our isolated galaxy simulation for \( R \gtrsim 20 \) kpc the correlation amplitude is negative and positive to the right and left of zero lag, respectively, which is interpreted as the displacement pattern leading the velocity. This behaviour may indicate non-linear dynamics or damping.
Figure 6.14: Normalized cross correlation (see equations 6.18 and 6.19) between $m = 1$ velocity and displacement bending waves. The column titles indicate which disc and simulation each panel corresponds to.
6.8 Discussion

There is now evidence for bending and breathing waves in the stellar disc of the MW from numerous astrometric surveys including GDR2. Nevertheless, the origin of these waves remains uncertain. One attractive proposal is that they are excited by interactions between the disc and satellite galaxies or dark matter subhaloes. This proposal is a natural extension of the hypothesis that satellite-disc encounters heat and thicken the disc (Toth and Ostriker 1992) and may also prompt the formation of a bar or spiral structure (Gauthier et al. 2006; Dubinski et al. 2008; Kazantzidis et al. 2008). In the simplest interpretation wave-like features in the disc can be associated with particular disc-satellite interactions. For example, Purcell et al. (2011) used $N$-body simulations to show that the passage of the Sagittarius dwarf galaxy through the disc may have been responsible for the MW’s bar and spiral structure. Gómez et al. (2013) then showed that the same encounter could generate vertical waves similar to the ones seen in the data.

The numerical experiments by Gauthier et al. (2006) and Dubinski et al. (2008) paint a more complicated picture. They simulated a disc galaxy with a clumpy halo and found that halo substructure vigourously excited spiral density waves. Moreover, even though their equilibrium (i.e., smooth halo) model was stable to bar formation for at least 10 Gyr, subhaloes could trigger the formation of a bar – though the timing of this depended on the particulars of the subhalo distribution. The implication is that the MW’s dynamical state may reflect both recent, singular encounters and the cumulative effect of many disc-satellite interactions over its lifetime.

A related question concerns the evolution of perturbations once they are excited. de la Vega et al. (2015) suggested that this evolution can be described as kinematic
phase mixing and that the self-gravity of the waves can be ignored. If true, this approximation would greatly simplify the analysis of perturbations in the disc. Indeed, Antoja et al. (2018) used this idea to interpret the spiral structure they found in \( v - z \) phase space projections of the GDR2 data. Simple kinematics allowed them to ‘unwind’ these structures and hence ‘date’ the event that gave rise to them.

Of course, similar arguments, when applied to the problem of spiral structure, lead to the so-called winding problem wherein spiral arms become so tightly wound in just a few orbital periods that they would dissolve into the rest of the disc. In an attempt to resolve the winding problem it was proposed that spiral arms are density waves, which are amplified and maintained by self-gravity (Lin and Shu 1964, but see also Lindblad 1963, Lin and Shu 1966, Lin et al. 1969, and Toomre 1969). Similar arguments have been made in the study of galactic warps. For example, the eigenmode analyses of bending waves by Hunter and Toomre (1969), Sparke (1984), and Sparke and Casertano (1988) explicitly include self-gravity by treating the disc as a system of gravitationally-coupled concentric rings. When one ring is displaced from its equilibrium position, it exerts a perturbing force on the unperturbed rings and also feels a restoring force due to them. It is the combination of these two effects that yields a discrete modified tilt mode, which provides an explanation of the global warp in the disc, and also the continuum of bending waves, which could describe the waves seen by Xu et al. (2015) and Schönrich and Dehnen (2017).

The simulations described in this paper allow us to explore these issues in more detail. In particular, we compare the evolution of a stellar-dynamical disc embedded in a clumpy dark halo with one that evolves in a smooth halo. In analyzing our
simulations we turn to the spectral techniques laid out in Sellwood and Athanas-
soula (1986), and extended to bending waves by Chequers and Widrow (2017) (c.f.
Fig. 6.12). We find that the dominant long-lived waves for both in-plane and vertical
disturbances with $m = 1$ symmetries lie mainly on or around two main branches in
the $R$-$\omega$ plane, which roughly coincide with the Lindblad resonances and their vertical
counterparts, and is in agreement with predictions from a linear eigenmode analysis of
bending waves (Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988,
but see Chequers and Widrow 2017). However, here we make a distinction between
‘modes’ of the disc and waves since the radial locations of the waves we observe in our
simulations change over time (c.f Figs 6.8-6.11) and only some of the waves appear to
have somewhat well-defined frequencies. Chequers and Widrow (2017) showed that
bending waves on the upper frequency branch are trailing and rotate prograde to that
of the disc while the lower branch corresponds to retrograde leading waves.

Rather astonishingly, we find the dominant waves to lie on these two branches for
both the isolated and satellite simulations. Therefore, the long-lived waves that the
subhalo collisions excite are not haphazard, but are the very same ones that arise
in the absence of halo substructure. This implies that the location of waves on the
$R$-$\omega$ plane is largely dictated by the structure of the disc, which is model dependent.
The $m = 1$ waves we observe in face-on maps of the galaxies (i.e. Figs. 6.8 and
6.9) manifest as a superposition of waves from the two frequency branches. Any
differences in bending wave morphology between the galaxies is attributed to the
relative weighting of the upper and lower branches.
6.9 Conclusions

In this paper, we explore the continual generation of bending waves by a system of satellites or dark matter subhaloes. (Though our focus here is on bending waves, we note that higher order waves, such as breathing waves, are also generated). In particular, we show that the long-lived bending waves that arise when the halo is clumpy are qualitatively similar to, though stronger in amplitude than, the waves that arise when the halo is smooth. Of course, the degree to which bending is excited depends on the characteristics of the subhalo distribution. Therefore, the study of vertical waves could provide insight into the properties of the MW’s subhalo population.

One of the main objectives of this paper is to develop analytic tools to study wave-like perturbations of the stellar disc, including those that might be observed by Gaia (See, for example, Gaia Collaboration et al. 2018b). Indeed, soon after the release of GDR2 Antoja et al. (2018) presented evidence of intriguing spiral structures in various phase space projections of the data. In that paper, the authors work under the assumption that the vertical waves are kinematic and generated by a single disc-substructure encounter. These assumptions allow them to ‘unwind’ the structure and infer an approximate epoch for when the perturbation arose.

Our simulations suggest a more complicated scenario. Consider first a single satellite that passes through the disc. Initially, the satellite transfers momentum to the disc stars thereby imprinting a local perturbation on the disc. An example can be found in Figs. 6.8 and 6.9. The perturbation is then sheared into roughly circular arcs and begins to act as a bending wave, which can propagate through the disc. Over time, energy in the wave migrates to the edge of the disc causing a warp. Energy is also dissipated into the random motions of the disc stars thus heating and
thickening the disc. Thus, the waves observed today will likely involve a superposition of kinematic waves from recent events and longer lived self-gravitating waves, which evolve on timescales much longer than kinematic phase wrapping.

Much of our analysis is based on the Fourier techniques developed by Sellwood and Athanassoula (1986) for the study of in-plane density waves, and later extended by Chequers and Widrow (2017) to the case of bending waves. The time series of bending waves in the $R - \omega$ plane (Fig. 6.13) supports our picture. In particular, at early times in the satellite simulation power is randomly distributed across the $R - \omega$ plane. These perturbations, which we interpret as a superposition of kinematic waves, quickly damp and shear. At late times, after most of the satellite interactions in the inner disc have occurred, the power is more closely aligned with the resonance curves indicating the existence of more organized bending waves.

Unsurprisingly, subhalo encounters excite bending waves to a larger degree than in the case of an isolated disc. However, the two cases display similar waves in that they tend to be tightly wound, morphologically leading, and rotate in a retrograde fashion compared to that of the disc, as is expected for neutrally stable bending waves (Sellwood 2013). Generally, the bending waves are dominant in the outer disc, and in particular just outside the edge of the thick disc ($R \sim 15$ kpc for our MW-like model). We attribute this to the increased (radial and vertical) velocity dispersion and self-gravity in the inner disc that act to resist bending (Debattista and Sellwood 1999).

A novel feature of our simulations is the inclusion of two dynamically distinct disc components. We find that vertical bending waves are significantly weaker in the kinematically hotter thick disc than the thin one. This observation has possible
ramifications for dynamical studies of the MW that assume vertical equilibrium. For example, the difference in response to compression and rarefaction (breathing wave) perturbations between the thin and thick discs could lead to systematic errors in an Oort-type analysis (e.g. Bovy and Rix 2013) for the vertical force and local surface density (see Banik et al. 2017).

The bending waves in our simulations are realized as spatial vertical displacements as well as bulk vertical motions with a phase offset (Gómez et al. 2013, 2016; Chequers and Widrow 2017; Gómez et al. 2017). We present a cross-correlation analysis between these two manifestations of bending and find that the waves are well described by simple plane waves, and therefore their rotational frequencies can be easily inferred from the amplitudes of spatial and velocity fluctuations as seen, for example, in Xu et al. (2015) and Schönrich and Dehnen (2017).

In summary, our simulations suggest that bending of the MW’s stellar disc can be understood as a superposition of waves, which lend themselves to various analysis tools such as linear theory, the WKB approximation, and spectral methods. The challenge is then to link theory and simulations of vertical waves with observations. The promise of Gaia and future projects such as the Large Synoptic Survey Telescope is that we will be able to make this linkage and gain a better understanding of the MW’s disc and the environment in which it lives.

6.10 Acknowledgements

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6.11 Appendix

6.11.1 Model parameters

In Table 6.1 we present the AGAMA model parameters used to construct the initial conditions of our Milky Way-like disc.
Table 6.1: The most sensitive AGAMA input file parameters for each component’s DF.

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<th>Component</th>
<th>Parameter</th>
<th>Value</th>
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<td>Central surface density $\Sigma_0$ [$M_\odot$ pc$^{-2}$]</td>
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<td></td>
<td>Exponential scale radius $R_d$ [kpc]</td>
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<td></td>
<td>Central radial velocity dispersion $\sigma_{R,0}$ [km s$^{-1}$]</td>
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<td></td>
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<td></td>
<td>Vertical velocity dispersion exponential scale radius $R_{\sigma_z}$ [kpc]</td>
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<tr>
<td>Thick disc</td>
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<td>Exponential scale radius $R_d$ [kpc]</td>
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Chapter 7

Summary and Conclusions
Throughout this thesis we have explored the dynamics of (vertical) density waves in disc galaxies using $N$-body simulations. The preceding chapters presented particular projects (publications) each with their own set of goals and conclusions, and were ordered in scenarios of increasing complexity to tell the story of this thesis in the most pedagogical way possible. We will now summarize and highlight the main conclusions of each chapter and unify them into a coherent set of research projects.

This thesis is mainly concerned with the dynamical formation and evolution of density waves in disc galaxies using theoretical tools. We particularly emphasized research on vertical bending waves in Milky Way-like (MW-like) discs, which was inspired by the exciting indications of these waves in the Solar Neighbourhood (SN; Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013). However, before diving into vertical (bending) waves, we started with a primer on more ‘classical’ in-plane density waves, such as bars, and introduced the main analysis tools extensively exploited throughout this thesis.

### 7.1 In-Plane Density Waves

In Chapter 3 we began the thesis by studying bar formation in a particular low surface brightness (LSB) galaxy, UGC 628. This chapter was included in the thesis to give an overview of the tools we use to study density waves in disc galaxies, namely Fourier and spectral analysis (Sellwood and Athanassoula 1986) and highlight interesting physics regarding in-plane density waves. The specific goals of the publication in Chapter 3 was to explore if LSB galaxies (that harbour a bar) are indeed dark matter dominated everywhere and answer how then can a bar form in such an environment? We used UGC 628 as an example since there exists comprehensive data
and measurements for this galaxy. This publication was borne out of a graduate class
tfinal project taught by Prof. Kristine Spekkens.

UGC 628 is quite unique for two main reasons. First, it is one of ~ 4 per cent of
LSB galaxies (Mihos et al. 1997) that harbours a bar (de Blok et al. 2001; de Blok and
Bosma 2002; Chemin and Hernandez 2009). This is very peculiar since LSB galaxies
are thought to be dark matter dominated in the inner disc (Bothun et al. 1997; de
Blok and McGaugh 1997; de Blok et al. 2001; Combes 2002; de Blok and Bosma
2002; Kuzio de Naray et al. 2008), and so any bar instability should be quelled by the
much larger mass density of the dark matter halo compared to that of the disc in that
region (see, for example, Mayer and Wadsley 2004). Second, the bar in UGC 628 is
‘snow’ (Chemin and Hernandez 2009) – that is, the length of the bar does not extend
to the corotation radius, past which the instability can no longer exist since it will
be sheared out by the differential rotation of the disc. In LSB galaxies, the slowness
of a bar is attributed to the fact that they are dark matter dominated, and therefore
dynamical friction from the halo (Chandrasekhar 1943; Mulder 1983; Weinberg 1985)
plays a significant role in the bar’s evolution (Chemin and Hernandez 2009). The
majority of bars in galaxies for which bar pattern speeds have been measured appear
to be ‘fast’ rotators (Rautiainen et al. 2008; Aguerri et al. 2015). Thus, UGC 628 is
one of only three galaxies, currently known, to possess a slow bar (see Bureau et al.
1999 and Banerjee et al. 2013).

We constructed $N$-body initial conditions (Kuijken and Dubinski 1995; Widrow
et al. 2008) of UGC 628 based on available photometric (Ahn et al. 2014) and rotation
curve (de Blok and Bosma 2002; Chemin and Hernandez 2009) data (Fig. 3.2). The
novelty of our model centered around deriving a surface density profile (see Fig. 3.1)
using a variable mass-to-light ratio profile computed from surface brightness profiles in multiple pass-bands (Into and Portinari 2013). Of course, discs are in general are comprised of multiple stellar populations and so there is no reason at all to believe a single mass-to-light ratio is valid for the entire disc. This surface density profile for UGC 628 served as the basis for our initial conditions model parameters. A variable mass-to-light ratio profile resulted in the disc having a steeper surface density profile and larger central density compared to previous studies (de Blok and Bosma 2002), and thus primarily contributed to the formation of a bar when the model was simulated. To the best of our knowledge, our simulation was the first to depict a bar forming in a LSB galaxy where the initial conditions were based on an actual galaxy. Indeed, a simulation of an exponential disc with parameters inferred by de Blok and Bosma (2002) had so little mass that it developed only flocculent spiral structure but no bar. This stemmed largely from the fact that de Blok and Bosma (2002) measured an exponential scale length using the outer part of the surface brightness profile (de Blok et al. 1995) and applied a constant mass-to-light ratio (see Table 3.1 for a list of parameters used to construct the models considered in this chapter).

Due to the increased surface density in the inner region of our initial conditions, it may not be surprising that our model formed a bar. However, our analysis implies that the situation is not so black and white and that the mass redistribution from the bar is also important. For example, our model had a disc mass fraction within 2.2 exponential scale lengths of $\sim 0.4$ throughout the simulation (Fig. 3.5d), and was therefore dark matter dominated according to that definition. However, contrary to the conclusion of Chemin and Hernandez (2009) that, given the low pattern speed of the bar, UGC 628 is dark matter dominated at all radii, our model suggests that
the disc mass fraction at 0.5 scale lengths (i.e. in the bar region) increased over time from $\sim 0.4$ to a value of $\sim 0.6$ at the onset of bar buckling. Thus our model of UGC 628 provides an example of the fact that, as argued by many authors in the past (Debattista and Sellwood 2000; Courteau and Dutton 2015), the disc mass fraction enclosed within the radius where its contribution peaks (2.2 scale lengths) is not a reliable indicator of its dynamical importance at all radii, and that pairing measurements at 2.2 scale lengths with those at smaller radii may be more informative.

This analysis leads to one of our main conclusions that the baryon dominated central regions of UGC 628 implied by our model distinguish it from most other LSB galaxies, where kinematics and photometry suggest a dark matter dominated inner disc. The disc in UGC 628 therefore has enough self-gravity to support a bar but also a sufficiently low central surface brightness ($\mu_{B,0} = 23.1$ mag arcsec$^{-2}$, Kim 2007) for the system to be classified as an LSB. Indeed, the rarity of bars in LSB galaxies (Mihos et al. 1997) suggests that this balance is a delicate one.

Our numerical measurements of bar pattern speed and the ratio of bar length to corotation radius at late times were largely in agreement with that measured by Chemin and Hernandez (2009), and therefore the bar was a slow rotator (see Figs. 3.5 and 3.6). Furthermore, based on comparisons between photometric projections (Fig. 3.3), rotation curves, and surface density profiles (Fig. 3.4) of our simulated disc at various times and UGC 628, we concluded that the bar in UGC 628 lies somewhere between peak strength and buckling in its evolution.

One of the key properties of our model is that the bar was birthed as a slow rotator (see Figs. 3.5 and 3.6). This contrasts with bars in most simulations which are born fast and can slow due to drag from dynamical friction (e.g. Debattista and Sellwood 2000; Courteau and Dutton 2015).
Therefore, similar to our conclusions about UGC 628 harbouring a bar at all, the disc in our model was dense enough to trigger a bar instability but at the same time lacked sufficient self-gravity to support a bar mode out to the corotation radius even before any braking from dynamical friction took place.

### 7.2 Fourier and Spectral Analysis

In addition to aiding in solving an interesting dynamical puzzle, Chapter 3 also served as an introduction to Fourier and spectral analysis, which played a central role in our analysis of vertical (bending) waves in later chapters. These methods, first laid out in the seminal work by Sellwood and Athanassoula (1986), formed the basis of analysis in deriving properties of the bar in our UGC 628 model, namely the bar pattern speed. From Chapter 3 it was clear that these techniques offer tremendous insight into density wave dynamics (see Figs. 3.5 and 3.6).

The main idea in Sellwood and Athanassoula (1986) is to decompose the surface density distribution into components of $m$-fold azimuthal symmetry terms as a function of radius – the total surface density is then approximated by summing up the contributions from all $m$ terms and truncating at some sufficiently large $m$. In particular, this allows one to pick out and focus on symmetries of interest, i.e., $m = 2$ for a bar and bisymmetric spiral structure, $m = 3$ for three-armed spiral structure, and so on.

If this is done for all time snapshots in a simulation then one can obtain a time series of the Fourier coefficients, allowing for the analysis of surface density as a function of radius, azimuth, time, and $m$. By (discrete) Fourier transforming this time series in time one can then consider the surface density waves as a function
of radius, azimuth, and rotational frequency. It is this spectral decomposition that is usually of greatest dynamical importance since it allows one to visualize an $m$-fold symmetric density distribution as a function of radius in the disc and rotational frequency.

As powerful as these methods are, they do come with some caveats. First, choosing a large time baseline to transform on gives superior frequency resolution but at the same time one ‘smears’ out waves on the radius-frequency plane – that is, any $m$-fold symmetric waveform that arises during the baseline will appear, at least to some degree, in the frequency power spectrum. Thus, this method is not particularly useful in determining instantaneous frequencies. This can somewhat be evaded by considering multiple Fourier transforms on chronological time baselines (they could even overlap). However, the cost of this is a decrease in frequency resolution. Nevertheless, the Fourier and spectral methods from Sellwood and Athanassoula (1986) allows one to gain tremendous insight into what, particularly long-lived, waves in the disc exist and where in the disc they transpire.

In our further development of these techniques we decompose two-dimensional (face-on) maps of vertical displacement and bulk vertical motion of the disc into its constituent $m$-fold symmetries. For the case of bending waves it is the $m = 1$ terms that are of utmost interest. A spectral analysis then gives information on the rotational frequency of vertical waves as a function of radius in the disc.

7.3 Vertical Waves in Milky Way-like Discs

An appreciable portion of this thesis is dedicated to researching vertical density waves in MW-like galaxies. In this Section we will summarize the chapters pertaining to this
topic. We begin with bending waves that seemingly arise spontaneously and progress through scenarios of increasing dynamical complexity to, finally, the (realistic) case of continual satellite bombardment from a system of halo substructure.

7.3.1 Bending Waves in Isolated Galaxies

In Chapter 4 we examined the evolution of two MW-like disc-bulge-halo systems evolved in isolation using $N$-body methods. The main goals of this chapter were to develop Fourier and spectral techniques in the context of bending waves and compare that analysis with linear theory, which in this case gave us physical insight into the nature of bending waves in the simulations. Although we focused on bending waves we note that breathing waves in our discs also arose.

The first model for the MW we considered was from Widrow et al. (2008) which was constructed to match kinematic and photometric observations of the MW (see Table 4.1 for the parameters used to construct our models). This model formed a relatively weak bar instability (see Fig. 4.2) at $\sim 2.5$ Gyr, and thus we labelled it the bar-forming model. We also considered a second model where we reduced the disc mass of the first model by roughly a third and increased the masses of the bulge and halo components to maintain a similar rotation curve to the bar-forming model (see Fig. 4.1). The disc in our second model was so light that it was stable to the formation of the bar for at least $\sim 10$ Gyr. Both of our disc models were evolved within a smooth dark matter halo. By considering two very distinct cases for driving the secular evolution of the discs (i.e., the presence of a bar) we were able to gain insight into the effect, if any, that a bar had on the evolution of bending waves.

It is there in Chapter 4 that we further developed Fourier and spectral techniques
(Sellwood and Athanassoula 1986) to the case of vertical bending waves (see equations 4.7 – 4.14). Using Fourier methods we constructed face-on maps of vertical displacement and found that bending waves rapidly formed in both of our models and were dominated by the $m = 1$ azimuthal Fourier component (Figs. 4.3 and 4.4).

At intermediate radii ($R \sim 8 - 15$ kpc for our MW-like galaxies) these waves developed into tightly wound and morphologically leading spirals that subtended large angles in azimuth. The corrugations also had an amplitude and radial wavelength that similar to those recently discovered in the MW (Xu et al. 2015). In contrast to the hypothesis of Xu et al. (2015) that the corrugated structures are associated with spiral arms, we found that they are independent, both in morphology and pattern speed.

The bending waves in our simulations persisted for many billions of years. Over time, the strength in bending migrated outward, and the corrugations at intermediate radii matched smoothly onto the warp that developed near the edge of the disc (see Figs. 4.3 and 4.5). Thus, the warp begins at $R \sim 10 - 15$ kpc with an amplitude of $\sim 200$ pc that increases to $\sim 300 - 500$ pc closer to the disc’s edge. These amplitudes are in agreement with recent measurements of the stellar warp in the MW (see for example Reylé et al. 2009, and references therein). The outward migration of power was more dramatic in our bar-forming model where the bar and/or more massive nature of the disc plays a dominant role in suppressing bending waves, especially at intermediate radii. Since our Galaxy has a central bar the implication is that bending wave corrugations are efficiently damped in the inner disc and should only be detected at Galactocentric radii greater than that of the Sun.

Upon a spectral decomposition of the $m = 1$ contribution to vertical displacement
(Fig. 4.6), we found that the frequency power lied along two main arcs, or branches, in the radius-rotational frequency plane. The upper branch corresponds to positive frequency, and therefore waves that are rotating prograde compared to that of the disc’s rotation. Furthermore, we found that these waves are morphologically trailing. In contrast, the lower branch corresponds to counter-rotating leading waves (see Fig. 4.7). In general, the waves that arise in face-on maps of the disc will be a superposition ‘frequency waves’ from the upper and lower branches. Since the frequency power in the lower branch is much greater than the upper, it is not surprising that we saw global patterns of bending that lead the rotation of the disc.

One of the key aspects of Chapter 4 was testing the connection between bending waves in three-dimensional $N$-body simulations and predictions borne out of linear perturbation theory (Hunter and Toomre 1969; Sparke 1984; Sparke and Casertano 1988, but see also Section 4.5 of this thesis). Following Sparke (1984) and Sparke and Casertano (1988) we divided a razor thin, dynamically cold disc into concentric rings and embedded it in a static halo. This disc was based on structural parameters of our bar-forming model. In this ‘$N$-ring’ model each ring is self-gravitating, thus when a ring is displaced from its midplane position it will impart a gravitational force on the other rings in addition to responding to the force from the rings. Using the second-order integro-differential equations for the local vertical displacement (Hunter and Toomre 1969), the problem then amounts to solving an eigenvalue equation for the frequency eigenmodes of the ring system (see quations 4.29 and 4.30). We found that the eigenfrequencies form a continuum that lie on two main branches in the radius-rotational frequency plane (Fig. 4.8). Furthermore, the locations of these branches in the linear ring model roughly coincided with predictions from the
WKB approximation (Binney and Tremaine 2008, section 6.6.1), where the waves are assumed to be tightly-wound. In particular, the eigenfrequencies were found to lie outside of the ‘forbidden region’ between the WKB vertical resonances (c.f. Fig. 4.8).

Quite remarkably, we found that the bending waves in our three-dimensional models were well-described by linear perturbation theory, especially during the early stages of the simulations (see Fig. 4.6). Of course, our $N$-body discs have a non-zero vertical velocity dispersion and thickness. Also, the subsequent evolution of our live models obviously differs from that in the linear regime, where dynamical friction from the live halo tends to efficiently damp bending waves on very short time-scales (Nelson and Tremaine 1995). Furthermore, as the disc evolves, in-plane and vertical (time-dependent) inhomogeneities and asymmetries emerge, driven by the bar, spiral structure, mass redistribution within the disc, and resolution effects, which the linear theory and WKB analysis does not incorporate. Nevertheless, power in the radius-frequency plane concentrated along the two resonances and lied, for the most part, outside the region between them (i.e., the forbidden region as predicted by WKB theory). Nevertheless, the implication of agreement between linear theory and our models was that the waves in our simulations were largely self-gravitating structures.

The self-gravitating nature of bending waves in our simulations contrasts with kinematic phase-wrapping explanations of the observed local bending and breathing waves in the SN as suggested by de la Vega et al. (2015). However, de la Vega et al. (2015) studied vertical waves that arose from satellite interactions with the disc and close flybys of a fairly massive Sagittarius dwarf model. Admittedly, we did not consider direct interactions between a disc and external perturbing agents. However, the vertical perturbations from de la Vega et al. (2015) do shear out into extended
ring-like structures similar in morphology to the bending waves we observed in our simulations (c.f. Fig. 4.3). Thus, it could very well be that the waves present at the early stages of a satellite encounter behave at least somewhat kinematically. Later on, when the perturbations phase-mix and shear out they can be described by a ring model, in which our analysis argues self-gravity is a key ingredient to the evolution of the (bending) waves.

It was a surprise (at least to us) that bending waves arose in a disc embedded in a smooth dark matter halo. This fact begs the question of what exactly is the origin of these perturbations? A likely culprit is the random noise of the halo and bulge particle distributions. To test this hypothesis, we re-simulated our bar-forming model with a fixed analytic potential for the bulge and halo and found that the vertical perturbations were negligible even though the disc developed spiral structure and a bar. As a further check, we evolved the live disc in a fixed potential that was generated by halo and bulge particles frozen in their initial positions. In this case, the disc developed a strong corrugation pattern of bending waves at intermediate radii and a large warp near the edge of the disc. Evidently, large-scale vertical waves can be excited by particle noise in the halo but not by structures in the disc itself, such as a bar or spiral arms.

Of course, dark matter haloes in Lambda cold dark matter cosmologies are predicted to be ‘clumpy’ and host a system of satellite galaxies and dark matter subhaloes (for example, see the early works of Klypin et al. 1999 and Moore et al. 1999). Using the shot noise in our simulations as a proxy for asymmetries and ‘clumpiness’ in haloes, it is tangible to imagine bending waves in MW-like (stellar) discs to be
continually excited, whether it be from direct interactions between the disc and sub-
haloes or, as we showed, long-range net torques imparted by the clumpy nature of
the host halo. Furthermore, the long-lived nature of bending waves in our simula-
tions is consistent with the conclusions of Binney et al. (1998) who showed that a
live halo responds to vertical displacements in the disc on relatively short time-scales.
In essence, bending waves are a phenomena of the disc-halo system. In the absence
of this coupling between the disc and live halo, dynamical friction would efficiently
damp the bending waves, as was suggested by Nelson and Tremaine (1995).

The bending waves observed in maps of vertical displacement in our simulations
also manifested in maps of bulk vertical motions (Fig. 4.3). This behaviour has been
observed in other simulations (Gómez et al. 2013, 2016, 2017) and is expected for
wave-like motion. Furthermore, Gómez et al. (2017) showed that vertical perturba-
tions in simulated disc galaxies are mirrored in both the gaseous and stellar compo-
nents. Thus, velocity corrugations should be observable in the line-of–sight kinematics
of nearly face-on external galaxies (see Figs. 4.9 and 4.10). Integral field spectroscopy
surveys hold great promise for detecting bending waves in external galaxies since they
gather spatially resolved information on both the stellar and gaseous components. To
this end, we also compiled a list of ‘minimum requirements’ needed to observe bending
waves in external galaxies using the results from our models. We found that bending
should be observable in external discs with inclinations as large as $\sim 20^\circ$. In addition
to an inclination constraint, a very large field of view, i.e. $\sim 4 - 5$ effective radii, is
needed to resolve corrugations in the outer disc. Furthermore, the spatial resolution
required to resolve corrugations similar to those found in our simulations should be
at least $\sim 2 - 3$ kpc.
7.3.2 Single Satellite Encounters

The work presented in Chapter 5 was motivated by the hypothesis of Widrow et al. (2012) that the bending and breathing wave motions of stars observed in the SN were the result of a recent interaction between the disc and a satellite galaxy or dark matter subhalo. As such, the aim of Chapter 5 was to explore such interactions in N-body simulations and, in particular, attempt to characterize the excitation of vertical waves (both bending and breathing) as a function of the satellite’s orbital parameters. We also presented a preliminary analysis of vertical waves that arose in a disc embedded in a halo with multiple orbiting subhaloes (Gauthier et al. 2006).

For the parent galaxy we chose the most stable disc-bulge-halo model for the MW from Widrow et al. (2008, but see also Table 5.1 for model parameters). We then considered a suite of N-body simulations with ‘light’ and ‘heavy’ models for the satellite. For each of these cases the satellite was initiated on prograde, retrograde, and polar orbits that intersected the disc at the Solar radius (R \sim 8 \text{kpc}). Then, for each of these orbits we further considered the vertical velocity of the satellite as is passed through the disc midplane being lesser and greater than that of the vertical velocity dispersion of disc particles.

To characterize the strength of bending and breathing waves in the disc we introduced a novel measure of wave strength (see equation 5.4). This measure effectively amounted to fitting the local distribution of particles in vertical phase space to a linear function. One could do this for multiple local volumes in the disc, perhaps on a grid, and therefore construct maps of wave strength as a function of position within the plane of the disc. This method was used in particular by Sun et al. (2015) in their analysis of LAMOST data. Of course, vertical waves correspond to a density in
phase space. Therefore, this linear fit measure of wave strength, especially the slope (breathing wave strength), is only approximate.

We found that in all of our simulations the passage of a satellite through the disc induced both bending and breathing waves in the disc (see Fig. 5.1). Over time the initial perturbations were sheared out into more extended arcs due to the differential rotation of the disc and persisted for many hundreds of Myrs. In particular, we found that perturbations were more robust and longer-lived for the case of massive satellites on prograde orbits (i.e. spin-orbit resonance, see Toomre and Toomre 1972). Furthermore, while both bending and breathing waves were excited by a given interaction, bending seemed to be stronger for the case of satellites with low vertical velocities relative to the disc stars, while breathing waves were excited to a greater degree when a satellite passed through the disc midplane more quickly.

We believe these results make physical sense since a slower moving satellite allows for the gravitational attraction the satellite imparts on the disc to more coherently bend the disc towards the satellite’s position. On the other, a faster satellite will attract disc particles on shorter time scales. Thus, as the satellite passes through the disc midplane disc particles feel a force that quickly changes in direction. The result is an expansion and compression of the disc since some fraction of the disc particles will continue in the direction that satellite entered the disc, and vice-versa for some other fraction of disc particles.

In Chapter 5 we also presented preliminary results for the case of multiple satellites incident on a disc. To this end, we analysed snapshots from the Gauthier et al. (2006) simulations in the context of vertical waves. Recall in that paper that their model M31 disc, when embedded in a smooth halo, was stable to the formation of a bar for at
least 10 Gyr and only formed weak and flocculent spiral structure. When 10 per cent of the halo mass was reinitialized in the form of orbiting halo substructure the disc formed greater amplitude spiral structure and a strong bar. Not surprisingly, bending and breathing wave perturbations were also found across the disc in this simulation (see Fig. 5.2). Prior to bar formation there were strong large-scale bending waves across the disc. The breathing waves had a somewhat smaller amplitude and varied on smaller scales. At later times, after the bar formed and the bulk of subhalo bombardment occurred, the bending wave perturbations diminished. Moreover, in the inner parts of the Galaxy, the breathing wave pattern mirrored that of the bar. Thus, while subhaloes may have triggered the formation of the bar, it is the bar that generates and maintains compression and rarefaction motions in the inner Galaxy. Of course, the Sun sits well beyond the region of the bar and it is therefore unlikely that the bar could cause the bulk motions observed in the solar neighbourhood (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013).

### 7.3.3 Towards a Realistic Scenario

In Chapter 6 we presented the ‘capstone’ $N$-body simulations of this thesis. There, we analysed bending wave phenomena that arose in a more realistic case of multiple orbiting subhaloes interacting with a MW-like disc and integrated all of the analysis and theory that was introduced and developed in previous chapters together. (We note that breathing wave perturbations were also induced). We also incorporated a two-component disc (thick and thin) into our model to test the response of these dynamically distinct populations to vertical perturbations. In addition to analysing a more realistic scenario for the generation of vertical waves, it was also one of the main
goals of this chapter to highlight the connections between bending waves as seen in vertical displacement as well as bulk vertical motions.

We used a method similar to Gauthier et al. (2006) to generate our system of subhaloes (or, satellite galaxies). That is, we first constructed an isolated disc-bulge-halo model. The parameters for the disc in this model were inspired by Bovy et al. (2012) and Bovy and Rix (2013). Parameters for the bulge and halo were chosen to yield a sub-maximal disc that does not form a bar or large amplitude spiral structure, which allowed us to isolate effects subhalo interactions had on the generation and evolution of vertical perturbations (see Table 6.1 for a list of the AGAMA model parameters). This model acted as the control experiment for the simulation with orbiting satellites and we labelled it our ‘isolated’ simulation (see Figs. 6.1 – 6.3 for properties of the model). We then constructed another model by replacing 10 per cent of the smooth halo mass in the isolated model with a system of 100 low-mass subhaloes (see Figs. 6.4 and 6.5 for properties of the subhalo system). We labelled this model as our ‘satellites’ simulation. Although the phase space distribution of subhaloes was not fully cosmological, it did allow for a heuristic study of vertical oscillations in a more realistic context than that of (particular) single subhalo encounters, which most authors have focused on in the past.

Over the course of our satellites simulation there are a few hundred disc-satellite interactions that occur (see Fig. 6.7). Not surprisingly, these interactions are not distributed uniformly in time (c.f. Fig. 6.6). There is an initial relatively short period of active bombardment, where the satellites initialized close to the disc fall to the centre of the galaxy potential. This is followed by a longer more quiescent phase, although significant disc-satellite interactions still do occur. In contrast to
the Gauthier et al. (2006) simulation, satellite interactions in our simulation do not trigger a bar instability in the disc (see Fig. 6.8), possibly due to the fact we only consider subhaloes with relatively low masses (c.f. Fig. 6.5).

Unsurprisingly, we found that subhalo encounters excited bending waves to a larger degree (by a factor of $\sim 5$) compared to the case of an isolated disc (see Figs. 6.9 and 6.10). However, the two cases display similar waves in that they tend to be tightly wound, morphologically leading, and rotate in a retrograde fashion compared to that of the disc (see Figs. 6.9 and 6.11). Generally, the bending waves were dominant in the outer disc, and in particular just outside the edge of the thick disc ($R \sim 15$ kpc for our Milky Way-like model). We attributed this to the increased (radial and vertical) velocity dispersion and self-gravity in the inner disc that act to resist bending (Debattista and Sellwood 1999).

Furthermore, we found that bending was significantly weaker (by a factor of $\sim 4-5$) in the kinematically hotter thick disc than the thin one (see Figs. 6.9 and 6.10). This has possible ramifications for dynamical studies of the MW that assume vertical equilibrium. For example, Bovy and Rix (2013) have suggested that to do an Oort type analysis (that is, to infer the surface density, vertical force, and density of dark matter in the disc) one should divide disc stars into bins based on [$\alpha$/Fe] and [Fe/H] – the idea being that the stars in each bin can be treated as a distinct isothermal tracer of the gravitational potential. On the other hand, if the disc is in a state of disequilibrium, and if the manifestation of disequilibrium varies from one population to the next, then population-dependent systematic errors might creep into this sort of analysis (see Banik et al. 2017 for the case of breathing waves). Put another way, discrepancies in the inferred potential might signal that the Galaxy is in a perturbed
As in Chapter 4, we once again turned to a spectral decomposition of $m = 1$ bending waves here to gain more insight into the difference in dynamics between the isolated and satellite simulations. Similar to Chapter 4, we found that the dominant long-lived bending waves lied mainly on or around two main branches in the radius-rotational frequency plane, which roughly corresponded with vertical resonances predicted from the WKB approximation and implies the waves were self-gravitating (c.f. Figs. 6.12 and 6.13). Again, the waves on the upper frequency branch are trailing and rotate prograde to that of the disc while the lower branch corresponds to retrograde leading waves.

Rather astonishingly, we found the dominant waves to lie on these two branches for both the isolated and satellite simulations. Therefore, the long-lived waves that the subhalo collisions excite were not haphazard, but are the very same ones that arose in the absence of halo substructure. This excitation of ‘intrinsic waves’ implies that the structure of waves in the radius-rotational frequency plane is largely dictated by the structure of the disc, which is model dependent. The $m = 1$ bending waves we observed in face-on maps of the galaxies (i.e. Fig. 6.9) manifest as a superposition of these ‘frequency waves’. Thus, any differences in bending wave morphology between our two models was attributed to the relative weighting of the upper and lower frequency branches.

In both of our control and satellite simulations the bulk vertical motions followed the same general patterns as the displacements (see Figs. 6.10 – 6.12) except for a phase offset. This behaviour is expected for wave-like motion and allows one to infer a rotational frequency from the respective amplitudes of each pattern. In Section 6.7
we presented a cross-correlation analysis between $m = 1$ vertical displacement and bulk motion and found that the waves are, for the most part, well described by simple monochromatic plane waves (see Fig. 6.14). Therefore, the rotational frequencies of the waves can be easily inferred from the amplitudes of spatial and velocity fluctuations as seen, for example, in the spatial corrugations and velocity ripples reported by Xu et al. (2015) and Schönrich and Dehnen (2017), respectively, observed in the SN.

We do note that for the case of the outer thin disc in our isolated galaxy simulation the cross-correlation analysis indicated that the displacement pattern leads the velocity, which is opposite to what is expected for wave-like motion. This behaviour is very intriguing since it possibly indicates non-linear dynamics and/or wave damping. Moreover, there is no reason to believe $m = 1$ velocity patterns should be related to the $m = 1$ displacement via a linear differential equation independent of higher order Fourier modes.

### 7.4 A New Outlook on Vertical (Bending) Wave Evolution

Perhaps the most important figure of this thesis is Fig. 6.13. Our interpretation of this figure incorporates all of the main ideas presented about vertical (bending) waves in this thesis into a new fundamental picture concerning the waves’ formation and evolution.

In Fig. 6.13 we show a sequence of $m = 1$ bending wave frequency power spectra over $\sim 2$ Gyr time intervals for the case of the thin disc in our simulations presented
in Chapter 6. For each of the time intervals in the isolated simulation, power lies along branches just outside the vertical resonance curves. The situation is more complicated in the case with satellites. At early times, there are numerous regions of power across the radius-frequency plane that are not particularly well correlated with the resonance curves. We posit that these waves are the kinematic waves studied by de la Vega et al. (2015). Over time, the positive frequencies diminish in power, although localized waves still exist, and the strength of low-frequency counter-rotating waves surges in strength and migrates outward. By the end of the simulation the branches of power roughly mirror that of the isolated simulation, though the relative weighting of the ‘frequency waves’ localized in radius is quite different. These results make intuitive sense. Bending waves in the control simulation involve the growth of linear perturbations, which are well described by a WKB analysis. The satellite-provoked waves, on the other hand, are more random in nature, though over time the most persistent ones also appear to coincide with predictions from linear theory.

The preeminent results of this thesis are epitomized in this figure, from which a new multi-phase framework for the excitation and evolution of vertical (bending) waves in MW-like disc galaxies emerges. First, consider a single satellite that passes through the disc. Initially, the satellite transfers a fraction of its orbital energy to the disc stars. Consequently, momentum is also transferred to disc stars and imprints a perturbation in phase space on a local region of the disc. The response of individual stars will depend on how well their vertical epicyclic motions match the time-dependence of the gravitational field (Sellwood et al. 1998; Widrow et al. 2014, but see also Chapter 5). It is in this preliminary stage of evolution that the perturbation behaves kinematically (de la Vega et al. 2015). The localized perturbation
disperses and is then sheared out into extended arcs due to differential rotation and phase-mixing (Widrow et al. 2014; de la Vega et al. 2015, but see also Chapter 5). Rather quickly, the sheared perturbation begins to behave as a tightly wound bending wave, with gravity providing the restoring force (i.e., the waves begin to self-gravitate, see Chapters 4 and 6), which can propagate through the disc. Over time, energy in the wave migrates to the edge of the disc causing a warp. Finally, energy is also dissipated into the random motions of the disc stars thus heating and thickening the disc (Lacey and Ostriker 1985; Toth and Ostriker 1992; Sellwood et al. 1998).

Thus, the life-cycle of bending waves induced by a satellite collision consists of three main phases: the collision generates an initial local perturbation that behaves kinematically, the local perturbation disperses and shears into roughly circular arcs quickly becoming self-gravitating, and the energy of the bending waves is dissipated causing the disc to thicken a little and become kinematically hotter.

For a realistic scenario with multiple disc-satellite collisions the situation becomes very complicated. The work in this thesis implies that bending wave features observed in astrometric surveys will comprise a superposition of kinematic waves from recent disc-satellite interactions, waves developing into self-gravitating structures, and long-lived waves from collisions that occurred in the distant past mixed with intrinsic oscillations of the disc. To make matters even more complex, satellite collisions will in general induce higher order vertical wave phenomena (i.e., breathing waves, see Widrow et al. (2014) and/or Chapter 5). We can only conjecture, at this point, that the evolution of breathing waves is similar to their bending wave counterparts.
CHAPTER 7. SUMMARY AND CONCLUSIONS

7.5 Future Work

In this thesis we mainly studied, albeit in great detail, bending wave phenomena in MW-like discs. Indeed, this is only one component to the perturbations observed in the SN (Widrow et al. 2012; Carlin et al. 2013; Williams et al. 2013). Thus, perhaps the most natural avenue for future research is to study the formation and evolution of breathing waves in $N$-body discs. Though only mentioned in Chapter 6, breathing waves also arise in our MW model with multiple orbiting substructure. Moreover, a preliminary analysis of that simulation indicates that the breathing waves present display differing oscillations in phase space. The spectral analysis techniques presented in this thesis could non-trivially be further extended to the study of breathing waves. In addition, it is also possible to further develop the linear perturbation theory in Chapter 4 to the case of breathing waves which might allow one to gain meaningful physical insight into the three-dimensional simulations, as was the case for bending waves, and answer questions such as are breathing waves also self-gravitating?

Another logical direction for research borne out of this thesis is exploring possible non-linear dynamics of wave motions. In Chapter 6 we only noted that we possibly saw evidence of this in the thin disc of our isolated MW-like simulation, where the patterns of vertical displacement led the velocity (see Section 6.7). In that chapter we postulated that this behaviour could possibly be explained by non-linear wave dynamics and/or damping. However, it is curious that we see this in an isolated two-component disc and not in the isolated single-component disc studied in Chapter 4. Perhaps there is an intricate interplay between various stellar populations. Nevertheless, in order to properly interpret any observations of vertical waves from astrometric surveys this mystery must investigated further.
One of the main goals of this thesis was to lay out tools and machinery one could use to analyse data from observing missions, such as \textit{Gaia}. Furthermore, we also wished to make predictions for what one might find hidden in that data. Indeed, our simulations predict we should see a complex, yet plentiful, assortment of bending (and perhaps breathing) wave perturbations in the MW disc. However, to really connect simulations of vertical waves with observations theorists must begin to study their simulations in the same manner we analyse astrometric data. Already in the early stages of the second \textit{Gaia} data release we see complex substructure in the phase space distribution of stars in the SN (Antoja et al. 2018). Furthermore, these observations indicate an interplay between patterns in the vertical phase space plane and in-plane motions, i.e., spiral-like structures in the \{z, v_R, v_\phi, v_z\} phase space. Thus, it is crucial that theorists track the evolution of the entire phase space in their simulations and examine how various perturbations might evolve in the phase space of more localized volumes.

We do note that this is no easy task. Apart from the particulars concerning tracking a volume as it rotates around the disc in simulations, there is also the complication of statistical noise. For example, Antoja et al. (2018) studied a sample of roughly one million stars in the immediate Solar vicinity. In comparison, our simulations contain only tens of thousands of particles \textit{in a ring} of width \(\sim 1\) kpc centred at the Solar radius. Thus, it is unclear exactly what useful physics can be learned by comparing such a low-resolution phase space distribution with the overabundance of astrometric data we now have access to (perhaps there is a clever way of increasing resolution around the Solar radius in simulations). Nevertheless, we believe this to be one of the key steps forward in connecting simulations with observations and will allow us
to explicitly track the imprints that perturbers, be it satellites or otherwise, leave in phase space. It is through this type of analysis that we can truly begin to study ‘galactoseismology’, and link perturbations with their origins.
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