Essays on the financial system and
the transmission of monetary policy

by

Junfeng Qiu

A thesis submitted to the Department of Economics
in conformity with the requirements for
the degree of Doctor of Philosophy

Queen’s University
Kingston, Ontario, Canada
July, 2007

Copyright ©Junfeng Qiu, 2007
Abstract

This thesis studies the role of the banking system in several aspects of the macroeconomy, including the likelihood of financial crises, volatility of asset prices and the transmission of monetary policy.

In chapter 2, I analyze the accumulation of international reserves by central banks as insurance against financial crises. In the model, private banks borrow from foreign creditors to invest in domestic projects. By lending to banks in response to liquidity shocks, the central bank can reduce the liquidation of bank assets and lower the probability of bank runs. I show that the central bank will hold more reserves when private banks hold lower reserves. I also find that if the central bank can borrow additional loans from external sources, then domestic banks will hold fewer reserves by themselves. If the borrowing cost of external loan is very high, then the central bank may actually want to accumulate more reserves in order to avoid borrowing from external sources at high costs.

In chapter 3, I show that the ability of banks to supply liquidity through money creation is important for financial stability. By supplying liquidity, banks can smooth the sale of assets and stabilize asset prices. I find that without elastic money, the attempt of non-bank mutual funds to raise cash by selling assets will only add more volatility into the market. Elastic money provided by banks can help mutual funds better smooth the consumption of their shareholders.

In chapter 4, we consider the role of elastic money in an different environment where liquidity shocks affect agents asymmetrically. We show how money growth and interest rate policy can be used to adjust the consumption level of households. We find that the optimal policy is affected by the sensitivity of the supply price to the interest rate. When the supply price is more sensitive to the interest rate, it would be better to adopt a higher inflation rate, and to make the zero-bound of nominal interest rate less likely to be binding.
Co-Authorship

Allen Head, for Chapter 4, “Elastic money, inflation and interest rate policy”.
Acknowledgements

I thank my supervisors Allen Head and Frank Milne for their patience and kindness. I thank Thorsten Koeppl, Gregor Smith, Beverly Lapham, Christopher Ferrall, Jan Zabojnik, Mei Li, Zheng Song, and Sharif Khan for their helpful comments.
Table of Contents

Abstract .................................................................................................................. i
Co-Authorship ........................................................................................................ ii
Acknowledgements ................................................................................................ iii
List of Tables ......................................................................................................... vii
List of Figures ........................................................................................................ viii

1 Introduction ........................................................................................................... 1

2 Endogenous probability of financial crises, lender of last resort,
and the accumulation of international reserves ...................................................... 5
  2.1 Introduction .................................................................................................... 5
  2.2 The environment ........................................................................................... 8
  2.3 The two-stage withdrawal game in period 2 .................................................. 13
  2.4 Bank’s choice in period 1 ............................................................................. 22
  2.5 Numerical example: basic effects of central bank reserves ............................ 25
  2.6 When the central bank is more cautious than the private banks ..................... 29
  2.7 An extension: external borrowing ................................................................ 32
  2.8 Summary and conclusion ............................................................................ 35

3 Bank money, aggregate liquidity and asset prices .................................................. 37
3.1 Introduction ........................................................................................................... 37
3.2 The environment ..................................................................................................... 40
3.3 The Equilibrium ...................................................................................................... 47
3.4 Numerical Results: \( U = \ln c \) ............................................................................... 57
3.5 The relationship between liquidity-risk sharing through coalitions and bank money creation .................................................................................................................. 61
3.6 Conclusion ............................................................................................................... 69

4 Elastic money, inflation and interest rate policy .......................................................... 70
4.1 Introduction ............................................................................................................... 70
4.2 The environment ...................................................................................................... 73
4.3 Optimal choices ....................................................................................................... 81
4.4 Equilibrium ............................................................................................................. 87
4.5 An example ............................................................................................................ 89
4.6 Interest rate policy ................................................................................................. 92
4.7 The optimal trend inflation rate ............................................................................ 100
4.8 Conclusion ............................................................................................................... 106

5 Summary and conclusion ............................................................................................ 108

Appendices .................................................................................................................... 111

A Proofs for Chapter 2 .................................................................................................... 111
A.1 Proofs ...................................................................................................................... 111

B Proofs and additional results for Chapter 3 ................................................................. 124
B.1 The detailed transaction steps .............................................................................. 124
List of Tables

2.1 The withdrawal decisions and the expected utilities ............................................. 16
2.2 Values of parameters ............................................................................................ 25
3.1 Values of parameters ............................................................................................ 57
3.2 Numerical example ............................................................................................... 58
4.1 Optimal $\gamma$ ..................................................................................................... 104
B.1 Monetary flows .................................................................................................... 125
B.2 Accumulated payment flows ................................................................................ 136
C.1 The creation, circulation and destruction of inside money .................................. 156
List of Figures

2.1 Events ................................................................. 10
2.2 The payoff differential function \( v(\theta, n) \) for a given level of \( \theta \) ...................... 18
2.3 Effects of exogenous changes in central bank reserves ....................... 27
2.4 Optimal central bank reserve ........................................... 29
2.5 The effects of central bank reserve on bank runs ............................... 31
2.6 Central bank’s optimal reserves ........................................... 31
2.7 The effects of external borrowing costs, \( \bar{b}_{ex} = 2 \) ................................. 34
2.8 The effects of external borrowing costs, \( \bar{b}_{ex} = 5 \) ................................. 34
2.9 Effects of high external borrowing cost ...................................... 35

3.1 Events ................................................................. 41
3.2 The flow of inside money .................................................. 44
3.3 The lending and settlement process ........................................ 46
3.4 \( Q_k \) in case 2 ........................................................... 58
3.5 \( Q_k, L, R, r^d \) and \( Eb \) in case 3a ....................................... 59
3.6 Compare \( Q_k \) in cases 1 and 3a ......................................... 59
3.7 Compare cases 3a and 3b .................................................. 60
3.8 The transmission of central bank interest rate policy ......................... 61
3.9 Payout policy and \( Q_k \) ..................................................... 65
3.10 Payout policy and $Q_k$ ......................................................... 66
4.1 Timing of the model ................................................................. 77
4.2 The flow of inside money. ......................................................... 80
4.3 The results when $i_c = 0$. ......................................................... 91
4.4 $Bu'(q_o)$ and $c'(q_s)$ when $i_c$ is fixed at zero. ..................... 95
4.5 The results of optimal interest rate policy when $i_s = 0.$ ........................ 96
4.6 The results of optimal interest rate policy when $i_s = i_d$ ................. 99
4.7 The effects of $\gamma$. $i_s = 0.$ ..................................................... 100
4.8 The effects of $\gamma$. $i_s = i_d$. ............................................... 102
4.9 The effects of $\gamma$ on social welfare for different $a.$ .................. 104
4.10 The result when there is no interest rate policy ($i_c = 0$) ............... 105
A.1 $v(\theta, n)$ for given levels of $\theta$. ........................................... 113
A.2 Functions $n(\theta, \theta^*)$ and $v(\theta, n(\theta, \theta^*))$ .......................... 117
B.1 Response curves ................................................................. 144
Chapter 1

Introduction

This thesis studies the role played by the banking system in several aspects of the macroeconomy. In particular, the likelihood of financial crises, the volatility of asset prices, and the transmission of monetary policy are studied in a sequence of related models.

In the first essay, I analyze the policy of the accumulation of international reserves by central banks as self-insurance against financial crises. I first use the global game method to build a model of bank runs caused by aggregate liquidity shocks. I then use the model to analyze how the central bank will optimally choose the reserve level. I show that central bank reserves can reduce the probability of financial crises. I also find that lower reserves held by private banks can cause the central bank to hold more reserves.

I first explain why the central bank may want to hold reserves. In the model, domestic banks collect deposits from foreign creditors and make investments in long-term projects. Banks face stochastic aggregate withdrawals. If the withdrawal needs are high, then banks may need to liquidate long-term projects. Liquidation is costly and less capital will be available for producing goods in the future, which reduces the wages of workers. When banks make their investments, they do not take into account the costs of financial crises to workers. As a result, their reserve holding will be lower than the socially optimal level. I assume that the central bank will take into account the social welfare costs of financial crises. As
a result, the central bank will have the incentive to hold reserves and lend them to banks during periods of high demand for liquidity.

The main findings are as follows. First, I find that higher central bank reserves can reduce the probability of bank runs. Second, the incentive of the central bank to accumulate reserves will be affected by the reserve level of private banks. If private banks are cautious and hold higher reserves, then the central bank can hold lower reserves. If the private sector has lower reserves, then the central bank will have a greater incentive to hold reserves.

I also extend the model to allow for external borrowing. If the central bank cannot commit not to lend to private banks at low costs, then the availability of external loans will cause the private sector to hold lower reserves, which will increase the probability for the central bank to borrow from external sources. If the cost for borrowing is high, then the central bank may actually want to accumulate more reserves in order to reduce the probability of borrowing.

The second essay analyzes how the ability of banks to create credit money can affect financial stability. In the model, people need to use money to make payments, and they may have to sell assets when their liquidity needs are high. By creating and lending out new deposits, which I refer to as inside money, banks can reduce the need for people to sell assets, and the asset price will be more stable.

I also show that without banks, non-bank institutions that cannot create deposits will not be able to provide adequate insurance against aggregate liquidity shocks to their shareholders. The reason is that without borrowing from banks, these institutions can acquire liquidity only by selling assets. If the aggregate money on the market is limited, then selling more assets will only lower the asset price, contributing to volatility. Elastic money of banks thus helps provide liquidity to their shareholders. By sharing the costs of borrowing, the presence of non-bank institutions also improves the effectiveness of the elastic liquidity provided by banks.

The third essay studies optimal monetary policy in an environment in which aggregate liquidity shocks affect individual agents asymmetrically and exchange may be conducted using
either bank deposits (inside money) or fiat currency (outside money).

In the model, in each period, some of the households will have to consume all their own money and leave the economy, and they cannot borrow from banks. The central bank controls the stock of outside money and chooses the trend inflation rate optimally. The central bank also pursues an interest rate policy that affects the rate at which private banks create inside money. Through its interest rate policy the central bank can affect the price level and adjust the consumption of agents, thus providing insurance against unfavorable liquidity shocks.

The long-run rate of inflation is important for determining the effectiveness of this policy. At a very low rate of inflation the zero bound on interest rates is likely to be binding, implying that policy is largely ineffective and the monetary authority cannot provide much insurance. The optimal inflation rate is therefore typically positive. As the trend inflation rate rises, however, a trade-off arises between being able to provide insurance and subjecting those agents without access to bank loans to a severe inflation tax.

At the same time, the optimal long-run rate of inflation depends on the responsiveness of prices to interest rate changes. The more responsive are prices to interest rates, the weaker is the inflation tax and the higher the optimal inflation rate. The essay thus links two principal components of monetary policy: the optimal interest rate policy (associated with short-run fluctuations) and the optimal long-run inflation rate.

The essays in this thesis employ two different methods of modelling the banking system.

In the first essay, I use the Diamond-Dybvig (1983) model of banking. In this type of model, banks act as intermediaries between depositors and borrowers. The liquidity lent by banks to borrowers is collected from depositors. In this case, banks facilitate investment rather than providing elastic aggregate liquidity. This model is useful for studying issues in which the ability of banks to supply elastic aggregate liquidity is not important.

The second essay considers the issue of liquidity and asset price. In this case, the elasticity of aggregate liquidity is crucial, and we allow banks to make loans by creating deposits.
Non-bank intermediaries now perform the function of Diamond-Dybvig banks. They collect resources from shareholders and make investments, and they also provide liquidity insurance to shareholders. I compare these two types of financial intermediaries, and I show that the monetary function of banks is actually very important because it gives banks the additional power to provide liquidity through deposit creation.

More specifically, because people can use deposits to make payments, banks can lend by creating new deposits which are not collected from depositors. This is because banks only need enough reserves to meet the settlement requirement, which is usually much lower than the gross loan level. By creating and lending out new deposits, banks can add liquidity to the economy.

In the third essay, this monetary model of banking is integrated with real consumption and production to study monetary policy. Again, bank lending is not limited by the stock of existing money or the deposits collected from depositors, because loanable funds can come from newly created bank deposits. Given the lending rate, bank money will be endogenously decided by the demand for bank loans. In this case, changes in deposits are not decided by the activities of depositors, but are largely decided by the lending activities of banks. When a loan is made, deposits will be created, and when a loan is repaid, deposits will be destroyed.

We also show that when the central bank uses the interest rate policy to affect the economy in the short run; although it does not directly control the money supply, it can still indirectly affect it by affecting the lending and deposit rate of banks. Given the interest rates, the money supply will be decided by demand for bank loans at those interest rates.

The above model provides a clearer way for understanding the aggregate changes in bank deposits in a general equilibrium framework. It also allows us to combine the explicit role of money as a medium-of-change and the interest rate policy of the central bank.
Chapter 2

Endogenous probability of financial crises, lender of last resort, and the accumulation of international reserves

“...As I argued in my K.B. Lall Lecture in 2001, following the Asian crises of the late 1990s, it was likely that countries might choose to build up large foreign exchange reserves in order to be able to act as a ‘do it yourself’ lender of last resort in US dollars. It is now clear that this is exactly what many Asian countries have done.... ” — Mervyn King, Governor of the Bank of England, New Delhi, India, February 20, 2006.

2.1 Introduction

During the past decade, there has been a rapid build-up of international reserves among developing countries. The causes and consequences of international reserve accumulation have become an important policy issue. This chapter analyzes the central bank’s policy to hold reserves as self-insurance against liquidity crises.

Traditional theories of international reserves focus on the role of reserves as a buffer stock when countries need to reduce their current account deficit. A rule-of-thumb is that reserves should be sufficient to pay for three to four months of imports. But in recent years, especially after the Asian financial crises, people have realized that the high mobility of capital is becoming increasingly important in evaluating the adequacy of international reserves. As pointed out by Fischer (2001):
We have also seen in the recent crises that countries that had big reserves by and large did better in withstanding contagion than those with smaller reserves - to an extent that is hard to account for through our usual analyses of the need for reserves.

It is therefore no surprise that the traditional current account approach has been viewed more skeptically in recent years. There has been a growing conviction that emerging market countries with open capital accounts need more reserves rather than less, and that we should look to the capital account in determining a country’s need for reserves.

This chapter analyzes how the central bank can accumulate reserves to insure against capital flight. I focus on liquidity shocks and I assume away current account problems and exchange rate risks.

The basic structure of the model is a two-generation one in which the central bank collects taxes from the first generation to build up reserves for the second generation. The second generation faces a three-period-bank-run model structure, which is modified from the Diamond-Dybvig (1983) bank-run model. In period 1, banks collect deposits from foreign depositors and divide them between liquid monetary reserves and illiquid long-term investments. In period 2, banks face stochastic aggregate withdrawal risks. Banks can borrow from the central bank to meet part of the withdrawal needs. If the withdrawal demand is high, banks need to liquidate long-term investments. Bank runs may also happen in this period. In period 3, returns from banks’ remaining investments are realized.

The central bank wants to hold reserves for the following reason: In the model, production uses both capital and labour. When investments are liquidated, less capital will be available for producing goods, which will lead to lower wage income of workers. However, when commercial banks make their optimal choice of reserves, they do not take into account the costs of liquidation to workers. Their optimal reserve level is thus lower than the socially optimal level. The central bank tries to maximize social welfare, so it has an incentive to accumulate reserves and lend them to banks to reduce the social welfare costs of asset liquidation.

The main results are as follows: First, I analyze the effects of central bank reserves on
financial crises. I analyze bank runs both with and without self-fulfilling panics. Although higher central bank reserves will cause commercial banks to choose lower reserves, overall, higher central bank reserves help to reduce the probability of bank runs.

Second, I analyze how the reserve level of commercial banks affects the central bank’s incentive to build up reserves. I find that when the private sector is cautious and holds high reserves, the central bank can hold less. If the private sector holds low reserves, the central bank will have a greater incentive to hold reserves.

Third, I extend the model to allow for external borrowing. I show that a re-evaluation of the external borrowing costs can cause countries to hold more reserves. An interesting result is that the possibility for the central bank to borrow from external sources may cause the private sector to hold fewer reserves, and the central bank may actually have to hold more reserves in order to avoid borrowing from external sources at high costs.

Aizenman and Lee (2005) also use liquidity shocks to analyze the effects of international reserves. They model international reserves as reserves held by commercial banks. The commercial banks use the reserves to meet random aggregate withdrawal needs and reduce the liquidation of assets. I instead focus on the central bank’s motive to hold reserves. The International Monetary Fund (IMF) defines international reserves as reserves held by the monetary authority. I analyze how the central bank policy will interact with the reserve level of the private sector. I also model bank runs and how their probability can be affected by the central bank’s policy.¹

I use the global game method to model bank runs with self-fulfilling panics. The global game method was originally developed by Carlsson and van Damme (1993).² Morris and Shin (1998) applied it to currency attacks. Goldstein and Pauzner (2005) and Rochet and

¹Some recent examples which model international reserves as an insurance against financial crises are Aizenman, Lee and Rhee (2004), Aizenman and Lee (2005), Garcia and Soto (2004), Jeannie and Ranciere (2005), and Kim, Li, Rajan, Sula and Willett (2005).
Vives (2005) applied it to bank runs caused by stochastic asset returns. I follow the method of Goldstein and Pauzner (2005) and adapt it for a more complicated payoff structure.

The basic idea of the global game method is that, by introducing a small noise into the private signal, players will be less certain about other players’ actions; this can reduce the complementarities in the payoff and may turn multiple equilibria into a unique equilibrium. As we know, a major problem of multiple equilibrium games is that the probability of each equilibrium is usually given exogenously by sunspots. This will prevent us from analyzing how policies would affect the probability of each equilibrium. With a unique equilibrium, the probability of bank runs is endogenously decided, and we can analyze explicitly how central bank policy would affect financial crises and social welfare. By using the global game method to model bank runs caused by aggregate liquidity shocks, this essay also contributes to the bank run literature.

This chapter is organized as follows: Section 2.2 describes the environment; Section 2.3 analyzes the withdrawal decisions in period 2; and Section 2.4 describes the choice of banks in period 1; Section 2.5 uses a numerical example to show the basic results of the model; Section 2.6 shows how a low reserve of the private sector may cause the central bank to hold more reserves; Section 2.7 extends the model to allow for borrowing from external sources; Section 2.8 concludes.

2.2 The environment

2.2.1 The basic structure

I use a two-generation model to analyze the costs and benefits of reserve accumulation. The income of the first generation is exogenously given. The central bank collects tax from this generation to build reserves for the second generation; higher reserves means lower consumption of the first generation. The central bank will use the reserves to provide temporary loans
to banks, and after the loan is repaid, the reserves will be consumed by the second generation. The central bank maximizes the sum of the expected utility of the two generations.

### 2.2.2 The three-period model for the second generation

The second generation faces a three-period bank-run model. Denote the time as \( t = 1, 2, 3 \).

There are two types of domestic agents: households and bankers; the population of each type of agent is normalized to 1. Households consume only in period 3, and the utility function is \( u^h = -\frac{1}{c^3} \). Each household also has endowment \( e_h \) in period 3, which is used to model the income that is not affected by financial crises. Each banker owns a bank and is endowed with \( e_b \) units of consumption goods in period 1. Bankers are risk-neutral and they maximize their profit.

One unit of consumption goods invested in period 1 will turn into 1 unit of capital goods in period 3, which can be combined with labour to produce new consumption goods in period 3. The production function is

\[
y = AK^\alpha L^{1-\alpha}
\]

where \( A \) is the productivity factor, \( K \) and \( L \) are capital and labour of each firm. Only domestic agents have access to the production technology, and only households can be workers.

Figure 2.1 shows the main events. We assume only bankers can make investments in period 1. Bankers can borrow deposits from foreign creditors. We assume the government sets a limit for aggregate foreign deposits and that banks then compete for the deposits. Every bank borrows from a continuum of foreign creditors of size 1. We use \( d_t \) to denote both the deposit of an individual depositor and the total deposit in a representative bank. Deposits must be paid in dollars, so the central bank cannot lend to banks by printing domestic money. Foreign creditors are risk-neutral.

\[3\] More generally, we can set \( u^h = \frac{1}{\sigma} - \frac{1}{\sigma c} \) with \( \sigma \geq 1 \). Here we set \( \sigma = 2 \), so \( u^h = -\frac{1}{c} \).
Banks collect deposits from foreign creditors, and divide the deposits between liquid reserves and long-term investments.

Two-Stage withdrawal game. Stage 1: Each agent observes the aggregate liquidity shock with a private noise. Each agent decides whether to withdraw.

<table>
<thead>
<tr>
<th>t=1</th>
<th>t=2</th>
<th>t=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage 2: The aggregate shock is known. Every depositor also knows whether he is a mover. Movers must withdraw, non-movers also decide whether to withdraw.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Production occurs in the remaining projects; Central bank loan is repaid; Remaining depositors are paid; Agents consume.

Figure 2.1: Events

Each bank then divides the deposits between monetary reserves in dollars and long-term investments. There is no official reserve requirement and banks can freely choose the reserve level. Let $γ$ denote the reserve ratio chosen by the bank, then the reserve is $γd_t$, and the investment is $e_b + (1 − γ)d_t$. There is no interest for holding monetary reserves.

Let $R$ denote the gross real return for the unliquidated assets in period 3. We design the model such that $R$ is not affected by the liquidations in period 2. More specifically, we assume that when banks invest in long-term projects, they create a continuum of small firms. Households are hired by those firms and each firm has the same capital-labour ratio, $k$.

Workers are hired in period 1 but work only in period 3. In period 2, if liquidation is needed, banks will liquidate assets firm by firm. Workers of liquidated firms will be unemployed in period 3. We assume that for the remaining firms, workers and capital will both be paid according to their marginal product. Let $δ$ denote the depreciation rate of capital, and we assume the re-selling price of un-depreciated capital in period 3 is 1. We get

$$R = AαK^{α−1}L^{1−α} + (1 − δ) = Aαk^{α−1} + (1 − δ)$$

(2.2)

The capital-labour ratio $k$ is decided in period 1 and is not affected by the liquidations.

After banks make investments, we enter period 2, where we will have a two-stage withdrawal game. At the beginning of the first stage, nature decides the aggregate liquidity shock. A random fraction $π(θ)$ of foreign creditors will be chosen as “movers” who must
withdraw their funds and leave the country.\textsuperscript{4} The reinvestment return for the funds withdrawn is zero. $\theta$ is uniformly distributed over $[0, 1]$, and $\pi(\theta) \in [0, 1]$ is strictly increasing and continuous in $\theta$. $\theta$ is independently and identically distributed; each depositor has the same probability of being a mover.

In the first stage of the game, everyone will receive a private signal $\theta_i$ of the true $\theta$, where $\theta_i$ is drawn uniformly from the interval $[\theta - \epsilon, \theta + \epsilon]$ for some small $\epsilon$. The distribution of $\theta_i$ is common knowledge. Note that everyone still does not know the personal liquidity shock yet. Based on $\theta_i$, each agent decides whether to withdraw in the first stage.

In the second stage, every agent is notified whether he is chosen to be a mover. In addition, the true value of $\theta$ becomes common knowledge, and people also observe the number of withdrawers in the first stage. Movers who have not withdrawn yet must withdraw in the second stage. Non-movers in the second stage will withdraw only if waiting until period 3 gives them a lower payoff.

Banks are subject to the sequential-service-constraint. The central bank will lend to banks only after they have used up their own reserves. If banks have used up both their reserves and the central bank loan, then they must liquidate assets.

The central bank follows a simple policy rule. It allocates a borrowing quota to each bank; the ratio between bank $i$’s quota and the aggregate quota is equal to the ratio between bank $i$’s deposit and the aggregate deposit. For example, if the aggregate deposit is 50, and the deposit in a particular bank is 1, then the borrowing quota of that bank is $\frac{1}{50}$ of the total central bank reserve. In the symmetric equilibrium, each bank has the same deposit and the same borrowing quota. We focus on how the quantity of international reserves affects the equilibrium and we assume the lending rate is fixed at zero.

The nominal price of capital goods and consumption goods are exogenously given by the

\textsuperscript{4}This setup in which some agents called “movers” must withdraw from banks is taken from Champ, Smith and Williamson (1996).
international markets and are fixed at 1 dollar per unit. Each unit of liquidated asset can be transformed into $\lambda$ units of consumption goods and sold in the international market for $\lambda$ dollars, where $\lambda < 1$. Banks are not allowed to default on the loans they borrowed from the central bank, and the liquidation must stop when the value of the remaining assets is just enough to repay the central bank loan. If a bank runs out of all resources, then the remaining depositors who have not withdrawn will lose all their deposits.

In period 3, production happens in firms that are not liquidated, and the wage rate is

$$w = (1 - \alpha)Ak^\alpha$$

(2.3)

If there are no liquidations, since the size of the labour force is normalized to 1, $w$ will be the aggregate wage income. When there are liquidations, let $\phi$ denote the share of unliquidated firms, then the aggregate wage income is $(1 - \phi)w$. We assume that workers fully share their income. Consumption goods and un-depreciated capital goods are then sold in the international market. After paying depositors, the remaining goods are consumed by domestic agents.

**Bank contract**

In order to simplify the analysis, we impose the following restriction on the deposit contract that can be offered by banks.

**Assumption 1.** Banks promise to pay gross rate $\bar{r}_m = 1$ to anyone who withdraws in $t = 2$, and gross rate $\bar{r}_n$ to anyone who waits until period 3. $\bar{r}_m$ and $\bar{r}_n$ are denominated in dollars; they are fixed at the beginning of $t = 1$ and are not contingent on $\pi(\theta)$.

The purpose of the above assumption is as follows. By assuming that the payment for early withdrawal $\bar{r}_m$ is 1, we only need to consider how banks will optimally set the long-term deposit rate $\bar{r}_n$. In addition, when the deposit rate is not contingent on the liquidity shock, banks cannot reduce or suspend the payment to avoid bank runs when the liquidity shock is high.
Since depositors are risk-neutral, banks will compete by choosing the interest rate $r$ and the reserve ratio $\gamma$ optimally to maximize the expected payoff of depositors subject to the zero expected-profit condition. Banks take the return for long-term capital, $R$, as given.

2.3 The two-stage withdrawal game in period 2

In this section, we analyze the two-stage withdrawal game in period 2. The commercial banks’ choice of $r$ and $\gamma$ in period 1 will be analyzed in section 2.4.

We will analyze two cases; each case has a different information structure in the first stage of the game. In the homogeneous information case, there is no noise in the private signal and all information is common knowledge. We assume that people will coordinate for the no-run equilibrium whenever it is a feasible choice. As a result, bank runs happen only when the fundamental factor (the liquidity shock) is high enough. We then compare the results by modelling self-fulfilling panics using the heterogeneous information structure, where the noise in the private signal is not zero. In this case, since people are no longer sure about other people’s actions, self-fulfilling panics are possible. We find that self-fulfilling panics tend to increase the probability of bank runs.

2.3.1 Homogenous information

When the private noise is zero ($\theta_i = \theta$), the aggregate liquidity shock is common knowledge. In stage 1, although people do not know their personal liquidity shock, they know the aggregate liquidity shock and how many resources banks will have after paying movers. Since people will coordinate for the no-bank-run equilibrium whenever it is feasible, if $\theta$ is sufficiently low that non-movers can get at least $\bar{r}$ in period 3, then people will not withdraw in stage 1. Then in stage 2, only those who turn out to be movers will withdraw. If $\theta$ is high and the actual payment in period 3 will be lower than $\bar{r}$, then all depositors will withdraw.
in stage 1.

Let $\kappa_e$ denote the highest level of $\pi(\theta)$ at which there is no bank run. $\kappa_e$ can be solved from:

$$\kappa_e d_t r^m = \gamma d_t + \bar{b} + \lambda \phi_e [(1 - \gamma)d_t + e_b] \quad (2.4)$$

$$(1 - \kappa_e) d_t r^m = (1 - \phi_e) [(1 - \gamma)d_t + e_b] R - \bar{b} \quad (2.5)$$

The meaning of equation (2.4) is that the withdrawal by movers at $\pi(\theta) = \kappa_e$ is met by the reserve of the bank $\gamma d_t$, the maximum loan from the central bank $\bar{b}$, and the proceeds from liquidating $\phi_e$ of the long-term investments. Equation (2.5) means that the remaining resources are just enough to give the remaining depositors the return $r^m$ in period 3. $1 - \phi_e$ is the proportion of unliquidated investments and $\bar{b}$ is the repayment for the central bank loan. The solutions of $\phi_e$ and $\kappa_e$ are

$$\phi_e = \frac{\gamma d_t + [(1 - \gamma)d_t + e_b] R - d_t r^m}{(R - \lambda)[(1 - \gamma)d_t + e_b]} \quad (2.6)$$

$$\kappa_e = \frac{\gamma d_t + \bar{b} + \lambda \phi_e [(1 - \gamma)d_t + e_b]}{d_t r^m} \quad (2.7)$$

Note that if we have $\kappa_e \geq 1$, then the bank will always be able to pay $r^m$, and no bank run will happen.

Suppose $\kappa_e < 1$. If $\pi(\theta) > \kappa_e$, then all depositors will run the bank in stage 1. Let $n_f$ denote the number of depositors who withdraw successfully. We have

$$n_f d_t r^m = \gamma d_t + \bar{b} + \lambda \phi_f [(1 - \gamma)d_t + e_b] \quad (2.8)$$

$$0 = (1 - \phi_f) [(1 - \gamma)d_t + e_b] R - \bar{b} \quad (2.9)$$

Equation (2.8) means that the maximum cash that can be paid by the bank is equal to the reserve of the bank, plus the loan from the central bank and the income from liquidating $\phi_f$ of the long-term assets. Equation (2.9) means that the liquidation will stop at $\phi_f$ when the value of the unliquidated assets is just enough to pay back the central bank loan. The
solutions for $\phi_f$ and $n_f$ are

$$
\phi_f = 1 - \frac{\bar{b}}{[(1 - \gamma)d_t + \epsilon_b]R} \quad (2.10)
$$

$$
n_f = \frac{\gamma d_t + \bar{b} + \lambda \phi_f[(1 - \gamma)d_t + \epsilon_b]}{d_t \bar{r}_m} \quad (2.11)
$$

In summary, if $\pi(\theta) \leq \kappa_e$, no bank run happens. Only movers withdraw in stage 2. If $\pi(\theta) > \kappa_e$, all people run the bank in stage 1, even if they still do not know whether they will be movers. Only $n_f$ of the depositors can actually withdraw their deposits.

### 2.3.2 Heterogenous Information

**The payoff differential function**

Now we assume that the private signal $\theta_i = \theta + \epsilon_i$ is uniformly distributed over $[\theta - \epsilon, \theta + \epsilon]$ with $\epsilon$ being a small positive number.

We will focus on stage 1 of the game. Let $n$ denote the number of depositors who decide to withdraw in stage 1. Since every depositor is equally likely to become a mover, in stage 2, $\pi(\theta)$ of the remaining $1 - n$ depositors will turn out to be movers, who must withdraw. So the minimum withdrawal in stage 2 is $(1 - n)\pi(\theta)$ and the minimum total withdrawal after two stages is $n + (1 - n)\pi(\theta)$, which we denote as $\kappa$. Let $n_e$ denote the $n$ at which $\kappa = \kappa_e$. The withdrawal decision and the expected payoff are summarized as follows (also see Table 2.1).

1. If $0 \leq n \leq n_e$, the bank will survive both stages. All depositors who withdraw in stage 1 will get $\bar{r}_m d_t$. If a depositor does not withdraw in stage 1, then in stage 2, with probability $\pi(\theta)$, he will be a mover, and he will withdraw $\bar{r}_m d_t$ from the bank. With probability $1 - \pi(\theta)$, he will be a non-mover. In that case, he will wait until period 3 and get $r^n d_t$ from the bank, where $r^n \in [\bar{r}_m, \bar{r}_m]$ is the actual interest rate paid in period 3.
Table 2.1: The withdrawal decisions and the expected utilities

<table>
<thead>
<tr>
<th>Withdraw in Stage 1</th>
<th>Do Not Withdraw in Stage 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq n \leq n_e$:</td>
<td>$r_{m}d_t$</td>
</tr>
<tr>
<td>$n_e &lt; n &lt; n_f$:</td>
<td>$r_{m}d_t$ : movers withdraw in stage 2</td>
</tr>
<tr>
<td></td>
<td>$r_{n}d_t$ : nonmovers wait until period 3</td>
</tr>
<tr>
<td>$n_f \leq n \leq 1$:</td>
<td>$egin{cases} r_{m}d_t : \text{with probability } \frac{n_f-n}{1-n} \ 0 : \text{with probability } 1 - \frac{n_f-n}{1-n} \end{cases}$</td>
</tr>
</tbody>
</table>

$Eu^W (Eu^{NW})$ is the expected utility if withdraw (not withdraw) in stage 1. The payoff differential function is defined as $v(\theta, n) = \frac{Eu^W - Eu^{NW}}{d_t}$.

<table>
<thead>
<tr>
<th>Withdraw in Stage 1</th>
<th>Do Not Withdraw in Stage 1</th>
<th>Payoff Differential Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq n \leq n_e$:</td>
<td>$Eu^W = r_{m}d_t$</td>
<td>$v(\theta, n) = (1 - \pi(\theta))(r_{m} - r_{n})$</td>
</tr>
<tr>
<td>$n_e &lt; n &lt; n_f$:</td>
<td>$Eu^W = r_{m}d_t$</td>
<td>$v(\theta, n) = 1 - \frac{n_f-n}{1-n} r_{m}$</td>
</tr>
<tr>
<td>$n_f \leq n \leq 1$:</td>
<td>$Eu^W = \frac{n_f-n}{n} r_{m}d_t$</td>
<td>$v(\theta, n) = \frac{n_f-n}{n} r_{m}$</td>
</tr>
</tbody>
</table>

2. If $n_e < n < n_f$, the bank does not fail in stage 1, and all people who withdraw in stage 1 can get $r_{m}d_t$. In stage 2, all remaining depositors will run the bank, a withdrawer gets $r_{m}d_t$ with probability $\frac{n_f-n}{1-n}$, and loses his money with probability $1 - \frac{n_f-n}{1-n}$.

3. If $n_f \leq n \leq 1$, the bank fails in the first stage. A withdrawer in the first stage will be able to withdraw $r_{m}d_t$ with probability $\frac{n_f-n}{n}$, and will lose his money with probability $1 - \frac{n_f-n}{n}$. People who do not withdraw in the first stage will lose their deposits.

What follows explains the above results.

First, if $n \geq n_f$, since the maximum amount of withdrawal that the bank can meet is $n_f$, then only $n_f$ depositors can get their money, and the bank fails in stage 1.

Second, if $n < n_f$, the bank does not fail in stage 1. Whether the bank fails in stage 2 depends on $\kappa$. If $\kappa = \kappa_e$, the remaining resources will be just enough to give depositors $r_{m}$ in period 3 (see equation 2.4 and 2.5). Since we assume that both the values of $\theta$ and $n$ become common knowledge in stage 2, people know the value of resources that banks will have in
period 3. Non-movers who have not withdrawn in stage 1 will choose to wait if \( \kappa \leq \kappa_e \). If \( \kappa > \kappa_e \), all remaining depositors will run the bank in stage 2.

Recall that \( n_e \) is the \( n \) at which \( \kappa = \kappa_e \), so
\[
\kappa_e = n_e + (1 - n_e)\pi(\theta) \quad \Rightarrow \quad n_e(\theta) = \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)}
\]  

Suppose \( n_e(\theta) \in (0, 1) \). If \( n \leq n_e \) (i.e., \( \kappa \leq \kappa_e \)), then there is no bank run in stage 2 and only movers withdraw in stage 2. If \( n_e < n < n_f \), then \( \kappa \geq \kappa_e \), and all remaining depositors will run the banks in stage 2. Since the banks can pay at most \( n_f \) depositors after two stages, only \( n_f - n \) depositors can get their money back in stage 2. So if there is a run in stage 2, the probability for successful withdrawal is \( \frac{n_f - n}{1 - \pi} \).

We define the payoff differential function \( v(\theta, n) \) as \( E_uW - E_uNW \), where \( E_uW (E_uNW) \) is the expected utility for withdrawing (not withdrawing) in stage 1. It is better to withdraw if \( v(\theta, n) > 0 \).

We consider only bank contracts in which banks will be able to pay the promised payment \( \bar{r}^N \) when they do not need to liquidate assets.\(^6\) Let \( \kappa_s(\theta) \) denote the value of \( \kappa \) above which the actual payment in period 3 will be lower than the promised payment \( \bar{r}^N \), and let \( n_s(\theta) \) denote the corresponding \( n \). Figure 2.2 illustrates \( v(\theta, n) \) when \( \theta \) is given and \( n \) changes from 0 to 1. The detailed derivation is shown in the appendix; what follows explains the intuition.

When \( n \) is lower than \( n_s(\theta) \), the bank will be able to pay the promised payment \( \bar{r}^N \) in period 3, and it is better not to withdraw in stage 1, so \( v(\theta, n) \) is negative. When \( n \) is higher than \( n_s(\theta) \), the actual payment in period 3 starts to decrease, and the value of \( v(\theta, n) \) starts to increase. When \( n \) reaches \( n_e(\theta) \), people will run the bank in stage 2. Since liquidation is

---

\(^5\)Note that \( n_e(\theta) \) is positive only if \( \pi(\theta) < \kappa_e \). If \( \pi(\theta) \geq \kappa_e \), the amount of movers, \( \pi(\theta) \), is higher than \( \kappa_e \). Since movers must withdraw, the bank must fail during the two-stage game even if the withdrawal in stage 1 is zero. In that case, we define \( n_e(\theta) = 0 \). \( n_e \) is smaller than 1 only if \( \kappa_e < 1 \). If \( \kappa_e \geq 1 \), no bank run would happen and we define \( n_e(\theta) = 1 \).

\(^6\)Theoretically, banks can set \( \bar{r}^N \) at extremely high levels such that the actual payment \( r^N \) is always lower than the promised payment \( \bar{r}^N \) even when there is no liquidation. We rule out such contract. In our simulations, we did not find optimal contracts in which banks cannot pay \( \bar{r}^N \) when there is no liquidation.
costly, there is a downward jump in the expected payoff for not withdrawing in stage 1; as a result, there is an upward jump in the value of \( v(\theta, n) \). When \( n > n_f \), the bank fails in stage 1 and people who choose to wait will get nothing. Since only \( n_f \) depositors can withdraw successfully, higher \( n \) will reduce the probability of successful withdrawal, \( \frac{n_f}{n} \). As a result, \( v(\theta, n) \) decreases in \( n \).

**The unique threshold equilibrium**

In this part, we define the equilibrium in the two-stage withdrawal game. If \( \kappa_e \geq 1 \), then no bank run will happen. So we will focus on the cases where \( \kappa_e < 1 \). Before we derive the equilibrium, we first impose a refinement condition to restrict the set of strategies that can be used by depositors:

**Assumption 2.** Agents will coordinate on an equilibrium in which, when agents observe signals that are extremely low (\( \theta_i \in [0, \epsilon] \)), they do not run. Agents will coordinate for this equilibrium as long as the following condition is satisfied: given that all agents \( j \neq i \) do not run for \( \theta_j \in [0, \epsilon] \), agent \( i \) also finds it is better not to run for \( \theta_i \in [0, \epsilon] \).

The meaning of this assumption is that people will not run when the private signal is extremely good (so that they know that the fundamental is excessively good, that is, the liquidity shock is extremely low).\(^7\) If the condition of Assumption 2 is not satisfied, that

\(^7\)See Goldstein and Pauzner (2005) for a discussion of this type of refinement condition in global games.
is, if we cannot find an equilibrium in which people do not run the bank when the private signal is extremely good ($\theta_i \in [0, \epsilon]$), then we simply assume that everyone will run the bank regardless of the private signal.

We have the following result:

**Proposition 1.** *In the two-stage withdrawal game, there is a unique threshold equilibrium. Depositors will withdraw in stage 1 if the private signal is higher than a threshold level $\theta^*$, and will not withdraw if the private signal is lower than $\theta^*$.*

Proof: see the Appendix

In this chapter, we focus only on threshold equilibrium in which each agent withdraws when the private signal is higher than the threshold and does not withdraw if the private signal is lower than the threshold. Proposition 1 says that there is only one such equilibrium $\theta^*$.  

In what follows, I define the equilibrium condition. The private signal received by each depositor is the true value of the fundamental variable plus a uniformly distributed noise: $\theta_i \in [\theta - \epsilon, \theta + \epsilon]$. Thus, in the equilibrium, the expected proportion of depositors who run the bank at each realized value of $\theta$ is

$$n(\theta, \theta^*) = \begin{cases} 1 & : \text{ if } \theta > \theta^* + \epsilon \\ \frac{1}{2} + \frac{\theta - \theta^*}{2\epsilon} & : \text{ if } \theta^* - \epsilon \leq \theta \leq \theta^* + \epsilon \\ 0 & : \text{ if } \theta < \theta^* - \epsilon \end{cases} \quad \text{(2.13)}$$

where $\theta$ in $n(\theta, \theta^*)$ is the realized value of $\theta$, and $\theta^*$ is the threshold level of the equilibrium strategy. When $\theta > \theta^* + \epsilon$, all private signals $\theta_i$ are higher than $\theta^*$, and everyone would withdraw. If $\theta < \theta^* - \epsilon$, all private signals $\theta_i$ are lower than $\theta^*$, and no one would withdraw. When $\theta^* - \epsilon \leq \theta \leq \theta^* + \epsilon$, $n(\theta, \theta^*)$ increases uniformly from 0 to 1.

---

8Actually, we can prove a stronger result: there is no equilibrium other than the unique threshold equilibrium. The proof is omitted here because it is technical and is not crucial to the main results of this essay. The proof can be found in an older version of this essay, which is available on the author’s website.
Denote the candidate for the threshold equilibrium as $\theta'$. A depositor $i$ would withdraw if the private signal $\theta_i$ is higher than $\theta'$. Let $\Phi(\theta_i, \theta')$ denote the depositor’s expected utility differential between withdrawing and not withdrawing in stage 1 when the private signal is $\theta_i$. Since $\theta_i$ is uniformly distributed between $[\theta - \epsilon, \theta + \epsilon]$, after the depositor observes $\theta_i$, the posterior belief of $\theta$ is a uniform distribution over $[\theta_i - \epsilon, \theta_i + \epsilon]$. For each value of $\theta$, the payoff differential function is $v(\theta, n(\theta, \theta'))$, the value of $n(\theta, \theta')$ is decided according to (2.13). $\Phi(\theta_i, \theta')$ is the average of $v(\theta, n(\theta, \theta'))$ over $[\theta_i - \epsilon, \theta_i + \epsilon]$.

$$\Phi(\theta_i, \theta') = \frac{1}{2\epsilon} \int_{\theta_i-\epsilon}^{\theta_i+\epsilon} v(\theta, n(\theta, \theta')) d\theta$$  \hspace{1cm} (2.14)

In the equilibrium, depositors will prefer to withdraw if $\theta_i > \theta'$ and will prefer to wait if $\theta_i < \theta'$. Depositors will be indifferent between these two choices if the private signal is equal to the threshold, $\theta_i = \theta'$. So the equilibrium is defined by $\Phi(\theta', \theta') = 0$. If we use $\theta^*$ to denote the equilibrium threshold, then the equilibrium condition can be written as

$$\Phi(\theta^*, \theta^*) = \frac{1}{2\epsilon} \int_{\theta^*-\epsilon}^{\theta^*+\epsilon} v(\theta, n(\theta, \theta^*)) d\theta = 0$$  \hspace{1cm} (2.15)

A simple way to compute the value of $\theta^*$ is to assume that the private noise is close to zero.

**Proposition 2.** When $\epsilon \to 0$, we can approximate $v(\theta, n(\theta, \theta^*))$ in equation (2.15) with $v(\theta^*, n(\theta, \theta^*))$. Since $n(\theta, \theta^*)$ changes uniformly from 0 to 1 when $\theta$ changes from $\theta^* - \epsilon$ to $\theta^* + \epsilon$, we get

$$\Phi(\theta^*, \theta^*) \approx \int_0^1 v(\theta^*, n) dn$$  \hspace{1cm} (2.16)

Since $\Phi(\theta^*, \theta^*) = 0$, $\theta^*$ is implicitly defined by

$$0 = \int_0^1 v(\theta^*, n) dn$$  \hspace{1cm} (2.17)

---

9We can assume that depositors choose to wait at $\theta_i = \theta^*$. This assumption does not affect the result in this essay.
Proof: see the Appendix.

The intuition of (2.16) is as follows. At the equilibrium, the private signal $\theta_i$ is equal to $\theta^*$, and the posterior belief of $\theta$ is uniform over $[\theta_i - \epsilon, \theta_i + \epsilon]$. $\theta$ affects the expected payoff through two channels. First, $\theta$ will affect the liquidity shock $\pi(\theta)$; this direct effect is described by the first $\theta$ in $v(\theta, n(\theta, \theta^*))$. Second, the difference between the true $\theta$ and $\theta_i$ will affect the value of $n$; this indirect effect is described by the $\theta$ in $n(\theta, \theta^*)$. When $\epsilon \to 0$, we have $\theta \to \theta^*$ and thus $\pi(\theta) \to \pi(\theta^*)$. Nevertheless, no matter how small $\epsilon$ is, when $\theta$ changes from $\theta_i - \epsilon$ to $\theta_i + \epsilon$, $n(\theta, \theta^*)$ always uniformly changes from 0 to 1. Thus, we have (2.16).

Equation (2.17) says that in order to compute $\theta^*$, all we need is to find out the $\theta$ at which the average value of $v(\theta, n)$ over $n \in [0, 1]$ is zero. $v(\theta, n)$ is defined in Table 2.1. The analytical results for $\int_0^1 v(\theta^*, n)dn$ are shown in the Appendix.
Bank runs in the 2-stage game when $\epsilon \to 0$

**Proposition 3.** In the limit when $\epsilon \to 0$, all people will run the bank in stage 1 if $\theta > \theta^*$. If $\theta < \theta^*$, then there is no bank run in either stages of the game, only people who turn out to be movers will withdraw in stage 2. The probability for a bank run in equilibrium is $1 - \theta^*$.

Proof: see the Appendix.

**Comparison to the homogeneous information case**

Recall that under homogeneous information, people withdraw only in stage 1 when $\pi(\theta) > \kappa_e$, that is, when the liquidity shock is high enough to make the bank fail. Under heterogeneous information, a bank run happens when $\theta > \theta^*$. In the proof for proposition 3, we show that $\pi(\theta^*) < \kappa_e$, which means $\theta^* < \theta_e$. So under heterogeneous information, a bank run happens before the fundamental factor (liquidity shock) is high enough to actually make the bank fail. The reason is that with noisy information, people can no longer coordinate perfectly.

For example, when $\epsilon \to 0$, the *fundamental uncertainty* (uncertainty concerning the value of the liquidity shock) almost disappears, but there is still the *strategic uncertainty* (uncertainty concerning the actions of other depositors, that is, the value of $n(\theta, \theta^*)$). On the one hand, bank runs will happen only when the fundamental factor $\theta$ is high enough (i.e., $\theta > \theta^*$). On the other hand, due to uncertainties about the actions of other people, agents may run the bank even when $\theta$ is lower than $\theta_e$. So bank runs in this case are caused by both fundamental factors and self-fulfilling panics. Agents run in stage 1 because they think the liquidity shock is high, and also because they worry that other agents may run the bank.

**2.4 Bank’s choice in period 1**

Section 2.3 describes the equilibrium of the two-stage game in period 2 given the choices of commercial banks in period 1. Given the expected result of the two-stage game in period 2,
banks will optimally choose their reserve level and the long-term deposit rate \( \bar{r} \) to maximize the expected payoff of depositors subject to the zero expected profit condition.

Banks will not try to keep positive profit because if the profit were positive, then banks could improve the expected payoff of depositors by increasing the level of \( \bar{r} \). Higher \( \bar{r} \) will increase the payoff for depositors if they wait until period 3, and it will also reduce the incentive for people to withdraw from banks in period 2, thus reducing the probability of bank runs. As a result, banks will increase \( \bar{r} \) until the expected profit is equal to zero.

The expected payoff of depositors and the expected profit of banks are as follows:

**Proposition 4.** The expected payoff of depositors under homogeneous information is

\[
EU^d = \int_{\theta_e}^{1} n_f d_t \bar{r} \, d\theta + \int_{0}^{\theta_e} [\pi(\theta) d_t \bar{r} + (1 - \pi(\theta)) d_t r^n(\theta)] \, d\theta
\]

Under heterogeneous information, it is

\[
EU^d = \int_{\theta^*}^{1} n_f d_t \bar{r} \, d\theta + \int_{0}^{\theta^*} [\pi(\theta) d_t \bar{r} + (1 - \pi(\theta)) d_t r^n(\theta)] \, d\theta
\]

The expected profit of banks under homogeneous information is

\[
E\Pi = \int_{0}^{\theta_s} [\max(\gamma d_t - \pi(\theta) d_t \bar{r}, 0) - b + (1 - \phi(\theta))((1 - \gamma) d_t + e_b)R - (1 - \pi(\theta)) r^n d_t] \, d\theta - e_b R
\]

Under heterogeneous information, if \( \theta^* > \theta_s \), then the profit is the same as (2.20). If \( \theta^* \leq \theta_s \), then we only need to replace the upper bound of the integral \( \theta_s \) with \( \theta^* \).

Proof: See the Appendix

The intuition is as follows. Under homogeneous information, people will run banks when \( \theta \in (\theta_e, 1] \), the probability for successful withdrawal is \( n_f \) and the expected payoff is \( n_f d_t \bar{r} \).

When \( \theta \leq \theta_e \), there is no bank run. With probability \( \pi \), the depositor will be a mover and get payoff \( d_t \bar{r} \) in period 2. With probability \( 1 - \pi \), the deposit will wait until period 3 and get payoff \( d_t r^n(\theta) \). Thus, we get (2.18).
Under heterogeneous information, people will run banks when $\theta \in (\theta^*, 1]$. We need only to replace $\theta_e$ in (2.18) with $\theta^*$, and we get (2.19).

In (2.20), the integrand in the first term is the gross income of banks in period 3 (see the Appendix for a detailed derivation). When $\theta > \theta_s$, banks will use all their resources to pay depositors and they will get zero income. So the expected profit is equal to the expected income over $[0, \theta_s]$, minus the opportunity cost of the equity $e_bR$.

Under heterogeneous information, a bank run happens when $\theta > \theta^*$. If $\theta_s < \theta^*$, then bank profit can be written in the same way as (2.20). If $\theta^* < \theta_s$, since the income of banks is zero when a bank run happens, we need only to consider the bank’s income over $[0, \theta^*]$. So we need to change the upper bound of the integral in (2.20) from $\theta_s$ to $\theta^*$.

The equilibrium

The equilibrium of the three-period model can be defined as follows: Given the lending quota of the central bank, the deposit rate $\tilde{\pi}$ and the reserve ratio $\gamma$, people will optimally decide their withdrawal decision in period 2. Under homogeneous information, all depositors will withdraw in stage 1 of period 2 when $\theta > \theta_e$. Otherwise, they will not withdraw in stage 1, and only movers withdraw in stage 2. All remaining depositors will wait until period 3. Under heterogeneous information, all depositors withdraw in stage 1 of period 2 when $\theta > \theta^*$. If $\theta < \theta^*$, only movers withdraw in stage 2, and all remaining depositors wait until period 3.

Given the expected outcome in period 2 and period 3, banks optimally decide the level of $\tilde{\pi}$ and $\gamma$ in period 1. In the equilibrium, the expected payoff of depositors are maximized, and the expected profit of banks is zero.

The total investment in the first period is equal to $e_b + (1 - \gamma)d_t$. In period 3, the return for unliquidated capital is $R$ (equation 2.2), the wage rate is (2.3) and the total wage income is $(1 - \phi)w$. 

24
2.5 Numerical example: basic effects of central bank reserves

In this section and the next two sections, we will show some numerical results of the model. This is not a calibration. The purpose of the example is to show the basic qualitative results of the model.

2.5.1 The function form and parameter values:

We use the following function form of $\pi(\theta)$:

$$\pi(\theta) = \theta^\eta$$  \hspace{1cm} (2.21)

Since $\theta$ is uniform over $[0, 1]$, higher $\eta > 0$ implies the liquidity shock $\pi(\theta)$ is more concentrated on low values. When $\eta$ is very low, $\pi$ tends to be high and banks will hold high reserves, and there may be no bank runs. We set $\eta$ high enough ($\eta = 4$) such that bank runs are possible.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\overline{f}$</th>
<th>$d_t$</th>
<th>$\lambda$</th>
<th>$e_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.02</td>
<td>12.5</td>
<td>50</td>
<td>0.9</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 2.2: Values of parameters

Other parameters are shown in Table 2.2. $\alpha$ and $\delta$ are set according to the literature of business cycles. $\overline{f}$ is the debt equity ratio $d_t/e_b$. We set its value according to banking regulations.\textsuperscript{10} We set the total deposit $d_t$ at 50; this value can be chosen freely. This gives the bank equity level $e_b = d_t/\overline{f} = 4$. $A$ is set such that the interest rate $R$ is neither too low nor too high. We set $\lambda$ at 0.9, lower $\lambda$ means higher liquidation cost, which will lead to slightly higher probability of bank runs, but the basic result is similar. We set household endowment $e_h$ at 50. At this level, when all assets are liquidated and there is no wage income,

\textsuperscript{10}The Basel accord requires that the risk-weighted loan-capital ratio of banks should not be higher than 12.5, and here we assume the debt-equity ratio of banks also takes a similar value.
the total household income will be reduced by approximately 10%, which corresponds to a mild financial crisis in reality.

2.5.2 Basic effects of central bank reserves

Figure 2.3 illustrates the effect of exogenous changes of the level of central bank reserves. The result can be summarized as follows: Increases in central bank reserves will cause banks to hold fewer reserves and make more investments. It will reduce the probability of bank runs and increase the utility of households. The details are as follows:

Figures (a) and (b) show that higher central bank reserves will cause commercial banks to hold fewer reserves, and will also reduce the probability of bank runs. When central bank reserves are high enough, the probability of bank runs can be reduced to zero, and the results for homogeneous and heterogeneous information will be the same because people do not run banks in stage 1 and will withdraw only in stage 2 if they turn out to be movers.

Recall that if the reserve level of commercial banks is given, bank runs are more likely to happen under heterogeneous information than under homogeneous information. But from (a) and (b), we can see that the probability of bank runs is very low under both types of information structure. This is because in this case the reserve level is not fixed, banks will choose to hold higher reserves under heterogeneous information, which will reduce the probability of bank runs to low levels. Intuitively, banks will not intentionally choose to have a high probability of bank runs because banks will lose all of their equity if a bank run happens. As a result, if banks take into account self-fulfilling panics and hold high reserves, then the probability of bank runs will still be at low levels.

Figure (c) shows that the investment level will be higher when banks reduce their reserve level. Figure (d) shows the probability that banks will have to liquidate assets. $\theta_b$ is the level of $\theta$ above which liquidation is needed. Under homogeneous information, higher central bank reserves will reduce the probability of liquidation. Under heterogeneous information, the
(a) Commercial bank reserve ratio $\gamma$

(b) Probability of bank runs

(c) Total investment $e_b + (1 - \gamma)d_t$

(d) Probability for liquidation $1 - \theta_b$

(e) The expected utility of households

Figure 2.3: The effects of exogenous changes of central bank reserves.
increase in central bank reserves may actually cause an increase in $\theta_b$ before lowering it. The reason is that if the reserve ratio decreases very quickly (see figure (a)), $\theta_b$ may temporarily increase.\footnote{\textit{\theta_b} is defined by $\pi(\theta_b) r_m d_t = \gamma d_t + \bar{b} \implies \pi(\theta_b) = \frac{\gamma d_t + \bar{b}}{r_m}$; if $\gamma$ decreases quickly enough, then the nominator $\gamma d_t + \bar{b}$ can be decreasing even when $\bar{b}$ is increasing, which means $\theta_b$ can be decreasing in $\bar{b}$.}

Figure (e) shows household utility. Here, we do not include the consumption of central bank reserves, because we try to isolate the effects of lending reserves to banks. By lending reserves to banks, the household wage income will be affected. We set the consumption at $e_h + (1 - \phi)w$, which is equal to the endowment plus the wage income. Higher central bank reserves can improve the wage income. First, the investment level will be higher, which means a higher capital-labour ratio $k$ and so a higher wage rate $w$. Second, higher reserves will generally lead to lower liquidations of assets. So households will enjoy higher utility.

2.5.3 The central bank’s decision to accumulate reserves

This part uses the two-generation model to analyze the choice of the central bank. The two generations are denoted as $T = 1, 2$. The tax for the first generation is $\tau_1$. For simplicity, we assume the initial reserve is zero. We try to find out what is the level of $\tau_1$ given different income levels of the first generation, the income level is given exogenously.

The central bank reserves will perform several functions: They will affect the investment level of banks, they will affect the probability of bank runs, and they can be lent to banks to reduce the liquidation of assets. At the end of $T = 2$, they will be consumed by households.

The central bank maximizes the sum of the expected utility of households

$$u_1 + Eu_2 = \frac{-1}{c_1} + E\frac{-1}{c_2}$$

(2.22)

where $c_1$ and $c_2$ are household consumption for the first and the second generation. $c_1$ is equal to the income of the first generation minus $\tau_1$, and

$$c_2 = e_h + (1 - \phi)w + \bar{b}$$

(2.23)
Figure 2.4: The optimal central bank reserve for generation 2 (optimal tax for generation 1) given different income levels of generation 1.

where $e_h$ is household endowment, $(1 - \phi)w$ is the wage income, and $\bar{r}$ is the central bank reserve.\textsuperscript{12}

We choose $[50, 55]$ as the range of the household income in $T = 1$. Figure 2.4 shows that the tax is higher when the income of the first generation is higher. The optimal reserve is higher under heterogeneous information than under homogeneous information. This is mainly because under heterogeneous information banks tend to hold higher reserves and make less investment, which will lead to lower utility of households. As a result, the central bank will build up more reserves; this can encourage banks to make more investments, and can also directly increase the consumption of the second generation.

### 2.6 When the central bank is more cautious than the private banks

In the previous analysis, banks and central banks are equally cautious. In particular, we showed that when self-fulfilling panics are possible (i.e., when the information structure is heterogeneous), if banks take into account this possibility, then they will hold enough reserves

\textsuperscript{12}We assume the central bank collects tax only from households and makes transfers to households. For simplicity, we do not include the utility of bankers in the central bank’s objective function. Since the expected profit is always zero, the banker’s utility is $e_bR$. Changes in $R$ are very small under different central bank policies.
to reduce the probability of bank runs to low levels. Compared to the case without self-fulfilling panics (homogeneous information structure), the central bank does not really need to hold a lot more reserves in order to reduce the probability of crises because banks themselves have already done that.

In this part, we analyze what will happen if the central bank is more cautious than private banks. We show that the central bank will have a higher incentive to hold reserves.

In the real world, governments and central banks sometimes are very cautious about financial crises because they are responsible for maintaining financial stability. Financial crises will not only cause damage to the economy, they will also hurt the reputation of the government and incur great political costs. People are likely to blame the government for not doing enough to prevent the crises, and government officials may be forced to resign (such as Suharto in Indonesia).

Here, we analyze a simple case. We assume that private banks think there are no self-fulfilling panics, and they will choose their reserve ratio accordingly. But the central bank wants to take into account the worst scenario, and it prefers to choose its reserve level according to the assumption that self-fulfilling panics are possible.

Figure 2.5 shows the effects of exogenous changes of central bank reserves. Banks will choose their reserve ratio according to the scenario in which there are no self-fulfilling panics. If that is true, then the reserves of banks would be enough to achieve low probabilities of bank runs. But if self-fulfilling panics are possible (the scenario used by the central bank), then the probability of bank runs could be much higher, especially at low levels of central bank reserves. In addition, increases in central bank reserves will have higher effects in reducing the probability of bank runs.

Because central bank reserves have higher effects in reducing the probability of financial crises, the central bank will want to hold more reserves. This is shown in Figure 2.6. The optimal central bank reserves are much higher compared to the case when private banks also
Figure 2.5: The effects of central bank reserve on bank runs. Private banks set the reserve ratio by assuming there are no self-fulfilling panics. If that is true, then the probability of bank runs will be very low. If self-fulfilling panics are possible (the scenario used by the central bank), then the probability of bank runs could be much higher, and central bank reserves will also have higher effects in reducing the probability of bank runs.

Figure 2.6: Central bank’s optimal reserves. The central bank uses the scenario that self-fulfilling panics are possible. If private banks use the same scenario and hold high reserves by themselves, then the central bank has low need to accumulate reserves. If private banks use the scenario that there is no self-fulfilling panics and hold low reserves, then the central bank will hold higher reserves.

An interesting finding in Figure 2.6 is that the optimal tax is still positive when household income is at the lowest level 50, which is the level of endowment. With positive $\tau_1$, the consumption of the second generation is at least $50+\tau_1$. Since the utility function is concave, if international reserves are used only to smooth consumption over generations, then the tax should not be positive at the lowest income. Here, the extra benefits of international reserves in reducing bank runs and liquidations of assets cause the central bank to accumulate more
2.7 An extension: external borrowing

One explanation for the accumulation of reserves after the Asia financial crises is that countries had learned that it is very costly to borrow from external sources such as the IMF during financial crises. In this part, we extend the model to allow for external borrowing. We show that a re-evaluation of borrowing costs could cause the central bank to accumulate more reserves. An interesting result is that when the borrowing cost is high, the possibility for the central bank to borrow external funds may actually cause the central bank to accumulate more reserves than if the central bank cannot borrow from external sources at all.

We assume that both the central bank and banks choose their reserve level according to the heterogeneous information structure. We also make the following assumption: After the central bank runs out of its own reserves, it can borrow up to $b_{ex}$ from external sources. The maximum central bank loan becomes

$$
\bar{b} = b_c + b_{ex}
$$

(2.24)

where $b_c$ now denotes the reserves accumulated by the central bank. We assume that the central bank cannot commit not to borrow from external sources. That is, as long as banks need more loans, the central bank will borrow up to $b_{ex}$ and lend the money to banks. The central bank cannot choose not to borrow. Domestic banks will take the additional central bank borrowing into account when they make their decisions. We still assume that the lending rate by the central bank to commercial banks is normalized to zero. So there are no changes to the three-period bank run model except the level of $\bar{b}$.

We analyze only external borrowing in $T = 2$. The loan must be repaid at the end of $T = 2$. The unit borrowing cost is $\psi$, which includes the interest cost and other costs due to the need to comply with the conditions of the loan (such as political costs). We assume the
total cost is equivalent to a reduction in household consumption by $\psi b_{ex}$, where $b_{ex} \leq \bar{b}_{ex}$ is the actual loan level.

The central bank still maximizes $\frac{1}{c_1} + E \frac{1}{c_2}$, where $c_2$ becomes

$$c_2 = e_h + w(1 - \phi) + b_c - \psi b_{ex}$$

(2.25)

c_2 is equal to the sum of household endowment, wage income and the central bank reserves, minus the cost of external borrowing.

**Result**

We show two cases: $\bar{b}_{ex} = 2$ and $\bar{b}_{ex} = 5$. We have two main results.

1. For the same external borrowing limit, the central bank will hold higher reserves when the borrowing cost is higher.

2. When the borrowing cost is low, the central bank will hold fewer reserves when it can borrow more. But if the borrowing cost is very high, the central bank may actually want to hold more reserves when it can borrow more.

The first result is shown in Figure 2.7 and 2.8. For the same $\bar{b}_{ex}$, higher $\psi$ causes the central bank to hold more reserves, which will in turn cause the commercial banks to hold lower reserves. An implication is that if the central bank suddenly realizes that the cost of external borrowing is higher than previously expected, then it will accumulate more reserves.

We can also see that increases in $\psi$ cause more changes in the optimal central bank reserve level under $\bar{b}_{ex} = 5$ than under $\bar{b}_{ex} = 2$. Intuitively, if there is no external borrowing (i.e., if $\bar{b}_{ex} = 0$), changes in the external borrowing cost $\psi$ will not affect the choice of the central bank. When $\bar{b}_{ex}$ is high, increases in $\psi$ will cause a greater increase in the borrowing cost.

The second result is shown in Figure 2.9. We can see that when $\psi = 0$, the reserve level is the highest for $\bar{b}_{ex} = 0$ and the lowest for $\bar{b}_{ex} = 5$, which means the central bank will hold
Figure 2.7: Central bank reserves when $b_{ex} = 2$. Higher external borrowing cost $\psi$ will cause the central bank to hold more reserves (figure a), which will in turn cause private banks to hold lower reserves (figure b).

Figure 2.8: Central bank reserves when $b_{ex} = 5$. Higher external borrowing cost $\psi$ will cause the central bank to hold more reserves (figure a), which will in turn cause private banks to hold lower reserves (figure b). Commercial banks hold lower reserves under $b_{ex} = 5$ than under $b_{ex} = 2$ (Figure 2.7(b)).

lower reserves if it can borrow more. But if the cost is very high (for example, $\psi = 0.7$), the reserves for $b_{ex} = 5$ are actually higher than the reserves for $b_{ex} = 0$ and $b_{ex} = 2$.

The intuition is as follows. If banks know that the central bank can borrow more from external sources, and if the central bank cannot commit not to borrow external loans and lend them to banks at low costs, then banks will hold lower reserves, and the central bank must be prepared to lend to banks during liquidity crises. If the external borrowing cost is
Figure 2.9: (a) shows that when the external borrowing cost is low, the central bank will hold lower reserves if they can borrow more. (b) shows that with a very high borrowing cost, the central bank may hold more reserves when it can borrow more (the reserves under $b_{ex} = 5$ are the highest).

very high, then the cost for accumulating more reserves to self-insure will be lower than the cost for borrowing external loans. As a result, the central bank will want to accumulate more reserves in order to reduce the probability of borrowing from external sources.

2.8 Summary and conclusion

In this chapter, I applied the global game approach to build a model of bank runs caused by aggregate liquidity shocks. I then used the model to analyze the central bank’s policy to accumulate international reserves as self-insurance against liquidity crises.

I found that higher central bank reserves can reduce the probability of financial crises. The incentive of the central bank to accumulate reserves depends on the policy of the private sector. Lower reserve holdings of the private sector will induce the central bank to hold more reserves. I also showed that central banks will hold more reserves if they find that the external borrowing costs are higher than they expected.

In this essay, I focus only on liquidity risks. In reality, central banks may also want to
build up reserves in order to maintain a stable exchange rate. I will explore this mechanism in future work.
Chapter 3

Bank money, aggregate liquidity and asset prices

3.1 Introduction

This chapter shows why the ability of banks to create money is important to financial stability. We show that bank money creation can help to meet the demand for liquid assets and stabilize the price of illiquid assets. We also show that the ability of banks to expand aggregate liquidity is essential when people try to create mutual-fund-like coalitions to share liquidity risks.

The main motivation of this chapter is to extend the recent literature of “liquidity and asset prices” to include money creation by banks. The studies in “liquidity and asset prices” focus on how the limited ability of the market to absorb sales of assets may cause assets to be sold at low prices. For example, in a series of works, Allen and Gale argue that if the amount of cash that buyers can use to buy assets is limited, then when people have to sell assets, the market price can deviate from the fundamental value. This is the so called “cash-in-the-market-pricing” (Allen and Gale (2005)).

Most of the current works in liquidity and asset prices are based on non-monetary models (usually the Diamond and Dybvig (1983) framework). Demand for liquidity is usually modelled as demand for consumption goods. If lots of people in the economy turn out to be impatient and need to consume in the short run, then long-term assets will be sold at low
prices relative to short-term consumption goods. Here the short-term consumption goods are used to model liquid assets or “cash”.

A major problem of this non-monetary framework is that it assumes the aggregate supply of liquid assets is predetermined and inelastic. But in the real world, the supply of liquid assets is not fixed. In this chapter, demand for liquidity takes the form of demand for cash to make payments, and banks can supply liquidity elastically by creating deposits. Elastic bank deposits can help to meet the demand for liquid assets, and can reduce the sales of assets and help stabilize asset prices. As a result, previous non-monetary models may underestimate the ability of the financial system to accommodate liquidity shocks.¹

We also compare two types of liquidity provision functions of financial intermediaries. The first is liquidity risk-sharing through coalitions such as mutual funds. The second is bank money creation. The first mechanism is included in many standard banking models using the Diamond and Dybvig (1983) framework. The basic idea is that by pooling resources into a coalition, people who need liquidity can have a higher consumption than if there were no coalitions. As pointed out by many people, this liquidity risk-sharing mechanism does not need to be carried out by a bank because the coalition described in the model is essentially a mutual fund, and it does not need to have the monetary functions of banks. Thus, in our model, we use mutual funds to model the risk-sharing function. We then use banks to model the liquidity provision function through money creation.

We show that the ability of banks to create money is important for mutual funds being able to insure against aggregate liquidity risks. The reason is that mutual funds rely on the market for liquidity; they need to sell assets to other people to raise cash in order to meet withdrawals. If the aggregate amount of cash in the economy is limited, then the amount of cash that mutual funds can raise by selling assets will not exceed the aggregate existing cash. ¹

¹Gale (2005) extends the framework to include outside money. He shows that if there is cash-in-advance constraint in the financial market, then sales of assets can lead to low asset price. Still, in his model, private institutions cannot supply elastic aggregate liquidity.
We show that the attempt of each individual mutual fund to raise more cash by selling assets will only push asset prices to lower levels, thus leading to more volatility in the financial market. But if there are banks which can provide elastic aggregate liquidity, then mutual funds will indeed be useful: they will help people to smooth consumption.

Freeman (1996a,b) also discussed how central banks and private banks can provide liquidity by issuing bank notes. In his model, buyers buy goods from sellers by issuing personal debt, buyers and sellers then go to a centralized location to settle the debt in fiat money. Sellers may have to leave before buyers arrive at the location, so sellers may have to sell their debt at low prices. Freeman showed that the social welfare can be improved if clearing house banks or central banks issue bank notes to buy that debt and then collect it when buyers arrive. This will also help to stabilize the asset price.

We generalize the results of Freeman. The function of banks is no longer limited to discounting existing private debt. Instead, banks lend money to borrowers who need cash now and can pay back the cash in the future. In addition, banks in our model are subject to a liquidity constraint due to the settlement requirement, while in Freeman’s model, there is no liquidity constraint for banks when they issue bank notes. Finally, the relationship between liquidity risk-sharing through coalition and liquidity provision through money creation is not analyzed in Freeman’s papers.

This chapter is organized as follows: Section 3.2 describes the environment; Section 3.3 characterizes the equilibrium; Section 3.4 derives numerical results for an example with a log utility function; Section 3.5 derives results for general utility functions, and also shows how the ability of banks to create money is important for the risk-sharing function of non-bank mutual funds; Section 3.6 concludes.
3.2 The environment

3.2.1 The basic events

We will first describe the basic environment, while the details of bank lending and settlement will be explained in section 3.2.2.

We consider an overlapping generations model with random relocation.\(^2\) Time is indexed by \(t = 1, 2, \ldots\). There are two locations in the economy. In each period, a new generation is born at each of the two locations. In each generation, there are three types of agents: “households”, “investment fund managers” and “bankers”. We normalize the measure of each type of agents to one. Each generation lives for two periods, and there is no population growth. The initial old generation of households in each location at time \(t = 1\) is endowed with outside money \(M\).

There is a single good per period. Agents care only about the consumption when they are old. Each household is endowed with \(e_h\) units of the good when young, and nothing when old. Young households thus save all their endowment. Households are risk-averse and they have the constant relative risk aversion (CRRA) utility function:

\[
U(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad \sigma \geq 1
\]  

Young investment fund managers can costlessly start new investment funds and young bankers can costlessly start new banks. There is free entry for investment funds and banks. Investment funds compete by offering the best contract to shareholders and banks compete by offering the best contract to depositors and borrowers.

Investment fund managers do not have endowments. And we assume away bankruptcy for banks using the following assumption: every banker has endowment \(e_b\) when old, which is sufficiently large to absorb the loss. Investment fund managers and bankers are risk neutral.

\(^2\)The basic random relocation setup follows Champ, Smith and Williamson (1996).
Consumption goods are non-storable but can be invested to produce new goods in the next period. Real risky investments can only be made by investment funds. The gross return rate for the risky project is

$$R_k = A$$

that is, one unit of consumption goods invested will turn into $A$ units of goods in the next period. $A$ is the aggregate productivity. There are two values of $A$: $A_H$ (high), $A_L$ (low) with equal probability. $A$ is i.i.d. in each period.

The main events are shown in Figure 3.1. The initial portfolio allocation is as follows. We assume that each young household can invest in at most one local investment fund and one local bank. Young households will sell part of their endowment to the old generation for money balance $M$. They will deposit money balance $aM$ ($0 \leq a \leq 1$) into banks. They then invest the remaining money balance $(1 - a)M$ and real goods $e_h - M$ into investment funds. Investment funds then use the real goods to make investments and deposit their money into banks. If $1 - a = 0$, then investment funds do not hold money. If $1 - a > 0$, then investment funds hold both real assets and bank deposits.

The actual number of households is higher than the number of investment funds and banks, and the actual number of investments is higher than the number of banks. This means each investment fund has many shareholders and each bank has many accounts of households and investment funds.
After real investments are made, we enter period $t + 1$. At the beginning of period $t + 1$, the productivity shock $A$ is publicly observed. The liquidity shock is also realized. A random fraction $\pi$ of the old households (denoted as “movers”) must move to the other location and consume there. $\pi$ is distributed over $[0, \bar{\pi}]$, where $\bar{\pi} < 1$ is the upper bound of the distribution. The distribution function is $F(\pi)$. $\pi$ is symmetric in the two locations. $\pi$ is independently and identically distributed, so each old household has the same *ex ante* probability to be a mover.

Movers cannot carry goods across locations. The value of bank deposits, however, can be verified across locations, so movers can use bank deposits to make payments. More specifically, when movers move from location $i$ to location $j$, they still keep their deposits in the banks in location $i$. And when they need to buy consumption goods in location $j$, they can pay using their deposits.

*The value of other assets cannot be verified across locations.* In particular, we assume that people cannot verify the value of investment fund shares, so investment fund shares are not accepted as means of payment across locations. We assume that movers must have their deposits ready in their banking account when they move to the other location. As a result, movers must redeem their investment fund shares into bank deposits before move.

In order to meet the redemption needs, investment funds may need to raise additional money by selling assets or by borrowing from banks. When assets are sold, only the ownership is transferred to the buyer, the production process is not stopped. The investment funds will collect the return and pay it to the buyer at the end of the period.

The redemption process is as follows: After shocks are realized, movers must send a withdrawal notice to their investment fund; Then the financial market opens. If the investment fund cannot meet the withdrawal needs with its own holdings of the riskless asset, then it can raise cash by selling assets to non-movers who have idle deposits. The transaction cost on the financial market is assumed to be zero. The fund can also choose to borrow from
banks. The fund then pays movers by transferring bank deposits to them. Movers receive the payment only after the transactions on the financial market are completed, so the cash received from the fund cannot be used to buy risky assets on the financial market.

After the redemption, movers move to the other location. At the end of $t+1$, risky projects are completed. Movers in each location use their bank deposits to buy consumption goods. Investment funds allocate returns to their shareholders and also repay the bank loan. Bankers consume the net income and old non-movers consume all their wealth.

There is no advantage in using currency in transactions, so we simply assume non-bank agents always use bank deposits to make payments. Although people can choose to withdraw their deposits, in the equilibrium there is no actual withdrawal of central bank currency.

Banks must settle the inter-bank balance with central bank money. Banks keep their reserves in the central bank deposit account, and the deposit rate paid by the central bank is normalized to zero. There is no official reserve requirement and banks can freely choose the reserve level. We assume only banks can borrow from the central bank, and the central bank is not allowed to use money to buy private risky assets.

### 3.2.2 The environment for bank lending and settlement

**The basic steps for bank lending**

In our model, the investment funds are the borrowers. When they borrow loan $L$ from banks, banks will credit their deposit account by $L$; this will increase the outstanding bank deposit by $L$. Investment funds then use their money to meet the withdrawal of movers. At the end of period $t+1$, investment funds sell their goods and then use bank deposit $L(1+r^t)$ to repay the bank loan, where $r^t$ is the lending rate. This will reduce the outstanding bank deposit by $L(1+r^t)$. Figure 3.2 illustrates the monetary flows. In the Appendix, we show the details of money flows using the balance sheet of banks.
Bank lending and settlement

We focus on the symmetric case and we assume that at the end of period $t$, depositors and investment funds are equally distributed among the banks. Also, the shareholders of each investment fund are equally distributed among all banks. There is no interest payment to depositors for holding deposits between the end of $t$ and the beginning of $t+1$.

At the beginning of period $t+1$, the productivity shock and the liquidity shock are realized, and then banks announce their deposit rate and lending rate. Banks are competitive and they take the market lending rate as given, and offer the highest deposit rate $r^d$ subject to the zero expected profit condition.

Each unit of bank loan incurs a management cost $\delta$ to the bank. Denote the net real lending rate as $r^l$, where $r^l \geq \delta$, and denote the gross lending rate as $R = 1 + r^l$.

An investment fund will borrow from the bank only when the borrowing cost is less than or equal to the cost for selling assets on the financial market. If the fund sells the asset, for each unit of asset with value $R_k$, the fund can get $Q_k$. If the fund borrows from the bank, for each unit of loan with future payment $R_k$, the fund can borrow $\frac{R_k}{1 + r^l}$. When bank loans are needed, we must have

$$Q_k = \frac{R_k}{1 + r^l}$$

(3.3)

that is, investment funds will borrow from banks only when the market price decreases to $\frac{R_k}{1 + r^l}$. Since $r^l \geq \delta$, so $R \geq 1 + \delta$ and we must have $Q_k \leq \frac{R_k}{1 + \delta}$ when bank loans are needed.
There is no transaction cost for non-movers to purchase assets. When investment funds need additional money, they will sell their assets to non-movers first. As long as non-movers’ cash is enough to absorb all the sales of assets, assets will be sold at their fundamental value \( Q_k = R_k \). When non-movers’ cash is not enough to absorb all the sales, then \( Q_k \) will be lower than \( R_k \). Once \( Q_k \) decreases to \( \frac{R_k}{1 + \delta} \), investment funds will start to borrow from banks. We use \( \pi_1 \) to denote the liquidity shock \( \pi \) at which the cash of non-movers is binding, and we use \( \pi_2 \) to denote the \( \pi \) at which investment funds start to borrow from banks.

When \( \pi \leq \pi_2 \), there is no need for bank loans. The deposit rate will be zero because there is no income for banks. (We assume there is no cost for managing the deposits and allowing the depositors to use the payment facility). Since there is no bank loan, the level of deposits is the same as the level of reserves. This means all deposits are backed by reserves, so banks can never run out of reserves during the settlement process because the maximum outflow of payment is equal to the level of deposits.

When \( \pi > \pi_2 \), investment funds need to borrow loans in addition to selling assets on the financial market. The main steps for lending and settlement are as follows (see Figure 3.3).

Let \( D_0 \) denote the deposit and reserve balance for each bank at the beginning of \( t + 1 \). After the shocks, the financial market opens, and the investment funds sell assets to non-movers. Non-movers will use all their own money holdings to buy assets because when bank loans are needed, the return for using money to buy assets, \( \frac{R_k}{Q_k} \), is equal to the lending rate \( 1 + r^l \), which is higher than the deposit rate. So after the transactions, non-movers transfer all their deposits to the investment funds. (This is “settlement 1” in Figure 3.3). Because all bank deposits are still backed by reserves, banks will not face any liquidity constraint in the settlement process. In the symmetric case, each fund sells the same amount of asset, and after the “settlement 1”, the deposit balance in each bank is still the same as the initial deposit \( D_0 \).

After all the payments by non-movers are completed, the financial market closes. We
then enter step 2 in which banks make loans to the investment funds. Investment funds will borrow from the banks they have the account with as long as the lending rate is not higher than the lending rate by other banks. If an investment fund borrows from bank $i$, it keeps the newly borrowed money with bank $i$ before making payments to movers.

When investment funds make payments to movers, we have another settlement, which is “settlement 2” in Figure 3.3. Since banks have created new deposits during lending, the payment may not be fully covered by initial reserves and banks may need to borrow from the central bank (the details will be explained later).

Movers can switch banks after receiving the payments; this will force banks to be competitive when offering deposit rates. In the symmetric case, all banks offer the same deposit rate and no mover will switch banks. As a result, in “settlement 2”, we need only to consider the inter-bank payments caused by investment funds paying their movers. After the redemption process is completed, movers move to the other location.

We ignore the liquidity constraint for banks during the transactions at the end of $t + 1$. We assume there is only one settlement based on net balance. In can be shown that after all transactions are completed, the deposits in each bank will be fully backed by reserves. This means the liquidity constraint will not be binding because the maximum payment by depositors is equal to the deposit level. As a result, we need only to consider the liquidity constraint in “settlement 2”.

Inter-bank payments in “settlement 2” are settled according to the “Real-Time Gross Settlement” method. More specifically, we assume that there are $N$ banks in the economy,
with \( N \) being a very large number. The redemption process will be separated into \( N \) subperiods. We normalize the total time length of the redemption process to 1. The time length of each subperiod is \( \frac{1}{N} \). In each subperiod, a bank will be randomly chosen to make payments to other banks, and the inter-bank balance is settled right away (i.e., the transfer of reserve happens right away). In the next subperiod, another bank is randomly chosen. During this process, any negative balance of reserve must be met by borrowing from the central bank.

There is no inter-bank loan market, and banks can borrow from the central bank when they need extra reserves. We also assume that the central bank consumes the interest by purchasing consumption goods, so the outside money is always restored to \( M \) at the end of each period.

### 3.3 The Equilibrium

#### 3.3.1 The portfolio choice

Young households optimally choose the share of wealth invested in bank deposits, which we denote as \( \omega \). The remaining wealth (share \( 1 - \omega \)) is invested in investment funds. Investment funds optimally choose the share of portfolio invested in bank deposits, which we denote as \( \alpha \). And the remaining \( 1 - \alpha \) of the portfolio is invested in risky assets. After shocks are realized, investment funds also optimally choose \( r_m \), the return rate paid to movers, and \( r_n \), the return rate paid the non-movers. \( \alpha, r_m \) and \( r_n \) should maximize the expected utility of shareholders.

Let \( v_m \) and \( v_n \) denote the value of the portfolio of movers and non-movers

\[
v_m = s \left[ \omega + (1 - \omega) r_m \right] (1 + r^d)
\]

\[
v_n = s \left[ \omega \frac{R_k}{Q_k} + (1 - \omega) r_n \right]
\]

\( s = e_h \) is the saving by each young household. For movers, \( s \omega \) is the initial bank deposit and
$s(1 - \omega)r_m$ is the deposit withdrawn from the investment fund. The deposit interest is paid at the end of the period.

For non-movers, the return for the initial deposit is $\frac{R_k}{Q_k}$. Non-movers can use part or all of their deposit to buy assets when investment funds sell assets. The market asset price is $Q_k$ and the future payment is $R_k$. When the deposit of non-movers is not binding, the market price will be equal to the fundamental price and we have $\frac{Q_k}{R_k} = 1$. If $Q_k < R_k$, then the return rate for buying assets is higher than 1. Non-movers are not affected by the deposit rate, as we will show later, whenever the deposit rate is positive, the return rate $\frac{R_k}{Q_k}$ is equal to the lending rate, which is higher than the deposit rate. So non-movers will use all their money to buy assets when $Q_k < R_k$.

The payout policy can be chosen after the shocks are realized. As a result, investment funds can choose the best $r_m$ contingent on the realized shocks. After the shocks are realized, investment funds try to maximize the expected welfare of shareholders

$$\pi \frac{(v_m)^{1-\sigma}}{1-\sigma} + (1-\pi) \frac{(v_n)^{1-\sigma}}{1-\sigma}$$

subject to its budget constraints and the constraint $r_m \leq r_n$ (payment to movers cannot be higher than non-movers, otherwise non-movers will pretend to be movers and withdraw.) We have the following result

**Proposition 5.** If $Q_k = R_k$, then it is optimal to set $v_m = v_n$ and $r_m = r_n = \alpha + (1-\alpha)R_k$.

When $Q_k < R_k$, if the constraint $r_m \leq r_n$ is not binding, then the optimal policy is to set

$$\frac{v_m}{v_n} = \frac{(\omega + (1 - \omega)r_m)(1 + r^d)}{\omega \frac{R_k}{Q_k} + (1 - \omega)r_n} = \left(\frac{Q_k(1 + r^d)}{R_k}\right)^{\frac{1}{\sigma}}$$

(3.7)

Given $\omega$, $Q_k$ and $r^d$, $\frac{r_m}{r_n}$ is increasing in $\sigma$. If the constraint $r_m \leq r_n$ is binding, then the optimal policy is $r_m = r_n$. For the log utility function ($\sigma = 1$), the optimal policy is $r_m = \alpha + (1 - \alpha)Q_k$ and $r_n = \alpha \frac{R_k}{Q_k} + (1 - \alpha)R_k$.

Proof: see the Appendix.
The intuition is as follows. When \( Q_k = R_k \), the investment fund can give movers and non-movers equal payment, and all shareholders will have the same consumption. When \( Q_k < R_k \), it is costly to raise cash, and the investment fund may not want to fully smooth shareholders’ consumption. When people are more risk averse (\( \sigma \) is higher), the fund will provide more consumption smoothing, and the value of \( \frac{r_m}{r_n} \) will tend to be higher. With the log utility function, the optimal \( r_m \) is equal to the the market value of the fund’s asset.

In the remaining part of this section and section 3.4, we consider the example of the log utility function. The purpose is to show more clearly the liquidity provision function of banks through money creation. The result for more general utility will be discussed in section 5, where we will discuss how different consumption smoothing due to different \( \sigma \) will affect the market’s volatility.

With the log utility function, we find that it is optimal to let the households hold all the riskless assets (\( \alpha = 0 \)). The basic reason is that when households hold more riskless assets by themselves, the money needed to be withdrawn is lower and the payment will be lower; as a result, banks are less likely to borrow from the central bank. The detailed explanation is provided in the Appendix. Since \( r_m = \alpha + (1 - \alpha)Q_k \) and \( \alpha = 0 \), we have \( r_m = Q_k \).

### 3.3.2 Bank’s problem

**The expected borrowing from the central bank**

Let \( L_i \) denote the total loan made by bank \( i \) and \( L_j = L \) denote the loan made by any of the other banks \( j \neq i \). When the loan is made, the deposit balance for bank \( i \) becomes \( D_0 + L_i \) and the deposit balance for all other banks \( j \neq i \) becomes \( D_0 + L_j \), where \( L_i \) and \( L_j \) are new deposits created during lending.

Then investment funds use all their deposits to pay the movers. We use \( X \) to denote the
payment made by each bank, then

$$X_i = (1 - \pi)D_0 + L_i$$  \hspace{1cm} (3.8)  
$$X_j = (1 - \pi)D_0 + L_j$$  \hspace{1cm} (3.9)

where \((1 - \pi)D_0\) is the deposits raised by investment funds by selling assets to non-movers. (The remaining deposits \(\pi D_0\) belong to movers.)

Recall that the settlement process is divided into \(N\) subperiods and in each subperiod a bank is chosen to make the payment to other banks. Since we assume there are \(N\) banks and the shareholders of each fund are evenly distributed among all banks, when bank \(i\) is chosen to make the payment, the payment made to the shareholders in the same bank is \(\frac{1}{N}X_i\), the payment made to each of the other \(N - 1\) banks is also \(\frac{1}{N}X_i\), and the total payment outflow is \(\frac{N-1}{N}X_i\). The pattern is symmetric for all other banks \(j \neq i\).

Let \(FL(k, n)\) denote the accumulated outflow of payments in subperiod \(k\) if a bank is chosen to make the payment to other banks in subperiod \(n\). Banks are required to borrow from the central bank as long as \(FL(k, n) > D_0\). Let \(b(k, n)\) denote the central bank loan.

$$b(k, n) = \max(0, FL(k, n) - D_0)$$  \hspace{1cm} (3.10)

Higher \(L_i\) will lead to higher payment \(X_i\), which will make \(b(k, n)\) more likely to be positive.

Suppose bank \(i\) makes the payment in period \(n\), we define the accumulated borrowing as

$$\overline{b}_n = \frac{1}{N} \Sigma_{k=1}^{N} b(k, n)$$  \hspace{1cm} (3.11)

The interest cost for the central bank loan is \(r^c \overline{b}_n\), where \(r^c\) is the central bank lending rate.

Before the settlement process starts, the expected future borrowing is defined as

$$Eb = \frac{1}{N} \Sigma_{n=1}^{N} \overline{b}_n$$  \hspace{1cm} (3.12)

When \(N\) is large, we can get a closed-form solution for \(Eb\); the result is as follows:
Proposition 6. Suppose $N$ is very large. Given $L_j$, when $Eb > 0$, it can be written as

$$Eb(L_i) = \frac{1}{6} \frac{(X_i - X_j - D_0)^3}{X^2_j} + \frac{1}{2} \frac{(X_i - X_j - D_0)^2}{X_j} + \frac{1}{2} (X_i - X_j - D_0) + \frac{1}{6} X_j$$

(3.13)

In the symmetric case ($L_i = L_j = L$), $Eb(L) > 0$ when $L > \pi D_0$ (when $X > D_0$).

Proof: See the Appendix ■

In the symmetric case, the payment $X$ is $(1 - \pi)D_0 + L$. If $L > \pi D_0$, then $X > D_0$ and the payments will not be fully covered by reserves.

Loan supply

After inter-bank payments are completed, bank $i$’s deposit balance is

$$\left(D_0 + L_i\right) - \frac{N - 1}{N} X_i + \frac{N - 1}{N} X_j \approx \left(D_0 + L_i\right) - X_i + X_j = D_0 + L_j$$

(3.14)

And the reserve balance is

$$D_0 - \frac{N - 1}{N} X_i + \frac{N - 1}{N} X_j \approx D_0 - (X_i - X_j) = D_0 - (L_i - L_j)$$

(3.15)

We assume that banks pay the interest of the central bank loan $r^c b_n$ at the end of the settlement process. For simplicity, we focus on the case in which the reserve balance after paying the interest, $D_0 - (L_i - L_j) - r^c b_n$, is positive. That is, $D_0$ is high enough to cover marginal increases in $L_i$ and the interest cost $r^c b_n$. (Since we focus on the symmetric case, $L_i - L_j$ means small marginal deviations of $L_i$ from $L_j$.) So banks do not need to borrow any central bank loan after the settlement process is completed.

\[\text{In order for this assumption to hold, we only need } D_0 \text{ to be higher than the maximum } r^c b_n. \text{ The bank which makes the payment in subperiod 1 will have the highest borrowing. In the symmetric case, the accumulated borrowing for the first bank is } (1 - \frac{D_0}{X})(-D_0) + \frac{1}{2} \left(\frac{D_0}{X}\right)^2 X < \frac{1}{2} X. \text{ So we need only } D_0 > \frac{r^c X}{2}. \text{ This condition can be easily met as long as } D_0 \text{ is not extremely low and the lending rate } r^c \text{ is very high.}\]
Bank profit is given by

$$\Pi = [D_0 - (L_i - L_j) - r^c b_n] + L_i(R - \delta) - (1 + r^d)(D_0 + L_j)$$  \hspace{1cm} (3.16)$$

The first term is the remaining reserves, the second term is the value of bank loan, and the third term is the gross payment to deposits, where the deposit level is given by (3.14).

The expected profit is

$$E\Pi = [D_0 - (L_i - L_j)] - r^c E b + L_i(R - \delta) - (1 + r^d)(D_0 + L_j)$$  \hspace{1cm} (3.17)$$

If the liquidity shock is low and no central bank loan is needed ($Eb = 0$), then (3.17) becomes

$$E\Pi = [D_0 - (L_i - L_j)] + L_i(R - \delta) - (1 + r^d)(D_0 + L_j)$$  \hspace{1cm} (3.18)$$

In the equilibrium, bank $i$ should not be able to increase the profit by changing $L_i$. The first order condition is

$$\frac{\partial E\Pi}{\partial L_i} = -1 + R - \delta = 0$$  \hspace{1cm} (3.19)$$

which gives the loan supply curve when $Eb = 0$

$$R = 1 + \delta$$  \hspace{1cm} (3.20)$$

In the symmetric case we have $L_i = L_j = L$, and the deposit rate can be computed by applying the zero expected profit condition ($E\Pi = 0$) to equation (3.18), the result is $r^d = 0$.

If $Eb > 0$, then the first order condition is

$$\frac{\partial E\Pi}{\partial L_i} = -1 - r^c \frac{\partial E b(L_i)}{\partial L_i} + (R - \delta) = 0$$  \hspace{1cm} (3.21)$$

Using (3.8) and (3.13), we have

$$\frac{\partial E b(L_i)}{\partial L_i} = \frac{1}{2} \left( 1 + \frac{X_i - X_j - D_0}{X_j} \right)^2$$  \hspace{1cm} (3.22)$$
and (3.21) becomes

$$R = 1 + \delta + r^c \frac{1}{2} \left( 1 + \frac{X_i - X_j - D_0}{X_j} \right)^2$$  \hspace{1cm} (3.23)$$

This is the loan supply curve of bank $i$ given $L_j$. In the symmetric case ($L_i = L_j = L$), the supply curve when $Eb > 0$ becomes:

$$R = 1 + \delta + r^c \frac{1}{2} \left( 1 - \frac{D_0}{X} \right)^2$$  \hspace{1cm} (3.24)$$

$$= 1 + \delta + r^c \frac{1}{2} \left( 1 - \frac{D_0}{(1 - \pi)D_0 + L} \right)^2$$  \hspace{1cm} (3.25)$$

$R$ is increasing in $L$. From (3.25), we can see that when $L > \pi D_0$, $\frac{D_0}{(1 - \pi)D_0 + L} < 1$ and so $R > 1 + \delta$.

### 3.3.3 The equilibrium lending rate and loan level

The demand curve for the bank loan can be derived from the payout policy of the investment fund. Since with the log utility function $\alpha = 0$, all assets of the investment fund are in the form of risky assets. Denote the level of risky assets as $Z_k$. Since investment funds pay movers the market value of the fund’s assets, the total payout is $\pi Z_k Q_k$. This should equal the cash collected from non-movers, $(1 - \pi)D_0$, plus the loan borrowed from the bank. Thus, we have

$$\pi Z_k Q_k(\pi) = (1 - \pi)D_0 + L(\pi)$$  \hspace{1cm} (3.26)$$

The market price for asset is

$$Q_k(\pi) = \frac{R_k}{R(\pi)}$$  \hspace{1cm} (3.27)$$

where $R(\pi)$ is the bank lending rate when the liquidity shock is $\pi$. Substitute $Q_k(\pi)$ into equation (3.26) and we have the demand curve for the bank loan

$$R(\pi) = \frac{\pi Z_k R_k}{(1 - \pi)D_0 + L(\pi)} = \frac{\pi Z_k R_k}{X}$$  \hspace{1cm} (3.28)$$
When $Eb = 0$, the loan supply curve is $R = 1 + \delta$, and $Q_k = \frac{R_k}{1+\delta}$. Using (3.26), the equilibrium loan level is

$$L = \pi Z_k Q_k - (1 - \pi) D_0$$

(3.29)

$$= \pi Z_k \frac{R_k}{1+\delta} - (1 - \pi) D_0 = S \left[ \pi (1 - \omega) \frac{R_k}{1+\delta} - (1 - \pi) \omega \right]$$

(3.30)

When $Eb > 0$, (3.25) and (3.28) give the following result:

**Proposition 7.** The equilibrium $L(\pi)$ and $R(\pi)$ are

$$L^*(\pi) = S \left( \frac{r^c \omega + \pi (1 - \omega) R_k + \sqrt{(r^c \omega + \pi (1 - \omega) R_k)^2 - 2(1 + \delta + \frac{r^c \omega^2}{2})}}{2(1 + \delta + \frac{r^c \omega^2}{2})} - (1 - \pi) \omega \right)$$

(3.31)

$$R^*(\pi) = \frac{2(1 + \delta + \frac{r^c \omega^2}{2}) \pi (1 - \omega) R_k}{r^c \omega + \pi (1 - \omega) R_k + \sqrt{(r^c \omega + \pi (1 - \omega) R_k)^2 - 2(1 + \delta + \frac{r^c \omega^2}{2})}}$$

(3.32)

Proof: see the Appendix.

Given the equilibrium $L(\pi)$ and $R(\pi)$, we can solve for $Eb(\pi)$ from equation (3.13) by setting $L_i = L_j = L^*(\pi)$, and then solve for the equilibrium deposit rate $r^d(\pi)$ from equation (3.17) by setting $E\Pi = 0$. The result is as follows:

**Proposition 8.** Suppose $N$ is large, when $Eb \geq 0$, the expected central bank loan is

$$Eb = \frac{-D_0^3}{6X^2} + \frac{D_0^2}{2X} - \frac{D_0}{2} + \frac{X}{6}$$

(3.33)

and the equilibrium deposit rate is

$$r^d(\pi) = \frac{Lr^c}{2} \left( 1 - \frac{D_0}{X} \right)^2 - r^c Eb = \frac{Lr^c}{2} \left( 1 - \frac{D_0}{X} \right)^2 - r^c \left( \frac{D_0^3}{6X^2} + \frac{D_0^2}{2X} - \frac{D_0}{2} + \frac{X}{6} \right)$$

(3.34)

where $X = (1 - \pi) D_0 + L$, and $L = L^*(\pi)$ is the equilibrium loan level.

Once the equilibrium loan level is decided, then the aggregate deposit level in the economy is also decided. The aggregate deposit (i.e., money supply) is $D_0 + L(\pi)$, where $D_0$ is the

---

4We replace $R$ in (3.17) using the loan supply curve (3.25).
initial deposit balance, and \( L(\pi) \) is the deposits that are created by banks during lending. Note that the aggregate bank loan level is elastic and is not limited by the monetary funds saved by depositors. Instead, it is the lending activities of banks that decide the aggregate deposits in the economy.

**Proposition 9.** The distribution of \( Q_k \) is

\[
Q_k(\pi) = \begin{cases} 
R_k & : \pi \leq \pi_1 = \frac{\omega}{\omega + (1-\omega)R_k} \\
\frac{\omega(1-\pi)}{\pi(1-\omega)} & : \pi_1 < \pi \leq \pi_2 = \frac{\omega}{\omega + (1-\omega)R_k} + \frac{\omega}{\pi(1-\omega)} \\
\frac{R_k}{1+\delta} & : \pi_2 \leq \pi \leq \pi_3 = \frac{\omega}{1-\omega}R_k \\
\frac{R_k}{\pi(1-\omega)} & : \pi > \pi_3 \end{cases} \tag{3.35}
\]

Proof: See the Appendix.

### 3.3.4 The first order condition for \( \omega \)

Using \( v_m \) and \( v_n \) from (3.4) and (3.5), the expected utility for a representative household \( i \) is

\[
EU_i = \frac{1}{2} \int_0^1 \left\{ \pi \ln \left[ (1 + r_d)s(\omega_i + (1 - \omega_i)Q_{k,H}) \right] + (1 - \pi) \ln \left[ \frac{R_{k,H}}{Q_{k,H}}(\omega_i + (1 - \omega_i)Q_{k,H}) \right] \right\} dF(\pi) + \frac{1}{2} \int_0^1 \left\{ \pi \ln \left[ (1 + r_d)s(\omega_i + (1 - \omega_i)Q_{k,L}) \right] + (1 - \pi) \ln \left[ \frac{R_{k,L}}{Q_{k,L}}(\omega_i + (1 - \omega_i)Q_{k,L}) \right] \right\} dF(\pi) \tag{3.36}
\]

The first order condition for \( \omega_i \) is

\[
\frac{\partial EU_i}{\partial \omega_i} = \frac{1}{2} \int_0^1 \frac{1 - Q_{k,H}}{\omega_i + (1 - \omega_i)Q_{k,H}} + \frac{1 - Q_{k,L}}{\omega_i + (1 - \omega_i)Q_{k,L}} dF(\pi) = 0 \tag{3.37}
\]

### 3.3.5 The equilibrium

The equilibrium of the model can be defined as follows. Given the initial portfolio choice in \( t \), after the shocks are realized in \( t + 1 \), investment funds optimally choose the payout policy. Banks optimally choose the lending rate to maximize the profit; they also choose the deposit
rate subject to the zero expected-profit condition. The asset market clears, the assets sold by investment funds are equal to the assets purchased by non-movers. The bank loan market clears. The loan borrowed by investment funds is equal to the loan lent by banks.

At the end of period $t + 1$, the goods market clears. The consumption of movers is $v_m$, and the consumption of non-movers is $v_n$. The aggregate consumption is equal to the total goods produced by investment funds $e_h(1 - \omega)R_k$ plus the goods $\frac{M}{P}$ purchased from the next young generation.

Given the expected outcome in period $t + 1$, at the end of period $t$, households and investment funds optimally choose their portfolios. With the log utility function, all riskless assets are held by households, the value of riskless assets is equal to the real balance of outside money:

\[
\frac{M}{P} = \omega e_h \tag{3.38}
\]

This equation determines the equilibrium price level. In the stationary equilibrium, $P$ is the same in each period.

### 3.3.6 When bank lending is exogenously shut-down

If there is no bank lending, then the cash in the economy will be limited to the outside money held by households. The market price will be lower than the fundamental price when non-movers’ cash is binding. In the Appendix, we show that with the log utility function, when there is no bank lending, we can still assume that all initial cash is held by households. The distribution of $Q_k$ is

\[
Q_k(\pi) = \begin{cases} 
R_k &: \pi \leq \pi_1 \text{ (Non-movers' cash is not binding)} \\
\frac{\omega(1-\pi)}{\pi(1-\omega)} &: \pi > \pi_1 \text{ (Non-movers' cash is binding)}
\end{cases} \tag{3.39}
\]

which is simply (3.35) without bank lending.
3.4 Numerical Results: $U = \ln c$

This section considers an example with $U = \ln c$. The purpose is to restrict $\sigma$ at 1 and then focus on liquidity provision by banks through money creation. (The effects of different $\sigma$ on consumption smoothing will be added in section 3.5.)

3.4.1 Parameter values

Table 3.1 shows the parameter values. The return of the risky assets are $R_{k,H} = A_H$ and $R_{k,L} = A_L$. We set the household endowment $e_h$ at 1, the loan management cost of banks $\delta$ at 3% and the central bank lending rate $r^c$ at 3%.

<table>
<thead>
<tr>
<th>$A_H$</th>
<th>$A_L$</th>
<th>$e_h$</th>
<th>$\delta$</th>
<th>$r^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.21</td>
<td>0.85</td>
<td>1</td>
<td>0.03</td>
<td>0.03</td>
</tr>
</tbody>
</table>

We assume that the liquidity shock is distributed according to

$$\pi = 0.9\theta^\varphi$$

where $\theta$ is uniform over $[0, 1]$. $\varphi$ is used to adjust the density of $\pi$. With higher $\varphi$, the density of $\pi$ will be more concentrated on low values and households will hold lower monetary balance. We use $\varphi = 6$. At this level, $D_0$ is low enough and we can see clearly the effects when banks borrow from the central bank.

3.4.2 Results

Table 3.2 shows $\omega^*$ and the expected utility of households. We show four cases. Case 1 is when all assets (including mutual fund shares) can be used to buy goods across locations. In this case, movers do not need to withdraw from investment funds. It can be shown that we will have a standard portfolio choice problem with one riskless and one risky asset. The riskless
Table 3.2: Numerical example

<table>
<thead>
<tr>
<th>Scenario</th>
<th>( \omega^* )</th>
<th>( E \ln c \text{(households)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) All assets can be used to make payments</td>
<td>0.0476</td>
<td>0.0141</td>
</tr>
<tr>
<td>(2) Fixed money</td>
<td>0.7013</td>
<td>0.0059</td>
</tr>
<tr>
<td>(3) With elastic inside money</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3a) Central bank sets ( r^c = 0.03 ) for all levels of ( \pi )</td>
<td>0.2460</td>
<td>0.0118989</td>
</tr>
<tr>
<td>(3b) CB sets ( r^c = 0.02 ) for ( \pi \geq 0.5 ). The policy is pre-announced</td>
<td>0.2438</td>
<td>0.0119093</td>
</tr>
</tbody>
</table>

Figure 3.4: \( Q_k \) in case 2: no bank loan.

Since the aggregate household wealth is set at 1, \( \omega^* \) is also equal to the aggregate real money balance \( \frac{M}{\pi} = D_0 \).

Table 3.2 shows that when money is more elastic, households will hold less money and enjoy higher utility.

\( \omega \) is slightly lower in 3b than in 3a because the lower interest rate will cause a lower bank lending rate and higher asset prices. The difference in \( \omega \) in these two cases is small because the density for \( \pi \geq 0.5 \) in our example is small.

Figure 3.4 shows the market price of risky assets in case 2 (equation 3.39) when the bank loan is shut down.
Figure 3.5: $Q_k$, $L$, $R$, $r^d$ and $Eb$ in case 3a (elastic money). $A = A_H$ is shown.

(a) Asset price $Q_k$
(b) Loan and money supply
(c) Bank lending and deposit rate
(d) Expected central bank loan

Figure 3.6: Compare $Q_k$ in cases 1(no inside money) and 3a(with inside money). $A = A_H$

Figure 3.5 shows the results for case 3a when bank lending is allowed. The results for $A = A_H$ are shown here (The results for $A = A_L$ are similar). $Q_k$ is decided according to equation (3.35). We can see that the aggregate money supply $D_0 + L(\pi)$ is stochastic. Higher
liquidity shock will cause investment funds to borrow more, which will in turn lead to higher money supply.

Figure 3.6 compares asset prices in case 2 (bank lending is shut down) and case 3a (bank lending is allowed) using the same scale. We can see that the asset price is more stable with elastic inside money.

Figure 3.7 compares case 3a, where the central bank lending rate $r^c$ is fixed, and case 3b, where $r^c$ is lower for $\pi > 0.5$. In case 3b, the lower central bank lending rate will lead to a lower bank lending rate, lower bank deposit rate, and higher asset price.

3.4.3 The transmission of central bank interest rate policy

The transmission of the interest rate policy is summarized in Figure 3.8. When the central bank adjusts the borrowing rate, it will affect the expected borrowing cost of banks, which will in turn affect the equilibrium bank lending rate and loan level. When the central bank sets a lower lending rate, the equilibrium lending rate will be lower and the loan level will be higher. The higher bank loan level will create a higher demand for settlement balances. Since the central bank meets the demand for settlement balances by providing loans to banks, the supply of reserve is thus endogenously decided by the demand.

Note that the aggregate money supply is not directly controlled by the central bank. Since the changes in money supply are simply the changes in bank deposits, the money supply is
The central bank reduces the interest rate.

Lower expected borrowing cost.

Expansion of bank credit.

Increased demand for reserves

Increased lending of reserves

Higher asset price

Figure 3.8: The transmission of central bank interest rate policy

decided by bank loan supply. Higher liquidity shock will cause higher demand for bank loans, which will in turn lead to higher money supply.

3.4.4 Implication

Our results have an important implication for the elasticity of liquidity. When liquidity needs take the form of demand for means of payment, it can be met with the creation of new inside money. The existing outside money can no longer measure the availability of liquidity, and the supply of liquidity is no longer fixed by liquid assets in the initial portfolio. Compared to models in which liquidity takes the form of consumption goods, the supply of liquidity here is more elastic. Thus, non-monetary models may underestimate the ability of the financial system to meet the increases in the demand for liquidity.

3.5 The relationship between liquidity-risk sharing through coalitions and bank money creation

Our model includes two types of liquidity provision mechanisms: liquidity risk-sharing through mutual funds and liquidity provision through money creation. Liquidity risk-sharing means early withdrawers of liquidity will get a higher payment than if there were no mutual funds. In the previous analysis, we use the log utility case to illustrate the second mechanism (liquidity provision through money creation). But with log utility, the first mechanism (liquidity risk-sharing) is not actually used by households. This is because when \( U = \ln c \), the payment
\( r_m \) is equal to the market price of the fund’s asset. If investment funds allocate all assets equally to shareholders and then movers sell assets directly to non-movers, then movers will get the same payment.

If \( \sigma > 1 \), \( r_m \) will be set above the market price of the fund’s asset. That is, given the market price \( Q_k \), movers will get more than what they would get if they sold the assets by themselves. In this case, investment funds perform risk-sharing (i.e., consumption smoothing) for shareholders.

We ask the following questions:

1. Can risk-sharing investment funds help movers to achieve higher consumption? How does the liquidity provision function of banks affect the liquidity-risk sharing function of investment funds?

2. How will the risk-sharing function of coalitions affect the volatility of assets on the market?

We show that if bank lending is exogenously shut down, since the aggregate money is limited to the money held by people in their initial portfolio, investment funds will not actually help movers to achieve higher consumption; the liquidity-risk-sharing function is useless. The attempt of investment funds to provide higher payment than the market value of the fund’s asset will cause only lower market prices and so add more volatility into the market. But with banks supplying elastic aggregate money, investment funds can indeed help movers to achieve higher consumption.

We will first illustrate the effects of \( \sigma \) on asset prices by assuming that the initial portfolios for different \( \sigma \) are the same. This assumption allows us to isolate the effects of different \( \sigma \) on risk-sharing. We will then discuss the outcome when people can choose different portfolios for different \( \sigma \).
3.5.1 When $\omega$ and $\alpha$ are exogenously fixed

We first consider the no-bank-lending case. We define $\kappa$ as the total share of bank deposit in a household’s portfolio (i.e., both the deposit held directly by the household and the deposit held indirectly through the investment fund).

$$\kappa = \omega + (1 - \omega)\alpha$$

(3.41)

We first show two basic results.

1. If there is no bank lending, then once total investment in the riskless asset, $\kappa$, is decided, the consumption level of movers and non-movers are determined. The consumption level is not affected by risk-sharing provided by investment funds.

2. Once the initial distribution of money among investment funds and households is given, then the distribution of $r_m$ is decided. $r_m$ does not depend on the risk-sharing policy, but risk-sharing will cause higher volatility in asset prices.

These two results are proved in the following two propositions. The results are then illustrated using a numerical example.

**Proposition 10.** In the symmetric equilibrium, the distribution of $v_m$ and $v_n$ only depends on $\kappa$. Suppose $e_h = 1$, when $\pi \leq \pi_1$, $v_m = v_n = \kappa + (1 - \kappa)R_k$. When $\pi > \pi_1$, $v_m = \frac{\kappa}{\pi}$ and $v_n = \frac{(1-\kappa)R_k}{1-\pi}$. $\pi_1 = \frac{\kappa}{\kappa + (1-\kappa)R_k}$.

Proof: See the Appendix.

The intuition is as follows. When we set $e_h = 1$, the total wealth of the economy can be written as $\kappa + (1 - \kappa)R_k$. When the aggregate cash constraint is not binding, everyone gets $\kappa + (1 - \kappa)R_k$. Once the cash constraint is binding, movers can only carry all the cash in the economy $\kappa$ with them to the other location, since the size of movers is $\pi$, so the consumption of every mover is $\frac{\kappa}{\pi}$. The aggregate consumption of non-movers will be the remaining risky
assets \((1 - \kappa)R_k\), and every non-mover consumes \(\frac{(1-\kappa)R_k}{1-\pi}\). At \(\pi_1\), each mover still receives the average wealth \(\kappa + (1 - \kappa)R_k\), so \(\pi_1\) is determined by the ratio between the aggregate cash \(\kappa\) and the aggregate wealth \(\kappa + (1 - \kappa)R_k\).

We can see that given \(\kappa\), the consumption level of movers is independent of the payout policy of the investment fund.

**Proposition 11.** In the symmetric case, if we fix the initial portfolio choice \(\omega\) and \(\alpha\), then the distribution of \(r_m\) will be the same for different \(\sigma\). But the distribution of \(Q_k\) will be different. Between \(\pi_1\) and \(\pi_{\text{bind}}\) (the \(\pi\) above which the constraint \(r_m \leq r_n\) is binding), \(Q_k\) is lower for higher \(\sigma\).

Proof: see the Appendix.

The intuition is as follows. Once the aggregate liquidity is decided by \(\omega\) and \(\alpha\), then the distribution of \(r_m\) will be uniquely decided. For \(\pi > \pi_1\), when \(\sigma\) is higher, the investment fund would like to set a higher \(r_m\) given the market value of the fund’s asset. But since the payment to movers is limited by the liquidity in the economy, it is the price \(Q_k\) (and the market value of the fund’s asset) that must adjust in order for the optimal payout policy to be satisfied. If \(\sigma\) is higher, then \(Q_k\) must decrease more in order to satisfy the optimal payout policy. On the microeconomic level, each investment fund takes the price \(Q_k\) as given and tries to liquidate assets to raise cash in order to provide liquidity insurance to movers. But if the aggregate liquidity is limited, then the effort of investment funds is self-defeating, it will lead to more volatile asset prices, without actually providing more liquidity to movers.

The results of proposition 11 are illustrated in Figure 3.9, where we use the example \([\omega = 0.4, \alpha = 0]\) (we set a low \(\omega\) so we can see clearly what will happen when \(Q_k\) goes to low levels.)

Figure 3.9 shows that given the initial portfolio, the actual payment to movers \(r_m\) will not be affected by \(\sigma\). In addition, when \(\sigma\) is higher, since investment funds try to liquidate more
assets, this will cause the asset price $Q_k$ to decrease more quickly (figure (b)). Since liquidations of assets are costly, this will also cause $r_n$, the payment to the remaining shareholders, to decrease more quickly (figure (a)).

In addition, once the constraint $r_m \leq r_n$ is binding, then $r_m$ is set equal to $r_n$, and the liquidation will no longer depend on $\sigma$, but is determined by the constraint $r_m = r_n$. As a result, once the constraint $r_m \leq r_n$ is binding, $Q_k$ will be the same for different $\sigma$. When $r_m \leq r_n$ is binding, given $Q_k$, the payment to movers will be lower than the optimal payment if $r_m$ were allowed to be higher than $r_n$. The lower payment to movers will reduce the need to liquidate assets. In this case, limiting the payment to movers does not really reduce the actual payment $r_m$ received by movers, but it helps stabilize the asset price.

**With bank lending**

With banks supplying elastic aggregate money, the aggregate liquidity is no longer limited by the existing cash on the market. In this case, investment funds can help people to smooth their consumption.

The results are shown in Figure 3.10, where the initial portfolio is fixed at $[\omega = 0.2460, \alpha = \ldots$
The findings can be summarized as follows:

1. \( r_m, r_n \) and the asset price \( Q_k \) are more stable compared to the no-bank-lending case. (Note that the scale for \( Q_k \) is smaller than that in Figure 3.10).

2. The levels of \( r_m \) are no longer the same for different \( \sigma \). When \( \sigma \) is higher, investment funds can provide higher \( r_m \) to movers.

The key figure is 3.10(d). Recall that \( r_m \) under the log utility function is the market price of the fund’s asset. This is also the payment that movers would get if all investment funds allocated their assets to shareholders and shareholders traded assets by themselves. That

---

It can be shown that this is the optimal equilibrium for \( \sigma = 1 \) when people can freely choose the portfolio.
is, if $\sigma > 1$ but there is no risk-sharing, movers would get $r_m(\sigma = 1)$ for the investments allocated into investment funds. But here, we can see that when $\sigma > 1$, the payment to movers is higher than $r_m(\sigma = 1)$, which means movers achieve higher consumption than if there were no mutual funds to provide risk-sharing.

### 3.5.2 When $\omega$ and $\alpha$ are chosen freely

We have the following results for the equilibrium portfolio choice

**Proposition 12.** People will choose $\alpha$ and $\omega$ such that $r_m \leq r_n$ is not binding. $r_m \leq r_n$ is more likely to be binding when $\alpha$ is lower. When $\alpha$ is high enough and $r_m \leq r_n$ is not binding, the maximization problem of both households and investment funds can be written as problems for choosing the optimal $\kappa$, and the response curves of households and investment funds will overlap with each other. The equilibrium portfolio will be defined by $\kappa$.

When there is no bank lending, there are many combinations of $\alpha$ and $\omega$ that will give the same equilibrium $\kappa$. When there is bank lending, the equilibrium with the highest $\omega$ is the most efficient equilibrium. For $U = \ln c$, the constraint $r_m \leq r_n$ is never binding, and it is most efficient for households to hold all the riskless assets.

Proof: see the Appendix.

Some of the intuitions are as follows. When there is bank lending, then it is better to let households hold more riskless assets because this can reduce the payment flow from investment funds to movers, and so banks are less likely to borrow from the central bank. We have proved in proposition 5 that when $U = \ln c$, we have $\frac{r_n}{r_m} = \frac{R_k}{Q_k} \geq 1$, so $r_m \leq r_n$ can never be binding. In this case, it is best to choose $\alpha = 0$ and $\omega = \kappa$.

We find that the basic results will not change when $\omega$ and $\alpha$ are chosen freely. For

---

6 The response curve of households is defined by the optimal $\omega$ chosen by households given the $\alpha$ chosen by investment funds. The response curve of investment funds is defined by the optimal $\alpha$ chosen by investment funds given the $\omega$ chosen by households.
example, when there is no bank lending, the distribution of consumption is determined by $\kappa$ and investment funds will not actually help people to smooth consumption. Once the cash constraint is binding, liquidations will cause only volatile asset prices. When there is bank lending, then asset price will be more stable and investment funds can help people better smooth consumption.

The only difference is that when portfolios are chosen freely, the equilibrium $\kappa$ tends to higher for higher $\sigma$. This is because when $\sigma$ is higher, once the cash is binding, investment funds will try to liquidate more assets (or to take more loans), which will lead to a lower asset price. As a result, people will hold fewer risky assets and more riskless assets in the initial portfolio. The detailed numerical results are omitted.

### 3.5.3 Summary and comparison with partial equilibrium models

Our results show that the ability of banks to relax the aggregate liquidity constraint is important for non-bank investment funds to provide insurance for aggregate liquidity risks. Without banks, investment funds cannot actually provide more liquidity to people who need liquidity early, and risk-sharing will only make asset prices more volatile. But with banks supplying elastic aggregate money, people who need liquidity early can indeed achieve higher consumption through the risk-sharing function provided by investment funds.

Note that we get the above results because we use a general equilibrium model to endogenously decide the asset price. Many partial equilibrium models usually assume that the liquidation price is exogenously given as a constant proportion of the fundamental value; an example is $Q_k = \lambda_q R_k$, where $\lambda_q < 1$ is a constant. These models essentially assume that risk-sharing coalitions can sell unlimited amounts of assets at the same price $\lambda_q R_k$. While in our model, if there is no bank lending, then once the aggregate liquidity is binding, sales of assets will not raise additional liquidity for investment funds, they will cause only lower asset prices. Our model shows more clearly why the function of banks to expand aggregate
liquidity is important to the financial market.

3.6 Conclusion

This chapter studied the role of banks in providing liquidity through inside money creation. We showed that the ability of banks to expand aggregate liquidity is important to financial stability.

We showed that elastic inside money can help to maintain stable asset prices. We also showed what happens when people use non-bank coalitions to share aggregate liquidity risks: If there is no elastic aggregate liquidity, then coalitions cannot actually provide higher consumption to shareholders who need liquidity early, and the risk-sharing function is useless and will add only more volatility to asset prices. But with banks supplying elastic aggregate liquidity, risk-sharing coalitions can help people to better smooth consumption.
Chapter 4

Elastic money, inflation and interest rate policy

4.1 Introduction

In this chapter, we consider monetary policy in an environment in which aggregate liquidity shocks affect individual agents differentially and exchange may be conducted using either bank deposits (inside money) or fiat currency (outside money). In our economy, a central bank conducts a monetary policy with two components: It controls the issuance of inside money by private banks by managing the interest rate and it sets the trend inflation rate by controlling the quantity of outside money. We find both components of the central bank’s policy to be useful for improving welfare. The effectiveness of interest rate policy depends on the trend inflation rate. Moreover, the optimal combination of interest rate management and trend inflation depends on the sensitivity of prices to the interest rate through sellers’ behaviour. In particular, the more sensitive is the supply price to the interest rate, the higher the optimal rate of trend inflation.

In models in which money plays an explicit role as the medium of exchange, monetary policy is typically modelled as direct control of the supply of fiat money. This type of analysis is not well suited for the study of either interest rate management policies such as those followed by actual central banks or the role played by private financial institutions in expanding
and contracting the supply of the medium of exchange. In a recent paper, Berentsen and Monnet (2006) model monetary policy implemented through a “channel” system in a version of the environment introduced by Lagos and Wright (2005). Our work extends theirs in two dimensions. First, we consider the role of private institutions which provide an elastic supply of money for exchange purposes in the transmission of monetary policy. Second, we consider jointly the roles of short-run interest rate policy and the long-run inflation rate in a setting where aggregate shocks affect different groups of agents asymmetrically.

Our model is an extension of that of Berentsen, Camera, and Waller (2005) in which anonymous agents have access to a number of identical private institutions (which we refer to as banks) which can both accept deposits and make loans. These private institutions can hold reserves in units of outside money at the central bank. Liquidity needs are random and the banking system can, at a cost, supply money through loans in excess of its collected deposits to individuals. By setting the rate at which it pays interest on reserves, the central bank can affect the loan rate charged by private banks in equilibrium. Changes to the interest rate, thus induce changes in the supply of inside money. These changes may be used to raise and lower output and consumption in response to fluctuations in both buyers’ marginal utilities and producers’ marginal costs. The associated fluctuations in the both the aggregate money stock and price level are, however, transitory. Through transfers the central bank determines the growth rate of the monetary base and the long-run rate of inflation.

We consider a stationary equilibrium in which aggregate shocks affect agents differently depending on whether they have access to the loan market. Each period a fraction of agents fall into the latter category—they learn they will die immediately after goods trading and therefore cannot borrow. The inability of agents to perfectly insure against these shocks provides a potential role for monetary policy to improve welfare. Interest rate movements affect prices as discussed above and thus redistribute wealth among households by changing the value of existing holdings of outside money.
The ability of the central bank to provide insurance through its interest rate policy is limited, however, by the zero bound on the nominal deposit rate. The central bank can minimize the effects of this bound by targeting a sufficiently high inflation rate and paying interest on reserves at a rate significantly above zero on average. Thus, a higher rate of inflation in the long-run tends to improve the effectiveness of the interest rate policy which is aimed at correcting short-run distortions. High inflation, by raising the price level is, however, costly for those agents who cannot make use of loans to support consumption.

There is then a complex relationship between the two components of the central bank’s monetary policy. High trend inflation is bad for the usual reason, the inflation tax. At the same time, however, it enables the central bank to run its interest rate management policy relatively effectively. That is, with a reduced probability of hitting the zero bound. The optimal mix of these short-run and long-run components of monetary policy depends on the responsiveness of prices to interest rate changes. In our model this depends on the extent of bank profits. If increases in interest rates to a large extent result in increases in bank profits, then prices are relatively unresponsive to interest rate changes and the central bank will prefer a lower long-run inflation rate and a higher probability of a binding zero bound. When bank profits are small and increases in rates are largely passed through to depositors, the implications for policy are the opposite.

As noted above, our work is related to that of Berentsen, Camera and Waller (2005), who extend Lagos and Wright (2005) to include lending and borrowing of idle cash in a credit market which they interpret as a banking system. In their model banks effectively observe a 100% reserve requirement as they can only lend out central bank issued currency collected from depositors. In our model borrowers have the ability to commit to repay loans to banks and as a result the banking system can create credit money in response to aggregate shocks.

Berentsen and Waller (2005) consider optimal stabilization policy in response to aggregate shocks in a model in which the central bank supplies an elastic currency and commits to a
long-run inflation rate. Our analysis differs from theirs in that we consider both the need for policy to address the asymmetric effects of shocks on consumers and the role of bank profits in determining the responsiveness of prices to interest rate movements. Our results differ principally with regard to the predictions of our model for the optimal long-run rate of inflation. Similarly, our analysis differs from that of Berentsen and Monnet (2006) also because we consider a private banking system and the distributional effects of both shocks and interest rate policies.

Sun (2006, 2007) studies private banking and the co-existence of both inside and outside money. Her analysis, however, focuses on the incentive problems associated with the issuance of inside money and the role of outside money produced by a central monetary authority in alleviating these problems. In our analysis, we abstract from all these issues by assuming that banks can commit to settling their obligations.

The remainder of the chapter is organized as follows. Section 4.2 describes the environment. Section 4.3 analyzes the optimal choices. Section 4.4 defines the stationary equilibrium. Section 4.5 provides an example in which the interest rate of the central bank is set at zero, which is equivalent to the case in which the central bank conducts no interest rate policy. Section 4.6 analyzes the optimal short-run interest rate management policy. Optimal trend inflation is considered in Section 4.7. Section 4.8 summarizes the results and describes future work.

### 4.2 The environment

Time is discrete and is indexed by \( t = 1, 2, \ldots \) etc. At each point in time there exists a unit measure of *ex ante* identical households of stochastic life-span. Each period fraction \( \lambda \) of these households *dies* and is replaced by an equal number of newly born agents who inherit the asset holdings of those whom they replace. The assets available in the economy
are described below.

Following Lagos and Wright (2005) each time period is divided into two distinct and consecutive sub-periods. Sub-periods differ with regard to both the goods that are available for consumption and the way in which agents interact. All goods are non-storable both between sub-periods and periods of time.

In the first sub-period of each period, households are divided by the realization of a shock. Fraction $\alpha + \lambda$ become consumers meaning that they have a desire to consume a good that can be produced during that sub-period of time. Of this group of consumers, fraction $\lambda / (\alpha + \lambda)$ become old. Old agents learn that they will die at the end of this sub-period and become recognizable as such to other agents. Consumers of both types, old and young (i.e. those who will not die this period) have preferences for the good produced in this sub-period given by

$$
\begin{align*}
\text{young agents:} & \quad u_y(q_y) = Au(q_y) \\
\text{old agents:} & \quad u_o(q_o) = Bu(q_o)
\end{align*}
$$

(4.1)

where $u(\cdot)$ satisfies $u'(q) > 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$, and $q_y$ and $q_o$ denote consumption by young and old households, respectively. $A$ and $B$ are preference shocks with affect all consumers of a particular type symmetrically and have compact support with $A, B \geq 0$. The cumulative distribution functions and densities for $A$ and $B$ are given by $F(A)$, $f(A)$ and $G(B)$, $g(B)$, respectively. For simplicity of exposition, for most of the analysis we assume that $A$ and $B$ are perfectly correlated and write $B(A)$.

Agents which are neither young or old consumers (fraction $1 - \alpha - \lambda$) become producers. That is, they acquire the ability to produce the available good at disutility $c(q)$ where $c'(q) > 0$ and $c''(q) > 0$. Producers derive no utility from consumption during this sub-period of time.

At the beginning of the second sub-period, $\lambda$ new agents are born to replace the old agents who died at the end of the previous sub-period. During this sub-period all agents can both consume and produce. All agents, regardless of whether they were consumers, producers, or
not yet born in the previous sub-period have preferences given by

\[ U(x) - h \]  \hspace{1cm} (4.2)\]

where \( x \) is the quantity of sub-period two good consumed, \( U'(0) = \infty, U'(\infty) = 0 \) and \( U''(x) \leq 0 \). The technology for producing this good is linear and one unit of the good can be produced with one unit of labour. Here \( h \) denotes the quantity of labour used to produce the good in the current period. Linear disutility of work plays the same role here as it does in Lagos and Wright (2005).

In both sub-periods exchange takes place in a centralized Walrasian market. In the first period all agents are anonymous (although their types are observable). In particular, buyers cannot be individually recognized by sellers and as a result cannot credibly commit to repay credit extended to them for the purchase of goods or any other purpose. In the second sub-period, agents are not anonymous and credit is in principle feasible.

In order to facilitate production and exchange in the first sub-period of period \( t \), there exists in the economy a stock of fiat money. This money is produced and circulated by a monetary authority which we will refer to as the central bank. Let \( M_t \) denote the quantity of fiat money in existence at the end of the second sub-period of period \( t - 1 \). Agents wishing to consume in the first sub-period of the subsequent time period \( (t + 1) \) may acquire some of this money by producing and selling goods in the second sub-period of this time period.

In addition to the central bank, there exist in the economy a large fixed number, \( N \), of private institutions which we will refer to as banks. Banks are owned by households and act so as to maximize the present value of their dividends, which are paid each period during the second sub-period. Banks have a technology which enables them to recognize individuals and for this reason may feasibly take deposits and make loans. While agents are anonymous in the first sub-period, through use of this technology they may effectively commit to repay in the second sub-period a loan taken out in the first.

For every dollar of loan lent out in sub-period 1, the management cost is \( \delta \) dollars of
second sub-period goods. Having paid this management fee, banks are able recognize in the second sub-period agents with whom they have contracted during the first. Loans can be repaid either with second sub-period goods or money, including bank money. We assume that the bank has access to a legal mechanism through which it can force its debtors to repay.

The institutions that we refer to as banks function much as the credit market in Berentsen, Camera, and Waller (2005). The key difference here is these institutions have the ability to make loans in excess of their deposits. They will do this by effectively creating deposits which function like a checking account. We will refer to these deposits as *inside money*. This is distinguished from the currency created by the central bank which we refer to as *outside money*.

In order to describe the way in which production and consumption take place in the economy, it is useful to first give a brief overview of the sequence of events within each sub-period. After that we describe the actions of the central bank in conducting monetary policy. We then describe the manner in which transactions take place, i.e. the interaction between individual agents and the banking system. At this point we describe explicitly the gross flows of money (both inside and outside) that occur within each period.

**Timing**

Agents enter the first sub-period of period \( t \) owning shares in banks and holding outside money acquired during the second half of period \( t - 1 \). At the beginning of sub-period 1, the shock which divides agents by type takes place and so they are divided into consumers (young and old) and producers.

Immediately following the realization of this shock banks open and agents may make deposits and/or take out a loan. At this time banks may also deposit reserves with the central bank. Banks then shut down and exchange takes place among consumers and sellers in an anonymous Walrasian market. Sellers realize their disutility of work and consumers
acquire the good for consumption. Banks remain closed to agents throughout trading and until later in sub-period 1, at which time they re-open and depositors have the option of moving their deposits to another bank. We will let the parameter $a$ denote the fraction of time in sub-period 1 remaining when banks re-open. After the settlement of these banking transactions, sub-period 1 ends and the economy moves to sub-period 2.

In sub-period 2 all agents have identical preferences and productive capacities but differ with regard to their asset holdings. Agents are not anonymous in this sub-period and so the sequence of events does not matter; all that matters are net flows of money and goods. In this sub-period banks collect interest from the central bank, pay interest to their depositors and pay dividends. Borrowers re-pay their loans and all agents exchange in a Walrasian market. Through exchange agents acquire both goods for consumption and currency to carry into period $t+1$.

**Monetary policy**

The central bank’s policy has two components. It pays interest on reserves of outside money held by private banks and makes lump-sum transfers of money to maintain a target growth rate of the stock of outside money and thus a target trend inflation rate. During the first sub-period, the central bank has no ability to identify and recognize individual agents, so it engages only in dealings with banks. Transfers take place during the second sub-period.

At the beginning of the first sub-period, the central bank sets an interest rate $i_c$, at which
it is willing to accept deposits (in units of outside money) from or make loans to banks. This rate is a policy choice and may be contingent on the realizations of shocks. Loans to banks are restricted, without loss of generality, to be repaid in sub-period 2. Interest is paid in sub-period 2 and is proportional to the fraction of sub-period 1 during which the central bank holds the deposits. For example, a bank which accepts a deposit from an agent at the beginning of sub-period 1 and then immediately transfers it to another bank through the settlement process following goods trading receives no interest on that deposit. If a bank receives a deposit from a seller later in sub-period one and deposits it in the central bank, they will receive only \( \alpha \) on this deposit when interest is paid in sub-period 2.

In sub-period 2, the central bank implements the following rule for the growth rate of the stock of outside money:

\[
M_{t+1} = \gamma M_t,
\]

where \( \gamma \) is the target trend rate of inflation. The central bank meets its inflation target by conducting equal lump-sum transfers to all agents. The total transfer is equal to \((\gamma - 1)M_t\) minus the total interest paid on reserve deposits to all banks. If reserve interest exceeds \((\gamma - 1)M_t\), then the transfer is negative—a lump-sum tax.

**Transactions, banking, and money flows**

As noted above the banking system opens at the very beginning of the first sub-period of every period. At this point agents may deposit their currency holdings in banks and/or take out loans. Interest on deposits is paid and loans are re-paid in sub-period 2. Thus, old agents may not borrow and have no incentive to deposit in banks. We assume, however, that all agents will do so, as there is no cost associated with administering deposits. When banks are open, agents can freely choose among them and may move their deposits from one to another without cost. Since, as described above, private banks may deposit their reserves at the central bank and earn net interest \( \hat{i}_c \), competition among banks (of which there are
a large number) implies that in any symmetric equilibrium the private bank deposit rate, $i_d$, satisfies $i_d = i_c$.

Agents may borrow from banks at rate $i_\ell$. There is no reserve requirement and banks are free to issue as many loans as they like. As with the deposit rate, competition among banks will force the loan rate charged to satisfy $i_\ell = i_d + \delta$, where $\delta$ is the management cost per unit associated with a loan.

Having collected deposits and made loans, banks close and do not re-open to agents until fraction $1 - a$ of the first sub-period has passed. Exchange then takes place in an anonymous Walrasian market. Since all agents deposit their holdings of outside money in banks, exchange takes place using bank deposits only (for example, by means of checks). Interbank settlement of net balances takes place immediately following exchange in outside money. Note that if an individual bank were to require more outside money than it has collected in deposits, it would be able to borrow from the central bank at rate $i_c$. Only after this initial settlement are private banks able to deposit reserves at the central bank.

Private banks will pay no interest to depositors on that part of their balance (typically all) which they spend in exchange. The reason for this is that banks will transfer these reserves to another bank almost immediately for settlement purposes. Of course, during settlement the bank will also receive reserves which will flow into sellers' accounts. The bank will not, however, have to pay interest on these reserves to the sellers whose accounts are credited, unless sellers have the ability to switch banks. Sellers' ability to do so has been limited by the assumption that the banking system remains closed for fraction $1 - a$ of the first sub-period. At the end of this time, banks re-open and agents can switch banks if they like in order to earn interest. As with trading that takes place earlier in the period, following switching settlement of net balances takes place immediately using outside money.

When banks re-open, they will again be forced by competition to pay interest, at rate $i_s$, on these increased deposits of sellers. Given that from this point on a competitor bank can earn
interest from the central bank only equal \( ai_c \), we have that in any equilibrium competition among banks will force \( i_s = ai_c \).

With \( a < 1 \), banks will derive profits of \((1 - a)i_c\) on each dollar of reserves that they receive in settlement after goods trading. As we will show below, the parameter \( a \) plays an important role in our results. The case of \( a = 0 \) corresponds in some aspects to the model of Berentsen and Waller (2005) in which exchange takes place using currency and the money stock is at all times limited by the quantity of outside money in circulation. The case of \( a = 1 \), similarly, corresponds to the case studied by Champ, Smith, and Williamson (1996). We view the parameter \( a \) as a simple way to introduce into the model a form of inefficiency associated with the market structure of the private banking system.

Following the re-opening of the banking system at time \( 1 - a \) and subsequent settlement associated with any switches of deposits among banks, sub-period 1 ends. In sub-period 2, agents are not anonymous and transactions may take place in any order. The following transactions take place: Loans are repaid by borrowers in units of sub-period 2 consumption good or bank deposit. Note that this is feasible given the technology implied by (4.2).

![Figure 4.2: The flow of inside money.](image)

We denote the initial deposits and loan balance of a representative bank \( D_0 \) and \( L \), respectively. The flows of inside money between buyers and sellers (through banks) in each period is depicted in Figure 4.2. A detailed description of gross monetary flows within each period is given in the appendix.
4.3 Optimal choices

We now consider agents’ optimal choices in a representative time period, \( t \). To economize on notation, we will omit the subscript “\( t \)” throughout. We use “\( t - 1 \)” and “\( t + 1 \)” to denote the previous and next periods, respectively. Let \( p_1 \) and \( p_2 \) denote the nominal price level in sub-period 1 and sub-period 2, respectively and let \( \phi = \frac{1}{p_2} \) denote the real value of money in sub-period 2.

Let \( V(m) \) denote the expected value of a representative household from entering sub-period 1 with \( m \) units of outside money carried over from the previous period. Since we restrict attention to situations in which all agents deposit their money holdings in banks, it is useful to introduce the notation \( d_0 \) to represent initial deposits and to treat it as the state variable. From this point on we will write the household’s value function at the beginning of sub-period 1 as \( V(d_0) \).

Let \( W(d_1, s, \ell) \) denote the expected value of a household entering the second sub-period. Here, \( d_1 \) denotes initial deposits that remain with the household at the end of sub-period 1 (i.e. unspent deposits), \( s \) denotes new bank deposits received by the household during settlement in sub-period 1 (this is relevant for sellers), and \( \ell \) denotes the loan, if any, taken out by the household in sub-period 1. All of these state variables are non-negative.

4.3.1 Sub-period 2

At the beginning of sub-period 2, a measure \( \lambda \) of new households are born into the economy and inherit the ownership shares in banks held by the previous generation. These agents, however, have no deposits and no loan balance. The value of any household entering sub-
period 2 may be written

$$W(d_1, s, \ell) = \max_{x, h, d_{0, t+1}} [U(x) - h + \beta E_t V(d_{0, t+1})]$$ (4.4)

subject to: \( x + \phi d_{0, t+1} = h + \phi \tau + \phi (1 + i_d) d_1 + \phi s (1 + i_s) + \phi \Pi - \phi (1 + i_\ell) \ell \) (4.5)

where (4.5) is written in units of sub-period 2 consumption good. Here \( \tau \) denotes the transfer from the central bank and \( \Pi \) is bank profits, also distributed lump-sum.

Using (4.5) to eliminate \( h \) in (4.4) we have

$$W(d_1, s, \ell) = \phi [\tau + \Pi + (1 + i_d) d_1 + s (1 + i_s) - (1 + i_\ell) \ell]$$

$$+ \max_{x, d_{0, t+1}} [U(x) - x - \phi d_{0, t+1} + \beta E_t V(d_{0, t+1})]$$ (4.6)

The first order conditions for \( x \) and \( d_{0, t+1} \) are given by

$$U'(x) = 1$$ (4.7)

$$\phi = \beta (E_t V'(d_{0, t+1}))$$ (4.8)

where \((E_t V(d_{0, t+1}))'\) is the expected marginal value of an additional unit of deposit carried into period \( t + 1 \). The envelope conditions are

$$W_d = \phi (1 + i_d)$$ (4.9)

$$W_s = \phi (1 + i_s)$$ (4.10)

$$W_\ell = -\phi (1 + i_\ell).$$ (4.11)

As in Lagos and Wright (2005) the optimal solution for \( x \) is the same for all households and the choice of \( d_{0, t+1} \) is independent of \( d_1, s \) and \( \ell \). As a result, all households choose to hold the same deposit balance \( d_{0, t+1} \) at the beginning of \( t + 1 \).

\(^1\text{We can adjust } U(x) \text{ such that people always produce positive amount of goods in sub-period 2. So we will not have the corner solution in which people select } h = 0. \text{ Please see the Appendix for more details.}\)
4.3.2 Sub-period 1

At the beginning of sub-period 1, households are divided into buyers (old and young) and sellers stochastically. We will denote their values using sub-scripts on the value function, $V$. That is let $V(d_0)_y$, $V(d_0)_o$, and $V(d_0)_s$ denote the value of a household that enters the current period with deposits $d_0$ and has become a young buyer, old buyer, or seller respectively.

Sellers

Sellers will not borrow from banks as in any equilibrium the lending rate will be at least equal to the deposit rate. Thus, sellers choose $\ell_s = 1$, and their optimization problem may be represented by the following Bellman equation:

$$V(d_0)_s = \max_{q_s} [-c(q_s) + W(d_0, p_1 q_s, 0)] \quad (4.12)$$

where $q_s$ is the goods sold in sub-period 1 and $p_1 q_s$ is the seller’s monetary income which is deposited into their bank on settlement following goods trading. The seller’s first order condition for choice of $q_s$ is

$$-c'(q_s) + p_1 W_s = 0 \quad (4.13)$$

Using (4.10), we may write

$$c'(q_s) = p_1 \phi (1 + i_s) = \frac{p_1}{p_2} (1 + i_s) \quad (4.14)$$

Since the marginal cost of producing in sub-period 2 is 1, sellers choose $q_s$ such that the ratio of marginal costs across markets $c'(q_s)/1$ is equal to the relative nominal price $p_1 \phi$, multiplied by the return rate $1 + i_s$. Given $\phi$ in sub-period 2, the price level in the first market is

$$p_1 = \frac{c'(q_s)}{\phi (1 + i_s)} \quad (4.15)$$

Young buyers
Young buyers receive preference shock $A$. There are three cases and so we will divide the support of $A$ into three connected sets, $\Omega_1$, $\Omega_2$ and $\Omega_3$ which correspond to these cases. At least one of the three sets is non-empty. If $A \in \Omega_1$, young buyers will find their initial deposits more than sufficient to finance consumption in the sub-period and so will choose $d_1 > 0$ and $\ell = 0$. When $A \in \Omega_2$, young buyers spend all of their initial deposit but will not find it worthwhile to borrow from banks. That is, they will choose $d_1 = 0$ and $\ell = 0$. Finally, for $A \in \Omega_3$, young buyers will spend all of their initial deposits and borrow as well so that $d_1 = 0$ and $\ell > 0$. The sets $\Omega_1$, $\Omega_2$ and $\Omega_3$ depend both on the support of $A$ and the interest rates faced by the buyers. We consider the decisions of young buyers in for $A$ in each of the three regions separately. Note that for all young buyers $s = 0$ in all states because they do not sell goods in sub-period 1.

For $(A \in \Omega_1)$, we may write

$$V(d_0)_y = \max_{q_y}[Au(q_y) + W(d_1, 0, 0)]$$

subject to: $d_1 = d_0 - p_1 q_y$ 

where $q_y$ is consumption purchases. The first order condition is

$$Au'(q_y) = p_1 W_d$$

Using (4.9) and (4.14), we get

$$Au'(q_y) = p_1 \phi(1 + i_d) = c'(q_s) \frac{1 + i_d}{1 + i_s}.$$  

Note that for $A \in \Omega_2$, buyers choose $d_1 = 0$ and so set $q_y = \frac{d_0}{p_1}$.

For $A \in \Omega_3$, buyers set $d_1 = 0$ but choose $\ell > 0$. In this case the Bellman equation may be written:

$$\max_{q_y}[Au(q_y) + W(0, 0, \ell)]$$

subject to: $p_1 q_y = d_0 + \ell$
The first order condition in this case is

\[ Au'(q_y) = -p_1 W_\ell \]  

(4.22)

and using (4.11) and (4.14), we have

\[ Au'(q_y) = p_1 \phi(1 + i_\ell) = c'(q_s) \frac{1 + i_\ell}{1 + i_s} \]  

(4.23)

Old buyers

Old buyers are unable to borrow as they will not be present during the second sub-period. As such their optimization problem is trivial and they simply consume the value of their deposits:

\[ q_o = \frac{d_0}{p_1} \]  

(4.24)

and their value is given by

\[ V(d_{0})_o = Bu \left( \frac{d_0}{p_1} \right) \]  

(4.25)

Banks

Since we will later restrict attention to symmetric equilibria, we consider now only cases in which all banks make the same decision. It will then be clear that no individual bank has incentive to deviate. The following proposition summarizes the optimizing choices of banks in sub-period 1:

**Proposition 13.** Symmetric banks will choose \( i_d = i_c \), \( i_\ell = i_d + \delta \), and \( i_s = ai_c \).

Recall that \( a \) is the fraction of time in sub-period 1 remaining when banks re-open following exchange. Proposition 13 is intuitive and the results stem (as described in the previous section) from competition among banks. For this reason we omit a formal proof. The following proposition states the implications of these choices for bank profits, which appear as \( \Pi \) in households sub-period 2 budget, (4.5):
Proposition 14. If young consumers do not borrow and spend only fraction \( \rho \) of their initial deposits then bank profits are given by:

\[
\Pi = (\rho a D_0 + \lambda D_0)[(1 - a)i_c].
\] (4.26)

In the case where young consumers spend all of their initial deposit and also borrow amount \( L \), bank profits are given by

\[
\Pi = [(\alpha + \lambda)D_0 + L][(1 - a)i_c].
\] (4.27)

Note that Proposition 2 implies that bank profits are positive if and only if \( a < 1 \).

4.3.3 Dynamic Optimization

The quantity of money to carry into the current period (i.e. the value of initial deposits in period \( t, d_0 \)) is chosen at the end of period \( t - 1 \) before either the household partition shock or the aggregate preference shock is realized. The Bellman equation may be written as

\[
E_{t-1}[V(d_0)] = (1 - \alpha - \lambda) \int_A [-c(q_s) + W(d_0, p_1 q_s, 0)] dF(A) + \alpha \int_A [Au(q_y) + W(d_1, 0, \ell)] dF(A) + \lambda \int_A B(A)u\left(\frac{d_0}{p_1}\right) dF(A)
\] (4.28)

In the appendix we show that the expected marginal value of a unit of money (again an additional unit of deposits) at the beginning of period \( t \) is given by:

\[
E_{t-1}[V'(d_0)] = (1 - \alpha - \lambda) \int_A [\phi(1 + i_d)] dF(A) + \alpha \int_{A \in \Omega_1} \phi(1 + i_d) dF(A) + \int_{A \in \Omega_2} A u'(d_0) \frac{1}{p_1} dF(A) + \int_{A \in \Omega_3} \phi(1 + i_\ell) dF(A)
\] + \lambda \int_A B(A)u'\left(\frac{d_0}{p_1}\right) \frac{1}{p_1} dF(A)
\] (4.29)

For sellers, the marginal value of an additional unit of deposits is the real value of \( 1 + i_d \) deposits in sub-period 2. For young buyers, if the deposit balance is not binding, the marginal value will be the same as that of sellers. When \( A \in \Omega_2 \), the deposit balance is binding, and
the marginal value of $d_0$ is the marginal utility. If $A \in \Omega_3$, the buyer borrows ($\ell > 0$) and the marginal value of $d_0$ is the real value of a reduction of the loan. Finally, for old buyers, the marginal value of deposits is the marginal utility of consumption.

4.4 Equilibrium

In this section, we define a symmetric stationary monetary equilibrium contingent on the central bank’s monetary policy (i.e. for a fixed profile of central bank deposit rates, $i_c$, which in general depends on $A$, and target inflation rate, $\gamma$). We will turn to the optimal selection of these policy variables later.

In a symmetric equilibrium all young buyers, old buyers, and sellers make the same choices. Similarly, all banks take in the same quantity of deposits, make the same loans, receive the same payments, and as a result earn the same profit. All choices, including the central bank policy (with the exception of the inflation target, $\gamma$) are implicitly functions of the aggregate state, $A$. In the exposition, however, dependence of quantities and prices etc. on $A$ are suppressed. We may then define a stationary monetary equilibrium as follows:

A stationary monetary equilibrium (SME) is a list of quantities, $q_y, q_o, q_s, \text{ and } x$; work efforts in sub-period 2 by surviving buyers, sellers, and newborn agents, $h_y, h_s, \text{ and } h_n$; prices $p_1$ and $\phi = 1/p_2$; interest rates, $i_d, i_\ell, \text{ and } i_s$; and a central bank policy $i_c$ (all of which are contingent on $A$) and $\gamma$ such that:

1. Taking the central bank policy and prices as given, households choose quantities to solve the optimization problems given by (4.12), (4.16)-(4.23), and (4.25).

2. Taking the central bank policy as given, banks set $i_d, i_\ell, \text{ and } i_s$ to maximize profits as described by Proposition 1.
3. Goods markets clear:

\[ \alpha q_y + \lambda q_o = (1 - \alpha - \lambda)q_s \Rightarrow q_s = \frac{\alpha q_y + \lambda q_o}{1 - \alpha - \lambda} \quad (4.30) \]

\[ \alpha h_y + (1 - \alpha - \lambda)h_s + \lambda h_n = x + \alpha \phi \ell \delta \quad (4.31) \]

where \( \alpha \phi \ell \delta \) is the real value of loan management costs.

4. The market for money clears:

\[ d_0 = D_0 = M_t \quad (4.32) \]

As noted above, in equilibrium at the end of sub-period 2 every agent adjusts their consumption and production so that at the end of the period, they will hold the same money balance, \( \omega_t \) where

\[ \omega_t \equiv \frac{d_0}{p_{2,t-1}} = d_0 \phi_{t-1}. \quad (4.33) \]

Using (4.8) and (4.29) it is possible to derive an equation which implicitly characterizes \( \omega \) in equilibrium. First, lag (4.8) by one period to get

\[ \phi_{t-1} = \beta(E_{t-1}V'(d_{0,t})) \quad (4.34) \]

Substituting for \( E_{t-1}V'(d_{0,t}) \) in (4.29), divide both sides of the equation by \( \phi \beta \), to get

\[
\frac{\gamma}{\beta} = \int_A (1 - \alpha - \lambda)(1 + i_d)dF(A) \\
+ \alpha \left[ \int_{A \in \Omega_1} (1 + i_d)dF(A) + \int_{A \in \Omega_2}Au' \left( \frac{d_0}{p_1} \right) \frac{1}{p_1 \phi}dF(A) + \int_{A \in \Omega_3} (1 + i_\ell)dF(A) \right] \\
+ \lambda \int_A B(A)u' \left( \frac{d_0}{p_1} \right) \frac{1}{p_1 \phi}dF(A) \quad (4.35)
\]

In an SME, the common real balance chosen by households, \( \omega_t = d_0 \phi_{t-1} \) will be constant over time. Characterizing an SME, then amounts to solving (4.35) for \( \omega \).
4.5 An example

In this section, we compute an equilibrium for a parameterized version of the economy in which the central bank fixes its interest rate, $i_c$, equal to zero for all values of $A$. This essentially shuts down the central bank’s interest rate policy. We do not intend for this example to be interpreted quantitatively, rather we have two goals in this section:

1. To illustrate the basic mechanisms at work in our economy.

2. To illustrate that the benchmark policy of $i_c = 0$ in all states is not optimal.

To begin with we parameterize the economy as follows: We assume logarithmic preferences $u(q) = \ln q$, and maintain the assumption that $A$ and $B$ are perfectly correlated. $A$ is uniformly distributed over $[0.4, 1.1]$, and

$$B = A - 0.15$$

The situation in which $B$ is low relative to $A$ has two implications. First it lowers the marginal utility of old agents and thereby reduces the differential effect of the inflation tax on them. Second, it induces all agents to hold lower real balances and makes it more likely that young agents will want to borrow. We set the discount factor, $\beta = 0.99$.

We set $\alpha = 0.6$ and $\lambda = 0.2$. The larger the share of young buyers in the economy, the higher bank lending will be in those states where young agents wish to borrow. The larger the share of old buyers, the more important is the distortion associated with their exclusion from borrowing.

Finally, we set $c(q) = q + \frac{1}{2}q^2$ and set the bank management cost to $\delta = 0$. The particular form of the cost function has only quantitative implications. The size of the management cost similarly has no significant qualitative effect. As noted above the central bank keeps the interest rate constant at $i_c = 0$. We also assume that it holds the stock of outside money
fixed \( (i.e. \gamma = 1) \). Later in this chapter we return to the case of no active interest rate policy at different trend inflation rates.

With \( i_c = 0 \), the bank lending rate is constant at \( i_t = \delta \) (\( = 0 \)) and the bank deposit rates are \( i_d = 0 \) and \( i_s = 0 \). With the interest rate constant, it may be easily shown that we may partition the support of the preference shock, \( \mathcal{A} \), into three ranges: \( \Omega_1 = [A, A_1] \), \( \Omega_2 = (A_1, A_2) \), and \( \Omega_3 = [A_2, \overline{A}] \). In the appendix we show that for this example we may write (4.35) as

\[
\frac{\gamma}{\beta} = (1 - \alpha - \lambda)(1 + i_d)
\]

\[
+ \alpha \left[ \int_{A_1}^{A} (1 + i_d) dF(A) + \int_{A_1}^{A_2} \frac{\gamma}{\omega} dF(A) + \int_{A_2}^{\overline{A}} (1 + i_t) dF(A) \right]
\]

(4.37)

\[
+ \lambda \int_{A}^{B} \frac{\gamma}{\omega} dF(A)
\]

Using (4.37) we then may establish the existence of an SME for this case:

**Proposition 15.** For the economy of this example, if \( \gamma > \beta(1 - \lambda) \) there exists a unique SME.

In the appendix we show how to calculate equilibrium quantities and prices as functions of the real money balance, \( \omega \). We then compute the unique \( \omega \), and thus the SME, numerically.

Figure 4.3 illustrates several aspects of the SME for this example:

1. Both production and the relative sub-period 1 price are increasing in the aggregate shock, \( A \).

2. The consumption of young buyers is increasing in \( A \), while that of old buyers decreases as the increase in the price level erodes the value of their savings.

3. For high values of \( A \), the loan balance is positive and the aggregate money balance exceeds the quantity of outside money.
Figure 4.3: The results when $i_c = 0.$
4. For high values of $A$ the banking system finances total payments in excess of the stock of outside money.

To understand these results, note first that the sub-period 1 goods price satisfies

$$p_1 = \frac{c'(q_s)}{\phi}.$$  \hspace{1cm} (4.38)

When agents’ demand for goods rises (because of an increase in $A$), marginal cost increases and $p_1$ must rise. An increase in both the quantity produced and the price is financed here by newly created inside money. Since the money supply is stochastic, $p_1$ is as well. Moreover, the response of the price level to a given shock depends on both the form of marginal cost, $c'(q_s)$ and the response (if any) of $i_s$ (Note that in this example $i_s = 0$ in all states). An increase in $p_1$, however, reduces the real value of old buyers’ deposits and so lowers their consumption. Inside money creation therefore redistributes wealth from old buyers to young ones.

Because the trend inflation rate is controlled by outside money growth $\gamma$, increases in the money stock have different implications for inflation depending on whether they are associated with increases in outside or inside money. An increase in $p_1/p_2$ results in a temporary increase in inflation during sub-period 1. Given, however that the central bank maintains a constant inflation rate (actually a constant price level ($\gamma = 1$) in this example), there must be a reduction in the price level between sub-periods 1 and 2. This occurs as all bank loans are repaid in sub-period 2. At that time, the inside money created in sub-period 1 is destroyed. Thus, creation of inside money has only a temporary effect on the price level while increases to the stock of outside money have a permanent effect.

### 4.6 Interest rate policy

In this section we characterize the central bank’s optimal interest rate management policy under the assumption that it cannot commit to an interest rate target in advance. Rather, it
chooses the interest rate, \( i_c \) optimally after the realization of \( A \) at the beginning of the current period. We allow the central bank to commit to its long-run inflation target, \( \gamma \), although as we show later, this assumption is unimportant.

At the end of sub-period 2 of period \( t-1 \), the expected utility of a representative household entering period \( t \) with deposits (outside money) \( d_0 \) is

\[
E_{t-1}V(d_0) = \int_A [\alpha Au(q_y) + \lambda Bu(q_o) - (1 - \alpha - \lambda)c(q_s)]dF(A) \\
+ \alpha \int_A \{[U(x) - h_y + \beta E_t V(d_{0t+1})]dF(A)\} \\
+ (1 - \alpha - \lambda) \int_A [U(x) - h_s] + \beta E_t V(d_{0t+1})dF(A). \tag{4.39}
\]

Note that utility from consumption in sub-period 2, \( U(x) \), is the same regardless of whether a household is a buyer or a seller in the previous sub-period 1.

The expected utility of the new born agents in sub-period 2 (of period \( t \)) is given by

\[
\int_A [U(x) - h_n]dF(A) + \beta E_t V(d_{0t+1}) \tag{4.40}
\]

where again, \( x \) is the same as for all other agents. Note that \( E_t V(d_{0t+1}) \) depends only policy in the future.\(^2\) Under the assumption that the central bank adopts the optimal policy in all future periods, \( E_t V(d_{0t+1}) \) will not be affected by period \( t \) policy choices. Thus, in choosing its current policy, the central bank can take this as given and consider only the effect of its choice on households’ utility in the current period.

Let \( \mathcal{W} \) denote social welfare defined as the sum of current (period \( t \)) utility for all agents entering from period \( t-1 \) and born at the beginning of sub-period 2 of period \( t \):

\[
\mathcal{W} = \int_A [\alpha Au(q_y) + \lambda Bu(q_o) - (1 - \alpha - \lambda)c(q_s)]dF(A) \\
+ \int_A \alpha[U(x) - h_y] + (1 - \alpha - \lambda)[U(x) - h_s]dF(A) \tag{4.41} \\
+ \int_A \lambda[U(x) - h_n]dF(A).
\]

\(^2\)In the stationary equilibrium, the level of \( d_0 \) does not depend on \( d_1, s \) and \( \ell \) carried into sub-period 2.
Social welfare (4.41) can be simplified using the sub-period 2 goods market clearing condition (4.31). Since $x$ is a constant, it can be ignored for policy analysis. Thus, we may rewrite social welfare as

$$W = \int_A \left[ \alpha Au(q_y) + \lambda Bu(q_o) - (1 - \alpha - \lambda)c\left(\frac{\alpha q_y + \lambda q_o}{1 - \alpha - \lambda}\right) \right] dF(A) + \int_A [-\alpha \phi \ell \delta] dF(A) \quad (4.42)$$

where we have eliminated $q_s$ using the sub-period 1 goods market clearing condition, (4.30). This is the measure of welfare we will use for comparing equilibria throughout this chapter.

At this stage we turn to the central bank’s optimal choice of $i_c$ contingent on the realization of $A$. The bank chooses $i_c$ to maximize (4.42) subject to the constraint that $i_c(A) \geq 0$ for all $A$. Since we assume the central bank cannot commit in advance, the central bank essentially maximizes the integrand of (4.42), $\alpha Au(q_y) + \lambda Bu(q_o) - (1 - \alpha - \lambda)c(q_s) - \alpha \phi \ell \delta$, for each realization of $A$. Note that the central bank cannot set an interest rate below zero, as banks are not required to deposit reserves. Of course, the central bank can choose to commit to a negative inflation rate, $\gamma < 1$. We will show later, however, that it will not in general be optimal for the bank to do so.

We assume from this point on that $\delta = 0$ as this simplifies calculations significantly. It has, however, no significant effect on our qualitative results. The case of $\delta > 0$ is useful, however, for quantitative analysis which is beyond the scope of the current version of this essay.

We consider optimal interest rate policy in three different cases with regard to the extent of bank profits. Recall that if $a = 0$ sellers do not receive interest on the deposits they receive in settlement following goods trade (i.e. $i_s = 0$ in all states). We will consider this case first. We then take up the opposite case in which $a = 1$ and $i_s = i_d$. Analysis of intermediate cases with $a \in (0, 1)$ is algebraically similar to the case of $a = 1$, but as we will see the implications for policy are more in line with those of the case in which $a = 0$.

Before characterizing the optimal policy, we will show that the equilibrium computed in the previous section does not maximize (4.42). From here it will be clear that it is not optimal
for the central bank to set $i_c = 0$ in all states regardless of the value of $a$ or the trend inflation rate $\gamma$. Under the assumptions that $\delta = 0$ and $i_c = 0$ in all states, (4.42) is maximized when

$$Au'(q_y) = c'(q_s) \quad (4.43)$$

$$Bu'(q_o) = c'(q_s). \quad (4.44)$$

Since we assume $\delta = 0$ and $i_c = 0$, we have $i_d = 0$, so (4.43) holds for all $A$. This corresponds to the findings of Berentsen and Waller (2005) in an environment with only one type of consumer who has access to loans with an elastic supply of money. In our economy, however, the marginal utility of old buyers will not in general be equated to the marginal production cost when $i_c = 0$. That is, (4.44) will not hold for all $A$. Figure 4.4 shows that when $A$ (and so $B(A)$) is low, $Bu'(q_o) < c'(q_s)$, and old buyers consume “too much”. In contrast, for high values of $A$, $Bu'(q_o) > c'(q_s)$ and old buyers consume too little.

### 4.6.1 Interest rate policy when $a = 0$ (bank profit)

In this case (4.15) becomes

$$\frac{\bar{p}_1}{\bar{p}_2} = c'({\bar{q}_s}) \quad (4.45)$$
Figure 4.5: The results of optimal interest rate policy when $i_s = 0$. Equilibrium $\omega = 0.6018$. 

(a) Interest rate policy 
(b) marginal utility and cost 
(c) Production $q_s$ 
(d) Price $\frac{p_1}{p_2} = \frac{c'(q_s)}{1+i_s}$ 
(e) Consumption, young buyer 
(f) Consumption, old buyer 
(g) Loan and aggregate money 
(h) Aggregate payment
where \( \tilde{q}_s \), for example, is the equilibrium value of \( q_s \) contingent on the interest rate \( i_c \) in state \( A \). Similarly, the first-order condition of young buyers in this case is

\[
Au'(\tilde{q}_y) = c'(\tilde{q}_s)(1 + i_c)
\]

(4.46)

where we have used the equilibrium condition that \( i_d = i_c \). From (4.46) we can see that any \( i_c > 0 \) will cause \( q_y \) to be lower than the optimal level.

The derivative of (4.42) with respect to \( i_c \) is

\[
\frac{\partial W}{\partial i_c} = \alpha[Au'(\tilde{q}_y) - c'(\tilde{q}_s)]\frac{\partial \tilde{q}_y}{\partial i_c} + \lambda[Bu'(\tilde{q}_o) - c'(\tilde{q}_s)]\frac{\partial \tilde{q}_o}{\partial i_c}
\]

(4.47)

Since higher \( i_c \) will reduce the equilibrium consumption of young buyers and increase the consumption of old buyers, we have \( \frac{\partial \tilde{q}_y}{\partial i_c} < 0 \) and \( \frac{\partial \tilde{q}_o}{\partial i_c} > 0 \).

Recall that with \( i_c = 0 \), \( Au'(\tilde{q}_y) - c'(\tilde{q}_s) = 0 \) and term \( I \) of (4.47) equals 0. So, the sign of (4.47) depends on the term \( II \). For sufficiently low values of \( A \), \( Bu'(\tilde{q}_o) - c'(\tilde{q}_s) < 0 \), and it is optimal for the central bank to set \( i_c = 0 \). Intuitively, with low \( A \) old consumers are consuming too much already, and \( i_c = 0 \) makes \( q_y \) to be set efficiently. In this situation the central bank would like to transfer wealth from the old to the young. The best it can do, however, is to set \( i_c = 0 \) because of the zero bound.

For higher \( A \), at some point \( Bu'(\tilde{q}_o) - c'(\tilde{q}_s) > 0 \). In this case the central bank will set \( i_c > 0 \), in order to transfer wealth from young to old. Raising \( i_c \) is costly, of course, as it will reduce the consumption of young buyers. It can be seen from (4.47) that it will not be optimal to raise \( i_c \) to the point that \( Bu'(\tilde{q}_o) = c'(\tilde{q}_s) \).

Figure 4.5 illustrates the optimal interest rate policy and equilibrium for the economy of the benchmark example with \( a = 0 \). In the figure it is clear that the optimal policy is to set \( i_c = 0 \) in states in which \( Bu'(\tilde{q}_o) - c'(\tilde{q}_s) < 0 \) (and old agents are consuming too much). For sufficiently high \( A \), however, the central bank raises the interest rate above zero to maintain the consumption of the old agents.
4.6.2 Interest rate policy when \( a = 1 \) (no bank profit)

In this case (4.15) becomes

\[
\frac{p_1}{p_2} = \frac{c'(\tilde{q}_s)}{1 + i_c} \quad (4.48)
\]

From (4.48) it is clear that for a given \( q_s \), \( p_1 \) will vary inversely with \( i_c \). In this case the first-order condition for young buyers is

\[
Au'(q_y) = c'(q_s) \quad (4.49)
\]

so that young buyers will consume the “correct” amount regardless of the interest rate. In this case, by choosing \( i_c \) the central bank can set (4.47) to zero (i.e. set \( Bu'(\tilde{q}_o) = c'(\tilde{q}_s) \) and attain the efficient outcome) unless the zero constraint is binding. If \( Bu'(\tilde{q}_o) < c'(\tilde{q}_s) \), and \( i_c > 0 \), then the central bank can improve welfare by lowering the interest rate to reduce \( \tilde{q}_o \).

Similarly the central bank will never choose an interest rate that results in \( Bu'(\tilde{q}_o) > c'(\tilde{q}_s) \) since in this case it could raise welfare by increasing \( i_c \).

With \( a = 1 \), for the case of logarithmic utility we can derive an analytic solution for the optimal interest rate policy.

**Proposition 16.** With \( u(q) = \ln q \) and \( a = 1 \), the optimal interest rate policy is given by:

\[
i_c(A) = \max \left[ 1, B(A)^\frac{\gamma}{\omega} \right] \quad (4.50)
\]

Given the support of \( A \), higher \( \gamma \) and/or lower \( \omega \) will induce the central bank to set higher interest rates and therefore the zero bound will be less likely to be binding.

For this case the optimal policy and equilibrium are depicted in Figure 4.6. The principal difference between this case and the situation with \( a = 0 \) (see Figure 4.5) is that here whenever the zero bound is not binding the optimal interest rate policy attains the global optimum. Again this occurs for high realizations of \( A \). The response of \( i_c \) to increases in \( A \) in this range is much greater with \( a = 1 \) because in this case interest rate increases do not redistribute
Figure 4.6: The results of optimal interest rate policy when $i_s = i_d$. Equilibrium $\omega = 0.7179$. 

(a) Interest rate policy

(b) marginal utility and cost

(c) Production $q_s$

(d) Price $\frac{p_1}{p_2} = \frac{c'(q_s)}{1 + i_s}$

(e) Consumption, young buyer

(f) Consumption, old buyer

(g) Loan and aggregate money

(h) Aggregate payment
wealth away from young agents. The central bank therefore increases the rate and reduces the price \((\frac{p_1}{p_2})\) until \(Bu'(q_o) = c'(q_s)\).

### 4.7 The optimal trend inflation rate

We now consider the central bank’s choice of the optimal inflation target under the assumption that it will subsequently choose an optimal discretionary interest rate policy. We begin with the cases where \(a = 0\) and \(a = 1\) and then illustrate the results for general \(a \in (0, 1)\).

#### 4.7.1 Optimal inflation \(a = 0\) (bank profit)

![Graphs showing the effects of \(\gamma\).](image)

(a) Social welfare  
(b) Equilibrium \(\omega\)  
(c) Interest rate policy  
(d) Consumption of old buyers

**Figure 4.7:** The effects of \(\gamma\). \(i_s = 0\).
Figure 4.7 illustrates welfare and the optimal interest rate policy for different trend levels of inflation. Panel (a) depicts social welfare under the optimal interest rate policy. At very low inflation rates (below $\gamma = 1.001$ in the figure) real balances are sufficiently high that for those agents who turn out to be old $Bu'(q_o) < c'(q)$ and the zero constraint on $i_c$ binds in most states (this can be seen in panel (c)). In this situation the central bank would like to lower the interest rate and effectively transfer wealth from old to young agents in almost all states. The zero bound, however, prevents this. Increases in $\gamma$ reduce the range of the state space in which the constraint binds, lessen this effect, and improve welfare.

At the same time, however, increases in the inflation rate reduce real balances (see panel (b)) and lower the consumption of old agents. This inflation tax dominates in those states the zero constraint is not binding. As inflation rises, these states occur more frequently and eventually these costs of inflation outweigh the gains associated with lowering the frequency with which the zero bound holds.

Overall, trend inflation has two effects in our economy. First, it levies the usual inflation tax. This is a particularly severe problem here because old agents neither borrow nor save. Second, the higher trend inflation, the less likely the zero constraint is to be binding. The latter effect of inflation enables the central bank to provide more insurance against liquidity shocks with its interest rate policy, thus improving welfare. The usefulness of trend inflation then, stems from the fact that it makes the interest rate policy, which is aimed at short-run fluctuations, more effective. Once the zero bound is rarely binding, the costs of the inflation tax outweigh the gains from further reducing the frequency with which it holds.

### 4.7.2 Optimal inflation with $a = 1$ (no bank profit)

Figure 4.8 illustrates welfare and the optimal interest rate policy for the case in which banks pay interest to sellers on deposits made during sub-period 1 after goods trading and settlement. The results reflect the finding from the previous section that the efficient outcome is
attained in all states in which the zero constraint on the interest rate is not binding.

As in the case with $a = 0$, when the inflation rate is very low, the zero bound often holds and old agents find themselves over-insured very frequently. Thus, increases in $\gamma$ which reduce the frequency with which the zero constraint binds improve welfare. The difference here is that because the central bank effectively pays interest on all holdings of fiat money through the functioning of the competitive banking system, there is no inflation tax to counteract this effect. This can be seen by noting that as shown in the previous section, (4.49) always holds when $a = 1$ and that the value of old buyers’ real balances is constant at rates of inflation for which the zero constraints on interest rates never binds.
Let $\gamma_{NB}$ denote the lowest money growth rate for which the zero bound is never binding. When $\gamma \geq \gamma_{NB}$, the efficient (i.e. first-best) allocation is attained because $Au'(q_y) = c'(q_s)$ and $Bu'(q_o) = c'(q_s)$ will hold for all $A$. These two equations will uniquely determine the equilibrium consumption and production, so consumption and output are independent of the inflation rate as long as it is above $\gamma_{NB}$.

When $\gamma \geq \gamma_{NB}$, we have $1 + i_d = 1 + i_\ell = B \frac{\gamma}{\omega}$ for all $A$, and (4.35) simplifies to

$$\frac{\gamma}{\beta} = (1 - \lambda) \int_A B \frac{\gamma}{\omega} dF(A) + \lambda \int_A B \frac{\gamma}{\omega} dF(A)$$

or

$$\omega = \omega_{NB} = \beta EB(A).$$

So the equilibrium $\omega$ is constant when $\gamma \geq \gamma_{NB}$. In this case, money is super-neutral if the zero constraint on interest rates is never binding.  

Using (4.53) we can calculate $\gamma_{NB}$, the minimal optimal inflation rate. At $\gamma_{NB}$, the zero bound is just binding at $A$, so $Bu'(q_o) = c'(q_s)$ must hold at $A = A$ for $i_c = 0$ and $\omega = \omega_{NB}$. Thus, we have

$$B(A) \frac{p_1}{d_0} = c'(q_s)$$

$i_c = 0$ implies $p_1 = \frac{c'(q_s)}{\phi}$, so we have

$$B(A) = d_0 \phi = \frac{\omega_{NB}}{\gamma} \text{ or } \gamma_{NB} = \frac{\omega_{NB}}{B(A)} = \frac{\beta EB}{B(A)}$$

As long as $\beta EB > B$, we have $\gamma_{NB} > 1$. In our example, $EB = 0.6$ and $B = 0.25$, so $\omega_{NB} = 0.594$ and $\gamma_{NB} = 2.376$.

---

3Under the optimal interest rate policy, the central bank pays higher interest on all deposits and offsets the effect of trend inflation on real balances. When inflation is too low, however, the zero bound holds for some $A$ and the central bank cannot use interest rate policy effectively to insure old agents.
4.7.3 Optimal inflation with $0 < a < 1$

We now consider the case in which banks earn some profits ($a < 1$) but competition limits their ability to avoid paying any interest to sellers on their post-settlement deposits ($a > 0$). Calculations are similar to those carried out in section 4.6 and so we omit them here.

![Figure 4.9: The effects of $\gamma$ on social welfare for different $a$.](image)

(a) Social welfare($0.85 \leq \gamma \leq 2.5$)  
(b) Social welfare($0.91 \leq \gamma \leq 1.08$)

Figure 4.9: The effects of $\gamma$ on social welfare for different $a$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0010</td>
</tr>
<tr>
<td>0.25</td>
<td>1.0235</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1078</td>
</tr>
<tr>
<td>1</td>
<td>$\geq 2.376$</td>
</tr>
</tbody>
</table>

Table 4.1: Optimal $\gamma$

Figure 4.9 illustrates social welfare for four different values of $a$ over a wide range of trend inflation rates with interest rate policy chosen optimally in each case. Table 4.1 lists the optimal trend inflation rates in each case. For any $a < 1$, the first best can never be attained and so eventually the inflation tax dominates and there exists a unique optimal inflation target, which increases monotonically with $a$. The higher are bank profits, the lower the optimal inflation rate, the less effective interest rate policy in providing insurance against liquidity shocks, and the lower social welfare in the SME.
4.7.4 Optimal inflation with no interest rate policy

So far we have considered optimal trend inflation under the assumption that the central bank chooses an optimal interest rate \textit{ex post}. We now return to our example of Section 4.5 and consider optimal inflation when the central bank commits to a constant interest rate of \( i_c = 0 \).

![Figure 4.10: The result when there is no interest rate policy \( i_c = 0 \)](image)

(a) Social welfare \((0.85 \leq \gamma \leq 2.5)\)  
(b) Social welfare \((0.925 \leq \gamma \leq 1.05)\)

Figure 4.10: The result when there is no interest rate policy \( i_c = 0 \)

Figure 4.10 illustrates the social welfare for different inflation rates with and without an interest rate policy. Optimal inflation is higher in the former case. And in our example, the highest welfare when there is no interest rate policy is lower than the highest welfare when interest rate policy is allowed. (The case with \( a = 0 \) is shown here. Note that the optimal welfare will be even higher for \( a > 0 \), see figure 4.9). In a neighborhood of the optimal inflation rate, however, the welfare gains associated with interest rate policy are small. The gains increase substantially, however, as the trend inflation rate increases above the optimum. In this economy, interest rate policy is particularly important when inflation is higher than optimal. This may be relevant for considering monetary policy in cases in which the monetary authority runs a higher than optimal inflation rate because, for example, of a need to generate a certain amount of seignorage revenue.
4.8 Conclusion

This chapter analyzes optimal monetary policy consisting of two components, a short-run interest rate management policy and a target trend inflation rate in an environment in which money plays an explicit role as the medium of exchange. The supply of money is elastic and is controlled by a private banking system into which agents make deposits and from which they borrow in the presence of aggregate liquidity shocks.

Through its interest rate policy, the central bank can improve welfare when private agents are affected asymmetrically by liquidity shocks by exploiting the effect of inside money creation on the distribution of wealth. The effectiveness of this interest rate policy depends on the trend rate of inflation. At low inflation rates, the zero bound on interest rates often holds and the central bank is thus constrained in its ability to provide insurance to agents who have no access to loans. Higher inflation reduces the frequency with which the zero bound is relevant and can therefore improve welfare. Typically, however, higher inflation also comes with the usual welfare costs associated with the inflation tax.

The result is an optimal monetary policy which is typically a positive trend inflation rate combined with an interest rate policy under which the central bank pays a zero interest rate on reserves in some states and a positive rate in others. The key problem facing the central bank is to prevent the banking system from creating too much inside money in states when demand is high, as this penalizes agents who are constrained by their monetary savings.

This essay does not include several phenomena which are clearly of importance to the role of the banking system in the transmission of monetary policy. For example, we ignored default risks associated with bank loans and bank capital constraints. We have also assumed that prices are not only perfectly flexible, but that they are determined in Walrasian markets. We have done this to simplify the analysis in order to more clearly illustrate the basic mechanism at work in our model. We will explore these potentially important additional frictions in
future work.
Chapter 5

Summary and conclusion

In this thesis, we studied the role of the banking system in several aspects of the macroeconomy, including the likelihood of financial crises, volatility of asset prices and the transmission of monetary policy.

In the first essay, we studied the policy of central banks to hold reserves as insurance against financial crises. We showed that higher reserves can help to reduce the probability of financial crises. We also found that the central bank will want to hold more reserves when the private sector holds lower reserves. In addition, the possibility of borrowing from external sources may actually cause the central bank to hold more reserves when the borrowing cost is high. This is because the central bank wants to reduce the probability of borrowing at high costs.

In the second essay, we showed that the ability of banks to provide liquidity through money creation can help stabilize asset prices and help non-bank financial intermediaries to provide liquidity insurance to shareholders.

In the third essay, we studied an environment in which agents are affected by liquidity shocks asymmetrically. The central bank can use the interest rate policy to adjust the price level and the consumption of households. We showed that both interest rate policy and inflation rate policy are useful in improving social welfare. The effectiveness of the interest
rate policy depends on the trend inflation rate. We also found that when the supply price is more sensitive to interest rates, it would be better to set a higher inflation rate and make the zero bound of nominal interest rate less likely to be binding.
Appendices
Appendix A

Proofs for Chapter 2

A.1 Proofs

A.1.1 Deriving the shape of $v(\theta, n)$

This part explains the shape of $v(\theta, n)$ that is shown in figure 2.2. We first solve for $\kappa_s$ and $n_s(\theta)$. The value of $\kappa_s$ can be computed from:

\[
\kappa_s d_t \bar{r}^m = \gamma d_t + \bar{b} + \lambda \phi_s[(1 - \gamma)d_t + e_b] \tag{A.1}
\]

\[
(1 - \kappa_s)d_t \bar{r}^n = (1 - \phi_s)[(1 - \gamma)d_t + e_b]R - \bar{b} \tag{A.2}
\]

where $\phi_s$ is the liquidation at $\kappa_s$. The first equation describes the payment in period 2 when $\kappa = \kappa_s$. The second equation means that the remaining resources of banks are just enough to give the promised payment in period 3. We get

\[
\kappa_s = \frac{[(1 - \gamma)d_t + e_b]R - \bar{b} + (\gamma d_t + \bar{b})\frac{R}{\lambda} - d_t \bar{r}^m}{\frac{R}{\lambda}d_t \bar{r}^m - d_t \bar{r}^n} \tag{A.3}
\]

Since $n_s(\theta) + (1 - n_s(\theta))\pi(\theta) = \kappa_s$, we have $n_s(\theta) = \frac{\kappa_s - \pi(\theta)}{1 - \pi(\theta)}$. If $\kappa_s < \pi(\theta)$, we define $n_s = 0$.

Figure 2.2 shows the case when $n_s(\theta)$ and $n_e(\theta)$ are between $(0, 1)$. Over $n \in [0, n_s(\theta)]$, $v(\theta, n)$ is equal to $(1 - \pi(\theta))(\bar{r}^m - \bar{r}^n)$, which is a constant.

Over $(n_s(\theta), n_e(\theta))$, we have the following result: $\bar{r}^m > r^n > \bar{r}^n$, and $v(\theta, n) = (1 -
\( \pi(\theta)(\bar{r}^m - r^n) \) is negative and is increasing in \( n \). The value of \( r^n \) can be computed as follows:

\[
\kappa d_t \bar{r}^m = \gamma d_t + \bar{b} + \lambda \phi[(1 - \gamma)d_t + e_b] \\
(1 - \kappa)d_t r^n = (1 - \phi)[(1 - \gamma)d_t + e_b]R - \bar{b}
\]

The solution is

\[
r^n(\kappa) = \frac{R}{\lambda} \bar{r}^m + \frac{1}{1 - \kappa} W
\]

where \( W \) is a constant

\[
W = \frac{[(1 - \gamma)d_t + e_b]R + \frac{R}{\lambda}(\gamma d_t + \bar{b} - d_t \bar{r}^m) - \bar{b}}{d_t}
\]

The derivative of \( r^n(\kappa) \) with respect to \( \kappa \) is

\[
-\frac{1}{(1 - \kappa)^2} W \quad \text{is negative and } W \text{ is a constant, which means } r^n \text{ is either decreasing or increasing over } [\kappa_s, \kappa_e].
\]

and \( r^n(\kappa = \kappa_e) = \bar{r}^m \). Since \( \bar{r}^m > \bar{r}^m \), \( r^n \) is decreasing in \( \kappa \). Because \( \kappa \) is strictly increasing in \( n \), \( r^n \) is decreasing in \( n \) over \([n_s, n_e]\), which means \( v(\theta, n) \) is increasing in \( n \).

At \( n = n_e(\theta) \), \( r^n = \bar{r}^m \) and \( v(\theta, n) = 0 \). Over \( n_e(\theta) < n < n_f \), we have \( v(\theta, n) = \frac{1 - n_f}{1 - n} \bar{r}^m \), which is positive and strictly increasing in \( n \). Over \( n_f \leq n \leq 1 \), we have \( v(\theta, n) = \frac{n_f}{n} \bar{r}^m \), which is positive and strictly decreasing in \( n \).

A.1.2 Proof of proposition 1

Some intermediate results

We first need to prove some intermediate results.

**Lemma 1.** When \( n_e(\theta) \) and \( n_s(\theta) \) are between \((0, 1)\), they are continuous and strictly decreasing in \( \theta \).

**Proof:** \( n_e = \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)} \), where \( \kappa_e \) is defined in equation (2.7) and is not a function of \( \theta \). The derivative of \( n_e \) with respect to \( \pi(\theta) \) is

\[
\frac{\kappa_e - 1}{(1 - \pi(\theta))^2}
\]

112
Figure A.1: $v(\theta, n)$ for given levels of $\theta$. $\theta_2 > \theta_1$. $v(\theta_1, n)$ is line $ABCD$, and $v(\theta_2, n)$ is line $A'B'C'DE$. $v(\theta_1, n) = v(\theta_2, n)$ for $n_e(\theta_1) < n \leq 1$, and $v(\theta_2, n) > v(\theta_1, n)$ for $0 \leq n \leq n_e(\theta_1)$.

When $n_e(\theta) \in (0, 1)$ (i.e., when $\pi(\theta) < \kappa_e < 1$), equation (A.8) is negative and so $n_e(\theta)$ is strictly decreasing in $\pi(\theta)$. Since $\pi(\theta)$ is strictly increasing in $\theta$, $n_e(\theta)$ is strictly decreasing in $\theta$. Similarly, since $n_s(\theta) = \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)}$, when $n_s(\theta)$ is between $[0, 1]$ (i.e., $\pi(\theta) < \kappa_s < 1$), the derivative of $n_s(\theta)$ with respect to $\theta$ is $\frac{\kappa_s - 1}{(1 - \pi(\theta))^2}$ and is negative, so $n_s$ is decreasing in $\theta$.

Since $\pi(\theta)$ is continuous in $\theta$, from the function form of $n_e(\theta)$ and $n_s(\theta)$, we can see that both $n_e(\theta)$ and $n_s(\theta)$ are continuous in $\theta$.

**Lemma 2.** Suppose $n_e(\theta) \in (0, 1)$. Then for $n \in (n_e(\theta), 1]$, $v(\theta)$ is positive and its value does not depend on $\theta$. For $0 \leq n \leq n_e(\theta)$, $v(\theta, n)$ is negative and strictly increasing in $\theta$.

Proof: (The intuition are shown in Figure A.1). Suppose $\theta$ increases from $\theta_1$ to $\theta_2$. Since $n_e(\theta)$ is strictly decreasing in $\theta$, we have $n_e(\theta_2) < n_e(\theta_1)$. For $n > n_e(\theta_1)$, $v(\theta_1, n) = v(\theta_2, n)$ (line $CDE$). $n_f$ is not a function of $\theta$. And from Table 2.1, we can see that for $n > n_e(\theta)$, $v(\theta, n)$ is not a function of $\theta$. For $n \in (n_e(\theta_2), n_e(\theta_1)]$, we have $v(\theta_2, n) > 0 \geq v(\theta_1, n)$. For $n \in [0, n_e(\theta_2)]$, the value of $v$ is $v(\theta, n) = (1 - \pi(\theta))(\bar{r}^m - r^n)$, $1 - \pi(\theta)$ is strictly increasing in $\theta$, so in order to prove that $v(\theta, n)$ is strictly increasing in $\theta$, we need to show that $r^n(\theta)$ is weakly deceasing in $\theta$ over $n \in [0, n_e(\theta_2)]$, i.e., $r^n(\theta_2, n) \leq r^n(\theta_1, n)$.

First, if $n_e(\theta_2) \leq n_s(\theta_1)$, since $r^n(\theta_1, n) = \bar{r}^m$ for $n \leq n_s(\theta_1)$, we must have $r^n(\theta_1, n) \geq r^n(\theta_2, n)$ over $n \in [0, n_e(\theta_2)]$ since $r^n(\theta_2, n)$ can not be higher than $\bar{r}^m$.

Second, if $n_s(\theta_1) \leq n_e(\theta_2)$, then we can separate $[0, n_e(\theta_2)]$ into two parts $[0, n_s(\theta_1)]$ and...
Lemma 3. The single crossing property of $v(\theta, n(\theta, \theta^*))$: There exists only one level of $\theta$, denoted as $\theta_c$, such that $v(\theta, n(\theta, \theta^*)) > 0$ if $\theta > \theta_c$, and $v(\theta, n(\theta, \theta^*)) < 0$ if $\theta \leq \theta_c$.

The meaning is that $v(\theta, n(\theta, \theta^*))$ crosses the zero line from below once and only once.

Proof: We can derive this property of $v(\theta, n(\theta, \theta^*))$ from the properties of $v(\theta, n)$. Let $v(\theta, n)$ denote the value function $v(\theta, n)$ with $\theta = \theta_e$, and suppose $n(\theta_e) \in (0, 1)$. Then from Figure A.1, we know that at $n = n(\theta_e)$, we have $v(\theta_e, n) = 0$, and $v(\theta_e, n) > 0$ for $n > n(\theta_e)$ and $v(\theta_e, n) < 0$ for $n < n(\theta_e)$.

Suppose the threshold of the equilibrium is $\theta^*$. In the equilibrium, $n(\theta, \theta^*)$ uniformly increases from 0 to 1 when $\theta$ increases from $\theta^* - \epsilon$ to $\theta^* + \epsilon$ (equation 2.13). On the other hand, we’ve shown in lemma 1 that if $0 < n_e(\theta) < 1$, then $n_e(\theta)$ is continuous and decreasing in $\theta$. Then there should exist a fixed point $\theta \in [\theta^* - \epsilon, \theta^* + \epsilon]$ at which $n(\theta, \theta^*) = n_e(\theta)$. Denote this $\theta$ as $\theta_c$. And we have $n(\theta_c, \theta^*) = n_e(\theta_c)$ and so $v(\theta_c, n(\theta_c, \theta^*)) = v(\theta_c, n_e(\theta_c)).$ \(^1\)

We’ve shown in lemma 2 that if $\theta$ increases from $\theta_1$ to $\theta_2$, for $n > n_e(\theta_1)$, the value of $v(\theta, n)$ is positive and does not depend on $\theta$, that is, $v(\theta, n) = v(\theta_1, n) > 0$ if $n > n_e(\theta_1)$ and $\theta > \theta_1$. Let $\theta_1 = \theta_e$, then for $\theta > \theta_e$, we have $n(\theta, \theta^*) > n_e(\theta)$, and so $v(\theta, n(\theta, \theta^*)) = v(\theta_e, n(\theta, \theta^*)) > 0$. That is, $v(\theta, n(\theta, \theta^*)) > 0$ for $\theta > \theta_e$. We also show in lemma 2 that if the value of $\theta$ decreases from $\theta_2$ to $\theta_1$, for $n < n_e(\theta_2)$, the value of $v(\theta, n)$ is negative and increasing in $\theta$, that is, $v(\theta_1, n) < v(\theta_2, n) < 0$. Let $\theta_2 = \theta_e$, then for $\theta < \theta_e$, we have $n(\theta, \theta^*) < n_e(\theta)$, and so $v(\theta, n(\theta, \theta^*)) < v(\theta_e, n(\theta, \theta^*)) < 0$. As a result, $v(\theta, n(\theta, \theta^*)) < 0$

\(^1\)We do not need all $n_e(\theta)$ to be located between (0, 1). But we know that there must exist some $\theta$ at which $n_e(\theta)$ is between (0, 1). If all $n_e(\theta)$ were equal to 0, then all $v(\theta, n(\theta, \theta^*))$ will be negative. And if all $n_e(\theta)$ were equal to 1, then all $v(\theta, n(\theta, \theta^*))$ will be positive. But according to the definition of $\theta^*$, the average value of $v(\theta, n(\theta, \theta^*))$ over $[\theta^* - \epsilon, \theta^* + \epsilon]$ should be zero.
for $\theta < \theta_c$. Thus, we proved that $v(\theta, n(\theta, \theta^*))$ changes from negative to positive values once and only once. (see Figure A.2)

**Dominance region**

Recall that $n_e = \frac{\kappa_e - \pi(\theta)}{1 - \pi(\theta)}$, and $n_e \in (0, 1)$ if $\pi(\theta) < \kappa_e < 1$. If $\kappa_e < \pi(\theta)$, $n_e$ is defined as 0, which means the bank would fail even if no one withdraw in stage 1. Let $\overline{\theta}$ denote the level of $\theta$ at which $\pi(\theta) = \kappa_e$. Then for $\theta > \overline{\theta}$, $n_e = 0$, so $v(\theta, n) > 0$ for all values of $n$ since $v(\theta, n)$ is positive for all $n > n_e(\theta)$. This means if $\theta > \overline{\theta}$, it would be optimal to withdraw no matter what other people’s decisions are. Since the private signal $\theta_i$ is uniformly distributed over $[\theta - \epsilon, \theta + \epsilon]$, if people observe signal $\theta_i > \overline{\theta} + \epsilon$, they would know that the true value of $\theta$ is higher than $\overline{\theta}$, and so they would choose to withdraw. We call the region $[\overline{\theta} + \epsilon, 1]$ as the upper dominance region.

But there is no lower dominance region when $n_e(\theta) < 1(\kappa_e < 1)$. If a depositor thinks that all other people will run (i.e., $n = 1$), then $v(\theta, n = 1)$ would be positive and the depositor would also run. We impose Assumption 2 such that people will not run the bank for extremely good signals.\footnote{Goldstein and Pauzner(2005) argue that we can use this type of assumptions to help analyzing the equilibrium selection process when there is no genuine upper or lower dominance region.} Denote $\epsilon$ as $\theta$. If Assumption 2 holds, then given all other agents $j \neq i$ do not run for $\theta_j \in [0, \epsilon]$, agent $i$ should also find that the expected payoff to run for $\theta_i \leq \theta$ is negative. In the limit, $\theta = \epsilon \rightarrow 0$. So as long as in the equilibrium, the threshold $\theta^*$ is above zero, then the equilibrium is consistent with Assumption 2. If we can not find a positive threshold equilibrium $\theta^*$, then we simply assume that people will run regardless of the private signal.

**Prove the unique threshold equilibrium**

With the above intermediate results, we can now prove the the result of proposition 1 that there is a unique threshold equilibrium. First, we need to show that the equilibrium condition
only holds at one value of $\theta^*$. Then we need to show that the depositors do not want to deviate from the equilibrium strategy.

The equilibrium is defined by $\Phi(\theta', \theta') = 0$. We will show that there is one and only one value of $\theta'$ which satisfies this condition.

First, assume that Assumption 2 holds, then it is strictly better not to withdraw when $\theta_i \leq \theta$, and we have $\Phi(\theta', \theta') < 0$ for $\theta' \leq \theta$. If the private signal is higher than $\bar{\theta} + \epsilon$, we are in the upper dominance region and all $v(\theta, n)$ in (2.14) is positive. So for $\theta' > \bar{\theta} + \epsilon$ we have $\Phi(\theta', \theta') > 0$. This means $\theta^*$ is between $[\theta, \bar{\theta} + \epsilon]$. From the function form of $\Phi(\theta', \theta')$, we can see that $\Phi(\theta', \theta')$ is continuous because the boundary of the integral is continues in $\theta_i$, and the value of $v(\theta, n)$ is bounded. So there exists at least one $\theta'$ such that $\Phi(\theta', \theta') = 0$. In addition, we can prove that $\Phi(\theta', \theta')$ is strictly increasing in $\theta'$ between $[\theta, \bar{\theta} + \epsilon]$. First, when the private signal (the first $\theta'$ in $\Phi(\theta', \theta')$) and the threshold (the second $\theta'$ in $\Phi(\theta', \theta')$) increase by the same amount, the expected distribution of $n$ does not change, it is still uniformly distributed over $[0, 1]$ for $\theta \in [\theta' - \epsilon, \theta' + \epsilon]$. Second, given the distribution of $n$, we can show that the average value of $v(\theta, n)$ is strictly increasing in $\theta$. The reason is as follows. We’ve already shown that $v(\theta, n)$ is weakly increasing in $\theta$, so we need to show that there are at least some $v(\theta, n)$ that are strictly increasing in $\theta$. Recall that in lemma 2, we show that for $n < n_e(\theta)$, $v(\theta, n)$ is strictly increasing in $\theta$. This means we need to show that there exist some $n$ for which we have $n < n_e(\theta)$. When the private signal $\theta_i$ reaches the upper bound $\bar{\theta} + \epsilon$, the lower boundary of the integral in equation (2.14) is $\theta_i - \epsilon = \bar{\theta}$, so we need to show that when $\theta < \bar{\theta}$, there are at least some $n$ that satisfy $n < n_e(\theta)$. For $\theta < \bar{\theta}$, we have $n_e(\theta) > 0$. Since $n$ is uniform over $[0, 1]$, we must have some $n$ which is smaller than $n_e(\theta)$. As a result, there are at least some $n$ that satisfy $n < n_e(\theta)$, and so there are at least some $v(\theta, n)$ which are strictly increasing in $\theta$. Thus, $\Phi(\theta', \theta')$ is strictly increasing in $\theta'$ when $\theta' \in [\theta, \bar{\theta} + \epsilon]$, which means there is only one level of $\theta' = \theta^*$ at which the equilibrium condition is satisfied.

The next step is to prove that depositors will not want to deviate from the equilibrium
strategy. That is, not only \( \Phi(\theta^*, \theta^*) = 0 \), given the private signal \( \theta_i \), \( \Phi(\theta_i, \theta^*) \) is smaller than zero if \( \theta_i < \theta^* \) and higher than zero if \( \theta_i > \theta^* \). This result can be proved using the “single crossing property” proved in lemma 3. As shown in Figure A.2, \( v(\theta, n(\theta, \theta^*)) \) changes from negative to positive value once and only once at \( \theta_c \).

According to the definition of the equilibrium, we have

\[
\Phi(\theta^*, \theta^*) = \frac{1}{2\epsilon} \int_{\theta^* - \epsilon}^{\theta^* + \epsilon} v(\theta, n(\theta, \theta^*)) d\theta = 0 \tag{A.9}
\]

which means that the average value of \( v(\theta, n(\theta, \theta^*)) \) over \([\theta^* - \epsilon, \theta^* + \epsilon]\) is zero.

Now suppose the private signal received by a depositor is \( \theta_i < \theta^* \). The value of \( \Phi(\theta_i, \theta^*) \) is

\[
\Phi(\theta_i, \theta^*) = \frac{1}{2\epsilon} \int_{\theta_i - \epsilon}^{\theta_i + \epsilon} v(\theta, n(\theta, \theta^*)) d\theta \tag{A.10}
\]

which is the average value of \( v(\theta, n(\theta, \theta^*)) \) over \([\theta_i - \epsilon, \theta_i + \epsilon]\). Since \( v(\theta, n(\theta, \theta^*)) \) changes from negative to positive values once and only once, from Figure A.2, we can see that compared with \( \Phi(\theta^*, \theta^*) \), \( \Phi(\theta_i, \theta^*) \) replaces the positive values of \( v(\theta, n(\theta, \theta^*)) \) between \([\theta_i + \epsilon, \theta^* + \epsilon]\) with negative values between \([\theta_i - \epsilon, \theta^* - \epsilon]\), so \( \Phi(\theta_i, \theta^*) < 0 \) and the depositors would prefer to wait. Similarly, if \( \theta_i > \theta^* \), \( \Phi(\theta_i, \theta^*) > 0 \) and the depositors would prefer to withdraw. So in the equilibrium, the depositors will not deviate from the equilibrium strategy.
A.1.3 Proof for proposition 3

Proof: We first show that $\kappa_e > \pi(\theta^*)$. Because the average value of $v(\theta^*, n)$ is zero over $n \in [0, 1]$, and because the value of $v(\theta^*, n)$ is negative over $[0, n_e(\theta^*))$ and positive over $(n_e(\theta^*), 1]$, we must have $n_e(\theta^*) > 0$, otherwise, the average value of $v(\theta^*, n)$ would be positive. $n_e(\theta^*) > 0$ means

$$\frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)} > 0 \implies \kappa_e > \pi(\theta^*) \quad (A.11)$$

In stage 1, according to equation (2.13), we have $n = 0$ if $\theta < \theta^* - \epsilon$ and $n = 1$ if $\theta > \theta^* + \epsilon$. If $\epsilon \to 0$, then $\theta^* + \epsilon \to \theta^*$, so all people will withdraw when $\theta > \theta^*$, and will not withdraw if $\theta < \theta^*$. Since $\theta$ is uniformly distributed between $[0, 1]$, bank runs in stage 1 happen with probability

$$\int_{\theta^*}^{1} d\theta = 1 - \theta^* \quad (A.12)$$

If $\theta < \theta^*$, depositors will not withdraw in stage 1, that is, $n = 0$. In stage 2, the minimum withdrawal is the amount of movers $\pi(\theta)$. Since $\pi(\theta^*) < \kappa_e$, so when $\theta < \theta^*$, we have $\pi(\theta) < \kappa_e$, which means if non-movers choose to wait together, their payment in period 3 is higher than $\overline{r_m}$. As a result, non-movers will choose to wait in stage 2, and no bank run will happen.\[3\]

A.1.4 Proof of Proposition 2

Other than affecting $n(\theta, \theta^*)$, $\theta$ only affects the value of $v(\theta, n(\theta, \theta^*))$ by affecting the value of $\pi(\theta)$. This can be seen from the function form of $v(\theta, n)$ in Table 2.1. Given $n$, $\theta$ only enters $v(\theta, n)$ through $\pi(\theta)$ and $r^n$, and $r^n$ is related with $\theta$ only through $\pi(\theta)(r^n)$ is related with $\kappa$, which is a function of $\pi(\theta)$. So when $\epsilon \to 0$, we can approximate the value of $v(\theta, n(\theta, \theta^*))$ with $v(\theta^*, n(\theta, \theta^*))$. And as explained in the main text, we get (2.16) and (2.17). (2.17) can be computed using the function form of $v(\theta, n)$ in Table 2.1. The result is as follows:

\[3\]n in stage 1 is located in $[0, 1]$ only when $\theta \in [\theta^* - \epsilon, \theta^* + \epsilon]$, this range is small in the limit.
If $n_s(\theta^*) > 0$, we have

$$
\int_0^1 v(\theta^*, n) dn = \int_0^{n_c(\theta^*)} (1 - \pi(\theta^*)) (\bar{\mu} - \bar{\mu}) dn \\
+ \int_{n_c(\theta^*)}^{n_e(\theta^*)} (1 - \pi(\theta^*)) (\bar{\mu} - r^n(\theta^*, n)) dn + \int_{n_e(\theta^*)}^{n_f(\theta^*)} \frac{1 - n_f}{1 - n} \bar{\mu} dn + \int_{n_f}^1 \frac{n_f}{1 - n} \bar{\mu} dn \\
= n_c(\theta^*) (1 - \pi(\theta^*)) \bar{\mu} - n_s(\theta^*) \bar{\mu} - (1 - \pi(\theta^*)) \int_{n_c(\theta^*)}^{n_e(\theta^*)} r^n(\theta^*, n) dn + \\
(1 - n_f) \bar{\mu} [\ln(1 - n_c(\theta^*)) - \ln(1 - n_f)] - n_f \bar{\mu} \ln n_f 
$$

(A.13)

Using $n_c(\theta^*) = \frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)}$, $n_s(\theta^*) = \frac{\kappa_s - \pi(\theta^*)}{1 - \pi(\theta^*)}$, and $r^n(\theta^*, n)$ from (A.6), we get

$$
(1 - \pi(\theta^*)) \int_{n_c(\theta^*)}^{n_e(\theta^*)} r^n(\theta^*, n) dn \\
= (1 - \pi(\theta^*)) \int_{\frac{\kappa_e - \pi(\theta^*)}{1 - \pi(\theta^*)}}^{\frac{\kappa_s - \pi(\theta^*)}{1 - \pi(\theta^*)}} \left( \frac{R}{\lambda} \bar{\mu} + \frac{1}{(1 - \pi(\theta^*)) (1 - n)} \bar{\mu} \right) dn \\
= \frac{R}{\lambda} \bar{\mu} (\kappa_e - \kappa_s) + W \ln \left( \frac{1 - \kappa_s}{1 - \kappa_e} \right) 
$$

Arranging terms, we get

$$
\int_0^1 v(\theta^*, n) dn = (\kappa_e - \pi(\theta^*)) \bar{\mu} - (\kappa_s - \pi(\theta^*)) \bar{\mu} - \left[ \frac{R}{\lambda} \bar{\mu} (\kappa_e - \kappa_s) + W \ln \left( \frac{1 - \kappa_s}{1 - \kappa_e} \right) \right] + \\
(1 - n_f) \bar{\mu} [\ln(1 - \kappa_e) - \ln(1 - n_f) - \ln(1 - \pi(\theta^*))] - n_f \bar{\mu} \ln n_f 
$$

(A.14)

with $W$ defined in equation (A.7). Using similar method, when $n_s(\theta^*) = 0$ (i.e., $\kappa_s < \pi(\theta^*)$),

$$
\int_0^1 v(\theta^*, n) dn \quad \text{is equal to} \\
(\kappa_e - \pi(\theta^*)) \bar{\mu} - \left[ \frac{R}{\lambda} \bar{\mu} (\kappa_e - \pi(\theta^*)) + W \ln \left( \frac{1 - \pi(\theta^*)}{1 - \kappa_e} \right) \right] + \\
(1 - n_f) \bar{\mu} [\ln(1 - \kappa_e) - \ln(1 - n_f) - \ln(1 - \pi(\theta^*))] - n_f \bar{\mu} \ln n_f 
$$

(A.15)

A.1.5 Proof of proposition 4

In this part, we prove the results of proposition 4. We also derive the analytical results when $\pi = \theta^n$. 

119
Homogenous information

Depositor’s expected payoff

Under homogeneous information, depositors will run the bank only when \( \pi(\theta) > \kappa_e (\theta > \theta_e) \).
If a bank run happens, the expected payoff is \( n_f d_t \bar{r} \). If \( \theta \leq \theta_e \), there is no bank run and only movers withdraw from the bank in period 2, non-movers will wait until period 3 and get the payoff \( d_t r^n \). We get

\[
E u^d = \int_{\theta_e}^{1} n_f d_t \bar{r} d\theta + \int_{0}^{\theta_e} [\pi(\theta) d_t \bar{r} + (1 - \pi(\theta)) d_t r^n(\theta)] d\theta
\]

\[
\Rightarrow \frac{E u^d}{d_t} = (1 - \theta_e) n_f \bar{r} + \int_{0}^{\theta_e} [\pi(\theta) \bar{r} + (1 - \pi(\theta)) r^n(\theta)] d\theta \tag{A.16}
\]

where \( \frac{E u^d}{d_t} \) is the expected return. \( r^n(\theta) \) can be solved from equations \( (A.4) \) and \( (A.5) \) by changing \( \kappa \) into \( \pi(\theta) \), the solution is \( R_\lambda r_m + \frac{1}{1 - \pi(\theta)} W \). Substituting it into \( (A.16) \), we have

\[
\frac{E u^d}{d_t} = (1 - \theta_e) n_f \bar{r} + \int_{0}^{\theta_e} [\pi(\theta) \bar{r} + (1 - \pi(\theta)) r^n(\theta)] d\theta
\]

\[
+ \int_{\theta_s}^{\theta_e} [\pi(\theta) \bar{r} + (1 - \pi(\theta)) \frac{R_\lambda \bar{r}}{R_\lambda \bar{r} + W}] d\theta \tag{A.17}
\]

For \( \pi(\theta) = \theta^\eta \), \( (A.17) \) becomes

\[
\frac{E u^d}{d_t} = (1 - \theta_e) n_f \bar{r} + \int_{0}^{\theta_s} [\theta^\eta \bar{r} + (1 - \theta^\eta) \bar{r}] d\theta + \int_{\theta_s}^{\theta_e} [\theta^\eta \bar{r} + (1 - \theta^\eta) \frac{R_\lambda \bar{r}}{R_\lambda \bar{r} + W}] d\theta
\]

\[
= (1 - \theta_e) n_f \bar{r} + \frac{\theta_s^{\eta+1}}{\eta+1} + \frac{\theta_e^{\eta+1}}{\eta+1} + \frac{R_\lambda \bar{r}}{R_\lambda \bar{r} + W} (\theta_e - \theta_s)
\]

\[
\frac{R_\lambda \bar{r} \theta_e^{\eta+1} - \theta_s^{\eta+1}}{\eta+1} \tag{A.18}
\]

where \( \theta_e = \kappa_e^\frac{1}{\eta} \) and \( \theta_s = \kappa_s^\frac{1}{\eta} \). \( \kappa_e \) and \( \kappa_s \) are defined in \( (2.7) \) and \( (A.3) \).

Bank’s expected profit

For each realized value of \( \theta \), the bank’s profit is equal to its gross payment minus the opportunity cost of its equity \( c_b R \).

Banks must use all their resources to make the payment to depositors. When \( \theta > \theta_s \), banks will not be able to make the promised payment, which means they lose all their equity.
So for $\theta > \theta_s$, the gross payment to banks is zero. Thus, we only need to focus on the gross payment for $\theta \in [0, \theta_s]$. Over $\theta \in [0, \theta_s]$, there is no bank runs, the payment to movers is $\bar{r}_m$ and the payment to non-movers in period 3 is $\bar{r}_n$.

Let $\theta_a$ denote the highest $\theta$ at which the bank can meet the withdrawal in period 2 with its own reserve. Then if $\theta \leq \theta_a$, the gross payment to the bank is

$$
(\gamma d_t - \pi(\theta) d_t \bar{r}_m) + [(1 - \gamma) d_t + e_b] R - (1 - \pi(\theta)) \bar{r}_m d_t
$$

where $\pi(\theta) d_t \bar{r}_m$ is the total withdrawal of movers in period 2 and $(\gamma d_t - \pi(\theta) d_t \bar{r}_m)$ is the reserve carried to period 3. $[(1 - \gamma) d_t + e_b] R$ is the return from investment and $(1 - \pi(\theta)) \bar{r}_m d_t$ is the total payment to non-movers.

Over $\theta \in [\theta_a, \theta_s]$, the bank needs to borrow from the central bank or liquidate assets. The budget constraint is

$$
\pi(\theta) d_t \bar{r}_m = \gamma d_t + b + \lambda \phi [(1 - \gamma) d_t + e_b] \\
(1 - \pi(\theta)) d_t \bar{r}_m \leq -b + (1 - \phi) [(1 - \gamma) d_t + e_b] R
$$

Equation (A.20) is the cash constraint in $t = 2$, $b$ is the actual borrowing from the central bank. (A.21) is the resource constraint in period 3. (A.21) gives the gross revenue of the bank:

$$
-b + (1 - \phi) [(1 - \gamma) d_t + e_b] R - (1 - \pi(\theta)) d_t \bar{r}_m
$$

Combining (A.19) and (A.22), the expected profit of the bank can be written as

$$
E\Pi = \int_0^{\theta_s} \left[ \max(\gamma d_t - \pi(\theta) d_t \bar{r}_m, 0) - b + (1 - \phi(\theta)) [(1 - \gamma) d_t + e_b] R - (1 - \pi(\theta)) \bar{r}_m d_t \right] d\theta
$$

where $e_b R$ which is equal to the expected remaining resources after all payments are made, minus the opportunity cost of bank’s equity $e_b R$. 

121
Equation (A.23) can be written as
\[
\begin{align*}
\mathcal{E}^\Pi & = \int_{0}^{\theta_a} (\gamma d_t - \pi(\theta) d_t r^{\mathcal{M}}) + [(1 - \gamma)d_t + e_b]Rd\theta + \int_{\theta_a}^{\theta_b} -b + [(1 - \gamma)d_t + e_b]Rd\theta \\
& + \int_{\theta_b}^{\theta} -\bar{b} + (1 - \phi(\theta))(1 - \gamma)d_t + e_b]Rd\theta - \int_{0}^{\theta} (1 - \pi(\theta))r^{\mathcal{M}}d\theta - e_bR \\
\end{align*}
\]
(A.24)

\[\theta_b\] is the \(\theta\) above which liquidation is needed. Over \(\theta \in [0, \theta_a]\), the withdrawal can be met with bank’s own reserve, we have \(b = 0\) and \(\phi = 0\). For \(\theta \in [\theta_a, \theta_b]\), the withdrawal can be met by borrowing from the central bank, we have \(b \in [0, \bar{b}]\) and \(\phi = 0\). Above \(\theta_b\), liquidation is needed and \(\phi > 0\).

For \(\pi(\theta) = \theta^n\), we have the following results. \(\theta_a\) and \(\theta_b\) can be computed from
\[
\begin{align*}
\pi(\theta_a) d_t r^{\mathcal{M}} &= \gamma d_t \quad \Rightarrow \quad \theta_a = \left(\frac{\gamma}{r^{\mathcal{M}}}\right)^{\frac{1}{n}} \quad (A.25) \\
\pi(\theta_b) d_t r^{\mathcal{M}} &= \gamma d_t + \bar{b} \quad \Rightarrow \quad \theta_b = \left(\frac{\gamma d_t + \bar{b}}{d_t r^{\mathcal{M}}}\right)^{\frac{1}{n}} \quad (A.26)
\end{align*}
\]
We set \(\theta_a\) or \(\theta_b\) to 1 if (A.25) or (A.26) is higher than 1.

The first integral in (A.24) is equal to \((\gamma d_t + [(1 - \gamma)d_t + e_b]R)	heta_a - d_t r^{\mathcal{M}}\theta_a^{\frac{n+1}{n+1}}\). The second integral is equal to \([(1 - \gamma)d_t + e_b]R(\theta_b - \theta_a) - \theta_b^{\frac{n+1}{n+1}} - \theta_a^{\frac{n+1}{n+1}} + \gamma d_t (\theta_b - \theta_a)\), where we use \(b = \pi(\theta) d_t r^{\mathcal{M}} - \gamma d_t\) for \(b \in [0, \bar{b}]\). The third integral is equal to \((-\bar{b} + [(1 - \gamma)d_t + e_b]R)(\theta_s - \theta_b) - R d_t r^{\mathcal{M}} - \theta_b^{\frac{n+1}{n+1}} + \bar{b}\) \((\theta_s - \theta_b)\), where we use \(\phi(\theta) = \frac{\pi(\theta) d_t r^{\mathcal{M}} - \gamma d_t - \bar{b}}{\lambda(1 - \gamma)d_t + e_b}\), which is solved from (A.20) and (A.21). And the forth integral is equal to \(-\frac{\pi(\theta) d_t r^{\mathcal{M}} - \gamma d_t - \bar{b}}{\lambda(1 - \gamma)d_t + e_b}\).

**Heterogenous information**

**Depositor’s expected payoff**

Under heterogeneous information, depositors will run the bank if \(\theta > \theta^*\), and the probability to get paid is \(n_f\). If \(\theta < \theta^*\), there is no bank run and only movers withdraw in stage 2. So the expected payoff is
\[
\begin{align*}
E u^d &= \int_{\theta_a}^{\theta^*} n_f d_t r^{\mathcal{M}}d\theta + \int_{0}^{\theta^*} [\pi(\theta) d_t r^{\mathcal{M}} + (1 - \pi(\theta)) d_t r^n(\theta)] d\theta \\
\Rightarrow \frac{E u^d}{d_t} &= (1 - \theta^*) n_f r^{\mathcal{M}} + \int_{0}^{\theta^*} [\pi(\theta) r^{\mathcal{M}} + (1 - \pi(\theta)) r^n(\theta)] d\theta \\
\end{align*}
\]
(A.27)
If \( \theta^* \geq \theta_s \), then for \( \theta \in [0, \theta^*] \), there are no bank runs and the payoff is the same as in the homogeneous information case. And we only need to change \( \theta_c \) in equation (A.18) into \( \theta^* \).

If \( \theta^* < \theta_s \), then \( r^n = \overline{r^n} \) over \( [0, \theta^*] \), and we have

\[
\frac{Eu^d}{dt} = (1 - \theta^*)n_f \overline{r^m} + \int_0^{\theta^*} [\pi(\theta) \overline{r^m} + (1 - \pi(\theta)) \overline{r^n}] d\theta
\]

\[
= (1 - \theta^*)n_f \overline{r^m} + \overline{r^n} \theta^* + \frac{(\theta^*)^{\eta+1}}{\eta + 1} \left( \frac{\overline{r^m} - \overline{r^n}}{\eta + 1} \right) \quad (A.28)
\]

**Bank’s expected profit**

If \( \theta^* > \theta_s \), since we only need to consider \( \theta \in [0, \theta_s] \), the expected profit can be written in the same way as (A.23), and result is equation (A.24).

If \( \theta^* < \theta_s \), since bank’s gross revenue is zero when \( \theta > \theta^* \) (due to the bank run), bank’s expected profit is

\[
E\Pi = \int_0^{\theta^*} \left\{ \max(\gamma d_t - \pi(\theta) d_t \overline{r^m}, 0) - b + (1 - \phi(\theta))(1 - \gamma) d_t + e_b R \right\} d\theta - e_b R 
\]

\[
= -(1 - \pi(\theta)) \overline{r^m} d_t \right\} d\theta - e_b R \quad (A.29)
\]

which is the same as equation (A.23) except the upper bound of the integral is changed from \( \theta_s \) to \( \theta^* \). There are three possibilities: \( 0 \leq \theta^* \leq \theta_a \), \( \theta_a < \theta^* \leq \theta_b \) and \( \theta_b < \theta^* < \theta_s \). We only need to change equation (A.24) accordingly. The details are omitted.
Appendix B

Proofs and additional results for Chapter 3

B.1 The detailed transaction steps

In this part, we use bank balance sheet to show the transactions and monetary flows in the model. Recall that the inter-bank settlement at the end of period $t+1$ is carried out based on net balance. But for illustrative purposes, we show the gross payment flows. We normalize the initial bank equity to zero. We put the interest costs and interest income of the bank under the entry “Bank Equity”.

The initial deposit and reserve balance is $D_0$, the total deposit held by movers is $\pi D_0$, and the total deposit held by non-movers is $(1 - \pi)D_0$. After the shocks, investment funds sell assets to non-movers and raise cash $(1 - \pi)D_0$. Investment funds also borrow $L$ from the bank. After the loan is made, ‘loan” and “deposit” on the balance sheet increase by the same amount $L$, and the new deposit $L$ is created. Investment funds then use all their deposits $L + (1 - \pi)D_0$ to meet the redemption needs of movers, and movers end up holding all the deposits $D_0 + L$. After the redemption process is completed, the bank pays the interest cost $r\tilde{b}_n$ of the central bank loan.

At the end of the period, the central bank consumes the interest income by using central bank money to buy goods from investment funds. The reserve balance of the bank and
### Table B.1: Monetary flows

<table>
<thead>
<tr>
<th></th>
<th>Asset Side</th>
<th>Liability side</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Balance sheet of a representative commercial bank in location i (when (L &gt; 0))</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Reserve (Fund)</strong></td>
<td>Balance before the liquidity shock</td>
<td></td>
</tr>
<tr>
<td><strong>Loan (movers)</strong></td>
<td>(D_0) = 0</td>
<td>(πD_0) = 0</td>
</tr>
<tr>
<td></td>
<td>((1− π)D_0) = 0</td>
<td>0 = 0</td>
</tr>
<tr>
<td><strong>Deposit (non-movers)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Deposit (Fund)</strong></td>
<td>(-L (1− π))</td>
<td>(+L(1− π))</td>
</tr>
<tr>
<td><strong>Bank Equity</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Changes in each account after the liquidity shock</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Investment fund sells assets to non-movers</td>
<td>(-L (1− π))</td>
<td>(+L(1− π))</td>
</tr>
<tr>
<td>Bank makes loans to the investment fund</td>
<td>(+L)</td>
<td>(+L)</td>
</tr>
<tr>
<td>Movers redeem fund shares</td>
<td>(+L + (1− π)D_0)</td>
<td>(-L) = ((-L - (1-π)D_0))</td>
</tr>
<tr>
<td>Bank pays the borrowing cost</td>
<td>(-r^d b_n)</td>
<td>(-r^d b_n)</td>
</tr>
<tr>
<td><strong>Changes in each account at the end of the Period</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The central bank consumes the interest</td>
<td>(+r^d b_n)</td>
<td>(+r^d b_n)</td>
</tr>
<tr>
<td>Bank pays the deposit interest</td>
<td>(+r^d(D_0 + L))</td>
<td>(-r^d(D_0 + L))</td>
</tr>
<tr>
<td>Movers(i) buy goods in location j</td>
<td>(-L + (1 + r^d))</td>
<td>(+L(1 + r^d))</td>
</tr>
<tr>
<td>Movers(j) buy goods in location i</td>
<td>(-L)</td>
<td>(-L(1 + r^d))</td>
</tr>
<tr>
<td>Investment fund repays the bank loan</td>
<td>(+)</td>
<td>(+Lr^d)</td>
</tr>
<tr>
<td>Bank spends the interest income</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td><strong>Final Balance</strong></td>
<td>(D_0) = 0</td>
<td>(0) = 0</td>
</tr>
</tbody>
</table>

the deposit of investment funds will increase by the same amount. Banks then pay deposit interest to movers. Movers then use their deposits to buy goods. When movers from location \(i\) buy goods in location \(j\), their deposit balance decreases by \((D_0 + L)(1 + r^d)\). And when movers from location \(j\) buy goods from the investment funds in location \(i\), the deposit balance of those investment funds will increase by \((D_0 + L)(1 + r^d)\).

Investment funds then repay the bank loan. The outstanding loan is reduced by \(L\), and the outstanding deposit is reduced by \(L(1 + r^d)\), and the interest income of the bank is \(Lr^d\). The bank then spends the income \([Lr^d - r^d b_n - r^d(D_0 + L)]\) to buy goods from investment funds.\(^1\) When the bank spends the income, the bank pays the sellers(investment funds) by

\(^1\)Note that the income is usually positive because the lending rate \(r^d\) includes the management cost for bank loans. If \(δ\) is not too small, then \(Lr^d\) should be higher than the interest costs of the bank. In case where the income is negative, then the bank can sell its endowment to absorb the loss.

---

125
crediting their deposit accounts.

After all the above steps are completed, investment funds will then transfer the deposit balance $D_0$ to non-movers, who will then use the deposit to purchase goods from the young generation.

### B.2 Additional results: the general case $\sigma \geq 1$, no bank lending

#### B.2.1 The optimal payout policy of the investment fund

**Lemma 4.** If $Q_k = R_k$, then the optimal policy is to set $v_m = v_n$, and $r_m = r_n = \alpha + (1 - \alpha)R_k$. When $Q_k < R_k$, if the constraint $r_m \leq r_n$ is not binding, then the optimal policy is to set

$$\frac{v_m}{v_n} = \frac{\omega + (1 - \omega)r_m}{\omega\frac{R_k}{Q_k} + (1 - \omega)r_n} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\sigma}}$$

(B.1)

Given $\omega$ and $Q_k$, $\frac{r_m}{r_n}$ is increasing in $\sigma$. If the constraint $r_m \leq r_n$ is binding, then the optimal policy is $r_m = r_n$. For the log utility function ($\sigma = 1$), we have

$$r_m = \alpha + (1 - \alpha)Q_k$$

(B.2)

$$r_n = \alpha\frac{R_k}{Q_k} + (1 - \alpha)R_k = \frac{r_m R_k}{Q_k}$$

(B.3)

The meaning of the proposition is as follows. First, as long as $Q_k = R_k$, there is no cost to raise cash by selling assets, and it is optimal to fully insure the liquidity risk and give movers and non-movers the same return. Second, if $Q_k < R_k$, since it is costly to raise cash, the investment fund may not provide full insurance. When people are more risk averse (higher $\sigma$), it is optimal to set a higher $\frac{r_m}{r_n}$, which means to give a higher payment to movers. When
the constraint $r_m \leq r_n$ is binding, it is optimal to set $r_m = r_n$. With the log utility function, the optimal $r_m$ is simply to pay the market value of the fund's asset.

**Proof:**

Let $v_m$ denote the value of movers’ portfolio and $v_n$ the value of non-movers’ portfolio. We have

\begin{align*}
v_m &= s \left[ \omega + (1 - \omega) r_m \right] \quad (B.4) \\
v_n &= s \left[ \omega \frac{R_k}{Q_k} + (1 - \omega) r_n \right] \quad (B.5)
\end{align*}

The expected utility of the household is

\begin{align*}
\frac{1}{2} \int_0^1 \left[ \frac{\pi (v_{m,H})^{1-\sigma}}{1-\sigma} + (1 - \pi) \frac{(v_{n,H})^{1-\sigma}}{1-\sigma} \right] dF(\pi) + \\
\frac{1}{2} \int_0^1 \left[ \frac{\pi (v_{m,L})^{1-\sigma}}{1-\sigma} + (1 - \pi) \frac{(v_{n,L})^{1-\sigma}}{1-\sigma} \right] dF(\pi) \quad (B.6)
\end{align*}

where $H$ and $L$ are productivity shocks. Since $r_m$ is chosen after the shocks are realized, the fund can choose the best $r_m$ for each level of the shock. So the fund maximizes

\begin{equation}
\pi \frac{(v_m)^{1-\sigma}}{1-\sigma} + (1 - \pi) \frac{(v_n)^{1-\sigma}}{1-\sigma} \quad (B.7)
\end{equation}

subject to its budget constraints and the constraint $r_m \leq r_n$ (payment to movers can not be higher than non-movers, otherwise non-movers will pretend to be movers and withdraw.)

We first analyze the case when the fund does not need to sell assets. The budget constraints are

\begin{align*}
\pi r_m &= \phi \alpha \quad (B.8) \\
(1 - \pi) r_n &= (1 - \phi) \alpha + (1 - \alpha) R_k \quad (B.9)
\end{align*}

where $\phi \alpha$ is the riskless asset used to pay movers, $(1 - \phi) \alpha$ is the unused riskless asset and $(1 - \alpha) R_k$ is the value of the risky assets. Using the budget constraints to replace $r_m$ and $r_n$ in $v_m$ and $v_n$ (equation (B.4) and (B.5)), the fund’s problem (B.7) can be written as (we...
eliminate the common “s” from \( v_m \) and \( v_n \)

\[
\pi \left( \omega + (1 - \omega) \frac{\phi \alpha}{\pi} \right)^{1-\sigma} + (1 - \pi) \left( \omega + (1 - \omega) \frac{(1 - \phi) \alpha + (1 - \alpha) R_k}{1 - \pi} \right)^{1-\sigma}
\]

(B.10)

Here, we use \( \frac{R_k}{Q_k} = 1 \) since there is no liquidation of assets. Taking the derivative with respect to \( \phi \) and simplifying the terms, we get

\[
\frac{1}{\left( \omega + (1 - \omega) \frac{\phi \alpha}{\pi} \right)^{\sigma}} - \frac{1}{\left( \omega + (1 - \omega) \frac{(1 - \phi) \alpha + (1 - \alpha) R_k}{1 - \pi} \right)^{\sigma}} = 0
\]

(B.11)

that is, \( \frac{1}{\nu_m} - \frac{1}{\nu_n} = 0 \), which means \( v_m = v_n \) and \( r_m = r_n = \alpha + (1 - \alpha) R_k \).

Next, suppose the fund needs to sell assets. Let \( \eta \) denote the share of risky assets that is liquidated. The budget constraints are

\[
\pi r_m = \alpha + (1 - \alpha) \eta Q_k
\]

(B.12)

\[
(1 - \pi) r_n = (1 - \alpha)(1 - \eta) R_k
\]

(B.13)

The fund maximizes

\[
\pi \left( \omega + (1 - \omega) \frac{\alpha + (1 - \alpha) \eta Q_k}{\pi} \right)^{1-\sigma} + (1 - \pi) \left( \omega \frac{R_k}{Q_k} + (1 - \omega) \frac{(1 - \alpha)(1 - \eta) R_k}{1 - \pi} \right)^{1-\sigma}
\]

(B.14)

Taking the derivative with respect to \( \eta \) and simplifying the terms, we get

\[
\frac{Q_k}{\left( \omega + (1 - \omega) \frac{\alpha + (1 - \alpha) \eta Q_k}{\pi} \right)^{\sigma}} - \frac{R_k}{\left( \omega \frac{R_k}{Q_k} + (1 - \omega) \frac{(1 - \alpha)(1 - \eta) R_k}{1 - \pi} \right)^{\sigma}} = 0
\]

(B.15)

which can be written as

\[
\frac{\omega + (1 - \omega) r_m}{\omega \frac{R_k}{Q_k} + (1 - \omega) r_n} = \left( \frac{Q_k}{R_k} \right)^{\frac{1}{\sigma}}
\]

(B.16)

When \( R_k = Q_k \), we have \( r_m = r_n \).

Now suppose \( \frac{Q_k}{R_k} < 1 \). When \( \sigma > 1 \), given \( \frac{Q_k}{R_k} \), \( \left( \frac{Q_k}{R_k} \right)^{\frac{1}{\sigma}} \) is increasing in \( \sigma \). So higher \( \sigma \) will increase the level of \( r_m \) relative to \( r_n \). That is, people share more liquidity risks when they are more risk averse. When \( \sigma \to \infty \), \( \left( \frac{Q_k}{R_k} \right)^{\frac{1}{\sigma}} \to 1 \), and equation (B.16) would imply that \( r_m > r_n \). Once the constraint \( r_m \leq r_n \) is binding, the fund sets \( r_m = r_n \).
When $\sigma = 1$, (B.16) becomes
\[
\frac{\omega + (1 - \omega)r_m}{\omega \frac{R_k}{Q_k} + (1 - \omega)r_n} = \frac{Q_k}{R_k}
\] (B.17)
which gives $r_n = r_m \frac{R_k}{Q_k}$. Substitute this into (B.12) and (B.13) and we get $\eta = \pi - \frac{\alpha(1-\pi)}{(1-\alpha)Q_k}$, which gives $r_m = \alpha + (1 - \alpha)Q_k$ and $r_n = \alpha \frac{R_k}{Q_k} + (1 - \alpha)R_k$. \[\square\]

**B.2.2 The values of $r_m$, $r_n$ and $Q_k$ in the symmetric equilibrium**

In this part, we take the initial portfolio choice $\alpha$ and $\omega$ as given and solve for $r_m$, $r_n$ and $Q_k$ in the symmetric equilibrium.

We can separate $\pi$ into three ranges: $[0, \pi_1]$, $[\pi_1, \pi_{bind}]$ and $[\pi_{bind}, \pi]$. Non-movers’ cash is binding for $\pi \geq \pi_1$. And for $\pi > \pi_{bind}$, the constraint $r_m \leq r_n$ is binding.

Below $\pi_1$, we have $Q_k = R_k$ and $r_m = r_n = \alpha + (1 - \alpha)R_k$. At $\pi_1$, the payment to movers is equal to the cash collected from non-movers plus the cash held by the fund, and we have
\[
\pi_1 Z_p r_m = Z_f + (1 - \pi_1)D^h
\]
\[
\Rightarrow \quad \pi_1 = \frac{D^h + Z_f}{D^h + Z_p r_m} = \frac{\omega + (1 - \omega)\alpha}{\omega + (1 - \omega)(\alpha + (1 - \alpha)R_k)} = \frac{\kappa}{\kappa + (1 - \kappa)R_k}
\] (B.18)
So $\pi_1$ only depends on $\kappa$.

For $\pi > \pi_1$, in the symmetric equilibrium, we have
\[
\pi Z_p r_m = Z_f + (1 - \pi)D^h \Rightarrow r_m = \frac{Z_f + (1 - \pi)D^h}{\pi Z_p} = \frac{(1 - \omega)\alpha + (1 - \pi)\omega}{\pi(1 - \omega)}
\] (B.20)
Having solved $r_m$, we can use (B.12), (B.13) and (B.16) to solve for equilibrium $Q_k$ and $r_n$.

Between $[\pi_1, \pi_{bind}]$, the constraint $r_m \leq r_n$ is not binding. Using (B.12) and (B.13), we can write $r_n$ as a function of $r_m$ and $Q_k$
\[
r_n = \frac{(1 - \alpha)R_k}{1 - \pi} - \frac{R_k(\pi r_m - \alpha)}{Q_k(1 - \pi)} = \frac{(1 - \alpha)R_k}{1 - \pi} - \frac{R_k}{Q_k} \frac{\omega}{1 - \omega}
\] (B.21)
Then substitute $r_m$ (B.20) and $r_n$ (B.21) into (B.16), we have
\[
\frac{\omega + (1 - \omega)\frac{(1-\omega)\alpha + (1-\pi)\omega}{\pi(1-\omega)}}{\omega \frac{R_k}{Q_k} + (1 - \omega) \left( \frac{(1-\alpha)R_k}{1-\pi} - \frac{R_k}{Q_k} \frac{\omega}{1-\omega} \right)} = \left( \frac{Q_k}{R_k} \right)^{\frac{1}{\sigma}}
\] (B.22)
Arranging terms, we get
\[
\frac{\omega + (1 - \omega)\alpha}{\pi} = \left(\frac{Q_k}{R_k}\right)^{\frac{1}{\pi}} \frac{(1 - \omega)(1 - \alpha)R_k}{1 - \pi} \implies Q_k = R_k^{1 - \sigma} \left(\frac{\kappa(1 - \pi)}{\pi(1 - \kappa)}\right)^{\sigma} \tag{B.23}
\]

Substitute $Q_k$ back to (B.21) and we get the solution for $r_n$
\[
r_n = \frac{(1 - \alpha)R_k}{1 - \pi} - \left(\frac{R_k \pi (1 - \kappa)}{\kappa(1 - \pi)}\right)^{\sigma} \frac{\omega}{1 - \omega} \tag{B.24}
\]

Also, from (B.23), we have
\[
\frac{Q_k}{R_k} = \left(\frac{\kappa(1 - \pi)}{R_k \pi (1 - \kappa)}\right)^{\sigma} \tag{B.25}
\]

This ratio is equal to 1 at $\pi_1$. For $\pi > \pi_1$, the RHS is lower than 1. So given $\kappa$, $R_k$ and $\pi$, $Q_k$ will be lower for higher $\sigma$.

At $\pi_{bind}$, we have $r_m = r_n$. Using (B.20) and (B.24), we have
\[
r_m = r_n \implies \frac{(1 - \omega)\alpha + (1 - \pi)\omega}{\pi(1 - \omega)} = \frac{(1 - \alpha)R_k}{1 - \pi} - \left(\frac{R_k \pi (1 - \kappa)}{\kappa(1 - \pi)}\right)^{\sigma} \frac{\omega}{1 - \omega} \tag{B.26}
\]

This equation implicitly defines $\pi_{bind}$.

Above $\pi_{bind}$, $r_m$ is still (B.20), and we have
\[
r_m = r_n = \frac{(1 - \omega)\alpha + (1 - \pi)\omega}{\pi(1 - \omega)} \tag{B.27}
\]

And using the budget constraints (B.12) and (B.13), we get
\[
Q_k = \frac{(1 - \pi)\omega}{(1 - \omega)(1 - \alpha) - \frac{(1 - \pi)(1 - \omega)\alpha + (1 - \pi)\omega}{\pi R_k}} \tag{B.28}
\]

So the distribution for $Q_k$ is
\[
Q_k(\pi) = \begin{cases} 
  R_k & : \pi \leq \pi_1 = \frac{\pi}{\kappa(1 - \kappa)R_k} \\
  R_k^{1 - \sigma} \left(\frac{\kappa(1 - \pi)}{\pi(1 - \kappa)}\right)^{\sigma} & : \pi_1 < \pi \leq \pi_{bind} \\
  \text{equation B.28} & : \pi_1 > \pi_{bind}
\end{cases} \tag{B.29}
\]

We can see that if the constraint $r_m \leq r_n$ is not binding, then the distribution of $Q_k$ only depends on $\kappa$. This can be seen from (B.28) where $Q_k$ for $\pi < \pi_{bind}$ only depends on $\kappa$ (remember that $\pi_1$ (B.19) only depends on $\kappa$).
With the log utility function, \( r_n = \frac{R_k}{Q_k} r_m \), so the constraint \( r_m \leq r_n \) is never binding. As a result, under the log utility function, the distribution of \( Q_k \) only depends on \( \kappa \).

### B.2.3 The first order conditions for \( \omega \) and \( \alpha \)

This part derives the first order conditions for the representative household and the investment fund. When deciding the optimal choice, the representative household and the investment fund will take the choices of other agents and the distribution of \( Q_k \) as given.

For notational convenience, we set \( s = 1 \), so \( v_m = \omega + (1 - \omega) r_m \) and \( v_n = \omega \frac{R_k}{Q_k} + (1 - \omega) r_n \).

The expected utility is

\[
EU = \frac{1}{2} \int_0^1 \left[ \pi \left( \frac{v_{m,H}}{1 - \sigma} \right)^{-\sigma} + (1 - \pi) \left( \frac{v_{n,H}}{1 - \sigma} \right)^{-\sigma} \right] dF(\pi)
\]

And the first order condition for \( \omega \) is

\[
\frac{\partial EU}{\partial \omega} = \frac{1}{2} \int_0^1 \left( \frac{\pi (1 - r_{m,H})}{(\omega + (1 - \omega) r_{m,H})^\sigma} + \frac{(1 - \pi) (\frac{R_k}{Q_k} r_{n,H} - r_{n,H})}{(\omega \frac{R_k}{Q_k} + (1 - \omega) r_{n,H})^\sigma} \right) dF(\pi) + \frac{1}{2} \int_0^1 \left( \frac{\pi (1 - r_{m,L})}{(\omega + (1 - \omega) r_{m,L})^\sigma} + \frac{(1 - \pi) (\frac{R_k}{Q_k} r_{n,L} - r_{n,L})}{(\omega \frac{R_k}{Q_k} + (1 - \omega) r_{n,L})^\sigma} \right) dF(\pi)
\]

The first order condition for \( \alpha \) is

\[
\frac{\partial EU}{\partial \alpha} = \frac{1 - \omega}{2} \int_0^1 \left( \frac{\partial r_{m,H}}{\partial \alpha} \frac{\pi}{v_{m,H}^\sigma} + (1 - \pi) \frac{\partial r_{m,H}}{\partial \alpha} \frac{1}{v_{n,H}^\sigma} + \frac{\partial r_{m,L}}{\partial \alpha} \frac{\pi}{v_{m,L}^\sigma} + (1 - \pi) \frac{\partial r_{m,L}}{\partial \alpha} \frac{1}{v_{n,L}^\sigma} \right) dF(\pi)
\]

We still need to decide \( \frac{\partial r_m}{\partial \alpha} \) and \( \frac{\partial r_n}{\partial \alpha} \). When \( \pi \leq \pi_1 \), since \( r_m = r_n = \alpha + (1 - \alpha) R_k \), we have

\[
\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = 1 - R_k
\]

For \( \pi > \pi_1 \), we first need to solve for \( r_m \) and \( r_n \) by taken \( Q_k \) as given.

First, for \( \pi \in [\pi_1, \pi_{bind}] \), using \( \text{(B.12)} \) and \( \text{(B.13)} \), we can write \( r_m \) as \( \frac{\alpha + (1 - \alpha) \eta Q_k}{\pi} \) and \( r_n \) as \( \frac{(1 - \alpha)(1 - \eta) R_k}{1 - \pi} \). Substituting them into \( \text{(B.16)} \) and arranging terms, we get

\[
\eta = \frac{-\omega - (1 - \omega) \frac{\alpha}{\pi} + (\frac{Q_k}{R_k})^{1/2} \left[ \frac{\omega R_k}{Q_k} + (1 - \omega)(1 - \alpha) R_k \right]}{(1 - \omega)(1 - \alpha) Q_k + (\frac{Q_k}{R_k})^{1/2} (1 - \omega)(1 - \alpha) R_k}
\]
Substitute $\eta$ into (B.12) and (B.13) and we get

$$r_m = \frac{1}{\pi} (\alpha + Q_k (1 - \alpha) \eta) = \frac{1}{\pi} \left( \alpha + Q_k \frac{-\omega - (1 - \omega) \alpha + \frac{Q_k}{R_k} \frac{1}{\pi} R_k + \frac{(1 - \omega)(1 - \alpha) R_k}{1 - \pi}}{1 - \pi Q_k + \frac{Q_k}{R_k} \frac{1}{\pi} R_k} \right)$$

(B.35)

$$r_n = \frac{R_k (1 - \alpha)(1 - \eta)}{1 - \pi} = \frac{R_k}{1 - \pi} \left( (1 - \alpha) - \frac{-\omega - (1 - \omega) \alpha + \frac{Q_k}{R_k} \frac{1}{\pi} R_k + \frac{(1 - \omega)(1 - \alpha) R_k}{1 - \pi}}{1 - \pi Q_k + \frac{Q_k}{R_k} \frac{1}{\pi} R_k} \right)$$

(B.36)

And so

$$\frac{\partial r_m}{\partial \alpha} = \frac{1}{\pi} \left( 1 + Q_k \frac{-\pi \frac{Q_k}{R_k} \frac{1}{\pi} R_k}{1 - \pi Q_k + \frac{Q_k}{R_k} \frac{1}{\pi} R_k} \right)$$

(B.37)

$$\frac{\partial r_n}{\partial \alpha} = \frac{R_k}{1 - \pi} \left( -1 + \frac{-\pi \frac{Q_k}{R_k} \frac{1}{\pi} R_k}{1 - \pi Q_k + \frac{Q_k}{R_k} \frac{1}{\pi} R_k} \right)$$

(B.38)

In the symmetric equilibrium, $(\frac{Q_k}{R_k})^\frac{1}{\pi} = \frac{\kappa(1-\pi)}{R_k(\pi(1-\pi))}$ (equation B.25). Rearranging terms, we get

$$\frac{\partial r_m}{\partial \alpha} = \frac{1}{\pi} \frac{\kappa(1-Q_k)}{\kappa + (1-\kappa)Q_k}$$

(B.39)

$$\frac{\partial r_n}{\partial \alpha} = \frac{R_k(1-\kappa)(1-Q_k)}{1-\pi \kappa + (1-\kappa)Q_k}$$

(B.40)

For $\pi > \pi_{\text{bind}}$, since $r_m = r_n$, using (B.12) and (B.13) and taking $Q_k$ as given, we get

$$r_m = r_n = \frac{R_k(\alpha + (1-\alpha)Q_k)}{(1-\pi)Q_k + \pi R_k}$$

(B.41)

and we have

$$\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = \frac{R_k(1-Q_k)}{(1-\pi)Q_k + \pi R_k}$$

(B.42)

**B.2.4 The equilibrium when $r_m \leq r_n$ is not binding**

This part considers the features of the equilibria when the constraint $r_m \leq r_n$ is not binding. We have the following result: First, the response curves of the household and the investment
fund overlap with each other. Second, the equilibrium is defined by $\kappa$. As long as $\kappa$ is equal to the equilibrium $\kappa$, then people can choose different combinations of $[\omega, \alpha]$.

The response curves

First, we explain why the response curves overlap with each other when the constraint $r_m \leq r_n$ is not binding. The response curves are simply the first order conditions of $\omega$ and $\alpha$ (equation B.30 and B.32). Denote the response curve of the household and the investment fund as $R_{household}(\alpha)$ and $R_{fund}(\omega)$. Let $\omega(\alpha)$ denote the optimal choice of the household by taken $\alpha$ as given and $\alpha(\omega)$ the optimal choice of the investment fund by taking $\omega$ as given. Then on the response curves, we have $\alpha(\omega(\alpha_0)) = \alpha_0$ and $\omega(\alpha(\omega_0)) = \omega_0$. The reason is that given the distribution of $Q_k$, the portfolio of movers and non-movers can be written as functions of $\kappa$. So when the investment fund chooses the best $\alpha$ given $\omega$, or when the household chooses $\omega$ given $\alpha$, they essentially choose the best $\kappa$.

The portfolio of movers is $v_m = \omega + (1 - \omega)r_m$ and the portfolio of non-movers is $v_n = \omega R_k Q_k + (1 - \omega)r_m$. For $\pi \leq \pi_1$, $r_m = r_n = \alpha + (1 - \alpha)R_k$, and so

$$v_m = v_n = \omega + (1 - \omega)(\alpha + (1 - \alpha)R_k) = \kappa + (1 - \kappa)R_k \quad (B.43)$$

For $\pi > \pi_1$, when $Q_k$ is given, the solutions for $r_m$ and $r_n$ are (B.35) and (B.36). After some arrangement of equations, we get

$$v_m = \omega + (1 - \omega)r_m = \frac{Q_k}{Q_k R_k} \left( \frac{1}{2} R_k Q_k \right) \left( \kappa + (1 - \kappa)Q_k \right) \frac{1}{(1 - \pi) + \pi(\kappa + (1 - \kappa)Q_k)} \quad (B.44)$$

$$v_n = \omega \frac{R_k}{Q_k} + (1 - \omega)r_m = \frac{R_k}{Q_k} \left( \frac{1}{2} R_k Q_k \right) \left( \kappa + (1 - \kappa)Q_k \right) \frac{1}{(1 - \pi) + \pi(\kappa + (1 - \kappa)Q_k)} \quad (B.45)$$

So given $Q_k$, $v_m$ and $v_n$ can be written as functions of $\kappa$.

The equilibrium $\kappa$

Given the equilibrium level of $\kappa$, different combinations of $[\omega, \alpha]$ which give the same $\kappa$ will also be the equilibrium.
We’ve already shown that in the symmetric equilibrium, when the constraint $r_m \leq r_n$ is not binding, $\pi_1$ and $Q_k$ only depend on $\kappa$. From the previous analysis, we know that households and investment funds try to maximize $EU$ given $v_m$ and $v_n$ specified in (B.44) and (B.45). We can see that $v_m$ and $v_n$ only depend on $\pi_1$ and $Q_k$. In the equilibrium, given $Q_k$, households and investment funds would find that the equilibrium $\kappa$ is optimal. And so if we keep the same $\kappa$ but change the combination of $\omega$ and $\alpha$, then since $Q_k$ does not change, households and investment funds will still choose the same $\kappa$ because they still face the same portfolio choice problem.

**B.3 Additional results: the general case $\sigma \geq 1$, with bank lending.**

**B.3.1 The optimal payout policy when bank loan is allowed**

The results are shown in Proposition 5. What follows is the proof.

**Proof of proposition 5:** When $Q_k = R_k$, there is no bank borrowing and $r^d = 0$. The problem is the same as in the no lending case.

Let $\eta_1$ denote the share of assets sold on the financial market and let $\eta_2$ denote the share of assets used as collateral to borrow from banks. Define $\eta = \eta_1 + \eta_2$. When $Q_k < R_k$, the budget constraint is

$$\pi r_m = \alpha + (1 - \alpha)\eta_1 Q_k + (1 - \alpha)\eta_2 Q_k = \alpha + (1 - \alpha)\eta Q_k$$

(B.46)

$$\pi n = \pi (1 - \alpha)(1 - \eta) R_k$$

(B.47)

Set $s = 1$. We have

$$v_m = [\omega + (1 - \omega) r_m] (1 + r^d)$$

(B.48)

$$v_n = \omega \frac{R_k}{Q_k} + (1 - \omega) r_n$$

(B.49)
and the fund’s problem (B.7) becomes
\[
\pi \left[ \frac{(\omega + (1 - \omega)\alpha(1 + r^d))Q_k(1 + r^d)}{1 - \sigma} \right]^{1-\sigma} + (1 - \pi) \frac{(\omega R_k/Q_k + (1 - \omega)(1 - \eta)R_k)}{1 - \sigma}^{1-\sigma} \tag{B.50}
\]
Taking the derivative with respect to \( \eta \) and simplifying the terms, we get
\[
\frac{Q_k(1 + r^d)}{(\omega + (1 - \omega)\alpha(1 + r^d))Q_k(1 + r^d)}^{\sigma} - \frac{R_k}{(\omega R_k/Q_k + (1 - \omega)(1 - \eta)R_k)^{\sigma}} = 0 \tag{B.51}
\]
which can be written as (3.7). We can also write it as
\[
\frac{\omega + (1 - \omega)r_m}{\omega R_k/Q_k + (1 - \omega)r_n} = \left( \frac{Q_k}{R_k} \right)^{\frac{1}{\sigma}} (1 + r^d)^{\frac{1}{\sigma} - 1} \tag{B.52}
\]
When \( Q_k/R_k < 1 \), if \( r^d = 0 \), it is clear that the RHS of (B.52) is increasing in \( \sigma \). \( r^d \) is positive only when investment funds borrow positive loans from banks. In this case, \( Q_k/R_k = \frac{1}{1 + r^d} \), and the RHS of (B.52) can be written as \( Q_k/R_k \left( \frac{1 + r^d}{1 + r^d} \right)^{1 - \frac{1}{\sigma}} \), which is increasing in \( \sigma \) since \( r^d > r^d \).

When \( \sigma = 1 \), it is easy to see that the result for \( r_m \) and \( r_n \) is the same as in the no-bank-lending case. ■

### B.3.2 The bank loan supply

We first prove Proposition 6.

**Proof:**

Table B.2 shows the accumulated flow of payments. For example, column 1 shows the result if bank \( i \) is chosen to make the payment in subperiod 1. In subperiod 1 (row 1), the outflow of payment is \( \frac{(N-1)X_i}{N} \). In each subperiod \( k > 1 \), bank \( i \) receives \( \frac{X_i}{N} \). Similarly, in column 2, bank \( i \) receives \( \frac{X_i}{N} \) in \( k = 1 \), makes the payment \( \frac{(N-1)X_i}{N} \) in \( k = 2 \), and receives \( \frac{X_i}{N} \) in each of the subperiods \( k > 2 \).

In Table B.2, in each column \( n \), the maximum accumulated payment is \( \frac{(N-1)X_i}{N} - \frac{(n-1)X_i}{N} \), which happens in period \( k = n \) (the diagonal of the matrix) when banks are chosen to make the payment. And for \( k > n \), the accumulated payment is \( \frac{(N-1)X_i}{N} - \frac{(k-1)X_i}{N} \). If \( N \) is very
Table B.2: Accumulated flow of payments. The accumulated flow in subperiod \( k \) (row \( k \)) is the total outflow minus the total inflow up to that subperiod. Column \( n \) shows the accumulated flow if bank \( i \) makes the payment in subperiod \( n \).

large, then \( \frac{(N-1)X_i}{N} \approx X_i \). We set \( 1 - \frac{n-1}{N} \) as \( \lambda_{\text{max}} \) and \( 1 - \frac{k-1}{N} \) as \( \lambda \), then for \( k \geq n \), we can write

\[
FL_{\text{max}} = X_i + (\lambda_{\text{max}} - 1)X_j \quad \text{(B.53)}
\]

\[
FL(k) = X_i + (\lambda - 1)X_j \quad \text{(B.54)}
\]

And the central bank loan is

\[
b(k) = \max(FL(k) - D_0, 0) \quad \text{(B.55)}
\]

Note that in Table B.2, in each column, the accumulated flow \( FL(k) \) for \( k \geq n \) is the same as the \( FL(k) \) in the previous column. Let \( \Lambda \) denote the level of \( \lambda \) at which \( b(k) = 0 \). Using (B.54) and (B.55), we get

\[
\Lambda = \frac{D_0 + X_j - X_i}{X_j}, \quad \Lambda \in [0, 1] \quad \text{(B.56)}
\]

\( b(k) > 0 \) if \( \lambda > \Lambda \).

When \( N \) is large, we can take \( \lambda \) as continuous, and the expected loan can be written as

\[
Eb(L_i) = \int_{\Lambda}^{\lambda_{\text{max}}} \int_{\Lambda}^{\lambda_{\text{max}}} b(k)d\lambda d\lambda_{\text{max}} = \int_{\Lambda}^{\lambda_{\text{max}}} \int_{\Lambda}^{\lambda_{\text{max}}} (X_i + (\lambda - 1)X_j - D_0) d\lambda d\lambda_{\text{max}} \quad \text{(B.57)}
\]
The integral of $b(k)$ over $[\lambda, \lambda_{\text{max}}]$ is the borrowing for each realized $n$ (i.e., each column of the matrix). The integral over $[\lambda, 1]$ denotes the changes in $\lambda_{\text{max}}$ caused by the changes in $n$ (i.e., different columns of the matrix). $b(k)$ is positive only when $\lambda$ and $\lambda_{\text{max}}$ are $> \lambda$.

$$Eb(L_i) = \int_{\lambda}^{1} \int_{\lambda}^{\lambda_{\text{max}}} [X_i - X_j - D_0 + \lambda X_j] d\lambda d\lambda_{\text{max}}$$

$$= \int_{\lambda}^{1} \left[ (\lambda_{\text{max}} - \lambda) (X_i - X_j - D_0) + \frac{\lambda^2_{\text{max}} - \lambda^2}{2} X_j \right] d\lambda_{\text{max}}$$

$$= \left( \frac{\lambda^2_{\text{max}}}{2} - \lambda_{\text{max}} \right) (X_i - X_j - D_0) + \frac{1}{2} \left( \lambda^3_{\text{max}} - \lambda_{\text{max}} \right) X_j$$

Repeating $\lambda$ with (B.56) and arranging terms, we get

$$Eb(L_i) = \frac{1}{6} \left( X_i - X_j - D_0 \right)^3 \frac{X^2}{X_j} + \frac{1}{2} \left( X_i - X_j - D_0 \right)^2 \frac{X}{X_j} + \frac{1}{2} (X_i - X_j - D_0) + \frac{1}{6} X_j \quad (B.59)$$

In the general case, we have

$$X_i = Z_f + (1 - \pi) D^h + L_i \quad (B.60)$$

$$X_j = Z_f + (1 - \pi) D^h + L_j \quad (B.61)$$

where $Z_f$ is the riskless asset of the investment fund, $(1 - \pi) D^h$ is the money collected from non-movers. The method is the same and it can be shown that in the symmetric case we still have

$$R = 1 + \delta + r^d \frac{1}{2} \left( 1 - \frac{D_0}{X} \right)^2 \quad (B.62)$$

$$r^d = \frac{L r^c (1 - \frac{D_0}{X})^2 - r^c Eb}{D_0 + L} = \frac{L r^c (1 - \frac{D_0}{X})^2 - r^c (\frac{D^3_0}{D_0 X} + \frac{D^2_0}{2 X} - \frac{D_0}{2} + \frac{X}{6})}{D_0 + L} \quad (B.63)$$

where $D_0$ is $D^h + Z_f$ and $X = Z_f + (1 - \pi) D^h + L$.

**B.3.3 The Equilibrium solutions for $r_m, r_n, Q_k, L, R$ and $r^d$.**

This part derives the equilibrium solutions in period $t + 1$ by taking $\omega$ and $\alpha$ as given. We first consider the case in which the constraint $r_m \leq r_n$ is not binding.
Recall that at $\pi_2$, investment funds start to borrow from banks. And at $\pi_3$, banks start to borrow from the central bank. Everything for $\pi < \pi_2$ is the same as in the non-bank lending case. In order to get the solution for $\pi \geq \pi_2$, we first decide $\pi_2$ and $\pi_3$.

**Derive $\pi_2$ and $\pi_3$**

$\pi_2$ can be decided as follows. At $\pi_2$, $R = 1 + \delta$, $Q_k = \frac{R}{1+\delta}$ and $L = 0$. At the same time, $\frac{Q_k}{R_k}$ should satisfy (B.25), so we have

$$\frac{Q_k}{R_k} = \frac{1}{1+\delta} = \left(\frac{\kappa(1-\pi)}{R_k \pi_2 (1-\kappa)}\right)^{\frac{1}{\gamma}}.$$  \hspace{1cm} (B.64)

$$\Rightarrow \pi_2 = \frac{\kappa}{R_k (1-\kappa) \left(\frac{1}{1+\delta}\right)^{\frac{1}{\gamma}} + \kappa}.$$  \hspace{1cm} (B.65)

$\pi_3$ can be decided as follows. At $\pi_3$, $R = 1 + \delta$, $r^d = 0$ and $Q_k = \frac{R_k}{1+\delta}$. At $\pi_3$, $X = D_0$.

Since $X = Z_f + (1-\pi)D^h + L$ and $D_0 = Z_f + D^h$, so $L = \pi D^h$. Thus, we have

$$\pi Z_f r_m = Z_f + (1-\pi)D^h + \pi D^h$$  \hspace{1cm} (B.66)

$$\Rightarrow r_m = \alpha + (1-\pi)\frac{\omega}{1-\omega} + \pi \frac{\omega}{1-\omega} = \alpha + \frac{\omega}{1-\omega}$$  \hspace{1cm} (B.67)

$$\Rightarrow r_m = \frac{\omega}{\pi (1 + \frac{\omega}{1-\omega})}.$$  \hspace{1cm} (B.68)

Also, comparing (B.46) and (B.67), we have

$$(1-\alpha)\eta Q_k = \frac{\omega}{1-\omega} \Rightarrow \eta = \frac{\omega}{(1-\omega)(1-\alpha)Q_k}.$$  \hspace{1cm} (B.69)

Substituting $\eta$ into (B.47) and we have

$$r_n = \frac{1}{1-\pi} \left( (1-\alpha) - \frac{\omega}{(1-\omega)Q_k} \right) R_k$$  \hspace{1cm} (B.70)

Then substitute (B.68) and (B.70) into the optimal payout policy (B.52) and we have

$$\frac{\omega + (1-\omega)\frac{\alpha}{1-\omega} + \frac{\omega}{(1-\omega)Q_k}}{\omega(1+\delta) + (1-\omega)\frac{1}{1-\pi} \left( (1-\alpha) - \frac{\omega}{(1-\omega)Q_k} \right) R_k} = \left(\frac{1}{1+\delta}\right)^{\frac{1}{\gamma}}.$$  \hspace{1cm} (B.71)

Arranging terms, we get

$$\omega \left( (1+\delta)^{\frac{1}{\gamma}} - 1 \right) \pi^2 - \pi \left[ \kappa - \omega + (1+\delta)^{\frac{1}{\gamma}} (1-\kappa) R_k \right] + \kappa = 0.$$  \hspace{1cm} (B.72)
When $\sigma = 1$, the solution is $\kappa \frac{\kappa - \omega}{\kappa + (1 - \kappa) R_k}$. When $\sigma > 1$, the smaller one of the two solutions is $\pi_3$. Note that given $\kappa$, $\pi_3$ is affected by $\omega$. For example, if $\omega = 0$ (all riskless assets are held by the investment fund), then $\pi_3 = \pi_2$.

The distribution for $Q_k$ takes the following form:

$$Q_k(\pi) = \begin{cases} R_k & : \pi \leq \pi_1 = \pi_1 = \frac{\kappa}{\kappa + (1 - \kappa) R_k} \\ R_k^{1-\sigma} \left( \frac{\kappa - \sigma}{\pi(1-\kappa)} \right)^{\sigma} & : \pi_1 < \pi < \pi_2 = \frac{\kappa}{R_k(1-\kappa)(1+\delta)} \prod + \kappa \\ \frac{R_k}{R(\pi)} & : \pi \geq \pi_3 \end{cases} \quad (B.73)$$

**Equilibrium solutions over $\pi_2$ and $\pi_3$**

Over $[\pi_2, \pi_3]$, $R = 1 + \delta$, $r^d = 0$ and $Q_k = \frac{R_k}{1+\delta}$. We still need to decide $r_m$, $r_n$ and $L$. Since $r^d = 0$, (B.52) is the same as (B.16), and the solution for $\eta$, $r_m$ and $r_n$ are simply (B.34), (B.35) and (B.36) with $Q_k = \frac{R_k}{1+\delta}$. Knowing $\eta$, we can decide $L$ from the budget constraint (B.46). Since $L$ is equal to the total external cash minus the cash from non-movers, so

$$L = Z_p(1-\alpha)\eta Q_k - (1-\pi)D^h = S[(1-\omega)(1-\alpha)\eta Q_k - (1-\pi)\omega] \quad (B.74)$$

**Equilibrium solutions for $\pi > \pi_3$**

We will set $L$ as the variable that we try to solve, and we express all other variables as a function of $L$. The equilibrium is defined by the following conditions. 1. The budget constraints (B.46) and (B.47); 2. The optimal payout policy (B.52); 3. Asset Price on the financial market: $Q_k = \frac{R_k}{R}$; 4. The cash paid to movers is equal to the fund’s own money plus the money raised from the financial market and the bank.

$$\pi Z_p r_m = Z_f + (1-\pi)D^h + L \quad (B.75)$$

5. The loan supply curve (B.62) which defines the relationship between $L$ and $R$; 6. $r^d$ (equation B.63) derived from the zero expected profit condition. We can write (B.62) as $R(L)$ and (B.63) as $r^d(L)$. Then $Q_k(L) = \frac{R_k}{R(L)}$. We can also write (B.75) as

$$r_m = \frac{1}{\pi Z_p} (Z_f + (1-\pi)D^h + L) = \frac{1}{\pi} \left( \alpha + (1-\pi) \frac{\omega}{1-\omega} + \frac{L}{S(1-\omega)} \right) \quad (B.76)$$
which we define as \( r_m(L) \). Then using the two budget constraints (B.46) and (B.47), we have

\[
r_n = \frac{R_k(1 - \alpha)}{1 - \pi} - \frac{R(L)(\pi r_m(L) - \alpha)}{1 - \pi}
\]

(B.77)

which we define as \( r_n(L) \). Substitute \( r_m(L), r_n(L), Q_k(L), R(L), \) and \( r^d(L) \) into the optimal payout policy (B.52), and we can get an equation in which the only unknown is \( L \):

\[
\frac{\omega + (1 - \omega)r_m(L)}{\omega R(L) + (1 - \omega)r_n(L)} = \left( \frac{1}{R(L)} \right)^{\frac{1}{\delta}} (1 + r^d(L))^{\frac{1}{\gamma} - 1}
\]

(B.78)

where \( R(L), r_d(L), r_m(L) \) and \( r_n(L) \) are (B.62), (B.63), (B.76), and (B.77). This equation implicitly defines the equilibrium \( L \). After deciding \( L \), all other variables can then be decided.

**When \( r_m \leq r_n \) is binding**

Let \( \pi_{bind} \) denote the \( \pi \) above which the constraint \( r_m \leq r_n \) is binding. We first consider the case when \( \pi_1 < \pi_{bind} < \pi_2 \). At \( \pi_2, r_m \) is still (B.20), and \( r_n \) is (B.41) with \( Q_k = \frac{R_k}{1+\delta} \), equating \( r_m \) and \( r_n \) gives the value of \( \pi_2 \). At \( \pi_3, Q_k = \frac{R_k}{1+\delta} \), \( r_m \) is (B.68) and \( r_n \) is (B.70).

Equating \( r_m \) and \( r_n \) gives \( \pi_3 \).

For equilibrium values of variables. For \( \pi \leq \pi_{bind} \), everything is the same as in the non-binding case. For \( [\pi_{bind}, \pi_2] \), we use (B.27) and (B.28). Over \( [\pi_2, \pi_3] \), \( r_m \) and \( r_n \) are (B.41) with \( Q = \frac{R_k}{1+\delta} \). For \( \pi > \pi_3, \) we can solve the equilibrium using the same method as in the non-binding case, the only difference is that instead of using condition (B.78), we use the condition \( r_m(L) = r_n(L) \).

If \( \pi_{bind} \in [\pi_2, \pi_3] \) or \( \pi_{bind} > \pi_3 \), then we can decide \( \pi_{bind} \) using simulation methods. The method for deciding the equilibrium values is the same as explained above.

**B.3.4 The first order conditions for \( \omega \) and \( \alpha \)**

This part derives the first order conditions for representative household and investment fund.
Using $EU(B.30)$, $v_m(B.48)$ and $v_n(B.49)$, we get

$$\frac{\partial EU}{\partial \omega} = \frac{1}{2} \int_0^1 \pi(1 - r_{m,H})(1 + r_H^d) \left( \frac{(1 - \pi)(R_k^H/Q_{k,H} - r_{n,H})}{(1 - \alpha)Q_{k,H}} + (1 - \pi)\frac{\partial r_m}{\partial \alpha}\frac{\partial r_n}{\partial \alpha} \right) dF(\pi)$$

$$+ \frac{1}{2} \int_0^1 \pi(1 - r_{m,L})(1 + r_L^d) \left( \frac{(1 - \pi)(R_k^L/Q_{k,L} - r_{n,L})}{(1 - \alpha)Q_{k,L}} + (1 - \pi)\frac{\partial r_m}{\partial \alpha}\frac{\partial r_n}{\partial \alpha} \right) dF(\pi)$$

The first order condition for $\alpha$ is

$$\frac{\partial EU}{\partial \alpha} = \frac{(1 - \omega)}{2} \int_0^1 \left( \pi - \frac{\partial r_m}{\partial \alpha} \frac{\partial r_n}{\partial \alpha} \right) dF(\pi)$$

We need to decide $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$. When $\pi \leq \pi_1$, since $r_m = r_n = \alpha + (1 - \alpha)R_k$, we get $\frac{\partial r_m}{\partial \alpha} = \frac{\partial r_n}{\partial \alpha} = 1 - R_k$. For $\pi > \pi_1$, we first need to solve for $r_m$ and $r_n$ by taken $Q_k$ and $r^d$ as given. Note that equations (B.46) and (B.47) are the same as (B.12) and (B.13), and the only difference between (B.16) and (B.52) is that the RHS is changed from $\left( \frac{Q_k}{R_k} \right)^{1/\sigma}$ into $\left( \frac{Q_k}{R_k} \right)^{1/\sigma} (1 + r^d)^{1/\sigma - 1}$. It turns out that we only need to modify the solutions of $\eta$, $r_m$, $r_n$ in the no-lending case (B.34, B.35 and B.36) by changing $\left( \frac{Q_k}{R_k} \right)^{1/\sigma}$ into $\left( \frac{Q_k}{R_k} \right)^{1/\sigma} (1 + r^d)^{1/\sigma - 1}$. And so $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$ are equations (B.37) and (B.38) with the term $\left( \frac{Q_k}{R_k} \right)^{1/\sigma}$ replaced by $\left( \frac{Q_k}{R_k} \right)^{1/\sigma} (1 + r^d)^{1/\sigma - 1}$.

If the constraint $r_m \leq r_n$ is binding, then for $\pi > \pi_{bind}$, $\frac{\partial r_m}{\partial \alpha}$ and $\frac{\partial r_n}{\partial \alpha}$ are the same as (B.42).

### B.4 The proof of the remaining propositions

**Proof of proposition 7:**

Eliminating $R$ from (3.25) and (3.28), we get a quadratic equation for $X$.

$$X^2 \left( 1 + \delta + \frac{r_c}{2} \right) - X \left( \frac{r_c}{2} D_0 + \pi Z_k R_k \right) + \frac{r_c}{2} D_0^2 = 0$$

(B.81)
After we solve for $X$, we have $L^*(\pi) = X - (1 - \pi)D_0$. $R^*(\pi)$ is decided according to (3.28). Finally, we can simplify the results using the relationship $D_0 = \omega S$ and $Z_k = (1 - \omega)S$. ■

**Proof of proposition 9:**

Below $\pi_1$, the cash constraint of non-movers is not binding and the cash of non-movers is more than enough to absorb the sale of assets, thus, $Q_k = R_k$. At $\pi_1$, non-movers use all their cash to buy assets, and $Q_k$ is still equal to $R_k$, we get

$$\pi_1 Z_k R_k = (1 - \pi_1)D_0 \quad (B.82)$$

Using $D_0 = \omega S$ and $Z_k = (1 - \omega)S$, we get the solution for $\pi_1$ in (3.35). Non-movers use all their deposits to buy assets if $\pi \geq \pi_1$.

Between $\pi_1$ and $\pi_2$, $Q_k$ is lower than $R_k$, but since $Q_k > \frac{R_k}{1+\delta}$, it is still not worthwhile for the investment funds to borrow from banks, and $Q_k$ is decided according to

$$\text{optimal payout} = \text{redemption} \Rightarrow \pi Z_k Q_k = (1 - \pi)D_0 \quad (B.83)$$

$Q_k$ decreases to $\frac{R_k}{1+\delta}$ at $\pi_2$, and investment funds start to borrow from banks when $\pi > \pi_2$. $\pi_2$ can be decided by replacing $Q_k$ in (B.83) with $Q_k = \frac{R_k}{1+\delta}$. Between $\pi_2$ and $\pi_3$, the lending rate is $R = 1 + \delta$ (equation 3.20), and $Q_k = \frac{R_k}{1+\delta}$.

$\pi_3$ is the level of $\pi$ above which the expected central bank loan is positive. At $\pi_3$, $Eb$ is exactly zero and $L(\pi_3) = \pi_3 D_0$. Since $Q_k(\pi_3)$ is still $\frac{R_k}{1+\delta}$, we have

$$\pi_3 = \frac{\text{Redemption}}{Z_k Q_k} = \frac{(1 - \pi_3)D_0 + L(\pi_3)}{Z_k \frac{R_k}{1+\delta}} = \frac{D_0}{Z_k \frac{R_k}{1+\delta}} = \frac{\omega}{(1 - \omega) \frac{R_k}{1+\delta}} \quad (B.84)$$

Above $\pi_3$, $Eb > 0$, and $Q_k = \frac{R_k}{R(\pi)}$, where $R(\pi)$ is defined in (3.32). ■

**Proof of proposition 10**

Proof: For notational convenience, we set household endowment at 1. First, given $\kappa$, the total wealth of the economy $\kappa + (1 - \kappa)R_k$ is decided. In addition, $\pi_1$ only depends on $\kappa$. At $\pi_1$, both $r_m$ and $r_n$ are still equal to the fundamental value of the fund: $r_m = r_n = \alpha + (1 - \alpha)R_k$, 142
and we also know that the payment to movers is equal to the cash of the investment fund $Z_f$ plus the cash collected from non-movers $(1 - \pi)D^h$, so we have

$$\pi_1 Z_p r_m = Z_f + (1 - \pi_1)D^h \quad \Rightarrow \quad \pi_1 = \frac{D^h + Z_f}{D^h + Z_p r_m} = \frac{\omega + (1 - \omega)\alpha}{\omega + (1 - \omega)(\alpha + (1 - \alpha)R_k)} = \frac{\kappa}{\kappa + (1 - \kappa)R_k} \quad (B.85)$$

For $\pi \leq \pi_1$,

$$v_m = v_n = \omega + (1 - \omega)(\alpha + (1 - \alpha)R_k) = \kappa + (1 - \kappa)R_k. \quad (B.86)$$

For $\pi > \pi_1$, movers carry all the cash $\kappa$ with them, which means the risky assets will become the wealth of non-movers.

$$\pi v_m = \kappa \Rightarrow v_m = \frac{\kappa}{\pi} \quad (B.87)$$

$$(1 - \pi)v_n = (1 - \kappa)R_k \Rightarrow v_n = \frac{(1 - \kappa)R_k}{1 - \pi} \quad (B.88)$$

Thus, the distribution of $v_m$ and $v_n$ only depends on $\kappa$.

**Proof of proposition 11**

Once $\omega$ and $\alpha$ are given, then $\kappa$ is given, and $\pi_1$ is uniquely decided. For $\pi \leq \pi_1$, we have $r_m = r_n = \alpha + (1 - \alpha)R_k$. For $\pi > \pi_1$, all the cash owned by investment funds and non-movers are used to pay movers, and we have

$$\pi Z_p r_m = Z_f + (1 - \pi)D^h \Rightarrow r_m = \frac{Z_f + (1 - \pi)D^h}{\pi Z_p} = \frac{(1 - \omega)\alpha + (1 - \pi)\omega}{\pi(1 - \omega)} \quad (B.89)$$

which is the same for different $\sigma$. We’ve shown that between $\pi_1$ and $\pi_{bind}$, we have

$$\frac{Q_k}{R_k} = \left( \frac{\kappa (1 - \pi)}{R_k \pi (1 - \kappa)} \right)^\sigma \quad (B.90)$$

This ratio is equal to 1 at $\pi_1$. For $\pi > \pi_1$, since $Q_k < R_k$, the right-hand-side should be lower than 1, which also implies that given $\kappa$, $R_k$ and $\pi$, $Q_k$ will be lower for higher $\sigma$.

**Proof of proposition 12**

Proof: Let $R_{\text{household}}(\alpha)$ denote the response curve of the household and $R_{\text{fund}}(\omega)$ the re-
response curve of the investment fund. In the previous analysis, we’ve already proved analytically that the two response curves will overlap with each other when \( r_m \leq r_n \) is not binding.

When \( r_m \leq r_n \) is binding, our numerical result shows that the two response curves will not intersect. The example for \( \sigma = 2 \) is shown in Figure B.1.

![Figure B.1: Response curves when \( \sigma = 2 \)](image)

There exists a level of \( \alpha = \alpha_{bind} \). When \( \alpha < \alpha_{bind} \), the constraint \( r_m \leq r_n \) is binding for positive probability. We find that it will cause \( R_{fund}(\omega) \) to be slightly higher than \( R_{household}(\alpha) \). So equilibrium will not be reached for \( \alpha < \alpha_{bind} \), because in this range, people will reduce \( \omega \) and increase \( \alpha \). For \( \alpha \geq \alpha_{bind} \), the constraint \( r_m \leq r_n \) is not binding. The two response curves overlap with each other. The results when there is bank lending are similar.

\[ \square \]
Appendix C

Proofs for Chapter 4

C.1 Proofs

C.1.1 Derive \((E(V(d_0)))'\)

(4.29) can be written as

\[
EV(d_0) = (1 - \alpha - \lambda) \int_A [-c(q_s) + W(d_0, p_1 q_s, 0)]dF(A)
+ \alpha \left[ \int_{A \in \Omega_1} (Au(q_y) + W(d_0 - p_1 q_y, 0, 0)f(A)dA \\
+ \int_{A \in \Omega_2} Au\left(\frac{d_0}{p_1}\right)f(A)dA + \int_{A \in \Omega_3} (Au(q_y) + W(0, 0, p_1 q_y - d_0))f(A)dA \right]
+ \lambda \int_A Bu\left(\frac{d_0}{p_1}\right)dF(A)
\]

(C.1)
Differentiating this equation with respect to \( d_0 \),

\[
(EV(d_0))' = (1 - \alpha - \lambda) \int_A [-c(q_s)\frac{\partial q_s}{\partial d_0} + W_d(1 + p_1 \frac{\partial q_s}{\partial d_0})]dF(A)
\]

\[+ \alpha \left[ \int_{A \in \Omega_1} (Au'(q_y)\frac{\partial q_y}{\partial d_0} + W_d(1 - p_1 \frac{\partial q_y}{\partial d_0}))f(A)dA \right. \\
\left. + \int_{A \in \Omega_2} Au'(\frac{d_0}{p_1}) \frac{1}{p_1} f(A)dA + \int_{A \in \Omega_3} (Au'(q_y)\frac{\partial q_y}{\partial d_0} + W\ell(p_1 \frac{\partial q_y}{\partial d_0} - 1))f(A)dA \right] \\
+ \frac{\partial A_{\Omega_{12}}}{\partial d_0} (A_{\Omega_{12}}u(q_y) + W(d_0 - p_1q_y, 0, 0)) f(A_{\Omega_{12}}) - \frac{\partial A_{\Omega_{12}}}{\partial d_0} (A_{\Omega_{12}}u(\frac{d_0}{p_1})f(A_{\Omega_{12}})) \\
+ \frac{\partial A_{\Omega_{23}}}{\partial d_0} (A_{\Omega_{23}}u(\frac{d_0}{p_1})f(A_{\Omega_{23}})) - \frac{\partial A_{\Omega_{23}}}{\partial d_0} (A_{\Omega_{23}}u(q_y) + W(0, 0, p_1q_y - d_0))f(A_{\Omega_{23}}) \\
+ \frac{\partial A_{\Omega_{13}}}{\partial d_0} (A_{\Omega_{13}}u(q_y) + W(d_0 - p_1q_y, 0, 0)) f(A_{\Omega_{13}}) \\
- \frac{\partial A_{\Omega_{13}}}{\partial d_0} (A_{\Omega_{13}}u(q_y) + W(0, 0, p_1q_y - d_0))f(A_{\Omega_{13}}) \\
+ \lambda \int_A Bu'(\frac{d_0}{p_1}) \frac{1}{p_1} dF(A) \tag{C.2}
\]

We use \( A_{\Omega_{12}} \) to denote the boundary between \( \Omega_1 \) and \( \Omega_2 \). The meaning for \( A_{\Omega_{23}} \) and \( A_{\Omega_{13}} \) is similar. If the integrand on the boundary is continuous, then the derivatives associated with \( A_{\Omega_{12}} \), \( A_{\Omega_{23}} \) and \( A_{\Omega_{13}} \) will be cancelled out. If the integrand on the boundary is not continuous, then marginal changes in \( d_0 \) will not change the boundary, and as a result, the derivatives of the boundary with respect to \( d_0 \) will be zero. In both cases, the derivatives
associated with $A_{\Omega_2}$, $A_{\Omega_3}$ and $A_{\Omega_4}$ can be ignored.\footnote{Here is an example. Suppose below $A_1$, the deposit is not binding, and above $A_1$, the deposit is binding. Then $A_{\Omega_2}$ is simply $A_1$. If at $A_1$, we have $Au(q_y) + W(d_0 - p_1 q_y, 0) = Au\left(\frac{d_0}{p_1}\right)$ (i.e., if the buyer exactly uses all his deposit and $q_y = \frac{d_0}{p_1}$), then the two terms $\frac{\partial A}{\partial d_0}(A_1 u(q_y) + W(d_0 - p_1 q_y, 0))f(A_1)$ and $-\frac{\partial A}{\partial d_0}(A_1 u(d_0))f(A_1)$ will offset each other. Now suppose the integrand is not continuous. For example, suppose at $A_1$, the deposit is not binding. Suppose at $A_1 + \epsilon$, the central bank sets a much lower interest rate, the buyer wants to consume more and the deposit is binding. In this case, marginal changes in $d_0$ will not change the level of $A_1$, the deposit is still non-binding at $A_1$ and binding at $A_1 + \epsilon$. As a result, $\frac{\partial A}{\partial d_0} = 0.$} As a result, (C.2) can simplified to

$$
(EV(d_0))' = (1 - \alpha - \lambda) \int_A [-c(q_s) \frac{\partial q_s}{\partial d_0} + W(d + p_1 \frac{\partial q_s}{\partial d_0})]dF(A)
$$

$$
+ \alpha \left[ \int_{A \in \Omega_1} (Au'(q_y) \frac{\partial q_y}{\partial d_0} + W(d - p_1 \frac{\partial q_y}{\partial d_0}))f(A)dA 
+ \int_{A \in \Omega_2} Au'(\frac{d_0}{p_1}) \frac{1}{p_1} f(A)dA + \int_{A \in \Omega_3} (Au'(q_y) \frac{\partial q_y}{\partial d_0} + W(p_1 \frac{\partial q_y}{\partial d_0} - 1))f(A)dA \right]

+ \lambda \int_A Bu'(\frac{d_0}{p_1}) \frac{1}{p_1} dF(A)
$$

(C.3)

For sellers, $\frac{\partial q_s}{\partial d_0} = 0$ because the choice of $q_s$ is independent of $d_0$. When $A \in \Omega_1$, the deposit balance is not binding and so $\frac{\partial q_s}{\partial d_0} = 0$. From (4.9) and (4.11), we know that $W_d = \phi(1 + i_d)$ and $W_\ell = -\phi(1 + i_\ell)$. When $A \in \Omega_3$, we have

$$
Au'(q_y) \frac{\partial q_y}{\partial d_0} + W_\ell(p_1 \frac{\partial q_y}{\partial d_0} - 1) = (Au'(q_y) - p_1\phi(1 + i_\ell)) \frac{\partial q_y}{\partial d_0} + \phi(1 + i_\ell) = \phi(1 + i_\ell)
$$

where we use (4.23). Arranging terms, equation (C.3) becomes

$$
(EV(d_0))' = \int_A (1 - \alpha - \lambda) \left[ \phi(1 + i_d) \right]dF(A)
$$

$$
+ \alpha \left[ \int_{A \in \Omega_1} \phi(1 + i_d)dF(A) + \int_{A \in \Omega_2} Au'(\frac{d_0}{p_1}) \frac{1}{p_1} dF(A) + \int_{A \in \Omega_3} \phi(1 + i_\ell)dF(A) \right]

+ \lambda \int_A Bu'(\frac{d_0}{p_1}) \frac{1}{p_1} dF(A)
$$

(C.4)

### C.1.2 Proof of Proposition 14

We first consider the case when bank loan is not needed. In this case, young buyers only use their own money. Let $\rho \in [0, 1]$ be the share of initial deposit used, then the total payment by buyers in a bank $i$ is

$$
X_i = \rho \alpha D_0 + \lambda D_0
$$

(C.5)
Since sellers are equally distributed among all banks, \( \frac{1}{N} X_i \) will be paid to sellers in the same bank, and \( \frac{N-1}{N} X_i \) will be paid to sellers in other banks. The pattern is symmetric for other banks. Denote the payment made by each of the other banks as \( X_j \). Then bank \( i \) will receive \( \frac{1}{N} X_j \) from each of the other \( N - 1 \) banks and the total inflow is \( \frac{N-1}{N} X_j \). When \( N \) is large, the outflow from bank \( i \) is approximately equal to \( X_i \), and the inflow is approximately equal to \( X_j \). The profit of a representative bank \( i \) will be

\[
\Pi = D_0(1 + i_c) + (X_j - X_i)(1 + i_c) - (1 + i_d)(1 - \alpha - \lambda)D_0
- (1 + i_d)(\alpha D_0 - \rho \alpha D_0) - X_j(1 + i_s) \quad (C.6)
\]

The first term is the value of the initial deposit. The second term means that if the interbank balance \( X_i - X_j \) is positive, the bank must pay reserve to other banks. The cost is the payment plus the interest that could have been earned. If \( X_i - X_j \) is negative, then the new reserve can be used to earn interest. The third term is the payment to the initial deposit of sellers. The fourth term is the payment to the unused deposit by buyers. \( X_j(1 + i_s) \) is the payment to the deposits earned by sellers in sub-period 1. The changes in the deposits caused by sellers switching banks are not included here because in the symmetric equilibrium, when \( i_s = a i_c \), there are no sellers switch banks.

In the symmetric equilibrium, all initial deposit \( D_0 \) are kept in the central bank’s deposit account. The payment flows among banks will offset each other and inter-bank balances are always zero. (C.6) becomes

\[
\Pi = (\rho \alpha D_0 + \lambda D_0)(i_d - i_s) = X(i_d - i_s) \quad (C.7)
\]

where \( X \) is the gross payment of buyers in the symmetric equilibrium.

Now suppose bank loan is needed. Let \( L_i \) denote the loan made by bank \( i \) and \( L_j \) the loan made by other banks \( j \neq i \). After the loan is made, the total deposit in bank \( i \) is \( D_0 + L_i \)
and the total deposit in bank \( j \) is \( D_0 + L_j \). The payment by bank \( i \) and banks \( j \neq i \) are

\[
X_i = (\alpha + \lambda)D_0 + L_i \quad \text{(C.8)}
\]

\[
X_j = (\alpha + \lambda)D_0 + L_j \quad \text{(C.9)}
\]

Again, the payment outflow and inflow of bank \( i \) are approximately \( X_i \) and \( X_j \), respectively. The net (negative) inter-bank balance for bank \( i \) is

\[
-X_i + X_j = -L_i + L_j
\]

The profit of bank \( i \) is

\[
\Pi = D_0(1 + i_c) - (X_i - X_j)(1 + i_c) + (1 + i_\ell - \delta)L_i - (1 + i_d)(1 - \alpha - \lambda)D_0 - X_j(1 + i_s) \quad \text{(C.10)}
\]

\((1 + i_\ell - \delta)L_i\) is the value of bank loan. The meanings of other terms are still the same.

The first order condition for \( L_i \) gives \( i_\ell = i_c + \delta \).

In the symmetric case, \( L_i = L_j = L \), and (C.10) becomes

\[
\Pi = [(\alpha + \lambda)D_0 + L](i_d - i_s) = X(i_d - i_s) \quad \text{(C.11)}
\]

### C.1.3 Solve the equilibrium and prove Proposition 15

In this part, we show how to solve the equilibrium numerically given \( \omega \) and the central bank’s interest rate policy. We also prove Proposition 15.

We first explain how to compute the equilibrium values in sub-period 1 given \( \omega \), then we explain how to compute \( \omega \). In the following illustration, we allow \( \delta \) to be positive.

Since the interest rate is constant, the deposit for young buyers will more likely to be binding for high values of \( A \). We can separate \( A \) into three ranges \( \Omega_1 = [A, A_1] \), \( \Omega_2 = (A_1, A_2) \),
and \( \Omega_3 = [A_2, \bar{A}] \). (4.35) can be written as

\[
\gamma = (1 - \alpha - \lambda)(1 + i_d)
\]

\[
+ \alpha \left[ \int_{A_1}^A (1 + i_d) dF(A) + \int_{A_2}^A A u'(\frac{d_0}{p_1}) \frac{1}{p_1 \phi} dF(A) + \int_{\bar{A}} (1 + i_d) dF(A) \right]
\]

\[
+ \lambda \int_A B u'(\frac{d_0}{p_1}) \frac{1}{p_1 \phi} dF(A)
\] (C.12)

When \( A \in [A, A_1] \), the deposit for young buyers is not binding. The equilibrium is decided by the following equations:

\[
A u'(q_y) = c'(q_s) \frac{1 + i_d}{1 + i_s}
\] (C.13)

\[
(1 - \alpha - \lambda)q_s = \alpha q_y + \lambda d_0
\] (C.14)

The first equation is the FOC of young buyers (4.19), the second equation is the clearing condition of the goods market (4.30).

We can write the real value of \( d_0 \) on sub-period 1 as

\[
d_0 = d_0 \phi \frac{1}{p_1 \phi} = \frac{d_0 \phi \gamma}{p_1 \phi} = \frac{\omega}{\gamma} \frac{1 + i_s}{c'(q_s)}
\] (C.15)

where we use \( p_1 \phi = c'(q_s)/(1 + i_s) \).

Using \( u'(q) = \frac{1}{q} \), \( c'(q) = 1 + q \) and (C.15), we can write (C.13) and (C.14) as

\[
\frac{A}{q_y} = (1 + q_s) \frac{1 + i_d}{1 + i_s}
\] (C.16)

\[
(1 - \alpha - \lambda)q_s = \alpha q_y + \lambda \frac{\omega}{\gamma} \frac{1 + i_s}{1 + q_s}
\] (C.17)

Using the first equation to replace \( q_s \) in the second equation, we have

\[
(1 - \alpha - \lambda) \left( \frac{A(1 + i_s)}{q_y (1 + i_d)} - 1 \right) = \alpha q_y + \lambda \frac{\omega}{\gamma} q_y (1 + i_d)
\]

\[
\Rightarrow \left( \alpha + \lambda \frac{\omega}{\gamma} \frac{(1 + i_d)}{A} \right) q_y^2 + (1 - \alpha - \lambda) q_y - \frac{A(1 + i_s)}{1 + i_d} (1 - \alpha - \lambda) = 0
\] (C.18)

The positive root is the solution.
Over $[A_1, A_2]$, both types of buyers consume $d_0/p_1$. This implies that both the consumption level and the price level $p_1$ will remain the same over $A \in [A_1, A_2]$. We have

$$ (1 - \alpha - \lambda)q_s = (\alpha + \lambda) \frac{d_0}{p_1} = (\alpha + \lambda) \frac{\omega}{\gamma} \frac{1 + i_s}{1 + q_s} \tag{C.19} $$

where we use (C.15). We have

$$ (1 - \alpha - \lambda)q_s^2 + (1 - \alpha - \lambda)q_s - (\alpha + \lambda) \frac{\omega}{\gamma} (1 + i_s) = 0 \tag{C.20} $$

The positive root is the solution. Denote the solution as $\hat{q}_s$. The corresponding $\hat{q}_y$ is $\frac{d_0}{p_1} = \frac{\omega}{\gamma} \frac{1 + i_s}{1 + q_s}$.

Over $[A_2, \bar{A}]$, borrowers borrow positive loans, and we have

$$ Au'(q_y) = c'(q_s) \frac{1 + i_\ell}{1 + i_s} \tag{C.21} $$

$$ (1 - \alpha - \lambda)q_s = \alpha q_y + \lambda \frac{d_0}{p_1} \tag{C.22} $$

The first equation comes from (4.23). The second equation is the same as (C.14). Arranging terms, we get

$$ \left( \alpha + \lambda \frac{\omega}{\gamma} \frac{1 + i_\ell}{A} \right) q_y^2 + (1 - \alpha - \lambda)q_y - A \frac{1 + i_s}{1 + i_\ell} (1 - \alpha - \lambda) = 0 \tag{C.23} $$

The nominal loan level for each young buyer is $\ell = q_y p_1 - d_0$. Using $\frac{Au'(q_y)}{\phi(1+i_\ell)}$ to replace $p_1$ and using $u'(q) = \frac{1}{q}$, we have

$$ \ell = q_y \frac{1}{\phi} \frac{Au'(q_y)}{1 + i_\ell} - d_0 = \frac{A}{\phi(1+i_\ell)} - d_0 \tag{C.24} $$

and the real level is

$$ \ell \phi = \frac{A}{(1 + i_\ell)} - d_0 \phi = \frac{A}{(1 + i_\ell)} - \frac{\omega}{\gamma} \tag{C.25} $$

The aggregate loan level is $L = \alpha \ell$.

$A_1$ and $A_2$ can be decided as follows. At $A_1$ and $A_2$, we should have $q_y = \hat{q}_y$ and $q_s = \hat{q}_s$. Substituting $\hat{q}_y$ and $\hat{q}_s$ into the first order condition for $A \in \Omega_1(C.13)$, we get $A_1$. Substituting
\( \hat{q}_y \) and \( \hat{q}_s \) into the first order condition for \( A \in \Omega_3 \) (C.21), we get \( A_2 \). The explicit solutions will be given below.

**Equilibrium condition for \( \omega \)**

For the log utility function, (C.12) can be simplified as follows. Using (C.15), we have

\[
u'(\frac{d_0}{p_1}) \frac{1}{p_1 \phi} = \frac{p_1}{d_0} \frac{1}{p_1 \phi} = \frac{1}{d_0 \phi} = \frac{\gamma}{\omega} \tag{C.26}\]

In addition, at \( A_1 \), (C.13) holds for \( \hat{q}_y = \frac{d_0}{p_1} \), we have

\[
A_1 u'(\frac{d_0}{p_1}) = c'(q_s) \frac{1 + i_d}{1 + i_s} = p_1 \phi(1 + i_d) \Rightarrow A_1 = \frac{p_1 \phi(1 + i_d)}{u'(\frac{d_0}{p_1})} = d_0 \phi(1 + i_d) = \frac{\omega}{\gamma}(1 + i_d) \tag{C.27}
\]

Similarly, at \( A_2 \), (C.21) holds for \( \hat{q}_y \), we have

\[
A_2 u'(\frac{d_0}{p_1}) = c'(q_s) \frac{1 + i_\ell}{1 + i_s} \Rightarrow A_2 = \frac{\omega}{\gamma}(1 + i_\ell) \tag{C.28}
\]

(C.12) becomes

\[
\frac{\gamma}{\beta} = (1 - \alpha - \lambda)(1 + i_d)
+ \alpha \left[ \int_{A_1}^{A_2} (1 + i_d) dF(A) + \int_{A_1}^{A_2} A \frac{\gamma}{\omega} dF(A) + \int_{A_2}^{\bar{A}} (1 + i_\ell) dF(A) \right]
+ \lambda \int_{A_2}^{\bar{A}} B(A) \frac{\omega}{\gamma} dF(A) \tag{C.29}
\]

with \( A_1 \) and \( A_2 \) being specified in (C.27) and (C.28).

**Proof of Proposition 15:** Given the function forms we have, the RHS of (C.29) will be continuous. The derivative of the RHS of (C.29) with respect to \( \omega \) is

\[
\alpha \int_{A_1}^{A_2} -A \frac{\gamma}{\omega^2} dF(A) + \lambda \int_{A_1}^{\bar{A}} -B(A) \frac{\gamma}{\omega^2} dF(A) < 0 \tag{C.30}
\]

This means the RHS of (C.29) is decreasing in \( \omega \). In addition, when \( \omega \to 0 \), the RHS of (C.29) approaches infinity due to the term \( \lambda \int_{A_1}^{\bar{A}} B(A) \frac{\omega}{\gamma} dF(A) \). When \( \omega \to +\infty \), the deposit balance will become non-binding for young buyers, that is, \( A_1 \) becomes \( \bar{A} \). At the same time, the term \( \int_{A} B(A) \frac{\omega}{\gamma} dF(A) \) will approach zero. The RHS of (C.29) then becomes

\[
(1 - \alpha - \lambda)(1 + i_d) + \alpha(1 + i_d) = (1 - \lambda)(1 + i_d) \tag{C.31}
\]
As a result, as long as \( \frac{\gamma}{\beta} > (1 - \lambda)(1 + i_d) \), there exists a unique equilibrium \( \omega \). For example, when \( i_d = 0 \), the lowest sustainable inflation rate in the monetary equilibrium is \( \gamma = \beta(1 - \lambda) \), higher \( \lambda \) means lower sustainable \( \gamma \).

What we’ve just shown is the solution method for the general case in which \( \delta \geq 0 \) and \( i_c \geq 0 \). In the special case when \( \delta = 0 \), we have \( i_d = i_\ell \) and the set \([A_1, A_2]\) shrinks to zero. In addition, when \( i_c = 0 \), we have \( i_d = i_\ell = 0 \). So (C.29) can be simplified to

\[
\frac{\gamma}{\beta} = (1 - \lambda) + \lambda \int_A B(A) \frac{\gamma}{\omega} dF(A) \quad \implies \omega = \frac{\gamma \lambda EB}{\frac{\gamma}{\beta} - (1 - \lambda)}
\]  

(C.32)

C.1.4 Proof of Proposition 16

The central bank only needs to care about \( Bu'(q_o) = c'(q_s) \). \( Bu'(q_o) \) can be written as

\[
Bu'(q_o) = B \frac{p_1}{d_0} = B \frac{1}{d_0 \phi} (p_1 \phi) = B \frac{\gamma}{\omega} \frac{c'(q_s)}{1 + i_s}
\]  

(C.33)

So we need \( B \frac{\gamma}{\omega} \frac{c'(q_s)}{1 + i_s} = c'(q_s) \). Since \( i_s = i_d \), we have

\[
1 + i_d = B \frac{\gamma}{\omega}
\]  

(C.34)

Due to the zero bound, the optimal interest rate policy is

\[
1 + i_c = \max(1, B \frac{\gamma}{\omega})
\]  

(C.35)

C.1.5 Production and monetary flows in the equilibrium

This part shows the details for production and consumption in the equilibrium. We will show the result for the general case in which \( i_s = a i_d \) with \( a \in [0,1] \). In the following example, we assume that the shock is high and the bank loan is positive. The analysis will be similar when there is no borrowing.
Consumption and production

Let \( d_0 \) denote the nominal money balance carried by every household into sub-period 1 in period \( t \). We have \( M = d_0 \).

In sub-period 1, each young buyer spends \( d_0 + \ell \), each old buyer spends \( d_0 \), and each seller earns \( \alpha (d_0 + \ell) + \lambda d_0 \). The consumption is \( q_y = \frac{d_0 + \ell}{p_1} \) for young buyers and \( q_o = \frac{d_0}{p_1} \) for old buyers, where \( p_1 = \frac{c'(q_s)}{\phi(1+i_s)} \), and the production is \( q_s = \frac{\alpha q_y + \lambda q_o}{1 - \alpha - \lambda} \). The equilibrium \( q_y \) is decided according to \( \frac{Au'(q_y)}{c'(q_s)} = 1 + i_\ell \) and \( \ell = q_y p_1 - d_0 \). In the symmetric equilibrium, all initial outside money \( d_0 \) is deposited in the central bank and earns interest \( i_c \). In the equilibrium, \( i_d = i_c \) and \( i_\ell = i_d + \delta \).

In sub-period 2, equation (4.7) gives \( x^* = U'^{-1}(1) \). The equilibrium transfer level is

\[
\tau = M(\gamma - 1) - i_c d_0 = (\gamma - 1)d_0 - i_d d_0 \tag{C.36}
\]

where \( i_d d_0 \) is the deposit interest paid by the central bank.

In the stationary equilibrium, since the price level in sub-period 2 grows according to \( \gamma \), agents need to carry deposit balance \( d_0 \gamma \) into the next period. The production of young buyer in the sub-period 2 is:

\[
h_y = x^* + \phi [d_0 \gamma - \tau + (1 + i_\ell)\ell] - \phi \Pi
\]

\[
= x^* + \phi [d_0 \gamma - ((\gamma - 1)d_0 - i_d d_0) + (1 + i_\ell)\ell] - \phi \Pi
\]

\[
= x^* + \phi [d_0 (1 + i_d) + (1 + i_\ell)\ell] - \phi \Pi \tag{C.37}
\]

For the seller, the initial deposit is \( d_0 \), and the newly earned money is \( s = \frac{\alpha (d_0 + \ell) + \lambda d_0}{1 - \alpha - \lambda} \). The production of the seller is

\[
h_s = x^* + \phi \left[ d_0 \gamma - \tau - d_0 (1 + i_d) - \frac{\alpha (d_0 + \ell) + \lambda d_0}{1 - \alpha - \lambda} (1 + i_s) \right] - \phi \Pi
\]

\[
= x^* + \phi \left[ d_0 \gamma - ((\gamma - 1)d_0 - i_d d_0) - d_0 (1 + i_d) - \frac{\alpha (d_0 + \ell) + \lambda d_0}{1 - \alpha - \lambda} (1 + i_s) \right] - \phi \Pi
\]

\[
= x^* - \phi \left[ \frac{\alpha (d_0 + \ell) + \lambda d_0}{1 - \alpha - \lambda} (1 + i_s) \right] - \phi \Pi \tag{C.38}
\]
The production of the newly born agents is

\[ h_n = x^* + \phi [d_0 \gamma - \tau] - \phi \Pi = x^* + \phi [d_0(1 + i_d)] - \phi \Pi \]

The expected \( h \) is

\[
\alpha h_y + (1 - \alpha - \lambda)h_s + \lambda h_n \\
= x^* + \phi [(\alpha + \lambda)d_0(i_d - i_s) + \alpha \ell (i_\ell - i_s)] - \phi \Pi
\]

(C.39)

In the symmetric case, as we have shown in (C.10), we have

\[
\Pi + \delta L = D_0(1 + i_c) + (1 + i_\ell)L - (1 + i_d)(1 - \alpha - \lambda)D_0 - X(1 + i_s)
\Rightarrow \Pi + \delta \alpha \ell = d_0(1 + i_c) + (1 + i_\ell)\alpha \ell - (1 + i_d)(1 - \alpha - \lambda)d_0 - (\alpha(d_0 + \ell) + \lambda d_0)(1 + i_s)
\]

(C.40)

Substitute this into (C.39), we get

\[
\alpha h_y + (1 - \alpha - \lambda)h_s + \lambda h_n = x^* + \phi \delta \alpha \ell
\]

(C.41)

So the total production equals the total consumption of households, plus the management cost of banks.

In sub-period 2, the net production of sellers will be the smallest. If sellers have positive \( h \), then young buyers and the new born old buyers will also have positive production.

In the equilibrium, \( \Pi = X(i_d - i_s) = [(\alpha + \lambda)d_0 + \alpha \ell](i_d - i_s) \). \( h_s \) (C.38) can be written as

\[
h_s = x^* - \phi \left[ \frac{\alpha(d_0 + \ell) + \lambda d_0}{1 - \alpha - \lambda} (1 + i_s) \right] - \phi [(\alpha + \lambda)d_0 + \alpha \ell](i_d - i_s)
\]

(C.42)

Since \( \phi = \frac{c'(q_y)}{p_1(1 + i_s)} \) and \( d_0 + \ell = p_1 q_y \), we have

\[
\phi(d_0 + \ell)(1 + i_s) = \frac{c'(q_y)}{p_1(1 + i_s)}(p_1 q_y)(1 + i_s) = c'(q_y) q_y
\]

(C.43)
In the equilibrium, when bank loan is needed, we have $c'(q_s) = \frac{Auu'(q_y)(1+i_s)}{1+i_e}$. Since $u'(q_y) = q_yu'(q_y)u(q_y) + q_yu'(q_y)q_y$, $c'(q_s)q_y$ can be written as $Ae(q_y)u(q_y)\frac{1+i_s}{1+i_e} \leq Ae(q_y)u(q_y)$. Similarly, $\phi(d_0 + \ell)(i_d - i_s)$ can be written as

$$c'(q_s)q_y \frac{i_d - i_s}{1 + i_s} = Ae(q_y)u(q_y)\frac{i_d - i_s}{1 + i_e} < Ae(q_y)u(q_y)$$

$e(q_y)$ is bounded, so we can scale $U(x)$ such that $x^* = U'^{-1}(1)$ is greater than $[\frac{\alpha + \lambda}{1 - \alpha - \lambda} + (\alpha + \lambda)]Ae(q_y)u(q_y)$ for all possible values of $q_y$, that is, $x^*$ is higher than the sum of the negative terms in (C.42), then $h_s$ will always be positive. As a result, we can design the model to avoid the corner solution $h^* = 0$.

### C.1.6 Monetary flows

#### Table C.1: The creation, circulation and destruction of inside money

<table>
<thead>
<tr>
<th>Commercial Bank Balance sheet (When $L &gt; 0$)</th>
<th>Asset Side</th>
<th>Liability side</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Asset Side</strong></td>
<td><strong>Deposit</strong></td>
<td><strong>Deposit</strong></td>
</tr>
<tr>
<td>Reserve (Young buyers)</td>
<td>Deposit (Young buyer)</td>
<td>Deposit (Type 2 buyers)</td>
</tr>
<tr>
<td>Bank makes loans</td>
<td>+L</td>
<td>+L</td>
</tr>
<tr>
<td>Buyers consume</td>
<td>-$\lambda$Dh</td>
<td>+L</td>
</tr>
<tr>
<td>Changes in each account: Sub-period 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank incurs the cost</td>
<td>+$\alpha$L$\delta$</td>
<td>+($1 - \alpha - \lambda$)L$\delta$</td>
</tr>
<tr>
<td>Bank distributes the profit</td>
<td>+$\alpha$H</td>
<td>+($1 - \alpha - \lambda$)H</td>
</tr>
<tr>
<td>Sellers spend the extra deposit</td>
<td>+SP - $\lambda$(D$_b$)</td>
<td>-SP</td>
</tr>
<tr>
<td>Buyers repay the bank loan</td>
<td>-$L(1 + i_t)$</td>
<td>-$L(1 + i_t)$</td>
</tr>
<tr>
<td><strong>Balance</strong></td>
<td>D$_b$</td>
<td>$\alpha$D$_b$</td>
</tr>
</tbody>
</table>

1. $\tau = D_b(\gamma - 1 - i_t)$. 2. $H = ((\alpha + \lambda)D_b + L)(i_d - i_s)$. 3. $SP$ is the aggregate spending of sellers, and $SP = [L + (\alpha + \lambda)D_b][1 + i_t] + (1 - \alpha - \lambda)(L \delta + H)$.

In this part, we use bank balance sheet to show the monetary flows in the economy. Table C.1 shows how bank deposit is created, how people use bank deposit to make payments.
and how bank deposit is destroyed. We use $D_0$ to denote the total initial deposit in the representative bank at the beginning of the period.

At the beginning of sub-period 1, the total reserve and deposit are equal to $D_0$. The total deposit held by young buyers is $\alpha D_0$, and the total deposit held by old buyers and sellers are $\lambda D_0$ and $(1 - \alpha - \lambda)D_0$, respectively.

In sub-period 1, when banks make loans, they simply credit the deposit account of borrowers by $L$. New deposit $L$ is created. Then buyers buy goods from sellers. Buyers’ deposit will be transferred to sellers.

Then we enter sub-period 2. In this market, only the final net money flow matters. But for illustrative purpose, we show the gross flows.

First, the central bank pays the deposit interest using central bank money. The central bank then makes the lump-sum transfer to households. (If $\tau$ is negative, then the central bank collects tax from people.)

Banks then pay interest to depositors. Banks simply credit the deposit account of depositors. As a result, an equivalent amount of new deposit is created. Banks then incur the management cost. Managing loans need to use real resources (i.e., real goods). Banks can buy different amount of goods from different individuals, the number of possible outcome is infinite. For illustrative purpose, we assume the goods purchased from each group is proportional to the size of each group. Banks buy goods by crediting the accounts of sellers. This will increase the level of outstanding bank deposit by the same amount. Banks then distribute the profit by crediting the deposit account of households. Again, an equivalent amount of new deposit is created.

Sellers of sub-period 1 then spend their extra deposit. We can use the final balance to determine the deposit that the newly born agents need to earn and the deposit that sellers of sub-period 1 will spend.

Finally, young buyers repay their bank loan. The deposit $L(1 + i_\ell)$ used to repay the loan
is destroyed, which reduces the outstanding bank deposit by the same amount.

Briefly speaking, before the repayment of bank loan, bank deposits are created in the following steps: when banks make loans, when the central bank makes the transfer, when banks pay deposit interest, when banks incur management cost and pay dividend. The deposits created in these steps will add up to \( D_0(\gamma - 1) + L(1 + i_\ell) \). \( L(1 + i_\ell) \) is destroyed when loan is paid back. As a result, the final accumulated change in bank deposit is \( D_0(\gamma - 1) \), which is equal to the change in outside money (bank reserve).

C.2 Solve the equilibrium when \( i_c \) is set optimally

1. Start with a value of \( \omega \).

2. For each \( A \), solve the equilibrium for different interest rate policy \( i_c \)
   
   a. Given \( i_c \), \( A_1 \) and \( A_2 \) can be computed using (C.27) and (C.28). Find out whether
   
   \( A < A_1 \), \( A_1 \leq A \leq A_2 \) or \( A > A_2 \). Solve the equilibrium accordingly.
   
   b. Select the equilibrium in which \( i_c \) maximizes the social welfare.

3. Iterate on \( \omega \) until (4.35) is satisfied.
Bibliography


