EXACT COEFFICIENTS FOR RIGID DUMBBELL SUSPENSIONS FOR STEADY SHEAR FLOW MATERIAL FUNCTION EXPANSIONS

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ABSTRACT

From kinetic molecular theory, we can attribute the elasticity of polymeric liquids to macromolecular orientation. For a suspension of rigid dumbbells, subject to a particular flow field, we must first solve the diffusion equation for the orientation distribution function. From this distribution, we then calculate physical properties such as the steady shear flow material functions. We thus arrive at power series expansions in the shear rate for both the orientation distribution function and for the steady shear flow material functions. Analytical work on many viscoelastic material functions must be checked for consistency, in their steady shear flow limits, against these power series. For instance, for large-amplitude oscillatory shear flow, we recover the coefficients of these expansions in the limits of low test frequency. The coefficients of the steady shear viscosity and the first normal stress coefficient functions are not known exactly beyond the fourth power. In this work, for both of these functions, we arrive at exact expressions for the first 20 coefficients. We close with five worked examples illustrating uses for our new coefficients.

Keywords: Rigid dumbbell suspensions, hydrodynamic interaction, Oldroyd 5-constant fluid, macromolecular theory, steady shear material function, steady shear viscosity, first normal stress coefficient, series analytic continuation, series reversion, series inversion, recursion.

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I. INTRODUCTION

From kinetic molecular theory, we can attribute the elasticity of polymeric liquids to macromolecular orientation. For a suspension of rigid dumbbells in any particular flow field, we must first solve the diffusion equation for the orientation distribution function (for any simple shear flow, see Eq. (15) of [1]). Figure 1 (b) and (c) illustrate the rigid dumbbell (see also FIG. 2 of [1]) and for the definition of any simple shearing flow see Eq. (1) of [1]. For instance, for steady simple shearing flow, we first solve the diffusion equation for the orientation distribution function as an expansion in powers of the Weissenberg number, $\lambda \dot{\gamma}$:

$$\psi = \frac{1}{4\pi} \sum_{k=0}^{\infty} \left(6\lambda \dot{\gamma}^k\right) \psi_k(\theta, \phi)$$

where:

$$\lambda \equiv \frac{\zeta L^2}{12kT} = \frac{\pi \eta_s b L^2}{4kT}$$

where all symbols in this paper are defined, along with their dimensions, in Table I and Table II, and where the friction factor is for Stokes flow around a bead [illustrated in Figure 1 (a)]. Moreover, for multibead-rods:

$$\lambda_N = \frac{\pi \eta_s b L^2 N(N+1)}{24(N-1)kT}$$

where the beads osculate at:

$$b_{max} = \frac{L}{N-1}$$

substituting Eq. (4) into Eq. (3) gives:

$$\lambda_N \equiv \frac{\pi \eta_s L^3 N(N+1)}{24(N-1)^2 kT}$$

so that:

$$\lambda_\infty \equiv \frac{\pi \eta_s L^3}{24kT} \lim_{N \to \infty} \frac{N(N+1)}{(N-1)^2} = \frac{\pi \eta_s L^3}{24kT}$$

which gives an upper bound for the possible diameter of all multibead-rods.

From the distribution in Eq. (1), we can calculate physical properties such as the steady shear flow material functions using the Giesekus expression for the stress (see Eq. (55) of [1]). We thus arrive at power series expansions in the shear rate for both the orientation distribution function [1], and then for the steady shear flow material functions (Eqs. (8) and (9) of [2]):
\[
\frac{\eta - \eta_s}{nkT\lambda} = \sum_{k=0}^{\infty} (-1)^k a_k \left( \lambda \dot{\gamma} \right)^{2k}
\]  
(7)

\[
\frac{5\Psi_1}{6nkT\lambda^2} = \sum_{k=0}^{\infty} (-1)^k b_k \left( \lambda \dot{\gamma} \right)^{2k}
\]  
(8)

For a history of these coefficient determinations, both correct and incorrect, see the footnote in [3] and also Subsection 6 of [4]. Since \( \eta(\dot{\gamma}) \) and \( \Psi_1(\dot{\gamma}) \) descend the \( \lambda \dot{\gamma} \), when truncating we prefer doing so after negative terms.

These steady shear flow material functions ignore hydrodynamic interaction.

For an illustration of solvent flow between the beads of a suspended rigid dumbbell, with and without hydrodynamic interaction, see Figure 1 (c) and (b), respectively. Including hydrodynamic interaction by use of the Rotne-Prager-Yamakawa hydrodynamic interaction tensor gives (see §14.6 of [5]):

\[
\frac{\eta - \eta_s}{nkT\lambda} = \frac{\lambda_2^{(2)}}{\lambda} \sum_{k=0}^{\infty} (-1)^k a_k \left( \lambda_2^{(2)} \dot{\gamma} \right)^{2k}
\]  
\[+ \frac{3}{5} \frac{\lambda_2^{(1)}}{\lambda} \left[ 1 - \frac{\lambda_2^{(2)}}{\lambda_2^{(1)}} \right] \sum_{k=0}^{\infty} (-1)^k b_k \left( \lambda_2^{(1)} \dot{\gamma} \right)^{2k}
\]  
(9)

where:

\[
\lambda_2^{(1)} = \frac{\pi \eta_s bL^2}{4kT} \left[ 1 - \frac{3}{8} \left( \frac{b}{L} \right) \left( 1 + \frac{1}{6} \left( \frac{b}{L} \right)^2 \right) \right]^{-1} = \lambda \left[ 1 - \frac{3}{8} \left( \frac{b}{L} \right) \left( 1 + \frac{1}{6} \left( \frac{b}{L} \right)^2 \right) \right]^{-1}
\]  
(10)

\[
\lambda_2^{(2)} = \frac{\pi \eta_s bL^2}{4kT} \left[ 1 - \frac{3}{4} \left( \frac{b}{L} \right) \left( 1 - \frac{1}{6} \left( \frac{b}{L} \right)^2 \right) \right]^{-1} = \lambda \left[ 1 - \frac{3}{4} \left( \frac{b}{L} \right) \left( 1 - \frac{1}{6} \left( \frac{b}{L} \right)^2 \right) \right]^{-1}
\]  
(11)

and, more generally, for multibead dumbbells (Eq. (14.6-23) of [5] which corrects Eq. (33) of [3]):

\[
\frac{\eta - \eta_s}{nkT\lambda} = \frac{\lambda_N^{(2)}}{\lambda} \sum_{k=0}^{\infty} (-1)^k a_k \left( \lambda_N^{(2)} \dot{\gamma} \right)^k
\]  
\[+ \frac{3}{5} \frac{\lambda_N^{(1)}}{\lambda} \left[ 1 - \frac{\lambda_N^{(2)}}{\lambda_N^{(1)}} \right] \sum_{k=0}^{\infty} (-1)^k b_k \left( \lambda_N^{(1)} \dot{\gamma} \right)^{2k}
\]  
(12)

where \( \lambda_N^{(1)} \) and \( \lambda_N^{(2)} \) are given by Eqs. (14.5-18) and (14.6-19) of [5]. Evaluating Eq. (9), in the limit of vanishingly small \( \lambda \dot{\gamma} \), we get:

\[
\frac{\eta - \eta_s}{nkT\lambda} = \frac{\lambda_2^{(2)}}{\lambda} + \frac{3 \lambda_2^{(1)}}{5 \lambda} \left[ 1 - \frac{\lambda_2^{(2)}}{\lambda_2^{(1)}} \right]
\]  
(13)
into which we substitute Eqs. (10) and (11) to get the polymer contribution to the zero-shear viscosity:

\[
\eta - \eta_s = \frac{-32 \left( (b/L)^3 - 12(b/L) + 20 \right)}{nkT\lambda} = \frac{5 \left( (b/L)^3 + 6(b/L) - 16 \right) \left( (b/L)^3 - 6(b/L) + 8 \right)}{nkT\lambda}
\]

\[
= 1 + \frac{21}{40} (b/L) + \frac{99}{320} (b/L)^2 + \ldots
\]

from which we learn that hydrodynamic interaction, from the Rotne-Prager-Yamakawa tensor, increases the polymer contribution to the zero-shear viscosity.

Similarly, when including hydrodynamic interaction by use of the Oseen hydrodynamic interaction tensor, the sums Eqs. (7) and (8) both arise (see Eq. (11.6-23) of [6] or Eq. (24) of [4]):

\[
\eta - \eta_s = \frac{1}{nkT\lambda} \frac{1 - \frac{3}{8} (b/L)}{1 - \frac{3}{4} (b/L)} \sum_{k=0}^{\infty} (-1)^k \frac{a_k (\lambda_h \gamma)^{2k}}{40 \left( 1 - \frac{3}{4} (b/L) \right)} \sum_{k=0}^{\infty} (-1)^k \frac{b_k (\lambda_h \gamma)^{2k}}{40 \left( 1 - \frac{3}{4} (b/L) \right)}
\]

or:

\[
\eta - \eta_s = \frac{1}{nkT\lambda} \frac{1 - \frac{3}{8} (b/L)}{1 - \frac{3}{4} (b/L)} \sum_{k=0}^{\infty} (-1)^k \left[ a_k \frac{9}{40 \left( 1 - \frac{3}{4} (b/L) \right)} b_k \right] (\lambda_h \gamma)^{2k}
\]

which, in the limit of vanishingly small \( \lambda_h \gamma \), gives:

\[
\eta - \eta_s = \frac{1}{nkT\lambda} \frac{1 - \frac{3}{8} (b/L)}{1 - \frac{3}{4} (b/L)}
\]

so that the polymer contribution to the zero-shear viscosity is given by:

\[
\eta_0 - \eta_s = \frac{1 - \frac{3}{5} (b/L)}{nkT\lambda} = 1 + \frac{3}{20} (b/L) + \frac{9}{80} (b/L)^2 \ldots
\]

from which we learn that hydrodynamic interaction, from the Oseen tensor, also increases the polymer contribution to the zero-shear viscosity. Comparing Eqs. (18) with (14), we find that this increase in \( \eta_0 - \eta_s \) for Oseen is just two sevenths that of Rotne-Prager-Yamakawa.

In Eq. (16), for Oseen hydrodynamic interaction, we define:

\[
\lambda_h = \frac{\lambda}{1 - \frac{3}{8} (b/L)}
\]

and for multibead:

\[
\lambda_h^{(N)} = \frac{\lambda_N}{1 - \frac{3}{8} (b/L)}
\]
From Eqs. (19) or (20) we learn that Oseen hydrodynamic interaction increases the relaxation time. Rewriting Eq. (16) as:

\[
\frac{\eta - \eta_s}{nkT\lambda} \left[ \frac{1 - \frac{3}{8}(b/L)}{1 - \frac{3}{8}(b/L)} \right] = 1 - \frac{3}{8}(b/L) \sum_{k=0}^{\infty} (-1)^k \left[ a_k - \frac{9}{40} \left( \frac{(b/L)}{1 - \frac{3}{8}(b/L)} \right) b_k \right] (\lambda_h \dot{\gamma})^{2k}
\]

which we will use below.

Substituting Eqs. (4) and (5) into Eq. (20) we get:

\[
\lambda_h^{(N)} = \frac{N(N+1)}{24kT} \frac{\pi \eta_s L^3}{1 - \frac{3}{8}(N-1)}
\]

so that:

\[
\lambda_h^{(\infty)} = \lim_{N \to \infty} \frac{N(N+1)}{24kT} \frac{\pi \eta_s L^3}{1 - \frac{3}{8}(N-1)} = \frac{\pi \eta_s L^3}{24kT}
\]

which is the relaxation time of an osculated multibead-rod with Oseen hydrodynamic interaction in the limit of an infinite number of beads.

Analytical work on a host of viscoelastic material functions must be checked for consistency against the power series expansions, Eqs. (7) and (8), in their limits of steady shear flow. For instance, for a rigid dumbbell suspension in large-amplitude oscillatory shear flow (LAOS), we must recover the coefficients of these expansions, \((a_k, b_k)\), in the limit of low test frequency (see Eq. (84) of [7]; §9 of [3]). For this same suspension, subjected to steady-state shear flow in parallel superposition with small-amplitude oscillatory shear flow, we must also recover the coefficients of these expansions in the limit of low test frequency (see Eq. (8.19) of [3]). For three other instances of such uses of \((a_k, b_k)\), on three other viscoelastic material functions, see §10 through §12 of [3]. Molecular dynamics simulation can also be checked for consistency against the power series expansions Eqs. (7) and (8) in their limits of steady shear flow. For instance, Brownian dynamics simulations of suspensions of rigid dumbbells in steady shear flow can be so checked (see Table 5.1 with Eqs. (5.76) and (5.77) of [8]).

Kim and Fan [2] found the radii of convergence to be \(\lambda_h \dot{\gamma} \approx 0.81082443 \equiv p\), for the series expansions in Eqs. (7) and (8). By the method of analytic continuation, they reported extending the radii of convergence to \(\lambda_h \dot{\gamma} \approx 1.5\) by replacing the series expansion, Eq. (7), with:
\[
\frac{\eta - \eta_s}{nkT \lambda} = \sum_{k=0}^{10} (-1)^k \left[ a_k - c_1 p^{-2(k+1)} \right] (\lambda \dot{\gamma})^{2k} + c_1 \left[ p^2 + (\lambda \dot{\gamma})^2 \right]^{-1}
\]
where 
\[
c_1 \equiv 0.18799972,
\]
and Eq. (8) with:
\[
\frac{5\Psi_1}{6nkT \lambda^2} = \sum_{k=0}^{10} (-1)^k \left[ b_k - c_2 p^{2(k+1)} \right] (\dot{\lambda \gamma})^{2k} + c_2 \left[ p^2 + (\dot{\lambda \gamma})^2 \right]^{-1}
\]
where 
\[
c_2 = 0.35822745.
\]
Eqs. (24) and (25) present two more uses for the main results of this work, \((a_k, b_k)\).

The summation on the right side of Eq. (7) appears as the first of two summations for the polymer contribution to the steady shear viscosity for a suspension of multibead-rods (Eq. (14.6-23) of [5] or Eq. (33) of [3]). Similarly, the summation on the right side of Eq. (8) appears as the first of two summations for the first normal stress coefficient for a suspension of multibead-rods (Eq. (14.6-24) of [5] or Eq. (34) of [9]). The exact values of the coefficients, \((a_k, b_k)\), are thus equally useful to those exploring the role of hydrodynamic interaction in macromolecular theory.

The coefficients, \((a_k, b_k)\), are also used to arrive at approximate constitutive equations for macromolecular theory ([10]; [11]; §9 of [12]; see problems 11B.9 and 11B.10 of [6]; TABLE 6.2-1 and TABLE 6.2-2 of [5]; Table 1 of [13]; Eq. (32) of [14]; §4 and §5 of [15]; see Example 12.5-2 of [5]; [16,17,18]). For instance, the first few coefficients \((a_k, b_k)\) have been used to arrive at approximate forms of the Oldroyd 6-constant model for suspensions of rigid dumbbells and of finitely extensible nonlinear dumbbells.

These coefficients \((a_k, b_k)\) are not known exactly beyond the fourth power (Eq. (6.8) of [3], Eq. (22) of [19], Eq. (11.4-20) of [6] or (11.6-14) of [12], Eq. (8.6-2) of [20]):
\[
\frac{\eta(\dot{\gamma}) - \eta_s}{nkT \lambda} \equiv - \frac{\tau_{yx} - \tau_{yx,s}}{nkT \lambda \dot{\gamma}} = 1 - \frac{18}{35} (\lambda \dot{\gamma})^2 + \frac{1326}{1925} (\lambda \dot{\gamma})^4 - \cdots
\]
Eq. (11) of [21] and Eq. (8.6-3) of [20]:
\[
\frac{5\Psi_1(\dot{\gamma})}{6nkT \lambda^2} \equiv - \frac{\tau_{xx} - \tau_{yy}}{nkT (\lambda \dot{\gamma})^2} = 1 - \frac{38}{35} (\lambda \dot{\gamma})^2 + \frac{38728}{25025} (\lambda \dot{\gamma})^4 - \cdots
\]
Whereas Eq. (26) gives the polymer contribution to the steady shear viscosity made dimensionless with \(nkT \lambda\), Eq. (27) gives the first normal stress coefficient made dimensionless with \(\frac{6}{5}nkT \lambda^2\). In this work, we arrive at exact expressions for the first 20 coefficients for both expansions, Eqs. (26) and (27).
II. METHOD

We begin with Eq. (1), the recursive formula for the terms of the expansion for the orientation distribution function, \( \psi_k \), in steady simple shear flow:

\[
\hat{\Lambda}(\psi_k) = \hat{\Omega}(\psi_{k-1}), \quad k = 2, 3, 4, \ldots \tag{28}
\]

with

\[
\psi_0 \equiv 1 \tag{29}
\]

and:

\[
\psi_1 = \frac{1}{12} P_2^2 \sin 2\phi \tag{30}
\]

where the lambda operator, from Eq. (13) of [9], is given by:

\[
\hat{\Lambda}(P_n^m \sin m\phi) \equiv -n(n + 1)P_n^m \sin m\phi \tag{31}
\]

\[
\hat{\Lambda}(P_n^m \cos m\phi) \equiv -n(n + 1)P_n^m \cos m\phi \tag{32}
\]

the omega operator, from Eq. (14) of [9], by:

\[
\hat{\Omega}(P_n^m \sin m\phi) = \sum_{j=m-2}^{m+2} \sum_{k=n-2}^{n+2} c_{n,k}^m j P_k^j \cos j\phi \tag{33}
\]

\[
\hat{\Omega}(P_n^m \cos m\phi) = -\sum_{j=m-2}^{m+2} \sum_{k=n-2}^{n+2} c_{n,k}^m j P_k^j \sin j\phi \tag{34}
\]

and where Table 1 of [9] defines \( c_{n,k}^m \). The associated Legendre polynomials in Eqs. (30) through (34) are given by (Eq. (19) of [1]):

\[
P_n^m(C) = \frac{(-1)^m}{2^n n!} \left(1 - C^2\right)^{m/2} \frac{d^{n+m}}{dC^{n+m}} \left(C^2 - 1\right)^n \tag{35}
\]

where \( C \equiv \cos \theta \). Furthermore:

\[
\hat{\Lambda}^{-1}(P_n^m \sin m\phi) \equiv \frac{1}{-n(n + 1)} P_n^m \sin m\phi \tag{36}
\]

\[
\hat{\Lambda}^{-1}(P_n^m \cos m\phi) \equiv \frac{1}{-n(n + 1)} P_n^m \cos m\phi \tag{37}
\]

To find \( a_k \) and \( b_k \), we use [2]:

\[
a_k = (-1)^k 6^{2k} \left[A(1,1,2k) - \frac{1}{5} A(1,2,2k) - \frac{6}{5} A(2,2,2k)\right] \tag{38}
\]

\[
b_k = (-1)^k 6^{2k} \left[12 A(2,2,2k + 1)\right] \tag{39}
\]
where \( A(m,n,k) \) is the coefficient of \( P_{2(n-1)}^{2(m-1)} \) for \( \psi_k \). To deepen the reader’s understanding of our method, APPENDIX VI details its first three steps.

We coded the recursions, Eq. (28) subject to Eq. (30) up to \( k = 20 \), into MATLAB (Version R2018a) (Mathworks, Natick, Massachusetts) on an ASUS Ux310UA Signature Edition Notebook (Intel® Core™ i5-6200U CPU @2.30GHz with 8.00 GB of RAM) employing the Windows 10 Home (Version 1709 Build 16288.431) operating system. Use of MATLAB’s Symbolic Toolkit is required for our code.

III. RESULTS

Following the method of Section II, we improve Eqs. (26) and (27) to:

\[
\eta(\dot{\gamma}) - \eta_s \over nkT \lambda = \left[ 1 - \frac{18}{35}(\lambda \dot{\gamma})^2 + \frac{1326}{1925}(\lambda \dot{\gamma})^4 - \frac{18186}{17875}(\lambda \dot{\gamma})^6 \right] \left[ 1 - \frac{2660632713537}{1732809203125}(\lambda \dot{\gamma})^4 - \frac{1710866325718131}{734163899218750}(\lambda \dot{\gamma})^6 + \ldots \right] (40)
\]

\[
\frac{5}{6}\Psi_1(\dot{\gamma}) \over 6nkT \lambda^2 = \left[ 1 - \frac{38}{35}(\lambda \dot{\gamma})^2 + \frac{38728}{25025}(\lambda \dot{\gamma})^4 - \frac{1206796407}{521145625}(\lambda \dot{\gamma})^6 \right] \left[ 1 + \frac{934753676387}{266586031250}(\lambda \dot{\gamma})^4 - \frac{673750174273478449243}{126488925451951562500}(\lambda \dot{\gamma})^6 + \ldots \right] (41)
\]

Table III and Table IV list our new exact coefficients up to \( (\lambda \dot{\gamma})^{20} \), prime factorized. Our new exact coefficients agree with the approximations of Kim and Fan [2], to within their reported eight significant figures, as they must.

Stewart and Sorensen (in columns 1 and 3 of TABLE II of [22]), to within four or five significant figures, and by the method of orthogonal collocation, report \( (\eta(\dot{\gamma}) - \eta_s) \over nkT \lambda \) at 23 values of \( \lambda \dot{\gamma} \) over four decades. Kirkwood and Plock (in columns 2 and 5 of TABLE I of [23] or [24]), to within three significant figures, report \( (\eta(\dot{\gamma}) - \eta_s) \over nkT \lambda \) at 33 values of \( \lambda \dot{\gamma} \) over three decades. These numerical results of Kirkwood and Plock [23,24] agree closely with those of Stewart and Sorensen [22]. Figure 2 compares both of these sets of numerical results with the prediction of Eq. (7) summed to \( k = 19 \) and plotted just beyond the radius of convergence. From Figure 2 we find that Eq. (7), summed to \( k = 19 \) (with our new exact coefficients, \( a_k \)), agrees closely with the numerical results of [22,23,24], within the radius of convergence.

Stewart and Sorensen (in columns 1 and 3 of TABLE II of [22]), to within four or five significant figures, and by the method of orthogonal collocation, report \( \frac{5}{6}\Psi_1(\dot{\gamma}) \over 6nkT \lambda^2 \) at 23 values of \( \lambda \dot{\gamma} \) over four decades. Figure 3 compares these numerical results with the prediction of Eq. (8) summed to \( k = 19 \) and plotted just beyond the radius of convergence. From Figure 3 we find that Eq. (8),
summed to \( k = 19 \) (with our new exact coefficients, \( b_k \)), agrees closely with the numerical results of [22], within the radius of convergence.

IV. WORKED EXAMPLES

In this section, we construct three examples of uses for we use both of our main results, \( a_k \) and \( b_k \) (in Table III and Table IV).

a. Hydrodynamic Interaction

In the next two subsections we use our main results, \( (a_k, b_k) \), to evaluate the two leading approaches to hydrodynamic interaction in a dilute suspension of rigid dumbbells in steady shear flow. We lead with the Rotne-Prager-Yamakawa tensor-valued description of the velocity disturbance, and then follow with the slightly less general Oseen description (see EXAMPLE 1.4-1 of [25] or EXAMPLE 1.2-4 of [12]; §14.6 a. and §13.6 of [5]).

i. Rotne-Prager-Yamakawa Tensor

In this first example, we explore the steady shear viscosity incorporating hydrodynamic interaction by use of the Rotne-Prager-Yamakawa tensor [Eq. (9)]. From Eq. (14), we already know that hydrodynamic interaction increases the polymer contribution to the zero-shear viscosity, and further, from Eqs. (10) and (11), that hydrodynamic interaction decreases the relaxation time. Figure 4 shows how Rotne-Prager-Yamakawa hydrodynamic interaction intensifies the shear thinning with \( b/L \). Interpolation of Figure 4, can thus be used to fit \( b/L \) to steady shear viscosity measurements. This fitted value of \( b/L \) will, of course, differ from the one obtained in Subsection IV.a.ii below.

ii. Oseen Tensor

In this second example, we explore the steady shear viscosity incorporating hydrodynamic interaction by use of the Oseen tensor [Eq. (21)]. From Eq. (18), we already know that hydrodynamic interaction increases the polymer contribution to the zero-shear viscosity, and from Eq. (19), that hydrodynamic interaction increases the relaxation time. Figure 5 shows how Oseen hydrodynamic interaction weakens the shear thinning. Interpolation of Figure 5, can thus be used to fit \( b/L \) to steady shear viscosity measurements. This fitted value of \( b/L \) will, of course, differ from the one obtained in Subsection IV.a.i above.
b. Molecularizing Oldroyd Fluid for Simple Shear Flows

We next follow our INTRODUCTION (see second to last paragraph), and use our new exact coefficients, \((a_k, b_k)\) to molecularize the Oldroyd 5-constant fluid [10]. By *molecularize*, we mean to assign macromolecular meaning to the continuum model parameters. To start molecularizing, we begin by obtaining:

\[
\eta - \eta_0 = n k T \lambda, \quad \lambda_i = \lambda, \quad \lambda_3 = \frac{3}{2} \lambda \tag{42),(43),(44)
\]

from a direct comparison of the analytical results for small-amplitude oscillatory shear flow. Specifically, we compare Eq. (9) of [1] of the rigid dumbbell model with Eq. (14) with Eqs. (72) and (73) of [26] for the Oldroyd 5-constant fluid:

\[
\tau + \lambda_i \frac{\partial \tau}{\partial t} + \mu_i \left[ \frac{\tau \cdot \gamma_i - 1}{2} \left\{ \tau \cdot \gamma_i + \gamma_i \cdot \tau \right\} \right] = -\eta_0 \left[ \gamma_i + \lambda_2 \frac{\partial \gamma_i}{\partial t} - \mu_2 \left\{ \gamma_i \cdot \gamma_i \right\} \right] \tag{45}
\]

which, when \(\mu_1 = \mu_2 = 0\), reduces to the corotational Jeffreys fluid.

The well-known result for the steady shear viscosity of the Oldroyd 5-constant fluid is (Eq. (14) of [27]):

\[
\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \sigma_1 \dot{\gamma}^2}{1 + \sigma_1' \dot{\gamma}^2} \tag{46}
\]

where:

\[
\sigma_1 = \lambda_2 - \frac{1}{2} \mu^2, \quad \sigma_2 = \lambda_1 \lambda_3 - \frac{1}{2} \mu_2 \mu_1 \tag{47),(48}
\]

Eq. (46) non-dimensionalizes as:

\[
\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \sigma_1 \dasheddot \dot{\gamma}^2}{1 + \sigma_1' \dasheddot \dot{\gamma}^2} \tag{49}
\]

which then expands as:

\[
\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{\sigma_2}{\sigma_1} + \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right) \sum_{k=0}^{\infty} (-1)^k \left( \frac{\sigma_1}{\lambda_1^2} \right)^{k+1} (\lambda_1 \dot{\gamma})^{2k} \tag{50}
\]

or:

\[
\frac{\eta(\dot{\gamma})}{\eta_0} = 1 - \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right)^2 (\lambda_1 \dot{\gamma})^2 + \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right) \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right)^2 (\lambda_1 \dot{\gamma})^4 - \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right)^3 \left( \frac{\sigma_1 - \sigma_2}{\lambda_1^2} \right)^3 (\lambda_1 \dot{\gamma})^6 + \cdots \tag{51}
\]

which simplifies as:

\[
\frac{\eta(\dot{\gamma})}{\eta_0} = 1 + (\sigma_1 - \sigma_2) \sum_{n=1}^{\infty} (-1)^n \sigma_1^{n-1} \dot{\gamma}^{2n} \tag{52}
\]

We will use Eq. (51) presently. Specifically, we compare Eq. (51) of the Oldroyd 5-constant fluid with Eq. (9) of [1] of the rigid dumbbell model.

Comparing \(a_1\) through \(a_3\) in Eq. (7) with the three coefficients in Eq. (51), and using Eq. (43), we get:
where $a_3$ is new, from the $k = 3$ row of Table III, and where $a_1$ and $a_2$ are well-known, from Eq. (26).

Substituting Eqs. (47) and (48) into Eqs. (53) and (54) gives:
\[
\left(\frac{x^2 - \frac{i}{2} \mu_1 - \frac{i}{2} \lambda_2}{\lambda^2} + \frac{i}{2} \mu_2 \mu_4 \right) = \frac{18}{35}
\]
\[
\left(\frac{\lambda_1^2 - \frac{i}{2} \mu_1^2}{\lambda^2} \left(\frac{\lambda_1^2 - \frac{i}{2} \mu_1^2}{\lambda^4} - \frac{i}{2} \lambda_1 \lambda_2 + \frac{i}{2} \mu_2 \mu_4 \right) \right) = \frac{1326}{1925}
\]
and substituting Eqs. (43) and (44) into these we get:
\[
\frac{\frac{3}{2} \lambda_1^2 - \frac{i}{2} \mu_1^2 + \frac{i}{2} \mu_2 \mu_4}{\lambda^2} = \frac{18}{35}
\]
\[
\left(\frac{\frac{3}{2} \lambda_1^2 - \frac{i}{2} \mu_1^2}{\lambda^4} \left(\frac{\frac{3}{2} \lambda_1^2 - \frac{i}{2} \mu_1^2}{\lambda^2} + \frac{i}{2} \mu_2 \mu_4 \right) \right) = \frac{1326}{1925}
\]

We next molecularize $\mu_1$ and $\mu_2$ by solving Eqs. (58) and (59), and then taking the positive roots:
\[
\mu_1 = i \frac{2\sqrt{770}}{55} \lambda, \quad \mu_2 = \frac{491}{392} \mu_1 = i \frac{491\sqrt{770}}{10780} \lambda
\]
\[
\sigma_1 = \frac{221}{165} \lambda^2, \quad \sigma_2 = \frac{953}{1155} \lambda^2
\]
Substituting these into Eq. (55), we estimate:
\[
\frac{a_3 = \sigma^2 (\sigma_1 - \sigma_2)}{\lambda^6} = \frac{97682}{105875}
\]
which falls just 9% below $a_3$, and thus, the molecularized Oldroyd 5-constant model is accurate to second order (fourth power of the Weissenberg number), and even accurate to third order, within 9%.

Substituting Eqs. (62) and (63) into Eq. (46):
\[
\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1 + \frac{953}{1155} (\lambda \dot{\gamma})^2}{1 + \frac{221}{165} (\lambda \dot{\gamma})^2}
\]

Figure 6 illustrates the accuracy of the molecularized Oldroyd 5-constant model.

Similarly, we can further evaluate the accuracy of our new molecularized Oldroyd 5-constant model by examining its implied first normal stress coefficient for which (see Eqs. (22) and (24) of [27]):
\[
\frac{\Psi_1(\dot{\gamma})}{2\eta_0(\dot{\lambda}_1 - \dot{\lambda}_2)} = \frac{\dot{\lambda}_2 - \eta(\dot{\gamma})}{\dot{\lambda}_1 - \eta_0 - \dot{\lambda}_2 - 1}
\]  

so that, using Eqs. (42) through (44) with (65):

\[
\frac{5\Psi_1(\dot{\gamma})}{6nkT\dot{\lambda}^2} = \left(1 + \frac{1}{nkT\dot{\lambda}}\right) \left[1 + \frac{557}{1155}(\dot{\lambda}\dot{\gamma})^2\right]^{-1}
\]

\[
\frac{5\Psi_1(\dot{\gamma})}{6nkT\dot{\lambda}^2} = \left[1 + \frac{221}{165}(\dot{\lambda}\dot{\gamma})^2\right]^{-1}
\]

which for \(\eta_s \ll nkT\dot{\lambda}\) gives:

\[
\frac{5\Psi_1(\dot{\gamma})}{6nkT\dot{\lambda}^2} = \frac{1}{\left[1 + \frac{221}{165}(\dot{\lambda}\dot{\gamma})^2\right]^{-1}}
\]

Figure 7 compares this with the numerical results of Stewart and Sorensen [22]. From this we learn that our new Oldroyd 5-constant model, molecularized with no consideration of \(\Psi_1(\dot{\gamma})\), still predicts \(\Psi_1(\dot{\gamma})\) accurately up to \(\dot{\lambda}\dot{\gamma} \approx 0.25\).

c. Analytic Continuation

We now exploit our new exact coefficients, \((a_k, b_k)\), to evaluate, and to then use Eqs. (24) and (25). We begin by correcting Eq. (24) to:

\[
\frac{\eta - \eta_s}{nkT\dot{\lambda}} = \sum_{k=0}^{9} (-1)^k \left[ a_k - c_1 p^{-2(k+1)} \right] (\dot{\lambda}\dot{\gamma})^{2k} + c_1 \left[ p^2 + (\dot{\lambda}\dot{\gamma})^2 \right]^{-1}
\]

(69)

where \(c_1\) is still about 0.18799972. Figure 8 compares the series expansion for the steady shear viscosity without hydrodynamic interaction, Eq. (7), with our analytic continuation, Eq. (69). Both Eqs. (7) and (69) hug the numerical results given by Stewart and Sorensen [22] or by Kirkwood and Plock [23,24]. From Figure 8 we discover that our corrected analytic continuation extends the original series expansion usefully, from \(\dot{\lambda}\dot{\gamma} \approx 0.81\) to just below \(\dot{\lambda}\dot{\gamma} \approx 1.5\). Our Eq. (69) corrects Eq. (24), and also corrects the end of the sum in Eq. (12) of [22] from “10” to “9”. Correspondingly, in TABLE II of [22], in the subheading of column 2, “\(N = 10\)” should be “\(N = 9\)”, and after Eq. (13) in [22], “\(N = 10\)” was the optimum value” should be “\(N = 9\)” was the optimum for the \((\eta(\dot{\gamma}) - \eta_s)/nkT\dot{\lambda}\)”, and \(N = 18\), for \(\Psi_1(\dot{\gamma})/nkT\dot{\lambda}^2\). Thus, our Table V corrects TABLE II of [22].

We continue by correcting Eq. (25) to:

\[
\frac{5\Psi_1}{6nkT\dot{\lambda}^2} = \sum_{k=0}^{18} (-1)^k \left[ b_k - c_2 p^{-2(k+1)} \right] (\dot{\lambda}\dot{\gamma})^{2k} + c_2 \left[ p^2 + (\dot{\lambda}\dot{\gamma})^2 \right]^{-1}
\]

(70)
where, to within twelve significant figures, \( c_2 \approx 0.429871827736660 \). Figure 9 compares the series expansion for the first normal stress coefficient without hydrodynamic interaction, Eq. (8), with our analytic continuation, Eq. (70). To construct Figure 9 accurately, we must carry all twelve significant figures of \( c_2 \).

Both Eqs. (8) and (70) hug the numerical results given by Stewart and Sorensen [22]. From Figure 9 we discover that the analytic continuation extends the original series expansion usefully, from \( \lambda \dot{\gamma} \approx 0.81 \) to just below \( \lambda \dot{\gamma} \approx 1.5 \). Our Eq. (70) corrects Eq. (25), and also corrects the end of the sum in Eq. (13) of [22] from “10” to “18”. Thus, mindful of our erratum to the text following Eq. (13) in [22] (see above), our Table V corrects all of column 4 in TABLE II of [22], including its subheading.

We can deepen our understanding of our analytic continuations by expanding the rational term in each of Eqs. (69) and (70) as power series, and then collapsing the results:

\[
\frac{\eta - \eta_s}{nkT\lambda} = \sum_{k=0}^{9} (-1)^k a_k (\lambda \dot{\gamma})^{2k} + c_1 \sum_{k=10}^{\infty} (-1)^k p^{-2(k+1)} (\lambda \dot{\gamma})^{2k} 
\]

\[
\frac{5\Psi_1}{6nkT\lambda^2} = \sum_{k=0}^{18} (-1)^k b_k (\lambda \dot{\gamma})^{2k} + c_2 \sum_{k=19}^{\infty} (-1)^k p^{-2(k+1)} (\lambda \dot{\gamma})^{2k} 
\]

which isolate the continuations on the right.

d. Series Reversion

We will next revert our main results, the series expansions for the steady shear material functions, Eqs. (7) and (8), to get \( \lambda \dot{\gamma} \) as functions of the steady shear material functions.

i. Steady Shear Viscosity

We first rewrite Eq. (7) in terms of the dimensionless shear stress:

\[
\bar{\tau}_{yx} \equiv \frac{\eta(\dot{\gamma}) - \eta_s}{nkT\lambda} = \lambda \dot{\gamma} \left[ 1 - \frac{18}{35} (\lambda \dot{\gamma})^2 + \frac{1326}{1925} (\lambda \dot{\gamma})^4 - \frac{18186}{17875} (\lambda \dot{\gamma})^6 + \ldots \right]
\]

which, by use of Eq. (4.5.12) of [28] or Eq (22) of [29], reverts as:

\[
\lambda \dot{\gamma} = \bar{\tau}_{yx} \left[ 1 + \frac{18}{35} \bar{\tau}_{yx}^2 + \frac{282}{695} \bar{\tau}_{yx}^4 - \frac{1130394}{613125} \bar{\tau}_{yx}^6 + \ldots \right]
\]

which Figure 10 plots, truncated after its third, fifth and seventh powers of \( \bar{\tau} \), and from which, we learn that Eq. (74) diverges after its \( \bar{\tau}^3 \) term. Taking the convergent part of Eq. (74), we next solve for the shear rate as a function of the polymer contribution to the viscosity:  

16
\[
\lambda \dot{\gamma} = \sqrt{\frac{35(1-\eta)}{18\eta^3}} 
\]

(75)

where:
\[
\bar{\eta} = \frac{\eta(\dot{\gamma}) - \eta_s}{nkT \lambda}
\]

(76)

From Figure 11 we discover that \( \bar{\eta} \), from this simple Eq. (75), inflects as it should, and thus does not fall below zero. Also, Eq. (75) agrees closely with the numerical solutions so long as \( \lambda \dot{\gamma} < 1.5 \).

**ii. First Normal Stress Coefficient**

Here, we begin by rewriting Eq. (8) as:
\[
\varphi \equiv \frac{5\Psi(\dot{\gamma})}{6nkT \lambda^2} \lambda \dot{\gamma} = \lambda \dot{\gamma} \left[ 1 - \frac{38}{35} (\lambda \dot{\gamma})^2 + \frac{38728}{25025} (\lambda \dot{\gamma})^4 - \frac{1206796407}{521145625} (\lambda \dot{\gamma})^6 \right] 
\]

(77)

which, by use of Eq. (4.5.12) of [30] or Eq (22) of [31], reverts as:
\[
\lambda \dot{\gamma} = \varphi \left[ 1 + \frac{38}{35} \varphi^2 + \frac{69676}{35035} \varphi^4 - \frac{169638899}{40088125} \varphi^6 + \cdots \right] 
\]

(78)

which we truncate after its \( \varphi^3 \) term, and then solve the resulting cubic to get:
\[
\lambda \dot{\gamma} = \sqrt{\frac{35(1-\bar{\Psi}_1)}{38\bar{\Psi}_1^3}} 
\]

(79)

where:
\[
\bar{\Psi}_1 \equiv \frac{5\Psi_1(\dot{\gamma})}{6nkT \lambda^2} 
\]

(80)

From Figure 12 we discover that \( \bar{\Psi}_1 \), from this simple Eq. (79), inflects as it should, and thus does not fall below zero. Also, Eq. (79) agrees closely with the numerical solutions so long as \( \lambda \dot{\gamma} < 0.12 \).

**e. Inversions**

For applications, when truncated after any positive term, we prefer the inverse forms of Eqs. (7) and (8) over the upright forms. This is because, whereas these inverse forms ensure that \( \eta(\dot{\gamma}) > 0 \) and \( \Psi_1(\dot{\gamma}) > 0 \) for all \( \dot{\gamma} \), the upright forms do not. Furthermore, whereas the inverse forms give proper inflections, the upright forms do not. We thus close our worked examples by inverting Eqs. (7) and (8) following Theorem 6 of §15.3. of [32], to get:
which, when truncated after any of their positive terms, are positive and inflect properly. Figure 13 and Figure 14 illustrate this.

V. CONCLUSION

This paper deepens our understanding of the power series expansions for the steady shear material functions for macromolecular theory, and specifically suspensions of rigid dumbbells, Eqs. (7) and (8). Our introduction explores the many uses of these series expansions, including checking analytical work for consistency on a host of viscoelastic material functions in their limits of steady shear flow. The coefficients of the steady shear viscosity and the first normal stress coefficient functions were not known exactly beyond the fourth power. In this work, for both of these functions, we arrive at exact expressions for the first 20 coefficients (Table III and Table IV; Table I and Table II of the Supplementary Materials).

We detail worked examples illustrating uses for our new coefficients. These examples begin with evaluations of two approaches to the hydrodynamic interaction between beads of rigid dumbbells in steady shear flow (Subsection IV.a). We next used our new coefficients to assign molecular meaning to the coefficients of a new constitutive equation from continuum mechanics, the Oldroyd 5-constant fluid (Subsection IV.b). We next used our new coefficients to correct and improve upon prior work on extending the power series by method of analytic continuation (Subsection IV.0). To get the shear rate as a function of the shear stress, we also reverted the power series expansions for the steady shear material functions (Subsection IV.d). Finally, to get power series expansions that are both positive and inflect properly, we inverted the power series expansions for the steady shear material functions (Subsection IV.e).

We close with one other for the power series expansions for the steady shear material functions, Eqs. (7) and (8). Consider the well-known result for the first normal stress coefficient with hydrodynamic interaction from the Oseen tensor (Eq. (11.6-24) of [6]):
\[ \frac{5\Psi_1}{6nkT\lambda_h^2} = \frac{1 - \frac{3}{8} b/L}{1 - \frac{3}{4} b/L} \sum_{k=0}^{\infty} (-1)^k \left[ b_k - \frac{3}{8} \frac{b/L}{1 - \frac{3}{8} b/L} c_k \right] \left( \lambda_h^2 \gamma \right)^{2k} \]  

in which the coefficients \( c_k \) are not yet known, even approximately, beyond the second power of \( \lambda h \gamma \). We leave the evaluation of these \( c_k \) for another day.

VI. APPENDIX: Three Recursions

We next detail our first three recursions.

a. First

Substituting Eq. (30) into Eq. (28) gives:

\[
\hat{\Lambda}(\psi_2) = \hat{\Omega} \left( \frac{1}{12} P_2^2 \sin 2\phi \right) = \frac{1}{12} \hat{\Omega} \left( P_2^2 \sin 2\phi \right)
\]

and then substituting this into Eq. (33), we get:

\[
\hat{\Omega} \left( P_2^2 \sin 2\phi \right) = \left[ c_{0,0}^2 P_0^0 \cos 0\phi + c_{2,2}^2 P_2^0 \cos 0\phi + c_{4,2}^2 P_4^0 \cos 0\phi \right] \\
+ c_{0,2}^2 P_2^2 \cos 2\phi + c_{2,2}^2 P_2^2 \cos 2\phi + c_{4,2}^2 P_4^2 \cos 2\phi \\
+ c_{0,2}^4 P_4^4 \cos 4\phi + c_{2,2}^4 P_4^4 \cos 4\phi + c_{4,2}^4 P_4^4 \cos 4\phi
\]

where we retain the \( \cos 0\phi \) terms since the \( \hat{\Lambda} \) and \( \hat{\Omega} \) operators of Eqs. (32) and (34) will receive these. Next, using Table 1 of [9] to get the values of \( c_{n,k}^{m,j} \):

\[
\hat{\Omega} \left( P_2^2 \sin 2\phi \right) = \begin{bmatrix}
\frac{6}{7} P_0^0 \cos 0\phi - \frac{6}{7} P_0^0 \cos 0\phi \\
- P_2^2 \cos 2\phi + \frac{1}{28} P_4^4 \cos 4\phi
\end{bmatrix}
\]

Substituting this into Eq. (84) gives:

\[
\hat{\Lambda}(\psi_2) = \frac{1}{12} \begin{bmatrix}
\frac{6}{7} P_0^0 \cos 0\phi - \frac{6}{7} P_0^0 \cos 0\phi \\
- P_2^2 \cos 2\phi + \frac{1}{28} P_4^4 \cos 4\phi
\end{bmatrix}
\]

so that:
\[
\psi_2 = \frac{1}{12} \hat{\Lambda}^{-1} \begin{bmatrix}
\frac{6}{7} P_2^0 \cos 0\phi - \frac{6}{7} P_4^0 \cos 0\phi \\
- P_2^2 \cos 2\phi + \frac{1}{28} P_4^4 \cos 4\phi
\end{bmatrix}
\]

(88)

\[
= \frac{1}{12} \begin{bmatrix}
\frac{6}{7} \hat{\Lambda}^{-1} (P_2^0 \cos 0\phi) - \frac{6}{7} \hat{\Lambda}^{-1} (P_4^0 \cos 0\phi) \\
- \hat{\Lambda}^{-1} (P_2^2 \cos 2\phi) + \frac{1}{28} \hat{\Lambda}^{-1} (P_4^4 \cos 4\phi)
\end{bmatrix}
\]

and inverting with Eq. (37):

\[
\psi_2 = \frac{1}{12} \begin{bmatrix}
\frac{6}{7} \left( - \frac{1}{6} P_2^0 \cos 0\phi \right) - \frac{6}{7} \left( - \frac{1}{20} P_4^0 \cos 0\phi \right) \\
- \left( - \frac{1}{6} P_2^2 \cos 2\phi \right) + \frac{1}{28} \left( - \frac{1}{20} P_4^4 \cos 4\phi \right)
\end{bmatrix}
\]

(89)

which expands as:

\[
\psi_2 = - \frac{1}{84} P_2^0 \cos 0\phi + \frac{1}{72} P_2^2 \cos 2\phi + \frac{1}{280} P_4^0 \cos 0\phi - \frac{1}{6720} P_4^4 \cos 4\phi
\]

(90)

Using this to evaluate Eq. (38) at \( k = 1 \) gives our result for the first recursion:

\[
a_1 = (-1)^1 6^2 \left[ A(1,1,2) - \frac{1}{5} A(1,2,2) - \frac{6}{5} A(2,2,2) \right]
\]

(91)

where:

\[
A(1,1,2) = 0, \quad A(1,2,2) = - \frac{1}{84}, \quad A(2,2,2) = \frac{1}{72}
\]

(92),(93),(94)

so that:

\[
a_1 = -6^2 \left[ 0 - \frac{1}{5} \left( - \frac{1}{84} \right) - \frac{6}{5} \left( \frac{1}{72} \right) \right] = \frac{18}{35}
\]

(95)

From Eq. (39) we see that \( b_1 \) requires \( \psi_3 \) and hence, this first recursion does not produce \( b_1 \).

b. Second

Following the pattern for the first recursion, we get:

\[
\hat{\Lambda}(\psi_3) = \hat{\Omega}(\psi_2)
\]

(96)
\[
\hat{\Lambda}(\psi_3) = \hat{\Omega} \left( -\frac{1}{84} P_2^0 \cos 0\phi + \frac{1}{72} P_2^2 \cos 2\phi \\
+ \frac{1}{280} P_4^0 \cos 0\phi - \frac{1}{6720} P_4^4 \cos 4\phi \right)
\]

\[
\hat{\Lambda}(\psi_3) = \left[ -\frac{1}{84} \left( \frac{1}{14} P_2^2 \sin 2\phi - \frac{1}{14} P_4^4 \sin 2\phi \right) \\
+ \frac{1}{72} \left( P_2^2 \sin 2\phi - \frac{1}{28} P_4^4 \sin 4\phi \right) \\
+ \frac{1}{280} \left( \frac{1}{63} P_2^2 \sin 2\phi + \frac{3}{154} P_4^4 \sin 2\phi - \frac{7}{198} P_6^6 \sin 6\phi \right) \\
- \frac{1}{6720} \left( \frac{40}{3} P_2^2 \sin 2\phi - \frac{12}{11} P_4^4 \sin 4\phi \\
+ \frac{14}{33} P_6^6 \sin 6\phi + 2 P_4^4 \sin 4\phi - \frac{7}{396} P_6^6 \sin 6\phi \right) \right]
\]

\[
\psi_3 = \hat{\Lambda}^{-1} \left[ -\frac{1}{84} \left( \frac{1}{14} P_2^2 \sin 2\phi - \frac{1}{14} P_4^4 \sin 2\phi \right) \\
+ \frac{1}{72} \left( P_2^2 \sin 2\phi - \frac{1}{28} P_4^4 \sin 4\phi \right) \\
+ \frac{1}{280} \left( \frac{1}{63} P_2^2 \sin 2\phi + \frac{3}{154} P_4^4 \sin 2\phi - \frac{7}{198} P_6^6 \sin 6\phi \right) \\
- \frac{1}{6720} \left( \frac{40}{3} P_2^2 \sin 2\phi - \frac{12}{11} P_4^4 \sin 4\phi \\
+ \frac{14}{33} P_6^6 \sin 6\phi + 2 P_4^4 \sin 4\phi - \frac{7}{396} P_6^6 \sin 6\phi \right) \right]
\]
Using this to evaluate Eq. (39) at $k = 1$ gives our result for the second recursion:

$$b_1 = (-1)^1 6^2 \left[ 12 A(2,2,3) \right]$$

where:

$$A(2,2,3) = -\frac{19}{7560}$$

so that:

$$b_1 = -6^2 \left[ 12 \left( -\frac{19}{7560} \right) \right] = \frac{38}{35}$$

From Eq. (38) we see that $a_2$ requires $\psi_4$ and hence, this second recursion does not produce $a_2$.

c. Third

$$\hat{\Lambda}(\psi_4) = \hat{\Omega}(\psi_3)$$
\[
\psi_4 = \left[ \begin{array}{c}
\frac{4}{10395} P^0 \cos \phi - \frac{23}{45360} P^2 \cos 2\phi - \frac{16039}{158558400} P^4 \cos 0\phi \\
- \frac{3}{616000} P^2 \cos 2\phi + \frac{7517}{86486400} P^4 \cos 4\phi \\
- \frac{1}{110880} P^6 \cos \phi + \frac{71}{139708800} P^2 \cos 2\phi \\
+ \frac{1}{39916800} P^4 \cos 4\phi - \frac{71}{3353011200} P^6 \cos 6\phi + \frac{1}{823680} P^8 \cos 0\phi \\
- \frac{1}{1037836800} P^8 \cos 4\phi + \frac{1}{99632332800} P^8 \cos 8\phi
\end{array} \right]
\] (106)

evaluate Eq. (38) at \( k = 2 \):
\[
a_2 = (-1)^2 6^4 \left[ A(1,1,4) - \frac{1}{5} A(1,2,4) - \frac{6}{5} A(2,2,4) \right]
\] (107)
\[
A(1,1,4) = 0, A(1,2,2) = \frac{4}{10395}, A(2,2,2) = -\frac{23}{45360}
\] (108),(109),(110)

Using this to evaluate Eq. (38) at \( k = 2 \) gives our result for the third recursion:
\[
a_2 = 6^4 \left[ -\frac{1}{5} \left( \frac{4}{10395} \right) - \frac{6}{5} \left( -\frac{23}{45360} \right) \right]
\] (111)
\[
a_2 = \frac{1326}{1925}
\] (112)

From Eq. (39) we see that \( b_2 \) requires \( \psi_5 \) and hence, this third recursion does not produce \( b_2 \).

VII. SUPPLEMENTARY MATERIAL

See supplementary material for the entire list of exact coefficients to \( k = 20 \) and for the MATLAB recursion code.

VIII. ACKNOWLEDGMENT

We acknowledge Professor Sangtae Kim of the chemical engineering department of Purdue University for his helpful correspondence on analytic continuation and with Table II of [2]. We thank Mr. Peter H. Gilbert for his help with series reversion. JHP also thanks the Faculty of Engineering and Applied Science of Queen’s University for the Charles Allan Thompson Research Award. AJG is indebted to the Faculty of Applied Science and Engineering of Queen’s University at Kingston, for its support through a Research Initiation Grant (RIG). This research was undertaken, in part, thanks to support from the Canada Research Chairs program of the Government of Canada for the Natural Sciences.
and Engineering Research Council of Canada (NSERC) Tier 1 Canada Research Chair in Rheology.
Table I: Dimensional Variables.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bead diameter</td>
<td>$b$</td>
<td>$L$</td>
</tr>
<tr>
<td>Bead diameter, maximum</td>
<td>$b_{\text{max}}$</td>
<td>$L$</td>
</tr>
<tr>
<td>Boltzmann constant</td>
<td>$k$</td>
<td>$M/Lt^2$; $M/Lt^2$ / dumbbell</td>
</tr>
<tr>
<td>Cartesian coordinate $x$</td>
<td>$x$</td>
<td>$L$</td>
</tr>
<tr>
<td>Cartesian coordinate $y$</td>
<td>$y$</td>
<td>$L$</td>
</tr>
<tr>
<td>Cartesian coordinate $z$</td>
<td>$z$</td>
<td>$L$</td>
</tr>
<tr>
<td>Extra stress tensor</td>
<td>$\tau_{yx}$</td>
<td>$M/Lt^2$</td>
</tr>
<tr>
<td>First normal stress coefficient</td>
<td>$\Psi_1(\dot{\gamma})$</td>
<td>$M/L$</td>
</tr>
<tr>
<td>Friction coefficient</td>
<td>$\zeta$</td>
<td>$M/t$</td>
</tr>
<tr>
<td>Lumped parameter, Oldroyd fluid, Eq. (47)</td>
<td>$\sigma_1$</td>
<td>$t^2$</td>
</tr>
<tr>
<td>Lumped parameter, Oldroyd fluid, Eq. (48)</td>
<td>$\sigma_2$</td>
<td>$t^2$</td>
</tr>
<tr>
<td>Relaxation time, Oldroyd fluid</td>
<td>$\lambda_1$</td>
<td>$t$</td>
</tr>
<tr>
<td>Relaxation time, rigid dumbbell</td>
<td>$\lambda$</td>
<td>$t$</td>
</tr>
<tr>
<td>Retardation time, Oldroyd fluid</td>
<td>$\lambda_2$</td>
<td>$t$</td>
</tr>
<tr>
<td>Shear rate, steady shear flow</td>
<td>$\dot{\gamma}$</td>
<td>$t^{-1}$</td>
</tr>
<tr>
<td>Solvent contribution to the stress tensor</td>
<td>$\tau_{yx,s}$</td>
<td>$M/Lt^2$</td>
</tr>
<tr>
<td>Solvent viscosity</td>
<td>$\eta_s$</td>
<td>$M/Lt$</td>
</tr>
<tr>
<td>Steady shear viscosity function</td>
<td>$\eta(\dot{\gamma})$</td>
<td>$M/Lt$</td>
</tr>
<tr>
<td>Length of the rod, rigid dumbbell</td>
<td>$L$</td>
<td>$L$</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>Time</td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td>Time constant, Oldroyd fluid</td>
<td>$\mu_1$</td>
<td>$t$</td>
</tr>
<tr>
<td>Time constant, Oldroyd fluid</td>
<td>$\mu_2$</td>
<td>$t$</td>
</tr>
<tr>
<td>Zero-shear viscosity</td>
<td>$\eta_0$</td>
<td>$M/Lt$</td>
</tr>
</tbody>
</table>

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature
**Table II:** Dimensionless Variables and Groups.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Azimuthal angle</td>
<td>$\phi$</td>
</tr>
<tr>
<td>Polar angle</td>
<td>$\theta$</td>
</tr>
<tr>
<td>Associated Legendre Polynomial</td>
<td>$P_{n}^{m}$</td>
</tr>
<tr>
<td>First normal stress coefficient</td>
<td>$\Psi_1$</td>
</tr>
<tr>
<td>Legendre Polynomial Coefficient</td>
<td>$A(m,n,k)$</td>
</tr>
<tr>
<td>Number of beads, multi-bead rod</td>
<td>$N$</td>
</tr>
<tr>
<td>Orientation Distribution</td>
<td>$\psi$</td>
</tr>
<tr>
<td>Orientation Distribution, $k^{th}$ term</td>
<td>$\psi_k$</td>
</tr>
<tr>
<td>Power-law index</td>
<td>$k$</td>
</tr>
<tr>
<td>Shear stress, steady shear flow</td>
<td>$\tau_{yx}$</td>
</tr>
<tr>
<td>Steady shear viscosity</td>
<td>$\eta$</td>
</tr>
<tr>
<td>Weissenberg number, steady shear flow</td>
<td>$\lambda\dot{\gamma}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$a_k$</td>
</tr>
<tr>
<td>-----</td>
<td>-------</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2 \cdot 3^2}{5 \cdot 7}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2 \cdot 3 \cdot 13 \cdot 17}{5^2 \cdot 7 \cdot 11}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2 \cdot 3 \cdot 7 \cdot 433}{5^3 \cdot 11 \cdot 13}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3 \cdot 31 \cdot 47 \cdot 191 \cdot 3,186,917}{5^6 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{3 \cdot 632,393 \cdot 901,794,889}{2 \cdot 5^8 \cdot 7^5 \cdot 11 \cdot 13 \cdot 17 \cdot 23}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1,117 \cdot 22,829,941,376,904,617,863}{2^2 \cdot 5^8 \cdot 7^8 \cdot 11^4 \cdot 13^2 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{43 \cdot 7,235,128,937 \cdot 1,117,491,369,200,783,185,338,581}{2^2 \cdot 3^2 \cdot 12^7 \cdot 11^6 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{19,477 \cdot 112,111 \cdot 11,239,843 \cdot 1,561,426,269,343,646,717,005,855,124,029}{2^6 \cdot 3^4 \cdot 5^{14} \cdot 7^{12} \cdot 11^7 \cdot 13^6 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>9</td>
<td>$\begin{bmatrix} 181,605,893 \cdot 1,847,609 \cdot 4,977,672,751 \ 65,007,894,781 \cdot 316,978,934,436,746,835,930,263 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$2^9 \cdot 3^6 \cdot 5^{15} \cdot 7^{13} \cdot 11^{10} \cdot 13^8 \cdot 17^3 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37$</td>
</tr>
<tr>
<td>10</td>
<td>$\begin{bmatrix} 3,851 \ 30,785,542,594,409,711,807,777,091,724,570,368,962,501,951,796,275,008,054,250,091,345,995,647,240,781 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$2^{12} \cdot 3^8 \cdot 5^{18} \cdot 7^{16} \cdot 11^{12} \cdot 13^{10} \cdot 17^6 \cdot 19^4 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43$</td>
</tr>
</tbody>
</table>
Table IV: List of First Normal Stress Coefficients in Rational Form.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$b_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2 \cdot 19}{5 \cdot 7}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2^3 \cdot 47 \cdot 103}{5^2 \cdot 7 \cdot 11 \cdot 13}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3 \cdot 23 \cdot 17,489,803}{5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{479 \cdot 1,951,469,053}{2 \cdot 5^6 \cdot 7 \cdot 11 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{927,769 \cdot 1,051,079 \cdot 690,913,493}{2^2 \cdot 5^8 \cdot 7^7 \cdot 11^3 \cdot 13^2 \cdot 19 \cdot 23}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{109 \cdot 84,731 \cdot 515,420,513 \cdot 3,679,905,447 \cdot 556,069}{2^3 \cdot 3^2 \cdot 5^9 \cdot 7^9 \cdot 11^5 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{3,803 \cdot 714,739 \cdot 2,329,075,788,349,523,039,608,263,578,845,877}{2^4 \cdot 3^4 \cdot 5^{12} \cdot 7^{11} \cdot 11^7 \cdot 13^5 \cdot 17^3 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{[53 \cdot 155,083 \cdot 162,473 \cdot 507,317 \cdot 4,031,071,678,790,551,071,799,710,464,846,335,043]}{2^5 \cdot 3^{16} \cdot 5^{14} \cdot 7^{11} \cdot 13^7 \cdot 17^3 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{[5,584,703 \cdot 322,231,423,949 \cdot 12,231,048,351,151,026,102,837,661,431,734,805,591,665,653,966,771]}{2^4 \cdot 3^{8} \cdot 5^{16} \cdot 7^{14} \cdot 11^{13} \cdot 13^8 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{[643 \cdot 6,842,873 \cdot 605,018,183,977,797,673 \cdot 178,078,353,778,824,465,638,912,294,107,471,970,499,924,244,378,017,016,201]}{2^5 \cdot 3^{10} \cdot 5^{18} \cdot 7^{17} \cdot 11^{13} \cdot 13^{10} \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43}$</td>
</tr>
</tbody>
</table>
Table V: Comparison of Material Functions.

<table>
<thead>
<tr>
<th>λ\hat{\gamma}</th>
<th>\left(\eta(\hat{\gamma}) - \eta_s\right)/nkT\lambda</th>
<th>\Psi_1(\hat{\gamma})/nkT\lambda^2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Eq. (69) Numerical Results</td>
<td>Eq. (70) Numerical Results</td>
</tr>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9949</td>
<td>0.9949</td>
</tr>
<tr>
<td>0.125</td>
<td>0.9921</td>
<td>0.9921</td>
</tr>
<tr>
<td>0.25</td>
<td>0.9703</td>
<td>0.9703</td>
</tr>
<tr>
<td>0.3333</td>
<td>0.9502</td>
<td>0.9502</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9029</td>
<td>0.9029</td>
</tr>
<tr>
<td>0.75</td>
<td>0.8306</td>
<td>0.8306</td>
</tr>
<tr>
<td>1</td>
<td>0.7676</td>
<td>0.7676</td>
</tr>
<tr>
<td>1.5</td>
<td>0.6604</td>
<td>0.6735</td>
</tr>
<tr>
<td>2.0833</td>
<td>-9.1876</td>
<td>0.5998</td>
</tr>
</tbody>
</table>
Figure 1: Velocity profiles near and away from beads, including beads of rigid dumbbells. (a) Stokes flow around one bead in sea of solvent. Stokes flows around each of two beads of rigid dumbbell (c) with (b) without interference between velocity profiles \( b/L = 2/7 \). Stokes flows around each of two beads of rigid dumbbell with interference \( b/L = 1/2 \).
Figure 2: Viscosity expansion, truncated after the thirty-eighth power, Eq.(7) truncated at $k = 19$. Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 3: First normal stress coefficient expansion, truncated after the thirty-eighth power, Eq. (8) at $k = 19$. Closed circles, calculated by Stewart and Sorensen [22].
Figure 4: Viscosity expansion with hydrodynamic interaction in the Rotne-Prager-Yamakawa tensor, truncated after the thirty-eighth power, Eq. (21) at $k = 19$. Black circles show the viscosity without hydrodynamic interaction calculated by Stewart and Sorensen [22].
Figure 5: Viscosity expansion with hydrodynamic interaction in the Oseen tensor, truncated after the thirty-eighth power, Eq. (21) at $k = 19$. Blue circles show the viscosity hydrodynamic interaction calculated by Stewart and Sorensen [22].
Figure 6: Molecularized Oldroyd 5-constant fluid steady shear viscosity function, Eq. (65), versus numerical results. Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 7: Molecularized Oldroyd 5-constant fluid first normal stress coefficient, Eq. (68), versus numerical results. Circles calculated by Stewart and Sorensen [22].
Figure 8: Viscosity expansion, blue line, truncated after the thirty-eighth power, Eq.(7) truncated at $k = 19$ and the red one is the analytic continuation plot of Eq. (24). Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 9: First normal stress expansion, blue line, truncated after the thirty-eighth power, Eq. (8) truncated at $k = 19$ and the red line is the analytic continuation of the first normal stress, Eq. (25). Closed circles calculated by Stewart and Sorensen [22].
Figure 10: Eq. (74) truncated after the third (red), fifth (green), seventh (blue) powers. Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 11: The reverted form of the steady shear viscosity, Eq. (75), in red, *versus* the plot of Eq. (7), in blue. Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 12: The reverted form of the first normal stress coefficient, Eq. (79), in red, *versus* the plot of Eq.(8), in blue. Circles calculated by Stewart and Sorensen [22].
Figure 13: Inverse steady shear viscosity series expansion, Eq. (81), truncated after its second, sixth and tenth powers in red, green and blue respectively. Closed circles, calculated by Stewart and Sorensen [22], and open ones, by Kirkwood and Plock [22,23].
Figure 14: Inverse steady shear first normal stress coefficient series expansion, Eq. (81), truncated after its second, sixth and tenth powers in red, green and blue respectively. Circles calculated by Stewart and Sorensen [22].
IX. REFERENCES


SUPPLEMENTARY MATERIAL TO:

EXACT COEFFICIENTS FOR RIGID DUMBBELL SUSPENSIONS
FOR STEADY SHEAR FLOW MATERIAL FUNCTION EXPANSIONS

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¹Chemical Engineering Department
²Mechanical and Materials Engineering Department
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Kingston, Ontario, CANADA K7L 3N6

ABSTRACT

From kinetic molecular theory, we can attribute the elasticity of polymeric liquids to macromolecular orientation. For a suspension of rigid dumbbells, subject to a particular flow field, we must first solve the diffusion equation for the orientation distribution function. From this distribution, we then calculate physical properties such as the steady shear flow material functions. We thus arrive at power series expansions in the shear rate for both the orientation distribution function and for the steady shear flow material functions. Analytical work on many viscoelastic material functions must be checked for consistency, in their steady shear flow limits, against these power series. For instance, for large-amplitude oscillatory shear flow, we recover the coefficients of these expansions in the limits of low test frequency. The coefficients of the steady shear viscosity and the first normal stress coefficient functions are not known exactly beyond the fourth power. In this work, for both of these functions, we arrive at exact expressions for the first 20 coefficients. We close with five worked examples illustrating uses for our new coefficients.

Keywords: Rigid dumbbell suspensions, hydrodynamic interaction, Oldroyd 5-constant fluid, macromolecular theory, steady shear material function, steady shear viscosity, first normal stress coefficient, series analytic continuation, series reversion, series inversion, recursion.

* Corresponding author (giacomin@queensu.ca).
I. EXACT COEFFICIENTS \( (a_k, b_k) \)

Table I and Table II of the Supplementary Materials list the results for \( (a_k, b_k) \) of the paper which this material supplements. We have prime factorized these results to \( k = 12 \) and partly prime factorized for \( 13 \leq k \leq 20 \). We leave the full prime factorization of \( (a_k, b_k) \) over \( 13 \leq k \leq 20 \) for another day.

II. MATLAB CODE FOR RECURSION

The MATLAB code used to arrive at the coefficients in Table I and Table II follows. This code executes the recursion of Eqs. (7) and (8) of the parent paper which details the recursion in its VI. APPENDIX.

a. Main Script

This section details the executable code used to generate the coefficients of Eq. (7) and Eq. (8).

```matlab
clear
clc
global b
b = 24;
psi2 = SCoefC(3,3)/12;
a1 = ((psi2(2,2) + psi2(2,3)/(-5) + psi2(3,3)/(-5/6))*6^2);
psi3 = zeros(b,b);
for i = 2:12
    for j = 2:12
        psi3 = psi3 + CfuncS(psi2(i,j),i,j);
    end
end
b2 = 12*(psi3(3,3))*6^2;
psi4 = zeros(b,b);
for i = 2:14
    for j = 2:14
        psi4 = psi4 + SfuncC(psi3(i,j),i,j);
    end
end
a2 = (psi4(2,2) + (-1/5)*psi4(2,3) + (-6/5)*psi4(3,3))*6^4;
psi5 = zeros(b,b);
for i = 2:12
    for j = 2:12
        psi5 = psi5 + CfuncS(psi4(i,j),i,j);
    end
end
b2 = 12*(psi5(3,3))*6^4;
psi6 = zeros(b,b);
for i = 2:14
```
for j = 2:14
    psi6 = psi6 + SfuncC(psi5(i,j),i,j);
end
end
a3 = (psi6(2,2) + psi6(2,3)/(-5) + psi6(3,3)/(-5/6))*6^6;
psi7 = zeros(b,b);
for i = 2:16
    for j = 2:16
        psi7 = psi7 + CfuncS(psi6(i,j),i,j);
    end
end
b3 = 12*(psi7(3,3))*6^6;
psi8 = zeros(b,b);
for i = 2:18
    for j = 2:18
        psi8 = psi8 + SfuncC(psi7(i,j),i,j);
    end
end
a4 = (psi8(2,2) + psi8(2,3)/(-5) + psi8(3,3)/(-5/6))*6^8;
psi9 = zeros(b,b);
for i = 2:20
    for j = 2:20
        psi9 = psi9 + CfuncS(psi8(i,j),i,j);
    end
end
b4 = 12*(psi8(3,3))*6^8;
psi10 = zeros(b,b);
for i = 2:22
    for j = 2:22
        psi10 = psi10 + SfuncC(psi9(i,j),i,j);
    end
end
a5 = (psi10(2,2) + psi10(2,3)/(-5) + psi10(3,3)/(-5/6))*6^10;
%Repeat for more Coefficients.

b. Functions Used
Below are the functions used in the main script:

i. SCoefC

function Coef = SCoefC(m,n)
global b
Coef = sym(zeros(b,b));
Coef(m,n)= 1;
m = sym(m); n = sym(n);
i = m; j = n;
%for i=3:4 %represents m in some cases
% for j=3:4 %represents n in some cases
if double(i*2-4) > double(j*2-4) || double(i*2-4) == 0 || Coef(double(i),double(j)) == 0
    Coef(i,j) = Coef(i,j);
else
    A = ACoef(i*2-4,j*2-4);
    Coef(i,j+1) = sym((Coef(i,j)*(A(2,3) + Coef(i,j+1)))*lambda((j+1)*2-4)); %m,n+1
    Coef(i,j-1) = sym((Coef(i,j)*(A(2,1) + Coef(i,j-1)))*lambda((j-1)*2-4)); %m,n-1
    Coef(i+1,j) = sym((Coef(i,j)*(A(3,2) + Coef(i+1,j)))*lambda((j)*2-4)); %m+1,n
    Coef(i+1,j+1) = sym((Coef(i,j)*(A(3,3) + Coef(i+1,j+1)))*lambda((j+1)*2-4)); %m+1,n+1
    Coef(i+1,j-1) = sym((Coef(i,j)*(A(3,1) + Coef(i+1,j-1)))*lambda((j-1)*2-4)); %m+1,n-1
    Coef(i,j) = sym(Coef(i,j)*(A(2,2))*lambda((j)*2-4)); %m,n
end
for i = 1:length(Coef)
    for j = 1:length(Coef)
        if double(i) > double(j)
            Coef(i,j) = 0;
        end
    end
end

% CCoeffS

function Coef = CCoeffS(m,n)
global b
Coef = sym(zeros(b,b));
Coef(m,n) = sym(1);
i = m; j = n;
m = sym(m); n = sym(n);

%for i=3:4 %represents m in some cases
% for j=3:4 %represents n in some cases
if double(i*2-4) > double(j*2-4) || double(Coef(i,j)) == 0
    Coef(i,j) = Coef(i,j);
    Coef(:,2) = 0;
elseif double((i*2-4)-2) == 0
    A = ACoef(i*2-4,j*2-4);
    Coef(i,j+1) = -(Coef(i,j)*(A(2,3) + Coef(i,j+1)))*lambda((j+1)*2-4); %m,n+1
    Coef(i,j-1) = -(Coef(i,j)*(A(2,1) + Coef(i,j-1)))*lambda((j-1)*2-4); %m,n-1
    Coef(i+1,j) = -(Coef(i,j)*(A(3,2) + Coef(i+1,j)))*lambda((j)*2-4); %m+1,n
    Coef(i+1,j+1) = -(Coef(i,j)*(A(3,3) + Coef(i+1,j+1)))*lambda((j+1)*2-4); %m+1,n+1
    Coef(i+1,j-1) = -(Coef(i,j)*(A(3,1) + Coef(i+1,j-1)))*lambda((j-1)*2-4); %m+1,n-1
    Coef(i-1,j) = 0; %m-1,n
%m - 1,n+1
Coef(i,j) = -Coef(i,j)*(A(2,2))*lambda(j*2-4); %m,n
Coef(:,2) = 0;
else
A = ACoef((i*2-4,j*2-4));

Coef(i,j+1) = -(Coef(i,j)*(A(2,3) + Coef(i,j+1)))*lambda(j+1)*2-4); %m,n+1
Coef(i,j-1) = -(Coef(i,j)*(A(2,1) + Coef(i,j-1)))*lambda(j-1)*2-4); %m,n-1
Coef(i+1,j) = -(Coef(i,j)*(A(3,2) + Coef(i+1,j)))*lambda(j*2-4); %m+1,n
Coef(i+1,j+1) = -(Coef(i,j)*(A(3,3) + Coef(i+1,j+1)))*lambda((j+1)*2-4); %m+1,n+1
Coef(i+1,j-1) = -(Coef(i,j)*(A(3,1) + Coef(i+1,j-1)))*lambda((j-1)*2-4); %m+1,n-1

Coef(i-1,j) = -(Coef(i,j)*(A(1,2) + Coef(i-1,j)))*lambda(j*2-4); %m-1,n
Coef(i-1,j+1) = -(Coef(i,j)*(A(1,3) + Coef(i-1,j+1)))*lambda((j+1)*2-4); %m-1,n+1
Coef(i-1,j-1) = -(Coef(i,j)*(A(1,1) + Coef(i-1,j-1)))*lambda((j-1)*2-4); %m-1,n-1

Coef(i,j) = -Coef(i,j)*(A(2,2))*lambda(j*2-4); %m,n
Coef(:,2) = 0;
end
for i = 1:length(Coef)
    for j = 1:length(Coef)
        if double(i) > double(j)
            Coef(i,j) = 0;
        end
    end
end

iii. SfuncC
function [D] = SfuncC(C,i,j)
    D = C .* SCoefC(i,j);
end

iv. CfuncS
function [D] = CfuncS(C,i,j)
    D = C .* CCoefS(i,j);
end

v. ACoef
function [Amn] = ACoef(m,n)
if m == 0
    sm = 1;
else
    sm = 0;
end
%Amn matrix represents the non zero terms of the sigma spherical
%harmonic
Amn = sym(zeros(3,3));
% Amn(1,1) represents A(m-2, n-2)
Amn(1,1) = ((n-2)*(n+m)*(n+m-1)*(n+m-2)*(n+m-3)*(1-sm))/(4*(2*n+1)*(2*n-1));

% Amn(1,2) represents A(m-2, n)
Amn(1,2) = (3*(n+m)*(n+m-1)*(n-m+1)*(n-m+2)*(1-sm))/(4*(2*n-1)*(2*n+3));

% Amn(1,3) represents A(m-2, n+2)
Amn(1,3) = ((n+3)*(n-m+1)*(n-m+2)*(n-m+3)*(1-sm))/(4*(2*n+1)*(2*n+3));

% Amn(2,1) and Amn(2,3) represents A(m,n-2) and A(m,n+2) respectively
Amn(2,1) = 0;
Amn(2,3) = 0;

% Amn(2,2) represents A(m,n)
Amn(2,2) = m/2;

% Amn(3,1) represents A(m+2, n-2)
Amn(3,1) = ((n-2)*(1+sm))/(4*(2*n+1)*(2*n-1));

% Amn(3,2) represents A(m+2, n)
Amn(3,2) = 3*(1+sm)/(4*(2*n-1)*(2*n+3));

% Amn(3,3) represents A(m+2, n+2)
Amn(3,3) = ((n+3)*(1+sm))/(4*(2*n+1)*(2*n+3));

end
**Table I**: List of shear stress coefficients in rational form

<table>
<thead>
<tr>
<th>$k$</th>
<th>$a_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2 \cdot 3^2}{5 \cdot 7}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2 \cdot 3 \cdot 13 \cdot 17}{5^2 \cdot 7 \cdot 11}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2 \cdot 3 \cdot 7 \cdot 433}{5^3 \cdot 11 \cdot 13}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3 \cdot 31 \cdot 47 \cdot 191 \cdot 3,186,917}{5^6 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{3 \cdot 632,393 \cdot 901,794,889}{2 \cdot 5^8 \cdot 7^5 \cdot 11 \cdot 13 \cdot 17 \cdot 23}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1,117 \cdot 22,829,941,376,904,617,863}{2^2 \cdot 5^8 \cdot 7^8 \cdot 11^4 \cdot 13^2 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{43 \cdot 7,235,128,937 \cdot 1,117,491,369,200,783,185,338,581}{2^3 \cdot 3^5 \cdot 7^{10} \cdot 11^6 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{19,477 \cdot 112,111 \cdot 11,239,843 \cdot 1,561,426,269,343,646,717,005,855,124,029}{2^6 \cdot 3^4 \cdot 5^{14} \cdot 7^{12} \cdot 11^7 \cdot 13^6 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{181 \cdot 605,893 \cdot 1,847,609 \cdot 4,977,672,751 \cdot 65,007,894,781 \cdot 316,978,934,436,746,835,930,2}{2^9 \cdot 3^6 \cdot 5^{15} \cdot 7^{13} \cdot 11^{10} \cdot 13^8 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{3,851}{2^{12} \cdot 3^8 \cdot 5^{18} \cdot 7^{16} \cdot 11^{12} \cdot 13^{10} \cdot 17^6 \cdot 19^4 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{78,892,973}{2^{15} \cdot 3^{10} \cdot 5^{20} \cdot 7^{18} \cdot 11^{14} \cdot 13^{12} \cdot 17^8 \cdot 19^6 \cdot 23 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47}$</td>
</tr>
<tr>
<td>12</td>
<td>$\begin{bmatrix} 109 &amp; 117119 &amp; 107097011830490887 &amp; 549,740,486,488,201,301 \ 1,477,835,501,085,222,882,094,325,463,228,049, \ 477,725,810,795,919,866,977,037,258,329,287,653,971 \ \end{bmatrix}$ $2^{18} \cdot 3^{12} \cdot 5^{12} \cdot 7^{20} \cdot 11^{16} \cdot 13^{14} \cdot 17^{10} \cdot 19^{8} \cdot 23^{4} \cdot 29^{3} \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$</td>
</tr>
<tr>
<td>13</td>
<td>$\begin{bmatrix} 829,208,209 \ \cdot 91,249,329,452,603,505,732,474,976,357,234,638,425,825, \ 466,887,475,231,380,228,169,860,587,100,095,977,216,412,247, \ 214,454,789,262,135,776,459,684,837,363,317,079 \ \end{bmatrix}$ $2^{21} \cdot 3^{14} \cdot 5^{24} \cdot 7^{22} \cdot 11^{18} \cdot 13^{16} \cdot 17^{12} \cdot 19^{9} \cdot 23^{6} \cdot 29^{3} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$</td>
</tr>
<tr>
<td>14</td>
<td>$\begin{bmatrix} 10453 &amp; 16097 &amp; 31231 &amp; 18379927 &amp; 394967549 \ \cdot 2,501,174,162,845,306,116,436,730,800,478,940,363,842,900, \ 727,395,381,300,476,990,263,654,768,871,901,365,114,889,586, \ 252,784,528,456,584,931,628,866,462,095,505,181 \ \end{bmatrix}$ $2^{24} \cdot 3^{16} \cdot 5^{26} \cdot 7^{24} \cdot 11^{18} \cdot 13^{16} \cdot 17^{14} \cdot 19^{12} \cdot 23^{8} \cdot 29^{2} \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59$</td>
</tr>
<tr>
<td>15</td>
<td>$\begin{bmatrix} 79309 \ \cdot 7,798,110,022,257,671,722,135,345,085,543,218,387,717,479, \ 135,353,797,752,552,647,592,935,512,776,257,903,064,830,068, \ 699,434,884,128,235,538,817,727,055,970,204,438,043,888, \ 432,783,173,869,758,587,833,597,713,199,958,322,413 \ \end{bmatrix}$ $2^{27} \cdot 3^{18} \cdot 5^{27} \cdot 7^{26} \cdot 11^{18} \cdot 13^{10} \cdot 17^{16} \cdot 19^{14} \cdot 23^{10} \cdot 29^{3} \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61$</td>
</tr>
<tr>
<td>16</td>
<td>$\begin{bmatrix} 167,543 &amp; 857,069 &amp; 363,581,463,983,18,823,472,162,141 \ \cdot 23,192,090,151,070,224,391,594,784,722,935,013,727,891,444, \ 340,695,843,624,605,056,026,363,640,914,974,645,672,147,153, \ 068,326,499,427,112,665,444,717,348,801,141,166,320,040, \ 221,015,071,462,570,040,325,042,027,267,360,501 \ \end{bmatrix}$ $2^{30} \cdot 3^{20} \cdot 5^{20} \cdot 7^{28} \cdot 11^{24} \cdot 13^{22} \cdot 17^{18} \cdot 19^{16} \cdot 23^{12} \cdot 29^{6} \cdot 31^{4} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67$</td>
</tr>
<tr>
<td></td>
<td>83 ⋅ 103 ⋅ 307 ⋅ 1,021 ⋅ 1,871 ⋅ 2,521 ⋅ 18,587 ⋅ 34,511 ⋅ 75,227</td>
</tr>
<tr>
<td>-----</td>
<td>------------------------------------------------------------------</td>
</tr>
<tr>
<td>17</td>
<td>⋅ 244,075,978,908,080,507,036,280,825,287,709,426,828,382,820,</td>
</tr>
<tr>
<td></td>
<td>221,629,224,442,235,324,440,466,031,838,745,309,766,745,072,092,</td>
</tr>
<tr>
<td></td>
<td>355,784,864,841,130,294,721,570,878,633,006,916,226,696,518,206,</td>
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<tr>
<td></td>
<td>639,668,855,270,146,820,485,770,745,695,626,812,790,409,014,141,</td>
</tr>
<tr>
<td></td>
<td>349,076,400,475,200,591,437</td>
</tr>
<tr>
<td></td>
<td>2^{30}3^{56}5^{32}7^{30}11^{26}13^{24}17^{20}19^{18}23^{14}29^{8}31^{6}37^{4}41^{4}43^{5}53^{5}61^{6}67^{7}1</td>
</tr>
<tr>
<td>18</td>
<td>229 ⋅ 1,889 ⋅ 220,240,285,159</td>
</tr>
<tr>
<td></td>
<td>⋅ 989,921,089,256,206,334,399,680,273,636,362,150,120,335,840,</td>
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<tr>
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<td>054,610,918,233,334,369,808,030,060,947,475,273,112,558,392,888,</td>
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<tr>
<td></td>
<td>480,813,136,610,794,840,816,721,011,974,174,458,254,169,022,860,</td>
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<td></td>
<td>382,477,622,678,464,381,484,111,418,338,035,682,364,002,964,</td>
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<tr>
<td></td>
<td>883,274,334,221,200,758,333,547,319,640,605,648,276,445,793,319,</td>
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<tr>
<td></td>
<td>353,819,559,221,837</td>
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<tr>
<td></td>
<td>2^{31}3^{59}5^{33}7^{11}2^{28}13^{6}17^{22}19^{20}23^{16}29^{10}31^{7}37^{2}41^{4}43^{7}53^{5}59^{6}61^{6}67^{7}1</td>
</tr>
<tr>
<td>19</td>
<td>179 ⋅ 12,619 ⋅ 33,317 ⋅ 898,172,684,623</td>
</tr>
<tr>
<td></td>
<td>⋅ 311,066,067,988,718,862,602,137,264,248,403,742,995,</td>
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<tr>
<td></td>
<td>096,501,022,977,894,713,335,614,851,993,119,051,725,978,830,</td>
</tr>
<tr>
<td></td>
<td>595,452,194,265,542,203,281,704,916,277,534,991,007,006,188,983,</td>
</tr>
<tr>
<td></td>
<td>616,169,671,427,928,877,406,193,139,589,379,983,825,194,678,</td>
</tr>
<tr>
<td></td>
<td>000,325,698,707,417,771,039,869,377,123,448,312,197,164,824,</td>
</tr>
<tr>
<td></td>
<td>885,131,567,313,491,718,848,314,640,574,194,031,875,946,923,033</td>
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<tr>
<td></td>
<td>2^{31}3^{64}5^{37}7^{34}11^{30}13^{28}17^{24}19^{22}23^{18}29^{12}31^{10}37^{3}41^{4}43^{5}53^{5}59^{6}61^{6}67^{7}1</td>
</tr>
<tr>
<td></td>
<td>73^{7}9</td>
</tr>
</tbody>
</table>

---

The table contains prime factorizations and powers of primes for different numbers. Each row represents a number followed by its prime factorization and powers of primes. The numbers are listed in a natural text format with proper spacing and formatting to ensure readability.
| \[2^{31} \cdot 3^{67} \cdot 5^{40} \cdot 7^{36} \cdot 11^{32} \cdot 13^{30} \cdot 17^{25} \cdot 19^{24} \cdot 23^{20} \cdot 29^{14} \cdot 31^{12} \cdot 37^{6} \cdot 41^{2} \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83] |
Table II: List of first normal stress coefficients in rational form.

<table>
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<th>$b_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2 \cdot 19}{5 \cdot 7}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2^3 \cdot 47 \cdot 103}{5^2 \cdot 7 \cdot 11 \cdot 13}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3 \cdot 23 \cdot 17,489,803}{5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{479 \cdot 1,951,469,053}{2 \cdot 5^6 \cdot 7 \cdot 11 \cdot 17 \cdot 19}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{927,769 \cdot 1,051,079 \cdot 690,913,493}{2^2 \cdot 5^8 \cdot 7^7 \cdot 11^3 \cdot 13^2 \cdot 19 \cdot 23}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{109 \cdot 84,731 \cdot 515,420,513 \cdot 3,679,905,447,556,069}{2^3 \cdot 3^2 \cdot 5^{10} \cdot 7^9 \cdot 11^5 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{3,803 \cdot 714,739 \cdot 2,329,075,788,349,523,039,60}{2^4 \cdot 3^4 \cdot 5^{12} \cdot 7^{11} \cdot 11^7 \cdot 13^5 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31}$</td>
</tr>
<tr>
<td>8</td>
<td>$\left\lfloor \frac{53 \cdot 155,083 \cdot 162,473 \cdot 507,317}{2^5 \cdot 3^6 \cdot 5^{14} \cdot 7^{13} \cdot 11^9 \cdot 13^7 \cdot 17^3 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37} \right\rfloor$</td>
</tr>
<tr>
<td>9</td>
<td>$\left\lfloor \frac{558,470,3 \cdot 322,231,423,949}{2^4 \cdot 3^5 \cdot 5^{16} \cdot 7^{14} \cdot 11^8 \cdot 13^8 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41} \right\rfloor$</td>
</tr>
<tr>
<td>10</td>
<td>$\left\lfloor \frac{643 \cdot 684,287,3 \cdot 605,018,183,977,779,767}{2^5 \cdot 3^7 \cdot 5^{18} \cdot 7^{17} \cdot 11^{13} \cdot 13^{10} \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43} \right\rfloor$</td>
</tr>
<tr>
<td>11</td>
<td>$\left\lfloor \frac{349 \cdot 154,656,720,488,712,648,381,485,076,438,846,151,653,484,978,894,358,545,896,844,398,724,029,240,185,694,416,305,733,233,121,601,353,483}{2^{12} \cdot 3^{12} \cdot 5^{20} \cdot 7^{19} \cdot 11^{15} \cdot 13^{12} \cdot 17^9 \cdot 19^7 \cdot 23^3 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47} \right\rfloor$</td>
</tr>
<tr>
<td>12</td>
<td>(239 \cdot 6,271,757)</td>
</tr>
<tr>
<td>----</td>
<td>------------------</td>
</tr>
<tr>
<td></td>
<td>(\cdot 249,961,287,957,043,256,527,533,487,197,743,059,038,934,852,450,689,597,628,299,400,056,258,075,906,703,081,303,711,292,872,504,346,378,949,098,813)</td>
</tr>
<tr>
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<td>(2^{13}3^{14}5^{22}7^{21}11^{17}13^{14}17^{9}19^{9}23^{5}29^{3}31^{3}37^{2}41^{3}43^{3}47^{2}53)</td>
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<table>
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<tr>
<th>13</th>
<th>(11,167,393)</th>
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<tbody>
<tr>
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<td>(2^{18}3^{16}5^{24}7^{23}11^{19}13^{17}13^{17}19^{11}23^{7}29^{2}31^{3}37^{2}41^{3}43^{3}47^{2}53)</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>14</th>
<th>(2,090,758,283)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\cdot 1,139,427,344,421,887,797,093,613,038,552,247,623,159,487,583,795,504,422,767,432,681,555,025,779,571,231,008,032,220,826,471,223,645,977,874,357,770,266,196,442,262,907,079,947,164,803,284,578,207,762,597)</td>
</tr>
<tr>
<td></td>
<td>(2^{19}3^{18}5^{26}7^{25}11^{21}13^{19}17^{15}19^{13}23^{9}29^{3}31^{3}37^{2}43^{2}47^{3}53^{2}59^{2}61)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>15</th>
<th>(79 \cdot 317 \cdot 347 \cdot 1217 \cdot 1354168681)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2^{24}3^{20}5^{28}7^{27}11^{23}13^{20}17^{19}15^{17}23^{11}29^{9}31^{3}37^{2}41^{4}43^{2}47^{3}53^{2}59^{2}61)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>16</th>
<th>(74323 \cdot 445799531 \cdot 131142150293818739)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2^{25}3^{22}5^{30}7^{29}11^{25}13^{23}17^{19}19^{17}23^{13}29^{7}31^{5}37^{3}41^{4}43^{3}47^{3}53^{2}59^{2}61^{2}67)</td>
</tr>
</tbody>
</table>
$$\begin{align*}
\text{17} & \quad \frac{103 \cdot 307 \cdot 1013 \cdot 1021 \cdot 1871 \cdot 34511 \cdot 3206175810977}{2^{832}021732718709880448383371120358254048169769918729506504850306920240298399764952185965156628978170204508029951156820962832993882067095730256237170741144283975663917889517081328328194797413910874457709996378118694962619714341} \\
\text{18} & \quad \frac{229 \cdot 3716754593 \cdot 220240285159}{2^{61}5^{34}7^{33}11^{29}13^{27}17^{23}19^{21}23^{17}29^{11}37^{3}41^{4}43^{14}53^{3}59^{6}61^{6}67^{7}73} \\
\text{19} & \quad \frac{44121036159971913761964066038377462887543485386679764204309937601157176795760746722778248792213266796965034886517912789211698085022256182689786994066159015806744624908656929157726403069791407857174289889622587328743941307081551090552438196910526944455630954823287967683439487432525815497189636514303279}{2^{28}3^{66}5^{13}7^{35}11^{31}13^{29}17^{25}19^{23}23^{18}29^{13}31^{11}37^{5}41^{2}43^{3}53^{3}59^{2}61^{6}67^{7}73^{7}79}.
\end{align*}$$
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<td>(2^{26} \cdot 3^{69} \cdot 5^{31} \cdot 7^{36} \cdot 11^{33} \cdot 13^{31} \cdot 17^{26} \cdot 19^{25} \cdot 23^{21} \cdot 29^{15} \cdot 31^{13} \cdot 37^{7} \cdot 41^{3} \cdot 43 \cdot 47 \cdot 59 \cdot 61 \cdot 67 \cdot 73 \cdot 79 \cdot 83)</td>
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