MARKED LENGTH SPECTRUM RIGIDITY OF ANOSOV MANIFOLDS

by

JON BRYAN

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Abstract

We provide a proof that, for compact Riemannian manifolds with Anosov geodesic flow and non-positive curvature, two sufficiently close metrics with equal marked length spectrum will necessarily be isometric. To be precise, there exists an $N > 0$ depending only on the dimension of the manifold where we have the result holding in a sufficiently small $C^N$-neighborhood of an arbitrary Anosov metric. The proof provided is a detailed version of Guillarmou and Lefeuvre’s 2019 proof including some additional calculations along the way. We also include a brief summary of some difficulties in generalizing the result. Chapters 2 and 3 provide the background on the topic as well as the relevant lemmas and propositions used in the proof with Chapter 4 providing the actual detailed proof of the result.
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Chapter 1

Introduction

Two Riemannian manifolds are called isometric if there exists a Riemannian metric-preserving diffeomorphism between them. All the Riemannian-geometric properties of isometric Riemannian manifolds are identical (up to the isometry), and this naturally leads to the question of characterizing classes of isometric Riemannian manifolds. When looking at compact Riemannian manifolds one would like to obtain a so-called rigidity result; that is to say, is there a less detailed mathematical object induced from the Riemannian manifold that uniquely determines its metric (up to the isometry)? In particular we will be looking at the rigidity question as it pertains to lengths on the manifold. Obviously, if one had information about the lengths between every pair of points on the manifold, this would uniquely identify the metric, so we must look at a subset of this.

Taking \((M, g)\) to be a compact Riemannian manifold. An obvious candidate for a length-based rigidity result would be the set of lengths of closed geodesics. This can be defined by taking the standard mapping of elements of the fundamental group \(\pi_1(M)\) to closed geodesics on \(M\) (discussed in Proposition 2.1) and, for each element \(c \in \pi_1(M)\), we look at the length of the corresponding geodesic. We define the length spectrum of \((M, g)\) to be the set of lengths of all closed geodesics. The problem is that, as discussed in [GL19], it has been shown that even in the case of restricting to manifolds with constant negative curvature the length spectrum is discrete but does not uniquely determine the metric. Thus one has one of two options, even further restrict the class of manifolds or define a more useful set of data. Doing the latter, one might think to define the marked length
1.1. MAIN THEOREM

spectrum, that is to say the following function:

\[ L_g : \pi_1(M) \to \mathbb{R}_{\geq 0} \]

\[ L_g(c) := \ell(\gamma_c) \]

where \( \ell(\gamma_c) \) is the length of the geodesic corresponding to the element \( c \in \pi_1(M) \). This function’s range would be the length spectrum; however, by looking at the function itself, our data is the pairing of both the length and the curve that generated it. This is, non-trivially, more information than just the length spectrum. Obviously if two metrics are isometric, then they would have equal marked length spectrum. But one struggles to think of metrics that are not isometric yet have the same marked length spectrum, suggesting that this may be rigid. This motivates the question of whether marked length spectrum is enough to guarantee isometry. In 1985, Burns and Katok conjectured the following (rewritten to match our notation).

**Conjecture 1.1 ([BK85], 3.1).** If \( M \) is a compact manifold with two Riemannian metrics \( g \) and \( g_0 \) that both have negative curvature and \( L_g = L_{g_0} \), then they are isometric\(^1\).

This conjecture was proved to be true in dimension 2; however it is still open in general. We will instead look at a modified version.

1.1 Main Theorem

In 2019, Guillarmou and Lefeuvre modified the conjecture to the following:

**Theorem 1.1.** Let \((M, g_0)\) be a smooth compact Riemannian manifold of dimension \( n \) with Anosov geodesic flow and let \( N > \frac{3n}{2} + 8 \). Additionally assume non-positive curvature for \( n \geq 3 \). There exists \( \varepsilon > 0 \) such that for all metrics \( g \in \Gamma^N(S^2_+ T^* M) \), if \( L_g = L_{g_0} \) and \( \| g - g_0 \|_{C^N} < \varepsilon \), then \((M, g)\) is isometric to \((M, g_0)\).

The definitions of \( \Gamma^N(S^2_+ T^* M) \) and \( \| \|_{C^N} \) are included later in Chapter 2; however, conceptually, Theorem 1.1 argues that manifolds which admit Anosov geodesic flow and have equal marked length spectrum will locally (under the \( C^N \) topology) have unique metrics (up to isometry). We note that,

\(^1\)In [BK85], they used the term length spectrum for what we call the marked length spectrum.
in [Ano67], it was shown that compact manifolds with constant negative curvature always admit Anosov geodesic flow. This means that Theorem 1.1 implies a local version of the Conjecture 1.1. Theorem 1.1 was originally proved in [GL19] and then later proved in an alternative way in [GKL21].

1.2 Organization and Motivation

This thesis aims to work towards a full understanding of the proof for Theorem 1.1 provided in [GL19] and to discuss some approaches and difficulties one might encounter when tackling the ambitious goal of finding an alternative proof for Theorem 1.1.

We proceed first in Chapter 2 by introducing some of the core background concepts that are necessary for understanding the results, as well as declaring the notation and terminology that will be used throughout the thesis. This chapter will also include some proofs of the background results that are of notable importance or that make frequent appearances. In Chapter 3 we will go through some of the lemmas and propositions used to prove the main theorem, as well as introduce some of the results from other related papers in the field that are not necessarily as common as those within the background section. In Chapter 4 we will prove the main theorem, mirroring the approach taken by [GL19] and utilizing the important details outlined in Chapters 2 and 3. Lastly, Chapter 5 will summarize the future directions one can go regarding this problem and conclude the thesis.
Chapter 2

Background and Notation

This chapter will be a discussion of various notation and core concepts that are used throughout the proof of Theorem 1.1. First we state some assumptions that will be made throughout all proofs. Most importantly, \( M \) will always be taken as compact with no boundary; so for all proofs, the notion of compactness may not be stated as an assumption, but is sometimes used. Similarly, \( n \) is always taken as the dimension of the manifold. Additionally, \( g_0 \) will always denote a smooth Riemannian metric on \( M \).

2.1 Space Definitions

Throughout the paper we will use the following conventions for various manifolds and spaces.

2.1.1 Space of Continuous and Continuously Differentiable Functions

We denote \( C^0(M) \) to be the space of continuous functions from \( M \) to \( \mathbb{R} \). For \( k \in \mathbb{N} \), we say that a function \( f \in C^0(M) \) is of class \( C^k \) at a point \( p \in M \) if there exists a chart \((\phi, U)\) with \( p \in U \) and \( f \circ \phi^{-1} \) is \( C^k \) in the normal sense on \( \mathbb{R}^n \). We say that a function \( f \) is \( C^k \) on \( M \) if it is \( C^k \) at all \( p \in M \). From this we define the space of space of continuously differentiable functions \( C^k(M) \) to be the set of functions of class \( C^k \) on \( M \). This directly lends itself to the relationship for \( k_1 \leq k_2 \)

\[ C^{k_2}(M) \subset C^{k_1}(M) \]
and thus we define

\[ C^\infty(M) := \bigcap_{k \in \mathbb{N}} C^k(M). \]

From here we create a standard construction of a norm on \( C^k(M) \) for compact \( M \). We use this particular construction as it mirrors the approach used to define \( H^s(M) \) later in Section 2.1.2.

Take \( f \in C^k(M) \) as defined above. Given we assume \( M \) to be compact, there exists a finite atlas \( \mathcal{A} = \{(U_1, \phi_1), \ldots, (U_m, \phi_m)\} \) for \( M \). From this we define a finite \( \mathcal{C}^\infty \) partition of unity \((\rho_i)_{i=1}^{m}\) subordinate to the atlas (i.e. \( \sum_{i=1}^{m} \rho_i = 1 \) and \( \text{supp}(\rho_i) \subset \subset (U_i) \) for \( i \in \{1, \ldots, m\} \)). From this we define the norm on \( C^k(M) \). For \( f \in C^k(M) \)

\[
\|f\|_{C^k} := \sum_{i=1}^{m} \max_{|\beta| \leq k} \sup_{p \in \text{supp}(\rho_i)} \left| \partial_\beta^2 (\rho_i \cdot f)(p) \right|
\]

where \( \beta = (b_1, \ldots, b_n) \) and for the coordinates \( \phi_i(p) = (x^1, \ldots, x^n) \), we have \( \partial_\beta = \left( \frac{\partial}{\partial x^1} \right)^{b_1} \cdots \left( \frac{\partial}{\partial x^n} \right)^{b_n} \).

One can verify that these functions are all defined on the correct domains and the supremum is of a continuous function on a compact set, so it is also finite. Note that while this definition of norm depends on our choice of atlas and partition of unity, the topology induced is independent of these choices.

For \( \alpha \in [0, 1] \), we additionally define the space of \( \alpha \)-Hölder continuous functions \( C^{0,\alpha}(M) \subset C^0(M) \) which has the seminorm

\[
|f|_{C^{0,\alpha}} := \sup_{x, y \in M, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}
\]

where \( d(x, y) \) is the distance metric on \( M \) that we define later in Section 2.2. For \( k \in \mathbb{N} \) we define the new space

\[
C^{k,\alpha}(M) := \left\{ f \in C^k(M) : \partial_\beta^2 (\rho_i \cdot f) \in C^{0,\alpha}(U_i) \quad |\beta| = k, \quad i \in \{1, \ldots, m\} \right\}.
\]
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This has the obvious norm

$$\|f\|_{C^k,\alpha} := \|f\|_{C^k} + \sum_{i=1}^{m} \max_{|\beta| = k} |\partial_\beta f|_{C^0,\alpha}$$

2.1.2 $L^2$ Sobolev Spaces

For an open set $U \subset \mathbb{R}^n$, we define the $L^2$-Sobolev space for $k \in \mathbb{N}$ by

$$H^k(U) := \{ f \in L^2(U) : \partial^\alpha f \in L^2(U) \quad \forall |\alpha| \leq k \},$$

where $\partial^\alpha(f)$ is the distributional derivative of $f$ of order $\alpha$ and $L^2(U)$ denotes the standard Hilbert space of square integrable functions. Here we take the distributional derivative in the sense defined by Schwartz in [Sch51]. One can verify that $H^k$ is a Hilbert space with respect to the following inner product:

$$\langle f_1, f_2 \rangle_{H^k} := \sum_{|\alpha| \leq k} \langle \partial^\alpha f_1, \partial^\alpha f_2 \rangle_{L^2}.$$

From the definition, clearly we have $H^k \subset H^l$ for $k > l > 0$. We now work towards the goal of defining $H^k$ on manifolds. We denote by $\mathcal{D}(U)$ to be the space of all $C^\infty$ compactly supported functions on $U$ and by $\mathcal{D}'(U)$ (space of distributions on $U$) its topological dual. This allows us to extend $H^k$ to all integers by

$$H^{-k}(U) := \{ f \in \mathcal{D}'(U) : f \text{ is continuous with respect to } \|\|_{H^k} \text{ on } \mathcal{D}(U) \}.$$  

From here, it is difficult to generalize further; however, in the case of $\mathbb{R}^n$, we can use the Fourier Transform to give an alternative definition. Let $\mathcal{F}$ be the Fourier transform. For $s \in \mathbb{R}$, we define

$$H^s(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : \left( \xi \mapsto (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \right) \in L^2(\mathbb{R}^n) \},$$
2.1. SPACE DEFINITIONS

where $\hat{f} = \mathcal{F}f$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$. From this, we define

$$H^s_0(\mathbb{R}^n) = H^s(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$$

where $\mathcal{E}'(\mathbb{R}^n)$ is the space of compactly supported distributions on $\mathbb{R}^n$. For the precise definitions of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$, see [Zim92]. This definition coincides with the original definition for $s \in \mathbb{Z}$ when restricted to compactly supported functions. Using the Fourier transform, we also define the norm

$$\|f\|_{H^s} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{f}(\xi)| d\xi.$$

Using this definition, we can now define the Sobolev spaces on manifolds. Given we assume $M$ to be compact, we take the same atlas $\mathcal{A} = \{(U_1, \phi_1), \ldots (U_m, \phi_m)\}$ and partition of unity $(\rho_i)_{1 \ldots m}$ used to define the $C^k$-norm for $M$. By compactness, we note that we have the non-trivial property $\phi_i(U_i)$ is compact in $\mathbb{R}^n$ for $i \in \{1, \ldots, m\}$. From this we define

$$H^s(M) := \{ f \in \mathcal{D}'(M) : (f \cdot \rho_i) \circ \phi_i^{-1} \in H^s_0(\mathbb{R}^n) \quad \forall i \in \{1, \ldots, m\} \}.$$

This is well defined as $\text{supp}(f \cdot \rho_i) \subset U_i$ so we can compose it with $\phi_i^{-1}$ and the compact support of $\rho_i$ ensures that $(f \cdot \rho_i) \circ \phi_i^{-1}$ will also be compactly supported in $\mathbb{R}^n$. Furthermore, by the compactness of the space, one can show that the set is defined independently of our choice of both atlas and partition of unity. This definition can be thought of as decomposing $f$ using the partition of unity and then taking each component and checking that its coordinate representative is a $H^s_0$ function. This has an induced norm defined by

$$\|f\|_{H^s} := \sum_{i=1}^m \| (\rho_i \cdot f) \circ \phi_i^{-1} \|_{H^s}.$$

where we are just summing the norms of the coordinate representatives of our decomposition. Again, this definition depends on choice of atlas and partition of unity; however, one can show that the topology is independent of our choice.
2.1.3 Sobolev Theorems

From our definition there is an immediate result allowing us to relate the norm of one Sobolev spaces to two other spaces.

**Lemma 2.1 (Sobolev Interpolation).** For all \( f \in H^s(M) \), let \( b > s \) and \( a \geq 0 \). We have,

\[
\|f\|_{H^s} \leq C \|f\|_{H^{-a}}^{1-\theta} \|f\|_{H^b}^\theta,
\]

where \( \theta = (s+a)/(b+a) \in (0,1) \) and \( C > 0 \) depends only on the manifold (specifically the minimum size of the atlas).

A proof is included given how extensively it is used to prove Theorem 1.1.

**Proof.** We work in the Fourier domain for. Let \( f \in H^s_0(\mathbb{R}^n) \)

\[
\|f\|_{H^s} = \int (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) d\xi = \int (1 + |\xi|^2)^{-a(1-\theta) + \theta} \hat{f}(\xi)^{1-\theta + \theta} d\xi = \int (1 + |\xi|^2)^{-a(1-\theta)} \hat{f}(\xi)^{1-\theta} (1 + |\xi|^2)^{\frac{b\theta}{2}} \hat{f}(\xi)^{\theta} d\xi = \int \left( (1 + |\xi|^2)^{-\frac{s}{2}} \hat{f}(\xi) \right)^{1-\theta} \left( (1 + |\xi|^2)^{\frac{b}{2}} \hat{f}(\xi) \right)^{\theta} d\xi \leq \left( \int (1 + |\xi|^2)^{-\frac{s}{2}} \hat{f}(\xi) d\xi \right)^{1-\theta} \left( \int (1 + |\xi|^2)^{\frac{b}{2}} \hat{f}(\xi) d\xi \right)^{\theta} = \|f\|_{H^{-a}}^{1-\theta} \|f\|_{H^b}^\theta.
\]

Note: The assumed inequalities ensure that \( \frac{1}{1-\theta} > 1 \) and \( \frac{1}{\theta} > 1 \) in order to use H"older’s Inequality.

When generalizing to a result on manifolds, we will have the result holding for each set for our finite partition of unity defined earlier meaning we could take \( C \) to be the size of the atlas, which makes the result hold on the whole manifold.

We additionally note the classical result that given enough weak derivatives we can ensure the function will be classically differentiable. This is summarized in the following lemma.
Lemma 2.2 (Sobolev Embedding). If $k > \frac{n}{2}$ and $N + \alpha = k + \frac{n}{2}$ then there is a continuous embedding of $H^k(M)$ into $C^{N,\alpha}(M)$, which means we have a $C > 0$ such that for all $f \in H^k(M)$

$$\|f\|_{C^{N,\alpha}} \leq C \|f\|_{H^k}$$

Proof. A proof of the result can be found in ([Zim92], Theorem 5.2.4).

2.1.4 Fiber and Vector Bundles

We define a fiber bundle as follows. Let $(E, M, \pi, F)$ be called a smooth fiber bundle with total space $E$, base space $M$, projection $\pi$ and typical fiber $F$ if $\pi$ is a smooth submersion\(^1\) from $E$ to $M$ and for every $p \in M$ there is a $U \subset M$ containing $p$ and a diffeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ such that $\text{proj}_1 \circ \varphi = \pi$ (where $\text{proj}_1(x, y) = x$). Said another way, $E$ locally looks like $U \times F$.

We similarly define a vector bundle $(E, M, \pi)$ to be a fiber bundle with the added property that $F$ is vector space isomorphic to $\mathbb{R}^n$ (so it is redundant to include) and that the diffeomorphism $\varphi$ must now act as a vector space isomorphism on each fiber. We will be using the following vector bundles:

1. The tangent bundle $(T^*M, M, \pi)$
2. The cotangent bundle $(T^*M, M, \pi)$
3. The space of symmetric $m$-tensors $(S^mT^*M, M, \pi)$

From these we define the following fiber bundles:

1. The unit tangent bundle for a Riemannian metric $g$ $(S^g M, M, \pi)$ defined as the restriction of $TM$ to elements where $g(v, v) = 1$

2. The space of positive-definite symmetric 2-tensors on $M$ $(S^2 T^* M, M, \pi)$. This is just the bundle $(S^2 T^* M, M, \pi)$ restricted to only positive-definite operators. Here we say an bilinear operator $A : E \times E \rightarrow \mathbb{R}$ is positive-definite, if $A(v, v) > 0$ for all $v \in E \setminus \{0\}$.

\(^1\)Submersion means that the differential of the map is surjective
2.1.5 Sections

Given a vector bundle \((E, M, \pi)\) we define a section of \(E\) to be a mapping \(f : M \to E\) such that \(\pi \circ f = \text{Id}\). We will use the notation \(\Gamma^k(E)\) to be \(C^k\) sections of \(E\). Similarly, we take \(\Gamma(E)\) to be smooth sections of \(E\). We will also use the notation \(\Gamma_{H^s}(E)\) to denote sections of \(E\) that have Sobolev regularity\(^2\) of order \(s\). From this we see the space of \(C^k\) Riemannian metrics is \(\Gamma^k(S^2_+ T^* M)\).

2.2 Length and Geodesics

Given a Riemannian manifold \((M, g_0)\) we recall, for a curve \(\gamma \in C^1([a, b]; M)\), the length function is given by

\[
\ell(\gamma) := \int_a^b g_0(\gamma(t), \dot{\gamma}(t))^{\frac{1}{2}} dt.
\]

We also note that the length function is independent of the parameterization. We define the set of \(C^1\) paths from \(p\) to \(q\) as \(P_{p,q} := \{\gamma \in C^1([0, 1], M) : \gamma(0) = p, \gamma(1) = q\}\). Using this, we define the distance metric on \(M\) given by:

\[
d(p, q) := \inf_{\gamma \in P_{p,q}} \ell(\gamma).
\]

Given the length function is positive for every non-trivial curve and our metric is defined as an infimum over possible curve, it is direct to verify that \(d\) is a metric on \(M\).

2.2.1 Geodesic Equation

From the distance metric we could define a notion of a geodesic by a curve that realizes the infimum; however, this definition is too restrictive and instead define geodesics in terms of the geodesic equation. Recall that, for a Riemannian metric \((M, g_0)\), there is a unique affine connection \(\nabla\) that is torsion free and metric preserving called the Levi–Civita connection. From this one can say that

\(^2\)Here the regularity of a section is defined by, for any coordinates, the coefficients of the section are functions of that regularity. (e.g. \(H^s(M)\) for Sobolev regularity of order \(s\))
a geodesic is defined as any curve satisfying the so-called geodesic equation:

\[ \nabla_{\dot{\gamma}} \dot{\gamma} = 0. \tag{2.1} \]

As it will be important later, for coordinates \((x_i)_{1,...,n}\), we have the geodesic equation is:

\[ \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \]

using the Einstein summation convention and where \(\Gamma^k_{ij}\) are the Christoffel symbols for the connection. The full definition of an affine connection, as well as the derivation for the Levi–Civita connection can be found in ([Jos02], Chapter 3).

### 2.2.2 Fundamental Group

One definition of the fundamental group of a manifold is:

\[ \pi_1(M) := \left( \bigcup_{p \in M} P_{p,p} \right) / \sim, \]

where \(\gamma_1 \sim \gamma_2\) if there is a continuous mapping \(F : [0, 1] \times [0, 1] \to M\) such that \(F(0, t) = \gamma_1(t)\) for all \(t \in [0, 1]\), \(F(1, t) = \gamma_2(t)\) for all \(t \in [0, 1]\), and \(F(s, 0) = F(s, 1)\) for all \(s \in [0, 1]\). This means that two closed loops are equivalent if they are freely-homotopic to each other. This is useful due to the following result.

**Proposition 2.1.** Let \((M, g_0)\) be a compact Riemannian manifold. Every homotopy class of closed curves in \(M\) contains a curve which is shortest (minimizes \(\ell\) over class) and geodesic (satisfies the geodesic equation).

*Proof.* A proof of this result can be found in ([Jos02], Theorem 1.5.1) \qed

### 2.2.3 Marked Length Spectrum

For an arbitrary Riemannian metric \(g\), as discussed in the introduction, the length spectrum of \((M, g)\) is not enough information to uniquely determine the metric so we now explicitly define the marked
length spectrum as:

\[ L_g : \pi_1(M) \to \mathbb{R}_{\geq 0} \]
\[ L_g(c) := \ell(\gamma_c) \quad (2.2) \]

where \( \gamma_c \) is the closed geodesic given by Proposition 2.1 for the homotopy class \( c \in \pi_1(M) \). Given that Proposition 2.1 ensures this geodesic is length minimizing, if two geodesics in the class both arose from the proposition, they would have the same length making the mapping well defined.

For \( g \) in a neighborhood \( U_0 \subset \Gamma^3(S^2_+T^*M) \) of \( g_0 \) we define the normalized marked length spectrum by

\[ \mathcal{L}_{g_0} : U_0 \to \ell^\infty(\pi_1(M)) \]
\[ \mathcal{L}_{g_0}(g) := \frac{L_g}{L_{g_0}}, \]

where the division is taken pointwise over elements of \( \pi_1(M) \). Here \( \ell^\infty(\pi_1(M)) \) indicates the Banach space of bounded functions from \( \pi_1(M) \) to \( \mathbb{R} \) (where the lowercase is used, as the underlying set is countable). By construction, we see that, if \( g \) is close enough to \( g_0 \), then \( \mathcal{L}_{g_0}(g) \) will be bounded over all of \( \pi_1(M) \) justifying the codomain. The idea is that \( \mathcal{L}_{g_0}(g) = 1 \) exactly when \( L_{g_0} = L_g \) so we have an single-function characterization of the key assumption in Theorem 1.1.

### 2.3 Geodesic Flow

The geodesic equation is a 2nd order differential equation with coefficients that are at least \( C^1 \) given our metric is at least \( C^2 \). Thus by the existence and uniqueness theorem for ODEs (Picard–Lindelöf Theorem), the geodesic equation has a unique solution \( \gamma : (-\epsilon, \epsilon) \to M \) for an initial condition \( (\gamma(0) = p, \dot{\gamma}(0) = v) \) for \( (p, v) \in S_{g_0}M \). Furthermore, this solution will be at unit speed by nature of the parallel transport rule for geodesics (i.e. \( g_0(\dot{\gamma}(t), \dot{\gamma}(t)) = 1 \) for \( t \in (-\epsilon, \epsilon) \)). This means that one has the function

\[ \varphi_t(p, v) = (\gamma(t), \dot{\gamma}(t)) \in S_{g_0}M. \]
2.3. GEODESIC FLOW

We call $\phi_t$ the geodesic flow on $S_{g_0}M$. Taking the derivative of this at $t = 0$, we obtain its generating vector field $X_{g_0}$. Call $X_{g_0}$ the geodesic vector field. Given local coordinates $(x_i)_{1,...,n}$ with the induced coordinates for fibers of $TM$ given by $(v_i)_{1,...,n}$, the coordinate representation of the geodesic vector field is given by

$$X_{g}(p, v) = \frac{d}{dt}\phi_t(p, v) = \frac{d\dot{\gamma}(t)}{dt} + \dot{\gamma}(t)\frac{\partial}{\partial x^i} - \Gamma^k_{ij}v^i v^j \frac{\partial}{\partial v^k},$$

following the summation convention. By compactness of our manifold, one can extend the solutions to the geodesic equation to be defined on $t \in \mathbb{R}$ making $\phi_t$ a global flow in our case.

2.3.1 Anosov Geodesic Flow

Let $g \in \Gamma(S^2 T^*M)$ with geodesic flow $\phi_t$ and geodesic vector field $X_g$. A flow is called Anosov if the following hold:

1. $T_z(S_{g}M) = \text{span}(X_g(z)) \oplus E_s(z) \oplus E_u(z)$ where the splitting is continuous with respect to $z \in S_{g}M$.

2. There exists $C, r > 0$ such that:

$$|T_z \phi_t(\xi)|_{\phi_t(z)} \leq Ce^{-rt}|\xi|_z \quad \forall t \geq 0, \xi \in E_s(z)$$

$$|T_z \phi_t(\xi)|_{\phi_t(z)} \leq Ce^{-rt}|\xi|_z \quad \forall t \leq 0, \xi \in E_u(z)$$

Here the distances are taken with respect to the Sasaki metric on $TM$ induced by $g_0$. A complete derivation of the Sasaki metric can be found in ([Pat99], Definition 1.17). The intuition for Anosov flow is that, $\text{span}(X_g(z))$ corresponds to part of $T_z(S_{g}M)$ in the direction of flow, $E_s(z)$ corresponds to the directions where flow is stable (i.e. converging for positive time), and $E_u(z)$ corresponds to the directions where the flow is unstable (i.e. converging for negative time).
Throughout the document we will take $g_0$ to be a smooth Riemannian metric that admits Anosov geodesic flow. This concept was first defined in [Ano67] and it was shown that the Anosov criterion is an open condition in $\Gamma(S^2_{+}T^*M)$, meaning that there is an $\varepsilon > 0$ such that if $\|g - g_0\|_{C^k} < \varepsilon$ then $g$ will also have Anosov geodesic flow. The intuition here is similar to that of a positive definite operator as the parameters $C$ and $r$ are both strictly positive and determined by $g$, so one can change $g$ a small amount and suspect that both $r$ and $C$ could remain positive so the metric $g$ would still be Anosov. Furthermore, using Anosov’s result, the geodesic defined by Proposition 2.1 is unique. We lastly note that Anosov geodesic flows are both ergodic and mixing.

### 2.3.2 Curvature

Recall that for a Riemannian manifold $(M, g_0)$ the Riemannian curvature tensor is defined as

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - [u, v]w,$$

where $[u, v]$ is the Lie bracket. From this we define the sectional curvature for $\sigma_p$ a 2-dimensional subspace of $T_pM$ as

$$K(\sigma_p) = K(u, v) := \frac{g_0(R(u, v)v, u)}{g_0(u, u)g_0(v, v) - g_0(u, v)^2},$$

where $u$ and $v$ are independent vectors in $\sigma_p$. One can verify that this is well defined as the output is equal for any choice of independent vectors in $\sigma_p$. The reason we introduce sectional curvature is that it was shown in [Ano67] that a compact manifold with constant negative cross sectional curvature has Anosov geodesic flow. This motivates why curvature plays a role in understanding Anosov flows.

### 2.4 X-ray Transform

A fairly important operator used throughout the paper is the X-ray transform, but in order to define it, we must first define the canonical bounded linear operator mapping symmetric m-linear forms to
2.4. X-RAY TRANSFORM

functionals on $S_{g_0}M$. For $k \in \mathbb{N}$, we call this operator $\pi^*_m$ and define it as

$$\pi^*_m : \Gamma^k(S^mT^*M) \to C^k(S_{g_0}M),$$

$$\pi^*_m f (p, v) := f(p) (v \otimes \cdots \otimes v), \quad (p, v) \in S_{g_0}M.$$ 

Using this operator, for $c \in \pi_1(M)$ we define the X-ray transform by the following. For $f \in \Gamma^0(S^mT^*M)$

$$I^g_m f(c) := \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} \pi^*_m f(\varphi_t(z)) \, dt$$

Where $z = (p, v) \in S_{g_0}M$ is any point and derivative pair on the geodesic for the class $c$ as discussed in Section 2.2.2 (the choice of $z$ does not change the result as we are only looking at closed geodesics of length $L_{g_0}(c)$ meaning we integrate back to the starting point). Note that we can also think of $\pi^*_m$ as a mapping from $\Gamma^s(S^mT^*M)$ to $H^s(S_{g_0}M)$ for $s \in \mathbb{R}$, which allows us to also define $\pi^*_{m,s}$ as the dual of $\pi^*_m$ (acting on the space of distributions).

One important thing to suggest that the X-ray transform is a key operator when working with the marked length spectrum is the following proposition.

**Proposition 2.2.** For $f \in \Gamma^3(S^2T^*M)$, the derivative of $g \mapsto \mathcal{L}_{g_0}(g)$ at $g_0$ is given by,

$$(D\mathcal{L}_{g_0})(f) = \frac{1}{2} I^g_2 f.$$ 

**Proof.** We will actually prove the differentiability of $\mathcal{L}_{g_0}$ later in Proposition 3.1 but below is a proof of the value of the derivative. Take $f \in \Gamma^3(S^2T^*M)$ and $s \in \mathbb{R}$ small enough such that $(g_0 + sf) \in \Gamma^3(S^2T^*M)$. We let $c \in \pi_1(M)$ be arbitrary with $\gamma_0$ and $\gamma_s$ as the corresponding geodesic for the metrics $g_0$ and $g_0 + sf$ respectively. We recall the following 3 facts:

1. Geodesics minimize the length of all curves in the class $c \in \pi_1(M)$. This means if we fix the metric and vary the curve $\gamma_s$ in a family $\gamma_{s+a}$ smoothly parameterized by $a$, we must be locally
2.4. X-RAY TRANSFORM

minimal at $a = 0$. This yields the following:

$$\ell_{g_0 + sf} (\gamma_{s+a}) = \ell_{g_0 + sf} (\gamma_s) + 0a + O(a^2)$$

(rearranging at $a = -s$)

$$\implies \ell_{g_0 + sf} (\gamma_s) = \ell_{g_0 + sf} (\gamma_0) + O(s^2)$$

2. The Taylor Series for the square root function is

$$\sqrt{a + bs} = \sqrt{a} + \frac{b}{2\sqrt{a}} s + O(s^2).$$

3. The length of a curve on a manifold is independent of the parameterization, so we can assume that $\gamma_0$ is parameterized on $t \in [0, L_{g_0(c)}]$ (so that $g_{0\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)) = 1$).

Thus we expand out $L_{g_0}$ for $c$ to get

$$L_{g_0} (g_0 + sf)(c) = \frac{L_{g_0 + sf}(c)}{L_{g_0}(c)} = \ell_{g_0 + sf} (\gamma_s)$$

(by fact 1) = \frac{1}{L_{g_0}(c)} \int_{0}^{L_{g_0}(c)} \left( (g_0 + sf)_{\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)) \right)^{\frac{1}{2}} dt + O(s^2)

(by fact 2) = \frac{1}{L_{g_0}(c)} \int_{0}^{L_{g_0}(c)} \left( (g_0)_{\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)) \right)^{\frac{1}{2}} dt + \frac{sf_{\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t))}{2 (g_{0\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)))^{\frac{1}{2}}} dt + O(s^2)

(by fact 3) = \frac{L_{g_0}(c)}{L_{g_0}(c)} + \frac{1}{2L_{g_0}(c)} \int_{0}^{L_{g_0}(c)} f_{\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt + O(s^2)

Taking the derivative at $s = 0$ we have:

$$DL_{g_0} (f)(c) = \frac{1}{2L_{g_0}(c)} \int_{0}^{L_{g_0}(c)} f_{\gamma_0(t)} (\dot{\gamma}_0(t), \dot{\gamma}_0(t)) dt = \frac{1}{2} I_{g_0}^2 f(c)$$

Note that $f$ was taken to be $C^3$ for the differentiability result in Proposition 3.1, which we prove later.
2.5 Pseudodifferential Operators

We will use pseudodifferential operators frequently throughout the paper, so a brief description of them is included; however, some of the details and motivation is omitted. A more detailed explanation can be found in [GS00]. We will only define these for open subsets of $\mathbb{R}^n$; however, the process can be generalized to compact manifolds. For an open set $U \subset \mathbb{R}^n$, $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, we define the space of symbols $S^{m}_{\rho,\delta}(U \times \mathbb{R}^n)$ to be the set of $a \in C^\infty(U \times \mathbb{R}^n)$ such that for all compact $K \subset U$ and multi-index $\alpha, \beta$, there exists a constant $C$ such that

$$\left| \partial_\alpha^\beta a(x, \theta) \right| \leq C(1 + |\theta|)^{m-\rho|\alpha|+\delta|\beta|}$$

for $(x, \theta) \in U \times \mathbb{R}^n$. Here, $m$ is called the order of the symbol and $(\rho, \delta)$ is the type of the symbol. From this we define the space of pseudodifferential operators as the set of Fourier integral operators $A : \mathcal{D}(U) \to \mathcal{D}'(U)$ defined by the following form.

$$Au(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a(x, y, \theta)u(y) dy d\theta, \quad u \in \mathcal{D}(U),$$

where $a \in S^m_{\rho,\delta}(U \times U \times \mathbb{R}^n)$ and similarly $A$ is said to be of order $m$ and type $(\rho, \delta)$.

One simple motivation as to why these are useful is that, if you take a standard differential operator $A_m = \sum_{|\alpha| \leq m} a_\alpha(x)\partial_x^\alpha$ where $a_\alpha \in C^\infty(U)$, we have that directly via the Fourier transform,

$$A_m u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} \left( \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha \right) u(y) dy d\xi$$

and $\sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha \in S^m_{1,0}(U \times \mathbb{R}^n)$. From here one can see that, differential operators have polynomial symbols and pseudodifferential operators are generalizations of differential operators to larger classes of symbols.

2.5.1 Properties of Pseudodifferential Operators

We note only a few of the useful properties of pseudodifferential operators; however, most of the detail including the algebra of composition can be found in [GS00]. We have that $S^{m_1}_{\rho,\delta} \subset S^{m_2}_{\rho,\delta}$ for $m_1 \leq m_2$. From this we can also define $\mathcal{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(U \times \mathbb{R}^n)$ which lends itself to
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Pseudodifferential operators of order $-\infty$.

Our construction of pseudodifferential operators generalizes to compact manifolds. This results in the codomain of pseudodifferential being $\mathcal{D}(M)$ making compositions well defined. By compactness we have the potentially most important relation to Sobolev spaces that, for a pseudodifferential operator $A$ of order $m > 0$, we have that $A : H^{k+m}(M) \to H^k(M)$. This lends itself to the earlier intuition that the order of the operator is like the order of a differential operator. Similarly, pseudodifferential of order $m < 0$ behave like a generalization of integral operators. As stated earlier, for more details on these results, see [GS00].

2.5.2 Elliptic Operator

A special subclass of pseudodifferential operators is elliptic operators. While we will never prove an operator is elliptic in any results, but we do make frequent use of their properties, so the definition is included. We say an operator $A$ is elliptic at a point $(x_0, \xi_0) \in U \times (\mathbb{R} \setminus \{0\})^n$ if there is a conical neighborhood $V$ of $(x_0, \xi_0)$ and a constant $C$, such that the symbol $a(x, \xi)$ for the operator satisfies

$$|a(x, \xi)| \geq \frac{1}{C}(1 + |\xi|)^m, \quad (x, \xi) \in V, |\xi| \geq C.$$ 

Here $V$ is a conical neighborhood of $(x_0, \xi_0)$ if $V$ contains an open neighborhood of $(x_0, \xi_0)$ and $(x, \xi) \in V$ implies $(x, \lambda \xi) \in V$ for all $\lambda > 0$. Intuitively, elliptic operators can be thought of as the symbol being non-zero near the point $(x_0, \xi_0)$ (so long as $\xi \neq 0$). From this we say that $A$ is elliptic if it elliptic at all points in $U \times (\mathbb{R} \setminus \{0\})^n$. Elliptic operators are important mostly due to the following result.

**Proposition 2.3.** Let $A$ be an elliptic operator of order $m$ of type $(\rho, \delta)$ with $\rho > \delta$. Then there exists an operator $Q$ of order $-m$ of type $(\rho, \delta)$ and operators $R$ and $R'$ of order $-\infty$ such that:

$$Q \circ A = \text{Id} + R$$

$$A \circ Q = \text{Id} + R'$$

**Proof.** A proof can be found in ([GS00], Theorem 4.1).
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The idea of this proposition is that elliptic operators have inverses modulo regularizing operators of order \(-\infty\).

2.5.3 Symmetric Derivative and Divergence

Let \(\nabla\) be the Levi–Civita connection for \(g_0\) we define the *symmetric derivative* as follows:

\[
D_{g_0} : \Gamma(S^m T^*M) \to \Gamma(S^{m+1} T^* M),
\]

\[
D_{g_0} f := \text{sym}(\nabla f).
\]

That is to say we take the symmetric part (projection onto \(\Gamma(S^{m+1} T^* M)\)) of the standard covariant derivative for tensors. From this we define the *divergence operator* as follows:

\[
D^*_{g_0} : \Gamma(S^{m+1} T^* M) \to \Gamma(S^m T^* M),
\]

\[
D^*_{g_0} f := -\text{Tr}(\nabla f),
\]

where

\[
\text{Tr} : \Gamma(S^{m+2} T^* M) \to \Gamma(S^m T^* M),
\]

\[
\text{Tr}(f)(v_1, \ldots, v_m) := \sum_{i=1}^n f(e_i, e_i, v_1, \ldots, v_m)
\]

for \(f \in \Gamma(S^{m+2} T^* M)\) and \(v_1, \ldots, v_m \in \Gamma(TM)\). Note that the choice of indices to sum over does not matter by the assumed symmetry. \(D^*_{g_0}\) is formal adjoint of \(D_{g_0}\). When looking at \(\mathbb{R}^n\), \(D^*_{g_0}\) directly reduces to the standard divergence operator for tensors (up to the sign). We say that a tensor \(f\) is *divergence-free* if \(D^*_{g_0} f = 0\). By nature of the covariant derivative, we have that \(D_{g_0}\) and \(D^*_{g_0}\) are both pseudodifferential operators of order 1. Additionally, we have that \(D^*_{g_0} D_{g_0}\) is elliptic (and is equal to the Laplacian on 1-forms). We have the following decomposition.

**Proposition 2.4 (Solenoidal Decomposition).** For \(f \in \Gamma^k(S^m T^* M)\), there exists \(f^s \in \Gamma^k(S^m T^* M)\)
and $u \in \Gamma^k(S^{m-1}T^*M)$ such that:

$$f = f^s + D_{g_0}u$$

where $D_{g_0}^* f^s = 0$.

Note: one can intuitively think of this as $\Gamma^k(S^mT^*M) \approx \ker D_{g_0}^* \oplus \text{Im} D_{g_0}$ so long as you are fixing the minimum regularity $k$ and order of the tensors $m$.

Proof. A proof of this can be found in ([CS98], Theorem 2.2).

**Lemma 2.3.** $\pi_{m+1}^* D_{g_0} f = X_{g_0} \pi_m^* f$ for $f \in \Gamma^k(S^mT^*M)$

Proof. This can be shown using microlocal analysis by verifying that the symbols of both sides are equal which was originally done within the argument of Theorem 3.3.2 in [Sha94]; however below is a explicit calculation using the coordinate definitions of $\nabla f$ and $X_{g_0}$ as a differential operator on $S_{g_0}M$.

Let $(p, v) \in S_{g_0}M$. Fix a coordinate chart $(x^i)$ and let $V$ be a vector field that is equal to $v$ at $p$ and constant with respect to the chart (i.e. $V = \sum v^i \frac{\partial}{\partial x^i}$).

$$(\pi_{m+1}^* D_{g_0} f)(V) = \nabla_V f(V, \ldots, V)$$

(i is symmetric) $\implies v^i \frac{\partial}{\partial x^i} (f(V, \ldots, V)) - m f(\nabla_V V, \ldots, V)$

$$= v^i \frac{\partial}{\partial x^i} (f(V, \ldots, V)) - m f \left( \left( v^i v^j \Gamma^k_{ij} + v^j \frac{\partial V^k}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \ldots, V \right)$$

$$\left( \frac{\partial V^k}{\partial x^j} = 0 \right) \implies v^i \frac{\partial}{\partial x^i} (f(V, \ldots, V)) - m v^i v^j \Gamma^k_{ij} f \left( \frac{\partial}{\partial x^k}, \ldots, V \right)$$
Conversely we can use the coordinate definition of the Geodesic Vector Field $X_{g_0}$ on $S_{g_0} M$:

$$(X_{g_0} \pi_m^* f)(p, v) = X_{g_0}((p, v) \mapsto f_p(v, \ldots, v)) = (v^i \frac{\partial}{\partial x^i} - \Gamma^{k}_{ij} v^i v^j \frac{\partial}{\partial v^k})((p, v) \mapsto f_p(v, \ldots, v))$$

$$= v^i \frac{\partial}{\partial x^i} (f(v, \ldots, v)) - m v^i v^j \Gamma^{k}_{ij} f_p(\frac{\partial}{\partial x^k}, \ldots, v)$$

As these two agree we have the desired equality.

**Proposition 2.5.** $I_{m+1}^{g_0} (D_{g_0} f) = 0$ for $f \in \Gamma^k (S^m T^* M)$.

**Proof.** We recall Lemma 2.3 stating $\pi_m^* D_{g_0} = X_{g_0} \pi_m^* f$. Let $f \in \Gamma^k (S^m T^* M)$, then

$$I_{m+1}^{g_0} (D_{g_0} f)(c) = \frac{1}{L_g(c)} \int_0^{L_g(c)} \pi_{m+1}^* (D_{g_0} f)(\varphi_t(z))dt$$

(Lemma 2.3) $\implies$ $= \frac{1}{L_g(c)} \int_0^{L_g(c)} (X_{g_0} \pi_m^* f)(\varphi_t(z))dt$

$$= 0.$$

This is all to say that, when we use solenoidal decomposition, we can never recover elements of the form $D_{g_0} u$, so we can only ask if a function is injective up to the elements of $\ker D_{g_0}^*$ (i.e. the solenoidal part). Thus, solenoidal-injectivity is defined to be injectivity when restricted to elements within $\ker D_{g_0}^*$.

**Proposition 2.6.** $I_{g_0}^2$ is solenoidal-injective for Anosov manifolds in dimension 2 and for dimension greater than 2 when they also have non-positive curvature.

**Proof.** The proof of the dimension 2 case can be found in ([Gui17], Theorem 3.12). Whereas the general non-positive curvature case can be found in ([CS98], Theorem 2.3).

Solenoidal injectivity was the best that could be achieved by Proposition 2.5, and Proposition
2.6 verifies that this is achieved for Anosov manifolds (so long as we have non-positive curvature in higher dimensions).
Chapter 3

Important Results

We now will go through some of the key ideas that are required to complete the proof of Theorem 1.1.

3.1 Normalized Length Spectrum and X-ray Transform

To prove Theorem 1.1, we will need various inequalities relating to the norm of \((g - g_0)\) under various topologies. One such relation, it turns out, comes from showing that \(L_{g_0}\) is \(C^2\) on a small enough neighborhood of \(g_0\). This is shown in the following proposition (which also proves the differentiability required for Proposition 2.2).

**Proposition 3.1.** There is a neighborhood \(U \subset \Gamma^3(S^2_+T^*M)\) of \(g_0\) such that \(g \mapsto L_{g_0}(g)\) is \(C^2\) on \(U\). This gives the following bound:

\[
\|L_{g_0}(g) - 1 - DL_{g_0}(g - g_0)\|_{\ell^\infty(\pi_1(M))} \leq C \|g - g_0\|^2_{C^3(M)} \quad \forall g \in U
\]

where \(C > 0\) only depends on \(g_0\).

**Proof.** Following the proof provided in [GL19], we will attempt to realize the mapping \(L_{g_0}\) as the composition of other mappings that are \(C^2\), thus ensuring that \(L_{g_0}\) is \(C^2\). One reason this is achievable is to note is that the length function can be constructed in terms of the length of closed orbits of the flow of the geodesic vector field \(X_{g_0}\). This means that we can look at the function \(g \mapsto X_g\) and then the function that maps vector fields on \(S_{g_0}M\) to the length of their orbit for a specific element \(c \in \pi_1(M)\).
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Given that we have assumed that \((M, g_0)\) is compact with Anosov geodesic flow, we have that the unit tangent bundle \(S_{g_0}M\) is also compact. As in [GL19], this means we can use to use the rigidity result from Appendix A in [dlLMM86]. This states that there is a neighborhood of vector fields \(\mathcal{X}_0 \subset \Gamma^2(TS_{g_0}M)\) around \(X_{g_0}\), and a \(C^2\) map \(\theta : \mathcal{X}_0 \to C^0(M)\) such that we have, for any fixed \(c \in \pi_1(M)\), the orbit of the vector field \(X \in \mathcal{X}_0\) equivalent to \(c\) has a length function given by the integral of \(\theta(X)\) along the equivalent orbit for \(X_{g_0}\). Call this mapping \(L_c : \mathcal{X}_0 \to \mathbb{R} \geq 0\). Stated explicitly, for closed orbits \(\gamma_{X_{g_0}}\) and \(\gamma_X\) in the class \(c \in \pi_1(M)\),

\[
L_c(X) = \ell(\gamma_X) = \int_{\gamma_{X_{g_0}}} \theta(X)
\]

and \(L_c\) is \(C^2\) due to \(\theta\) being \(C^2\). From this we can define a normalized version \(L_c(X) := L_c(X)/\ell(\gamma_{X_{g_0}})\).

Recalling that for a fixed \(c \in \pi_1(M)\), \(\ell(\gamma_{X_{g_0}})\) is constant. Thus we have

\[
\left\| \frac{\partial L_c}{\partial X} \right\| = \left\| \frac{1}{\ell(\gamma_{X_{g_0}})} \frac{\partial L_c}{\partial X} \right\|
= \left\| \frac{1}{\ell(\gamma_{X_{g_0}})} \int_{\gamma_{X_{g_0}}} \frac{\partial \theta(X)}{\partial X} \right\|
\leq \frac{1}{\ell(\gamma_{X_{g_0}})} \left\| \frac{\ell(\gamma_{X_{g_0}})}{\ell(\gamma_{X_{g_0}})} \sum_{X \in \gamma_{X_{g_0}}} \frac{\partial \theta(X)}{\partial X}(x) \right\|
\leq \sup_{X \in \mathcal{X}_0} \left\| \frac{\partial \theta(X)}{\partial X} \right\|
\leq C
\]

for some \(C > 0\) dependent only on our neighborhood \(\mathcal{X}_0\) and importantly not depending on our choice of \(c\). We can repeat this bound with the second derivative as \(X \mapsto \theta_X\) is \(C^2\). Thus, we have a uniform bound on our derivatives in the region \(\mathcal{X}_0\) for each \(c \in \pi_1(M)\).

Now one might naively think that we can finish the proof by simply taking the map \(g \mapsto X_g\) where \(X_g\) is the geodesic vector field for \(g\); however there are two problems here: the output vector field has to be a vector field on \(S_{g_0}M\), where the geodesic vector field is on \(S_gM\) (which changes with the input making the regularity of the mapping ambiguous to even define), and the output vector field has to be \(C^2\) (for the sake of composing it with \(L_c\)), which in general is not true for geodesic vector
fields. To resolve the first problem, we note that there is a \( C^\infty \) mapping \( \nu : S_gM \to S_{g_0}M \) defined by \( \nu(p, v) = \frac{p}{\|v\|_{g_0}} \) which is just simply scaling back the fiber on \( S_gM \). This is well defined as, if \((p, v) \in S_gM\), then \( v \neq 0 \) so \( \|v\|_{g_0} \neq 0 \). Thus, when we take the push-forward of this mapping, it will map the vector field on \( S_gM \) to a vector field on \( S_{g_0}M \) but importantly the map will preserve orbits but at different parameterizations (which does not affect length). We now tackle the problem of verifying the mapping of \( g \mapsto \nu_*X_g \) is \( C^2 \) and that the output is a \( C^2 \) vector field. We note that the geodesic vector field formula only contains \( g \) within the Christoffel symbols, which at most uses the first derivatives of \( g \). Thus if we take \( g \in \Gamma^3(S_2^2 + T^*_M) \), we will have \( X_g \in \Gamma^2(TS_gM) \) and thus \( \nu_*X_g \in \Gamma^2(TS_{g_0}M) \). Thus we define the mapping \( \mathcal{X}(g) = \nu_*X_g \). The relationship is summarized in the following commutative diagram

\[
\begin{array}{ccc}
\Gamma^2(TS_gM) & \xrightarrow{\nu_*} & \Gamma^2(TS_{g_0}M) \\
X_g & & X_g \circ \mathcal{L}_{g_0}(c) \\
\Gamma^3(S_2^2 + T^*_M) & \xrightarrow{\mathcal{L}_{g_0}(c)} & \mathbb{R}_{\geq 0}
\end{array}
\]

Putting this all together we have, for a fixed \( c, \mathcal{L}_{g_0}(c) = \mathcal{L}_c \circ \mathcal{X} \) which is the composition of two \( C^2 \) operators whose derivatives are uniformly bounded over \( c \in \pi_1(M) \). We then take the mapping as a whole on all of \( \pi_1(M) \). From this the inequality holds as a result of \( g \mapsto \mathcal{L}_{g_0}(g) \) being \( C^2 \).

**Corollary 3.1.** There is a neighborhood \( U \subset \Gamma^3(S_2^2 + T^*_M) \) of \( g_0 \) such that, for all \( g \in U \), if \( L_g = L_{g_0} \), then the following bound holds:

\[
\|I^0_2(g - g_0)\|_{L^\infty(\pi_1(M))} \leq 2C \|g - g_0\|_{C^1(M)}^2, \quad \forall g \in U,
\]

where \( C > 0 \) only depends on \( g_0 \).

**Proof.** \( L_g = L_{g_0} \) directly implies \( \mathcal{L}_{g_0}(g) = 1 \), so simply taking the results of Propositions 3.1 and 2.2 we immediately have the result after multiplying both sides by 2. \( \square \)

One other important relationship that we can exploit is the following lemma showing that having equal marked length spectrum yields a positive X-ray transform.

\[\text{1 the diagram only commutes if we take a small enough neighborhood } U \text{ of } g_0 \text{ such that } X \text{ lands within } X_0\]
Lemma 3.1. If $L_g = L_{g_0}$ then for all $c \in \pi_1(M)$:

$$I_2^{g_0}(g - g_0)(c) \geq 0$$

Proof. For a fixed $c \in \pi_1(M)$ let $\gamma_0$ be the geodesic for $c$ with respect to $g_0$. By Cauchy–Schwarz we have:

$$\ell_g(\gamma_0)^2 = \left( \int_{\gamma_0} \sqrt{\pi_2^* g} \cdot 1 \right)^2 \leq \left( \int_{\gamma_0} \pi_2^* g \right) \left( \int_{\gamma_0} 1 \right) = \left( \int_{\gamma_0} \pi_2^* g \right) L_{g_0}(c)$$

Thus:

$$\frac{1}{L_{g_0}(c)} \int_{\gamma_0} \pi_2^* (g - g_0) = \frac{1}{L_{g_0}(c)} \int_{\gamma_0} \pi_2^* g - 1$$

$$\geq \frac{\ell_g(\gamma_0)^2}{L_{g_0}(c)} - 1$$

$$\geq \frac{L_g(c)^2}{L_{g_0}(c)^2} - 1 = 0$$

This is especially important, as we will show that there are many theorems that rely on positive integral operators of some kind in order to obtain very strong results.

3.2 Reducing Problem to Divergence-Free Slice

The next most important relationship is the fact that we can reduce the problem from not just looking at metrics who are near $g_0$ to instead looking only at metrics who are divergence-free with respect to $g_0$. The reason we can do this is summarized in the following proposition.

Proposition 3.2. There exist $C > 0$ and $\varepsilon > 0$ such that if $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$, there exists a $\tilde{g}$ where the following hold:

1. There exists a unique diffeomorphism $\phi$ close to the identity in the $C^{N+1,\alpha}$ topology such that $\tilde{g} = \phi^* g$ (i.e. $(M, g)$ is isometric to $(M, \tilde{g})$);

2. The metric $\tilde{g}$ is divergence-free with respect to $g_0$ ($D^*_g g = 0$);
3. The mapping $g \mapsto \phi$ is smooth and $\|\tilde{g} - g_0\|_{C^{N,\alpha}} < C \|g - g_0\|_{C^{N,\alpha}}$.

Proof. We again follow the proof provided in [GL19] with additional detail. We aim to use the Implicit Function Theorem for Banach spaces to solve the problem; thus we must construct a differentiable function that, when equal to zero, solves our problem. We define the following function taking vector fields to diffeomorphisms:

$$e : \Gamma^{N+1,\alpha}(TM) \to \text{Diff}^{N+1,\alpha}(M),$$

$$e_V(x) = \exp_{x}^{g_0}(V(x)), \quad V \in \Gamma^{N+1,\alpha}(TM), \quad x \in M$$

where $\text{Diff}^{N+1,\alpha}(M)$ is the group of diffeomorphisms homotopic to the identity. One can think of $e_V$ as the diffeomorphism on $M$ obtained by flowing by $V$ for one unit of time. We take $U_0$ to be a open subset of $\Gamma^{N+1,\alpha}(TM)$ such that for all $V \in U_0$, $V$ is close enough to 0 such that, by the nature of the exponential mapping, $e_V$ is a diffeomorphism close to Id in the $C^{N+1,\alpha}$ topology. We define our function $F$ by

$$F : U_0 \times \Gamma^{N,\alpha}(S^2T^*M) \to \Gamma^{N-1,\alpha}(S^2T^*M)$$

$$F(V, f) := D_{g_0}^* (e_V^*(g_0 + f))$$

where $f = g - g_0$. Framing the problem in this way, we see that, if for every $f \in U_0$ there is a unique $V_f$ such that $F(V_f, f) = 0$, then we have $D_{g_0}^* (e_{V_f}^*(g_0 + f)) = 0$; thus, $\phi = e_{V_f}$ and $\tilde{g} = e_{V_f}^* g$ would prove fact 1 and 2 of the proposition. As mentioned earlier, we attempt to use the Implicit Function Theorem; however, one can verify that that $F$ is differentiable but not $C^\infty$. As such, a solution $V(f)$ arising from the Implicit Function Theorem will only be $C^1$. Without smooth dependence, we would not be able to conclude fact 3 of the proposition given $g \mapsto \phi^* g = e_V^* (g - g_0) g$ would only be $C^0$, where we need it to be $C^{N,\alpha}$. As such, we will have to construct a different (but equivalent) $F$ that is smooth. This is done by noting

$$e_{-V}^* D_{g_0}^* \circ e_V^* = D_{e_{-V}^* g_0}^*$$
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and so we can construct the operator $\tilde{F}$ defined as follows:

$$
\tilde{F} : U_0 \times \Gamma^{N,\alpha}(S^2T^*M) \to \Gamma^{N-1,\alpha}(S^2T^*M)
$$

$$
\tilde{F}(V, f) = D^*_{e^V g_0}(g_0 + f).
$$

From this we note that $\tilde{F}(V, f) = e_{-V^*} \circ F(V, f) = 0$ is equivalent to $F(V, f) = 0$ (as $e_{-V^*}$ is invertible). So we instead apply the Implicit Function Theorem for $\tilde{F}$. The advantage to doing this is that $\tilde{F}$ is now a smooth function (as the $V$ arguments are only within the divergence operator and not incorporated into a function applied to $f$). This means solutions to the Implicit Function Theorem on $\tilde{F}$ would be smooth.

To invoke the Implicit Function Theorem, we must simply prove that $\frac{\partial}{\partial V} \tilde{F}(V, f) \mid_{(0, 0)}$ is an isomorphism. Recall that $e_0 = \text{Id}$ and $\frac{\partial}{\partial V} e_V \mid_0 = \text{Id}$ based on the definition of the exponential map. Additionally recall that $F(0, 0) = D^*_{g_0} g_0 = 0$. Using these, we proceed as follows. We have

$$
\frac{\partial}{\partial V} \tilde{F}(V, f) \mid_{(0, 0)} = \frac{\partial}{\partial V} (e_{-V^*} \circ F(V, f)) \mid_{(0, 0)}
$$

$$
= \left( \frac{\partial}{\partial V} e_{-V^*} \right) \mid_{F(0, 0)} \circ \frac{\partial}{\partial V} F(V, f) \mid_{(0, 0)}
$$

$$
= \text{Id} \circ \frac{\partial}{\partial V} D^*_{g_0} (e_V g_0 + f) \mid_{(0, 0)}
$$

$$
= D^*_{g_0} \frac{\partial}{\partial V} (e_V g_0) \mid_{0}
$$

Taking a coordinate system $(x^1, \ldots, x^n)$, we have the coefficients on the 2-form defined by:

$$
(e_V^* g_0)_{ij} = \frac{\partial e^p_V}{\partial x^i} \frac{\partial e^q_V}{\partial x^j} g_{pq} \circ e_V
$$

Thus, there are three dependencies on the $V$. We proceed by taking the derivative with respect to $V$ (and then set $V = 0$). Let $Y \in U_0$ and recall the isomorphism $\flat : TM \to T^*M$ defined by
3.2. REducing Problem to Divergence-Free Slice

\( Y^g_j = g_{0ij}Y^i. \)

\[
\frac{\partial}{\partial V} (e^V g_0 Y)_{ij} \bigg|_0 = \frac{\partial \text{Id} Y}{\partial x^i} \frac{\partial e^q_0}{\partial x^j} g_{0pq} \circ e_0 + \frac{\partial e^q_0}{\partial x^j} \frac{\partial \text{Id} Y}{\partial x^i} g_{0pq} \circ e_0 + Y^p \frac{\partial e^p_0}{\partial x^j} \frac{\partial g_{0pq}}{\partial x^r} \circ e_0
\]

\[
= g_{0ip} \frac{\partial Y^p}{\partial x^j} + g_{0jp} \frac{\partial Y^p}{\partial x^i} + \frac{\partial g_{0ij}}{\partial x^p} Y^p
\]

\[
= g_{0ip} (\nabla_j Y^p - \Gamma^p_{jq} Y^q) + g_{0jp}(\nabla_i Y^p - \Gamma^p_{iq} Y^q) + \frac{\partial g_{0ij}}{\partial x^p} Y^p
\]

\[
= g_{0ip} \nabla_j Y^p + g_{0jp} \nabla_i Y^p + \sum_{p} g_{0ij} Y^p
\]

\[
= \nabla_j Y^i + \nabla_i Y^j
\]

(by symmetry) = \((2D_{g_0} Y^g)_{ij}\)

Putting this back together, we have \( \frac{\partial}{\partial V} \hat{F}(V, f) \bigg|_{(0,0)} = 2D^*_g \circ D_{g_0} \circ \phi. \) We now argue this is an isomorphism which maps \( U_0 \) to \( \Gamma^{N-1,\alpha}(S^2 T^*M). \) As \( \phi \) is already an isomorphism, we look at the elliptic operator \( 2D^*_g \circ D_{g_0} \) from \( U_0 \subseteq \Gamma^{N+1,\alpha}(T^*M) \) to \( \Gamma^{N-1,\alpha}(S^2 T^*M). \) One can quickly note that, given we have solenoidal decomposition from Proposition 2.4, \( D^*_g \cap \text{Im} D_{g_0} = 0. \) This means \( \ker(2D^*_g \circ D_{g_0}) = \ker D_{g_0}. \) Given we are looking at elements of \( \Gamma(T^*M), \) showing \( \ker(D_{g_0}) = \{0\} \) is a known result proved in ([CS98] Lemma 2.1 with odd \( m). \) To apply this result, we use the fact that our geodesic flow is Anosov, so has dense orbits in \( S_{g_0} M \) (which we discuss later in Section 3.3.2).

By the Implicit Function Theorem (a version of the theorem can be found in [AG07]), we have the smooth function \( f \mapsto V_f \) that satisfies \( \hat{F}(V_f, f) = 0 \) proving fact 1 and 2 of the lemma.

To conclude fact 3 we define the mapping \( \phi_g = e^g_{V_g-g_0} \) and thus by continuity of the solution, we can take \( g \) close enough to \( g_0 \) such that, as an operator, \( \|\phi^* - \text{Id}\|_{C^{N,\alpha}} \leq C \|g-g_0\|_{C^{N,\alpha}} \) for \( C' > 0 \) depending only on \( g_0. \) This directly yields:

\[
\|\phi^*_g g - g_0\|_{C^{N,\alpha}} = \|\phi^*_g g - g - g_0 + g_0 - g_0\|_{C^{N,\alpha}}
\]

\[
(\phi_{g_0} = \text{Id}) \implies \|\phi^* - \text{Id}\|_{C^{N,\alpha}} + \|g-g_0\|_{C^{N,\alpha}} \leq (C' + 1) \|g-g_0\|_{C^{N,\alpha}},
\]

proving fact 3 for any \( C' > 1 \) depending only on \( g_0. \) \( \square \)
As stated, the takeaway of this proposition is that, for any $g$ satisfying the conditions for Theorem 1.1, we can instead solve the problem for $\tilde{g}$ defined by Proposition 3.2 without loss of generality (reducing $\varepsilon$ if necessary).

3.3 Additional Results From Other Related Papers

3.3.1 $\Pi$ and $\Pi_m$ Operators

A core piece of the proof relies on the use of the following $\Pi$ operator, which was first explored in [Gui17]. $\Pi$ is most conveniently defined via the resolvent operators

$$R^\pm : L^2(S_{g_0}M) \to L^2(S_{g_0}M)$$

$$R^\pm(\lambda) := (-X \pm \lambda)^{-1} \Re(\lambda) > 0$$

or alternatively,

$$R^\pm(\lambda) := \pm \int_0^\infty e^{-\lambda t} e^{\pm tX} dt \Re(\lambda) > 0$$

where $X$ is a vector field generating an Anosov flow (the geodesic vector field $X_{g_0}$ in our case).

Taking the Laurent series one finds,

$$R^\pm(\lambda) = \pm \langle \cdot, 1 \rangle_\lambda \pm R^\pm_0 + O(\lambda), \quad (3.1)$$

Note that in ([Gui17], Lemma 2.5), Guillarmou showed that flows being mixing (and all Anosov geodesic flows are mixing, as proved in [Ano67]) is an equivalent condition to stating the only imaginary pole of $R^\pm(\lambda)$ is at $\lambda = 0$ with residue $\pm \langle \cdot, 1 \rangle$, which is where the Laurent series formula arose. We then define

$$\Pi := R^+_0 + R^-_0. \quad (3.2)$$

In particular we will note some facts about this operator $\Pi$ taken from [Gui17].

**Proposition 3.3.** When restricting the domain of $\Pi$ to various Sobolev spaces, we have the following:
### 3.3. ADDITIONAL RESULTS FROM OTHER RELATED PAPERS

1. \( \Pi \) is bounded as mapping from \( H^s(S_{g_0}, M) \) to \( H^r(S_{g_0}, M) \) (\( s > 0, r < 0 \)).

2. \( \Pi \) is self-adjoint as mapping from \( H^s(S_{g_0}, M) \) to \( H^{-s}(S_{g_0}, M) \) (\( s > 0 \)).

3. For all \( f \in H^s(S_{g_0}, M) \), \( X_{g_0} \Pi f = 0 \) (\( s > 0 \)).

4. For all \( f \in H^s(S_{g_0}, M) \) with \( X_{g_0} f \in H^s(S_{g_0}, M) \), \( \Pi X_{g_0} f = 0 \) (\( s > 0 \)).

5. For all \( f \in H^s(S_{g_0}, M) \) with \( \langle f, 1 \rangle = 0 \), \( f \in \ker \Pi \) iff \( X_{g_0} u = f \) for some unique (modulo constants) \( u \in H^s(S_{g_0}, M) \).

6. \( \Pi(1) = 0 \)

**Proof.** All of these properties were proved between Theorem 1.1 and Theorem 2.4 within [Gui17]; however, we will quickly explain some of the simple results that were used in the proof. The boundedness is proved by showing \( R^\pm(\lambda) \) is a Fredholm operator of order 0, which makes \( \Pi \) similarly bounded as a consequence.

The self-adjoint nature comes from the symmetry of the resolvents. Namely, one can see \( R^+(\bar{\lambda})^* = -R^-(\lambda) \), so we have the constant operators from the Laurent series can be shown to similarly have \( R_0^+ = R_0^- \) which immediately implies \( \Pi \) is self-adjoint when looking at the correct spaces.

The other properties have more involved logic and are proved in ([Gui17], Theorem 2.2).

An important addition to the core properties of the operator is that it can be conjugated with our \( \pi_m^* \) and \( \pi_{-m} \) to create a new operator. The main value of this is summarized in the following proposition again taken from [Gui17].

**Proposition 3.4.** The operator \( \Pi_m := \pi_m^* \Pi \pi_m^* \) is a pseudodifferential operator of order -1 that is elliptic when restricted to the space of solenoidal tensors. Thus when restricted to \( \ker D_{g_0}^* \) we have there exists pseudodifferential operators \( Q \) of order 1 and \( R \) of order \(-\infty\) such that:

\[
Q \Pi_m = \text{Id} + R
\]

**Proof.** A proof can be found in ([Gui17], Theorem 3.5).
We also have the following fact.

**Proposition 3.5.** If \( I_m \) is solenoidal injective then \( \Pi \pi^*_m \) is injective on the set

\[
\{ f \in \Gamma_{H^{-s}}(S^m T^* M) \cap \ker D^*_g : \langle \pi^*_m f, 1 \rangle_{L^2} = 0 \}.
\]

**Proof.** A proof of this can be found at ([Gui17], Lemma 3.6). One can see intuitively that

\[
\langle \pi^*_m f, 1 \rangle_{L^2} = 0
\]

is very similar to the idea that the X-ray transform is 0 with the difference being that one is integrating on \( S_{g_0} M \), where the other is just integrating along all closed geodesics. If there were some kind of density argument between closed geodesics and the whole of \( S_{g_0} M \), then one could make the connection of this result. We will see later in Proposition 3.6 that such a result does exist for Anosov manifolds.

The \( \Pi \) operator is introduced for the explicit goal of being able to use the following lemma in order to create a second inequality of norms.

**Lemma 3.2.** If \( I_m \) is solenoidal injective for all \( s > 0 \), then there is \( C_{g_0, s} > 0 \) such that, for all \( f \in \Gamma_{H^{-s}}(S^m T^* M) \cap \ker D^*_g \),

\[
\|f\|_{H^{-1-s}} \leq C_{g_0, s} \left( \|\Pi \pi^*_m f\|_{H^{-s}} + |\langle \pi^*_m f, 1 \rangle_{L^2}| \right).
\]

**Proof.** We work to a contradiction, assuming that, for all \( C > 0 \),

\[
\|f\|_{H^{-1-s}} > C \left( \|\Pi \pi^*_m f\|_{H^{-s}} + |\langle \pi^*_m f, 1 \rangle_{L^2}| \right).
\]

Take a sequence \((f_n) \in \Gamma_{H^{-1-s}}(S^m T^* M) \cap \ker D^*_g\) such that \( \|f_n\|_{H^{-1-s}} = 1 \). Thus for each \( n \), we
have the inequality for $C = n$ and thus

\[
\|f_n\|_{H^{-1-s}} > n (\|\Pi \pi_m^* f_n\|_{H^{-1-s}} + |\langle \pi_m^* f_n, 1 \rangle|_{L^2})
\]

\[\implies \|\Pi \pi_m^* f_n\|_{H^{-1-s}} + |\langle \pi_m^* f_n, 1 \rangle|_{L^2} < \frac{1}{n}\]

\[\implies \lim_{n \to \infty} \Pi \pi_m^* f_n = 0, \quad \lim_{n \to \infty} |\langle \pi_m^* f_n, 1 \rangle|_{L^2} = 0.\]

We see from Proposition 3.5, we have that $\Pi \pi_m^*$ must be solenoidal-injective and we have, by Proposition 3.4, for $f \in \Gamma_{H^{-1-s}}(S^m T^* M) \cap \ker D^*_g$,

\[
\|f_n\|_{H^{-1-s}} \leq C (\|\Pi \pi_m^* f_n\|_{H^{-1-s}} + \|Rf_n\|_{H^{-1-s}})
\]

\[Q, \pi_m^* \text{ are bounded} \leq C' (\|\Pi \pi_m^* f_n\|_{H^{-1-s}} + \|Rf_n\|_{H^{-1-s}}) \tag{3.3}\]

for some $C' > 0$. But we know $R$ is a pseudodifferential operator of order $-\infty$, so we have that, as $(f_n)$ is bounded, $Rf_n$ is a bounded sequence. This means there is a subsequence $Rf_{n_k}$ that converges. In particular this means $Rf_{n_k}$ and $\Pi \pi_m^* f_{n_k}$ are Cauchy, so Equation (3.3) directly gives that $f_{n_k}$ is Cauchy. Thus we take an additional subsequence $f_{n_{k_l}}$ that converges to some $f$ (that must have $\|f\|_{H^{-1-s}} = 1$). By Proposition 3.3 (fact 1), we know that $\Pi$ is bounded, so we have $\Pi \pi_m^* f_{n_{k_l}}$ must converge to $\Pi \pi_m^* f$. But we already showed that it converges to 0. Given, $\Pi \pi_m^*$ is solenoidal-injective, $f = 0$ and we have a contradiction (to $\|f\|_{H^{-1-s}} = 1$).

\[\square\]

3.3.2 Parry’s Formula

We introduce a very important formula that relates the integrals over each closed geodesic to the integral over the entire compact manifold $S_{g_0} M$.

**Proposition 3.6** (Parry’s Formula). For all $F \in C^0(S_{g_0} M)$

\[
\lim_{T \to \infty} \frac{1}{N(T)} \sum_{c \in \pi_1(M), L_{g_0}(c) \leq T} \frac{e^{f_{c \cdot c}}}{L_{g_0}(c)} \int_{c \cdot c} F = \frac{1}{Vol(S_{g_0} M)} \int_{S_{g_0} M} F d\mu,
\]

where $N(T)$ is a normalization function counting $c \in \pi_1(M)$ with $L_{g_0}(c) \leq T$, $\mu$ being the Liouville measure and $J^u$ being the unstable Jacobian of the flow defined by $J^u(z) := -\partial_t \left( \text{det} d\varphi_t(z) |_{E^u(z)} \right) \bigg|_{t=0}$.
3.3. ADDITIONAL RESULTS FROM OTHER RELATED PAPERS

Proof. See [Par88] for the proof which relies on the fact that Anosov flows have topological mixing in order to generate a Gibbs-type argument.

The reason Parry’s Formula was introduced was so that we could combine it with Proposition 3.1 to create a bound that was related to all of $S_{g_0}M$ and not simply geodesics classes in $\pi_1(M)$. Our use of this is summarized in the following lemma.

**Lemma 3.3.** If $I_{g_0}^2(g - g_0)(c) \geq 0$ for all $c \in \pi_1(M)$, then there is a constant $C > 0$ depending only on $g_0$ such that,

$$0 \leq \int_{S_{g_0}M} \pi_2^*(g - g_0) \leq C \left( \|\mathcal{L}_{g_0}(g) - 1\|_{\ell^\infty(\pi_1(M))} + \|g - g_0\|^2 \right).$$

Proof. Given the assumption $0 \leq I_{g_0}^2(g - g_0)$, we apply Parry’s Formula with $F = \pi_2^*(g - g_0)$ and see that starting with a fixed $T > 0$ (and recalling that the sum is normalized via $N(T)$),

$$0 \leq \frac{1}{N(T)} \sum_{c \in \pi_1(M), \mathcal{L}_{g_0}(c) \leq T} e^{\int_{c} J_n} I_{g_0}^2(g - g_0)(c) \leq \|I_{g_0}^2(g - g_0)\|_{\ell^\infty(\pi_1(M))}$$

taking $T \to \infty$ and multiplying by $\text{Vol}(S_{g_0}M)$

$$0 \leq \int_{S_{g_0}M} \pi_2^*(g - g_0) \, d\mu \leq C \|I_{g_0}^2(g - g_0)\|_{\ell^\infty(\pi_1(M))} \quad (3.4)$$

Rewriting the result of Proposition 3.1, we have

$$\left\| \frac{1}{2} I_{g_0}^2(g - g_0) - (\mathcal{L}_{g_0}(g) - 1) \right\|_{\ell^\infty(\pi_1(M))} \leq C' \|g - g_0\|^2 \|\pi_1(M)\|$$

$$\implies \|I_{g_0}^2(g - g_0)\|_{\ell^\infty(\pi_1(M))} \leq 2 \|\mathcal{L}_{g_0}(g) - 1\|_{\ell^\infty(\pi_1(M))} + 2C' \|g - g_0\|^2 \|\pi_1(M)\|$$

Applying this back to Equation (3.4), we have the desired result after adjusting the constant. 

3.3.3 Positive Livsic Theorem

The Livisic Theorem is a result that relates functions whose integral is zero over each closed loop geodesic and the functions on $S_{g_0}M$ that could have this property. This was originally proved in
1986 and can be used to explicitly characterize the kernel of the X-ray transform; however, for our purposes, we need a modified version of the Livsic theorem that allows for positive integrals over the closed loop geodesics. This version of the theorem is a case of ([LT05], Theorem 1), where it is written below in the notation relevant to our proof.

**Proposition 3.7** (Positive Livsic Theorem). For $\alpha \in (0, 1]$, there exists a constant $C > 0$ (dependent only on $g_0$) and $\beta \in (0, 1)$ such that for all $u \in C^\alpha(S_{g_0} M)$ satisfying

$$\forall c \in \pi_1(m), \int_{\gamma_c} u \geq 0,$$

there exists $h, F \in C^{\alpha \beta}(S_{g_0} M)$, such that

1. $F \geq 0$ (equality if equality in assumption),
2. $u + X_{g_0} h = F$, and
3. $\|F\|_{C^{\alpha \beta}} \leq C \|u\|_{C^\alpha}$.

One can think of $\beta$ as reducing the regularity, but allowing a powerful decomposition.
Chapter 4

Main Result

4.1 Proof of Main Theorem

We will begin by using the definitions and results from Chapters 2 and 3 to work through a detailed version of the proof originally provided in [GL19]. Recall the main theorem.

**Theorem 1.1.** Let \((M, g_0)\) be a smooth compact Riemannian manifold of dimension \(n\) with Anosov geodesic flow and let \(N > \frac{3n}{2} + 8\). Additionally assume non-positive curvature for \(n \geq 3\). There exists \(\varepsilon > 0\) such that for all metrics \(g \in \Gamma^N(S^2_+ T^* M)\), if \(L_g = L_{g_0}\) and \(\|g - g_0\|_{CN} < \varepsilon\), then \((M, g)\) is isometric to \((M, g_0)\).

**Proof of Theorem 1.1.** We proceed by assuming that \((M, g_0)\) is a smooth compact Riemannian manifold with Anosov geodesic flow. Let \(\varepsilon > 0\) small enough to satisfy the various lemmas that we will use. We take \(g\) to be another metric (in \(\Gamma^{N, \alpha}(S^2_+ T^* M)\)) admitting Anosov geodesic flow with \(\|g - g_0\|_{CN, \alpha} < \varepsilon\) where \(N\) and \(\alpha\) are fixed (to be determined later). Furthermore, assume \(L_g = L_{g_0}\). By Lemma 3.2, we can assume \(g\) is divergence free with respect to \(g_0\) (reducing \(\varepsilon\) if necessary) as there will always be a divergence free \(\tilde{g}\) isometric to \(g\) that we can work with instead.

Take \(f = g - g_0\). Our goal is to show that \(f = 0\), which would show that \(g = g_0\) for divergence-free metrics (meaning that we have isometry in general as discussed above via Lemma 3.2). As \(L_g = L_{g_0}\), we can apply Lemma 3.1 getting \(I_2^{g_0} f \geq 0\), which directly allows us to apply Lemma 3.7 to the
functional $\pi_2^* f$. This means there exist $\beta \in (0, 1)$ and $h \in C^{\alpha \beta}(S_{g_0} M)$ such that,

$$
\pi_2^* f + X_{g_0} h \geq 0,
$$

$$
\|\pi_2^* f + X_{g_0} h\|_{C^{\alpha \beta}} \leq C_{g_0} \|\pi_2^* f\|_{C^\alpha}.
$$

Furthermore, we can see that clearly $\pi_2^*$ is a bounded linear operator, so we have

$$
\|\pi_2^* f + X_{g_0} h\|_{C^{\alpha \beta}} \leq C_{g_0} \|\pi_2^* f\|_{C^\alpha} = C \|f\|_{C^\alpha}
$$

for some $C > 0$.

Take $s$ satisfying $0 < s < \alpha \beta$, to be chosen later. We now assume that we have non-positive curvature (in dimension greater than 2) so that, by Proposition 2.6, we have $I_2$ is solenoidal injective. Thus we begin with Lemma 3.2. We have (for our chosen $s$)

$$
\|f\|_{H^{s-1}} \leq C_{g_0} \|\Pi \pi_m^* f\|_{H^{-s}} + |\langle \pi_m^* f, 1 \rangle_{L^2}|.
$$

This is the baseline for the proof as we will now proceed by manipulating the right hand side of this inequality into something that is just a constant times some non-trivial power of $\|f\|$ so that, when we take $\varepsilon$ to be small, we arrive at a contradiction. While each step is difficult to individually motivate, one should keep in mind that the goal is to reduce any terms into homogenous products of norms (preferably of $f$).

We note that, given Proposition 3.3 (fact 4), $\Pi X_{g_0} h = 0$, and thus we can add $\Pi X_{g_0} h$ with no change to the equation. Similarly, we have $\int_{S_{g_0} M} X_{g_0} h \, d\mu = \langle X_{g_0} h, 1 \rangle_{L^2} = 0$, thus we have

$$
\|\Pi \pi_m^* f\|_{H^{-s}} + |\langle \pi_m^* f, 1 \rangle_{L^2}| = \|\Pi (\pi_m^* f + X_{g_0} h)\|_{H^{-s}} + |\langle \pi_m^* f + X_{g_0} h, 1 \rangle_{L^2}|.
$$

By Proposition 3.3 fact 1, we have that $\Pi$ is bounded as a mapping from $H^s(S_{g_0} M)$ to $H^{-s}(S_{g_0} M)$. Furthermore, the mapping $A(w) = \langle w, 1 \rangle_{L^2}$ is clearly a bounded linear operator when restricted to
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Thus we have

\[ \|\Pi(\pi^*_m f + X_{g_0} h)\|_{H^{-s}} + \|\pi^*_m f + X_{g_0} h, 1\|_{L^2} \leq \|\Pi\|\|\pi^*_m f + X_{g_0} h\|_{H^s} + \|A\|\|\pi^*_m f + X_{g_0} h\|_{H^s} \]

\[ \leq (\|\Pi\| + \|A\|)\|\pi^*_2 f + X_{g_0} h\|_{H^s} \]

\[ = C\|\pi^*_2 f + X_{g_0} h\|_{H^s} \]

for some \( C > 0 \).

From here, one might think to simply use our Livsic result and bound this by \( C\|f\|_{C^n} \); however, this will result in both sides of the inequality (when subbed back into (4.2)) having norms to the power of one making it difficult to arrive at any contradictions. We will instead make use of the the S Wolfe interpolation process detailed in Lemma 2.1. We interpolate \( H^s \) with the spaces \( H^0 = L^2 \) and \( H^b \) for \( b \) satisfying \( 0 < s < b < \alpha \beta \). Thus, our interpolation exponent is \( \frac{s}{b} \) and we have

\[ \|\pi^*_2 f + X_{g_0} h\|_{H^s} \leq \|\pi^*_2 f + X_{g_0} h\|_{L^2}^{1 - \frac{s}{b}} \|\pi^*_2 f + X_{g_0} h\|_{H^b}^{\frac{s}{b}} . \]

Combining this back with Equation (4.2), we have

\[ \|f\|_{H^{-s-1}} \leq C\|\pi^*_2 f + X_{g_0} h\|_{L^2}^{1 - \frac{s}{b}} \|\pi^*_2 f + X_{g_0} h\|_{H^b}^{\frac{s}{b}} . \]

The advantage here is that, while the exponents still add to one, the \( L^2 \) norm in the inequality will prove to be useful. From here we want to manipulate the two norms on the right hand side to something only containing \( f \) and not \( \pi^*_2 f + X_{g_0} h \). The \( H^b \) norm can be done quickly by recalling that we assumed \( b < \alpha \beta \), which allows us to lower bound the Livsic result in Equation (4.1):

\[ \|\pi^*_2 f + X_{g_0} h\|_{H^b} \leq \|\pi^*_2 f + X_{g_0} h\|_{C^n, \beta} \leq C\|f\|_{C^n} . \]

This yields

\[ \|f\|_{H^{-s-1}} \leq C\|\pi^*_2 f + X_{g_0} h\|_{L^2}^{1 - \frac{s}{b}} \|f\|_{C^n} \]  

(4.3)

for some \( C > 0 \). We now have the goal of converting \( \|\pi^*_2 f + X_{g_0} h\|_{L^2} \) into something of just \( f \)
4.1. PROOF OF MAIN THEOREM

(and hopefully in a way that changes its exponent non-trivially). To do this, we recall that we have $L_g = L_{g_0}$, so Lemma 3.1 gives $I_{g_0}^g f \geq 0$. We thus can apply Lemma 3.3, which gives the following relationship ($\mathcal{L}_{g_0}(g) = 1$ in this case),

$$0 \leq \int_{S_{g_0}M} \pi_2^* f \, d\mu \leq C \|f\|_{C_3}^2$$

But, as discussed earlier, $\int_{S_{g_0}M} X_{g_0} h \, d\mu = 0$, so we can add this in with no change. Furthermore, we have $\pi_2^* f + X_{g_0} h \geq 0$ from our Livsic result. This means we have

$$\|\pi_2^* f + X_{g_0} h\|_{L_1} = \int \pi_2^* f + X_{g_0} h \, d\mu$$

$$= \int \pi_2^* f \, d\mu$$

$$\leq C \|f\|_{C_3}^2.$$ (4.4)

This relationship is key to the proof as our upper bound is to the power of two while the lower bound is only to the power of one. Pairing this with Equation (4.1), we can proceed by rewriting the $L^2$-norm in terms of $C^{\alpha\beta}$ and $L^1$. This is done as follows:

$$\|\pi_2^* f + X_{g_0} h\|_{L^2} = \left\| (\pi_2^* f + X_{g_0} h)^2 \right\|_{L^1}^{\frac{1}{2}}$$

$$\leq \|\pi_2^* f + X_{g_0} h\|_{L^1}^{\frac{1}{2}} \|\pi_2^* f + X_{g_0} h\|_{L^\infty}^{\frac{1}{2}}$$

($L^\infty$ is bounded by $C^{\alpha\beta}$)

$$\leq \|\pi_2^* f + X_{g_0} h\|_{L^1}^{\frac{1}{2}} \|\pi_2^* f + X_{g_0} h\|_{C^{\alpha\beta}}^{\frac{1}{2}}$$

Equations (4.1) and (4.4)

$$\leq C \|f\|_{C_3} \|f\|_{C_3}^{\frac{1}{2}}$$

Combining this with Equation (4.3) (and simplifying the exponents), we have

$$\|f\|_{H^{-\alpha-1}} \leq C \|f\|_{C_3}^{1-\frac{\alpha}{2}} \|f\|_{C_3}^{\frac{1}{2}(1+\frac{\alpha}{2})},$$

which is finally of the form that both sides are only in terms of norms of $f$. Dividing by the right
hand side gives

\[ 1 \leq C \left( \left\| f \right\|_{H^{-s-1}}^{r-1} \left\| f \right\|_{C^3}^{1+rac{1}{s}} \left\| f \right\|_{H^N}^{\frac{1}{\theta_3}} \right). \tag{4.5} \]

From here, we have the goal of converting the right hand side to norms that are related to our assumed \( \left\| f \right\|_{C^{N,\alpha}} \leq \varepsilon \) and importantly, they must be all positive powers to proceed by contradiction. This means that we must also find a way to cancel out the negative exponent on the \( \left\| f \right\|_{H^{-1-s}} \). Given that \(-1-s < 0\), we want to interpolate to \( H^{-1-s} \) and \( H^{N_0} \) for some \( N_0 \) that will end up being a lower bound on our \( N \). To do this we will have to interpolate away from our \( C^3 \) and \( C^\alpha \) norms.

Interpolation can only be done on Sobolev spaces not on Hölder spaces so we use Sobolev Embedding (Lemma 2.2) to first bound by Sobolev norms.

Take \( k_\alpha = \frac{n}{2} + \alpha + s \) and \( k_3 = \frac{n}{2} + 3 + s \). Thus by Sobolev Embedding (Lemma 2.2):

\[
\begin{align*}
\left\| f \right\|_{C^\alpha} &\leq \left\| f \right\|_{H^{k_\alpha}} \\
\left\| f \right\|_{C^3} &\leq \left\| f \right\|_{H^{k_3}}
\end{align*}
\]

Take \( N_0 > k_3 > k_\alpha > 0 \). We can now interpolate both \( \left\| f \right\|_{H^{k_\alpha}} \) and \( \left\| f \right\|_{H^{k_3}} \) in the same manner with the same reference spaces: \( H^{-1-s} \) and \( H^{N_0} \).

\( H^\alpha \): Take our interpolation exponent to be \( \theta_\alpha = \frac{k_\alpha + s + 1}{N_0 + s + 1} \), so we have

\[
\left\| f \right\|_{H^{k_\alpha}} \leq \left\| f \right\|_{H^{-1-s}}^{1-\theta_\alpha} \left\| f \right\|_{H^{N_0}}^{\theta_\alpha}.
\]

\( H^3 \): Take our interpolation exponent to be \( \theta_3 = \frac{k_3 + s + 1}{N_0 + s + 1} \), so we have

\[
\left\| f \right\|_{H^{k_3}} \leq \left\| f \right\|_{H^{-1-s}}^{1-\theta_3} \left\| f \right\|_{H^{N_0}}^{\theta_3}.
\]

Putting this all back into Equation (4.5), we have

\[
1 \leq C \left\| f \right\|_{H^{-s-1}}^{r-1} \left\| f \right\|_{H^{N_0}}^{r'}. 
\]
where
\[ r = \frac{1}{2} (1 - \theta_\alpha) \left( 1 + \frac{s}{b} \right) + (1 - \theta_3) \left( 1 - \frac{s}{b} \right), \]
\[ r' = \frac{1}{2} \theta_\alpha \left( 1 + \frac{s}{b} \right) + \theta_3 \left( 1 - \frac{s}{b} \right). \]

From here we proceed by contradiction. Taking \( f \neq 0 \) and \( N + \alpha > N_0 \) we have
\[
1 \leq C \| f \|_{H^{-1},-1} \| f \|_{H^{N_0}}^{-1}
\leq C \| f \|_{C^{N_0,0}}^{-1+r'}
\leq C \varepsilon^{-r-1+r'},
\]
but if \( r-1+r' > 0 \) we can take \( \varepsilon \) small enough (\( g \) close enough to \( g_0 \)) so that we have a contradiction. Thus \( f = 0 \).

In order to do this we skipped over the step where we ensured that \( r-1+r' > 0 \). To find when this is true, we first note that \( r' \) is always positive so we can solve for when \( r > 1 \). This is mostly an effort in algebra and recalling we have some freedom over our variables within the ordering \( 0 < s < b < \alpha < 1 \). We take \( \alpha \) to be very small and we take \( s \) to be very small such that \( \frac{s}{b} \) is very small. Then we have
\[
\frac{1}{2} \left( 1 - \frac{n/2 + \alpha + s + 1 + s}{N_0 + s + 1} \right) \left( 1 + \frac{s}{b} \right) + \left( 1 - \frac{n/2 + 3 + s + 1 + s}{N_0 + s + 1} \right) \left( 1 - \frac{s}{b} \right) > 1
\]
\[
\frac{1}{2} (N_0 - n/2 - \alpha - \beta) \left( 1 + \frac{s}{b} \right) + (N_0 - n/2 - 3 - \beta) \left( 1 - \frac{s}{b} \right) > N_0 + \beta + 1
\]
\[
N_0 > \frac{3n}{2} + 8.
\]
Thus, as long as we take \( N_0 > 3n/2 + 8 \), then the result holds for \( N + \alpha > N_0 \)

4.2 Discussion on Generalization of Result

As discussed in the introduction, an alternative proof of Theorem 1.1 was provided in [GKL21], where they on the surface appeared to have generalized the result by looking not at two metrics who
had equal marked length spectrum but instead at two metrics with their marked length spectrum equal at infinity. This is defined as, for a sequence $c_j \in \pi_1(M)$ ordered by increasing lengths $L_{g_0}(c_j)$, they assumed that $\lim_{j \to \infty} L_{g_0}(g)(c_j) \to 1$. This appears to be a weaker assumption, but in their appendix, they show it is actually equivalent under the same other assumptions. While this might suggest the existence of more alternatives proofs; one will see, after looking through the details, the second proof still relies heavily on the $\Pi_m$ operator from [Gui17], as well as many other results from the same paper. This suggest that, while it is an alternative proof, it is not as substantially different as one might have hoped.

Looking from another point of view, the locality of the result is similar to that of something out of an Inverse or Implicit Function Theorem that we utilized in Proposition 3.2. The problem with this approach is in the formulation of the derivative to the space. One could look at the set

$$A := \{ g \in \Gamma^N(S^2_+ T^* M) : g \text{ admits Anosov geodesic flow}, L_{g_0} = L_g \} \quad (4.6)$$

for a fixed $g_0$. This set is effectively what we are obtaining a local result on; however, it is difficult to formulate $A$ as a manifold or Fréchet space due to it not being a vector bundle (or an obvious open subset of one). In Proposition 3.2 we saw that $S^2_+ T^* M$ is an open subset of $S^2 T^* M$ which is a vector space, so identification of the derivatives was direct. In our case, the addition of the equal marked length spectrum makes no such obvious identification. The result of Theorem 1.1 does at least ensure that when restricting $A$ to the neighborhood of size $\varepsilon$ prescribed by Theorem 1.1, it must be a connected manifold as we can continuously deform the identity to our diffeomorphism, giving a continuous path in $A$ from $g_0$ to $g$. 
Chapter 5

Summary and Conclusions

In this thesis, a deeper understanding of the local rigidity of the marked length spectrum on Anosov manifolds was achieved. A complete description of the original proof from [GL19] was detailed with a brief explanation of the difficulties with generalizing the result further.

Obviously the main open area of improvement is still to work towards a proof (or counter example) of Conjecture 1.1; however, the less ambitious goal of improving the result of Theorem 1.1 can be accomplished through the following open problems. One improvement would be to show the solenoidal-injectivity for the X-ray transform on Anosov manifolds in higher dimensions in a way that does not rely on the assumption of non-positive curvature as this would cascade directly into removing the assumption from the existing proof. Similarly, one could work to get a deeper understanding of the II operator as it is currently necessary in all existing proofs for the result. Lastly, one could work to find a way to use the local result with stronger assumptions such as constant negative curvature to potentially obtain a non-local result on the more restrictive class but clearly it is an open problem to whether this is possible.
BIBLIOGRAPHY

Bibliography


