COOPERATIVE DIVERSITY in WIRELESS TRANSMISSION:

Multi-Hop Amplify-and-Forward Relay Systems

by

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Abstract

A multi-hop, amplify-and-forward (AF), cooperative diversity system with K relays is studied. An accurate approximate expression for the symbol-error-rate (SER) is derived for the multi-hop system. Also, a lower bound for the outage probability of the system, that is tight throughout nearly the entire signal-to-noise ratio (SNR) range, is presented. Neither an SER expression nor an outage probability expression had previously been reported in the literature for the multi-hop system. To assist in the derivation of the SER expression, the cumulative density function (CDF), probability density function (PDF), and moment generating function (MGF) are found for the random variable (RV), $Z = X \cdot Y/(X + Y + c)$, where X and Y are RVs which have PDFs that are sums of terms of the form $x^n e^{-\beta x}$. It is shown that with the CDF, PDF, and MGF of this type of RV, it is possible to derive an expression for the SER of the multi-hop system for several important scenarios with respect to what type of fading is present in the channels of the system. To assist in the derivation of the lower bound of the outage probability, the CDF is found for an interesting new RV, presented in a recursive formula, that is used to represent the upper bound of the instantaneous end-to-end SNR of the multi-hop system. These mathematical results are useful beyond the scope of the multi-hop system researched in this thesis. Also, many of the results found in this thesis for the previously-scarcely-studied multi-hop sytem are shown to be generalizations of results that had been found for the previously-often-studied *two-hop*, AF, cooperative diversity system with K relays.

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Acronyms and Abbreviations

AF	amplify and forward
AWGN	additive white Gaussian noise
BER	bit error rate
bps	bits per second
BPSK	binary phase-shift keying
CDF	cumulative distribution function
CSCG	circularly symmetric complex Gaussian
CSI	channel state information
dB	decibels
DF	decode and forward
DSTBC	distributed space-time block codes
Hz	Hertz
LOS	line-of-sight
MGF	moment generating function
MIMO	multiple-input multiple-output
MISO	multiple-input single-output
ML	maximum likelihood
MRC	maximum ratio combining
PAM	pulse amplitude modulation
PDF	probability density function
QAM	quadrature amplitude modulation
RV	random variable

SER	symbol error rate
SIMO	single-input multiple-output
SISO	single-input single-output
SNR	signal-to-noise ratio

List of Important Symbols

$\approx$	approximately equal to
N_o	AWGN variance
$F_X(x)$	CDF of the RV X
$\mathcal{T}_c\{\cdot\}$	CDF transform
$\otimes$	CDF transform multiplication operator
$\mathcal{CN}(\mu, \sigma^2)$	complex Gaussian RV with mean μ and variance σ^2
$*$	convolution
$X \sim (\cdot)$	distribution – i.e., the RV X has distribution $(\cdot)$
$\arg\min_{s \in S} f(s)$	element s in the set S that minimizes the function $f(s)$
$\triangleq$	equal to by definition
$\equiv$	equivalent to
$\mathbb{E}[\cdot]$	expectation
$\mathbb{E}_X[\cdot]$	expectation with respect to the RV X
$\hat{\mathbb{E}}$	expectation operator
$Q(\cdot)$	Gaussian Q-function
$\lvert \cdot \rvert$	magnitude of a complex number
$\overline{X}$	mean of the RV X
$M_X(s)$	MGF of the RV X
$\mathcal{T}_m\{\cdot\}$	MGF transform
$\odot$	MGF transform multiplication operator
$K_\nu(\cdot)$	modified Bessel function of the second kind and order ν

I	mutual information
K	number of relays
P_o	outage probability
$f_X(x)$	PDF of the RV X
$P_b(E)$	probability of bit error
P_e	probability of error
$\mathbb{P}(A)$	probability of event A
$\mathbb{P}(A\|B)$	probability of event A given event B has occured
$P_s(E)$	probability of symbol error
ε_o	source transmit power
R	spectral efficiency
$U(\cdot)$	unit step function

Chapter 1

Introduction

1.1 Introduction

Communication over wireless channels is made difficult due to multipath scattering, which causes rapid fluctuations in the amplitude and phase of the signal received at the destination, a phenomenon known as fading [1]. Finding methods to combat this fading in order to communicate information reliably over a wireless channel has provided the motivation for much research work in this area and has led to many research publications, including this thesis.

Consider a typical model for the wireless channel. A single-input single-output (SISO) wireless channel is described using a discrete-time baseband equivalent model. Mathematically, the model is given as

$$y[m] = h[m] * x[m] + n[m] = \sum_{k=0}^{M} h[k]x[m-k] + n[m] \ , \tag{1.1}$$

where $x[m]$ is the transmitted symbol at time m, $y[m]$ is the received symbol, $h[m]$ is the channel impulse response, $n[m]$ is additive noise, and * denotes convolution [1, 2]. For a frequency nonselective, or flat fading channel, the impulse response has only one tap, $h[0]$, and the system equation becomes

$$y = hx + n \ , \tag{1.2}$$

where x, y, h, and n are all complex numbers, and the dependence on time can be suppressed in this case. For a Rayleigh fading channel with additive white Gaussian noise (AWGN), the channel coefficient h and the AWGN variable n are both zero-mean circularly symmetric, complex Gaussian (CSCG) random variables (RVs), $|h|$ is a Rayleigh RV, and $|h|^2$ is an exponential RV. The Rayleigh model represents a channel in which there is no direct, line-of-sight (LOS) component between the transmitter and the receiver. For a wireless channel *with* a LOS component, a Rician fading model is used. For Rician fading, h is a *non*-zero-mean CSCG RV, $|h|$ is a Rician RV, and $|h|^2$ is a non-central chi-squared RV with two degrees of freedom. Other system models, such as the Nakagami-m fading model, where $|h|$ is a Nakagami-m RV and $|h|^2$ is a Gamma RV, are also commonly studied. It is the Rayleigh fading channel model that will be used predominantly throughout this thesis.

To demonstrate the detrimental effects of the fading that is present in wireless channels, consider the following comparison between a channel with AWGN only and a channel with AWGN *and* fading. For a SISO system that uses binary phase-shift-keying (BPSK) modulation over an AWGN channel, it is well known that the symbol-error-rate (SER), also known as the probability of symbol error, $P_s(E)$, is given by $P_s(E) = Q(\sqrt{2\phi})$, where $\phi = \varepsilon_o/N_o$ is the receive signal-to-noise ratio (SNR), ε_o is the source's transmit power, N_o is the variance of n, that is, the noise power, and

$Q(\cdot)$ is the Gaussian Q-function. For high SNR, $P_s(E) \approx \frac{1}{2}e^{-\phi}$, showing that the error probability decays *exponentially* with SNR [1]. For the fading channel with AWGN, the error probability *given* the channel coefficient h is $P_s(E|h) = Q(\sqrt{2\gamma})$, where $\gamma = |h|^2\phi$ is defined as the instantaneous SNR. The average error probability is found by averaging this expression over the channel coefficient. For the Rayleigh fading case, at high SNR, using BPSK modulation, the SER [1, 3] is $P_s(E) \approx (4\phi)^{-1}$, showing that the error probability only decays *inversely* with SNR in this case. This result leads to the conclusion that the system performance is much poorer for the fading channel as compared to the AWGN channel.

In a commonly used technique to combat the detrimental effects caused by the fading, multiple copies of a symbol are transmitted over independent paths in order to reduce the probability of symbol error. When multiple, independent copies of a symbol are transmitted, it is said that the *diversity* of the system increases [1], where the mathematical definition of diversity will be given shortly. When multiple antennas are used at the transmitter and/or receiver, as is the case in multiple-input multiple-output (MIMO) systems, symbols are transmitted over multiple independent paths in *space*, and the diversity introduced is referred to as *spatial diversity.* As an example of the performance of such a system, if two receive antennas are used in a Rayleigh fading channel with AWGN, and BPSK modulation is used, then at high SNR the error probability [3] is $P_s(E) \approx (\frac{4}{\sqrt{3}}\phi)^{-2}$, which holds as long as there is sufficient spacing between antennas at the receiver to ensure that the two transmission paths are independent. Now the error probability decreases as the *square* of the inverse of the SNR, which is a significant improvement over the single-antenna case.

Mathematically, the diversity d can be defined as [3, eq. (3)]

$$d \triangleq - \lim_{\phi \to \infty} \frac{\log_{10} P_s(E)}{\log_{10} \phi} , \tag{1.3}$$

which is the negative of the slope of the $\log_{10} P_s(E)$ vs. $\log_{10} \phi$ curve, in the high SNR region. The interpretation of Equation (1.3) is that, for a system with diversity d, the SER will decrease as ϕ^d in the high SNR region, demonstrating that increasing the diversity of a system is a powerful method for decreasing the SER. A more fundamental, information theoretic approach to the analysis of the system leads to the concept of outage probability, P_o. With this approach, the diversity d is more accurately defined as [4, eq. (15)]

$$d \triangleq - \lim_{\phi \to \infty} \frac{\log_{10} P_o}{\log_{10} \phi} , \tag{1.4}$$

The concept of outage probability is discussed in detail in Chapter 3. For a MIMO system with n_T transmit antennas and n_R receive antennas, the diversity order [2] can be as high as $d = n_T \cdot n_R$. The substantial reduction in SER that is achieved by increasing the diversity is the reason why so many researchers have tried to design and analyze systems that provide high diversity gains.

Much research has been done on the topic of MIMO systems due to the potential of high diversity gains that they provide. However, many practical devices are too small to allow for the use of multiple antennas on a single device, and MIMO systems cannot be used for these situations. For that reason, recently many researchers have studied a relatively new concept, called *cooperative diversity*, in which single-antenna sources transmit their information to other single-antenna users, as well as to a destination, and the other users in turn relay the symbols that they receive to the destination. This has the effect of providing spatial diversity for the system, since the source's symbols are transmitted over multiple paths in space ($K+1$ paths for the case where K relays are used), similar to the MIMO system. This increase in diversity once again provides a substantial decrease in the SER. An important difference between the cooperative diversity system and the MIMO system, however, is that the relays in the cooperative diversity system only have noisy copies of the source's symbols to forward to the destination, as opposed to the actual symbols themselves. For this reason, the performance of the cooperative diversity system will always be upper bounded by the performance of a multiple-input single-output (MISO) system that uses the same number of transmit antennas.

Many cooperative diversity systems with different numbers of relays, different transmit protocols, different levels of channel-state information (CSI) available at the receivers and the transmitters, and different combining techniques at the receivers have been designed, considered, and analyzed. Two important protocols that are very often used are the amplify-and-forward (AF) and the decode-and-forward (DF) protocols first introduced in [5]. In the AF scenario, the relays multiply the noisy copies of the symbols that they receive from the source by an amplifying gain, and then forward the amplified versions of the noisy symbols to the destination. (When the relays transmit the same (or a noisy version of the same) symbol that the source transmits, the scheme is known as *repetition-based* transmission.) In the DF scenario, the relays first decode the symbols that they receive from the source, and then forward the estimate of the symbol to the destination. Furthermore, it is common for cooperative diversity systems to use an *orthogonal* transmitting scheme. For example, when the transmitting scheme is orthogonal in time, as will be the case for the systems analyzed in this thesis, each device (source or relay) is allotted its own time slot in which it is the only device that transmits (thereby making all transmitted signals orthogonal to one another). For a system that uses K relays, $K + 1$ time slots are used to transmit each symbol. In the first time slot, the source transmits its symbol to the K relays and the destination, and in the next K consecutive times slots, each relay takes a turn transmitting its version of the symbol. Also, another issue to consider is that the relays can transmit either to the destination only, or to the destination *and* to the other relays that have not yet transmitted. In this thesis, a system in which the relays transmit only to the destination is called a *two-hop* system, whereas a system in which the relays also transmit to other relays is called a *multi-hop* system.

1.2 Motivation and Thesis Overview

In this thesis, multi-hop, AF, cooperative diversity systems with K relays, that use repetition-based, time-orthogonal transmission and maximum-ratio-combining (MRC) at the receivers (which are assumed to have complete CSI), are analyzed. While a lot of research has previously been done on the two-hop system, very little work has been done on the multi-hop system prior to the writing of this thesis. In fact, in [6, p. 223], during a discussion on the performance analysis of the multi-hop system, the authors state that, "The SER analysis of the [multi-hop] protocol is very complicated and a close-form analysis is not tractable." An analysis of the outage probability of the multi-hop system is not even attempted in [6], which was published very recently (2009). This provided the motivation to research this system and develop analytical tools for it, such as accurate, approximate SER and outage probability expressions.

In [6], a thorough discussion of the two-hop system (called the source-only system in that publication) was presented, where many of the previous results for that system were summarized. The discussion included the system model and description, the instantaneous SNR expressions, the SER analysis, the mutual information and outage probability results, and optimal power allocation schemes. An informative discussion describing some scenarios in which the multi-hop system (called the MRC-based system in [6]) outperforms the two-hop system (typically when the relays are close together and/or close to the source), and some scenarios in which the two-hop system performs better than, or at least nearly as well as, the multi-hop system (typically when the relays are close to the destination for which case the problem of noise propagation diminishes the performance of the multi-hop system), was also included. Also, a brief discussion of the multi-hop system was included. For the multi-hop system, the system model and description, and the instantaneous SNR expressions were given. However, as noted earlier, the SER and outage probabilities were *not* analyzed. In [7], an approximate SER expression was derived for the special case of the multi-hop system where the number of relays was limited to $K = 2$ and a relay ordering algorithm was used. To the best of our knowledge, this thesis will be the first paper to consider the SER analysis for a general number of relays, K, for the multi-hop system, and also the first to consider the outage probability analysis for the multi-hop system.

In Chapter 2, *the SER analysis of the multi-hop system* is carried out. Then, in Chapter 3, *the outage probability analysis of the multi-hop system* is carried out. In both chapters, some interesting new mathematical results that are used in the analyses are presented. It will be seen that many of the results for the multi-hop system presented in this thesis are generalizations of previously found results for the two-hop system. Furthermore, more detailed introductory material, including in-depth literature surveys, will be included in Chapter 2 for the SER analysis and in Chapter 3 for the outage probability analysis. Concluding remarks for the overall thesis are given in Chapter 4.

1.3 Thesis Contribution

The primary contributions of this thesis are summarized as follows:

- The cumulative distribution function (CDF) and the probability density function (PDF) are derived for the random variable (RV), $Z = X \cdot Y/(X + Y + c)$, where the PDFs of the RVs, X and Y, are given by sums of terms of the form $Cx^n e^{-\beta x}$, where C is a constant, n is an integer, β is a positive real number, and c is also a constant. In particular, these results are valid when X and Y are sums of any number of exponential and/or Erlang RVs (Gamma RVs with shape parameters restricted to be integers). These results are a generalization of previously published results for some special cases of Z.

- An accurate, approximate SER expression is found for the multi-hop AF relay system (for any number of relays) that uses repetition-based transmission and MRC at the receivers. (The results for the CDF and PDF of Z are used to find the SER expression.)

- An important new RV that is expressed in a recursive formula, and which is used to represent the upper bound of the end-to-end instantaneous SNR of the multi-hop system, is introduced and its CDF is found.

- A lower bound of the outage probability of the multi-hop system, which is very tight to the actual outage probability throughout most of the SNR range for most practical system configurations, is found. (The new RV representing the upper bound of the end-to-end SNR is used to find the lower bound of the outage probability.)

Chapter 2

SER and PDF Analysis for Multihop Relay Systems

In this chapter, an amplify-and-forward, multi-branch, multi-hop relay system with K relays, in which the relays broadcast to other relays, as well as the destination, is analyzed. The exact end-to-end instantaneous signal-to-noise ratio (SNR) of the system is found. A novel, conceptual approach is used to develop an approximate expression for the SNR, one which can be used in the derivation of a symbol-error-rate (SER) expression. Also, the cumulative distribution function, probability density function, and moment generating function are found for the random variable $Z = XY/(X + Y + c)$, where X and Y are sums of independent, exponential random variables (RVs) and/or Erlang RVs, and c is a constant. It is demonstrated that these results are generalizations of previously published results for special cases of Z. An approximate expression for the SER of the multi-hop system is found by using the approximate SNR expression, which contains random variables in the form of Z, and by using the functions that were found for Z. It is shown that the approximate SER expression is very accurate throughout the entire SNR range, for a practical choice of power allocation between the source and relays and for a typical choice of channel variances (all equal), as compared to the SER obtained from simulations.

2.1 Introduction

A wireless communications system with multiple single-antenna radios can achieve spatial diversity, called cooperative diversity, by having the radios relay each other's information to the desired destinations. It has been shown that a system that uses cooperative diversity has a higher capacity than one in which the users do not cooperate with one another [14, 15]. In [5], Laneman *et al.* developed several protocols for the cooperative diversity network, including the amplify-and-forward (AF) protocol, in which the relays simply amplify the noisy symbols that they receive and then re-transmit them. It was shown in [5] that an AF relay system with one relay provides full transmit diversity order of two (the two transmit antennas being the source and the relay). The fact that user cooperation systems can provide full diversity and have an increased capacity, when compared to wireless systems that do not use cooperation, provides a strong motivation for researching these systems. The user cooperation system that will be discussed in this chapter uses an AF scheme in which the source broadcasts its symbol to K relays and the destination and, in turn, the K relays amplify the noisy copies of the symbol that they receive and then broadcast them to other relays and the destination. The relays take turns broadcasting, one at a time, over K consecutive time slots.

AF relay systems have been analyzed in many previous publications for various numbers of relays and various configurations of *branches* and *hops*, where the number of branches refers to the number of parallel paths from the source to the destination, and the number of hops refers to the number of serial jumps along a branch. (For example, a source-to-relay-to-destination transmission path is a single branch with two hops). Single-branch two-hop networks were considered in [16]–[18], multi-branch two-hop networks were considered in [19], and multi-branch multi-hop networks were considered in [20]–[7]. The system analyzed in this chapter consists of a multi-branch multi-hop network with K relays. When relay 1 transmits in time slot 1, it broadcasts to relay $2, \ldots,$ relay K, and the destination; when relay 2 transmits in time slot 2, it broadcasts to relay $3, \ldots,$ relay K, and the destination; and so on. Therefore, the number of hops along the paths ranges from 1, for the direct source-destination path, to anywhere from 2 to $K + 1$ for the paths that go through the relays. The network considered in [19] also uses K relays in an AF system, but differs in that the relays transmit only to the destination and *not* to other relays, thereby making it a multi-branch *two-hop* system. The system in [21] uses K relays that broadcast to other relays as well as the destination, but uses decode-and-forward (DF) protocols, which were also presented in [5], as opposed to an AF protocol. The system that uses what Boyer *et al.* call *multi-hop channels with diversity* in [22] is actually identical to the system considered here, as are the systems in [23, 7].

The motivation for studying the multi-branch multi-hop system, in which the relays broadcast to other relays and not just the destination, is provided by the fact that an improvement in performance can be achieved when compared to systems in which the relays transmit only to the destination, such as those in [19] and many other publications. The fact that antennas are omnidirectional and naturally broadcast to all other terminals in the system means that the signals that the relays transmit are made available at the other relays freely, without any additional costs in terms of power requirements. Since, in the multi-hop system, the relays are now using these extra copies of the transmitted symbols that come from other relays, instead of ignoring them as was the case for the multi-branch two-hop system, the performance of the multi-hop system will improve (at least in most cases) as compared to the two-hop system. Since the relays now have multiple copies of the symbols that they receive, they will combine them, as the destination does, by using maximum-ratio-combining (MRC).

In [23], where the multi-hop system that is being analyzed in this thesis was also considered, the system equations and an expression for the exact instantaneous signal-to-noise ratio (SNR) were developed. In [7], two relay-ordering schemes were developed for the multi-hop system, and approximate expressions for the symbol error rate (SER) were found for the special case where the system uses a relay-ordering scheme and the number of relays is restricted to $K = 2$. To the best of our knowledge, however, a general approximate SER expression for an arbitrary number of relays, K, has never been developed for this system. That provided the motivation for the work in this chapter, where an accurate approximate expression for the SER of the multi-hop system, for arbitrary K, is indeed developed.

An important aspect that arises during the analysis of the multi-hop system is that the instantaneous SNR expressions that are considered and used to calculate the SER contain random variables (RVs) of the form $Z = XY/(X + Y + c)$, where X and Y are sums of exponential RVs, and c is a constant. In this chapter, for the most general case where X and Y are sums of arbitrary numbers of independent exponential RVs with no limitations on their means (when RVs with identical means are present, then X and Y become equivalent to sums of Erlang RVs), the cumulative distribution function (CDF) and the probability density function (PDF) of Z for any value of $c \geq 0$, as well as the moment generating function (MGF) of Z for the special case where $c = 0$, are derived. These functions are then used in the process of deriving an SER expression for the multi-hop system.

While the derivations of the CDF, PDF, and MGF of Z are a byproduct of the primary objective of this chapter, which is to find the SER of the multi-hop system, these results are significant in their own right and they are valuable beyond the scope of the system considered in this thesis. Indeed, RVs of the form given by Z have been analyzed in many other publications including [5], [16]–[19], [24], and [37], since SNRs of this form arise in various different relay systems. However, in all other publications, special cases for Z were considered, whereas in this paper, the most general case for Z is considered. Similar derivations as those carried out in this paper were carried out in [17]–[19] for the special case where X and Y are both single exponential RVs; in [24] for the special case where all of the RVs in the summation of X have identical means, as do all of the RVs in the summation of Y, making X and Y to be single Erlang RVs; in [16] for the case where X and Y are single Gamma RVs and $c = 0$; and in [37] for the case where X and Y are single Erlang RVs (Gamma RVs with shape parameters restricted to being integers) and $c = 0$. The results in this paper, concerning the CDF, PDF, and MGF of Z, are a generalization of the results in [17]–[19], [24] and [37].

The rest of the chapter will be organized in the following manner: In Section 2.2, the model for the multi-hop system is described and the system equations are developed. The exact instantaneous end-to-end SNR of the system, as well as approximate

SNR expressions that are simple enough to make the SER calculations manageable, are developed in Section 2.3. Expressions for the CDF, PDF, and MGF of Z are found in Section 2.4. An approximate SER expression for the system is developed in Section 2.5. Numerical results, with plots showing that the approximate SER expression is very accurate when compared to the SER results obtained by Monte Carlo simulations, are presented in Section 2.6. Concluding remarks are given in Section 2.7.

2.2 System Model

The block diagram of the system is shown in Fig. 2.1. A discrete-time, baseband equivalent model will be used to describe this system. The source transmits symbols to the destination with the help of the K relays, using an AF protocol over $K+1$ time slots (one for the source and one for each of the relays), as described in the introduction. All channels are Rayleigh fading channels with additive white Gaussian noise (AWGN). The channel coefficient for the channel from transmitting terminal i to receiving terminal j is denoted by $h_{i,j}$ where $i = s$ for the source; $i = 1, 2, \ldots, K$ for relay i; $j = 1, 2, \ldots, K$ for relay j; and $j = d$ for the destination. Channel coefficient $h_{i,j}$ is a zero-mean complex Gaussian RV with variance $\Omega_{i,j}/2$ per dimension, and this distribution is denoted by $h_{i,j} \sim \mathcal{CN}(0, \Omega_{i,j})$. Since $h_{i,j}$ is complex Gaussian with variance $\Omega_{i,j}$, it follows that $|h_{i,j}|^2$ is an exponential RV with a mean of $\Omega_{i,j}$. The channels are considered to be slow fading, so that the coefficients are constant over the time duration (called a time slot) of one symbol. The AWGN component at terminal j, added to the signal transmitted from terminal i, is denoted by $n_{i,j}$. It is also a zero-mean complex Gaussian RV, its variance per dimension is $N_{i,j}/2$, and its

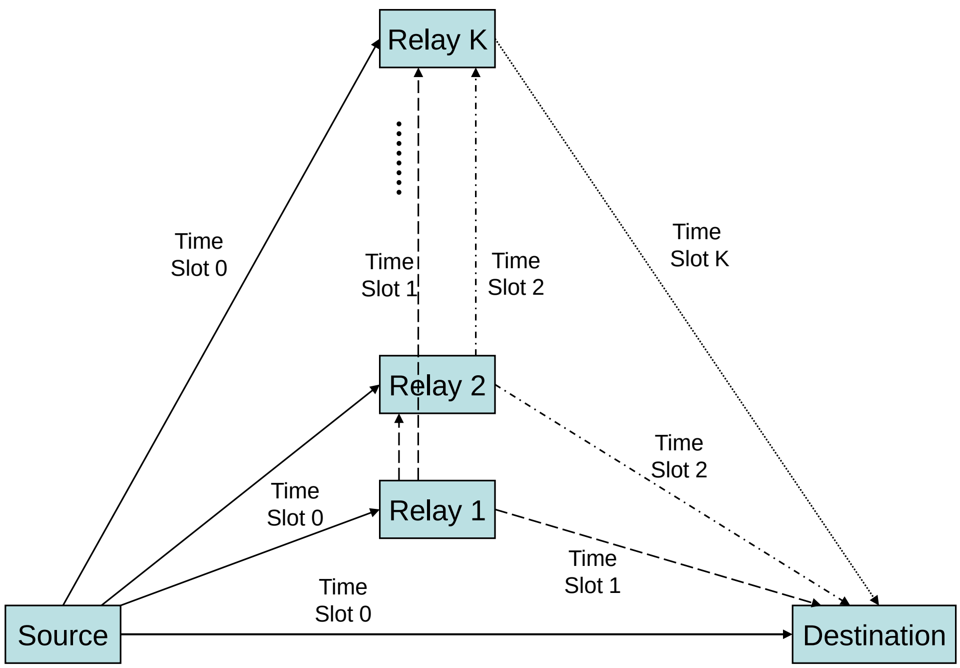

Figure 2.1: Block Diagram of the Multi-Hop System. The terminals involved, the signals that they transmit, and the order in which they transmit them are shown. Each point-to-point channel from terminal i to terminal j is a Rayleigh fading channel with fading coefficient $h_{i,j}$ and additive noise $n_{i,j}$.

distribution is denoted by $\mathcal{CN}(0, N_{i,j})$. (There is a subtle, but important, difference between $n_{i,j}$ and $h_{i,j}$ that needs to be pointed out. While they are both RVs, $h_{i,j}$ is considered to be constant throughout the time slot, as was mentioned earlier, whereas $n_{i,j}$ is always considered to be a RV.) The channel coefficients and AWGN signals are all mutually independent.

In time slot 0, the source broadcasts the symbol $\sqrt{\varepsilon_o}x$, where ε_o is the average source power, $x \in S$, and S is the constellation or, in other words, the set of possible signals for the source to transmit. The relays ($j = 1, 2, \ldots, K$) and destination ($j = d$) receive the signals $y_{s,j} = \sqrt{\varepsilon_o}h_{s,j}x + n_{s,j}$ for $j = 1, 2, \ldots, K, d$. It is convenient to define the parameters, $\beta_{s,j} = \sqrt{\varepsilon_o}h_{s,j}/N_{s,j}$ for $j = 1, 2, \ldots, K, d$. Then the received signals can be expressed as $y_{s,j} = N_{s,j}\beta_{s,j}x + n_{s,j}$.

In time slot 1, relay 1 broadcasts $x_1 = \alpha_1 y_{s,1}$ to terminals $j = 2, \ldots, K, d$, where α_1 is chosen so that the average power used by relay 1 is equal to ε_1, a predetermined desired value of power. The average transmit power of relay 1 is $\varepsilon_1 = \mathbb{E}\left[|x_1|^2\right] = \alpha_1^2\,\mathbb{E}\left[|y_{s,1}|^2\right] = \alpha_1^2(|h_{s,1}|^2\varepsilon_o + N_{s,1}) = \alpha_1^2(N_{s,1}^2|\beta_{s,1}|^2 + N_{s,1})$, where $\mathbb{E}\left[\cdot\right]$ denotes an expectation. The amplifying gain, α_1, is given by

$$\alpha_1 = \sqrt{\frac{\varepsilon_1}{|h_{s,1}|^2\varepsilon_o + N_{s,1}}} = \sqrt{\frac{\varepsilon_1}{N_{s,1}^2|\beta_{s,1}|^2 + N_{s,1}}}. \tag{2.1}$$

The relays and destination receive $y_{1,j} = h_{1,j}x_1 + n_{1,j} = \alpha_1 N_{s,1}\beta_{s,1}h_{1,j}x + \eta_{1,j}$ for $j = 2, \ldots, K, d$, where the total received noise signal at terminal j is $\eta_{1,j} = h_{1,j}\alpha_1 n_{s,1} + n_{1,j}$. It will again be convenient to define the parameters, $\beta_{1,j} = \alpha_1 N_{s,1}\beta_{s,1}h_{1,j}/\mathcal{N}_{1,j}$ for $j = 2, \ldots, K, d$, where $\mathcal{N}_{1,j} = \mathbb{E}[|\eta_{1,j}|^2] = \alpha_1^2|h_{1,j}|^2 N_{s,1} + N_{1,j}$ is the variance of the noise signal $\eta_{1,j}$. The received signals can now be expressed as $y_{1,j} = \mathcal{N}_{1,j}\beta_{1,j}x + \eta_{1,j}$.

In time slot j, for $j = 2, \ldots, K$, relay j broadcasts to terminals $k = j+1, \ldots, K+1$, where the destination is represented here by $k = K + 1$, instead of $k = d$, for convenience. Before transmission, relay j first combines the signals that it has received using MRC. At time slot j, relay j has received $y_{s,j} = N_{s,j}\beta_{s,j}x + n_{s,j}$ and $y_{i,j} = \mathcal{N}_{i,j}\beta_{i,j}x + \eta_{i,j}$ for $i = 1, 2, \ldots, j-1$. The signal obtained after combining is $y_j = \sum_{i=0}^{j-1} \xi_{i,j} y_{i,j} = \sum_{i=0}^{j-1} \beta_{i,j}^* y_{i,j}$, where the source is represented here by $i = 0$, instead of $i = s$, for convenience. The MRC coefficient $\xi_{i,j}$ is the ratio of the conjugate of the signal component coefficient to the noise power, that is, $\xi_{s,j} = N_{s,j}\beta_{s,j}^*/N_{s,j} = \beta_{s,j}^*$, and $\xi_{i,j} = \mathcal{N}_{i,j}\beta_{i,j}^*/\mathcal{N}_{i,j} = \beta_{i,j}^*$. Substituting the equations for $y_{s,j}$ and $y_{i,j}$ into the equation for y_j yields $y_j = \left(N_{s,j}|\beta_{s,j}|^2 + \sum_{i=1}^{j-1} \mathcal{N}_{i,j}|\beta_{i,j}|^2\right) x + \left(\beta_{s,j}^* n_{s,j} + \sum_{i=1}^{j-1} \beta_{i,j}^* \eta_{i,j}\right)$. Then relay j broadcasts $x_j = \alpha_j y_j$ to terminals $k = j+1, \ldots, K+1$, where α_j is chosen so that the average power used by relay j is ε_j. The transmit power of relay j is

$$\varepsilon_j = \mathbb{E}\left[|x_j|^2\right] = \alpha_j^2 \left\{ \left(N_{s,j}|\beta_{s,j}|^2 + \sum_{i=1}^{j-1} \mathcal{N}_{i,j}|\beta_{i,j}|^2 \right)^2 + \mathbb{E}\left[\left| \beta_{s,j}^* n_{s,j} + \sum_{i=1}^{j-1} \beta_{i,j}^* \eta_{i,j} \right|^2 \right] \right\}. \tag{2.2}$$

The amplifying gain used for relay j is $\alpha_j = \sqrt{\frac{\varepsilon_j}{A_j^2 + A_j}}$, where $A_j = N_{s,j}|\beta_{s,j}|^2 + \sum_{i=1}^{j-1} \mathcal{N}_{i,j}|\beta_{i,j}|^2$. The received signals at terminals $k = j+1, \ldots, K+1$ are $y_{j,k} = h_{j,k}x_j + n_{j,k} = h_{j,k}\alpha_j A_j x + \eta_{j,k}$, where the noise signals are given by $\eta_{j,k} = h_{j,k}\alpha_j \cdot \left(\beta_{s,j}^* n_{s,j} + \sum_{i=1}^{j-1} \beta_{i,j}^* \eta_{i,j}\right) + n_{j,k}$. By letting $\beta_{j,k} = \alpha_j A_j h_{j,k}/\mathcal{N}_{j,k}$, where $\mathcal{N}_{j,k} = \mathbb{E}[|\eta_{j,k}|^2]$, the received signals can then be expressed as $y_{j,k} = \mathcal{N}_{j,k}\beta_{j,k}x + \eta_{j,k}$.

2.3 Exact and Approximate Instantaneous End-to-End SNR Expressions

Now that the transmitted signals, noise signals, and received signals have been determined for all terminals in the system, the next step in the analysis is to determine the instantaneous end-to-end SNR, or the overall SNR of the system *given* the channel coefficients. After time slot K, the destination has received the following signals: $y_{s,d} = N_{s,d}\beta_{s,d}x + n_{s,d}$ and $y_{m,d} = \mathcal{N}_{m,d}\beta_{m,d}x + \eta_{m,d}$ for $m = 1, 2, \ldots, K$. The destination combines the signals using MRC to obtain $y_d = \beta^*_{s,d}y_{s,d} + \sum_{m=1}^{K}\beta^*_{m,d}y_{m,d} = A_d x + \eta_d$, where $A_d = N_{s,d}|\beta_{s,d}|^2 + \sum_{m=1}^{K}\mathcal{N}_{m,d}|\beta_{m,d}|^2$ and the overall noise signal component is $\eta_d = \beta^*_{s,d}n_{s,d} + \sum_{m=1}^{K}\beta^*_{m,d}\eta_{m,d}$. The estimate, $\hat{x}$, of the transmitted signal is obtained using maximum likelihood (ML) detection. That is, $\hat{x} = \arg\min_{s\in S}|y_d - A_d s|$.

Theorem 1 ([23])**.** The exact, instantaneous, end-to-end SNR, Γ, is

$$\Gamma = \frac{A_d^2}{\mathcal{N}_d} = \frac{\left[N_{s,d}|\beta_{s,d}|^2 + \sum_{m=1}^{K}\mathcal{N}_{m,d}|\beta_{m,d}|^2\right]^2}{\mathcal{N}_d}. \tag{2.3}$$

To use Equation (2.3), it is necessary to find expressions for $\mathcal{N}_{m,d}$ and $\mathcal{N}_d$. The variance, $\mathcal{N}_{m,k}$, of the total noise signal at terminal k, due to transmission from terminal m, is given by

$$\mathcal{N}_{m,k} = \mathbb{E}[|\eta_{m,k}|^2] = N_{m,k} + \sum_{j=1}^{m}\sum_{i=0}^{j-1}|\lambda(i,j,m,k)|^2 N_{i,j} \tag{2.4}$$

for $m = 1, \ldots, K$, and $k = m+1, \ldots, K+1$. The $\lambda(i,j,m,k)$ parameters are given by

$$\lambda(i,j,m,k) = \begin{cases} \sum_{l=j}^{m-1} \mu_{l,m}^{k} \lambda(i,j,l,m) & \textit{for} \quad j = 1,\ldots,m-1, \\ \\ \mu_{i,m}^{k} & \textit{for} \quad j = m, \end{cases} \tag{2.5}$$

where $k = m+1,\ldots,K+1$ and $i = 0,\ldots,j-1$. The $\mu_{i,m}^{k}$ parameters are defined as $\mu_{0,1}^{k} = h_{1,k}\alpha_1$ for $m = 1$ and $k = 2,\ldots,K+1$; and $\mu_{i,m}^{k} = h_{m,k}\alpha_m\beta_{i,m}^{*}$ for $m = 2,\ldots,K$; $k = m+1,\ldots,K+1$; and $i = 0,1,\ldots,m-1$. As before, it is convenient here to let $i = 0$ replace $i = s$ and $k = K+1$ replace $k = d$.

Also, the variance, $\mathcal{N}_d$, of the overall noise signal at the destination is given by

$$\mathcal{N}_d = \mathbb{E}[|\eta_d|^2] = |\beta_{s,d}|^2 N_{s,d} + \sum_{m=1}^{K} |\beta_{m,d}|^2 N_{m,d} + \sum_{m=1}^{K}\sum_{j=1}^{m}\sum_{i=0}^{j-1} |\beta_{m,d}|^2 |\lambda(i,j,m,d)|^2 N_{i,j}. \tag{2.6}$$

Proof. See Appendix A.1.[1] □

It is worth mentioning here that by letting $h_{i,j} = 0$ for $1 \leq i < j \leq K$, the system reduces to the multi-branch *two-hop* system that was analyzed in [19]. It was shown in [23] that by setting $h_{i,j} = 0$ for $1 \leq i < j \leq K$ in the system equations, the end-to-end SNR expression given in Equation (2.3) reduces to the end-to-end SNR expression given in previous publications (see [17], [19], [20]) for the multi-branch two-hop system. Therefore, the result for the SNR given in Equation (2.3) is a generalization of previously published results. See Section III-A in [23] for details.

[1] In [23], where the result of Theorem 1 was first presented, it was stated that the proof could be found in the paper, "Exact SNR analysis and relay ordering in multi-hop cooperative diversity" by I.-M. Kim *et al.*, which had been submitted for publication. However, that paper was revised such that it focused only on the relay ordering, while excluding the exact SNR analysis. In the revised version [7], the result and proof of Theorem 1 do not appear and therefore, they are included here.

The exact expression for Γ, found in Equation (2.3), is too complicated for use in the derivation of an SER expression. Therefore, in this section, approximate expressions for the instantaneous end-to-end SNR that *can* be used for deriving SER expressions will be found. Since the SNR expressions will be approximations, it will follow that the SER expressions found will also be approximations. However, it will be seen that the SER expressions obtained provide very accurate approximations, throughout the entire SNR range considered, as compared to SER results obtained from Monte Carlo simulations.

A useful approximate expression for Γ of Equation (2.3) can be found by first neglecting the dependency among the $\eta_{m,d}$ terms found in the expression for η_d. Then the variance of the overall noise signal at the destination becomes $\mathcal{N}_d \approx N_{s,d}|\beta_{s,d}|^2 + \sum_{m=1}^{K} \mathcal{N}_{m,d}|\beta_{m,d}|^2$, which is equal to A_d. Substituting this expression into Equation (2.3) leads to the following approximate, instantaneous, end-to-end SNR expression: $\Gamma \approx N_{s,d}|\beta_{s,d}|^2 + \sum_{m=1}^{K} \mathcal{N}_{m,d}|\beta_{m,d}|^2$.

Corollary 1 ([23])**.** Ignoring the dependency among the $\eta_{m,d}$ noise variables, the approximate, instantaneous, end-to-end SNR expression, Γ_{app}, can be written as

$$\Gamma \approx \Gamma_{app} = \gamma_{s,d} + \sum_{m=1}^{K} \frac{A_m \gamma_{m,d}}{A_m + \gamma_{m,d} + 1}, \tag{2.7}$$

where $\gamma_{m,k}$ is the individual channel SNR for the channel from terminal m to terminal k, and is defined as

$$\gamma_{m,k} = \frac{\varepsilon_m |h_{m,k}|^2}{N_{m,k}} \qquad \textit{for} \quad m = 0, 1 \ldots, K; \quad k = m+1, \ldots, K+1. \tag{2.8}$$

Also, A_m can be written as

$$A_m = \gamma_{s,m} + \sum_{i=1}^{m-1} \frac{A_i \gamma_{i,m}}{A_i + \gamma_{i,m} + 1} \qquad \textit{for} \quad m \geq 2, \tag{2.9}$$

and A_1 is defined as $A_1 = \gamma_{s,1}$.

Proof. See Appendix A.2. [2] □

Note that, since $|h_{m,k}|^2$ is an exponential RV with mean equal to $\Omega_{m,k}$, it follows that $\gamma_{m,k}$ is also an exponential RV with mean equal to $\overline{\gamma}_{m,k} = \varepsilon_m \Omega_{m,k}/N_{m,k}$. Also note that, because of Equation (2.9), Equation (2.7) is recursive in nature. It is identical to the recursive equation given for the end-to-end SNR in [22, eq. (13)].

The expression given by Γ_{app} is still too complicated to be useful in the derivation of the SER. Therefore, a further approximation for Γ, which will be in a form that *can* be used to derive an approximate SER expression, is presented next.

Theorem 2. The instantaneous SNR, Γ, can be further approximated by Γ':

$$\Gamma \approx \Gamma' = \gamma_{s,d} + \sum_{m=1}^{K} \gamma_{s,m,d} \ , \tag{2.10}$$

where

$$\gamma_{s,m,d} = \frac{\gamma_{s,rm}\gamma_{m,d}}{\gamma_{s,rm} + \gamma_{m,d} + 1} \qquad \textit{for} \quad m = 1, 2, \ldots, K, \tag{2.11}$$

$$\gamma_{s,rm} = \sum_{i=1}^{m} \gamma_{s,i,m} \qquad \textit{for} \quad m = 1, 2, \ldots, K, \tag{2.12}$$

$$\gamma_{s,i,m} = \min(\gamma_{s,i}, \gamma_{i,i+1}, \ldots, \gamma_{m-1,m}) \qquad \textit{for} \quad i = 1, \ldots, m-1, \tag{2.13}$$

[2]While an outline of the proof was given in [23], a detailed and exact proof is given in App. A.2.

and $\gamma_{s,i,m} = \gamma_{s,m}$ for $i = m$. See Fig. 2.2 for a visual representation of the SNRs involved in the expression for Γ'.

Proof. See Appendix A.3. □

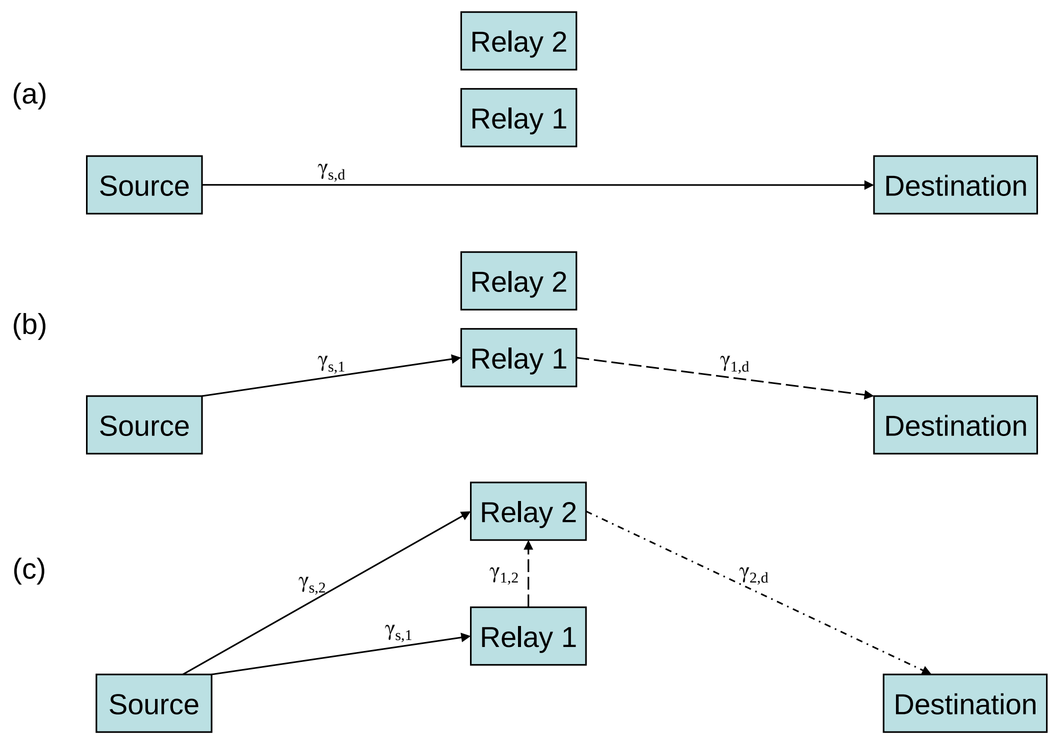

Figure 2.2: Visual representation of the channel SNRs used to develop the approximate end-to-end SNR expression, Γ'. For $K = 2$, the transmitted signals take three different paths on their way from the source to the destination. (a) The $S - D$ path. (b) The $S - R_1 - D$ path. (c) The $S - R_2 - D$ path.

It will be shown that this approximation, Γ', of the instantaneous SNR leads to the derivation of an accurate SER expression. Before deriving the SER expression, the CDF, PDF, and MGF of an RV that is of the same form as that given by Equation

(2.11), will be derived in the next section. Those results will then be used in Section 2.5 to obtain the desired SER expression.

2.4 Development of CDF, PDF, and MGF of an Important RV, Z

In this section, the CDF, PDF, and MGF will be developed for RVs that have the following general form:

$$Z = \frac{XY}{X + Y + c}, \tag{2.14}$$

where $X = \sum_{m=1}^{M} X_m$, $Y = \sum_{n=1}^{N} Y_n$, and c is a constant. The PDFs of the individual, independent, exponential RVs, X_m and Y_n, are $f_{x_m}(x_m) = c_m e^{-c_m x_m} U(x_m)$ for $m = 1, \cdots, M$ and $f_{y_n}(y_n) = r_n e^{-r_n y_n} U(y_n)$ for $n = 1, \cdots, N$; their MGFs are $M_{x_m}(s) = (1 - s\overline{x}_m)^{-1}$ and $M_{y_n}(s) = (1 - s\overline{y}_n)^{-1}$; their means are $\overline{x}_m = 1/c_m$ and $\overline{y}_n = 1/r_n$; and $U(\cdot)$ denotes the unit step function. It will be shown in Section 2.5 that the $\gamma_{s,i,m}$ terms of Equation (2.13) are exponential RVs and that, therefore, the $\gamma_{s,m,d}$ terms of Equation (2.11) are all of the form given by Equation (2.14) with $Z = \gamma_{s,m,d}$, $X = \gamma_{s,rm}$, $M = m$, $Y = \gamma_{m,d}$, and $N = 1$ for $m = 1, 2, \ldots, K$. This provides the motivation for analyzing the RV, Z, in detail. The CDF, PDF, and MGF of Z, which are developed in this section, will be applied to the $\gamma_{s,m,d}$ terms during the process of finding the SER of the multi-hop relay system with instantaneous SNR approximated by Γ'.

In [24], the CDF and PDF were found for an RV with the same form as Z, for the case where the $\overline{x}_m$ are all identical and the $\overline{y}_n$ are also all identical (but $\overline{x}_m \neq \overline{y}_n$ in general), which was appropriate for the conditions and the system analyzed there. For the multi-hop system discussed in this thesis, however, the means of the exponential RVs representing the individual channel SNRs between two terminals are dissimilar in general (although some may be identical to one another), as are the means of the RVs, $\gamma_{s,i,m}$. For that reason, in this section, the CDF, PDF, and MGF of Z will be developed for the case in which the means, $\overline{x}_m$ and $\overline{y}_n$, of the exponential RVs are dissimilar in general.

2.4.1 The PDFs for X and Y

The first step in determining the CDF of Z is to find the PDFs of X and Y. Let J be the number of distinct means in the group of the M individual RVs found in the summation for X. Denote these means as $\overline{x}_1, \ldots, \overline{x}_j, \ldots, \overline{x}_J$. Let r_j be the number of RVs that have mean $\overline{x}_j$, for $j = 1, \ldots, J$. The total number of RVs is given by $M = \sum_{j=1}^{J} r_j$. Similarly, let Q be the number of distinct means in the group of the N individual RVs found in the summation for Y. Denote these means as $\overline{y}_1, \ldots, \overline{y}_q, \ldots, \overline{y}_Q$. Let t_q be the number of RVs that have mean $\overline{y}_q$, for $q = 1, \ldots, Q$. The total number of RVs is given by $N = \sum_{q=1}^{Q} t_q$.

Lemma 1. The PDFs of X and Y are

$$f_X(x) = \sum_{j=1}^{J} \sum_{i=1}^{r_j} \frac{k_{x,i,j}}{(i-1)!} x^{i-1} e^{-x/\overline{x}_j}, \qquad f_Y(y) = \sum_{q=1}^{Q} \sum_{p=1}^{t_q} \frac{k_{y,p,q}}{(p-1)!} y^{p-1} e^{-y/\overline{y}_q}, \tag{2.15}$$

where the expressions for the $k_{x,i,j}$ and $k_{y,p,q}$ constants are given by

$$k_{x,i,j} = \frac{1}{(-1)^n n!} \frac{d^n}{ds^n}\left[M_X(s)\cdot(c_{x,j}-s)^{r_j}\right]_{s=c_{x,j}}$$
$$\textit{for} \quad i=1,\ldots,r_j; \quad j=1,\ldots,J; \quad n=r_j-i\ , \tag{2.16}$$
$$k_{y,p,q} = \frac{1}{(-1)^n n!} \frac{d^n}{ds^n}\left[M_Y(s)\cdot(c_{y,q}-s)^{t_q}\right]_{s=c_{y,q}}$$
$$\textit{for} \quad p=1,\ldots,t_q; \quad q=1,\ldots,Q; \quad n=t_q-p\ , \tag{2.17}$$

and $M_X(s)$ and $M_Y(s)$ are the MGFs of X and Y, respectively.

Proof. See Appendix A.4. □

2.4.2 The CDF and PDF of Z

With the PDFs for X and Y, the CDF of Z can be found. The PDF of Z can always be found by differentiating the CDF of Z.

Theorem 3. The CDF, $F_z(\gamma)$, for the most general form of Z, is given by

$$F_z(\gamma) = 1 - 2\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{k=0}^{i-1}\sum_{q=1}^{Q}\sum_{p=1}^{t_q}\sum_{j'=0}^{p-1}\sum_{i'=0}^{k}\binom{p-1}{j'}\binom{k}{i'}\frac{k_{x,i,j}k_{y,p,q}}{k!(p-1)!}\overline{x}_j^{[i-(i'+j'+k+1)/2]}\overline{y}_q^{\nu/2}$$
$$\gamma^{[p+(i'-j'+k-1)/2]}(\gamma+c)^{(-i'+j'+k+1)/2}\exp\left(-\frac{\sigma_{j,q}\gamma}{\rho_{j,q}}\right)K_\nu\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right), \tag{2.18}$$

where $\sigma_{j,q} = \overline{x}_j + \overline{y}_q$, $\rho_{j,q} = \overline{x}_j \cdot \overline{y}_q$, $\nu = i' + j' - k + 1$, and $K_\nu(\cdot)$ is the modified Bessel function of the second kind and order ν. The PDF, $f_z(\gamma)$, is given by

$$f_z(\gamma) = 2\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{k=0}^{i-1}\sum_{q=1}^{Q}\sum_{p=1}^{t_q}\sum_{j'=0}^{p-1}\sum_{i'=0}^{k}\binom{p-1}{j'}\binom{k}{i'}\frac{k_{x,i,j}k_{y,p,q}}{k!(p-1)!}\times$$
$$\overline{x}_j^{[i-(i'+j'+k+1)/2]}\overline{y}_q^{\nu/2}\gamma^{[p+(i'-j'+k-3)/2]}(\gamma+c)^{(-i'+j'+k-1)/2}e^{-\sigma_{j,q}\gamma/\rho_{j,q}}\times$$
$$\left[\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}(2\gamma+c)K_{\nu-1}\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right)+g(\gamma)K_{\nu}\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right)\right], \tag{2.19}$$

where $g(\gamma) = (i'-k)\gamma + (-p+j'-k+1)(\gamma+c) + \frac{\sigma_{j,q}}{\rho_{j,q}}\gamma(\gamma+c)$.

Proof. See Appendix A.5. □

Corollary 2. The CDF, $F_z(\gamma)$, of Z for the special case where all of the means in X are dissimilar and all of the means in Y are also dissimilar, is given by

$$F_z(\gamma) = 1 - 2\sum_{j=1}^{M}\sum_{q=1}^{N}k_{x,1,j}k_{y,1,q}\sqrt{\rho_{j,q}\gamma(\gamma+c)}\exp\left(-\frac{\sigma_{j,q}\gamma}{\rho_{j,q}}\right)K_1\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right). \tag{2.20}$$

The PDF, $f_z(\gamma)$, of Z for this special case, is given by

$$f_z(\gamma) = 2\sum_{j=1}^{M}\sum_{q=1}^{N}k_{x,1,j}k_{y,1,q}e^{-\sigma_{j,q}\gamma/\rho_{j,q}}\left[(2\gamma+c)K_0\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right)\right.$$
$$\left.+\sigma_{j,q}\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}K_1\left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho_{j,q}}}\right)\right]. \tag{2.21}$$

Proof. Set $J = M$; $r_j = 1$ for $j = 1, \dots M$; $Q = N$; and $t_q = 1$ for $q = 1, \dots N$ into Equations (2.18) and (2.19). □

The special case considered in [24], where all of the exponential RVs in the summation for X have the same mean, $\overline{x}$, and all of the exponential RVs in the summation for Y have the same mean, $\overline{y}$, is represented by letting $J = 1$, $r_1 = M$, $Q = 1$, and $t_1 = N$. In this case, the RVs X and Y become Erlang RVs with the following distributions [25]:

$$f_X(x) = \frac{1/\overline{x}^M}{(M-1)!} x^{M-1} e^{-x/\overline{x}}, \qquad f_Y(y) = \frac{1/\overline{y}^N}{(N-1)!} y^{N-1} e^{-y/\overline{y}}. \tag{2.22}$$

By comparing Equations (2.15) with Equations (2.22), it can be seen that, in this case, $k_{x,i,1} = 0$ for $i \neq M$, $k_{x,M,1} = \overline{x}^{\,-M}$, $k_{y,p,1} = 0$ for $p \neq N$, and $k_{y,N,1} = \overline{y}^{\,-N}$.

Corollary 3. The CDF, $F_z(\gamma)$, of Z for the special case where X is a sum of exponential RVs with identical means and Y is also a sum of exponential RVs with identical means, is given by

$$F_z(\gamma) = 1 - \frac{2}{(N-1)!} \sum_{k=0}^{M-1} \sum_{j'=0}^{N-1} \sum_{i'=0}^{k} \binom{N-1}{j'} \binom{k}{i'} \frac{1}{k!} \overline{x}^{[-(i'+j'+k+1)/2]} \overline{y}^{[-N+(\nu/2)]} \times \\ \gamma^{[N+(i'-j'+k-1)/2]} (\gamma + c)^{(-i'+j'+k+1)/2} \exp\left(-\frac{\sigma\gamma}{\rho}\right) K_\nu \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}}\right), \tag{2.23}$$

where $\sigma = \overline{x} + \overline{y}$, and $\rho = \overline{x} \cdot \overline{y}$. The PDF, $f_z(\gamma)$, of Z for this special case, is given by

$$f_z(\gamma) = \frac{2}{(N-1)!} \sum_{k=0}^{M-1} \sum_{j'=0}^{N-1} \sum_{i'=0}^{k} \binom{N-1}{j'} \binom{k}{i'} \frac{1}{k!} \overline{x}^{[-(i'+j'+k+1)/2]} \overline{y}^{[-N+(\nu/2)]} \times$$
$$\gamma^{[N+(i'-j'+k-3)/2]} (\gamma + c)^{(-i'+j'+k-1)/2} e^{-\sigma\gamma/\rho} \times$$
$$\left[\sqrt{\frac{\gamma(\gamma+c)}{\rho}} (2\gamma + c) K_{\nu-1} \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \right) + g(\gamma) K_\nu \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \right) \right], \tag{2.24}$$

where $g(\gamma) = (i' - k)\gamma + (-N + j' - k + 1)(\gamma + c) + \frac{\sigma}{\rho}\gamma(\gamma + c)$.

Proof. Set $J = 1$, $r_1 = M$, $\overline{x}_1 = \overline{x}$, $Q = 1$, $t_1 = N$, $\overline{y}_1 = \overline{y}$, $k_{x,i,1} = 0$ for $i \neq M$, $k_{x,M,1} = \overline{x}^{-M}$, $k_{y,p,1} = 0$ for $p \neq N$, and $k_{y,N,1} = \overline{y}^{-N}$ into Equations (2.18) and (2.19). □

The CDF and PDF given in Corollary 3 are identical to the CDF and PDF derived in [24, eqs. (11) and (12)] for the same RV.

Corollary 4. The CDF, $F_z(\gamma)$, of Z for the special case where X and Y are both single exponential RVs with means equal to $\overline{x}$ and $\overline{y}$, respectively, is given by

$$F_z(\gamma) = 1 - 2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \exp\left(-\frac{\sigma\gamma}{\rho}\right) K_1 \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \right), \tag{2.25}$$

where $\sigma = \overline{x} + \overline{y}$, and $\rho = \overline{x} \cdot \overline{y}$. The PDF, $f_z(\gamma)$, of Z for this special case, is given by

$$f_z(\gamma) = \frac{2(2\gamma + c)}{\rho} e^{-\sigma\gamma/\rho} K_0 \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \right) + \frac{2\sigma}{\rho} \sqrt{\frac{\gamma(\gamma+c)}{\rho}} e^{-\sigma\gamma/\rho} K_1 \left(2\sqrt{\frac{\gamma(\gamma+c)}{\rho}} \right). \tag{2.26}$$

Proof. Set $M = 1$ and $N = 1$ into Equations (2.20) and (2.21) and note that $k_{x,1,1} = \overline{x}^{\,-1}$ and $k_{y,1,1} = \overline{y}^{\,-1}$ for this special case; or, set $M = 1$ and $N = 1$ into Equations (2.23) and (2.24). □

For the special case where $c = 0$, the CDF and PDF given in Corollary 4 are identical to the CDF and PDF derived in [19, eqs. (8) and (12)] for the same RV.

2.4.3 The MGF of Z

The MGF, $M_z(s)$, of Z is given in the following theorem for the special case where $c = 0$.

Theorem 4. The MGF, $M_z(s)$, of Z for the special case where $c = 0$, is given by

$$M_z(s) = 2\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{k=0}^{i-1}\sum_{q=1}^{Q}\sum_{p=1}^{t_q}\sum_{j'=0}^{p-1}\sum_{i'=0}^{k}\binom{p-1}{j'}\binom{k}{i'}\frac{k_{x,i,j}k_{y,p,q}}{k!(p-1)!}\overline{x}_j^{[i-(i'+j'+k+1)/2]}\overline{y}_q^{\;\nu/2} \times\left[\frac{2}{\sqrt{\rho_{j,q}}}\mathcal{I}_1 + \frac{\sigma_{j,q}}{\rho_{j,q}}\mathcal{I}_2 + (i'+j'-2k-p+1)\mathcal{I}_3\right], \tag{2.27}$$

where the integrals $\mathcal{I}_1 = \int_0^\infty \gamma^{p+k}e^{-\alpha\gamma}K_{\nu-1}(\beta\gamma)\,\mathrm{d}\gamma$, $\mathcal{I}_2 = \int_0^\infty \gamma^{p+k}e^{-\alpha\gamma}K_{\nu}(\beta\gamma)\,\mathrm{d}\gamma$, and $\mathcal{I}_3 = \int_0^\infty \gamma^{p+k-1}e^{-\alpha\gamma}K_{\nu}(\beta\gamma)\,\mathrm{d}\gamma$, with $\alpha = (\sigma_{j,q}/\rho_{j,q}) - s$ and $\beta = 2/\sqrt{\rho_{j,q}}$, can be solved by using [26, eqs. 6.611.3, 6.611.9, 6.621.3, or 6.624.1]. The forms of the integrals depend on the values of p, k, and ν.

Proof. Set $c = 0$ in Equation (2.19) and use that form of the PDF in the equation, $M_z(s) = \int_0^\infty e^{s\gamma}f_z(\gamma)\,\mathrm{d}\gamma$. □

Corollary 5. The MGF, $M_z(s)$, of Z for the special case where all of the means in

X are dissimilar, all of the means in Y are also dissimilar, and $c = 0$, is given by

$$M_z(s) = \sum_{j=1}^{M}\sum_{q=1}^{N} k_{x,1,j}k_{y,1,q}\left[\frac{4\rho_{j,q}^2 - \sigma_{j,q}^2\rho_{j,q} + \sigma_{j,q}\rho_{j,q}^2 s}{4\rho_{j,q} - (\sigma_{j,q} - \rho_{j,q}s)^2} + \frac{4\rho_{j,q}^3 s\ \arccos\left(\frac{\sigma_{j,q}-\rho_{j,q}s}{2\sqrt{\rho_{j,q}}}\right)}{[4\rho_{j,q} - (\sigma_{j,q} - \rho_{j,q}s)^2]^{3/2}}\right]. \tag{2.28}$$

Proof. See Appendix A.6. □

Corollary 6. The MGF, $M_z(s)$, of Z for the special case where X and Y are both single exponential RVs with means equal to $\overline{x}$ and $\overline{y}$, respectively, and $c = 0$, is given by

$$M_z(s) = \frac{4\rho - \sigma^2 + \sigma\rho s}{4\rho - (\sigma - \rho s)^2} + \frac{4\rho^2 s}{[4\rho - (\sigma - \rho s)^2]^{3/2}} \arccos\left(\frac{\sigma - \rho s}{2\sqrt{\rho}}\right). \tag{2.29}$$

Proof. Set $M = 1$ and $N = 1$ into Equation (2.28) and note that $k_{x,1,1} = \overline{x}^{\,-1}$ and $k_{y,1,1} = \overline{y}^{\,-1}$ for this special case. □

The result given in Corollary 6 is identical to the result derived in [19, eq. (7)].

2.5 Approximate SER Expression for Multi-Hop AF System with K Relays

The approximate, instantaneous, end-to-end SNR, Γ', which will be used to find the SER of the multi-hop system, is summarized by the equations given in Theorem 2. The first step in finding an approximate SER expression for the multi-hop system is to recognize that the $\gamma_{s,m,d}$ terms from Equation (2.11) are RVs of the form given by Z. From [19], if γ_1 and γ_2 are independent exponential RVs, then $\gamma_{min,2} = \min(\gamma_1, \gamma_2)$

is also an exponential RV with mean $\overline{\gamma}_{min,2} = \frac{\overline{\gamma}_1\overline{\gamma}_2}{\overline{\gamma}_1+\overline{\gamma}_2}$. It is easy to show that for P independent exponential RVs, $\gamma_1, \gamma_2, \ldots, \gamma_P$, the minimum of these RVs, $\gamma_{min,P} = \min(\gamma_1, \gamma_2, \ldots, \gamma_P)$ is also an exponential RV. Its mean is

$$\overline{\gamma}_{min,P} = \frac{\overline{\gamma}_{min,P-1}\overline{\gamma}_P}{\overline{\gamma}_{min,P-1} + \overline{\gamma}_P} = \frac{\prod_{p=1}^{P} \overline{\gamma}_p}{\sum_{q=1}^{P} \prod_{\substack{p=1,\\ p\neq q}}^{P} \overline{\gamma}_p} \qquad \text{for} \quad P \geq 2\ , \tag{2.30}$$

where $\gamma_{min,P-1} = \min(\gamma_1, \gamma_2, \ldots, \gamma_{P-1})$, and $\overline{x}$ denotes the mean of the RV, x. From Equation (2.13), $\gamma_{s,i,m}$ is the minimum of a set of exponential RVs for the $i < m$ case, which makes it an exponential RV as well. By using Equation (2.30), its mean is found to be

$$\overline{\gamma}_{s,i,m} = \frac{\overline{\gamma}_{s,i} \prod_{p=i}^{m-1} \overline{\gamma}_{p,p+1}}{\overline{\gamma}_{s,i} \left(\sum_{q=i}^{m-1} \prod_{\substack{p=i,\\ p\neq q}}^{m-1} \overline{\gamma}_{p,p+1} \right) + \prod_{p=i}^{m-1} \overline{\gamma}_{p,p+1}} \qquad \text{for} \quad i \leq m-2, \tag{2.31}$$

and $\frac{\overline{\gamma}_{s,i}\overline{\gamma}_{i,m}}{\overline{\gamma}_{s,i}+\overline{\gamma}_{i,m}}$ for $i = m-1$. From Theorem 2, $\gamma_{s,i,m}$ is also an exponential RV for the $i = m$ case, with mean $\overline{\gamma}_{s,i,m} = \overline{\gamma}_{s,m}$, since $\gamma_{s,m,m} = \gamma_{s,m}$.

From Equation (2.12), it follows that $\gamma_{s,rm}$ is the sum of m exponential RVs for $m = 1, 2, \ldots, K$. It then follows that the $\gamma_{s,m,d}$ terms are of the same form as Z given in Equation (2.14), with $Z = \gamma_{s,m,d}$; $X = \gamma_{s,rm}$; $Y = \gamma_{m,d}$; $M = m$; $N = 1$; $X_i = \gamma_{s,i,m}$ for $i = 1, \ldots, m$; $Y_1 = \gamma_{m,d}$; and $c = 1$. Note that it is necessary here to neglect the dependency between the $\gamma_{s,i,m}$ terms and treat them as though they were independent since the results for Z in the previous section were derived under the restriction that the variables in the summations of X and Y all be independent.

The next step in finding the approximate SER expression of the multi-hop system is to obtain the MGF of $\gamma_{s,m,d}$ for $m = 1, \ldots, K$. In the expression for $\gamma_{s,m,d}$ in

Equation (2.11), the constant c is equal to 1. In this case, it is not possible to solve for the MGF of $\gamma_{s,m,d}$ in closed form. For that reason, at this point the approximation $c = 0$ is used. This is the same type of approximation that was used in Appendix A.3 and in many other publications as well [16]–[19]. As mentioned in [7] and [23], it leads to an SER approximation that is very tight to the actual SER throughout the entire SNR range.

Lemma 2. The MGF, $M_{\gamma_{s,m,d}}(s)$, of $\gamma_{s,m,d}$ for the special case where $c = 0$, is given by

$$M_{\gamma_{s,m,d}}(s) = \frac{1}{\overline{\gamma}_{m,d}} \sum_{i=1}^{m} k_{x,1,i} \left[\frac{4\rho_{i,1}^2 - \sigma_{i,1}^2 \rho_{i,1} + \sigma_{i,1} \rho_{i,1}^2 s}{4\rho_{i,1} - (\sigma_{i,1} - \rho_{i,1} s)^2} + \frac{4\rho_{i,1}^3 s \ \arccos\left(\frac{\sigma_{i,1} - \rho_{i,1} s}{2\sqrt{\rho_{i,1}}}\right)}{[4\rho_{i,1} - (\sigma_{i,1} - \rho_{i,1} s)^2]^{3/2}} \right], \tag{2.32}$$

where $k_{x,1,1} = \frac{1}{\overline{\gamma}_{s,1}}$ when $m = 1$; $k_{x,1,i} = \frac{1}{\overline{\gamma}_{s,i,m}} \prod_{l=1, l \neq i}^{m} \frac{\overline{\gamma}_{s,i,m}}{\overline{\gamma}_{s,i,m} - \overline{\gamma}_{s,l,m}}$ for $i = 1, \ldots, m$ when $m \geq 2$; $\sigma_{i,1} = \overline{\gamma}_{s,i,m} + \overline{\gamma}_{m,d}$; and $\rho_{i,1} = \overline{\gamma}_{s,i,m} \overline{\gamma}_{m,d}$.

Proof. Set $Z = \gamma_{s,m,d}$, $M = m$, and $N = 1$ into Equation (2.28), and note that $k_{y,1,1} = \overline{\gamma}_{m,d}^{\ -1}$ for this special case. The $k_{x,1,i}$ terms are solved using Equation (2.16). □

With the MGFs for the $\gamma_{s,m,d}$ terms, it is now possible to find an approximate SER expression for the multi-hop system, as long as the correlation between the RVs, $\gamma_{s,m,d}$, are ignored. Note that the assumption that the correlation can be ignored is used to make the analysis mathematically tractable, and therefore its use may not necessarily be justified. However, it is shown in Section 2.6 that, for typical values of channel variances, the results for the analytical SER expressions obtained by using the following theorem are very accurate when compared to the simulation results, thereby justifying the use of this assumption.

Theorem 5. An approximate SER expression, $\mathrm{P}_s(E)$, for the multi-hop system is given by

$$P_s(E) = \frac{a}{\pi} \int_{\theta=0}^{\pi/2} \frac{\sin^2\theta}{\sin^2\theta + b\overline{\gamma}_{s,d}} \prod_{m=1}^{K} M_{\gamma_{s,m,d}}\left(\frac{-b}{\sin^2\theta}\right) d\theta, \tag{2.33}$$

where $M_{\gamma_{s,m,d}}(s)$ is given by Equation (2.32), and a and b are modulation-dependent constants. [3]

Proof. See Appendix A.7. □

The SER can now be solved with only one numerical integration for any value of K by using Equations (2.32) and (2.33). (It is worth mentioning here that the CDF of the most general form of Z given in Theorem 3 can always be used to find the *exact* SER of a system with end-to-end SNR represented by Z, for any possible combination of individual channel SNR means, in closed form for the case where $c = 0$ and with one numerical integration for the case where $c \neq 0$. The equation that makes this possible is $\mathrm{P}_s(E) = \frac{a\sqrt{b}}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-b\gamma}}{\sqrt{\gamma}} F_z(\gamma)\, \mathrm{d}\gamma$, which is found in [24, eq. (15)] and [27, eq. (20)]. While this result is noteworthy, it does not apply to the multi-hop system being discussed in this chapter, since the SNR of the multi-hop system consists of a *sum* of RVs of the form given by Z, and not just a single Z term.)

[3]For binary phase-shift-keying (BPSK), $a = b = 1$; for M-ary pulse-amplitude-modulation (M-PAM), $a = 2(M-1)/M, b = 3/(M^2-1)$; for M-ary phase-shift-keying (M-PSK), an approximate expression for the SER is found by using Equation (2.33) with $a = 2, b = \sin^2(\pi/M)$; and for M-ary quadrature-amplitude modulation (M-QAM), an approximate expression for the SER is found by using Equation (2.33) with $a = 4(\sqrt{M}-1)/\sqrt{M}, b = 1.5/(M-1)$. See [13, 24].

2.6 Numerical Analysis

Plots of the SER versus the single-channel SNR, ε_o/N_o, where ε_o is the source power and N_o is the noise power, were generated for various values of K relays and for two different choices of channel variances. For each value of K, the SER was found by using Equations (2.32) and (2.33), and the SER was also found by running Monte Carlo simulations. (For all cases in this thesis, the number of iterations for a Monte Carlo simulation was chosen so that an average of at least 100 errors would occur (and often much more than that). For example, for an expected SER of 10^{-6}, the number of iterations was chosen to be 10^8.) BPSK modulation was used in all cases. Also, the AWGN variances, $N_{i,j}$, are the same and equal to N_o for all terminals i and j in all cases. In Fig. 2.3, the channel variances, $\Omega_{i,j}$, were chosen to be $\Omega_{i,j} = 1$ for all terminals i and j, for all values of K. In Fig. 2.4, the source-to-relay and relay-to-relay channel variances, $\Omega_{i,j}$, were chosen to be $\Omega_{i,j} = 1$, and the relay-to-destination channel variances, $\Omega_{i,j}$, were chosen to be $\Omega_{i,j} = 0.5$ for all values of K. This choice of channel variances models the system in which the relays are closer to each other and closer to the source than to the destination. As can be seen, the analytic results for the SER found from the equations are very accurate approximations of the SER found from the simulations. This is true for both Fig. 2.3 and Fig. 2.4, and for all values of K, throughout the entire SNR range. For example, in Fig. 2.3, at SER $= 10^{-4}$ the difference between the SNR of the analytic plot and the SNR of the simulation plot ranges from 0.12 dB to 0.35 dB, depending on the value of K. In Fig. 2.4, at SER $= 10^{-4}$ the difference between the SNR of the analytic plot and the SNR of the simulation plot ranges from 0.14 dB to 0.27 dB.

It should be pointed out that for the results shown in both Fig. 2.3 and Fig.

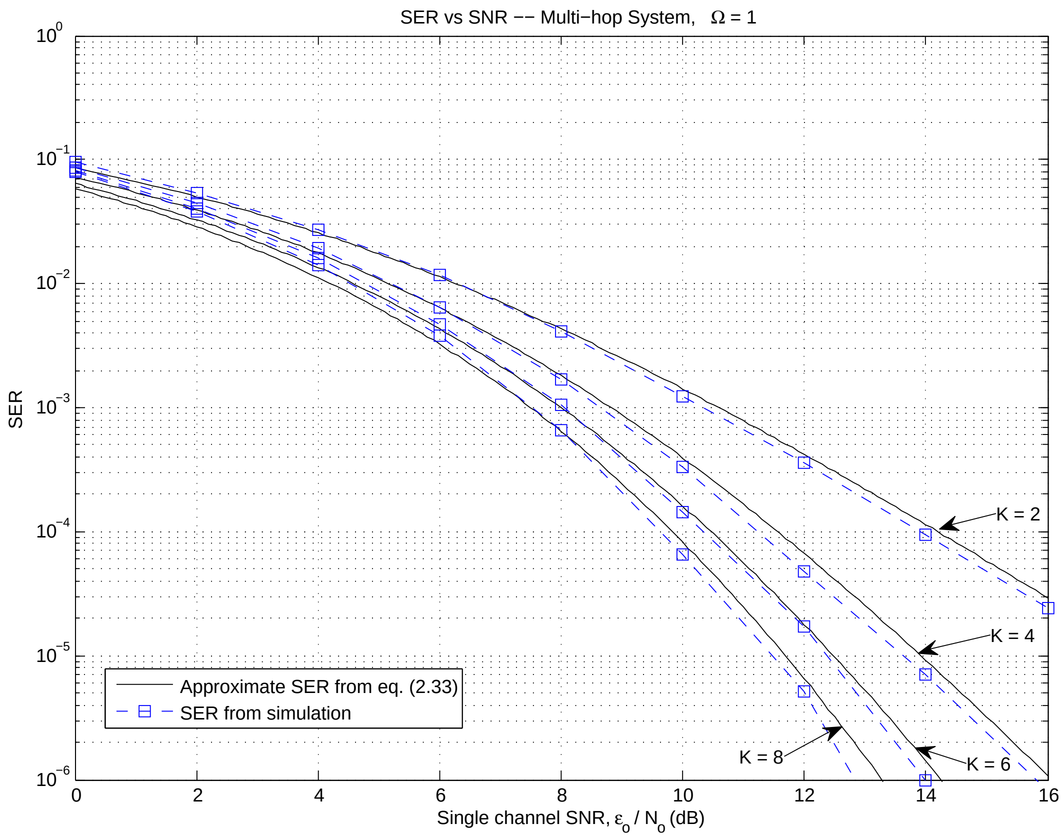

Figure 2.3: SER for $K = 2, 4, 6,$ and 8, determined from (i) the approximate SER equation, and (ii) simulations. BPSK modulation is used, the source power is ε_o, the relay powers are ε_o/K, and the channel variances are $\Omega = 1$ for all channels.

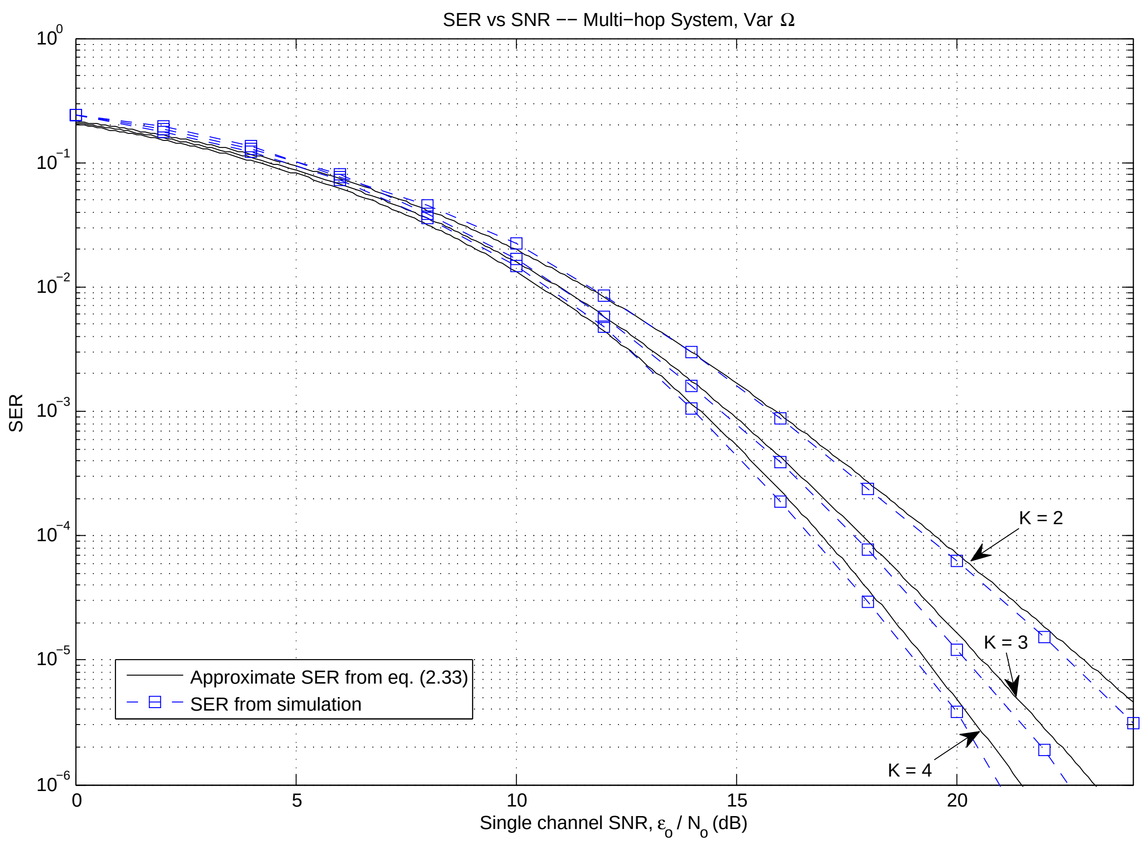

Figure 2.4: SER for the case where the source-to-relay and relay-to-relay channel variances ($\Omega = 1$) are twice as strong as the relay-to-destination channel variances ($\Omega = 0.5$). BPSK modulation is used, and the relay powers are ε_o/K.

2.4, the choice of power for each relay was ε_o/K. While finding an optimal power allocation scheme for the multi-hop system is very important and could be an area of future research, the optimal choice for power allocation was not determined at this time. Power allocation schemes were found in the literature for similar systems with only one relay and for systems with multiple relays that used full transmit channel state information (CSI) at the relays (by using feedback from the destination to the relays). Neither of those situations are applicable to the multi-hop system being discussed here. In [28], it was shown that setting the relay power to ε_o/K, as was done in this chapter, under a total system power constraint of $P = 2\varepsilon_o$, maximized the SNR of a two-hop system with K relays that used distributed space-time block codes (DSTBCs) at the relays. While the system discussed in [28] has important differences with the one studied in this thesis, the choice of that power allocation scheme nevertheless seemed to be a reasonable choice for the multi-hop system discussed here. Recently, optimal power allocation schemes were presented for the *two-hop* system in [6], and those schemes are comparable to the one used here.

2.7 Conclusion

A multi-branch, multi-hop AF relay system with K relays was analyzed. An expression for the exact end-to-end instantaneous SNR of the system was presented. It was noted that this result is a generalization of previously published results for the end-to-end SNR of multi-branch, *two-hop* systems. An approximate expression for the end-to-end SNR was developed from the exact SNR expression, in the form of a recursive formula, by neglecting the correlation between noise variables. This recursive formula was shown to be consistent with a previously published result. A conceptual approach was used to develop a more useful approximate SNR expression, one that can be used in SER derivations. The CDF, PDF, and MGF of the RV, $Z = XY/(X + Y + c)$, were developed. In this chapter, X and Y are sums of independent, exponential RVs with no limitations placed on the number of RVs or the parameters of the RVs. It was shown that the results in this chapter are generalizations of previously published results regarding special cases of Z. The results for the general form of Z, along with the approximate instantaneous SNR expression, were used to find an approximate expression for the SER of the multi-hop system for an arbitrary number of relays, K. Plots showed that the approximate SER expression was very accurate when compared to the SER results obtained from simulations.

Chapter 3

Outage Probability of Multi-Hop AF Relay Systems

In this chapter, the outage probability of multi-hop, amplify-and-forward relay systems with multiple relays is analyzed. Previously, the outage probability of the two-hop system with multiple relays had been analyzed. The results found for the multi-hop system in this chapter are shown to be a generalization of results previously found for the two-hop system. (That is, the results for the two-hop system are a special case of the results found in this chapter for the multi-hop system.) Both a lower bound of the outage probability, which is reasonably tight to the actual outage probability in the entire signal-to-noise ratio (SNR) range and asymptotically tight in the high-SNR region, as well as a high-SNR straight-line approximation of the outage probability that is very tight to the actual outage probability at moderately high to very high SNRs, are derived.

3.1 Introduction

In this chapter, the outage probability is analyzed for the case of multi-hop AF cooperative diversity systems with K relays that use repetition-based transmission and maximum-ratio combining (MRC) at the receivers, and where the transmission is done over wireless Rayleigh fading channels. In the multi-hop system, the relays transmit to the other relays as well as the destination, whereas in the two-hop system, which has previously been analyzed in detail, relays transmit only to the destination. In [6], an important recent publication where these systems were discussed, the two-hop system is called the *source-only* system since the relays receive signals from the source only, and the multi-hop system is called the *MRC-based* system since the relays receive signals from other relays as well as the source and then combine those signals using MRC.

Two-hop AF relay systems with repetition-based transmission were analyzed for $K = 1$ relay in [5], [16]–[18], two-hop systems for multiple relays ($K \geq 1$) were studied in [32]–[35], and multi-hop systems for multiple relays ($K \geq 2$) were studied in [7], [20]–[23]. The exact instantaneous signal-to-noise ratio (SNR) (the SNR conditioned on the event that the values of the fading channel coefficients are given) for the multi-hop system (of which the exact SNR for the two-hop system is a special case) was found in [23]; an approximate instantaneous SNR expression for the multi-hop system, given in a recursive form and used in this chapter, was first given in [22] and was also used in [23] and in Chapter 2; an approximate symbol-error-rate (SER) expression for the multi-hop system was derived for the special case of $K = 2$ relays (and where a relay-ordering algorithm was used) in [7]; and an approximate SER expression for the multi-hop system was developed for the general case of K relays in Chapter 2.

The outage probability for AF relay systems with repetition-based transmission was studied in [5], [16]–[18] for $K = 1$ relay, and in [32] (focused on high SNR regime) and in [34, 35] (focused on low SNR regime) for the two-hop system with $K \geq 1$ relays. Recently, in [6], a thorough discussion of the two-hop system was presented, where many of the previous results for this system were summarized. Also, a brief discussion of the multi-hop system was included. However, the SER and outage probabilities of the multi-hop system were *not* analyzed in [6]. [1] To the best of our knowledge, the outage probability for the multi-hop system has not previously been analyzed.

In this chapter, the work in [6] (regarding the outage probability of the two-hop system) is generalized by analyzing the outage probability of the multi-hop system, of which the two-hop system is a special case. Specifically, for the multi-hop system with K relays, a lower bound for the outage probability is found and a high-SNR straight line asymptotic approximation is found. The lower bound will be seen to be tight to the exact outage probability for nearly the entire SNR region, and the high-SNR approximation will be seen to be very tight to the exact curve for SNRs that are moderately high and greater. The outage probability, P_o, of the systems considered in this thesis are given by $P_o = \mathbb{P}\left(\frac{1}{K+1}\log_2(1+\Gamma) < R\right)$, where Γ is the exact instantaneous end-to-end SNR and R is the spectral efficiency in bits-per-second per Hertz (bps/Hz). Since the outage probability can also be expressed as $P_o = \mathbb{P}\left(\Gamma < 2^{(K+1)R} - 1\right)$, the strategy for finding the outage probability will be to first find an expression for Γ, and then find its cumulative distribution function (CDF), $\mathbb{P}(\Gamma < x)$. Because the exact expression of Γ is too complicated for calculating P_o

[1] In [6, p. 223], during a discussion on the performance analysis of the multi-hop system, the authors state that, "The SER analysis of the [multi-hop] protocol is very complicated and a close-form analysis is not tractable." An analysis of the outage probability of the multi-hop system is not even attempted in [6].

analytically (since its CDF cannot be found), an expression for the upper bound, Γ_U, of the instantaneous SNR is given in an interesting and new recursive formula that has not been previously published, and of which its CDF *can* be found. Then the CDF of Γ_U will be derived and used to find the lower bound of P_o. This, the lower bound of the outage probability of the multi-hop system, is the main result of this chapter. The high-SNR straight-line approximation will be found as an approximation of the expression for the lower bound of P_o. It will also be shown that the lower bound and the high-SNR approximation of P_o for the two-hop system are special cases of those found in this chapter for the multi-hop system.

The rest of this chapter is organized in the following manner. The system model and instantaneous end-to-end SNR expressions are given in Section 3.2. The outage probability analysis for the multi-hop system, including the presentation of the new expression for the upper bound of the SNR, Γ_U, is carried out in Section 3.3. Numerical results with plots of the outage probability expressions for the two-hop and multi-hop systems are given in Section 3.4. Concluding remarks are presented in Section 3.5.

Notation: $A := B$ is used to denote that A, by definition, equals B; and $A =: B$ is used to denote that B, by definition, equals A. Also, $h \sim \mathcal{CN}(\mu, \Omega)$ denotes that h is a circularly symmetric complex-valued Gaussian random variable (RV) with mean μ and variance Ω.

3.2 System Model and Instantaneous SNR Expressions

The block diagram of the multi-hop system model is shown in Fig. 3.1. A discrete-time, baseband equivalent model is used to describe this system. It will be convenient to refer to the source, the K relays, and the destination as terminals m, where $m = 0$ or $m = s$ for the source, $m = 1, \ldots, K$ for the K relays, and $m = K + 1$ or $m = d$ for the destination. In time slot m, for $m = 0, \ldots, K$, terminal m transmits to terminal(s) k for $k = m + 1, \ldots, K + 1$. (The transmitting scheme is orthogonal in time since each device (source or relay) is allotted its own time slot in which it is the only device that transmits.) Note that the source transmits in time slot 0 and that $K + 1$ time slots are used to transmit one symbol. The receiving terminals use MRC to combine all of the signals that they have previously received. All channels are Rayleigh fading channels with additive white Gaussian noise (AWGN). The channel coefficient for the channel from transmitting terminal m to receiving terminal k is denoted by $h_{m,k}$ where $h_{m,k} \sim \mathcal{CN}(0, \Omega_{m,k})$. It follows that $|h_{m,k}|^2$ is an exponential RV with a mean of $\Omega_{m,k}$. The channel coefficients are constant over one time slot. The AWGN component at terminal k, added to the signal transmitted from terminal m, is denoted by $n_{m,k}$ where $n_{m,k} \sim \mathcal{CN}(0, N_{m,k})$. The channel coefficients and AWGN signals are all mutually independent. The individual channel SNR for the channel from terminal m to terminal k is given by $\gamma_{m,k} = \frac{\varepsilon_m |h_{m,k}|^2}{N_{m,k}}$, where ε_m is the power used by terminal m. The RV, $\gamma_{m,k}$, is exponentially distributed with mean equal to $\overline{\gamma}_{m,k} = \varepsilon_m \Omega_{m,k} / N_{m,k}$. For the *two-hop* system, the relays transmit only to the destination and not to other relays. It follows that the two-hop system can be considered a special case of the multi-hop system, with $\gamma_{m,k} = 0$ for $1 \leq m < k \leq K$.

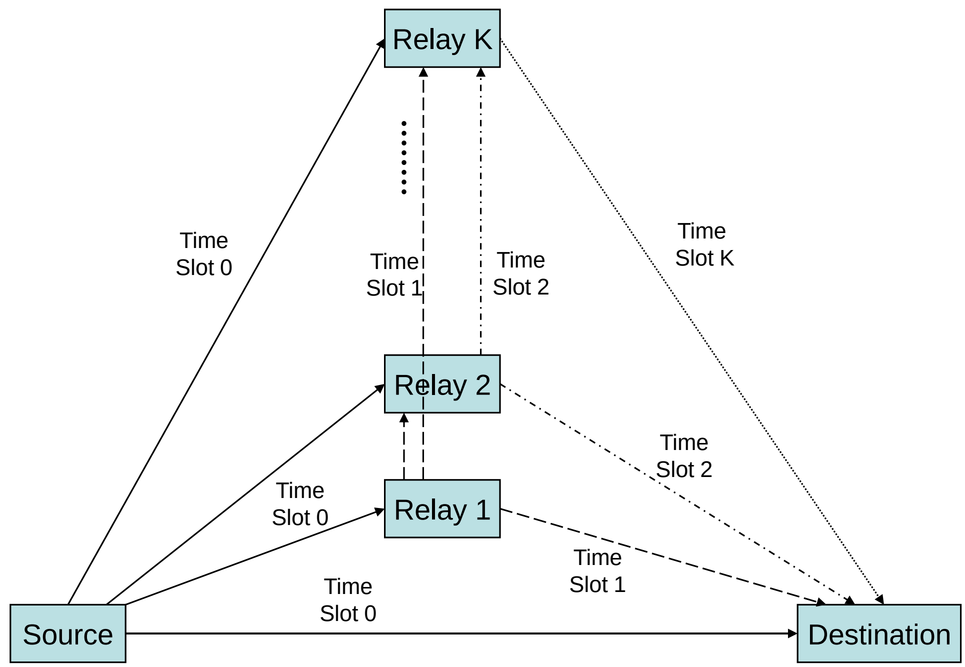

Figure 3.1: Block Diagram of the Multi-Hop System. The terminals involved, the signals that they transmit, and the order in which they transmit them are shown. Each point-to-point channel from terminal i to terminal j is a Rayleigh fading channel with fading coefficient $h_{i,j}$ and additive white Gaussian noise $n_{i,j}$.

The exact expression for the end-to-end instantaneous SNR, Γ_{exact}, which is given in Chapter 2 and was also given in [23], is too complicated for use in the derivation of the outage probability, P_o. Therefore, in order to calculate the outage probability, the following very accurate, approximate, instantaneous, end-to-end SNR expression,

Γ_{multi}, for the multi-hop system is used (see [22, eq. (13)], [23], and also Chapter 2):

$$\Gamma_{exact} \approx \Gamma_{multi} = \gamma_{s,d} + \sum_{m=1}^{K} \frac{\gamma_{sRm}\gamma_{m,d}}{\gamma_{sRm} + \gamma_{m,d} + 1} , \tag{3.1}$$

where γ_{sR1} is defined as $\gamma_{sR1} = \gamma_{s,1}$, and γ_{sRm} is given by

$$\gamma_{sRm} = \gamma_{s,m} + \sum_{i=1}^{m-1} \frac{\gamma_{sRi}\gamma_{i,m}}{\gamma_{sRi} + \gamma_{i,m} + 1} \qquad \text{for} \quad m \geq 2 . \tag{3.2}$$

For the two-hop system, the exact end-to-end SNR expression is given by [23, eq. 31]

$$\Gamma_{2-hop} = \gamma_{s,d} + \sum_{m=1}^{K} \frac{\gamma_{s,m}\gamma_{m,d}}{\gamma_{s,m} + \gamma_{m,d} + 1} . \tag{3.3}$$

(See also [17], [19], [20].) It can be seen that Equation (3.3) is a special case of Equation (3.1) since, for the two-hop system, $\gamma_{i,m} = 0$ for $1 \leq i < m \leq K$, and substituting these results into (3.2) leads to $\gamma_{sRm} = \gamma_{s,m}$ for $1 \leq m \leq K$. It follows that (3.1) reduces to (3.3) for the two-hop system. (It is also interesting to note that both Γ_{exact} and Γ_{multi} reduce to Γ_{2-hop}, which is the *exact* end-to-end SNR for the two-hop system, when the appropriate channel SNRs are set to zero.)

3.3 Outage Probability for the Multi-hop System

For the multi-hop system, a lower bound for P_o that is valid in the entire SNR range and asymptotically tight in the high-SNR region, and a high-SNR straight-line approximation for P_o that is very accurate in the medium-to-high-SNR region, are derived in this section. By using MRC to combine the received signals at the destination, transmitting information with the AF relay system is equivalent to transmitting over a scalar, single-input single-output (SISO) Gaussian channel. Therefore, the instantaneous mutual information, I, can be given as [8, 36]

$$I = \frac{1}{K+1} \log_2(1 + \Gamma_{exact}) \, , \tag{3.4}$$

where Γ_{exact} is the instantaneous end-to-end SNR, $\log_2(1 + \Gamma_{exact})$ is the capacity of a SISO Gaussian channel, and the factor, $\frac{1}{K+1}$, is due to the fact that, because of the repetition-based transmission and half duplex constraints (terminals cannot transmit and receive at the same time), it takes $K + 1$ time slots to transmit one symbol. (Note that the two-hop system discussed in this paper will also use $K + 1$ time slots to transmit one symbol in order to make the comparison fair. Therefore, for the two-hop system, the additional restriction that the AF protocol must be an *orthogonal* scheme (only one relay is allowed to transmit per time slot) is also imposed.) The outage probability, P_o, is given by

$$P_o = \mathbb{P}(I < R) = \mathbb{P}\left(\frac{1}{K+1} \log_2(1 + \Gamma_{exact}) < R\right) = \mathbb{P}\left(\Gamma_{exact} < 2^{(K+1)R} - 1\right) \, , \tag{3.5}$$

where R is the spectral efficiency, that is, the data rate per unit bandwidth of the system in bps/Hz, and $\mathbb{P}\left(\Gamma_{exact} < x\right)$ is the CDF of Γ_{exact}.

3.3.1 Upper Bound of End-to-End SNR

It is clear that it would be necessary to find the CDF of Γ_{exact} in order to find P_o. However, since the expression for Γ_{exact} and the expression for Γ_{multi} given in (3.1) are too complicated, the first step will be to find an upper bound, Γ_U, of the end-to-end

SNR, and then *its* CDF will be found. Then the lower bound, $P_{o,L}^{multi}$, of the outage probability of the multi-hop system is found as $P_{o,L}^{multi} = \mathbb{P}\left(\Gamma_U < 2^{(K+1)R} - 1\right)$. In the following theorem, Γ_U is given by an interesting and new recursive formula.

Theorem 6. The upper bound, Γ_U, of the instantaneous SNR, Γ_{multi}, as defined by Equation (3.1), is given by

$$\Gamma_U = \gamma_{s,d} + \sum_{m=1}^{K} \min(\gamma_{sRm,up}, \gamma_{m,d}) \geq \gamma_{s,d} + \sum_{m=1}^{K} \frac{\gamma_{sRm,up}\gamma_{m,d}}{\gamma_{sRm,up} + \gamma_{m,d}} \geq \Gamma_{multi} , \qquad (3.6)$$

where $\gamma_{sRm,up} = \gamma_{s,1}$ for $m = 1$ and $\gamma_{sRm,up} = \gamma_{s,m} + \sum_{i=1}^{m-1} \min(\gamma_{sRi,up}, \gamma_{i,m})$ for $m = 2, \ldots, K$. Furthermore, $\gamma_{sRm,up} \geq \gamma_{sRm}$ for all m; that is, $\gamma_{sRm,up}$ is the upper bound, or maximum, of the expression γ_{sRm}.

Proof. See Appendix B.1. □

To find the CDF for Γ_U of (3.6), for the general case of $K \geq 2$ relays, it is first necessary to recognize which terms in Γ_U repeat. Then the conditional CDF for Γ_U, given that the repeated terms are held constant, is found. Finally, the CDF of Γ_U is found by taking the expected value of the conditional CDF with respect to all of the RVs that were held constant. In the next subsection, several sets will be defined in order to help in the formulation of the expressions required to solve for the CDF of Γ_U.

3.3.2 Sets of RVs, Constants, and Events

First, it will be convenient to define the following sets of RVs.

Definition 1. *Let $\mathcal{R}$ be the set of RVs that repeat in the expression for Γ_U given in* (3.6)*, in terms of the individual channel SNRs, $\gamma_{i,j}$, for a given value of K:*

$$\mathcal{R} := \{\gamma_{i,j} : 0 \leq i < j \leq K-1\} \ . \tag{3.7}$$

Let $\widetilde{\mathcal{R}}$ be the set of RVs that repeat in the expression for Γ_U given in (3.6)*, in terms of the source-to-relay multi-hop SNRs, $\gamma_{sRm,up}$, for a given value of K:*

$$\widetilde{\mathcal{R}} := \{\gamma_{sRm,up} : 1 \leq m \leq K-1\} \ . \tag{3.8}$$

□

It will also be convenient to define the following sets of constants.

Definition 2. *Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be the following sets of constants:*

$$\mathcal{C} := \{z_{i,j} : 0 \leq i < j \leq K-1\} \ , \tag{3.9}$$

$$\widetilde{\mathcal{C}} := \{y_m : 1 \leq m \leq K-1\} =: \{y_1, \ldots, y_{K-1}\} \ , \tag{3.10}$$

where $z_{i,j}$ and y_m are positive real constants.

□

Let the event that the RV, $\gamma_{i,j}$, is equal to the constant, $z_{i,j}$, be denoted by $\mathcal{E}_{i,j}$. Also, let the event that the RV, $\gamma_{sRm,up}$, is equal to the constant, y_m, be denoted by $\widetilde{\mathcal{E}}_{i,j}$. It will be convenient to define the following sets of events:

Definition 3. *Let $\mathcal{S}_{eq}$ be the set of events that describe the fact that the repeating terms are held constant, in terms of the individual channel SNRs, $\gamma_{i,j}$, for a given* *value of K:*

$$\mathcal{S}_{eq} := \{\mathcal{E}_{i,j} : \forall \gamma_{i,j} \in \mathcal{R}, \forall z_{i,j} \in \mathcal{C}\} \ . \tag{3.11}$$

Let $\widetilde{\mathcal{S}}_{eq}$ be the set of events that describe the fact that the repeating terms are held constant, in terms of the source-to-relay multi-hop SNRs, $\gamma_{sRm,up}$, for a given value of K:

$$\widetilde{\mathcal{S}}_{eq} := \{\widetilde{\mathcal{E}}_{i,j} : \forall \gamma_{sRm,up} \in \widetilde{\mathcal{R}}, \forall y_m \in \widetilde{\mathcal{C}}\} \ . \tag{3.12}$$

□

Note that, because of the relationship between $\gamma_{sRm,up}$ and $\gamma_{i,m}$ given in Theorem 6, it follows that the constants, y_m, and the constants, $z_{i,j}$, are related by $y_m = z_{0,1}$ for $m = 1$, and $y_m = z_{0,m} + \sum_{i=1}^{m-1} \min(y_i, z_{i,m})$ for $m = 2, \ldots, K-1$. Also, the cardinalities of the sets, $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{S}_{eq}$ are the same and are given by $n_R = \sum_{n=1}^{K-1} n$; and the cardinalities of the sets, $\widetilde{\mathcal{R}}$, $\widetilde{\mathcal{C}}$, and $\widetilde{\mathcal{S}}_{eq}$ are the same and are given by $\widetilde{n}_R = K - 1$.

In the following, a procedure for deriving the conditional CDF of Γ_U given that the repeating terms are held constant, which is denoted by $\mathbb{P}\left(\Gamma_U < x | \widetilde{\mathcal{S}}_{eq}\right)$, will be given. The procedure begins by presenting the conditional end-to-end SNR, $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$, that is, the end-to-end SNR given the event, $\widetilde{\mathcal{S}}_{eq}$:

Lemma 3. The end-to-end SNR, $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$, conditioned on the event $\widetilde{\mathcal{S}}_{eq}$, is

$$\Gamma_{U|\widetilde{\mathcal{S}}_{eq}} = \gamma_{s,d} + \sum_{m=1}^{K-1} \min(y_m, \gamma_{m,d}) + \min(\gamma_{sRK,up}, \gamma_{K,d}) \ , \tag{3.13}$$

where

$$\gamma_{sRK,up} = \gamma_{s,K} + \sum_{m=1}^{K-1} \min(y_m, \gamma_{m,K}) \ . \tag{3.14}$$

Proof. Substitute $y_m \in \widetilde{\mathcal{C}}$ for $\gamma_{sRm,up} \in \widetilde{\mathcal{R}}$ into (3.6) for $1 \leq m \leq K-1$. □

The end-to-end SNR given that the repeating terms are held constant, $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$, takes on different forms for different values of the constants, $y_m \in \widetilde{\mathcal{C}}$, which determine whether it is a RV or a constant that emerges from each of the $\min(\cdot,\cdot)$ functions in the expression for $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$. Therefore, these different cases need to be treated separately when solving for $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$, and the law of total probability needs to be used to add up the probabilities of all the different cases. Whether $\min(y_i, \gamma_{i,j}) = y_i$ or $\min(y_i, \gamma_{i,j}) = \gamma_{i,j}$ depends on whether $\gamma_{i,j} < y_i$ or $\gamma_{i,j} > y_i$, which in turn determines what form $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$ will take. For that reason, the following notation and definition is introduced. Let the event that $\gamma_{i,j} < y_i$ be denoted by $\mathcal{I}_{i,j}^{(0)}$; and let the event that $\gamma_{i,j} > y_i$ be denoted by $\mathcal{I}_{i,j}^{(1)}$. It can be determined from Theorem 6 and Definition 3 that there are $n_c = \sum_{n=2}^{K} n$ terms of the form $\min(y_i, \gamma_{i,j})$ which determine which of the $N_c = 2^{n_c}$ possible expressions that $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$ will take on. Therefore, there are N_c cases to consider, each case being determined by a set of n_c inequality events, $\mathcal{I}_{i,j}^{(b)}$, where $b \in \{0, 1\}$.

Definition 4. *Let $\mathcal{S}_{ineq}^{(r)}$ be an ordered set of inequality events that determines which of the N_c forms that $\Gamma_{U|\widetilde{\mathcal{S}}_{eq}}$ will take on:*

$$\mathcal{S}_{ineq}^{(r)} := \left\{\mathcal{I}_{i,j}^{(b_n)} : 1 \leq i \leq K-1,\ i < j \leq K+1,\ 1 \leq n \leq n_c\right\} = \left\{\mathcal{I}_{1,2}^{(b_1)}, \ldots, \mathcal{I}_{K-1,K+1}^{(b_{n_c})}\right\} \ , \tag{3.15}$$

for $r = 0, 1, \ldots, N_c - 1$; and where the superscripts, b_n, come from the binary representation of r, which is given by $r_b = [b_1, \ldots, b_{n_c}]$; $b_n \in \{0, 1\}$ for $1 \le n \le n_c$; and the $\mathcal{I}_{i,j}^{(b_n)}$ terms are ordered primarily in terms of ascending i, and then, for a given i, in terms of ascending j.

□

In the next subsection, the conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$ is expressed in a form that contains terms that can be and will be calculated in subsequent subsections.

3.3.3 Expressions for the Conditional CDF of the End-to-End SNR

The conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$, can be expressed in terms of the N_C different cases by using the law of total probability, as shown in the following lemma.

Lemma 4. The conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$, is given by

$$\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right) = \sum_{r=0}^{N_c-1} \mathbb{P}\left(\Gamma_U < x,\ \mathcal{S}_{ineq}^{(r)}|\widetilde{\mathcal{S}}_{eq}\right) = \sum_{r=0}^{N_c-1} P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)\ , \tag{3.16}$$

where $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right) = \mathbb{P}\left(\Gamma_U < x,\ \mathcal{S}_{ineq}^{(r)}|\widetilde{\mathcal{S}}_{eq}\right)$ is a function of x and all $y_m \in \widetilde{\mathcal{C}}$.

Proof. The law of total probability is used. Note that $\{\mathcal{S}_{ineq}^{(r)} : 0 \le r < N_c\}$ forms a partition of the entire probability space. □

The expression for the conditional CDF, $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$, is given in the following lemma.

Lemma 5. The conditional CDF, $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$, of the compound event $\left(\Gamma_U < x \text{ and } \mathcal{S}_{ineq}^{(r)}\right)$

given the event $\widetilde{\mathcal{S}}_{eq}$, is given by

$$P^{(r)}\left(x,\widetilde{\mathcal{C}}\right)=\mathbb{P}\left(\Gamma_U<x,\ \mathcal{S}_{ineq}^{(r)}|\widetilde{\mathcal{S}}_{eq}\right)=\mathbb{P}\left(\Gamma_U<x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)\cdot\prod_{\mathcal{S}_{ineq}^{(r)}}\mathbb{P}\left(\mathcal{I}_{i,j}^{(b_n)}\right) \quad (3.17)$$

for $r=0,1,\ldots,N_c-1$; and where b_n is related to r according to Definition 5; and the product is done over all $\mathbb{P}\left(\mathcal{I}_{i,j}^{(b_n)}\right)$ such that $\mathcal{I}_{i,j}^{(b_n)}\in\mathcal{S}_{ineq}^{(r)}$.

Proof. The fact that $\mathbb{P}(A,\ B|C)=\mathbb{P}(A|B,\ C)\cdot\mathbb{P}(B)$ is used. Also note that $\mathbb{P}(B)$ represents $\mathbb{P}\left(\mathcal{S}_{ineq}^{(r)}\right)=\prod_{\mathcal{S}_{ineq}^{(r)}}\mathbb{P}\left(\mathcal{I}_{i,j}^{(b_n)}\right)$. □

As long as the probabilities, $\mathbb{P}\left(\Gamma_U<x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)$ from (3.17), are expressed in terms of x and $y_1,\ldots,y_{K-1}$, they are only affected by $n_{c,init}=2K-2\leq n_c$ inequality events: $K-1$ from the terms in the sum of (3.13), and the other $K-1$ from the terms in the sum of (3.14). The other $n_\Delta:=n_c-n_{c,init}$ inequality events of the set, $\mathcal{S}_{ineq}^{(r)}$, can initially be ignored until future steps in which the probabilities are expressed as functions of x and the constants, $z_{i,j}\in\mathcal{C}$. Therefore, it is useful to introduce another set of inequality events.

Definition 5. *Let $\widetilde{\mathcal{S}}_{ineq}^{(r')}$ be an ordered set of events:*

$$\widetilde{\mathcal{S}}_{ineq}^{(r')}=\left\{\mathcal{I}_{i,j}^{(b_n)}:1\leq i\leq K-1,\ K\leq j\leq K+1,\ 1\leq n\leq n_{c,init}\right\}\ , \quad (3.18)$$

for $r'=0,1,\ldots,N_{c,init}-1$; and where $N_{c,init}=2^{n_{c,init}}$; the superscripts, b_n, come from the binary representation of r', which is given by $r_b=[b_1,\ldots,b_{n_{c,init}}]$; $b_n\in\{0,1\}$ for $1\leq n\leq n_{c,init}$; and the $\mathcal{I}_{i,j}^{(b_n)}$ terms are ordered primarily in terms of ascending i, and then, for a given i, in the order of $j=\{K,K+1\}$.

□

Consider now the conditional probability, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$.

Lemma 6. $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$ is related to $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)$ as follows:

$$\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right) = \mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)\ , \tag{3.19}$$

where $r = i \cdot N_{c,init} + r'$, for $0 \leq i < N_\Delta := 2^{n_\Delta}$; $0 \leq r' < N_{c,init}$; and $0 \leq r < N_c$.

Proof. It follows from the way that the sets $\widetilde{\mathcal{S}}_{ineq}^{(r')}$ and $\mathcal{S}_{ineq}^{(r)}$ are defined; and from the fact that n_Δ out of the n_c inequality events in the set $\mathcal{S}_{ineq}^{(r)}$ have no effect on $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)$. □

Note that each of the $N_{c,init}$ probabilities, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$, is equal to N_Δ probabilities, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(i\cdot N_{c,init}+r')}\right)$, for $0 \leq i < N_\Delta$, for a total of $N_c = N_{c,init} \cdot N_\Delta$ probabilities of the form, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)$. In the next subsection, the process of solving for $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$ begins.

3.3.4 MGF of the SNR Conditioned on Equality and Inequality Events

At this stage, the goal is to solve the probabilities, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$. In proceeding towards this goal, begin by denoting the end-to-end SNR, conditioned on the events $\widetilde{\mathcal{S}}_{eq}$ and $\widetilde{\mathcal{S}}_{ineq}^{(r')}$, as $\Gamma_{r'}$; and by denoting the conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$, as $\mathbb{P}\left(\Gamma_{r'} < x\right)$, for simplicity of notation:

$$\Gamma_{r'} := \Gamma_{U|\widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}} =: V_{i'} + W_{j'}\ , \tag{3.20}$$

where

$$V_{i'} = \gamma_{s,d} + \sum_{m=1}^{K-1} \min(y_m, \gamma_{m,d}) \tag{3.21}$$

$$W_{j'} = \min\left(\gamma_{sRK,up};\ \gamma_{K,d}\right) = \min\left(\gamma_{s,K} + \sum_{m=1}^{K-1} \min(y_m, \gamma_{m,K});\ \gamma_{K,d}\right) \tag{3.22}$$

for $0 \leq i' < 2^{K-1}$; and $0 \leq j' < 2^{K-1}$. The subscripts, r', can be expressed in terms of i' and j' by using the binary representations of all the subscripts. The RVs, $V_{i'}$, are of the form, $V = \sum_{m=1}^{M} \gamma_m + \sum_{n=1}^{N} y_n$, and the RVs, $W_{j'}$, are of the form, $W = \min(V; \gamma_d)$, where $\gamma_1, \ldots, \gamma_M$, and γ_d are independent exponential RVs, and $y_1, \ldots, y_N$ are nonnegative constants (real numbers). It will be necessary to find the moment generating functions (MGFs) of these types of RVs, and this is done in the following lemma.

Lemma 7. The MGFs, $M_v(s)$ and $M_w(s)$, of $V = \sum_{m=1}^{M} \gamma_m + \sum_{n=1}^{N} y_n$ and $W = \min(V; \gamma_d)$, respectively, for the case in which all of the independent exponential RVs, γ_m and γ_d, have *distinct* parameters, are given by

$$M_v(s) = \left(\prod_{n=1}^{N} e^{sy_n}\right) \sum_{m=1}^{M} \frac{c_m \zeta_m}{\zeta_m - s} \tag{3.23}$$

$$M_w(s) = \frac{\zeta_d}{\zeta_d - s} - \left(\prod_{n=1}^{N} e^{-\zeta_d y_n} e^{sy_n}\right) \frac{\zeta_d}{\zeta_d - s} + \left(\prod_{n=1}^{N} e^{-\zeta_d y_n} e^{sy_n}\right) \sum_{m=1}^{M} \frac{c_m(\zeta_m + \zeta_d)}{(\zeta_m + \zeta_d) - s}, \tag{3.24}$$

where ζ_m is the parameter of γ_m; ζ_d is the parameter of γ_d; and $c_m = \prod_{\substack{i=1,\\ i \neq m}}^{M} \frac{\zeta_i}{\zeta_i - \zeta_m}$.

Proof. See Appendix B.2. □

By using Lemma 7, the MGFs of $V_{i'}$, $W_{j'}$, and $\Gamma_{r'}$ can be expressed in the following general form:

$$M_a(s) = \sum_{n=1}^{n_a} \left(\prod_{l=1}^{K-1} e^{u_{a,n}^{(l)} s y_l} e^{\lambda_{a,n}^{(l)} y_l} \right) \sum_{m=1}^{m_a} \frac{\alpha_{a,n}^{(m)} \beta_{a,n}^{(m)}}{\beta_{a,n}^{(m)} - s} , \tag{3.25}$$

where $u_{a,n}^{(l)}$ is a nonnegative integer; $\lambda_{a,n}^{(l)}$ and $\alpha_{a,n}^{(m)}$ are real numbers; and $\beta_{a,n}^{(m)}$ is a nonegative real number. Also, $\beta_{a,n}^{(m)} = 0$ if and only if $\alpha_{a,n}^{(m)} = 0$.

There are a total of 2^K RVs of the form $V_{i'}$ or $W_{j'}$ that need to be solved, and it is assumed that solving 2^K MGFs individually, without the use of a general formula, is manageable. Therefore, no general formula is given for the terms, $\alpha_{a,n}^{(m)}$, $\beta_{a,n}^{(m)}$, $u_{a,n}^{(i)}$, $\lambda_{a,n}^{(i)}$, m_a, and n_a of (3.25), since it will be assumed that the required MGFs have already been solved and that, therefore, all of these parameters will already be known. For the same reason, no general formula is given for the relationship between the subscripts, r', i', and j' of (3.20). From this point on, however, it will be necessary to come up with a set of *general and systematic* equations that can be solved with a computer program for any value of K, since the number of equations that are required to be solved, in order to solve for the CDF of Γ_U, grows very large for very small values of K already. (There are $N_c = 2^{\sum_{i=2}^{K} i}$ terms that need to be solved in (3.16), each of which requires numerous calculations. Then, after those terms are expressed in terms of the constants, $z_{i,j} \in \mathcal{C}$, they need to be averaged out over the corresponding n_R RVs, $\gamma_{i,j} \in \mathcal{R}$, with each expectation (averaging out) also requiring many calculations, and where $n_R = \sum_{i=1}^{K-1} i$.)

To develop a systematic approach for solving all of these equations, begin by noting that the MGFs of (3.25) are completely defined by their parameters, $\alpha_{a,n}^{(m)}$, $\beta_{a,n}^{(m)}$, $u_{a,n}^{(l)}$, $\lambda_{a,n}^{(l)}$, m_a, and n_a. Therefore, these MGFs can be represented by matrices consisting of their parameters. Then the operations required to calculate the CDF of

Γ_U, such as multiplying MGFs, obtaining CDFs from the corresponding MGFs, and taking the expectation of these conditional CDFs with respect to the RVs that have been held constant, can be carried out by using properly defined matrix operations which can be calculated using a computer program.

Definition 6. *Let the* MGF transform *of $M_a(s)$ be denoted by $\hat{M}_a := \mathcal{T}_m\{M_a(s)\}$, where $M_a(s)$ is the MGF given in* (3.25)*. Let $\hat{M}_a$ be defined as*

$$\hat{M}_a = \begin{bmatrix} u_{a,1}^{(1)}, & \cdots, & u_{a,1}^{(K-1)}, & \lambda_{a,1}^{(1)}, & \cdots, & \lambda_{a,1}^{(K-1)}, & \alpha_{a,1}^{(1)}, & \beta_{a,1}^{(1)}, & \cdots & \alpha_{a,1}^{(m_a)}, & \beta_{a,1}^{(m_a)} \\ & \vdots & & & & & & & & & \\ u_{a,n_a}^{(1)}, & \cdots, & u_{a,n_a}^{(K-1)}, & \lambda_{a,n_a}^{(1)}, & \cdots, & \lambda_{a,n_a}^{(K-1)}, & \alpha_{a,n_a}^{(1)}, & \beta_{a,n_a}^{(1)}, & \cdots & \alpha_{a,n_a}^{(m_a)}, & \beta_{a,n_a}^{(m_a)} \end{bmatrix} \tag{3.26}$$

□

Note that the size of $\hat{M}_a$ is $n_a \times 2(K-1+m_a)$. Now, a multiplication operator that represents the multiplication of two MGFs of the form given by (3.25), but carries out the multiplication by using the MGF transforms of the form given by (3.26), is introduced.

Definition 7. *Let the* MGF transform multiplication operator*, $\odot$, be defined as follows: If $\hat{M}_1 = \mathcal{T}_m\{M_1(s)\}$; $\hat{M}_2 = \mathcal{T}_m\{M_2(s)\}$; $\hat{M}_3 = \mathcal{T}_m\{M_3(s)\}$; and $M_3(s) = M_1(s) \cdot M_2(s)$; then $\hat{M}_3 := \hat{M}_1 \odot \hat{M}_2$.*

□

The following lemma demonstrates how the MGF transforms of Definition 7 are related.

Lemma 8. If $\hat{M}_3 = \hat{M}_1 \odot \hat{M}_2$, where $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$ are defined by (3.26), then the parameters of $\hat{M}_3$ are given, in terms of the parameters of $\hat{M}_1$ and $\hat{M}_2$, by the following equations:

$$u_{3,n}^{(l)} = u_{1,i}^{(l)} + u_{2,j}^{(l)} \qquad \lambda_{3,n}^{(l)} = \lambda_{1,i}^{(l)} + \lambda_{2,j}^{(l)}$$

$$\alpha_{3,n}^{(m)} = \begin{cases} \sum_{k=1}^{m_2} \alpha_{1,i}^{(m)} \alpha_{2,j}^{(k)} \frac{\beta_{2,j}^{(k)}}{\beta_{2,j}^{(k)} - \beta_{1,i}^{(m)}} & , \quad 1 \le m \le m_1 \\ \sum_{k=1}^{m_1} \alpha_{1,i}^{(k)} \alpha_{2,j}^{(m-m_1)} \frac{\beta_{1,i}^{(k)}}{\beta_{1,i}^{(k)} - \beta_{2,j}^{(m-m_1)}} & , \quad m_1 + 1 \le m \le m_1 + m_2 = m_3 \end{cases}$$

$$\beta_{3,n}^{(m)} = \begin{cases} \beta_{1,i}^{(m)} & , \quad 1 \le m \le m_1 \\ \beta_{2,j}^{(m-m_1)} & , \quad m_1 + 1 \le m \le m_1 + m_2 = m_3 \end{cases}$$

where $1 \le l \le K-1$; $1 \le i \le n_1$; $1 \le j \le n_2$; and $1 \le n \le n_3$. Row $n = (i-1)n_2 + j$ of $\hat{M}_3$ corresponds to the *MGF multiplication* of row i from $\hat{M}_1$ and row j from $\hat{M}_2$. The size of $\hat{M}_1$ is $n_1 \times 2(K-1+m_1)$; the size of $\hat{M}_2$ is $n_2 \times 2(K-1+m_2)$; the size of $\hat{M}_3$ is $n_3 \times 2(K-1+m_3)$; $n_3 = n_1 \cdot n_2$; and $m_3 = m_1 + m_2$.

Proof. It is straightforward to calculate $M_3(s) = M_1(s) \cdot M_2(s)$ using partial fraction techniques; and then find $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$ from $M_1(s)$, $M_2(s)$, and $M_3(s)$, respectively, by using Definition 6; and then show that the entries of $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$ are indeed related by the equations given in this lemma. □

Definitions 6 and 7, as well as Lemma 8, are used to find the MGFs, $M_{\Gamma_{r'}}(s)$, of the conditional SNRs, $\Gamma_{r'} = V_{i'} + W_{j'}$, for $0 \le r' < N_{c,init}$. The MGF transforms, $\hat{M}_{V_{i'}} = \mathcal{T}_m \left\{ M_{V_{i'}}(s) \right\}$ and $\hat{M}_{W_{j'}} = \mathcal{T}_m \left\{ M_{W_{j'}}(s) \right\}$, of the corresponding RVs, $V_{i'}$ and $W_{j'}$, respectively, are calculated first. Since the RVs, $V_{i'}$ and $W_{j'}$, are independent, it follows that $M_{\Gamma_{r'}}(s) = M_{V_{i'}}(s) \cdot M_{W_{j'}}(s)$ (meaning that the MGF transform multiplication operator is applicable in this case). Therefore, the MGF transform of $\Gamma_{r'}$ is found as $\hat{M}_{\Gamma_{r'}} = \hat{M}_{V_{i'}} \odot \hat{M}_{W_{j'}}$. Then, $M_{\Gamma_{r'}}(s) = \mathcal{T}_m^{-1}\left\{\hat{M}_{\Gamma_{r'}}\right\}$, that is, $M_{\Gamma_{r'}}(s)$ is found as the *inverse MGF transform* of $\hat{M}_{\Gamma_{r'}}$, according to Definition 6. In the next subsection, the conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$, of (3.16) is found. The derivation begins by finding the CDF of $\mathbb{P}\left(\Gamma_{r'} < x\right) = \mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq},\, \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$, and proceeds by multiplying the CDFs in (3.17).

3.3.5 Calculating Conditional CDF of End-to-End SNR – Multiplication of CDFs

It is now necessary to find the CDF, $F_{\Gamma_{r'}}(x) = \mathbb{P}\left(\Gamma_{r'} < x\right)$, from the MGF, $M_{\Gamma_{r'}}(s)$, which can be accomplished by using the following lemma.

Lemma 9. For the MGF, $M_a(s)$, given in (3.25), the CDF, $F_{a'}(x)$, of the same RV is given by

$$\begin{aligned} F_{a'}(x) &= \sum_{n=1}^{n_a}\sum_{m=1}^{m_a}\left\{\alpha_{a,n}^{(m)}\left(\prod_{l=1}^{K-1} e^{\lambda_{a,n}^{(l)} y_l}\right)\right. \\ &\qquad \left. - \alpha_{a,n}^{(m)} e^{-[\beta_{a,n}^{(m)}]x}\left(\prod_{l=1}^{K-1} e^{[\lambda_{a,n}^{(l)} + u_{a,n}^{(l)}\beta_{a,n}^{(m)}]y_l}\right)\right\}\cdot U\left(x - \sum_{l=1}^{K-1} u_{a,n}^{(l)} y_l\right) \\ &= \sum_{n=1}^{n_{a'}}\sum_{k=1}^{m_{a'}} \alpha_{a',n}^{(k)} e^{-[\beta_{x,a',n}^{(k)}]x}\left(\prod_{l=1}^{K-1} e^{[\beta_{y_l,a',n}^{(k)}]y_l}\right)\cdot U\left(x - \sum_{l=1}^{K-1} u_{a',n}^{(l)} y_l\right), \end{aligned} \tag{3.27}$$

where $n_{a'} = n_a$, $m_{a'} = 2m_a$, and $U(\cdot)$ is the unit step function. The parameters of the CDF are related to the parameters of the MGF by the following equations:

$$\begin{aligned} &\alpha_{a',n}^{(2m-1)} = \alpha_{a,n}^{(m)} && \beta_{x,a',n}^{(2m-1)} = 0 && \beta_{y_l,a',n}^{(2m-1)} = \lambda_{a,n}^{(l)} && u_{a',n}^{(l)} = u_{a,n}^{(l)} \\ &\alpha_{a',n}^{(2m)} = -\alpha_{a,n}^{(m)} && \beta_{x,a',n}^{(2m)} = \beta_{a,n}^{(m)} && \beta_{y_l,a',n}^{(2m)} = \lambda_{a,n}^{(l)} + u_{a,n}^{(l)}\beta_{a,n}^{(m)} && \end{aligned}$$

for $1 \leq l \leq K-1$; $1 \leq m \leq m_a$; and $1 \leq n \leq n_a$.

Proof. The probability density function (PDF), $f_a(x)$, corresponding to the MGF, $M_a(s)$, is found using the inverse Laplace transform, and the CDF, $F_{a'}(x)$, is found by integrating $f_a(x)$. □

Now that the conditional CDFs $\mathbb{P}\left(\Gamma_{r'} < x\right) = \mathbb{P}\left(\Gamma_U < x | \widetilde{\mathcal{S}}_{eq},\ \widetilde{\mathcal{S}}_{ineq}^{(r')}\right)$ have been solved, the conditional CDFs $\mathbb{P}\left(\Gamma_r < x\right) = \mathbb{P}\left(\Gamma_U < x | \widetilde{\mathcal{S}}_{eq},\ \mathcal{S}_{ineq}^{(r)}\right)$ are found from (3.19) and given by $\mathbb{P}\left(\Gamma_r < x\right) = \mathbb{P}\left(\Gamma_{r'} < x\right)$, where $r = i \cdot N_{c,init} + r'$ for $0 \leq i < N_\Delta$; $0 \leq r' < N_{c,init}$; and $0 \leq r < N_c$.

It can be seen from (3.17) that the next step is to multiply the CDFs, $\mathbb{P}\left(\Gamma_r < x\right)$, by terms of the form $\mathbb{P}\left(\mathcal{I}_{i,j}^{(0)}\right) = \mathbb{P}\left(\gamma_{i,j} < y_i\right) = 1 - e^{-\beta_{i,j} y_i}$ and / or $\mathbb{P}\left(\mathcal{I}_{i,j}^{(1)}\right) = \mathbb{P}\left(\gamma_{i,j} > y_i\right) = e^{-\beta_{i,j} y_i}$, in order to solve for the terms, $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$. The CDFs, $\mathbb{P}\left(\Gamma_r < x\right)$ and $\mathbb{P}\left(\mathcal{I}_{i,j}^{(0)}\right)$, and the reliability functions, $\mathbb{P}\left(\mathcal{I}_{i,j}^{(1)}\right)$, will be represented by matrices consisting of their parameters, and a properly defined matrix operation will be applied to these matrices in order to carry out the multiplications.

Definition 8. *Let the* CDF transform *of $F_{a'}(x)$ be denoted by $\hat{F}_{a'} := \mathcal{T}_c\left\{F_{a'}(x)\right\}$, where $F_{a'}(x)$ is the CDF given in* (3.27). *Let $\hat{F}_{a'}$ be defined as*

$$\hat{F}_{a'} := \begin{bmatrix} u^{(1)}_{a',1}, & \cdots, & u^{(K-1)}_{a',1}, & \alpha^{(1)}_{a',1}, & \beta^{(1)}_{x,a',1}, & \beta^{(1)}_{y_1,a',1}, & \cdots, & \beta^{(1)}_{y_{K-1},a',1}, \\ & & \cdots, & \alpha^{(m_{a'})}_{a',1}, & \beta^{(m_{a'})}_{x,a',1}, & \beta^{(m_{a'})}_{y_1,a',1}, & \cdots, & \beta^{(m_{a'})}_{y_{K-1},a',1} \\ & \vdots & & & & & & \\ u^{(1)}_{a',n_{a'}}, & \cdots, & u^{(K-1)}_{a',n_{a'}}, & \alpha^{(1)}_{a',n_{a'}}, & \beta^{(1)}_{x,a',n_{a'}}, & \beta^{(1)}_{y_1,a',n_{a'}}, & \cdots, & \beta^{(1)}_{y_{K-1},a',n_{a'}}, \\ & & \cdots, & \alpha^{(m_{a'})}_{a',n_{a'}}, & \beta^{(m_{a'})}_{x,a',n_{a'}}, & \beta^{(m_{a'})}_{y_1,a',n_{a'}}, & \cdots, & \beta^{(m_{a'})}_{y_{K-1},a',n_{a'}} \end{bmatrix}. \tag{3.28}$$

□

Note that the size of $\hat{F}_{a'}$ is $n_{a'} \times [K - 1 + m_{a'}(K + 1)]$. Also note that the CDF transform, $\hat{F}_{a'}$, can be found directly from the corresponding MGF transform, $\hat{M}_a$, by using the equations given in Lemma 9 that relate the parameters of the CDF, $F_{a'}(x)$, to the parameters of the MGF, $M_a(s)$, without having to solve the actual CDF and MGF. Now, a multiplication operator that represents the multiplication of two CDFs of the form given by (3.27), but carries out the multiplication by using the CDF transforms of the form given by (3.28), is introduced.

Definition 9. *Let the* CDF transform multiplication operator*,* $\otimes$*, be defined as follows: If* $\hat{F}_1 = \mathcal{T}_c\{F_1(x)\}$*,* $\hat{F}_2 = \mathcal{T}_c\{F_2(x)\}$*,* $\hat{F}_3 = \mathcal{T}_c\{F_3(x)\}$*, and* $F_3(x) = F_1(x) \cdot F_2(x)$*, then* $\hat{F}_3 := \hat{F}_1 \otimes \hat{F}_2$*. To complete the definition of the* $\otimes$ *operator, impose the restriction that* $\hat{F}_2$ *must have only* 1 *row with the first* $K-1$ *entries of the row equal to* 0*.*

□

The restriction imposed in Definition 9 will prove to be convenient, and the $\otimes$ operator will still be general enough to solve the multiplications that are required in this section. The following lemma demonstrates how the CDF transforms of Definition 9 are related.

Lemma 10. If $\hat{F}_3 = \hat{F}_1 \otimes \hat{F}_2$, where $\hat{F}_1$, $\hat{F}_2$, and $\hat{F}_3$ are defined by (3.28), and $\hat{F}_2$ is restricted to having 1 row with the first $K-1$ entries of the row equal to 0, then the parameters of $\hat{F}_3$ are given, in terms of the parameters of $\hat{F}_1$ and $\hat{F}_2$, by the following equations:

$$u_{3,n}^{(i)} = u_{1,n}^{(i)} \qquad \alpha_{3,n}^{(k)} = \alpha_{1,n}^{(m)}\alpha_{2,1}^{(l)} \qquad \beta_{x,3,n}^{(k)} = \beta_{x,1,n}^{(m)} + \beta_{x,2,1}^{(l)} \qquad \beta_{y_i,3,n}^{(k)} = \beta_{y_i,1,n}^{(m)} + \beta_{y_i,2,1}^{(l)}$$

for $1 \leq i \leq K-1$; $1 \leq k \leq m_3$; $1 \leq l \leq m_2$; $1 \leq m \leq m_1$; and $1 \leq n \leq n_1$; and where $k = (m-1)m_2 + l$. Row n of $\hat{F}_3$ corresponds to the *CDF multiplication* of row n from $\hat{F}_1$ and row 1 from $\hat{F}_2$ for $1 \leq n \leq n_1$. The size of $\hat{F}_1$ is $n_1 \times [K-1+m_1(K+1)]$; the size of $\hat{F}_2$ is $1 \times [K-1+m_2(K+1)]$; the size of $\hat{F}_3$ is $n_1 \times [K-1+m_3(K+1)]$; and $m_3 = m_1 \cdot m_2$.

Proof. It is straightforward to calculate $F_3(x) = F_1(x) \cdot F_2(x)$; and then find $\hat{F}_1$, $\hat{F}_2$, and $\hat{F}_3$ from $F_1(x)$, $F_2(x)$, and $F_3(x)$, respectively, by using Definition 8; and then show that the entries of $\hat{F}_1$, $\hat{F}_2$, and $\hat{F}_3$ are indeed related by the equations given in this lemma. □

Now, let $\hat{P}^{(r)}_{\{x,\widetilde{\mathcal{C}}\}}$, $\hat{F}_{\Gamma_r}$, $\hat{P}^{(0)}_{\gamma_{i,j}}$, and $\hat{P}^{(1)}_{\gamma_{i,j}}$ be the CDF transforms of $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$, $\mathbb{P}\left(\Gamma_r < x\right)$, $\mathbb{P}\left(\mathcal{I}^{(0)}_{i,j}\right)$, and $\mathbb{P}\left(\mathcal{I}^{(1)}_{i,j}\right)$, respectively. Note that $\hat{P}^{(0)}_{\gamma_{i,j}}$ and $\hat{P}^{(1)}_{\gamma_{i,j}}$ are special cases of (3.28) with $n_{a'} = 1$ row and the first $K-1$ entries of the row equal to 0. The CDF transform, $\hat{P}^{(r)}_{\{x,\widetilde{\mathcal{C}}\}}$, is solved by using Definition 9 to take the CDF transform of (3.17) as follows:

$$\hat{P}^{(r)}_{\{x,\widetilde{\mathcal{C}}\}} = \hat{F}_{\Gamma_r} \bigotimes_{n=1}^{n_c} \hat{P}^{(b_n)}_{\gamma_{i,j}} = \hat{F}_{\Gamma_r} \otimes \hat{P}^{(b_1)}_{\gamma_{1,2}} \otimes \cdots \otimes \hat{P}^{(b_{n_c})}_{\gamma_{K-1,K+1}} , \tag{3.29}$$

where $\{b_n : 1 \leq n \leq n_c\}$ is related to r according to Definition 5; the $\bigotimes$ operator denotes the *CDF transform multiplication product* of all of the $\hat{P}_{\gamma_{i,j}}^{(b_n)}$ terms corresponding to $\mathbb{P}\left(\mathcal{I}_{i,j}^{(b_n)}\right) \forall\, \mathcal{I}_{i,j}^{(b_n)} \in \mathcal{S}_{ineq}^{(r)}$; and the terms are multiplied according to the definition of the $\otimes$ operator. Then, $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$ is found for $0 \leq r < N_c$ by taking the inverse CDF transform of $\hat{P}_{\{x,\widetilde{\mathcal{C}}\}}^{(r)}$; and the conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$, is found by using (3.16). In the next subsection, the CDF of Γ_U is found by taking the expectation of $\mathbb{P}\left(\Gamma_U < x|\widetilde{\mathcal{S}}_{eq}\right)$ with respect to all of the RVs that were originally held constant.

3.3.6 Derivation of Unconditional CDF – Taking Expectation of Conditional CDF

The next step is to convert the conditional CDFs, $P^{(r)}\left(x, \widetilde{\mathcal{C}}\right)$, which are functions of x and the terms, $y_m \in \widetilde{\mathcal{C}}$, into the conditional CDFs, $P^{(r)}\left(x, \mathcal{C}\right) = \mathbb{P}\left(\Gamma_U < x,\ \mathcal{S}_{ineq}^{(r)}|\mathcal{S}_{eq}\right)$, which are functions of x and the terms, $z_{i,j} \in \mathcal{C}$. This is done by using the relationships, $y_m = z_{0,m}$ for $m = 1$ and $y_m = z_{0,m} + \sum_{i=1}^{m-1} \min(y_i, z_{i,m})$ for $m \geq 2$, in a recursive manner. At this stage, it is the $n_\Delta = n_c - n_{c,init}$ inequality events of the sets, $\mathcal{S}_{ineq}^{(r)}$, that were initially ignored that determine whether the $\min(y_i, z_{i,m})$ terms are equal to y_i or $z_{i,m}$, and therefore determine the final form of $P^{(r)}\left(x, \mathcal{C}\right)$. The conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\mathcal{S}_{eq}\right)$, a function of x and the terms, $z_{i,j} \in \mathcal{C}$, can now be solved as the sum of all the $P^{(r)}\left(x, \mathcal{C}\right)$ terms, that is, by using the law of total probability. Finally, the CDF of Γ_U, denoted by $\mathbb{P}(\Gamma_U < x)$, is solved by taking the expected value of the conditional CDF, $\mathbb{P}\left(\Gamma_U < x|\mathcal{S}_{eq}\right)$, with respect to the n_R RVs, $\gamma_{i,j} \in \mathcal{R}$, one at a time.

Lemma 11. The CDF of Γ_U is given by

$$\mathbb{P}(\Gamma_U < x) = \mathbb{E}_{\mathcal{R}}\left[\mathbb{P}\left(\Gamma_U < x|\mathcal{S}_{eq}\right)\right] = \sum_{r=0}^{N_c-1} \mathbb{E}_{\mathcal{R}}\left[P^{(r)}\left(x,\mathcal{C}\right)\right] , \tag{3.30}$$

where $\mathbb{E}_{\mathcal{R}}[\cdot]$ denotes an expectation that is done with respect to all RVs, $\gamma_{i,j} \in \mathcal{R}$.

□

The CDF transforms of the terms on the right-hand side of (3.30) will be taken and then another properly defined matrix operation will be introduced in order to calculate all of the expectations in (3.30). To begin the calculations, consider the conditional CDF, $P^{(r)}\left(x, z_1, \ldots, z_N\right) = \mathbb{P}\left(\Gamma_U < x, \mathcal{S}^{(r)}_{ineq}|\mathcal{S}^{*}_{eq}\right)$, which is given by

$$P^{(r)}\left(x, z_1, \ldots, z_N\right) = \sum_{n=1}^{n_r}\sum_{m=1}^{m_r} \alpha^{(m)}_{r,N,n} e^{-[\beta^{(m)}_{x,r,N,n}]x} \left(\prod_{l=1}^{N} e^{[\beta^{(m)}_{z_l,r,N,n}]z_l}\right) \cdot U\left(x - \sum_{l=1}^{N} u^{(l)}_{r,n} z_l\right) , \tag{3.31}$$

where $\mathcal{S}^{*}_{eq} = \{\mathcal{E}_l : 1 \leq l \leq N\} \subseteq \mathcal{S}_{eq}$; $\mathcal{E}_l$ represents the event, $\gamma_l = z_l$; each γ_l is equal to some $\gamma_{i,j} \in \mathcal{R}$; each z_l is equal to some $z_{i,j} \in \mathcal{C}$; and $N \leq n_R$ is the number of RVs, $\gamma_{i,j}$, being held constant. For $N = n_R$, $P^{(r)}\left(x, z_1, \ldots, z_N\right) = P^{(r)}\left(x, \mathcal{C}\right)$. The CDF transform, $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_N\}} = \mathcal{T}_c\left\{P^{(r)}\left(x,z_1,\ldots,z_N\right)\right\}$ is given by

$$\hat{P}^{(r)}_{\{x,z_1,\ldots,z_N\}} = \begin{bmatrix} u^{(1)}_{r,1}, & \cdots, & u^{(N)}_{r,1}, & \alpha^{(1)}_{r,N,1}, & \beta^{(1)}_{x,r,N,1}, & \beta^{(1)}_{z_1,r,N,1}, & \cdots, & \beta^{(1)}_{z_N,r,N,1}, \\ & & \cdots, & \alpha^{(m_r)}_{r,N,1}, & \beta^{(m_r)}_{x,r,N,1}, & \beta^{(m_r)}_{z_1,r,N,1}, & \cdots, & \beta^{(m_r)}_{z_N,r,N,1} \\ & \vdots & & & & & & \\ u^{(1)}_{r,n_r}, & \cdots, & u^{(N)}_{r,n_r}, & \alpha^{(1)}_{r,N,n_r}, & \beta^{(1)}_{x,r,N,n_r}, & \beta^{(1)}_{z_1,r,N,n_r}, & \cdots, & \beta^{(1)}_{z_N,r,N,n_r}, \\ & & \cdots, & \alpha^{(m_r)}_{r,N,n_r}, & \beta^{(m_r)}_{x,r,N,n_r}, & \beta^{(m_r)}_{z_1,r,N,n_r}, & \cdots, & \beta^{(m_r)}_{z_N,r,N,n_r} \end{bmatrix}. \tag{3.32}$$

Note that the size of $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_N\}}$ is $n_r \times [N + m_r(N+2)]$. Now, an expectation operator that represents taking the expectation of a conditional CDF of the form given by (3.31), where the expectation is carried out with respect to one of the RVs being held constant, but where this procedure is carried out by using CDF transforms of the form given by (3.32), is introduced.

Definition 10. *Let the* expectation operator, $\hat{\mathbb{E}}_{\gamma_{i,j}}$, *be defined as follows:*
If $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}} = \mathcal{T}_c\left\{P^{(r)}\left(x,z_1,\ldots,z_{N-1},z_{i,j}\right)\right\}$; $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}} = \mathcal{T}_c\left\{P^{(r)}\left(x,z_1,\ldots,z_{N-1}\right)\right\}$; *and* $P^{(r)}\left(x,z_1,\ldots,z_{N-1}\right) = \mathbb{E}_{\gamma_{i,j}}\left[P^{(r)}\left(x,z_1,\ldots,z_{N-1},z_{i,j}\right)\right]$, *where the expectation is done with respect to the RV,* $\gamma_{i,j} = z_{i,j}$*; then* $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}} := \hat{\mathbb{E}}_{\gamma_{i,j}}\left[\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}}\right]$.

□

The following lemma demonstrates how the CDF transforms of Definition 10 are related.

Lemma 12. If $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}} = \hat{\mathbb{E}}_{\gamma_{i,j}}\left[\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}}\right]$, and $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}}$ is defined by (3.32) with $z_N = z_{i,j}$, then $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}}$ is also defined by (3.32) with N replaced by $N-1$ and m_r replaced by $2m_r$. The size of $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}}$ is $n_r \times [N-1+2m_r(N+1)]$.

The relationships between the parameters in $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}}$ and the parameters in $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}}$ are given by the following equations:

$$\begin{aligned}
\alpha^{(2m-1)}_{r,N-1,n} &= \frac{\alpha^{(m)}_{r,N,n}\beta_{i,j}}{\beta_{i,j}-\beta^{(m)}_{z_{i,j},r,N,n}} & \alpha^{(2m)}_{r,N-1,n} &= -\alpha^{(2m-1)}_{r,N-1,n} \\
\beta^{(2m-1)}_{x,r,N-1,n} &= \beta^{(m)}_{x,r,N,n} & \beta^{(2m)}_{x,r,N-1,n} &= \beta^{(m)}_{x,r,N,n} + \frac{\beta_{i,j}-\beta^{(m)}_{z_{i,j},r,N,n}}{u^{(N)}_{r,n}} \\
\beta^{(2m-1)}_{z_l,r,N-1,n} &= \beta^{(m)}_{z_l,r,N,n} & \beta^{(2m)}_{z_l,r,N-1,n} &= \beta^{(m)}_{z_l,r,N,n} + \frac{u^{(i)}_{r,n}(\beta_{i,j}-\beta^{(m)}_{z_{i,j},r,N,n})}{u^{(N)}_{r,n}}
\end{aligned}$$

for $1 \le l \le N-1$; $1 \le m \le m_r$; $1 \le n \le n_r$; and $0 \le r < N_c$; as long as $u^{(N)}_{r,n} > 0$ for the given value of n. If $u^{(N)}_{r,n} = 0$, then the expressions for $\alpha^{(2m-1)}_{r,N-1,n}$, $\beta^{(2m-1)}_{x,r,N-1,n}$, and $\beta^{(2m-1)}_{z_l,r,N-1,n}$ are the same as above, but $\alpha^{(2m)}_{r,N-1,n} = \beta^{(2m)}_{x,r,N-1,n} = \beta^{(2m)}_{z_l,r,N-1,n} = 0$.

Proof. See Appendix B.3. □

In order to calculate $\mathbb{P}(\Gamma_U < x)$ from $\mathbb{P}(\Gamma_U < x|\mathcal{S}_{eq})$, the $\hat{\mathbb{E}}_{\gamma_{i,j}}$ operator needs to be applied n_R times, once for every RV, $\gamma_{i,j} \in \mathcal{R}$, to each of the $P^{(r)}(x,\mathcal{C})$ terms. In the first step, before any of the RVs, $\gamma_{i,j} \in \mathcal{R}$, have yet to be averaged out, $N = n_R$, and $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_N\}} = \hat{P}^{(r)}_{\{x,\mathcal{C}\}}$, where $\hat{P}^{(r)}_{\{x,\mathcal{C}\}}$ is the CDF transform of $P^{(r)}(x,\mathcal{C})$. At the final step, after the $\hat{\mathbb{E}}_{\gamma_{i,j}}$ operator has been applied n_R times, once for every RV, $\gamma_{i,j} \in \mathcal{R}$, the end result is $\hat{P}^{(r)}_{\{x\}} = \hat{\mathbb{E}}_{\gamma_{s,1}}\left[\hat{P}^{(r)}_{\{x,z_{01}\}}\right]$ (assuming that the last RV to be averaged out is $\gamma_{s,1}$, which is not strictly necessary). The general form of $\hat{P}^{(r)}_{\{x\}}$ is given by

$$\hat{P}^{(r)}_{\{x\}} = \begin{bmatrix} \alpha^{(1)}_{r,1}, & \beta^{(1)}_{x,r,1}, & \cdots, & \alpha^{(M_r)}_{r,1}, & \beta^{(M_r)}_{x,r,1} \\ \vdots & & & & \\ \alpha^{(1)}_{r,N_r}, & \beta^{(1)}_{x,r,N_r}, & \cdots, & \alpha^{(M_r)}_{r,N_r}, & \beta^{(M_r)}_{x,r,N_r} \end{bmatrix}. \quad (3.33)$$

The inverse CDF transform of $\hat{P}^{(r)}_{\{x\}}$ is given by

$$P^{(r)}(x) = \mathbb{E}_{\mathcal{R}}\left[P^{(r)}(x, \mathcal{C})\right] = \mathcal{T}_c^{-1}\left\{\hat{P}^{(r)}_{\{x\}}\right\} = \sum_{n=1}^{N_r}\sum_{m=1}^{M_r} \alpha^{(m)}_{r,n} e^{-[\beta^{(m)}_{x,r,n}]x} \cdot U(x) \ . \qquad (3.34)$$

Once $P^{(r)}(x)$ has been found for $0 \leq r < N_c$, the CDF of Γ_U is solved as shown in the following lemma.

Lemma 13. The CDF of Γ_U, that is, $\mathbb{P}(\Gamma_U < x)$, is given by

$$\mathbb{P}(\Gamma_U < x) = \sum_{r=0}^{N_c-1} P^{(r)}(x) = \sum_{r=0}^{N_c-1}\sum_{n=1}^{N_r}\sum_{m=1}^{M_r} \alpha^{(m)}_{r,n} e^{-[\beta^{(m)}_{x,r,n}]x} \cdot U(x) \ . \qquad (3.35)$$

□

Remark 1. It should be pointed out that the SER calculations in Chapter 2 and the outage probability calculations in this chapter both consist of finding the CDFs of SNR terms, and therefore may appear similar. However, in Chapter 2 the *exact* CDF was found for terms of the type, $\frac{\gamma_{sRm}\gamma_{m,d}}{\gamma_{sRm}+\gamma_{m,d}+1}$, *within* the overall SNR, where γ_{sRm} was expressed as a sum of exponential RVs and the end result of that CDF was given by (2.18). That approach was preferable for finding the SER. However, in order to solve for the outage probability, as is done in this chapter, the CDF for the *overall* end-to-end SNR is required. Since finding the CDF for the overall SNR is not possible, it was necessary to introduce the new *upper bound* expression, Γ_{up}, of the overall SNR, for which it *was* possible to find the CDF. For that reason, a substantially different approach needed to be used in this chapter as compared to the approach that was used in Chapter 2.

□

3.3.7 Lower Bound and High-SNR Approximation of Outage Probability

Now, a very tight lower bound of the outage probability is found and given in the following theorem.

Theorem 7. The lower bound of the outage probability, $P_{o,L}^{multi}$, for the multi-hop system, for a general value of $K \geq 2$ relays, is given by

$$P_{o,L}^{multi} := \mathbb{P}(\Gamma_U < 2^{(K+1)R} - 1) = \sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \alpha_{r,n}^{(m)} e^{-[\beta_{x,r,n}^{(m)}][2^{(K+1)R}-1]} \ , \tag{3.36}$$

where $N_c = 2^{n_c}$; $n_c = \sum_{i=2}^{K} i$; the $\alpha_{r,n}^{(m)}$ and $\beta_{x,r,n}^{(m)}$ parameters, as well as the values for M_r and N_r, are found by using the procedure described in previous subsections; and R is the spectral efficiency.

Proof. Substitute $x = 2^{(K+1)R} - 1$ into (3.35). □

The upper bound of the end-to-end SNR for the two-hop system can be found from the upper bound of the end-to-end SNR for the multi-hop system, Γ_U, by setting $\gamma_{i,m} = 0$ for $1 \leq i < m \leq K$ in (3.6). It follows that the expression for the lower bound of the outage probability given in (3.36) is valid for the two-hop system as well, in which case it reduces to [6, eq. (6.122)] and is given in the following corollary.

Corollary 7. The lower bound of the outage probability, $P_{o,L}^{2-hop}$, for the two-hop system, for a general value of $K \geq 1$ relays, is given by

$$P_{o,L}^{2-hop} = 1 - \sum_{m=0}^{K} c_m e^{-\beta_m[2^{(K+1)R}-1]} \ , \tag{3.37}$$

where $c_m = \prod_{i=0,i\neq m}^{K} \frac{\beta_i}{\beta_i - \beta_m}$ for $0 \leq m \leq K$; $\beta_0 = \overline{\gamma}_{s,d}^{-1}$; $\beta_m = \overline{\gamma}_{s,m,d}^{-1}$; $\overline{\gamma}_{s,m,d} = \frac{\overline{\gamma}_{s,m}\overline{\gamma}_{m,d}}{\overline{\gamma}_{s,m}+\overline{\gamma}_{m,d}}$ for $1 \leq m \leq K$; $\overline{\gamma}_{i,j}$ is the mean of the RV, $\gamma_{i,j}$; and R is the spectral efficiency.

Proof. For the two-hop system, the upper bound of the end-to-end SNR is given by $\Gamma_U^{2-hop} = \gamma_{s,d} + \sum_{m=1}^{K} \min(\gamma_{s,m}, \gamma_{m,d})$. Since this end-to-end SNR is simply a sum of exponential RVs, it is straightforward to find its CDF, $\mathbb{P}(\Gamma_U^{2-hop} < x)$. Then, the equation $P_{o,L}^{2-hop} = \mathbb{P}(\Gamma_U^{2-hop} < 2^{(K+1)R} - 1)$ leads to (3.37). □

Since the expression for $P_{o,L}^{2-hop}$ in (3.37) can be expressed in the form of (3.36), and since Γ_U^{2-hop} is a special case of Γ_U with $\gamma_{i,m} = 0$ for $1 \leq i < m \leq K$, the result in (3.37) can be considered a special case of (3.36), even though the lengthy procedure that was used for solving $P_{o,L}^{multi}$ in this section does not apply for the case of the two-hop system. Also, since (3.37) is identical to [6, eq. (6.122)], it follows that (3.36) can be considered to be a generalized version of the previous results found for the two-hop system in [6].

In the following lemma, a high-SNR straight-line approximation of the outage probability is given.

Lemma 14. A high-SNR straight-line approximation of the outage probability, $P_{o,H}^{multi}$, for the multi-hop system is given by

$$P_{o,H}^{multi} = \frac{1}{(K+1)!} \left(\sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \alpha_{r,n}^{(m)} [\beta_{x,r,n}^{(m)}]^{K+1} \right) (-1)^{K+1} \left(2^{(K+1)R} - 1 \right)^{K+1} , \tag{3.38}$$

where the parameters are described in Theorem 7.

Proof. See Appendix B.4. □

Note that (3.38) is derived as an approximation of (3.36), and, since (3.36) is valid for the two-hop system as well as the multi-hop system, it follows that the high-SNR approximation given in (3.38) is valid for the two-hop system as well, in which case it reduces to [6, eq. (6.133)] and is given in the following corollary.

Corollary 8. The high-SNR straight-line approximation of the outage probability, $P_{o,H}^{2-hop}$, for the two-hop system is given by

$$P_{o,H}^{2-hop} = \frac{1}{(K+1)!} \left(\prod_{m=0}^{K} \beta_m \right) \left(2^{(K+1)R} - 1 \right)^{K+1} , \tag{3.39}$$

where the parameters are described in Corollary 7.

Proof. See [6, Chapter 6]. The proof begins with (3.37), expresses the exponential functions in terms of Taylor series, ignores the higher order terms, and then demonstrates that many of the remaining terms are equal to 0. □

Note that (3.39) is identical to [6, eq. (6.133)]. It follows that (3.38) is a generalized version of the previous results found for the two-hop system in [6].

3.4 Numerical Results

The numerical results presented in this section demonstrate the accuracy of the expressions for the lower bound of the outage probability, given by (3.37) for the two-hop system and (3.36) for the multi-hop system. They also demonstrate the accuracy of the high-SNR straight-line approximations of the outage probability, given in [6] and repeated in (3.39) for the two-hop system, and given by (3.38) for the multi-hop system. Fig. 3.2 provides results for the two-hop system for $K = 1$, $K = 2$, and

$K = 3$ relays; Figs. 3.3 and 3.4 provide results for the multi-hop system for $K = 2$ and $K = 3$ relays; and Fig. 3.5 compares results between the two-hop system and the multi-hop system for $K = 3$ relays. In all cases, the spectral efficiency is $R = 1$ bps/Hz, and the power allocation scheme, under a total power constraint of P Watts, is given by $\varepsilon_o = P/2$ and $\varepsilon_m = \varepsilon_o/K$ for $1 \leq m \leq K$. Also, the AWGN variance, $N_{m,k}$, is the same and equal to N_o for all terminals m and k. In Figs. 3.2 and 3.3, all of the channel variances, $\Omega_{m,k}$, are equal to 1. In Fig. 3.4, the channel variances are: $\Omega_{s,d} = 1$; $\Omega_{s,m} = 5$ for $1 \leq m \leq K$; $\Omega_{m,k} = 10$ for $1 \leq m < k \leq K$; and $\Omega_{m,d} = 2$ for $1 \leq m \leq K$. In Fig. 3.5, the channel variances are: $\Omega_{s,d} = 0.1$; $\Omega_{s,1} = 10$; $\Omega_{s,2} = 12$; $\Omega_{s,3} = 0.12$; $\Omega_{1,2} = 16$; $\Omega_{1,3} = 18$; $\Omega_{2,3} = 20$; $\Omega_{1,d} = 0.14$; $\Omega_{2,d} = 0.16$; and $\Omega_{3,d} = 14$.

The most important result from the figures to note is that the lower bound is extremely tight to the actual outage probability, for most values of outage probability, for the cases shown in Figs. 3.4 and 3.5 (here the focus is on the results for the multi-hop system). In Fig. 3.5, the analytical results for the lower bound of the outage probability is almost exactly the same as the simulation results for outage probabilities as high as about 3×10^{-2}, and the analytical curve is right on top of the simulations curve for all outage probabilities less than that. In Fig. 3.4, the lower bound is not quite as tight to the actual outage probability as it was for the case of Fig. 3.5, but it is nevertheless very, very tight for all outage probabilities less than about 10^{-3}. The fact that the results are outstanding for these two cases is significant because, as it will be discussed shortly, these cases represent scenarios for which the multi-hop system performs very well. Therefore, they are cases for which it is desirable to use the multi-hop system and desirable to have accurate analytical results for it.

It can also be seen from Figs. 3.4 and 3.5 that the high-SNR straight-line approxi-

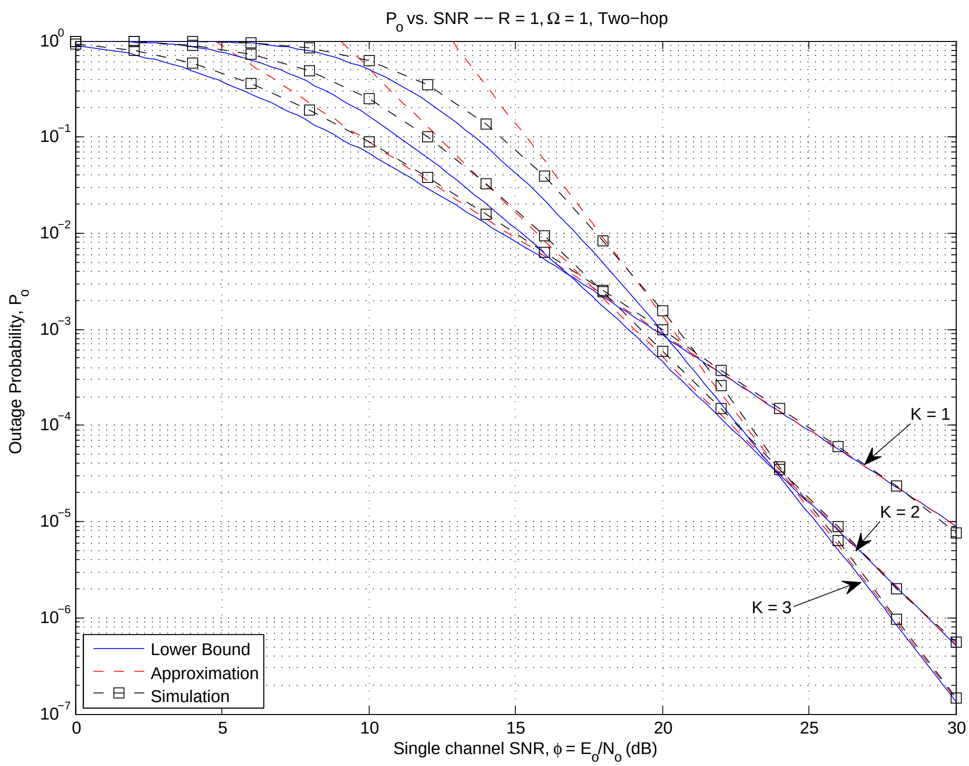

Figure 3.2: Outage probability of *two-hop* system with spectral efficiency of $R = 1$ bps/Hz, and channel variances of $\Omega_{i,j} = 1$ for all terminals i and j. The curves labeled 'Lower Bound' are generated by using (3.37), and the curves labeled 'Approximation' are generated by using (3.39).

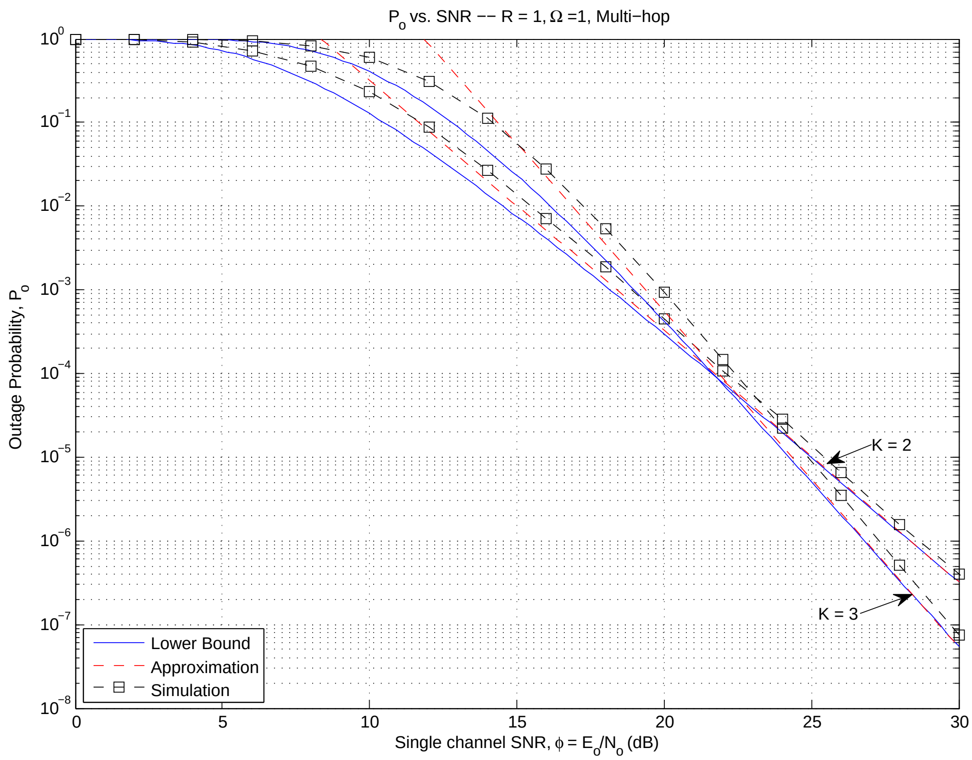

Figure 3.3: Outage probability of *multi-hop* system with spectral efficiency of $R = 1$ bps/Hz, and channel variances of $\Omega_{i,j} = 1$ for all terminals i and j. The curves labeled 'Lower Bound' are generated by using (3.36), and the curves labeled 'Approximation' are generated by using (3.38).

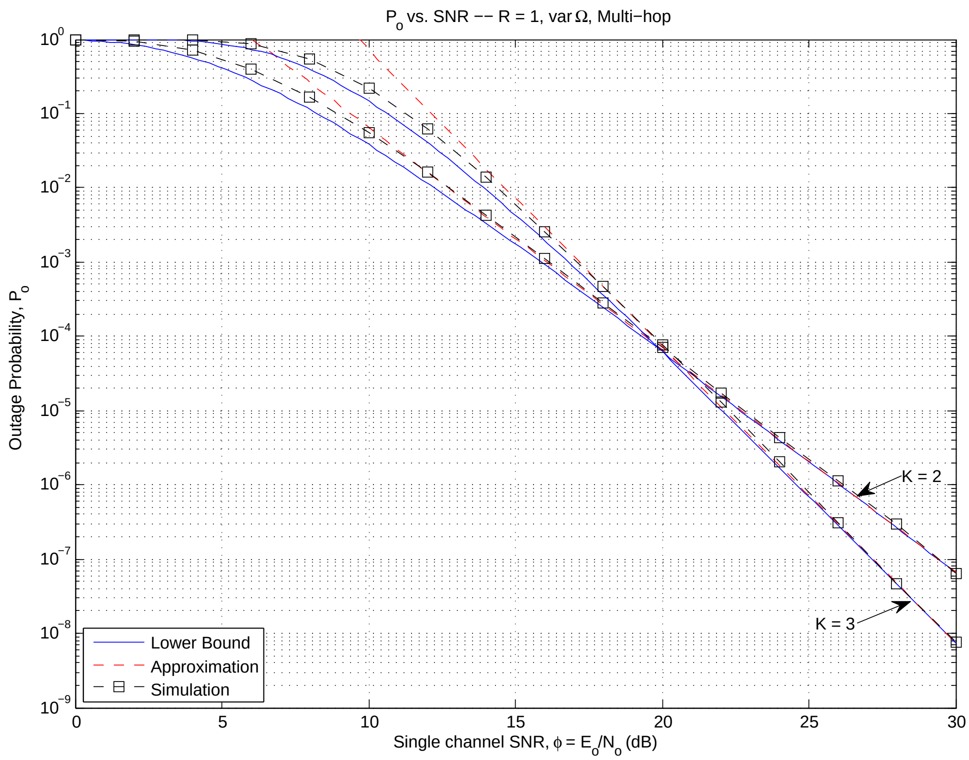

Figure 3.4: Outage probability of *multi-hop* system with spectral efficiency of $R = 1$ bps/Hz. The channel variances are $\Omega_{s,d} = 1$, $\Omega_{s,1} = 5$, $\Omega_{s,2} = 5$, $\Omega_{s,3} = 5$, $\Omega_{1,2} = 10$, $\Omega_{1,3} = 10$, $\Omega_{2,3} = 10$, $\Omega_{1,d} = 2$, $\Omega_{2,d} = 2$, and $\Omega_{3,d} = 2$. The curves labeled 'Lower Bound' are generated by using (3.36), and the curves labeled 'Approximation' are generated by using (3.38).

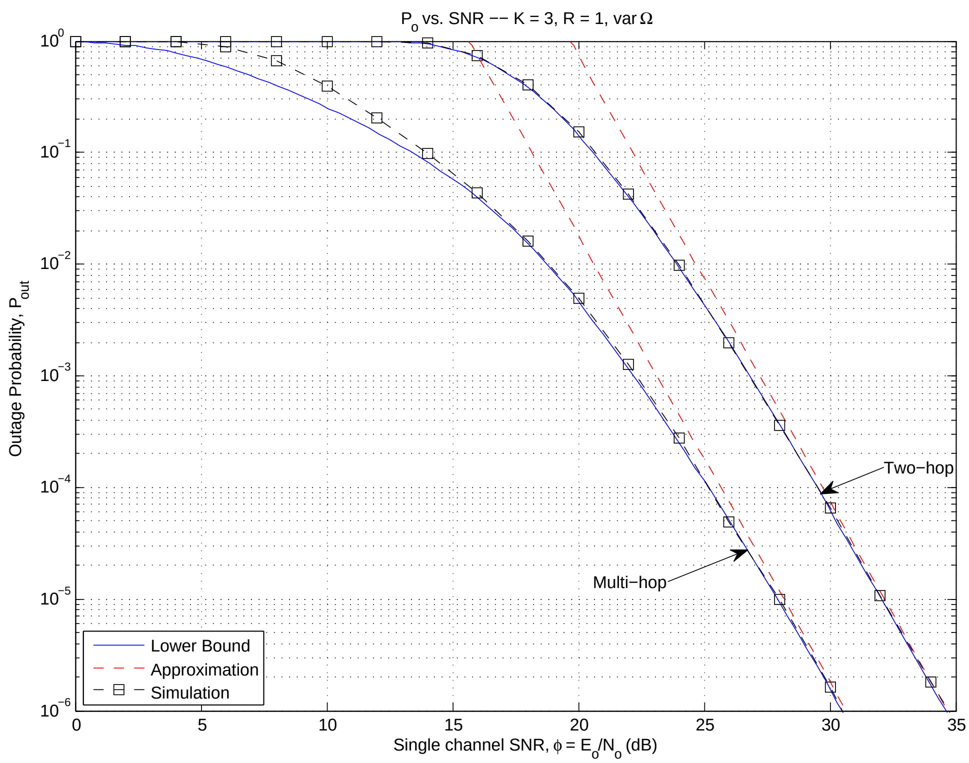

Figure 3.5: Outage probability of both the *two-hop* system and the *multi-hop* system with spectral efficiency of $R = 1$ bps/Hz. The channel variances are $\Omega_{s,d} = 0.1$, $\Omega_{s,1} = 10$, $\Omega_{s,2} = 12$, $\Omega_{s,3} = 0.12$, $\Omega_{1,2} = 16$, $\Omega_{1,3} = 18$, $\Omega_{2,3} = 20$, $\Omega_{1,d} = 0.14$, $\Omega_{2,d} = 0.16$, and $\Omega_{3,d} = 14$. The curves labeled 'Lower Bound' are generated by using (3.36), and the curves labeled 'Approximation' are generated by using (3.38).

mations are asymptotically tight to the simulation curves (the phrase, "asymptotically tight", is used to describe the observation that the analytical curve becomes identical to the simulation curve as the SNR goes to infinity). For the case of Fig. 3.4, the high-SNR straight-line approximation is extremely tight to the simulation curve (for the $K = 3$ case, the two curves are identical for outage probabilities as high as 5×10^{-2}), demonstrating that, for certain cases, the expression for the high-SNR approximation can also be very useful.

Another important result can be obtained by comparing Figs. 3.3 and 3.4. It can be seen from Fig. 3.3 that, for the multi-hop system where the channel variances are all equal to one (indicating that the distances between pairs of terminals are all equal), there is a slight offset (about 0.4 dB for the $K = 3$ case) in the high-SNR region between the lower bound and the simulation curve. However, in Fig. 3.4, for the multi-hop system in which the channel variances for the source-to-relay channels and relay-to-relay channels are greater than those for the relay-to-destination channels (indicating that the relays are close to each other and closer to the source than to the destination), it has already been seen that the lower bound of the outage probability is very tight to the simulation curve. Therefore, it can be concluded that the lower bound expression given in (3.36) for the outage probability of the multi-hop system is *very accurate when the relays are closer to the source than they are to the destination*, and that it becomes slightly less accurate as the relays get closer to the destination. It should also be pointed out that it was demonstrated in [6] that the *performance of the multi-hop system improves as the relays move close together and closer to the source than the destination*, whereas the performance worsens as the relays move closer to the destination, due to the fact that the received signals at the destination are strongly correlated for the latter case, where the correlation is due to noise propagation. (Note that noise propagation is always a problem, even for the two-hop case, in an AF system because the protocol calls for the relays to transmit noisy versions of their received signals. However, for the case of the multi-hop system, the additional problem, which is that the signals received by the destination are correlated, arises because the noise signals get propagated in several different paths due to the broadcasting nature of the relays). Therefore, it seems that the lower bound expression is very accurate for those cases in which the mutli-hop system performs very well.

The fact that the lower bound expression is very accurate for those cases in which the mutli-hop system performs very well is not a coincidence and it can be explained in the following way. It was shown in [23] that the approximate, instantaneous end-to-end SNR, Γ_{multi}, for the multi-hop system, that was used in this chapter in order to find the lower bound of the outage probability, was found by neglecting the correlation between the noise variables at the destination. Therefore, it follows that the analytical expression for the lower bound of the outage probability that was derived is very accurate when it is acceptable to neglect this correlation, that is, when the correlation between the noise variables at the destination is negligible. But, as noted, it is precisely when the correlation is negligible that the multi-hop system performs very well. Therefore, the factors that cause the multi-hop system to perform very well are the same factors that lead to our results being very accurate. This is an important point that enhances the usefulness of the lower bound expression.

On another note, the usefulness of the multi-hop system can best be seen from Fig. 3.5, which demonstrates that, for certain cases, the performance of the multi-hop system can be substantially superior to that of the two-hop system. For Fig. 3.5, the channel variances were chosen such that the only strong end-to-end paths from the source to the destination go through the relay-to-relay links, a situation that is represented visually in Fig. 3.6. As mentioned, this case is an example for which the multi-hop system's performance is substantially superior (an improvement of over 4 dB at an SER of 10^{-5}) to that of the two-hop system, which can be seen from Fig. 3.5. It also demonstrates the importance of having good analytical tools for the multi-hop system, such as those that are developed in this thesis, as well as for the two-hop system, since the performance of the two systems can be substantially different.

Another advantage of the high-SNR straight-line approximation that should be pointed out is that the diversity, $d = K+1$, of the system can be obtained directly from its slope. This can be seen by inspection of Figs. 3.2 through 3.5. The diversity can also be obtained from the equations for the high-SNR straight-line approximations, given by (3.38) for the multi-hop system and by (3.39) for the two-hop system, since the β parameters in those expressions, $\beta_{x,r,n}^{(m)}$ in (3.38) and β_m in (3.39), are inversely related to the single-channel SNR, ε_o/N_o, and, therefore, the overall expressions for the straight-line approximations are inversely related to the factor, $(\mathrm{SNR})^{K+1}$. Also note that the diversity is $d = K + 1$ for both the two-hop system and the multi-hop system, meaning that the diversity cannot be improved by introducing relay-to-relay transmission into the system.

3.5 Conclusion

New and accurate outage probability results were presented for the multi-hop AF system in the form of a lower bound and also in the form of a high-SNR straight-line approximation for the general case of $K \geq 2$ relays. A lower bound for the outage

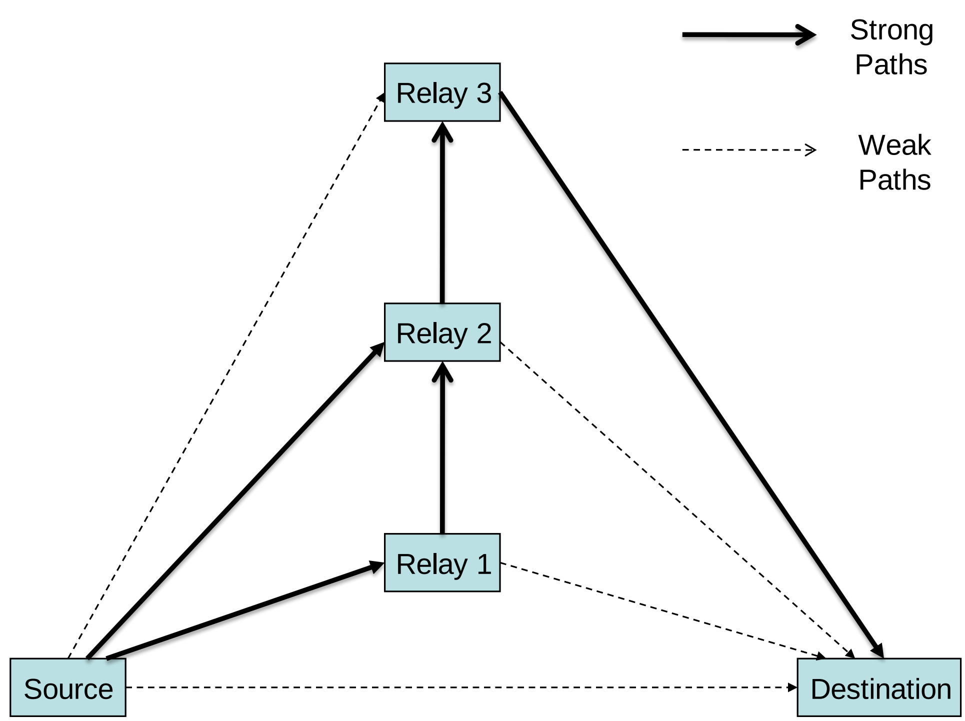

Figure 3.6: Block diagram of the multi-hop system for $K = 3$ relays. The labeling of the channels as either *strong* or *weak* corresponds to the channel variances chosen for the plots in Fig. 3.5. This visual representation of the system provides an intuitive explanation as to why the multi-hop system performs much better than the two-hop system for the case of Fig. 3.5.

probability of the two-hop system was also presented, as was a previously published high-SNR straight-line approximation [6]. It was shown that the results for the multi-hop system are generalized versions of those for the two-hop system. Plots of these results along side outage probability curves from simulations demonstrate that both the lower bounds and the high-SNR approximations are very accurate – the lower bounds are more useful in the low-SNR region, while the straight-line approximations are (often, but not always) tighter to the simulation curves in the medium-SNR range – and both are asymptotically tight (for most cases) to the actual outage probability as the SNR gets high. It was also seen that the results for the multi-hop system are most accurate for the case when the relays are closer to the source than to the destination, which coincides with the case for which it is most beneficial to use the multi-hop system.

Chapter 4

Conclusion and Future Work

4.1 Conclusion

A multi-hop, amplify-and-forward (AF), cooperative diversity system with K relays, which used repetition-based transmission and maximum-ratio-combining (MRC) at the receivers, was researched. This system, which had scarcely been studied previously in the literature, is a generalization of the two-hop system which had previously been studied in depth. In this thesis, four main new important results related to the multi-hop system were presented.

First, the cumulative distribution function (CDF) and the probability density function (PDF) were derived for the random variable (RV), $Z = X \cdot Y/(X + Y + c)$, where X and Y are sums of any amounts of exponential, m-Erlang, or chi-squared RVs (all of which arise in systems with Rayleigh fading channels), with no restrictions on any of their parameters (or means), and these RVs, as well as X and Y, represent signal-to-noise ratio (SNR) expressions within the system. These results are a generalization of previously found results for special cases of the RV, Z. The moment generating function (MGF) of Z for the special case where $c = 0$ was also found. The details of these results were given in Section 2.4.

Second, by using the results from Section 2.4, the MGF-based approach [13] for calculating the symbol-error-rate (SER), Craig's formula [31], and an intuitive approach for finding approximate signal-to-noise ratio (SNR) expressions that accurately model the SNR expressions of the multi-hop system, an accurate approximate SER expression was found for the multi-hop system for any value of K relays. The details of this result was given in Section 2.5. Figure 2.3 demonstrates the high level of accuracy of this result. It is worth pointing out that previous researchers considered the multi-hop system to be too complicated [6, p. 223] to allow for good SER results to be obtained.

Third, the RV, Γ_U, which represents the upper bound of the end-to-end instantaneous SNR of the multi-hop system for any value of K relays, was introduced in (3.6) and its CDF was derived. The result for this CDF, which was obtained through a lengthy process as described in Sections 3.3.1 to 3.3.6, was given in (3.35).

Fourth, it was straightforward to find the lower bound of the outage probability of the multi-hop system from the CDF of Γ_U, as shown in Section 3.3.7. The result for the lower bound of the outage probability was given in (3.36). Figures 3.4 and 3.5 demonstrate that this bound is very tight in nearly the entire SNR range. As was the case for the SER expression, an accurate expression for the outage probability of the multi-hop system had not previously been reported.

The significance of these results and the impact that they have on the field of wireless communications are three-fold. First, the accurate analytical expressions for both the SER and the outage probability of the multi-hop system can be used to optimize this system's parameters without having to run simulations for each possible set of parameters (which would take forever). For example, the analytic expressions could be used to find the system performance for many different choices of channel variances, and an optimal set of channel variances could be found (the set that optimizes the system performance). Since the channel variances are closely related to the distances between the terminals (source, relays, and destination), the optimal set of channel variances would lead to the optimal locations for the terminals. Second, the analytical expressions found for the multi-hop system in this thesis can be used in conjunction with previously found results for the two-hop system in order to compare the performances of the two systems for many different scenarios. Then this information can be used to determine for which scenarios it is preferable to use the multi-hop system and for which scenarios it is preferable to use the two-hop system, which is equivalent to determining whether the relays should broadcast or whether they should transmit only to the destination. Third, since the mathematical results for the RVs, Z and Γ_U, are quite general, they may be used in the analysis of other wireles relay systems where similar SNR expressions arise.

4.2 Future Work

Since the mathematical results derived in Chapters 2 and 3 are quite general, the work in those chapters can be repeated for various different scenarios and various different types of channel fading for both the two-hop system and the multi-hop system. Several of those scenarios are considered in the following discussion.

It should be pointed out that while the results for the CDF, PDF, and MGF of Z found in Section 2.4 were found under the premise that X and Y were sums of exponential, m-Erlang, or chi-squared RVs, they are actually valid for the general case where the PDFs of X and Y can be represented by sums of terms of the form $x^n e^{-\beta x}$. (The case of X and Y being sums of exponential, m-Erlang, or chi-squared RVs is a special case of this more general case.) For the case where a channel has Nakagami-m fading, the PDF of the SNR, γ, of the channel [13, 37] is a Gamma RV with MGF, $M_\gamma(s) = \left(\frac{1}{1-s\bar{\gamma}/m}\right)^m$. If m is restricted to being an integer, $m \geq 1$, then γ becomes an Erlang RV and its PDF is indeed in the form of the general case mentioned above. Also, for the case of a Rician (Nakagami-n) channel, the SNR is represented by a non-central chi-squared RV with two degrees of freedom, which contains a modified Bessel function of the first kind and order zero. For the case of a Hoyt (Nakagami-q) channel, the PDF of the SNR also contains an exponential function and a modified Bessel function of the first kind and order zero [13, Table 2.2]. If these Bessel functions are expressed in terms of their infinite sum representations [26, eq. 8.447.1], then the PDFs of the SNRs for these cases are also in the desired form of the general case. Therefore, the results in Chapter 2 are easily generalized so that an SER expression can be found for the multi-hop system in which all of the source-to-relay channels and all of the relay-to-relay channels are Rayleigh (which ensures that the RV, X, which represents the source-to-relay multi-hop SNR, has a PDF in the desired form), but in which the relay-to-destination channels can be either Rayleigh, Rician (Nakagami-n), Hoyt (Nakagami-q), or Nakagami-m, (which ensures that the RV, Y, which represents the relay-to-destination single-hop SNR, has a PDF in the desired form).

Furthermore, since the CDF of Γ_U given in (3.35) is a sum of exponentials, it follows that the PDF of Γ_U will also be a sum of exponentials, and that this form for the PDF matches the desired form of the general case mentioned above. And, it is easy to show that the source-to-relay multi-hop SNRs are of the exact same form as Γ_U. Therefore, expressions for the PDFs of the source-to-relay multi-hop SNRs (which are represented by the RV, X, in the expression for Z), can be found by using the mathematical techniques in Chapter 3, instead of using the intuitive approach that was used in Chapter 2. By doing so, the PDF of the RV, X, will still be in the desired form (since it will be a sum of exponentials), and this provides a different (and probably improved) approach for finding the SER expression.

It should also be pointed out that, since the PDF of the upper bound, Γ_U, of the end-to-end SNR can readily be derived from the CDF of Γ_U that was found in Chapter 3, it follows that it (the PDF of the end-to-end SNR expression, Γ_U) can be used with the MGF-based approach and Craig's Formula in order to obtain an approximate SER expression. (The integral that results from using this technique can actually be solved in closed form since the PDF is a sum of exponentials.) Therefore, the results of Chapter 3 can be used in the SER analysis as well as the outage probability analysis, and this provides yet another method for finding an approximate SER expression.

From the above discussion, it also follows that for the two-hop system, the results in Section 2.4 can be used to find an SER expression for the cases where the individual channels can be any of Rayleigh, Rician (Nakagami-n), Hoyt (Nakagami-q), or Nakagami-m (with integer m) fading (although infinite sums are required for the Rician and Hoyt cases). Indeed, the approximate SER was found in [37] for the two-hop system for the case where all channels are Nakagami-m (with integer m), a result which can also be found by using the special case of the PDF, $f_Z(\gamma)$, of Z given in

Corollary 3 of Chapter 2 (this result was also used in [24], as noted earlier, for an entirely different system).

Also, for the case where only one relay is considered, and there is no direct path from the source to the destination, the end-to-end SNR of the system is represented by the RV, Z, from Section 2.4. Since the CDF of Z was found in that section, the CDF of the end-to-end SNR is known for this special case; and since the outage probability of a system can always be found directly from the CDF of the instantaneous end-to-end SNR of the system, it follows that the results in Section 2.4 can always be used to find the outage probability for this special case (only one relay and no source-to-destination path) for any of the types of channel fading mentioned above, that is, Rayleigh, Rician (Nakagami-n), Hoyt (Nakagami-q), or Nakagami-m. Indeed, the outage probability was found in [38] for this special case (only one relay and no source-to-destination path) for the case where one channel (either source-to-relay or relay-to-destination) is Rayleigh and the other is Rician, using the exact same technique that is being discussed here.

It can also be shown that all of the derivations required in both Chapter 2 and Chapter 3, which were carried out for the case of the multi-hop system where all individual channel SNRs were exponential RVs (due to Rayleigh fading), are also solvable for the case of the multi-hop system where all individual channel SNRs are Erlang RVs, which arise for the case of Nakagami-m fading (as long as m is restricted to being an integer). Therefore, the SER analysis that was carried out in Chapter 2 for the multi-hop system with Rayleigh channels can be repeated for the multi-hop system with Nakagami-m channels by using exactly the same techniques that were used in that chapter. (Note that this is not just another special case of the work done in Chapter 2, since some of the steps are slightly more involved.) Also, for the outage probability analysis that was carried out in Chapter 3, the fact that all of the calculations are solvable for Erlang RVs as well as exponential RVs means that the lower bound of the outage probability can be found for the case where all channels have Nakagami-m fading, or for the case where all channels have Rayleigh fading and where the exponential RVs that arise may have identical means.

It can be concluded that the mathematical results found in this thesis – the CDF for the most general form of Z found in Section 2.4, and the CDF for the upper bound of the end-to-end SNR of the multi-hop system found in Section 3.3 – are useful beyond the scope for which they were used in this thesis.

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Appendix A

Proofs for Chapter 2

A.1 Proof of Theorem 1

Equation (2.3) follows directly from the fact that the received signal is $y_d = A_d x + \eta_d$, so that the signal power is A_d^2 and the noise power is $\mathcal{N}_d = \mathrm{E}[|\eta_d|^2]$.

In Section 2.2, it was found that the total noise signal, $\eta_{m,k}$, at terminal k, added to the signal transmitted from terminal m, was given by $\eta_{1,k} = h_{1,k}\alpha_1 n_{s,1} + n_{1,k}$ for $m = 1$; and $\eta_{m,k} = h_{m,k}\alpha_m \left(\beta_{s,m}^* n_{s,m} + \sum_{i=1}^{m-1} \beta_{i,m}^* \eta_{i,m}\right) + n_{m,k}$ for $m > 1$. The noise signals $\eta_{m,k}$ are in general mutually dependent since, for example, η_{m,k_1} and η_{m,k_2} (with $k_1 \neq k_2$) both contain the term $n_{s,m}$. Therefore, the noise signals $\eta_{m,k}$ will be rewritten as sums of the independent, complex Gaussian noise signals $n_{i,j}$. From these new expressions for the noise signals, $\eta_{m,k}$, it will be a simple matter to solve for their variances, $\mathcal{N}_{m,k}$. By using $\mu_{0,1}^k$ and $\mu_{i,m}^k$ as defined in Theorem 1, the noise signals can be written as $\eta_{m,k} = n_{1,k} + \mu_{0,1}^k n_{0,1}$ for $m = 1$, and $\eta_{m,k} = n_{m,k} + \sum_{l=0}^{m-1} \mu_{l,m}^k \eta_{l,m}$ for $m = 2, \ldots, K$, where $k = m+1, \ldots, K+1$. Since $\eta_{m,k}$ consists of the terms $n_{i,j}$ for $i = 0, \ldots, j-1$ and $j = 1, \ldots, m$, as well as the term $n_{m,k}$, the noise signals can be rewritten as

$$\eta_{m,k} = n_{m,k} + \sum_{j=1}^{m} \sum_{i=0}^{j-1} \lambda(i,j,m,k) n_{i,j} \quad \text{for} \quad m = 1,2,\ldots,K; \quad k = m+1,\ldots,K+1, \tag{A.1}$$

where the $\lambda(i,j,m,k)$ coefficients will be determined next.

For $m = 1$, equating $\eta_{1,k} = n_{1,k} + \mu_{0,1}^{k} n_{0,1}$ with (A.1) leads to $\lambda(0,1,1,k) = \mu_{0,1}^{k}$ for $k = 2,\ldots,K+1$. For $m = 2,\ldots,K$, using (A.1) to substitute for the $\eta_{l,m}$ terms in the equation, $\eta_{m,k} = n_{m,k} + \sum_{l=0}^{m-1} \mu_{l,m}^{k} \eta_{l,m}$, leads to

$$\eta_{m,k} = n_{m,k} + \mu_{s,m}^{k} n_{s,m} + \sum_{l=1}^{m-1} \mu_{l,m}^{k} n_{l,m} + \sum_{l=1}^{m-1} \sum_{j=1}^{l} \sum_{i=0}^{j-1} \mu_{l,m}^{k} \lambda(i,j,l,m) n_{i,j}, \tag{A.2}$$

where $k = m+1,\ldots,K+1$. Then, by equating (A.1) with (A.2) and matching the coefficients of the noise variable terms, the $\lambda(i,j,m,k)$ terms can be found. In this procedure, it was found that it is convenient to set the value of j first. Using this technique with $j = 1$ leads to $\lambda(0,1,m,k) = \sum_{l=1}^{m-1} \mu_{l,m}^{k} \lambda(0,1,l,m)$ for $m = 2,\ldots,K$, and $k = m+1,\ldots,K+1$. For $j = 2$ (but $j < m$, since the results for the $j = m$ case are different), equating (A.1) with (A.2) leads to $\sum_{i=0}^{1} \lambda(i,2,m,k) n_{i,2} = \sum_{l=2}^{m-1} \sum_{i=0}^{1} \mu_{l,m}^{k} \lambda(i,2,l,m) n_{i,2}$. Matching coefficients for the two variables, $n_{0,2}$ and $n_{1,2}$, leads to $\lambda(i,2,m,k) = \sum_{l=2}^{m-1} \mu_{l,m}^{k} \lambda(i,2,l,m)$ for $m = 3,\cdots,K$; $k = m+1,\cdots,K+1$; and $i = 0,1$. This procedure is repeated for all values of j up to $j = m$. When $j = m$, the same procedure is used, but the result is given by $\lambda(i,m,m,k) = \mu_{i,m}^{k}$. Using this procedure leads to the complete results for the $\lambda(i,j,m,k)$ terms as they are given in Theorem 1.

The overall noise signal at the destination can be expressed in terms of the independent AWGN variables by substituting the expression from (A.1) into the expression for η_d given in Section 3.2. The result is

$$\eta_d = \beta_{s,d}^* n_{s,d} + \sum_{m=1}^{K} \beta_{m,d}^* \eta_{m,d} = \beta_{s,d}^* n_{s,d} + \sum_{m=1}^{K} \beta_{m,d}^* n_{m,d} + \sum_{m=1}^{K} \sum_{j=1}^{m} \sum_{i=0}^{j-1} \beta_{m,d}^* \lambda(i,j,m,d) n_{i,j}. \tag{A.3}$$

The noise variances are then found from $\mathcal{N}_{m,k} = \mathrm{E}[|\eta_{m,k}|^2]$ and $\mathcal{N}_d = \mathrm{E}[|\eta_d|^2]$. The results are given in (2.4) and (2.6), respectively.

A.2 Proof of Corollary 1

The derivation of the approximate expression for the end-to-end instantaneous SNR given in (2.7) is described as follows. By using $\beta_{s,d} = \sqrt{\varepsilon_o} h_{s,d}/N_{s,d}$ and $\gamma_{s,d} = \varepsilon_o |h_{s,d}|^2/N_{s,d}$, it follows that $N_{s,d}|\beta_{s,d}|^2 = \gamma_{s,d}$. By using $\beta_{1,d} = \alpha_1 N_{s,1} \beta_{s,1} h_{1,d}/\mathcal{N}_{1,d}$, $\mathcal{N}_{1,d} = \alpha_1^2 |h_{1,d}|^2 N_{s,1} + N_{1,d}$, the expression for α_1 given in (2.1), and $\gamma_{m,k} = \frac{\varepsilon_m |h_{m,k}|^2}{N_{m,k}}$, it follows that $\mathcal{N}_{1,d}|\beta_{1,d}|^2 = \frac{\gamma_{s,1}\gamma_{1,d}}{\gamma_{s,1}+\gamma_{1,d}+1} = \frac{A_1 \gamma_{1,d}}{A_1+\gamma_{1,d}+1}$. For $m \geq 2$, an approximation for $\mathcal{N}_{m,k}$ is found by neglecting the dependency of the $\eta_{i,m}$ terms in the equation $\eta_{m,k} = h_{m,k}\alpha_m \left(\beta_{s,m}^* n_{s,m} + \sum_{i=1}^{m-1} \beta_{i,m}^* \eta_{i,m}\right) + n_{m,k}$ when calculating $\mathcal{N}_{m,k} = \mathrm{E}[|\eta_{m,k}|^2]$. The result is $\mathcal{N}_{m,k} \approx |h_{m,k}|^2 \alpha_m^2 A_m + N_{m,k} = \frac{\varepsilon_m |h_{m,k}|^2}{A_m+1} + N_{m,k}$, where $k = m+1, \ldots, K+1$, and the equations $\alpha_m = \sqrt{\frac{\varepsilon_m}{A_m^2 + A_m}}$ and $A_m = N_{s,m}|\beta_{s,m}|^2 + \sum_{i=1}^{m-1} \mathcal{N}_{i,m}|\beta_{i,m}|^2$ were used. Then, by using $\beta_{m,d} = \alpha_m A_m h_{m,d}/\mathcal{N}_{m,d}$, along with the approximation for $\mathcal{N}_{m,d}$, it follows that

$$\mathcal{N}_{m,d}|\beta_{m,d}|^2 = \frac{\alpha_m^2 A_m^2 |h_{m,d}|^2}{\mathcal{N}_{m,d}} \approx \frac{\varepsilon_m A_m |h_{m,d}|^2}{\varepsilon_m |h_{m,d}|^2 + N_{m,d}(A_m+1)} = \frac{A_m \gamma_{m,d}}{A_m + \gamma_{m,d} + 1}. \tag{A.4}$$

Equation (2.7) is obtained by substituting these results into the approximate expression for Γ given by $\Gamma \approx N_{s,d}|\beta_{s,d}|^2 + \sum_{m=1}^{K} \mathcal{N}_{m,d}|\beta_{m,d}|^2$.

Using similar arguments, it follows that $N_{s,m}|\beta_{s,m}|^2 = \gamma_{s,m}$ and $\mathcal{N}_{i,m}|\beta_{i,m}|^2 = \frac{A_i \gamma_{i,m}}{A_i+\gamma_{i,m}+1}$ for $i = 1, \ldots, m-1$ and $m = 2, \ldots, K$. Equation (2.9) is obtained by substituting these results into the expression for A_m given earlier for $m \geq 2$.

A.3 Proof of Theorem 2

For $K = 2$, the approximate, instantaneous, end-to-end SNR, Γ_{app}, of the multi-hop system can be upper bounded by

$$\Gamma_{app} < \Gamma' = \gamma_{s,d} + \frac{\gamma_{s,1}\gamma_{1,d}}{\gamma_{s,1}+\gamma_{1,d}+1} + \frac{(\gamma_{s,2}+\gamma_{s,1,2})\,\gamma_{2,d}}{(\gamma_{s,2}+\gamma_{s,1,2})+\gamma_{2,d}+1}, \tag{A.5}$$

where $\gamma_{s,1,2} = \min(\gamma_{s,1}, \gamma_{1,2})$. This can be seen by noting that, for $K = 2$, Equation (2.7) yields $\Gamma_{app} = \gamma_{s,d} + \frac{\gamma_{s,1}\gamma_{1,d}}{\gamma_{s,1}+\gamma_{1,d}+1} + \frac{A_2\gamma_{2,d}}{A_2+\gamma_{2,d}+1}$, where $A_2 = \gamma_{s,2} + \frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}+1}$ is found from (2.9). Clearly, $\frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}+1} < \frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}}$. Also, since $\frac{\gamma_{s,1}}{\gamma_{s,1}+\gamma_{1,2}} < 1$ and $\frac{\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}} < 1$, it follows that $\frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}} < \min(\gamma_{s,1}, \gamma_{1,2}) = \gamma_{s,1,2}$, which is also reported in [19]. The chain of inequalities $\frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}+1} < \frac{\gamma_{s,1}\gamma_{1,2}}{\gamma_{s,1}+\gamma_{1,2}} < \gamma_{s,1,2}$ shows that $A_2 < \gamma_{s,2} + \gamma_{s,1,2}$. Now, consider the following inequalities: $\frac{1}{A_2} > \frac{1}{\gamma_{s,2}+\gamma_{s,1,2}} \rightarrow \frac{\gamma_{2,d}+1}{A_2} > \frac{\gamma_{2,d}+1}{\gamma_{s,2}+\gamma_{s,1,2}} \rightarrow 1 + \frac{\gamma_{2,d}+1}{A_2} > 1 + \frac{\gamma_{2,d}+1}{\gamma_{s,2}+\gamma_{s,1,2}} \rightarrow \left(1 + \frac{\gamma_{2,d}+1}{A_2}\right)^{-1} < \left(1 + \frac{\gamma_{2,d}+1}{\gamma_{s,2}+\gamma_{s,1,2}}\right)^{-1} \rightarrow \frac{A_2}{A_2+\gamma_{2,d}+1} < \frac{\gamma_{s,2}+\gamma_{s,1,2}}{(\gamma_{s,2}+\gamma_{s,1,2})+\gamma_{2,d}+1} \rightarrow \frac{A_2\gamma_{2,d}}{A_2+\gamma_{2,d}+1} < \frac{(\gamma_{s,2}+\gamma_{s,1,2})\gamma_{2,d}}{(\gamma_{s,2}+\gamma_{s,1,2})+\gamma_{2,d}+1}$. The upper bound given by Equation (A.5) follows directly from the last inequality. This upper bound for Γ_{app} is taken as the approximation for Γ given in (2.10).

For $K > 2$, Equation (2.7) becomes very complicated due to its recursive nature, and reducing it to a desired form becomes too difficult. Therefore, a more intuitive, conceptual approach will be taken in order to find the desired, approximate SNR expression for $K > 2$. The SNR expression given by (A.5) can be written as $\Gamma' = \gamma_{s,d} + \gamma_{s,1,d} + \gamma_{s,2,d}$, where $\gamma_{s,m,d} = \frac{\gamma_{s,rm}\gamma_{m,d}}{\gamma_{s,rm}+\gamma_{m,d}+1}$ for $m = 1, 2$. Also, $\gamma_{s,rm}$ can be expressed as $\gamma_{s,rm} = \sum_{i=1}^{m} \gamma_{s,i,m}$ for $m = 1, 2$, where $\gamma_{s,1,1} = \gamma_{s,1}$, $\gamma_{s,2,2} = \gamma_{s,2}$, and $\gamma_{s,1,2} = \min(\gamma_{s,1}, \gamma_{1,2})$. The first term in the expression for Γ' is the SNR of the source-destination $(S - D)$ channel depicted in Fig. 2.2(a). The second term is due to the two-hop $S - R_1 - D$ path depicted in Fig. 2.2(b), where R_1 refers to relay 1. The third term represents the SNR of the multi-hop path of the signals that go through relay R_2, as shown in Fig. 2.2(c). This conceptual view of the approximate SNR expression, Γ', can be expanded for $K > 2$ by using the same pattern for the additional relays as that given in Fig. 2.2 and given in the above equations for the $K = 2$ case. This leads to the result given in Theorem 2.

A.4 Proof of Lemma 1

The MGF of X is given by:

$$M_X(s) = \prod_{j=1}^{J} \left(\frac{1}{1 - s\overline{x}_j} \right)^{r_j} = \prod_{j=1}^{J} \left(\frac{c_{x,j}}{c_{x,j} - s} \right)^{r_j} = \sum_{j=1}^{J} \sum_{i=1}^{r_j} \frac{k_{x,i,j}}{(c_{x,j} - s)^i} = \sum_{j=1}^{J} \sum_{i=1}^{r_j} \frac{k_{x,i,j}}{c_{x,j}^i} \left(\frac{c_{x,j}}{c_{x,j} - s} \right)^i, \tag{A.6}$$

where $c_{x,j} = 1/\overline{x}_j$. Each term in the double sum of (A.6) corresponds to the MGF of an exponential RV when $i = 1$ (within a constant) and an Erlang RV when $i > 1$ (again, within a constant). The relationships between the MGFs and PDFs (see [25, 29]) of these RVs are applied to each term in the expression for the MGF of X in order to find the corresponding term in the expression for the PDF of X. The result is given in (2.15).

Similarly, the MGF of Y is given by

$$M_Y(s) = \prod_{q=1}^{Q} \left(\frac{1}{1 - s\overline{y}_q} \right)^{t_q} = \prod_{q=1}^{Q} \left(\frac{c_{y,q}}{c_{y,q} - s} \right)^{t_q} = \sum_{q=1}^{Q} \sum_{p=1}^{t_q} \frac{k_{y,p,q}}{(c_{y,q} - s)^p} = \sum_{q=1}^{Q} \sum_{p=1}^{t_q} \frac{k_{y,p,q}}{c_{y,q}^p} \left(\frac{c_{y,q}}{c_{y,q} - s} \right)^p, \tag{A.7}$$

where $c_{y,q} = 1/\overline{y}_q$. The PDF of Y, also given in (2.15), is found from the MGF of Y in the same way that the PDF of X is found from the MGF of X. The constants $k_{x,i,j}$ and $k_{y,p,q}$ are found by using standard partial fraction techniques.

A.5 Proof of Theorem 3

The CDF of Z, $F_z(\gamma)$, given in (2.18), is found as follows: $F_z(\gamma) = \mathrm{P}(Z \leq \gamma) = \mathrm{P}\left(\frac{XY}{X+Y+c} \leq \gamma\right) = \int_{y=0}^{\infty} \mathrm{P}\left(\frac{Xy}{X+y+c} \leq \gamma\right) f_y(y)\,\mathrm{d}y$, where $\mathrm{P}(\cdot)$ denotes a probability. The argument of the probability function can be expressed as $X \leq \frac{(y+c)\gamma}{y-\gamma}$ if $y > \gamma$ and $X \geq \frac{(y+c)\gamma}{y-\gamma}$ if $y < \gamma$. Then, the CDF is evaluated as

$$F_z(\gamma) = \int_{y=0}^{\gamma} \mathrm{P}\left(X \geq \frac{(y+c)\gamma}{y-\gamma} \right) f_y(y)\,\mathrm{d}y + \int_{y=\gamma}^{\infty} \mathrm{P}\left(X \leq \frac{(y+c)\gamma}{y-\gamma} \right) f_y(y)\,\mathrm{d}y. \tag{A.8}$$

By using the PDFs of X and Y from (2.15) to solve for the integrals in (A.8), the following expression for the CDF of Z is found with straightforward integrations and algebraic manipulations:

$$F_z(\gamma) = 1 - \sum_{j=1}^{J}\sum_{i=1}^{r_j} k_{x,i,j} \sum_{k=0}^{i-1} \frac{\overline{x}_j^{i-k}\gamma^k}{k!} \sum_{q=1}^{Q}\sum_{p=1}^{t_q} \frac{k_{y,p,q}}{(p-1)!} \exp\left(-\frac{\sigma_{j,q}\gamma}{\rho_{j,q}}\right)$$
$$\int_{w=0}^{\infty} \frac{(w+\gamma+c)^k (w+\gamma)^{p-1}}{w^k} \exp\left(-\frac{\gamma(\gamma+c)/\overline{x}_j}{w} - \frac{w}{\overline{y}_q}\right) \mathrm{d}w. \quad \text{(A.9)}$$

The integral of (A.9) is solved in the same manner as in the appendix of [24], where the binomial theorem is applied to obtain $(w+\gamma+c)^k = \sum_{i'=0}^{k}\binom{k}{i'} w^{i'}(\gamma+c)^{k-i'}$ and $(w+\gamma)^{p-1} = \sum_{j'=0}^{p-1}\binom{p-1}{j'} w^{j'}\gamma^{p-1-j'}$. Then, the remaining integral is solved using [26, eq. 3.471.9]. The CDF given by (2.18) is found by substituting these results for the integral in (A.9).

The PDF of Z, $f_z(\gamma)$, is found by differentiating Equation (2.18) and using $\frac{\partial K_\nu(z)}{\partial z} = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z)$. (See [30].)

A.6 Proof of Corollary 5

Set $J = M$ and $r_j = 1$ for $j = 1, \ldots M$, and set $Q = N$ and $t_q = 1$ for $q = 1, \ldots N$ into Equation (2.27). For this special case, $M_z(s) = \sum_{j=1}^{M}\sum_{q=1}^{N} k_{x,1,j}k_{y,1,q}\left[4\,\mathcal{I}_1 + \frac{2\sigma_{j,q}}{\sqrt{\rho_{j,q}}}\mathcal{I}_2\right]$. The integrals were solved using the Symbolic Toolbox ® from Matlab ®. The results are

$$\mathcal{I}_1 = \int_0^\infty \gamma e^{-\alpha\gamma} K_0(\beta\gamma)\,\mathrm{d}\gamma = \frac{1}{\beta^2-\alpha^2} - \frac{\alpha}{(\beta^2-\alpha^2)^{3/2}} \arccos\left(\frac{\alpha}{\beta}\right), \quad \text{(A.10)}$$

$$\mathcal{I}_2 = \int_0^\infty \gamma e^{-\alpha\gamma} K_1(\beta\gamma)\,\mathrm{d}\gamma = -\frac{\alpha/\beta}{\beta^2-\alpha^2} + \frac{\beta}{(\beta^2-\alpha^2)^{3/2}} \arccos\left(\frac{\alpha}{\beta}\right). \quad \text{(A.11)}$$

After substituting these results into the expression for $M_z(s)$ and rearranging terms, the result in Corollary 5 is obtained.

A.7 Proof of Theorem 5

The SER, $\mathrm{P}_s(E)$, is found from $\mathrm{P}_s(E) = \mathrm{E}\left[aQ\left(\sqrt{2b\Gamma'}\right)\right] = \mathrm{E}_{\Gamma'}\left[\frac{a}{\pi}\int_0^{\pi/2}\exp\left(\frac{-b\Gamma'}{\sin^2\theta}\right)\,\mathrm{d}\theta\right]$, where $Q(\cdot)$ is the Gaussian Q-function, Γ' is the approximate instantaneous end-to-end SNR from (2.10), $\mathrm{E}_{\Gamma'}[\cdot]$ denotes that the expectation is done with respect to Γ', and the expression within the expectation on the right-hand side of the equation is Craig's Formula from [31]. The following result is obtained:

$$
\begin{aligned}
\mathrm{P}_s(E) &= \int_{\gamma=0}^{\infty}\frac{a}{\pi}\int_{\theta=0}^{\pi/2}\exp^{s\gamma}\,\mathrm{d}\theta\; f_{\Gamma'}(\gamma)\,\mathrm{d}\gamma = \frac{a}{\pi}\int_{\theta=0}^{\pi/2}\int_{\gamma=0}^{\infty}\exp\left(s\left[\gamma_{s,d}+\sum_{m=1}^{K}\gamma_{s,m,d}\right]\right)f_{\Gamma'}(\gamma)\,\mathrm{d}\gamma\;\mathrm{d}\theta \\
&= \frac{a}{\pi}\int_{\theta=0}^{\pi/2}\underbrace{\int_0^{\infty}\cdots\int_0^{\infty}}_{K+1\text{ integrals}} e^{s\gamma_{s,d}}\prod_{m=1}^{K}e^{s\gamma_{s,m,d}}f_{\gamma_{s,d}}(\gamma_{s,d})\prod_{m=1}^{K}f_{\gamma_{s,m,d}}(\gamma_{s,m,d})\,\mathrm{d}\gamma_{s,d}\prod_{m=1}^{K}\mathrm{d}\gamma_{s,m,d}\;\mathrm{d}\theta \\
&= \frac{a}{\pi}\int_{\theta=0}^{\pi/2}M_{\gamma_{s,d}}(s)\prod_{m=1}^{K}M_{\gamma_{s,m,d}}(s)\;\mathrm{d}\theta = \frac{a}{\pi}\int_{\theta=0}^{\pi/2}\frac{\sin^2\theta}{\sin^2\theta+b\overline{\gamma}_{s,d}}\prod_{m=1}^{K}M_{\gamma_{s,m,d}}\left(\frac{-b}{\sin^2\theta}\right)\,\mathrm{d}\theta,
\end{aligned}
\tag{A.12}
$$

where $s = -b/\sin^2\theta$ was used. Note that it was assumed that $f_{\Gamma'}(\gamma) = f(\gamma_{s,d})\prod_{m=1}^{K}f(\gamma_{s,m,d})$, implying that any dependencies between the RVs, $\gamma_{s,m,d}$, were once again neglected.

Appendix B

Proofs for Chapter 3

B.1 Proof of Theorem 6

The inequality $\frac{(X+a)Y}{(X+a)+Y} > \frac{XY}{X+Y}$ is true for all positive values of a. This shows that $\frac{XY}{X+Y}$ can be maximized by maximizing X, that is, $\left(\frac{XY}{X+Y}\right)_{max} = \frac{X_{max}Y}{X_{max}+Y} \geq \frac{XY}{X+Y}$ where $X_{max} \geq X$. Furthermore, it is a well known fact that $\min(X,Y) \geq \frac{XY}{X+Y}$ (see [19], for example). These two facts lead to the expressions for $\gamma_{sRm,up}$ and Γ_U in Theorem 6.

B.2 Proof of Lemma 7

The RV, V, can be expressed as $V = X + a$, where $X = \sum_{m=1}^{M} \gamma_m$ is an RV and $a = \sum_{n=1}^{N} y_n$ is a constant. The MGF of X is given by $M_X(s) = \sum_{m=1}^{M} \frac{c_m \zeta_m}{\zeta_m - s}$, where $c_m = \prod_{\substack{i=1,\\ i\neq m}}^{M} \frac{\zeta_i}{\zeta_i - \zeta_m}$. It is straightforward to find the PDF, $f_X(x)$, of X from its MGF, $M_X(s)$. Since $V = X + a$, its PDF is $f_V(x) = f_X(x-a)$. It is straightforward to find the MGF, $M_V(s)$, of V from its PDF, $f_V(x)$.

Since V and γ_d are independent RVs, the PDF, $f_W(x)$, of $W = \min(V;\gamma_d)$ is given by $f_W(x) = f_V(x)R_{\gamma_d}(x) + f_{\gamma_d}(x)R_V(x)$, where $f_{\gamma_d}(x)$ is the PDF of γ_d, $R_{\gamma_d}(x) = \mathbb{P}(\gamma_d > x)$, and $R_V(x) = \mathbb{P}(V > x)$. Again, it is straightforward to find the MGF, $M_W(s)$, of W from its PDF, $f_W(x)$.

B.3 Proof of Lemma 12

Consider the following expectation:

$$\mathbb{E}_{\gamma_{i,j}}\left[e^{\beta z_{i,j}}U\left(x-\sum_{n=1}^{N-1}u_n z_n - u_N z_{i,j}\right)\right] = \begin{cases}\left[\alpha-\alpha e^{-\beta_x x}\left(\prod_{n=1}^{N-1}e^{\beta_{z_n}z_n}\right)\right]U\left(x-\sum_{n=1}^{N-1}u_n z_n\right) & ,u_N>0\\ \alpha U\left(x-\sum_{n=1}^{N-1}u_n z_n\right) & ,u_N=0\end{cases} \tag{B.1}$$

where $\alpha = \frac{\beta_{i,j}}{\beta_{i,j}-\beta}$; $\beta_x = \frac{\beta_{i,j}-\beta}{u_N}$; and $\beta_{z_n} = \frac{u_n(\beta_{i,j}-\beta)}{u_N}$ for $1 \le n \le N-1$. By using (B.1) to calculate $P^{(r)}(x, z_1, \ldots, z_{N-1}) = \mathbb{E}_{\gamma_{i,j}}\left[P^{(r)}(x, z_1, \ldots, z_{N-1}, z_{i,j})\right]$; and by using the definition of the CDF transform to find $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}} = \mathcal{T}_c\left\{P^{(r)}(x, z_1, \ldots, z_{N-1})\right\}$ and $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}} = \mathcal{T}_c\left\{P^{(r)}(x, z_1, \ldots, z_{N-1}, z_{i,j})\right\}$; it is straightforward to show that the entries of $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1}\}}$ and $\hat{P}^{(r)}_{\{x,z_1,\ldots,z_{N-1},z_{i,j}\}}$ are indeed related by the equations given in Lemma 12.

B.4 Proof of Lemma 14

A high-SNR straight-line approximation for the outage probability of the multi-hop system is found in the same manner that it was found in [6] for the two-hop system, that is, by using a Taylor series expansion for the exponential functions. Then, $P_{o,L}^{multi}$

can be expressed as

$$\begin{aligned}
P_{o,L}^{multi} &= \sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \left(-\alpha_{r,n}^{(m)}\right) \left\{1 - \exp\left(-[\beta_{x,r,n}^{(m)}][2^{(K+1)R} - 1]\right) \right\} \\
&= \sum_{l=1}^{\infty} \sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} -\alpha_{r,n}^{(m)} [\beta_{x,r,n}^{(m)}]^l \, \frac{(-1)^{l+1} \left(2^{(K+1)R} - 1\right)^l}{l!} \\
&\approx \sum_{l=1}^{K+1} \left(\sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \alpha_{r,n}^{(m)} [\beta_{x,r,n}^{(m)}]^l \right) \frac{(-1)^{l} \left(2^{(K+1)R} - 1\right)^l}{l!} \; ,
\end{aligned}$$

where the fact that $\sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \alpha_{r,n}^{(m)} = 0$ was used in the first equation; the Taylor series expansion for the exponential term was introduced in the second equation; and the higher order terms of the Taylor series were ignored in the third equation. Similar to the results found for the two-hop system in [6], it turns out that $\sum_{r=0}^{N_c-1} \sum_{n=1}^{N_r} \sum_{m=1}^{M_r} \alpha_{r,n}^{(m)} [\beta_{x,r,n}^{(m)}]^l = 0$ for $1 \leq l \leq K$, and this leads to the result in (3.38).