SIMULATIONS OF SCALE-FREE COSMOLOGIES FOR THE SMALL-SCALE COLD DARK MATTER UNIVERSE

by

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ABSTRACT

Cosmological simulations show that dark matter halos contain a wealth of substructure. These subhalos are assumed have a mass distribution that extends down to the smallest mass in the Cold Dark Matter (CDM) hierarchy, which lies below the current resolution limit of simulations. Substructure has important ramifications for indirect dark matter detection experiments as the signal depends sensitively on the small-scale density distribution of dark matter in the Galactic halo. A clumpy halo produces a stronger signal than halos where the density is a smooth function of radius.

However, the small-scale Universe presents a daunting challenge for models of structure formation. In the CDM paradigm, structures form in a hierarchical fashion, with small-scale perturbations collapsing first to form halos that then grow via mergers. However, near the bottom of the hierarchy, dark matter structures form nearly simultaneously across a wide range of scales.

To explore these small scales, I use a series of simulations of scale-free cosmological models, where the initial density power spectrum is a power-law. I can effectively examine various scales in the Universe by using the index in these artificial cosmologies as a proxy for scale. This approach is not new, but my simulations are larger than previous such simulations by a factor of 3 or more.

My results call into question the often made assumption that the subhalo population is scale-free. The subhalo population does depend on the mass of the host. By combining my study with others, I construct a phenomenological model for the subhalo mass function. This model shows that the full subhalo hierarchy does not greatly boost the dark matter annihilation flux of a host halo. Thus, the enhancement of the Galactic halo signature due to substructure can not alone account the
observed flux of cosmic rays produced by annihilating dark matter.

Finally, I examine the nonlinear power spectrum, which is used to determine cosmological parameters based on large-scale, observational surveys. I find that in this nonlinear regime, my results are not consistent with currently used fitting formulae and present my own empirical formula.
STATEMENT OF CO-AUTHORSHIP

The work present in thesis thesis was performed under the supervision of L.M. Widrow at Queen’s University. The work presented in Chapter 4 is based on a paper written in conjunction with L.M. Widrow, R.J. Thacker and E. Scannapieco (published in the *Monthly Notices of the Royal Astronomical Society*). The work presented in Chapter 5 is based on a paper written in conjunction with L.M. Widrow and R.J. Thacker submitted to *Physical Review Letters*. The work presented in Chapter 6 is based on a paper written in conjunction with L.M. Widrow, R.J. Thacker, M. Richardson and E. Scannapieco (published in the *Monthly Notices of the Royal Astronomical Society*).
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STATEMENT OF ORIGINALITY

All the work described here was done by the author (P.J. Elahi) except where explicitly stated otherwise. The N-body GADGET-2 code used in this work, a massively parallel TreePM code, was written by V. Springel and is described in Springel (2005). The 6DFOF code used to identify subhalos was based on a massively parallel FOF group finder written by the Virgo Consortium. In Chapter 6 the integrator used to calculate the one loop perturbation equations as well as the partial differential equation solver used to renormalize these perturbation equations were written by Widrow.
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Chapter 1

Introduction

The ultimate goal of this thesis is to explore the substructure of dark matter halos and examine the ramifications for dark matter detection. I also study a related topic, the nonlinear evolution of the matter power spectrum using numerical simulations. To this end, I will present two manuscripts, Elahi et al. (2009a) and Elahi et al. (2009b) (Papers I & II), as published and submitted respectively, and discuss material based on Widrow et al. (2009), Paper III.

The first two papers attempt to answer the main questions posed in this thesis, “What is the small-scale substructure of a dark matter halo and what does it mean for dark matter detection?”. The various searches for the dark matter particle depend on the distribution of dark matter in the local solar neighbourhood and our Galaxy at large, and are sensitive to the small-scale distribution of dark matter (e.g. Jungman et al. 1996, Bergström et al. 1998, Bertone et al. 2005). Direct detection experiments measure the recoil energy distribution from dark matter particles scattering off nuclei. This observation depends on the local density and speed distribution. Indirect detection experiments have the best chance of observing evidence for dark matter if the distribution is clumpy. Thus, it is vital we understand dark matter structures down to very small scales.

In modern cosmology the distribution of matter in the early Universe, both non-baryonic “dark matter” and baryonic “normal” matter, is almost perfectly smooth. The tiny ripples in density, put in place in the very early Universe by cosmic inflation, are the seeds of structure formation.
These perturbations grow through gravitational instability. At early times, when the ripples are "linear", that is the size of the density perturbations relative to the mean density are much smaller than unity, we can use analytic techniques to model their evolution. At later times when these perturbations exceed unity, the assumptions used in analytic techniques begin to break down. At this point numerical N-body simulations are used.

These simulations show that the initial dark matter perturbations collapse to form virialized dark matter halos. In the Cold Dark Matter paradigm, halos continuously accrete matter and merge with other halos, forming a hierarchy of structure. These dark matter halos are the potential wells in which baryons coalesce to form stars, galaxies, and galaxies clusters, the observable Universe. Therefore, dark matter halos drive galaxy formation. Our own Galaxy is embedded in a such a dark matter halo. At the largest scales, these halos collect in filaments and sheets, forming a cosmic web (Springel et al. 2005). At the smallest scales, simulations have uncovered a wealth of substructure (Moore et al. 1999). Halos are clumpy, not smooth.

Naively, it might seem to be a simple matter of using N-body methods to follow the evolution of the smallest density perturbations and thereby determine the dark matter detection signal. For instance Diemand et al. (2005) investigated the formation of the first parsec size earth mass dark matter objects. Diemand et al. (2006) also examined the substructure near these scales. However, these studies should be treated with caution as these small scales are the most difficult to simulate. At these scales, perturbations become nonlinear and structures begin to form almost simultaneously across a wide range of scales and masses. Furthermore, large-scale tidal fields become increasingly important. N-body simulations, by their very nature can only include perturbations up to the simulation volume. This makes the dynamically complex environment at small scales computationally difficult to accurately simulate.

Instead of simulating volumes with enough dynamic range to follow the evolution of the small-scale Universe, I appeal to scale-free simulations. A scale-free cosmology is one which only contains dark matter and where the ensemble average variation of density perturbations, given by the
power spectrum, is a simple-power law function of scale. As a consequence, such an artificial cosmology has only one physically distinct scale, the scale that separates the linear regime from the nonlinear. This simplicity means that scale-free simulations have useful mathematical properties which serve as a guide in interpreting results. By using the index of the power-law in these artificial universes as a proxy for scale, I can effectively sample different scales in the Universe. I am not the first to use scale-free simulations to study the properties of halos at different scales in the Universe (e.g. Cole and Lacey [1996], Reed et al. [2005a], Knollmann et al. [2008]), although the series of simulations in this thesis are larger by a factor of 3 or more than any previously run. By examining substructure in these cosmologies, I am effectively working my way down to smaller and smaller scales in the Universe and thereby explore the consequences of small-scale structure on dark matter detection.

I also examine the nonlinear evolution of the power spectrum of the density perturbations, again with a focus on the evolution at the smallest scales. In particular, I am interested in how far analytic techniques, some of which are used to initialize N-body simulations, can be pushed into the nonlinear regime. The motivation here is that large, extragalactic surveys probe the nonlinear regime. These surveys, such as the Sloan Digital Sky Survey (Percival et al. [2007]) are excellent probes of the cosmological parameters, although they have not reached the same level of accuracy as measurements of the Cosmic Microwave Background (Komatsu et al. [2009]). Such surveys estimate parameters by determining the linear power spectrum from the measured nonlinear one. Therefore, improving these estimates crucially relies on improving our understanding of the power spectrum’s nonlinear evolution.

Scale-free simulations offer a unique window into this nonlinear evolution. Due to the form of the power spectrum, these simulations evolve in a “self-similar” fashion (Jain and Bertschinger [1998]). Numerous other studies have also examined the power spectrum in these artificial cosmologies (Peacock and Dodds [1996], Smith et al. [2003]). My simulations build on this previous work, examining the power spectrum with higher resolution and at more negative spectral indices than any prior study.
The thesis is broken up into 7 chapters. I start in Chapter 2 by presenting some general background on cosmology, dark matter, the CDM paradigm and dark matter detection experiments. I present a more technical literature review in Chapter 3. This review contains a discussion of the power spectrum followed by a short outline of the N-body techniques. I then review the properties of dark matter halos and how the indirect detection signal from dark matter self-annihilation is affected the properties of these halos. This is followed by three chapters describing the primary work of this thesis: Paper I concerning subhalos in scale free cosmologies, Paper II concerning substructure and indirect dark matter detection, and material based on Paper III concerning the power spectrum. I conclude with a summary of my results, a general discussion, and some final thoughts regarding future work.
Chapter 2

Background

The purpose of this chapter is to give some general background regarding to cosmology, dark matter, Cold Dark Matter (CDM) structures and dark matter detection are in order. Readers with a background knowledge of cosmology and dark matter can skip directly to Chap. 3 and brush up on the more technical background information.

2.1 The Expanding Universe

The standard cosmological paradigm is the so-called “Big Bang” scenario. It has its roots in the discovery by Hubble (1929) that all galaxies save those in the Local Group (our nearest galactic neighbours) are receding away from us at a rate that is proportional to their distance from us. This is the famous Hubble Law, $v = dH_o$, where $v$ is the recession velocity and $d$ is the distance, and $H_o$ is referred to as Hubble’s constant. At the same time, Friedmann (1924), Lemaître (1931), Robertson (1933) & Walker (1933) all contributed to the discovery of a solution to Einstein’s equations of General Relativity describing an expanding homogeneous and isotropic universe.

The metric for such a universe, the FLRW metric, can be written as

$$ds^2 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right),$$

(2.1)
where $a(t)$ is the scale factor, $r$ is the comoving coordinate, and $K$, which describes the spatial curvature, can take on values of $-1, 0, 1$. The physical coordinate $x = a(t)r$ is observed to expand by $Hx$, where $H(t) = \dot{a}(t)/a(t)$ is the Hubble expansion. The unitless scale factor $a$ is generally defined such that $a(t_o) = 1$ at the current time $t_o$ and $a(t = 0) = 0$ at the “Big Bang”. It is also common to write the Hubble expansion at the current epoch, the Hubble constant, $H_o = 100h$ km s$^{-1}$ Mpc$^{-1}$, where $h$ is the Hubble parameter. In this text I will also refer to the redshift, which is $z(t) = a(t)^{-1} - 1$.

The rate of expansion depends on the energy budget of the Universe via the Friedmann equation,

$$H^2(t) = \frac{8\pi G_N \rho_{\text{tot}}(t)}{3} - \frac{K}{a^2(t)}, \quad (2.2)$$

where $G_N$ is Newton’s constant and $\rho_{\text{tot}}$ is the total energy density of the universe. This equation shows that for a flat universe, $K = 0$, there is a critical density,

$$\rho_c(t) \equiv \frac{3H(t)^2}{8\pi G_N}, \quad (2.3)$$

It is common to define the amount of a substance at the current epoch ($a = 1$) in units of $\rho_c$,

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}. \quad (2.4)$$

Each substances’ relative contribution to the total energy density of the universe will evolve according to their equation of state. Radiation, where the pressure $p = \rho/3$, has $\rho_r(t) \propto a(t)^{-4}$ for example. For pressureless matter, $\rho_m(t) = \rho_{m,0}/a(t)^3$, where $\rho_{m,0}$ is the present-day density.

As a universe expands, different substances dominate the energy budget. Consider a universe containing only matter and radiation. At early times, regardless of $\Omega_r$ and $\Omega_m$, radiation was the dominant component of the energy budget. At this early epoch, $H = \frac{H_o}{2a^2}$. The epoch of matter-radiation equality, $\rho_r(a) = \rho_m(a)$, occurs at $a_{eq} = \Omega_r/\Omega_m$, after which the universe begins expanding at a rate of $H = \frac{2H_o}{3a_{eq}^{3/2}}$. Our Universe has both matter and radiation and also appears to have a non-zero cosmological constant [Komatsu et al. 2009].

It follows that, since as the universe ages, it expands and cools, then at early times it must have been very dense and hot. Little is actually known about the very early universe and is based on the
extrapolation of known physics back to the Planck epoch, when the Universe was only $t = 10^{-43}$ seconds old. I briefly outline the standard inflationary model of the early Universe below, though there are several scenarios regarding the early universe, some of which differ radically. For more detail the reader should see Peacock (1999), Ryden (2003), and Dodelson (2003), for example.

- Between $10^{-43} - 10^{-36}$ s, gravity separates from the other forces. At $10^{-36}$ s and energies of $\sim 10^{16}$ GeV, grand unification breaks down and the strong nuclear force separates from the electroweak force.

- The inflationary epoch is thought to occur between $10^{-36}-10^{-32}$ s. During this period the universe enters rapid homogeneous and isotropic expansion in which the seeds of structure formation are laid down in the form of a primordial spectrum of nearly-scale-invariant fluctuations in the energy density (e.g. Guth [1981], Linde [1982]).

- At $t \sim 10^{-9}$ s and $10^2$ GeV the electroweak force breaks into the electromagnetic and weak nuclear force. Any massive weakly interacting particles with masses in the GeV to TeV range will drop out of thermal equilibrium and freeze-out.

- As the universe cools further to temperatures of $\sim 100 - 1000$ keV ($t \sim 100 - 100$ s), neutrons begin to bind to protons, forming light elements. This is referred to as Big Bang Nucleosynthesis.

- The epoch of matter-radiation equality occurs at $t \sim 70,000$ years and $\sim 1$ eV. Prior to this the universe was radiation dominated. Density perturbations, instead of being wiped out by free-streaming photons, begin to grow as they enter the causal horizon.

- Between 240,000-310,000 years, recombination occurs, that is free electrons cool enough to bind with hydrogen and helium ions forming neutral atoms. After this epoch, photons free-stream, that is they propagate without scattering. These relic photons form the Cosmic Microwave Background (CMB).
The CMB, one of the main predictions of the Big Bang scenario, was found accidentally by Penzias and Wilson (1965). It was well described by a homogeneous, isotropic, blackbody spectrum as predicted by Gamow (1948) and Alpher et al. (1948). Observations by the Cosmic Background Explorer (COBE) a space satellite launched in 1989, showed that the CMB did in fact contains small deviations from homogeneity. These perturbations, which are on the order of $10^{-5}$, are the seeds of structure formation.

2.2 The Power Spectrum of Density Perturbations

Once matter dominates the universe gravity begins to amplify the tiny inhomogeneities left by cosmic inflation, causing matter to fall towards dense regions and making underdense regions more rarefied. The matter density is $\rho_m(x, a) = \rho_{m,o}(a)[1 + \delta(x, a)]$, where $a$ is the scale factor, $\rho_{m,o}$ is the mean background density and $\delta(x, a)$ is the perturbation. The initial perturbations laid down by inflation have the property that at any point, $x$, $\delta(x, a)$ is sampled from an isotropic, homogeneous Gaussian field with a mean of zero. The spatial Fourier transform of the linear perturbations, $\tilde{\delta}(k)$, where $k$ is the wavenumber, is given by

$$\tilde{\delta}(k, a) = \mathcal{F}[\delta(x, a)] = \int d^3 x \delta(x, a) e^{-ik \cdot x}. \quad (2.5)$$

These perturbations have an ensemble average of

$$\langle \tilde{\delta}(k, a) \tilde{\delta}^*(k', a) \rangle = (2\pi)^3 \delta^D(k - k') P(k, a), \quad (2.6)$$

where $\langle \cdots \rangle$ denotes the ensemble average, $\delta^D$ is the Dirac delta function and $P(k)$ is the power spectrum. The power spectrum is the variance of the Gaussian density field.

In the early epoch, the density inhomogeneities are small, $\delta \ll 1$, and perturbations with different wavelengths evolve independently. Perturbations only evolve when their wavelength is smaller than the causal horizon. As the causal horizon or Hubble radius grows due to the expansion of the Universe, it encompasses perturbations at progressively longer wavelengths. Perturbations also grow only in the matter dominated epoch. Consequently, small-scale perturbations which were
already in the horizon during the radiation dominated epoch, have their growth is suppressed relative to larger modes which entered after the epoch of equality. This mode-dependent evolution, which depends on $\Omega_m, \Omega_r$ and the scattering process that occur between the various constituents, is summarized in the transfer function, $T(k)$. The linear growth of the perturbations right after the transfer function regime due to gravitational evolution in an expanding universe is given by the growth function,

$$D(a) \propto H(a) \int_0^a [a'H(a')]^{-3} da'.$$  \hfill (2.7)

The linear matter power spectrum is the combination of three components: the initial nearly scale-invariant spectrum, $k^{n_s}$, growth function, $D(a)$, and transfer function, $T(k)$,

$$P(k, a) \propto k^{n_s} T^2(k) D^2(a).$$  \hfill (2.8)

For more details regarding the derivation of the power spectrum, I invite the reader to see Peebles (1980), Peacock (1999), and Dodelson (2003). I show a sample power spectrum in Figure 2.1 on the following page.

### 2.3 Dark Matter and the Energy Budget of the Universe

The evidence for the existence of a non-luminous form of matter actually dates back to early observations of the Coma galaxy cluster. Zwicky (1933) proposed that if the cluster was gravitationally bound, the high velocities dispersions measured indicated the cluster must contain a great deal more matter than was visible. Velocity measurements by Rubin and Ford (1970) of the Andromeda galaxy, a spiral disk galaxy, also indicated the existence of “dark” matter. They observed a flat rotation curve that was in disagreement with visible mass distribution, which predicted a rotation velocity that decline with distance from the centre of the galaxy. The rotation curve suggested that Andromeda was embedded in a massive dark halo. Later surveys showed that most disk galaxies appeared to be embedded in dark halos that are several times the size of the optical disks.
There were two theories as to what this non-luminous matter might be. It was either baryonic matter in the form of gas, dust, planets, dead stars, that is objects too faint to be detected, or a new form of matter that did not interact with light. Various lines of evidence strongly disfavour baryonic dark matter. Searches for Massive Compact Halo Objects (MACHOS), such as dead stars or black holes, in the Milky Way using gravitational lensing indicate that MACHOS account for at most 20% of the Galaxy’s missing mass (Alcock et al., 2000) and possibly less than 1% (Afonso et al., 2003). Furthermore, a purely baryonic universe fails to predict the correct amount of deuterium observed (Wagoner, 1973; Iocco et al., 2009). Measurements of the CMB by Wilkinson Microwave Anisotropy Probe (WMAP) (Komatsu et al., 2009), the large-scale distribution of galaxies measured by surveys like Sloan Digital Sky Survey (SDSS) (Percival et al., 2007), and other observations indicate that the energy budget of the Universe to be roughly 25% matter, 75% cosmological constant, with approximately 80% of the matter being nonbaryonic dark matter.
2.4 Dark Matter Particle

There have been several candidates for dark matter. The neutrino, first proposed by Pauli in 1933 to account for the missing energy and angular momentum in certain nuclear decays, was the first dark matter candidate. Neutrinos interact only through the weak nuclear force and are now known to have mass.\(^1\) Neutrinos, however, have a small mass. WMAP data constrains the sum of the masses of all three flavours of neutrinos to be \(\sum m_{\nu,i} < 0.62\,\text{eV}\) (Komatsu et al., 2009). Their low mass, combined with production rate estimates during Big Bang nucleosynthesis, means that neutrinos are a small component of the Universe’s energy budget and cannot account for all the dark matter. Furthermore, neutrinos at decoupling are ultra-relativistic, and are thus a form of Hot Dark Matter (HDM). However, WMAP’s measurement of an early reionization epoch, the epoch of the first stars, strongly disfavours HDM.

There are numerous proposed particles: axions, wimpzillas, Q-balls, mirror particles, etc (Bertone et al., 2005). However, observations of the large-scale structure of the Universe indicate that most of the dark matter must be non-relativistic Cold Dark Matter (CDM) that does not interact strongly with normal baryonic matter. What is needed is a Weakly Interacting Massive Particle (WIMP). Here I will focus on supersymmetric (SUSY) particles and Kaluza Klein (KK) particles as both types of particles are theoretically well-motivated (e.g. Jungman et al., 1996; Bertone et al., 2005). Note that in the text I will denote the dark matter particle by \(\chi\) regardless of type, whether SUSY or KK dark matter.

2.4.1 Supersymmetry and the Neutralino

Supersymmetry was developed primarily to solve the “hierarchy” problem in particle physics, that is to account for the enormous difference between the electroweak energy scale (245 GeV) and the Planck energy scale (\(10^{19}\) GeV). Another way to phrase this problem is why is gravity so much weaker than the other forces. The problem arises due to the radiative corrections to the mass of

\(^1\)Measurements by the Sudbury Neutrino Observatory (Ahmad et al., 2002) and Super K (Fukuda et al., 1998) show neutrinos oscillate, which can only occur if they have mass.
CHAPTER 2. BACKGROUND

the Higgs boson, the particle that gives mass to every elementary particle in the Standard Model with mass. Another motivation for this framework is it provides a symmetry which relates the two fundamental types of particles, fermions (fractional spin particles) and bosons (integer spin particles). This symmetry is in fact how this theory solves the hierarchy problem. It proposes the existence of a supersymmetric partner to all the particles in the standard model that differ in spin by $1/2$. The result is that the divergent radiative corrections to the Higgs mass are canceled out at all orders. Finally the introduction of such a symmetry ensures that the coupling constants of the various fields naturally unify at a scale of $10^{16}$ GeV, which is not the case in the Standard Model. One drawback to this model is the enormous number of free parameters, well over a hundred. However, with a few key assumptions it is possible to reduce the number of free parameters and give rise to all the fields in the Standard Model with the smallest possible number of additional fields. This framework is referred to as the Minimal Supersymmetric extension to the Standard Model (MSSM). I refer the interested reader to reviews by Jungman et al. (1996) and Bertone et al. (2005) and references therein for more information on supersymmetry.

MSSM predicts the existence of super particles or sparticles, some of which have no electric or colour charge. This means that these sparticles interact with baryonic matter via the weak force and do not scatter light, making them ideal dark matter candidates. Here I will focus on the neutralino, which is a combination of the bino $\tilde{B}$, wino $\tilde{W}^3$, and higgsino doublet ($\tilde{H}^0_1$, $\tilde{H}^0_1$). These sfermions mix to form four mass eigenstates. The lightest neutralino in most MSSM models predict masses such that for the $\Omega_m$ observed, it is non-relativistic or Cold Dark Matter (CDM) when it decouples from baryonic matter and drops out of thermal equilibrium. This mass also means that the neutralino is expected to be non-relativistic now and as such its annihilation cross section can be written as $\sigma v =$constant. At these low velocities, the neutralino annihilates primarily to heavy fermion-antifermion pairs and $W^\pm$ and $Z$ gauge bosons. Full calculations of annihilation cross sections for various processes can be found in Bertone et al. (2005) and references therein. A few of the key results can be summarized as follows:
• In fermionic annihilations, neutralinos are more likely to decay to heavier fermions and in most models, the dominant fermionic secondaries are either bottom-antibottom quarks or \( \tau \) leptons. These particles themselves will annihilate or decay and produce a cascade of lighter particles.

• There is no tree-level annihilation process for photons, that is \( \chi \chi \to \gamma \gamma \) is a one-loop process (see Bergström et al. (1999) for examples of the one-loop diagrams and calculations of the cross sections). As a result, the direct production of photons is suppressed relative to other gauge bosons (\( W^+W^- \) & \( Z^0Z^0 \)).

Thus, standard MSSM neutralinos are unlikely to directly annihilate to photons or \( e^+e^- \) pairs.

### 2.4.2 Extra Dimensions and Kaluza-Klein Particles

Kaluza initially extended general relativity to five dimensions in an attempt to unify gravity and electromagnetism in 1921. The idea of extra spatial dimensions, so called Kaluza-Klein theory, can also address the hierarchy problem. In these models, the 3+1 spacetime we observe called the \textit{brane} is embedded in a \( 3 + \delta + 1 \) spacetime called the \textit{bulk}. The hierarchy problem is addressed by assuming that these extra dimensions are compactified to a scale \( R \), thereby reducing the effective Planck energy scale. Gravity travels through the bulk but the other fundamental particles can be confined to the brane. The topology and scale of compactification varies from model to model as does the number of particles that travel through the bulk (see for instance Arkani-Hamed et al. (1998), Randall and Sundrum (1999)). In the Universal Extra Dimensions (UED) models all particles can travel through the bulk (Appelquist \textit{et al}. 2001). Due to the compactification of the higher dimensions and the conservation of momentum, all particles traveling through the bulk have their momentum quantized. For each particle, there is a discrete set of excitations, KK states, in the higher dimensional space. From our perspective in the brane, these KK states form a ladder of particles with masses \( m_l = lE_R \) where \( E_R \) is the energy scale of the compactification and \( l \) is a natural number denoting the ladder number. These KK particles have the same quantum numbers.
as the corresponding particle but differ in mass.

In the case of one extra dimension, UED models constrain $E_R \gtrsim 300 \text{ GeV}$, setting a lower limit on the mass of the lightest KK particle. In models where the lightest KK particle is the first excitation of the photon, the $B^{(1)}$, it must have a mass of $400 - 1200 \text{ GeV}$ if it accounts for the dark matter relic density. Thus, UED models predict the existence of an electrically neutral, colour neutral particle with roughly the correct mass required to explain the dark matter relic density. Interestingly, the annihilation cross section for the fermionic channel does not favour the production of heavier quarks or leptons and tends to favour the production of light leptons. Thus, KK particles can produce monoenergetic $e^+e^-$ pairs.

2.4.3 The Dark Matter Particle and the Power Spectrum

It is important to note that the exact properties of the dark matter particle only effect the initial form of the matter density perturbations. These particles, while still in chemical equilibrium with the rest of the universe, will scatter and collisionally damp density perturbations. Furthermore, when these particles are no longer in chemical equilibrium but still kinetically coupled with the other constituents, baryonic matter and radiation, they will free-stream and wipe-out anisotropies below the free-streaming scale. The result of these processes is the exponential damping of the power spectrum below these scales (see, for example, Green et al. 2004, 2005 for more details; also Figure 2.1 on page 10 for an example of the power spectrum). However, the gravitational evolution of these perturbations on astrophysical scales for cold dark matter is unaffected the details of the dark matter particle.

2.5 Modeling Nonlinear Gravitational Growth

There are several methods one can use to follow the nonlinear evolution of the initial power spectrum. Zel'Dovich (1970) for instance outlined a methodology for evolving a collisionless self-gravitating fluid until the perturbations are on the same order as the background ($\delta \lesssim 1$), at which
stage a perturbation might undergo gravitational collapse. Once the perturbations have grown sufficiently, $\delta \sim 1$, their evolution can no longer be described by simple linear differential equations. Another way to say this is that mode-mode coupling becomes important and the Fourier transform of $\delta(x)$ is not simply $\tilde{\delta}(k)$ but contains terms like $\int d^3k_1 \alpha(k_1, k) \tilde{\delta}(k)$ where $\alpha$ is the mode coupling. There are several analytic techniques that model the evolution of the power spectrum into the nonlinear regime, such as 2nd-order Lagrangian Perturbation Theory (2LPT) (e.g. Scoccimarro and Frieman, 1996a, Scoccimarro and Frieman, 1996b, Scoccimarro, 1998). It is possible to calculate higher order Lagrangian Perturbation Theory, however due to the ever increasing complexity, this is rarely done.

Structures, such as dark matter halos, in the CDM paradigm form from the bottom-up in a hierarchical fashion (Larson, 1969): gravitational instabilities cause small-scale matter density perturbations to collapse first, these object then merge to form larger objects. This process gives rise to a hierarchy of nonlinear structures. The run-away nature of collapse means that an overdensity does not have to be large for it to become a distinct virialized bound object or halo. Press and Schechter (1974) put forward an analytic framework to describe the perturbations in a collisionless self-gravitating fluid as they underwent spherical collapse. They assumed that, so long as the large-scale tidal fields can be neglected, a spherical volume with radius $R$ and mass $M$ will gravitationally collapse into a bound object, if at some time $t$, its average overdensity exceeds a critical overdensity, $\delta_{sc}(t)$ that is of order unity. A prediction of the resulting mass distribution of objects is made by using the statistical properties of the density field. The theory is limited due to its assumption of spherical collapse and makes no predictions about the internal structure of halos. Subsequent work by Bond et al. (1991) Lacey and Cole (1993) expanded the Press-Schechter framework, allowing the semi-analytic construction of halo merger trees. Another study by Sheth et al. (2001) generalized Press-Schechter to ellipsoidal collapse.

However, the highly nonlinear dynamics are generally best understood using N-body simulations. While simple in principle as the only force acting on the particles is gravity, in practice N-body simulations are difficult to implement. To determine the force experienced by one particle,
one must sum over the contributions from all other particles and this must be done for every particle. This inefficient particle-particle (PP) calculation scales as $O(N^2)$, where $N$ is the number of particles, making it computationally prohibitive to run simulations with a large number of particles. For example, an early study using direct summation by Peebles (1971) had only 90 particles. However, the development of algorithms such as tree-codes (see, for example, Barnes and Hut (1986) and Dehnen (2001) which outline tree-codes which scale as $O(N \ln N)$ and $O(N)$ respectively) and mesh codes adapted from plasma physics (see, for example, Couchman (1991) which can scale as $O(N \ln N)$) have lead to dramatic increases in the size of N-body simulations.

2.6 The Cold Dark Matter Hierarchy

The current favoured cosmology is often referred to as a $\Lambda$-Cold Dark Matter ($\Lambda$CDM) cosmology. Simulations of $\Lambda$CDM cosmologies show that matter condenses into large filaments and halos that form intricate web-like structures which surround voids (Springel et al., 2005). An example of these dark matter filaments and halos is shown in Figure 2.2 on the following page.

This CDM hierarchy should extend all the way down to the minimum cold dark matter halo mass $m_o$. The minimum halo mass depends on the microphysics of the dark matter particle as discussed in Section 2.4.3. Due to the damping of the power spectrum at scales smaller than the dark matter free-streaming scale, there are effectively no perturbations from which halos can form. For a neutralino with a mass of $\sim 100$ GeV typical values for $m_o$ are $10^{-6}$ M$\odot$ (Martinez et al., 2009), though it can range between $10^{-12} - 10^{-4}$ M$\odot$ (Profumo et al., 2006).

Unfortunately, studying these very small scales in a CDM-dominated universe is analytically and computationally challenging due to the nature of the power spectrum. The CDM power spectrum has a logarithmic slope or effective index $n_{\text{eff}} \equiv d \ln P/d \ln k$ which monotonically decreases from the inflationary spectral index $n_s$ to $\approx -3$ as $k \to \infty$ as shown in Figure 2.1 on page 10. As the spectral index becomes increasingly negative, the analytic framework of Press-Schechter suggests that structures over a wide range in mass will begin to form almost simultaneously, i.e.,
Figure 2.2 A slice from one of the cosmological simulations examined in this thesis and a zoom in of a halo (upper right inset). Colour code shows the logarithmic density, with dark blue regions being high density and light regions being low.
structure formation is no longer hierarchical. Hierarchical clustering, the fundamental principle on which theoretical analysis rests, no longer applies. Furthermore, due to the finite size of N-body simulations, the computation time required to carry out credible simulations, that is simulations with the required size and resolution, is prohibitive (Elahi et al., 2009a; Widrow et al., 2009). These requirements arise because simulations cannot contain modes larger than the simulation volume. This leads to finite volume effects from missing power that become progressively larger as $n_{\text{eff}} \to -3$ (e.g. Bagla and Prasad, 2006; Power and Knebe, 2006).

This is the motivation for using scale-free cosmologies in this work. These are flat $\Omega_m = 1.0$ (Einstein-de Sitter) cosmologies with power-law power spectra, $P(k) = Ak^n$. Their simplicity makes them easier to analyze. By using the spectral index as a proxy for scale one examines the nonlinear evolution of various scales in the $\Lambda$CDM hierarchy. An example of how the spectral index affects dark matter structures is shown in Figure 2.3.

Figure 2.3 Slices from two scale-free cosmological simulations with indices of $n = -1$ (left) and $-2.5$ (right) examined in this thesis. Colour code shows the logarithmic overdensity, with dark blue regions being high density and light regions being low. This colour scale is the same for both simulations.
2.7 Searching for Dark Matter

The evidence for dark matter is still indirect, based on gravitational effects and the CMB. The detection of a dark matter particle would be conclusive, hence the ever growing number of experiments searching for it. Some of these experiments are searching for the particles directly by looking for evidence of WIMPs scattering off target nuclei. Others might discover convincing indirect evidence in the form of secondary particles produced either by the decay or self-annihilation of the dark matter particle.

2.7.1 Direct Detection

The idea behind direct detection experiments is simple: if the Galaxy is filled with WIMPs, then many of them should be streaming through the earth, making it possible to look for the nuclear recoil caused by a WIMP scattering off a target nucleus. The detection rate depends on the scattering cross section, the detector target mass and the volume filling factor of the local dark matter, i.e., its local density and velocity relative to the earth. There are a large number of experiments running or in development (see Baudis (2007) and references therein for a review). Many of these experiments have already produced quite strong limits on the elastic scattering cross section of potential dark matter particles with nucleons. These limits will be improved by several orders of magnitude in the coming years as new experiments with ever increasing target masses come online. However, the published cross section limits, which are based on the measured event rate, scale inversely with the local dark matter density $\sigma_{\text{lim}} \propto \rho_{\odot}^{-1}$. The fiducial value used in the literature is $\rho_{\odot} = 0.3 \text{ GeV/cm}^3$, though if the solar neighbourhood resides in a local underdensity of dark matter, the published limits are overestimates.

2.7.2 Indirect Detection

Most dark matter candidates can either self-annihilate or decay, producing secondary particles which in turn can be detected directly. The case for dark matter would be strengthened by the detection
of these secondaries, though such detections still might not be considered conclusive. Furthermore, the spectrum and composition of the secondaries could strongly disfavour one dark matter model over another. For annihilating dark matter the flux of such radiation is proportional to the square of the dark matter density. Therefore, the optimal places to look at are regions where large dark matter densities are expected, such as the dark matter halos surrounding satellites of the Milky Way and the Galactic center.

There are several telescopes in operation capable of detecting $\gamma$-ray secondaries. The Fermi Gamma-ray Large Area Space Telescope (GLAST) launched in June of last year can detect $\gamma$-rays at energies of 30 MeV to 300 GeV. In just the first three months of data collection, GLAST has already found a number of new $\gamma$-ray sources, though none appear to be from annihilating dark matter (Abdo, 2009). There are other ground based telescopes that are also examining the $\gamma$-ray sky, albeit indirectly. At the energy range of interest for annihilating dark matter (GeV to TeV), photons interact with matter via $e^+e^-$ pair production. As a consequence, their interaction length that is much shorter than the thickness of the Earth’s atmosphere and these photons cannot reach the ground. Ground based telescopes, such as the Whipple Gamma-ray observatory (Wood et al., 2008), instead detect the electromagnetic cascade and the resulting Čerenkov radiation produced by these photons. However, GLAST provides the best chance of detecting $\gamma$-ray secondaries.

It is also possible to detect the cosmic ray particles produced by annihilating dark matter. In fact, it is from cosmic ray detectors that the most intriguing result have come. The Advanced Thin Ionization Calorimeter (ATIC), a balloon borne instrument capable of differentiating between protons and leptons that flew in 2000 and 2003 detected an intriguing anomalous cosmic ray flux (Chang et al., 2008). The instrument found a larger flux of electrons in the range of 300-800 GeV than expected. The ATIC collaboration, using the cosmic ray propagation code GALPROP (Strong and Moskalenko, 1998; Moskalenko and Strong, 1998) found that the excess and its spectral shape could be reproduced by a source that traced the dark matter halo capable of injecting monochromatic $e^+e^-$ pairs at 620 GeV. They claim that an annihilating KK particle with a mass of $m_\chi = 620$ GeV could produce this flux but it would need a cross section $\sim 200$ times larger than that predicted for a
smooth dark matter halo and a thermal dark matter particle, that is \( \langle \sigma v \rangle \sim 10^{-23} \text{ cm}^3/\text{s} \) as opposed to \( \sim 4 \times 10^{-26} \text{ cm}^3/\text{s} \).

A similar excess was observed with Polar Patrol Balloon (PPB) (Torii et al., 2008). However, the High Energy Stereoscopic System (HESS) (H. E. S. S. Collaboration: F. Aharonian, 2009) and Fermi GLAST (Abdo et al., 2009) found different though not necessarily contradictory results. Both instruments detected an excess, though not as large as that observed by either ATIC or PPB. Furthermore, HESS did not observe the sharp peak at a \( \sim 700 \) GeV seen by ATIC. GLAST did observe such a feature but it is not nearly as sharp. Despite the different results, neither instrument can rule out the large excess observed by the balloon instruments because of large systematic uncertainties introduced by the analysis.

Another experiment that reported intriguing results was PAMELA (Payload for AntiMatter Exploration and Light-nuclei Astrophysics) (Adriani et al. 2009). This satellite borne instrument detected an anomalously high positron fraction relative to the standard prediction at energies of 10-100 GeV. This positron fraction suggests there may be an unaccounted for source within the Galaxy capable of directly producing positrons, such as annihilating dark matter.

All these experiments point to the injection of high energy electrons, however, the flux is generally substantially higher than one would predict for a thermal dark matter particle and a smooth Galactic halo.
CHAPTER 3

LITERATURE REVIEW AND TECHNICAL BACKGROUND

This section provides relevant technical background information. Some of the review overlaps with the introductions given in Chapters 4-6, however, here I go into more detail when it is informative to do so. This chapter begins in 3.1 with a review of the statistics of the density field and a discussion of the power spectrum and initial conditions for N-body simulations. Section 3.2 briefly reviews N-body methods. This is followed by a discussion on CDM halos, how the substructure in these halos is identified and the properties of this substructure in 3.3. I end in 3.4 with a discussion of the indirect dark matter detection signal.

3.1 The Power Spectrum

Here I present information that is primarily relevant to Paper III (Chap. 6) though also useful elsewhere. As mentioned in Chap. 2 the perturbations laid down by cosmic inflation are the seeds of structure formation and are described by their statistical properties, such as the power spectrum.
3.1.1 Statistical Quantities

Recall that the ensemble average of the density perturbations is given by the power spectrum\(^1\), which is, strictly speaking, a spectral density,

\[
\langle \tilde{\delta}(k, a)\tilde{\delta}^*(k', a) \rangle = (2\pi)^3\delta^D(k - k')P(k, a),
\]

(3.1)

where \(\tilde{\delta}\) is given by Eq. (2.5). The time dependence of the power spectrum is given by the growth function, see Eq. (2.7) and Eq. (2.8). Note that though \(\delta(x)\) is purely real, \(\delta(k)\) can be complex, and both \(\delta^D(k)\) and \(P(k)\) have units of volume. So long as the density field is statistically isotropic, \(k = |k|\) encapsulates the wavenumber dependence.

Before discussing the evolution of the power spectrum I will introduce several useful quantities. One such quantity is the dimensionless power spectrum, \(\Delta^2(k, a) \equiv \frac{4\pi}{(2\pi)^3}k^3P(k, a)\), which measures the power per \(d\ln k\). Given \(\Delta^2 \propto \delta^2\), the wavenumber at which \(\Delta^2\) is of order unity can be considered the dividing line between the linear and nonlinear scales. The nonlinear scale \(L_{NL} = 2\pi/k_{NL}\) is defined by the condition \(\Delta^2(k = k_{NL}, a) = 1\). For power-law power spectra, \(P(k) = D^2(a)Ak^n\), and an Einstein-de Sitter cosmology (flat \(\Omega_m = 1.0\), \(D(a) = a\)), \(\Delta^2 = a^2Ak^{n+3}/2\pi^2\), and \(k_{NL} = (a^2A/2\pi^2)^{-1/(n+3)}\). This indicates that \(n \to -3\) represents a singular case where all modes become nonlinear simultaneously.

Another useful quantity is the two-point autocorrelation function of the density field, \(\xi\),

\[
\xi(r, a) = \langle \delta(x, a)\delta(x + r, a) \rangle
= \int \frac{d^3k}{(2\pi)^3}P(k)e^{ik\cdot r}.
\]

(3.3)

This is also referred to as the correlation function and is simply the inverse 3 dimensional Fourier transform of the power spectrum.

\(^1\)It is important to note that \(P(k)\) describes the statistical properties of an ensemble, that is the expectation value of variance between many different realizations. Measuring the power spectrum requires many different realizations of a universe. However, by sampling large enough volumes which contain many different realizations of modes that are smaller than the volume, one approaches the true ensemble expectation value. Thus, when reporting measurements of the power spectrum, one is in fact taking volume averages and relying on the density field being ergodic, that is volume average \(\leftrightarrow\) ensemble average.
One can also filter the density field. The ensemble average of the convolution of the density field with a window function $W(r, R)$, is

$$\sigma^2(R, a) = \langle |\delta(x, a) \ast W(r, R)|^2 \rangle, \quad (3.4)$$

$$= \int \frac{k^2 dk}{(2\pi)^3} P(k, a) \tilde{W}^2(k, R), \quad (3.5)$$

where $\tilde{W}(k, R)$ is the Fourier transform of the window function. For a spherical top-hat window function with radius $R$ in physical comoving space, the window function is

$$W(r, R) = \frac{3}{4\pi R^3} \Theta(R - r) \iff \tilde{W}(k, R) = \frac{3}{(kR)^3} (\sin(kR) - kR \cos(kR)), \quad (3.6)$$

where $\Theta(x)$ is the Heaviside function. The quantity $\sigma(R)$ is simply the average rms overdensity in a sphere enclosing a mass $M = \frac{4\pi}{3} \rho_{m,0} R^3$, where $\rho_{m,0}$ is the background matter density. The quantity $\sigma^2(M)$, also referred to as the mass variance, is central to the spherical infall model of Press and Schechter (1974). This model, based on hierarchical formation assumes that spherical perturbations will, on average, collapse if the overdensity exceeds a critical value, $\sigma(M, a) \geq \delta_{sc}$. I discuss this model in more detail below in Section 3.1.4. For power-law power spectra with index $n$, the mass variance is $\sigma^2(M) \propto M^{-(n+3)/3}$. This also indicates $n = -3$ to be a special case where structures over a wide range of masses begin to collapse simultaneously.

In Figure 3.1 on the following page I show the variances $\Delta^2(k)$ and $\sigma^2(M)$ at $a = 1$ for the CDM power spectrum based on the transfer function of Hu and Eisenstein (1998). This transfer function is given by

$$T(q) = \frac{L_o}{L_o + C_o q^2}, \quad (3.7)$$

where

$$L_o = \ln(2e + 1.8q), \quad C_o = 14.2 + \frac{731}{1 + 62.5q}, \quad (3.8)$$

$q \equiv \frac{k}{\text{Mpc} h^{-1}} \frac{\Theta^2}{\Omega_m h^2}$, and $\Omega_{2.7}$ is defined by $\Theta_{2.7} = 2.7 K/T_o$ where $T_o$ is the CMB temperature at the current epoch. Here as usual $h$ is the Hubble parameter and $\Omega_m$ is the matter density ratio. I also
Figure 3.1 Left: Dimensionless power spectrum $\Delta^2$ (solid red curve) and mass variance $\sigma^2(M)$ (dashed green curve) for the CDM power spectrum $P(k)$ based on Hu and Eisenstein (1998) using $n_s = 1$, $\Omega_m = 0.25$ and $\Omega_\Lambda = 0.75$ at $a = 1$. Right: Related effective spectral indices of $n_{\text{eff}}(k)$ and $n_{\text{eff}}(M)$. Here the power spectrum has been damped at scales of $R_f = 2$ pc via $P(k) = P_{\text{CDM}}(k) \exp \left[ -(kR_f/2) - (kR_f/2)^2 \right]$.

show the effective spectral indices as a function of scale and mass, which are given by

$$n_{\text{eff}}(k) = \frac{d \ln P}{d \ln k} = \frac{d \ln \Delta^2}{d \ln k} - 3,$$

$$n_{\text{eff}}(M) = -3 \frac{d \ln \sigma^2}{d \ln M} - 3.$$  \hspace{1cm} (3.9)

The linearity of a system can be determined using these quantities. A system can be considered linear at scales $> k^{-1}$ and for masses $< M$ when $\Delta^2(k) \ll 1$ and $\sigma^2(M) \ll 1$, and nonlinear when $\Delta^2(k) \gg 1$, $\sigma^2(M) \gg 1$. As seen in Figure 3.1, the CDM power spectrum has an effective spectral index that flattens off to $n_{\text{eff}} = n_s \approx 1$ at scales of $\sim 100$ Mpc but steepens to $n \sim -2.8$ at parsec scales, implying that structure begins to form on multiple scales at the low mass end.
3.1.2 The Linear Regime

Before discussing the nonlinear evolution of the power spectrum, I will describe the evolution in the linear regime. The relevant equations for pressureless matter in the linear regime are the conservation of mass, the conservation of energy, and the Poisson equation (Peebles [1980], which in comoving coordinates using the conformal time $d\tau = dt/a$ and $\nabla$ operator in the Eulerian coordinate $x$ are

$$\frac{\partial \delta(x, \tau)}{\partial \tau} + \nabla \cdot \{(1 + \delta(x, \tau))v(x, \tau)\} = 0,$$

(3.11)

$$\frac{\partial v(x, \tau)}{\partial \tau} + aH(a)v(x, \tau) + [v(x, \tau) \cdot \nabla]v(x, \tau) = -\nabla \Phi(x, \tau),$$

(3.12)

$$\nabla^2 \Phi(x, \tau) = \frac{3}{2} \Omega_m a^2 H^2(a) \delta(x, \tau).$$

(3.13)

Using linearized versions of these equations it is possible to generate initial conditions for N-body simulations.

Zel’Dovich Approximation (ZA)

Zel’Dovich (1970) presented a kinematical approach to structure formation that works well for discrete systems in the linear regime and is thus often used to initialize N-body simulations. His ansatz was that the position of a perturbed particle in comoving Eulerian coordinates at conformal time $\tau$ was given by

$$x(\tau) = q + b(\tau) \Psi(q),$$

(3.14)

where $q$ is the particle’s initial unperturbed position, $b(\tau)$ is a time-dependent function and $\Psi(q)$ is the time-independent displacement field. This approximation is first-order Lagrangian perturbation theory as the displacement field depends only on the Lagrangian coordinate $q$. By using $dx/d\tau = v$ and assuming that perturbations depend only on the Lagrangian coordinate $\delta(q)$, one can rewrite Eq. (3.11) to first order, i.e., dropping the $\nabla(\delta v)$ term, as

$$\delta(q, \tau) = -b(\tau) \nabla_q \cdot \Psi(q),$$

(3.15)
where here $\nabla_q$ is the divergence with respect to the Lagrangian coordinate $q$. Since the linear evolution of the matter perturbations is given by the growth function, $b(\tau) = D(\tau)$. By taking the Fourier transform of the above equation one finds that the mode-dependent evolution is given by

$$\tilde{\Psi}(k) = -i \frac{k}{k^2} \tilde{\delta}(k).$$

The amplitude of a particular mode is sampled from a Gaussian distribution and it also has a direction thus $\tilde{\delta}(k) = c_k \sqrt{P_L(k)}$, where $c_k$ follows a Rayleigh distribution and has the property that $c_k = c_{-k}^{*}$ since $\delta^*(k) = \delta(-k)$. Note that here I have explicitly denoted the linear power spectrum with the subscript $L$. A discrete N-body system with period $L_B$ and $N^3$ particles, giving an inter-particle spacing of $L_B/N$, contains modes from the fundamental wavenumber, $k_B = 2\pi/L_B$, to the Nyquist wavelength $k_{Ny} = \pi N/L_B$. For such a system the displacement field is then

$$\tilde{\Psi}(q) = -\sum_{l,m,n} i \frac{k}{k^2} c_k \sqrt{P_L(k)} e^{ik \cdot q}, \quad k = \frac{2\pi}{L} (l^2 + m^2 + n^2)^{1/2},$$

$$l, m, n = 0, \pm 1, \ldots, \pm \frac{N}{2},$$ excluding $l = m = n = 0$.

The resulting perturbed positions and peculiar velocities are given by

$$x = q - D(\tau) \sum_{l,m,n} i \frac{k}{k^2} c_k \sqrt{P_L(k)} e^{ik \cdot q},$$

$$v = -\frac{dD(\tau)}{d\tau} \sum_{l,m,n} i \frac{k}{k^2} c_k \sqrt{P_L(k)} e^{ik \cdot q}.$$

2nd order Lagrangian Perturbation Theory (2LPT)

However, there is an issue with ZA initial conditions. Although the ZA correctly reproduces the linear growing modes of density and velocity perturbations, nonlinear growth is inaccurately represented, particularly for velocity perturbations because the ZA fails to conserve momentum. Consequently, it takes time for the actual dynamics to establish the correct statistical properties of density and velocity fields. Scoccimarro (1998), based on previous work (see, for instance, Buchert et al., 1994; Bouchet et al., 1995), developed a new formulation for...
generating initial conditions. This 2nd order Lagrangian Perturbation Theory approach interpolates between the initial conditions and the late-time solutions given by the exact nonlinear dynamics.

Unlike ZA which uses Eq. (3.11), the basis of 2LPT is Eq. (3.12). Therefore 2LPT explicitly conserves momentum. By applying the Eulerian \(\nabla\) operator to Eq. (3.12), using the Poisson equation and again keeping terms only to first order, one has

\[
\nabla \cdot \left[ \frac{\partial^2 x}{\partial \tau^2} + aH(a) \frac{\partial x}{\partial \tau} \right] = \frac{3}{2} \Omega_m a^2 H^2(a) \delta(x, \tau).
\] 

(3.20)

By using the ZA ansatz, \(x = q + \Psi(x, \tau)\), the resulting nonlinear equation for \(\Psi\) is solved perturbatively about its linear ZA solution, \(\nabla_q \Psi^{(1)} = -D(\tau)\delta(q)\). The solution to second order that corrects the ZA displacement for gravitational tidal effects is

\[
\nabla_q \Psi^{(2)} = \frac{1}{2} D^{(2)}(\tau) \sum_{i \neq j} \left[ \frac{\partial \Psi^{(1)}_i}{\partial q_j} \frac{\partial \Psi^{(1)}_j}{\partial q_i} - \frac{\partial \Psi^{(1)}_i}{\partial q_i} \frac{\partial \Psi^{(1)}_j}{\partial q_j} \right],
\] 

(3.21)

where \(D^{(2)}(\tau) \approx -\frac{2}{3} D^2(\tau)\) is the second order growth factor. The displacement is then

\[
x(\tau) = q - D(\tau)\Psi^{(1)}(q) - \frac{3}{7} D^2(\tau)\Psi^{(2)}(q).
\] 

(3.22)

Operationally, 2LPT initial conditions are generated in a similar fashion to ZA, however, it requires two additional Fourier transforms and the differencing of a potential (see Scoccimarro 1998 for more details). One of the benefits of 2LPT initial conditions is the reduced impact of spurious transient modes which arise from the truncation of the perturbative expansion. Since these modes decay, their impact is to delay the time in the simulation at which credible statistics can be calculated (Scoccimarro 1998). For example, Crocce et al. (2006) examined the size of transients in the power spectrum and the bispectrum using \(\Lambda\)CDM simulations at scales of several. This study found that ZA initial conditions had to be started at \(z = 49\) for the transients to be negligible at \(z = 0\) unlike 2LPT initial conditions which could started at \(z = 11.5\).

The reason for wanting to start a simulation later as opposed to earlier is that the introduced power must be larger than the power originating from the numerical noise of the simulation. For instance, a simulation where particles are initially placed on a grid effectively has a shot noise power
spectrum with an amplitude $\propto N^3$, the number of particles in the simulation. If the power spectrum of the density perturbations that one wishes to study with this simulation has at some scale and time an amplitude smaller than this shot noise, there will be spurious artificial structures and power at smaller scales. One way of circumventing this problem is by starting the simulation later where the power spectrum will have a greater amplitude, which requires higher order PT to generate accurate initial conditions. However, the added computational cost and mathematical complexity means that PT is rarely calculated beyond 2LPT.

**N-Body Initial Conditions**

I use 2LPT to generate the initial conditions for all the N-body simulations presented in this work. At this epoch the density field is accurately described by linear perturbation theory, $P_L$. Due to the finite nature of N-body simulations, technically the power spectrum is given by

$$P(k, a) = P_L(k, a) f_{UV}(k, k_{Ny}) f_{IR}(k, k_B),$$

where $f_{IR}$ and $f_{UV}$ are low and high $k$ cutoffs. I will follow Kudlicki *et al.* (1996) and use the truncation functions

$$f_{UV}(k, k_e) = e^{-(k/0.8k_e)^{16}} \quad f_{IR}(k, k_B) = \Theta(k - k_B).$$

### 3.1.3 The Quasilinear Regime

**One Loop Perturbation Theory (PT)**

The evolution of the power spectrum from the linear regime to the mildly nonlinear or quasilinear regime can be estimated via PT (see, for example, Bernardeau *et al.* (2002)). By taking the Fourier transform of Eq. (3.11) and the Fourier transform of the gradient of Eq. (3.12) combined with the
Poisson equation, one has
\[
\frac{\partial \tilde{\delta}(k, \tau)}{\partial \tau} + \tilde{\vartheta}(k, \tau) = - \int d^3k_1 \int d^3k_2 \delta^D(k - k_1 - k_2) \alpha(k, k_1) \tilde{\delta}(k_1, \tau) \tilde{\delta}(k_2, \tau), \tag{3.25}
\]
\[
\frac{\partial \tilde{\vartheta}(k, \tau)}{\partial \tau} + aH(a) \tilde{\vartheta}(k, \tau) + \frac{3}{2} \Omega_m a^2 H^2(a) \tilde{\delta}(k, \tau) = - \int d^3k_1 \int d^3k_2 \delta^D(k - k_1 - k_2) \beta(k, k_1, k_2) \tilde{\vartheta}(k_1, \tau) \tilde{\vartheta}(k_2, \tau), \tag{3.26}
\]
where \( \tilde{\varphi} \equiv \hat{\mathbf{f}} \cdot \hat{\mathbf{v}} \), and \( \alpha(k, k_1) = k \cdot k_1/k_1^2 \) and \( \beta(k, k_1, k_2) = k^2(k_1 \cdot k_2)/2k_1^2k_2^2 \) are mode coupling terms. These are coupled equations. In linear PT, the mode coupling is assumed to be negligible, giving \( \frac{\partial \tilde{\delta}(k, \tau)}{\partial \tau} = - \tilde{\vartheta}(k, \tau) \). The starting point for higher order PT is an expansion for the density perturbation field of the form:
\[
\tilde{\delta}(k, \tau) = \sum_{n=1}^{\infty} D_n(\tau) \delta_n(k), \quad \tilde{\vartheta}(k, \tau) = aH(a) \sum_{n=1}^{\infty} D_n(\tau) \vartheta_n(k), \tag{3.27}
\]
where \( D \) is again the growth function. In this expansion \( \delta_1 \) characterizes linear density fluctuations, and \( \delta_n \) denote terms of order \((\delta_1)^n\). The same applies for the divergence of the velocity field, \( \tilde{\vartheta} \). So long as the density field is statistically isotropic, the perturbative expansion can be written in terms of the power spectrum:
\[
P_{PT}(k, \tau) = P_L(k, \tau) + P^{(1)}(k, \tau) + \ldots, \tag{3.28}
\]
where \( P^{(1)} = \mathcal{O}(k^3 P_L^2) \).

By expanding about the linear solution it is possible to construct recursion relations describing the higher order terms. [Scoccimarro and Frieman (1996a)] developed a diagrammatic scheme in which these higher-order terms are described as “loop corrections” to the “tree-level” term, \( P_L \). This framework greatly aided the calculation of PT. However, due to the complexity, in practice the series is rarely carried beyond second order in \( P_L \) or one-loop PT. The one-loop correction, \( P^{(1)} \), comprises two distinct terms or diagrams, \( P_{13} \) and \( P_{22} \), which involve integrals over the linear power spectrum, \( P_L \). These integrals are difficult to evaluate without imposing infrared and ultraviolet \( k \) cutoffs. Imposing low and high wavenumber cutoffs \( k_{IR} \) and \( k_{UV} \) respectively, the first loop term
$P_{22}$ is (see, for example, Scoccimarro and Frieman [1996a]; Jeong and Komatsu [2006])

$$P_{22}(k, \tau) = 2 \int \frac{d^3q}{(2\pi)^3} P_L(|k - q|, \tau) P_L(q, \tau) |F_2(q, k - q)|^2,$$

$$= D^4(\tau) \frac{k^4}{98(2\pi)^2} \int_{k_{IR}}^{k_{UV}} dq P_L(q) \int d\mu P_L(|k - q|) \frac{(3q + 7k\mu - 10q\mu^2)^2}{(k^2 + q^2 - 2qk\mu)^2}, \quad (3.29)$$

where $F_2$ describes the mode coupling, $\mu \equiv k \cdot q / qk$ and the limits on $\mu$

$$\mu_{\text{max}} = \min \left\{ 1, \frac{k^2 + q^2 - k^2_{IR}}{2kq} \right\}, \quad \mu_{\text{min}} = \max \left\{ -1, \frac{k^2 + q^2 - k^2_{UV}}{2kq} \right\}, \quad (3.30)$$

arise due to the fact that $k_{IR} < |k - q| < k_{UV}$. The second loop term $P_{13}$ is

$$P_{13}(k, \tau) = 6 \int \frac{d^3q}{(2\pi)^3} P_L(k, \tau) P_L(q, \tau) F_3(k, q, -q),$$

$$= D^4(\tau) \frac{k^2}{126(2\pi)^2} P_L(k) \int_{k_{IR}}^{k_{UV}} dq P_L(q)$$

$$\times \left[ 100 \frac{q^2}{k^2} - 158 + 12 \frac{k^2}{q^2} - 42 \frac{q^4}{k^4} + 3 \frac{k^5}{k^4} (q^2 - k^2)(2k^2 + 7q^2) \ln \left( \frac{k + q}{|k - q|} \right) \right]. \quad (3.31)$$

For scale-free models, these two terms combine to give to first order for $k \ll k_{UV}$

$$\Delta^2_{PT}(k, a) = \Delta^2_{L}(k, a) \left( 1 + \lambda(n) \Delta^2_{L}(k, a) \right) + O \left( \Delta^2_{L}^3 \right), \quad (3.32)$$

where $\lambda(n)$, which can be calculated analytically, is positive for $n > -1.4$ and negative for $n < -1.4$ (Scoccimarro and Frieman [1996b]; Bernardeau et al., 2002). Hence, $n \simeq -1.4$ represents a “critical index” where nonlinear corrections are vanishingly small (Scoccimarro [1997]). For example, these loop terms for a power-law power spectrum with $n = -2$, $P_L(k, a) = a^2 Ak^{-2}$ and $k \ll k_{UV}$, are

$$\frac{2\pi^2 k_{UV}^2}{D^4(a)A^2} P_{13} = \frac{4}{3} \frac{k_{UV}}{k_{IR}} + \frac{5\pi^3}{28} \frac{k_{UV}}{k} + O(\frac{(k/k_{UV})^0}{k}), \quad (3.33)$$

$$\frac{2\pi^2 k_{UV}^2}{D^4(a)A^2} P_{22} = \frac{4}{3} \frac{k_{UV}}{k_{IR}} + \frac{75\pi^2}{196} \frac{k_{UV}}{k} + O(\frac{(k/k_{UV})^0}{k}), \quad (3.34)$$
where here the loop terms have been written in a dimensionless form. The dominant term in each loop correction is $k_{UV}/k_{IR}$ which cancels in the sum. For power spectra with loop corrections that cannot be determined analytically, the challenge, numerically, is to determine the surviving terms which can be much smaller than the terms that cancel, especially for large $k$ and steep $n$.

One-loop PT has been used in several instances to examine the nonlinear evolution of the Baryonic Acoustic Oscillations (BAO) in the $\Lambda$CDM power spectrum. These oscillations in the matter power spectrum occur at wavenumbers of $\sim 0.1 - 0.4 \hMpc$ where the effective spectral index (neglecting the oscillations) is $\sim -2$ to $-2.5$. For example [Jeong and Komatsu (2006)] compared one-loop PT with the nonlinear matter power spectrum from N-body simulations of the BAO in the weakly nonlinear regime ($\Delta^2(k) < 0.4$) and found that the one-loop PT prediction agreed with the simulation to better than $1\%$. Despite its usefulness of one-loop PT, it breaks down when the loop corrections, which are formally divergent, become comparable to the tree-level terms. The exact scale $k$ at which this occurs is still an open question. Does this scale depend for instance on the slope of the power spectrum?

Higher order PT obviously would be an improvement over one-loop PT but the complexity of even the third order (two-loop) terms makes such calculations daunting (see for instance [Tatekawa 2005a,b]). The improvement of two-loop PT initial conditions has not been explored, but the increasing computational cost of such initial conditions may outweigh the computational cost of simply starting a simulation earlier. In regards to predicting the nonlinear evolution, often renormalization schemes are used instead as they can be mathematically or computationally simpler.

**Renormalization Group (RG) Approach**

One method of expanding the regime where one-loop PT works is to use renormalization techniques, which remove the divergences in the PT expansion, and leave behind a well-behaved, renormalized power spectrum. One such scheme proposed by [McDonald (2007)] renormalizes the power spectrum essentially by updating the perturbative expansion as the system evolves.

Operationally, one begins with an initial power spectrum and takes a small step forward in time.
using one-loop perturbation theory,

\[ P(k, \tau) = \frac{D^2}{D^4} P(k) + \frac{D^4}{D^6} \left[ P^2_L \right](k), \tag{3.35} \]

where \( \left[ P^2_L \right](k) \) is \((P_{22} + P_{11})\) integrals evaluated with \( P_L \) as in Eq. (3.29) & (3.31). Writing \( \hat{P} = P/{D^2} \) and setting \( D^2 = D^2_\ast + (D^2 - D^2_\ast) \) the equation can be recast as

\[ \hat{P}(k, \tau) = P_L(k) + \frac{D^2}{D^4} \left[ P^2_L \right](k) + \left( D^2(\tau) - D^2(\tau) \right) \left[ P^2_L \right](k). \tag{3.36} \]

By absorbing \( D^2_\ast(\tau) \left[ P^2_L \right](k) \) into the new initial conditions \( P_\ast = P_L(k) + D^2_\ast(\tau) \left[ P^2_L \right](k) \) the above equation can be recast into

\[ \hat{P}(k, \tau) = P_\ast(k) + \left( D^2(\tau) - D^2_\ast(\tau) \right) \left[ P^2_\ast \right](k), \tag{3.37} \]

where \( P_L \) can be replaced with \( P_\ast \) in the second term because the change is formally 3rd order. Since \( D^2_\ast \) is an arbitrary constant \( \hat{P} \) cannot have any dependence on it giving the differential equation

\[ \frac{d\hat{P}}{dD^2_\ast} = 0 = \frac{dP_\ast}{dD^2} - \left[ P^2_\ast \right](k) + \left( D^2(\tau) - D^2_\ast(\tau) \right) \frac{d\left[ P^2_\ast \right](k)}{dD^2_\ast}. \tag{3.38} \]

Here again one can drop the derivative of \( \left[ P^2_\ast \right](k) \) because it is higher order. This gives

\[ \frac{dP_\ast}{dD^2} \propto \left[ P^2_\ast \right], \tag{3.39} \]

where \( D_\ast \) has been set to \( D \) since it is arbitrary. Thus, the power spectrum can be stepped forward in time by solving the above integro-differential equation with the initial condition that \( P_\ast = P_L \).

McDonald (2007) found that this RG method extends the validity of PT to more nonlinear scales when comparing to fitting formula based on N-body simulations. However, the author indicated that this method has a number of limitations. In particular, Eq. (3.39) ignores higher-order terms (two-loop and beyond) in \( P_L \). Moreover, decaying mode solutions are not included.

This approach is not the only one. Crocce and Scoccimarro (2006, 2008) outline an alternative method to study the nonlinear evolution of large-scale structure which also employs RG techniques. Their formalism is conveniently represented in terms of Feynman diagrams and is closer in spirit to
RG applications in high energy and statistical physics. Another RG method based on the Wilsonian Renormalization Group was outlined by Matarrese and Pietroni (2007).

In general, these various techniques appear to improve the agreement of PT predictions with N-body simulations. However, most studies have focused on the mildly nonlinear regime of the low-redshift CDM power spectrum, and in particular the evolution of the BAO. But, due to the complexity of the true CDM power spectrum, it is not known under what conditions these methods break down and how far they can be extend into the nonlinear regime.

3.1.4 The Nonlinear Regime

N-body simulations are the most direct means of examining the nonlinear power spectrum where $\Delta^2(k) \gg 1$. Understanding the nonlinear regime is a different story.

HKLM and Stable Clustering

Hamilton et al. (1991) (hereafter referred to as HKLM) presented one of the first avenues for understanding the nonlinear regime on a theoretical basis. They proposed that there is a unique function that smoothly maps the linear power spectrum to the nonlinear one. The mapping for the power spectrum can be understood by looking at the real-space analogue. Consider a particle initially at a linear radius $r_L$. Its nonlinear position due to a spherical overdensity $(1 + \delta)$ can be related to its linear position by simple conservation of mass, $(1 + \delta)r_{NL}^3 = r_L^3$. For an ensemble of density perturbations, one appeal to the correlation function $\xi(r)$. In this case one is interested in

$$\bar{\xi}_{NL}(r) = \frac{3}{4\pi r^3} \int_0^r \xi(r') 4\pi r'^2 dr'$$

(3.40)

the volume averaged correlation function which measures of average density contrast interior to a separation $r$. Thus, a dark matter particle at an initial linear position $r_L$ will be moved to a new nonlinear position $r_{NL}$ such that

$$r_{NL} = [1 + \bar{\xi}_{NL}(r_{NL})]^{-1/3} r_L.$$ 

(3.41)
With this mapping, the nonlinear correlation function is then a simple function of the linear one:

\[ \xi_{NL}(r_{NL}) = f_{NL}[\xi_L(r_L)], \]  

(3.42)

where \( f \) is an appropriately chosen fitting formula.

The assumption is that a similar mapping can be applied to wavenumbers, that is

\[ k_{NL} = [1 + \Delta^2_{NL}(k_{NL})]^{1/3} k_L, \]  

(3.43)

and where the nonlinear power spectrum may then be written as a function of the linear power spectrum,

\[ \Delta^2_{NL}(k_{NL}) = f_{NL}[\Delta^2_L(k_L)]. \]  

(3.44)

In the linear regime \( f(x) \) must be the identity function, i.e., \( f(x) = x \), and for Einstein-de Sitter cosmologies, the power spectrum scales as \( a^2 \). In the nonlinear regime, HKLM appealed to the “stable clustering hypothesis”, which holds that highly nonlinear structures decouple from the expansion. Under this assumption, \( \xi_{NL} \), and thus \( \Delta_{NL} \), should scale as the inverse of proper physical density, that is \( a^3 \). Therefore, the asymptotic behaviour of \( f(x) \) must be given by \( f \propto x^{3/2} \). HKLM, however, did not predict the functional form \( f(x) \) in the transition region between the linear regime and the nonlinear regime.

The stable clustering hypothesis was examined in follow-up work by Jain et al. (1995) (hereafter JMW95) and Peacock and Dodds (1996) (hereafter PD96), who turned to numerical simulations to determine the form of the mapping in this quasilinear regime. JMW95, using simulations with \( 100^3 \) particles of scale-free cosmologies with \( n = 0, -0.5, -1, -1.5, -2 \), found that

\[ f_{JM}(x) = x \left( \frac{1 + 0.6x + x^2 + 0.2x^3 - 1.5x^{3.5} + x^4}{1 + 0.0037x^3} \right)^{1/2}, \]  

(3.45)

where \( x = \Delta^2[3/(3 + n)]^{1/3} \), fit the measured power spectrum to within 15 – 20%.

PD96 using simulations with \( 80^3 \) particles and a similar set of cosmologies, advocated the fitting formula

\[ f_{PD}(x) = x \left\{ \frac{1 + B\beta x + (Ax)^\gamma}{1 + (Ax)^\gamma / (V x^{1/2})^\beta} \right\}^{1/\beta}, \]  

(3.46)
where the parameters $\gamma$, $\beta$, $V$, $A$, and $B$ are determined by fitting Eq. 3.46 to power spectra measured in simulations. The parameters are understood as follows: $B$ determines a second-order departure from linear growth, $A$ and $\gamma$ control the behaviour in the quasilinear regime, $V$ controls the amplitude of the asymptote and $\beta$ shapes the transition between the two regimes. Best-fit parameters are expressed as functions of $1 + n/3$, for example, $\gamma = 3.310 (1 + n/3)^{-0.244}$. The expressions for the other parameters similarly diverge as $n \to -3$ though one must bear in mind that they are based on results from simulations with $n \geq -2$. PD96 found their formula worked to within $\sim 14\%$.

However, a later study by Smith et al. (2003), using simulations with $256^3$ particles as opposed to $\leq 100^3$, found neither fitting formula accurately described the nonlinear power spectrum. This study found the JMW95 and PD96 fitting formulae were only accurate to $\approx 56\%$ and $\approx 54\%$ respectively. They presented another fitting formula following the HKLM prescription:

$$f_{\text{EdS}}(x) = x \left[ \frac{1 + x/a + (x/b)^2 + (x/c)^{\alpha - 1}}{(1 + (x/d)^{(\alpha - \beta)\gamma})^{1/\gamma}} \right].$$

Their seven parameter fitting function was accurate to within $9\%$ for scale-free cosmologies, although the coefficients did not display any systematic dependence on the spectral index $n$ like those in the fitting formulae of JMW95 and PD96.

**Halo Model**

The assumption of stable clustering has been challenged by various groups (Ma and Fry, 2000; Peacock and Smith, 2000; Seljak, 2000; Smith et al., 2003) because dark matter halos continually accrete matter and never fully decouple from the rest of the Universe. An alternative approach is provided by the halo model in which the density field is given as a distribution of mass concentrations (halos) which evolve and have their own internal structure. The two-point correlation function comprises a one-halo term, which is associated with the correlation of mass within a single halo, and a two-halo term, which is associated with the correlation between different halos (Ma and Fry, 2000; Peacock and Smith, 2000; Seljak, 2000; Smith et al., 2003). Since the power spectrum is
the Fourier transform of the two-point correlation function, the associated components of the power spectrum, $P_{1h}$ and $P_{2h}$, involve integrals over the halo mass function and Fourier-transformed halo density profile.

The halo-halo term, $P_{2h}$, should dominate in the quasilinear regime and is the distribution of halos convolved with their density profiles

$$P_{2h}(k) = P_L(k) \left[ \frac{1}{\rho_m} \int dM b(M) \frac{dn(M)}{dM} \tilde{\rho}(k, M) \right]^2,$$

where $\rho_m$ is the mean background density. Here $b(M)$ is the bias of halos relative to the density field $P_L(k)$, $dn(M)/dM$ is the halo mass function, which gives the number of halos in a mass range of $M$ to $M + dM$, and $\tilde{\rho}(k, M)$ is the Fourier transform of halo density profile. The halo-halo term is only expected to apply for $k$ such that halos are well separated.

At smaller scales and higher wavenumbers, the correlation function and power spectrum should only sample the internal density profile of a halo. The single halo term, which arises from the density profiles of halos of all masses, is

$$P_{1h}(k) = \frac{1}{\rho_m^2} \int dM \frac{dn(M)}{dM} |\tilde{\rho}(k, M)|^2.$$

The single halo term is what determines the asymptotic slope. The slope depends on the low $r$ behaviour of halo density profiles, how this behaviour scales with mass, and what mass scales dominate the halo mass function. In the CDM hierarchy, the low mass halos should dominate the mass function.

Ma and Fry (2000) and Seljak (2000) calculated an analytic expression of the power spectrum based on the halo model using the Press-Schechter halo mass function and the density profiles from numerical simulations. Both found good agreement with numerical simulations of CDM cosmologies with $256^3$ particles. There were some discrepancies between the model and the N-body simulations in the quasilinear regime and at extremely high $k$, where numerical effects might be an issue.

Motivated in part by the separation of components used in the halo model, Smith et al. (2003)
constructed another fitting formula with the form

$$\Delta^2 (k, a) = \Delta^2_Q (k, a) + \Delta^2_{NL} (k, a).$$

(3.50)

By construction, $\Delta^2_Q$ dominates the power spectrum in the quasilinear regime and is meant to account for halo-halo correlations while $\Delta^2_{NL}$ dominates the power spectrum in the nonlinear regime and is meant to account for single halo correlations. However, this model is purely empirical and not calculated directly from the halo model. Smith et al. (2003) found this function fitted their data to within 8.6%, a slight improvement over their HKLM motivated fitting function shown in Eq. (3.47). Furthermore, the coefficients of Eq. (3.50) depended systematically on the spectral index, unlike Eq. (3.47). It is also worth noting that their formula has eight free parameters, three more than that of PD96.

**Press-Schechter (PS) Formalism**

Once a perturbation has become nonlinear, it should collapse and form a gravitationally bound object. Press and Schechter (1974), (hereafter PS) proposed a framework that combined linear theory with the spherical collapse model to bridge gap between the power spectrum of density perturbations and virialized structures that form out of these perturbations. The model assumes that once a spherical perturbation of radius $R$ and mass $M$ exceeds a critical overdensity $\delta_{sc}$, it undergoes runaway collapse. This critical density depends on the cosmology and evolves with time. For an Einstein-de Sitter cosmology which contains only dark matter, $\delta_{sc}(a) = 1.686/a$. Critically, the theory assumes that, even if the field on scales $R$ is nonlinear, the large-scale field is still linear and tidal effects can be neglected. This assumption is valid so long as the average overdensity of a volume decrease as the volume increases. The PS framework effectively assumes that objects form in a hierarchical fashion.

Key to the using the PS framework is the filtered density field or mass variance $\sigma^2(M)$. With the appropriate choice of window function, such as a top-hat or Gaussian window, this quantity measures the average overdensity enclosed in a spherical volume. Due to the Gaussian nature of the
density field, the probability that a given point lies in a volume enclosing a mass $M$ where $\delta > \delta_{sc}$
is given by

$$p(\delta > \delta_{sc} | M) = \frac{1}{2} \text{erfc}\left(\frac{\delta_{sc}}{\sqrt{2} \sigma(M)}\right).$$  \hspace{2cm} (3.51)

Due to underdense regions with $\delta < 0$, the initial framework outlined by PS predicts only $1/2$ of
the mass in the universe is locked up in gravitationally bound objects. To resolve this problem, the
authors simply multiplied the probability by 2 in an *ad hoc* fashion. I will return to this issue later.
This probability is also the fraction of mass bound in halos of mass $> M$. The halo mass function
or the comoving number density of objects in a mass interval $M$ to $M + dM$, $\frac{dn(M)}{dM}$ is determined
by differentiating this probability,

$$\frac{dn_{PS}}{dM} dM = \frac{\rho_{m,o}}{M} \frac{dF}{dM} dM$$

$$= \sqrt{2} \rho_{m,o} \frac{\delta_{sc}}{\pi M^2 \sigma(M)} \left| \frac{d\ln \sigma(M)}{d\ln M} \right| \exp\left(-\frac{\delta_{sc}^2}{2\sigma^2(M)}\right) dM. \hspace{2cm} (3.52)$$

The quantity $\delta_{sc}/\sigma(M)$, often referred to as the peak height $\nu$, is a measure of the rarity of halos with
mass $M$ and denotes the mass scale above which the halo mass function is exponentially damped.
The characteristic mass scale, $M_*(a)$, is defined to be the mass at which the $\sigma(M) = \delta_{sc}(a)$. For
an Einstein-de Sitter universe, with a power-law spectrum where $\sigma(M) \propto M^{-(n+3)/6}$ and $\delta_{sc} = 1.686/a(t)$, this mass scale is

$$M_*(a) = \left(\frac{a}{1.686}\right)^{\frac{6}{n+3}}. \hspace{2cm} (3.53)$$

As $n \to -3$, the mass variance becomes independent of scale and the characteristic mass scale,
below which objects are forming, approaches infinity. This signifies a break down in the basic
assumptions of the PS framework, thus as mentioned before, $n = -3$ appears to represent a singular

case where analytic theory can no longer be used as a guide.

The PS formalism was further developed by Bond et al. (1991) and Lacey and Cole (1993). The
key difference in these studies was that they allowed regions to undergo random walks in density
space with time where the mass variance defined a barrier. These groups realized that in the absence
of tidal forces, a region will never decrease in density. Therefore, a region with $\delta < 0$ could have at a later time have $\delta > 0$ and subsequently form a halo. This accounts for the factor of 2 that PS added in an ad hoc fashion. The semi-analytic technique outlined in Lacey and Cole (1993), referred to as Extended Press-Schechter (EPS) technique, has been used to estimate merger rates and even construct merger trees (Cole et al., 2000).

Sheth and Tormen (1999) and Sheth et al. (2001) (hereafter ST) generalized EPS to ellipsoidal collapse using the random walk approach. A similar approach was outlined by Lee and Shandarin (1998). Ellipsoidal collapse alters the critical overdensity and its evolution, and consequently the functional form of the halo mass function. The resulting mass function is similar to PS with a few key differences and is given by

$$\frac{dn_{ST}}{dM} dM = A \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M^2} \left| \frac{d \ln \sigma(M)}{d \ln M} \right| \nu' \left[ 1 + (\nu')^{-2q} \right] \exp\left[ -\frac{(\nu')^2}{2} \right] dM.$$  \hspace{1cm} (3.54)

Here, $\nu' = a \nu$, and $a = 0.707, q = 0.3, A = 0.322$ were calibrated using cosmological simulations. In comparison to the PS mass function, it predicts a greater number of massive, rare halos and predicts few low mass, common halos. Sheth and Tormen (2002) showed this revised mass function to be in better agreement with N-body results, not surprising considering halos are not perfectly spherical overdensities.

### 3.2 N-body methods

Though there are a number of analytic techniques that can be used to follow the evolution of the initial density perturbations, but all of these techniques involve some level of approximation. In the end, the highly nonlinear gravitational system is best modeled using numerical techniques. Here I briefly outline the three main algorithms used to determine the gravitational force in discrete N-body simulations. I then discuss GADGET-2, the N-body code used in this work.
3.2.1 Calculating Gravitational Forces

The three main algorithms used are simple Particle-Particle (PP) calculations, so-called Tree algorithms which use hierarchical multipole expansions, and Particle-Mesh (PM) based algorithms (these were briefly discussed in Section 2.5).

**Particle-Particle code**

A PP code calculates the gravitational acceleration for each particle using a direct sum

\[ a_i = -G_N \sum_{j \neq i}^N m_j \frac{x_i - x_j}{|x_j - x_i|^3}, \]  

(3.55)

where \( G_N \) is Newton’s constant. The computational time of PP codes scale as \( \mathcal{O}(N_p^2) \), where \( N_p \) is the number of particles in the sum. This scaling is quite prohibitive and the PP method is only suited to simulations with a small number of particles. Only the earliest cosmological simulations exclusively used PP code [Peebles, 1971]. Now, such calculations tend to be used in tandem with the algorithms discussed below or are limited to the modeling of stellar systems, such as globular clusters, on specialized hardware (see for example [Makino et al., 2003]).

**Tree codes**

Tree codes in an astrophysical context were first outlined by [Barnes and Hut, 1986] (hereafter BH). A tree code divides up the simulation volume into cubic cells, so that only particles from nearby cells need to be treated individually. Distant particles are grouped into ever larger cells, thereby allowing their gravity to be accounted for by means of a single multipole expansion force. This drastically reduces the number of particle-particle computations. When calculating the force on the \( i^{th} \) particle, an opening angle \( \theta \) is used to determine when and how distant particles are grouped together. This approach allows the gravitational force to be computed with \( \mathcal{O}(N_p \log N_p) \) interactions.

As regions become more and more clustered, the cells are refined to smaller cells in the denser regions to prevent the simulation from being slowed down by particle-particle computations. The
approach of BH is to recursively subdivide the simulation volume to encompass the full mass distribution. These nodes are repeatedly subdivided into eight daughter nodes of half the side-length each, until one ends up with leaf nodes containing single particles. Forces are then obtained by “climbing” the tree. That is for the \( i \)th particle or root node, a decision is made whether or not the multipole expansion of the next node is considered to provide an accurate enough partial force. If at the \( j \)th node (or branch) the answer is yes, then a multipole expansion is used and the climb along this branch is terminated. If no, then each of the daughter nodes are considered in turn.

The final result of the tree algorithm is only an approximation of the true force. However, the error can be conveniently controlled by modifying the opening angle criterion or the order of the multipole expansion. Higher order expansions are naturally more accurate than lower order expansions and therefore allow larger cell-opening angles to be used for the same level of accuracy as lower order expansions. However, higher order expansions are also more computationally intensive. Thus, a balance must be achieved between the order of the multipole expansion and the opening angle used to optimize the computational efficiency of the code for a desired accuracy. BH advocated the use of quadpole order, though other codes have used octopole even hexadecapole order.

**Particle-Mesh codes**

Mesh codes, initially borrowed from plasma physics, smooth particles onto a discrete grid to determine the density. The gravitational potential is calculated by using the Fourier transform of the density \( \tilde{\rho}(k) \) and the Poisson equation, \( \tilde{\Phi} = 4\pi G N \tilde{\rho} k^2 \). The gravitational force is found by computing the inverse Fourier transform of \( \tilde{\Phi} \) and calculating the gradient of the potential.

Explicitly, this is done by using a assignment function \( U \) to assign particle masses to the mesh and calculate the density of a cell. The density of the \( j \)th cell of size \( \Delta_m \) is

\[
\rho(x_{m,j}) = \Delta_m^3 \sum_{i=1}^{N_p} m_i U(x_i, x_{m,j}).
\] (3.56)
The potential for a periodic system of size $L_B$ at the $j^{th}$ mesh point is

$$\Phi(x_j) = \frac{4\pi G N}{L_B^3} \sum_k \tilde{\rho}(k) \frac{k^2}{k^2} e^{ik \cdot x_j},$$  \hspace{1cm} (3.57)

where the sum over $k$ is the discrete inverse Fourier transform. Typically, the potential is calculated using Fast Fourier Transforms (FFT). The force is obtained by differencing the potential using a differencing operator $D(x)$. In general one can write at the $j^{th}$ cell

$$F(x_j) = -\frac{4\pi G N}{L_B^3} D(x_j) \sum_k \tilde{\rho}(k) \frac{k^2}{k^2} e^{ik \cdot x_j}.$$ \hspace{1cm} (3.58)

The force field is then interpolated to the $i^{th}$ particle’s position using the same smoothing kernel. Note that here the equation has been divided by Fourier transform of the kernel $\tilde{U}$ twice. This de-convolves the potential for the smoothing kernel twice, once for the mass assignment, and the other for the force interpolation. The computational cost of the PM method scales as $a N_p + b N_m \log N_m$ where $a$ and $b$ are constants and $N_m$ is the size of the mesh used. Typically, with a simulation of $N_p$ particles, $N_m \sim 8 N_p$ which gives a scaling of $O(N_p \log N_p)$.

The main disadvantage of this method is that it cannot reproduce structures on scales smaller than the Nyquist wavelength of the mesh. To alleviate this problem, adaptive meshes in which the denser regions are subdivide into another mesh of the simulation, or simple PP codes are used to compute the small-scale forces. PM codes also requires rapidly increasing amounts of memory as one increases the resolution of the mesh in three dimensions since most implementations of FFT require some padding. For a mesh with $N_m = N^3_g$ cells in three dimensions the typical memory requirements correspond to $(2 N_g)^3$ cells.

### 3.2.2 Gadget-2: A Tree-PM N-body Code

All the simulations in this work were carried out using GADGET-2 [Springel 2005], a massively parallel TreePM N-body code with Smoothed Particle Hydrodynamics (SPH) written in the C language using the Message Passing Interface (MPI) and the open source libraries GSL and FFTW. As the work present in this thesis does not involve gas particles, I will ignore the SPH component of
the code and focus solely on the gravitational algorithms. The \texttt{GADGET-2} N-body code uses the TreePM algorithm which combines all three methods listed above.

**The TreePM algorithm**

In the TreePM algorithm implemented in \texttt{GADGET-2}, the gravitational force is split up into long- and short-range forces according to

$$\tilde{\Phi}(k) = \tilde{\Phi}^{lr}(k) + \tilde{\Phi}^{sr}(k),$$  \hspace{1cm} (3.59)

where $\tilde{\Phi}^{lr}(k) = \Phi(k) \exp \left( -k^2 r_f^2 \right)$ with $r_f$ denoting the scale of the force split. (For details on the other TreePM algorithms and their performance characteristics see Bagla and Ray\cite{BaglaRay2003}.)

The long-range force is computed using the PM method. By setting $r_f$ to values larger than the mesh scale, force anisotropies introduced by the mesh can be suppressed. \texttt{GADGET-2} uses a Cloud-in-cell assignment scheme to smooth particles onto the mesh \cite{HockneyEastwood1981},

$$U(x, x_m; \Delta_m) = u(x, x_m)u(y, y_m)u(z, z_m),$$  \hspace{1cm} (3.60)

where $x_{m,j}$ is the position of the $j$th cell, $\Delta_m$ is the size of a cell and

$$u(x) = \begin{cases} 
1 - \frac{|x - x_m|}{\Delta_m}, & |x - x_m| \leq \Delta_m \\
0, & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (3.61)

The mesh component is computed very efficiently using the parallelized FFTs of the \texttt{FFTW} library. The force is calculated using a four-point differencing operator which offers order $O(\Delta_m^4)$ accuracy.

The short range force is calculated using a tree code and for a periodic system in real space is given by

$$\Phi^{sr}(x) = G_N \sum_{i=1}^{N_p} \frac{m_i}{r_i} \text{erfc} \left( \frac{r_i}{2r_f} \right),$$  \hspace{1cm} (3.62)

where \text{erfc} is the complimentary error function and $r_i$ is the shortest distance of any of the images of particle $i$ to the point $x$. The tree forces are calculated using monopole moments, that is all
distant particles are treated as one large particle located at the center of mass. To construct the tree
GADGET–2 uses a relative opening criterion optimized for monopole moments. Specifically, a node
of mass \( M \) and width \( l \) at distance \( r \) is considered for usage if

\[
\frac{G_N M}{r^2} \left( \frac{l}{r} \right)^2 \leq \alpha_{\text{tol}} |a|_{\text{old}},
\]

where \(|a|_{\text{old}}\) is the size of the total acceleration obtained in the last time step, and \( \alpha_{\text{tol}} \) is the user
defined tolerance parameter. This criterion attempts to limit the absolute force error introduced in
each particle-node interaction by comparing a rough estimate of the truncation error with the size of
the total expected force. This ensures that the typical relative force error is kept roughly constant.

Now to remove the effects of extremely large forces due to close encounters, a softening length,
\( \epsilon \) is used in calculating the force, i.e., the \( r^{-2} \) term in gravity is effectively replaced by \( (r^2 + \epsilon^2)^{-1} \).
The choice of softening used in GADGET–2 amounts to replacing the point like particles with Plum-
mer spheres of size \( \epsilon \) so that the corresponding potential of particle \( i \) is \( \Phi_i = \frac{G_N m_i}{\sqrt{r^2 + \epsilon^2}} \).

Note that GADGET–2 uses both physical and comoving softening lengths in cosmological simula-
tions. Using a physical softening ensures that the binding energies do not change versus time, but
the first halos to form tend to have a “soft” potential. The opposite is true for comoving softening
lengths. GADGET–2 uses whichever softening length is smaller in physical space, \( \min(\epsilon_{\text{comove}}, \epsilon_{\text{phys}}) \). Therefore, as the scale factor increases and the simulation becomes more clumpy, the code
switches from using the comoving softening length to the physical softening length. This is a subtle
point: which softening length should be used has not been definitively settled from a theoretical
perspective, though the compromise used by GADGET–2 is generally accepted.

**Time integration**

Evolving the positions and velocities in time is not a simple matter. Cosmological simulations
have large dynamic ranges in time-scales. High density regions, such as the cores of halos, require
small changes in time for accurate integration of particle positions and velocities. Low density
regions can be evolved with comparatively large time steps, hence evolving the system with a fixed
time step set by the densest regions is a waste of computational resources. GADGET-2 reduces the number of computational cycles by employing adaptive time steps and makes use of the fact that the gravitational force is broken into short- and long-range components.

The positions and velocities are updated using a leapfrog or Kick-Drift-Kick method. The Kick and Drift operators update the momentum and position respectively via

\[
K_t(\Delta t) = \begin{cases} 
  x_i \mapsto x_i \\
  p_i \mapsto p_i + f_i \int_t^{t+\Delta t} \frac{dt}{a} 
\end{cases} \quad D_t(\Delta t) = \begin{cases} 
  p_i \mapsto p_i \\
  x_i \mapsto x_i + \frac{p_i}{m_i} \int_t^{t+\Delta t} \frac{dt}{a^2} 
\end{cases},
\]

where \( f_i \) is the force on particle \( i \). For fixed time-steps and no force separation, the standard time evolution operator \( T(\Delta t) = K(\Delta t/2)D(\Delta t)K(\Delta t/2) \) is formally symplectic, i.e., volume conserving in phase-space. In other words this method conserves energy, ensuring that the gravitational evolution of a particle is time reversible. Such a time operator introduces errors in the Hamiltonian of the system of order \( O(\Delta t^2) \). The specific time operator used in GADGET-2 is a simple modification of the KDK scheme,

\[
T(\Delta t) = K^{lr}(\Delta t/2)[K^{sr}(\Delta t/2m)D(\Delta t/m)K^{sr}(\Delta t/2m)]^m K^{lr}(\Delta t/2),
\]

where \( m \) is a positive integer that indicates the number of times short range forces are evaluated.

The adaptive time step criterion for an individual particle based on the short-range forces is

\[
\Delta t_{\text{grav, sr}} = \min\left[ \Delta t_{\text{max}}, \left( \frac{2\eta\epsilon}{|a_{\text{int}}|} \right)^{1/2} \right],
\]

where \( \eta \) is the user defined accuracy parameter, \( \epsilon \) is again the softening length and \( \Delta t_{\text{max}} \) is a user defined fixed fraction of the instantaneous Hubble time.

It should be noted that with the force written into two components and the individual time steps, this time integration is no longer formally symplectic. However, for collisionless systems with a large number of particles where the force between two particles \( i \) and \( j \) is much less than the total force experienced by particle \( i \) due to all other particles, particles can be considered to move quasi-independently in a collective potential. Springel (2005) showed that so long as the evolution of
each particle is accurately followed in this potential, the resulting integration can reach accuracies comparable to a phase-space conserving symplectic integration.

3.3 Dark Matter Structures

In this section I review the properties of dark matter halos from cosmological simulations relevant to Papers I & II (Chap. 4 & 5). There have been numerous numerical studies of the formation and properties of dark matter halos, with early N-body simulations having a few thousand particles (e.g. Efstathiou and Eastwood 1981; Dekel 1983) to the latest multi-billion particle simulations (e.g. Diemand et al. 2008; Springel et al. 2008; Elahi et al. 2009a; Widrow et al. 2009). Improvements in computing power have generally been followed by improvements in our understanding of the hierarchical structure formation process and the properties of virialized dark matter halos. Here I discuss the bulk properties of dark matter halos from cosmological simulations followed by a discussion of subhalos.

3.3.1 Halos

As dark matter is effectively collisionless, dark matter halos are defined by their phase-space structure, that is their density profile and velocity distribution. Here I focus on their density distribution and a related quantity, the concentration parameter. I also quickly review other bulk halo properties like their morphology.

Before I begin the review, it is informative to discuss how halos are identified in cosmological simulations. There are several algorithms used, the most common algorithm being Friends-Of-Friends (FOF) (Davis et al. 1985). The FOF algorithm links particles that are closer than a linking length, \( \ell \). It will continue to link particles until no within the group have a neighbour closer than \( \ell \). The primary disadvantage of this conceptually simplicity is that FOF will return a string as a group, which is unlikely to be a gravitationally bound object (known as the “string of pearls” effect). It will also artificially group gravitationally distinct objects together if they are connected by filaments.
Halos are typically identified using a $\ell = 0.2$ times the interparticle spacing of the simulation which will tend to find objects with a mean overdensity of $\approx \frac{3}{2}\ell^{-3} = 188$, roughly the virialization overdensity.

**The radial density profile**

Preliminary analytic work which assumed spherical symmetry and radial infall suggested dark matter halos had power-law radial density profiles that did not differ drastically from that of an isothermal sphere, where $\rho \propto r^{-2}$ (e.g. Gunn and Gott, 1972; Bertschinger, 1985; Ryden and Gunn, 1987). Early numerical simulations with limited resolution indicated that the hierarchical formation process gave rise to dark matter halos with a “universal” radial density profile, i.e., the profile of one halo could simply be rescaled to match the profile of another halo with a different mass. Dubinski and Carlberg (1991) found that simulations of the isolated collapse of spherical overdensities produced halos that tended to have density profiles with an inner logarithmic slope of $\approx -1$ that gradually steepened to $\approx -4$ in the outer regions. Another study by Navarro et al. (1996) investigated the structure of 19 dark matter halos in a CDM cosmology ranging in mass between $10^{11} - 10^{15} M_\odot$. This study found that halos had radial density profiles that were approximately isothermal, in agreement with earlier analytic work. In a follow up study, Navarro et al. (1997) (hereafter NFW) found halos were well fit by a two parameter radial density profile given by

$$\rho_{\text{NFW}}(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}. \quad (3.67)$$

Here $\rho_s$ the characteristic density of the halo and $r_s$ is the scale radius where the density profile has a logarithmic slope of $-2$. It appeared that halos had cuspy profiles, that is density continued to increase sharply as $r \to 0$.

A similar study by Moore et al. (2001) lead to an alternative

$$\rho_{\text{M}}(r) = \frac{\rho_M}{(r/r_M)^{1.5}[1 + (r/r_M)^{1.5}]} \quad (3.68)$$

Though different in form, both this profile and NFW profile, have two scaling parameters, characteristic density and radius, but a fixed shape. As a consequence, two different halos can, in principle,
be rescaled to be indistinguishable from each other. The key difference between these profiles is the central logarithmic slope. The Moore profile is more cuspy than the NFW profile and in fact gives an divergent mass and thus is often truncated by “core” (a constant density profile). A similar study by Diemand et al. (2004) found yet a different well-defined central logarithmic slopes of $-1.2$.

More recently, Navarro et al. (2004) found halos did not appear to have a well-defined central logarithmic slope and that the Moore profile was too steep. They proposed that the controversy surrounding the interior logarithmic slope lay in the assumption of a “universal” profile and the use of a two parameter model. They suggested that three parameters are needed to accurately describe the density profile. The authors found that the average halo density was well fit by a three parameter profile first introduced by Einasto (1965) to describe the distribution of old stars within the Milky Way. The Einasto profile is given by

$$\ln \left( \frac{\rho_E(r)}{\rho_{-2}} \right) = -\left( \frac{2}{\alpha_E} \right) \left[ \left( \frac{r}{r_{-2}} \right)^{\alpha_E} - 1 \right]. \quad (3.69)$$

Here, $\rho_{-2}$ is the density at $r = r_{-2}$, the radius where the logarithmic slope of the density profile has the “isothermal” value of -2. In this sense, $r_{-2}$ is equivalent to the NFW scale length, $r_s$ as it marks the location of the maximum in $r^2 \rho$. Unlike previous profiles, the Einasto profile does not have a constant asymptotic logarithmic slope. Instead $d \ln \rho / d \ln r = -2(r/r_s)^{\alpha_E}$ is a power-law and is zero at $r = 0$. The quantities of the different profiles relate as follows: for the NFW profile, $r_{-2} = r_s$ and $\rho_{-2} = \rho_s / 4$, while for the Moore profile, $\rho_{-2} = (4/3)\rho_M$ and $r_{-2} = 2^{-2/3}r_M$.

It is also intriguing to note that a high redshift study by Gao et al. (2005) found that the halo density profile changed quite drastically with mass and redshift. At the two smallest masses examined in the study, $5.2 \times 10^7 M_\odot$ at $z = 29$ and $1.2 \times 10^5$ at $z = 49.5$, the shape of radial density profile was not well characterized by NFW profile, in contrast to the halos with masses $> 10^{10} M_\odot$.

Later studies by Gao et al. (2008) and Navarro et al. (2008) with higher mass resolution also showed that the Einasto profile was a much better fit to the data than the two parameter models. It is possible that the improved fits may have been simply due to the addition of the extra parameter. However, both studies presented evidence that mass profiles of halos are not strictly self similar.
example, [Navarro et al. (2008)] found even among similar mass halos with similar formation histories, the shape parameter $\alpha_E$ showed halo-to-halo scatter, enough that there was no rescaling would map one halo’s profile to another’s. [Gao et al. (2008)], by stacked density profiles of many halos of similar mass, also showed that the mean Einasto shape parameter $\alpha_E$ depended systematically on halo mass.

The controversy over the central slope may be traced to the fact that earlier work lacked the numerical resolution and the representative halo sample needed to settle the debate. Work still needs to be done to conclusively pin down the density profile of dark matter halos and to quantify the halo-to-halo scatter in the shape. Due to the uncertainty in the exact interior slope, the [NFW profile is still in common use. It is important to note that the exact shape of the profile has important consequences for dark matter detection. Cuspy halos with steep central logarithmic slopes will naturally have higher annihilation and decay rates than less cuspy or cored halos since these rates scale as $\rho^2$ and $\rho$ respectively.

The concentration parameter

The concentration parameter, $c$, quantifies how centrally concentrated is a halo’s density profile and is defined as the ratio of the radius of a halo to its scale radius, $c \equiv R_{\text{vir}}/r_s$. Here $r_s$ is taken to be the [NFW scale radius and the virial radius is to be given by the relation $M_{\text{vir}} = 4\pi \Delta_{\text{vir}}(z)\rho_m(z)R_{\text{vir}}^3/3$ where $\Delta_{\text{vir}}$ is the virial overdensity of a halo, and $\rho_m(z)$ is the mean mass density of the universe at a redshift $z$. The virial mass is also occasionally defined as $M_{\text{vir}} = 4\pi \Delta_{\text{vir},c}\rho_c(z)R_{\text{vir}}^3/3$, where $\rho_c$ is the critical density. The virial overdensity is often set equal to 200.

[NFW] found that concentration showed a mass and redshift dependence. Less massive halos were more concentrated than more massive ones and for a given mass interval, $c$ increased with redshift. A follow up study by [Bullock et al. (2001)] (hereafter [B01]) examined about 5000 halos with masses in the range of $10^{11} - 10^{14} h^{-1} M_\odot$ in a $\Lambda$CDM cosmology and found similar results. Both [NFW] and [B01] proposed that the central density of a halo reflects the mean density of the universe at a time when the core was accreting matter at a high rate. This naturally explains why
halos that formed earlier have higher concentrations than similar mass halos that formed later and why lower mass halos have higher concentrations than larger ones since lower mass halos on average form early.

As I make use of the $\text{B01}$ model to determine the $c - M$ relation with some modifications, I will outline it in detail here. The first step is to determine the collapse redshift of a halo of mass $M_{\text{vir}}$. This redshift is at which the characteristic mass is equal to a fraction $F$ of the halo mass at the observed redshift $z$,

$$M_*(z_{\text{coll}}) = F M_{\text{vir}}(z),$$  \hspace{1cm} (3.70)

where the characteristic mass $M_*$ is defined via

$$\sigma(M_*, z_{\text{coll}}) = \sigma(M_*, 0) D(z_{\text{coll}}) = \delta_{\text{sc}}.$$  \hspace{1cm} (3.71)

Here $\sigma(M, z)$ is the mass variance, that is the rms overdensity on mass scale $M$ at redshift $z$, $D(z)$ is the growth function, and $\delta_{\text{sc}}$ is the critical density for spherical collapse. The quantity $F$ is a free parameter that must be calibrated using simulations. $\text{B01}$ found setting $F = 0.01$ reproduced the observed $c - M$ relation. The next step is to associate the characteristic density of a halo, $\bar{\rho}_c = 3 M_{\text{vir}}/4 \pi r_s^3 = \Delta \rho_m(z) c(M, z)^3$, with the mean density of the universe at the collapse epoch via

$$\bar{\rho}_c = K^3 \Delta_{\text{vir}} \rho_m(z_{\text{coll}}),$$  \hspace{1cm} (3.72)

where $K$ is a proportionality constant that must be calibrated against simulations. This gives

$$c(M, z) = K (1 + z_{\text{coll}})/(1 + z).$$  \hspace{1cm} (3.73)

Another model for the concentration-mass relation was proposed by $\text{Eke et al} \ (2001)$ (hereafter $\text{ENS}$). This model also assumes that a halo’s central density reflects the mean density at its collapse time. The key difference between these two models relates to the fact that $\text{B01}$ only examined one cosmology, whereas $\text{ENS}$ examined halos in $\Lambda$CDM cosmologies with several different power
spectra. These authors found that the slope of the concentration-mass relation depended on the shape of the power-spectrum and proposed to define the collapse redshift by

\[ D(z_{\text{coll}})\sigma(M_c, 0) \left( \frac{d \ln \sigma}{d \ln M} \right)_{M=M_c} = \frac{1}{C_{\sigma}}, \quad (3.74) \]

where \( M_c \) is the mass within 2.17\( r_s \), \( C_{\sigma} \) is a constant that must be calibrated and \( d \ln \sigma / d \ln M \) is the shape of the mass variance, which depends on the shape of the power spectrum. Otherwise, the model is similar to the [B01] model.

Later studies by [Macciò et al. (2007)] and [Neto et al. (2007)] found that neither model fully captured the observed \( c - M \) relation. [Macciò et al. (2007)] examined \( \Lambda \)CDM halos in a mass range of \( 3 \times 10^9 - 3 \times 10^{13} \, h^{-1} M_\odot \) whereas [Neto et al. (2007)], using the Millennium Simulation, examined halos with masses in the range of \( 10^{12} - 10^{15} \, h^{-1} M_\odot \). The former study found the [B01] model with \( F = 0.01 \) agreed with the observed \( c - M \) relation at the low mass end whereas [ENS] disagreed, predicting a significantly shallower slope. The latter study, which looked at higher masses, found the opposite was true, [ENS] worked but [B01] did not. [Macciò et al. (2007)] found that a better fit to the concentration of high-mass halos (\( M \gtrsim 10^{13} \, h^{-1} M_\odot \)) could be obtained using the [B01] model with \( F = 0.001 \), although this is at the expense of under predicting the concentrations of low-mass halos (\( M \lesssim 10^{11} \, h^{-1} M_\odot \)). Both studies also found that for a given mass interval \( c \) was logarithmically scattered about a central value and that there appeared to be two distinct populations. One population corresponded to relaxed halos, the other to unrelaxed or tidally disrupted halos. Relaxed halos had higher concentrations and a smaller dispersion than unrelaxed ones, though both populations were well characterized by lognormal distributions.

In a follow-up study, [Macciò et al. (2008)] suggested a slight modification to the [B01] model to account for the discrepancies observed. They assumed that the characteristic density was independent of redshift, unlike the previous models. This alters the model such that

\[ c(M, z) = K \left[ \frac{\Delta_{\text{vir}}(z_{\text{coll}})\rho_m(z_{\text{coll}})}{\Delta_{\text{vir}}(z)\rho_m(z)} \right]^{1/3}. \quad (3.75) \]

The modification essentially takes into account the redshift dependence of the halo density contrast and is similar to the [ENS] model except for the different definition of \( z_{\text{coll}} \). The authors found this
new model with $F = 0.01$ fit the observed $c - M$ relation over a mass range of $10^{12} - 10^{15} \, h^{-1} M_\odot$ for three slightly different $\Lambda$CDM cosmologies based on WMAP 1$^{st}$ (Spergel et al., 2003), 3$^{rd}$ (Spergel et al., 2007), and 5$^{th}$ year (Komatsu et al., 2009) data.

Unfortunately, these studies have only examined the $c - M$ relation over a small fraction of the masses in the CDM hierarchy, and have primarily focused on galactic to cluster scales where the effective spectral index is $\gtrsim -2.2$. A study by Knollmann et al. (2008) attempted to bridge the gap using scale-free cosmologies, i.e., the power spectrum and resulting mass variance are a power-laws, $P(k) \propto k^n$ and $\sigma^2(M) \propto M^{-\frac{n+3}{3}}$. By examining cosmologies with $n < -2$ they are effectively examining smaller scales in the CDM cosmology. They used $n = [-0.5, -1.5, -2.25, -2.5, -2.75]$ and found that steeper-$n$ cosmologies have less concentrated halos. However, it should be noted that the simulations with $n < -2$ should be considered suspect due to missing power and finite volume effects (e.g. Bagla and Prasad, 2006; Power and Knebe, 2006; Elahi et al., 2009a).

### 3.3.2 Subhalos

Substructure appears to be a natural outcome of the hierarchical formation process, whereby halos grow through mergers and the accretion of smaller halos. As objects merge, their central regions might be dense enough to survive infall (Rees and Ostriker, 1977). However it has also been argued that mergers and the virialization process actually destroy substructure in collisionless systems (White and Rees, 1978). Observational evidence seems to indicate that objects can and do survive infall into larger systems. For example, clusters contain numerous galaxies.

However, early simulations of dark matter halos ended with no substructure in clusters mass halos ($\sim 10^{15} \, M_\odot$) (e.g. White et al., 1987; Frenk et al., 1988). Simulations showed galactic mass halos ($\sim 10^{12} \, M_\odot$) were not surviving infall into the cluster environment. This “overmerging” problem was investigated in detail by Moore et al. (1996). They concluded that the loss of substructure was due to numerical and physical effects. The poor force resolution and artificial particle-halo
heating combined with physical effects such as tidal and halo-halo heating were leading to premature evaporation of cluster substructure. Improvements in computing power and the use of nested higher resolution regions inside cosmological simulations lead to substantial improvements in force resolution. With a mass and force resolution of $10^8 - 10^9 \, M_\odot$ and $\sim 1 - 3 \, \text{kpc}$ respectively, simulations shows galactic mass halos surviving infall into clusters (Klypin et al., 1999).

Before discussing some of the key results, I will review how subhalos are identified.

**Identifying subhalos**

Identifying subhalos or “halos within halos”, the self-bound local overdense regions that lie within the virial radius of a larger host halo as shown in Figure 4.4 on page 88, is a non-trivial exercise. A variety of algorithms have been developed to solve this problem. For example, using the FOF algorithm with a smaller linking length than that used to identify halos will identify regions at higher density within halos. However, in this nonlinear environment there is a higher probability that FOF will artificially link structures with transient particle bridges. Gottlöber (1999) proposed a hierarchical FOF algorithm, which reduces the FOF linking length in discrete steps to remove this artificial linking. Though this process does reduce the likelihood of artificially linked groups, it does not alleviate the problem altogether. Furthermore, such algorithms will also return overdense “streams” which may be purely artificial. Other techniques include Bound Density Maxima (BDM) (Klypin et al., 1999), DENMAX (Gelb and Bertschinger, 1994), Adaptive HOP (AdaptaHOP) (Aubert et al., 2004), and MLAPM Halo Tracker (MHT) (Gill et al., 2004). I discuss in detail a couple of widely used algorithms that primarily use density information below.

One widely used algorithm is SKID (Stadel, 2001), the offspring of DENMAX. This algorithm first calculates the density for each particle using the standard SPH method, i.e., the local density is calculated using a smoothing kernel over a particles nearest $N_{\text{SPH}}$ neighbours, where the local smoothing scale is set to the distance that encloses these neighbours. It then applies a minimum density criterion and moves all particles that pass this density threshold along density gradients towards higher density regions. It continues moving these particles in steps of $\ell_{\text{SKID}}$, a user-specified
distance, till they are all confined in density regions $\ell_{\text{SKID}}$ in size. Particles are then grouped using a FOF algorithm with the same linking length $\ell_{\text{SKID}}$. An unbinding routine is then applied to each group. This removes any particles that are unbound relative to the group center from the group. Finally, only groups containing more than a certain number of members are considered subhalos.

Another commonly used algorithm is SUBFIND (Springel et al., 2001). The algorithm first calculates the local density of all particles in a halo using the standard SPH method. It then essentially searches for local density maxima that lie above a density threshold, where a local region is defined as being enclosed by an isodensity contour that traverses a saddle point. The search for these isodensity contours amounts to lowering a global density threshold and examining the regions that lie above it. As the threshold is lowered, these regions will grow in size till they meet and merge into a common domain. At these intersections, the density field forms a saddle point.

Technically, the algorithm implements this search by first finding all particles that lie above the threshold and locates their nearest $N$ neighbours, where $N$ is user defined. Each set of near neighbours associated to particle $i$ is then searched for the two closest neighbour particles that have densities greater than particle $i$. This subset may be empty or contain one or two particles. If the set is empty, that is particle $i$ has no $N$ near neighbours with a higher density, particle $i$ is considered a local maximum and the code begins to grow a group around it. If there is a single particle or two particles that are currently attached to the same group, then that particle is associated with the same group. If the subset contains two particles but they do not belong to the same group then particle $i$ is considered a saddle point and the two groups are registered as subhalo candidates. Particles below the saddle point density are associated with the large group, as is the saddle point particle. Thus SUBFIND naturally finds subhalos within larger subhalos but at the expense reducing the mass associated with the larger halo. Generally, this is not an issue as one halo tends to be substantially larger than the other. These subhalo candidates are then check for self-boundness and again only groups containing more than a certain number of particles are considered subhalos.

The algorithms discussed so far only use density information to establish group membership.
However, subhalos are overdensities not just in physical space but in phase-space as well (see Figure 4.4 on page 88 for example). The first algorithm to make use of the full phase-space information to find groups was 6D Friends-of-Friends (6DFOF) outlined in Diemand et al. (2006). This is the algorithm used in this work and is discussed in detail in Section 4 so I will not go into detail here. It is important to note that this algorithm does not inherently construct a subhalo tree while producing a subhalo candidate list as SUBFIND does.

The 6DFOF algorithm is not the only phase-space algorithm in use. The Hierarchical Structure Finder (HSF) by Maciejewski et al. (2009) is very similar to SUBFIND but instead of searching for saddle points in the physical density field, it searches for them in phase-space density. However, HSF does redistribute particles below the saddle point in a slightly different fashion than SUBFIND. Once a saddle point is found, the algorithm tests to see if the structures to either side of the saddle point are statistically significant when compared to Poisson noise. If the saddle point is not significant, the particles in the smaller group are merged into the larger one. If this criterion is satisfied and one of the structures is significantly less massive than the other, all particles with phase-space densities below the saddle point’s are associated with the larger of the two groups. However, if the groups are of comparable mass a cut or grow criterion is applied. Both groups are grown normally according to the rules outline in the SUBFIND discussion until one group is substantially larger than the other. That is a particle $i$ is assigned to the group that its densest near neighbours belong to unless it is found to be a saddle point, in which case, the mass ratios of the two groups are compared and if one is larger than the other, all particles below the saddle point density are associated with the larger group.

Subhalo distribution

A surprising result from Moore et al. (1999), one of the first studies capable of resolving substructure, was remarkable visual similarity between a galactic mass halo and a cluster mass considering these objects differed by 2 orders of magnitude in mass. These authors simulated a $4.6 \times 10^{14} \text{ h}^{-1} M_\odot$ halo and a $1.2 \times 10^{12} \text{ h}^{-1} M_\odot$ halo both containing $\approx 6 \times 10^5$ particles in their virial
radius. The authors used SKID to find subhalos and calculated a velocity scale for each subhalo given by $v = (Gm/r)^{1/2}$, where $m$ is the mass of the subhalo and the $r$ is the distance to the least bound particle from the subhalo center. They found that the cumulative number of subhalos with velocities greater than $v$, $N(> v)$, once scaled by the velocity of the host $V_{\text{host}} = (GM_{\text{host}}/R_{\text{host}})^{1/2}$, showed little difference between the two different mass scales and appeared to be characterized by a power-law.

A follow-up study by Ghigna et al. (2000) of a cluster mass object with $\approx 8 \times 10^6$ particles within the virial radius also showed that the velocity distribution, the number of subhalos with velocities in an interval $\ln v$ to $\ln v + d\ln v$, was well characterized by a power-law, $dn(v)/d\ln v \propto v^{-\gamma}$. This study also showed that the subhalo mass function, the number of subhalos in a mass interval $\ln m$ to $\ln m + d\ln m$ was characterized by a power-law, $dN(m)/d\ln m \propto m^{-\alpha}$. However, they did note that the index of the power-law for subhalos with $m \gtrsim 10^{11} \, M_\odot$ was $\approx 1$ but then appeared to decrease to 0.7 for smaller masses. Later studies found the mass function appeared to be well characterized by a single power-law with the index $\alpha \approx 1$ independent of the host halo mass $M_h$ for $10^{12} \, M_\odot \lesssim M_h \lesssim 10^{15} \, M_\odot$ (e.g. Stoehr et al., 2003; Kravtsov et al., 2004; Gao et al., 2004).

Another study by Reed et al. (2005a) examined the subhalo velocity function for a variety of host masses in a $\Lambda$CDM cosmology and hosts in several scale-free cosmologies ($n = 0, -1, -2, -2.7$). The authors examined $\Lambda$CDM halos with $2 \times 10^{11} \, M_\odot \lesssim M_h \lesssim 2 \times 10^{14} \, M_\odot$, most composed of $\approx 5 \times 10^5$ particles but a few had up to $\sim 10^7$ particles. SKID, was used to identify subhalos and the maximum circular velocity was determined for each subhalo. The resulting velocity distribution, once normalized by the hosts maximum circular velocity, appeared on average to be well characterized by a power-law with $\gamma \approx 3$. However, the distribution function showed quite a bit of halo to halo scatter. In scale-free cosmologies, the amplitude showed a strong dependence on the spectral index $n$, decreasing with decreasing $n$. The slope of the velocity function also appeared to decrease with decreasing $n$, although the steepest $n$, where the effect is the greatest, may be suspect due to finite volume effects and the fact that the halo examined was composed of just $10^4$ particles.
Although most studies have only examined subhalos in ΛCDM halos with \( M_h \gtrsim 10^{11} M_\odot \), there are two studies, Gao et al. (2005) and Diemand et al. (2006), that did probe smaller scales. Gao et al. (2005) examined the subhalos of 5 halos at different masses and redshifts: 8.1 × 10^{14} h^{-1} M_\odot at \( z = 0 \), 4.6 × 10^{12} h^{-1} M_\odot at \( z = 5 \), 2.0 × 10^{10} at \( z = 12.4 \), 5.2 × 10^{7} h^{-1} M_\odot at \( z = 29 \), and \( 1.2 \times 10^5 \) at \( z = 49.5 \). The hosts were generally composed of \( \approx 2 \times 10^6 \) particles except for the highest-\( z \) halo, which was composed of 1/10 this number. In order to examine the low mass, high redshift halos, the study had to use very rare peaks in the density field. Consequently, their results may not be representative of the general population. They found the subhalos, identified using SUBFIND, had a power-law cumulative subhalo mass function with an index that did not show a strong dependence on host mass. The amplitude of the cumulative mass function, unlike the index did appear to show a strong dependence on host mass, increasing with increasing host mass.

Diemand et al. (2006) examined a 0.014 solar mass halo at \( z = 75 \) composed of \( \approx 14 \times 10^6 \) particles. With this resolution they were able to sample the subhalo mass function down to the minimum CDM halo mass of \( 10^{-6} \) defined by SUSY dampening scale (Green et al., 2004, 2005). To find subhalos, these authors introduced the 6DFOF algorithm, which was used to find local phase-space density peaks. In order to determine a subhalo’s maximum circular velocity, the circular velocity profile of these peaks were fit with the velocity profile corresponding to an NFW halo embedded in a constant background density. They did not apply an unbinding routine to these subhalos as is generally done. The resulting velocity function was well characterized by a power-law, however, the index \( \gamma \) was smaller than that of a cluster mass host. They also used SKID to examine the subhalo mass function and found it was similar to that of galaxy and cluster mass halos. However, it is important to note that the mass function of the SUSY halo was very sensitive to the parameters used in SKID. This is in sharp contrast to the mass functions of low redshift cluster halos, which appear insensitive to the parameters used. The authors also noted that 20 – 40\% of the subhalos identified at \( z = 75 \) were destroyed within 1.3 expansion factors, again in sharp contrast to \( \approx 1\% \) destroyed in a simulation of a cluster mass halo over the same expansion factors.

The latest multi-billion particle simulations of galactic mass halos, VL-II (Diemand et al. 2008)
and the Aquarius project [Springel et al.] (2008), have examined the subhalo mass function down to mass fractions of $10^{-7}$ and subhalo masses of $\sim 10^5 \ M_\odot$. Even down to these low mass fractions, the subhalo mass and velocity functions still appear to be well characterized by power-laws. Diemand et al. (2008), using 6DFOF, found the velocity function index $\gamma = 3$. Springel et al. (2008), using SUBFIND, found $\alpha = 0.9$ and $\gamma \approx 3$.

Both these projects also had enough resolution to examine the velocity and mass function of subsubhalos, that is subhalos that lie within the boundary of another subhalo. When examining subsubhalos, there is a subtle issue of defining the exact boundary of a subhalo so as to compare subhalos on equal footing. Diemand et al. (2008) defined the boundary to be the radius at which the average density interior to that radius was 1000 times the mass density, $\rho_m$, whereas Springel et al. (2008) used an average density of 250 times the critical density, $\rho_c$. These are essentially equivalent definitions as $\rho_c \approx 4 \rho_m$. The choice of overdensity is somewhat arbitrary so long as it is higher than the overdensity defining the boundary of the halo, which was effectively $50 \rho_c$ in both studies.

The VL-II simulation had six subhalos which were host to their own subhalos. These subhalo hosts had circular velocities ranging from 30 to 40 km/s, corresponding to masses of $5 \times 10^9 \ M_\odot \lesssim M_{1000\rho_m} \lesssim 2 \times 10^{10} \ M_\odot$. Diemand et al. (2008) claimed that the subsubhalo velocity function is simply a scaled version of that of field halos. This does not appear to be the case. The velocity functions of the subhalo hosts appear to have shallower slope (smaller $\gamma$) than the host field halo, though the poor statistics means it is not possible to draw any strong conclusions.

The highest Aquarius simulation had 12 subhalo hosts with masses in the range of $10^9 \ M_\odot \lesssim M_{250\rho_c} \lesssim 2 \times 10^{10} \ M_\odot$. Due to the higher resolution of the Aquarius run, the measured mass functions had better statistics than the corresponding velocity functions of VL-II. The subsubhalo mass functions appeared well characterized by power-laws, although the slopes exhibited quite a bit of scatter and tended to be shallower (smaller $\alpha$) than those of the field halos. The authors noted that the subhalos contained less substructure than field halos did at the same overdensity. These subhalos also had radial density profiles that were well fit by Einasto profiles, though the fitted Einasto slope parameter showed greater (sub)halo-to-(sub)halo scatter than field halos. Springel
et al. (2008) concluded that subhalos are not simply scale versions of halos as they contain less substructure, though they did not make any conclusive statements about the apparent differences in slope of the mass function.

3.3.3 The current consensus

The current consensus is that the hierarchical assembly of galactic and cluster mass CDM halos yields:

- halo mass profiles that are approximately “universal” (aside from simple physical scalings and $c$ dependence on cosmological parameters such as $\sigma_8$) with “cuspy” inner profiles.

- abundant but non-dominant substructure (subhalos account for $\lesssim 50\%$ of the mass of the halo) that has a scale-free mass distribution.

- halos with triaxial morphologies, which tend to be prolate rather than oblate, and are more triaxial in the central regions and more spherical in the other regions (e.g. Allgood et al. 2006; Kuhlen et al. 2007; Hayashi et al. 2007).

3.4 Indirect Dark Matter Detection Signals

This section discusses information relevant to Paper II (Chap. 5) regarding indirect dark matter detection. As mentioned in the introduction, there are numerous dark matter candidates. Most candidates can either self-annihilate or decay, producing secondary particles which in turn can be detected directly. Thus, the existence of dark matter would be placed on even more solid footing if these secondary particles can be detected. Furthermore, the composition and energy spectrum of these secondaries would also constrain the model parameter space. The annihilation or decay rate depends not only on the properties of the dark matter particle but also the distribution of dark matter. Here I discuss how astrophysical quantities affect the annihilation rate and the resulting
indirect detection signal. The annihilation rate at a position $x$ is given by

$$\Gamma(x) = \frac{\langle \sigma v \rangle}{\langle m_\chi \rangle^2 \rho^2(x)},$$

(3.76)

where $\langle \sigma v \rangle$ is the velocity averaged cross section, $m_\chi$ is particle’s mass, and $\varsigma$ is 2 if the particle is Majorana (taking into account that particles are not discernible) and 4 for Dirac particles (taking into account the fact that the density of particles and antiparticles is $\rho/2$). It should be noted that it is a simple matter of altering the discussion presented for decaying dark matter by replacing $\langle \sigma v \rangle \rho^2 / \varsigma m_\chi^2$ with $\rho / m_\chi \tau_\chi$ where $\tau_\chi$ is the decay time. I first start with a discussion of $\gamma$-rays secondaries and then cosmic rays.

### 3.4.1 Gamma rays

Gamma rays can be produced directly by annihilating dark matter, resulting in 2 monoenergetic photons. Or they can be produced through the annihilation and decay of short lived secondaries. For example, annihilations to heavy charged fermions, such as $\chi \chi \rightarrow \bar{b}b$, will produce $\pi^0$ mesons which will in turn decay to $2\gamma$, resulting in a continuum \cite{Bengtsson:1990}. It is also possible to produce a continuum of $\gamma$-rays via Internal Bremsstrahlung (IB) \cite{Bringmann:2008}. This process corresponds to the non-zero probability that the production of a charged particle pair will be accompanied by the emission of a photon. Finally, another continuum of high energy photons can be produced when charged secondaries Inverse Compton scatter off CMB photons, however, since these photons do not necessarily originate in the immediate vicinity of the annihilation source I will neglect it here. The luminosity (in number of photons per unit volume per unit energy per second) can be written as

$$L(x, E) = \left[ \frac{\langle \sigma v \rangle}{\langle m_\chi \rangle^2 \rho^2(x)} \right] \times \left[ \mathcal{P}_\gamma \delta^D(E - m_\chi) + \sum_i \mathcal{P}_i \left( \frac{dN_{\gamma, i}^{\pi^0}}{dE} + \frac{dN_{\gamma, i}^{\text{IB}}}{dE} \right) \right] = \Gamma(x) \times \mathcal{E}(E).$$

(3.77)

Here $\mathcal{P}$ is the branching ratio for a given annihilation channel and encapsulates the particle physics. For example, $\mathcal{P}_\gamma$ corresponds to the annihilation cross of $\chi \chi \rightarrow \gamma \gamma$, which produces monoenergetic...
photons, relative to the total annihilation cross section. The sum is over all other annihilation channels and their related continuum. For the interested reader, the spectrum for the first two process can be computed for neutralinos with codes such as DarkSUSY \cite{Gondolo:2004sc}. For simplicity, I will not explicitly concern myself with the energy spectrum $E$ from here on.

Since photons simply free-stream from their source, the flux from a point source $x_d$ a distance $d$ away is simply $\phi(E) = \mathbb{I}_s(x_d, E)/4\pi d^2$. The observed flux by detectors, like GLAST, is a little more complicated as these instruments collect the photons from all sources along the Line-Of-Sight (LOS) within an angular window $\Delta \Omega$ \cite{Kuhlen:2008vb},

$$\phi(E) = \frac{E}{4\pi} \frac{\langle \sigma v \rangle}{m_X^2} \int_{\Delta \Omega} d\Omega \int_{\text{LOS}} \rho^2(x) d\Omega.$$

The situation is simplified somewhat if one is just interest in the flux contribution from a (sub)halo of mass $M_h$ and a distance $d$ away which is much smaller than the angular window. In this case, the flux can be written as

$$\phi(E) = \frac{E}{4\pi} \frac{\langle \sigma v \rangle}{m_X^2} \mathcal{L}(M_h),$$

where

$$\mathcal{L}(M_h) = \int \rho^2(x) dV_h,$$

the integral is over the volume of a (sub)halo. It is a simple matter of calculating this quantity for the smooth component of a halo using the appropriate radial density profile, which is denoted here as $\bar{\rho}(r)$. However, halos are host to subhalos which in turn are host to their own subhalos. In this work I follow Strigari et al. \cite{Strigari:2007}, and quantify the enhancement due to density spikes from subhalos by the boost factor $B(M_h)$. The above equation is then

$$\mathcal{L}(M_h) = [1 + B(M_h)]\tilde{\mathcal{L}}(M_h),$$

where $\tilde{\mathcal{L}} = 4\pi \int \bar{\rho}^2(r) r^2 dr$. The boost factor for subhalos with a mass function $dN/d\ln m$ is given by the recursive equation

$$B(M_h) = \frac{1}{\mathcal{L}(M_h)} \int \frac{dN}{d\ln m} [1 + B(m)]\tilde{\mathcal{L}}(m)d\ln m.$$
With a little algebra and by using the fact that a (sub)halo’s mass is given by \( M \equiv \frac{4\pi}{3} \rho_v R_{\text{vir}}^3 \) (here \( \rho_v \equiv \Delta_{\text{vir}} \rho_m \)) and \( M = 4\pi \int \bar{\rho}(r)r^2dr \), it is a simple matter to show that

\[
\bar{L}(M_h) = \frac{\rho_v}{3} M_h c^3(M_h)g(c).
\]  
(3.83)

The function \( g(c) = f_2(c)/f_1^2(c) \) where

\[
f_1(c) = \int_0^c \rho^i(u)u^2du,
\]
(3.84)

and where \( \rho(u) \) is the radial density profile with the characteristic density \( \rho_s \) and radius \( r_s \) are set to 1. For the often used NFW profile, this function is

\[
g_{\text{NFW}}(c) = \frac{1}{3} \left( 1 - \frac{1}{(1+c)^2} \right) \left[ \ln(1+c) - \frac{c}{1+c} \right]^{-2}.
\]
(3.85)

This function behaves approximately as \( c^{-1.4} \) for low \( c \) and gradually becomes less steep at large \( c \), going as \( c^{-0.6} \). This gives \( \bar{L} \propto M c^{1.6 \text{ to } 2.4} \). For the Einasto profile, one has

\[
g_{\text{E}}(c, \alpha_E) = \frac{\alpha_E^{1-3/\alpha_E}}{\Gamma(3/\alpha_E)} \left\{ \frac{1 - Q(3/\alpha_E, 4c^{\alpha_E}/\alpha_E)}{[1 - Q(3/\alpha_E, 2c^{\alpha_E}/\alpha_E)]^2} \right\},
\]
(3.86)

where \( \Gamma(a) \) is the complete Gamma function and \( Q(a, z) \equiv \Gamma(a, z)/\Gamma(a) \) is the regularized, upper, incomplete Gamma function. This function naturally depends on \( \alpha_E \), for example with \( \alpha_E = 0.15 \), \( g_{\text{E}}(c) \propto c^{-2} \) for low \( c \) and \( c^{-0.2} \) for large \( c \). Thus, the boost factor depends on not only the subhalo mass function but on the concentration-mass relation as well.

Strigari et al. (2007) first examined the boost factor for dwarf spheroidal (dSph) satellite galaxies of the Milky Way, such as Segue 1 and Draco. They did not solve the boost equation numerically, instead they analytically calculated an upper limit. To calculate \( B \), they assumed halos which surrounded the satellites and all the subhalos in the hierarchy had NFW profiles, while the subhalo mass distribution was a simple power-law and used the B01 model for \( c(M) \). Based on the subhalo mass function of Diemand et al. (2007) (VL-I), they determined \( B(M_{\text{dSph}}) < 100 \). However, they did not explore the dependence of the upper limit on their assumptions, nor did they calculate the flux for a given dark matter model.
A follow-up study by Martinez et al. (2009) to Strigari et al. (2007) outlined a more comprehensive framework for estimating the flux from satellite galaxies, focusing on the same two satellites, Segue 1 and Draco, as examples. They used Markov-Chain Monte Carlo (MCMC) analysis to estimate the properties of the dark matter halos surrounding these satellites based on published radial velocity data. An independent MCMC analysis was also done using the SuperBayeS code to explore the Constrained Minimal Supersymmetric Standard Model (CMSSM) parameter space and determine the most probable minimum CDM halo mass. They showed the boost factors from halo substructure in these galaxies depended strongly on the extrapolation of $c(M)$ for subhalos down to the minimum possible mass. For example, a power-law extrapolation of the relation observed at $10^{10} - 10^{15} \, \text{M}_\odot$ leads to boost factors of $\sim 20$ for these satellites whereas using the B01 model, the boost factors were of order unity.

Another study by Pieri et al. (2008) examined the number of Galactic satellites that might have their $\gamma$-ray annihilation flux detected at the $5\sigma$ level by GLAST after 5 years of operation. The authors assumed a thermal dark matter particle ($\langle \sigma v \rangle = 3 \times 10^{-26} \, \text{cm}^3/\text{s}$) with a mass of $m_\chi = 40 \, \text{GeV}$ that annihilates only to bottom-antibottom quarks ($\chi\chi \rightarrow \bar{b}b$). They calculated the flux from all the contributions along the line-of-sight as given by Eq. (3.78). In their calculations, they accounted for increase in the background $\gamma$-ray flux due to unresolved subhalos of the Galactic halo. They did not include the boost in flux to a resolved subhalo from its own substructure. The variation in the number of detectable satellites was estimated by using different realizations of the spatial distribution of resolved subhalos and several different observer locations. They also examined several models for the $c - M$ relation. They found that there are very few detectable subhalos and that detectable subhalos have masses $\gtrsim 10^8 \, \text{M}_\odot$. For example, there were $3.51 \pm 2.11$ detections using the B01 model for $c(M)$. However, large boosts factors would significantly increase the detectable number of resolved subhalos.

A similar study was performed by Kuhlen et al. (2008) using the VL-II simulation. This study used a similar dark matter model ($\langle \sigma v \rangle = 3 \times 10^{-26} \, \text{cm}^3/\text{s}$, $m_\chi = 50 \, \text{GeV}$, $\chi\chi \rightarrow \bar{b}b$). Unlike Pieri et al. (2008), this study included the boost factor due to (sub)subhalos that lay below the resolution
limit of the VL-II simulation \( (10^5 \, M_{\odot}) \). They calculated boost factors of \( \sim 2 - 10 \) using NFW profiles, B01 model for \( c(M) \) and a mass function index of \( \alpha = -1.0 \). With these large boost factors, they found \( \approx 20 \) subhalos could be detected at the \( 5\sigma \) level, though again only the largest satellites \( (\gtrsim 10^8) \) have a high probability of being detected.

### 3.4.2 Cosmic Rays

Calculating the flux of cosmic rays such as electrons, positrons, protons and heavy nuclei is more complicated than it is for \( \gamma \)-rays. Though the luminosity can be written in the same fashion, cosmic rays do not free-stream. Instead they undergo a variety of interactions as they propagate through the interstellar space: they interact with the Galactic magnetic field, scatter of the interstellar medium, fragment, decay, emit synchrotron radiation and inverse Compton scatter off photons. The propagation equation for cosmic rays is (Strong and Moskalenko [1998])

\[
\frac{\partial}{\partial t} \psi = Q(x, p) + \nabla \cdot (D_{xx} \nabla \psi - V_{\text{conv}} \psi) + \frac{\partial}{\partial p} \left( p^2 D_{pp} \frac{\partial}{\partial p} \frac{1}{p^2} \psi \right) - \frac{\partial}{\partial p} \left[ \frac{dp}{dt} \psi - \frac{1}{3} \left( \nabla \cdot V_{\text{conv}} \right) \psi \right] - \left( \frac{1}{\tau_f} + \frac{1}{\tau_r} \right) \psi. \tag{3.87}
\]

Here \( \psi \equiv \frac{dn}{dp} \) is the number density per unit momentum and \( Q \) is the source term. The second term on the right hand side corresponds to diffusion and convection. The spatial diffusion term \( D_{xx} \) arises from the scattering of particles off magnetic turbulence in the Galactic magnetic field. The convection arises from the Galactic wind, \( V_{\text{conv}} \), which pushes particles out of the Galactic disk. The third term corresponds to re-acceleration which is described by diffusion in momentum space with a coefficient \( D_{pp} \). In the fourth term, \( \dot{p} \) is the momentum loss rate. The last terms, \( \tau_f \) and \( \tau_r \), are the fragmentation/spallation and radioactive decay time scales respectively.

The goal is to solve for the steady-state equation with the source term simply being the luminosity of cosmic rays from dark matter annihilation. Since the propagation depends on the injected energy spectrum through \( Q(E) \), it is not possible to disentangle the properties of the dark matter particle from the density distribution as is the case for \( \gamma \)-rays. In general, the propagation equation
must be solved numerically. The most commonly used code is \textsc{GALPROP} \cite{Moskalenko:1998}. However, since I am interested in the flux of high energy electrons and positrons produced solely by dark matter annihilation, there are a few simplifying assumptions which allow the propagation equation to be solved analytically. Here I outline the approach taken by \cite{Moskalenko:1999} and \cite{Lavalle:2007} and write the propagation equation in terms of \( N = \frac{dn}{dE} \) the number density per unit energy. First, the source term is simply the luminosity (in number density per unit energy per second)

\[
Q(x, E) = L(x, E) = \Gamma(x) \left[ P_{e^\pm} \delta^D(E - m_\chi) + \sum_i P_i \frac{dN_{e^+ e^-}}{dE} \right].
\]

(3.88)

Second, the fragmentation and radioactive decay terms do not apply and the convective velocity can be safely ignored \cite{Moskalenko:1998}. It is also assumed that the diffusion term is spatially independent and has a simple power-law dependence on energy, \( D_{xx} = D_o \left( \frac{E}{E_o} \right)^a \), where \( E_o \) is an energy scale usually set to 1 GeV. The momentum or energy loss for electrons is dominated by synchrotron emission and inverse Compton scattering off CMB photons. Both mechanisms scale as \( E^2 \) and the net loss rate can be written as \( dE/dt = -\frac{E^2}{\tau_E E_o} \). With these simplifications, the propagation equation is

\[
-D_o \left( \frac{E}{E_o} \right)^a \nabla^2 N + \frac{\partial}{\partial E} \left( -\frac{E^2}{\tau_E E_o} N \right) = Q(x, E).
\]

(3.89)

By defining a pseudo-time variable \( \hat{t} = \tau_E \left( \frac{E}{E_o} \right)^{a-1} \), \( \hat{N} = \left( \frac{E}{E_o} \right)^2 N \) and \( \hat{Q} = \left( \frac{E}{E_o} \right)^{2-a} Q \) along with a little algebra, it is possible to write

\[
-D_o \nabla^2 \hat{N} + \frac{\partial}{\partial \hat{t}} \hat{N} = \hat{Q}(x, \hat{t}).
\]

(3.90)

This is the time dependent diffusion equation which can be solved with well known Green’s functions, that is

\[
\hat{N}(x, \hat{t}) = \int d\hat{t}_s \int d^3 x_s \hat{G}(x, \hat{t} \leftarrow x_s, \hat{t}_s) \hat{Q}(x_s, \hat{t}_s).
\]

(3.91)
The pseudo-time Green function can be transformed to the more physical, energy dependent one by
\[ G(x, E \leftarrow x_s, E_s) = \frac{\tau E E_o}{E^2} \tilde{G}(\tilde{\tau} \leftarrow \tilde{x}_s, \tilde{E}_s). \] (3.92)

Determining the flux is simply a matter of specifying the geometry and boundary conditions. For example, the retarded Green function for a one dimensional system with no boundary is
\[ \tilde{G}_{1D}(\tilde{x}, \tilde{t} \leftarrow \tilde{x}_s, \tilde{t}_s) = \Theta(\tilde{\tau}) \left[ \pi \lambda^2_{D}(\tilde{\tau}) \right]^{1/2} \exp \left( -\frac{(\tilde{x} - \tilde{x}_s)^2}{\lambda^2_{D}(\tilde{\tau})} \right), \] (3.93)
where \( \tilde{\tau} = \tilde{t} - \tilde{t}_s \) and \( \lambda^2_{D}(\tilde{\tau}) = 4D_o \tilde{\tau} \) is the diffusion scale.

Most solutions will have this exponential dependence. Of course, the actual Galactic magnetic field is more complicated. It is most often modeled as a leaky cylindrical slab with a vertical height \( L \) and an infinite radius extent. The Green function for a vertical leaky slab is outlined in Lavalle et al. (2007, 2008b). The choice of Green function is also quite sensitive to the diffusion parameters used. There is actually a great deal of uncertainty in the various diffusion parameters and they are correlated (Maurin et al., 2001). Typical values are \( D_o = 0.0112 \text{ kpc}^2 \text{ Myr}^{-1} \), \( a = 0.7 \) and \( L = 4 \text{ kpc} \). With these diffusion parameters and high energy (\( \gtrsim 100 \text{ GeV} \)) electrons, it is possible to make further simplifying assumptions. The diffusion scale, shown in Figure 3.2 on page 69 is smaller than the vertical height of the magnetic disk and for an observer located in the disk plane, the system can be accurately modeled by
\[ \tilde{G}_{3D}(\tilde{x}, \tilde{t} \leftarrow \tilde{x}_s, \tilde{t}_s) = \frac{\Theta(\tilde{\tau})}{[\pi \lambda^2_{D}(\tilde{\tau})]^{3/2}} \exp \left( -\frac{|\tilde{x} - \tilde{x}_s|^2}{\lambda^2_{D}(\tilde{\tau})} \right). \] (3.94)
Thus, for an observer located at the solar neighbourhood \( \mathbf{x}_\odot \), one has
\[ N(E) = \frac{(\pi m_e^2)}{4} \int \int d^3\mathbf{x}_s G_{3D}(\mathbf{x}_\odot, E \leftarrow \mathbf{x}_s, E_s) \rho^2(\mathbf{x}_s). \] (3.95)
The flux (in \( \text{cm}^{-2}\text{s}^{-1}\text{sr}^{-1}\text{GeV}^{-1} \)) is simply \( \phi(E) = \frac{v(E)}{4\pi} N(E) \), where \( v(E) \) is the velocity of the electron/positron with energy \( E \). I will now consider the monoenergetic production of \( e^+e^- \) pairs since it simplifies the calculation greatly. The flux from the smooth background component of the
Galactic halo is then
\[
\phi_{\text{halo}}(E) = \left( \frac{v(E) \langle \sigma v \rangle}{4\pi} \right) \times \left[ \int d^3x_s G_{3D}(x_\odot, E \leftarrow x_s, m_\chi) \bar{\rho}^2(x_s) \right],
\]
\[= S(E) \times A(E; \rho_s, r_s),
\]
where \(S(E)\) represents the first bracket and \(A(E)\) is the integral of the propagation Green function in the second bracket.

To quantify the effect of subhalos on the flux I will follow [Lavalle et al. (2008a,b)] and assume that the diffusion scale is much larger than the size of a subhalo such that \(\int dV G \rho^2 \rightarrow G \int \rho^2 dV\). Effectively, subhalos are treated as point sources. This is not a bad approximation as most of the flux originates from the region interior to \(r_s\), \((\approx 7/8\) for an NFW profile, and \(\approx 90\%\) for the Einasto profile), and for \(a \lesssim 10^9\) M\(_\odot\) halo \(r_s \lesssim 1\) kpc. The contribution of subhalo \(i\) with mass \(m_i\) located at \(x_i\) is then
\[
\phi_i(E) = S(E) \bar{L}(m_i)[1 + B(m_i)]G(x_\odot, E \leftarrow x_i, m_\chi).
\]
Assuming the subhalos follow a volume distribution \(dP_V/dV\) that is independent of mass, the contributions from all subhalos is
\[
\phi_{\text{sub}}(E) = S(E) \left\{ \int \frac{dN}{d\ln m} \bar{L}[1 + B(m)] \right\} \times \left[ \int G_{3D}(x_\odot, E \leftarrow x_s, m_\chi) \frac{dP_V}{dV} d^3x_s \right],
\]
\[= S(E) C_{\text{sub}} \times B(E; dP_v/dV),
\]
where \(C_{\text{sub}}\) is the integral over all subhalos in the first bracket and \(B(E)\) is the integral of the propagation Green’s function in the second bracket. The net effective boost in flux from subhalos is
\[
B_{\text{eff}} = \left( (1 - f_t)^2 \phi_{\text{halo}} + \phi_{\text{sub}} \right) / \phi_{\text{halo}},
\]
\[= (1 - f_t)^2 + C_{\text{sub}} B(E; dP_V/dV) / A(E; \rho_s, r_s).
\]
This equation accounts for the fact that a fraction \(f_t\) of the Galactic halos mass is in subhalos. I show an example of the resulting energy dependence from \(B(E)/A(E)\) in Figure 3.2 on the next page.
Figure 3.2  Top: The diffusion scale for several different sets of diffusion parameters as a function of energy. Here $\tau_E = 10^{16}$ s, $m_\chi = 700$ GeV and diffusion lengths for $(D_0 [\text{kpc}^2 \text{Myr}^{-1}], a) = [(0.0112, 0.70), (0.0765, 0.46), (0.0016, 0.85)$ are shown in solid (red), dashed (blue), and dotted (blue) lines. The thin horizontal lines show the corresponding magnetic disk scale heights $L$. (See Lavalle et al. 2008b and Maurin et al. 2001 for details about the range of each quantity.) Bottom: The ratio of the propagation terms $B(E)/A(E)$ for subhalos which have a volume distribution that follows the host halos density profile, i.e., $B \propto \int G_{3D}(x_\odot, E \leftarrow x_s, m_\chi) \rho(x_s) d^3 x_s$. Solid (red) and dashed (blue) lines are for an NFW and Einasto profile respectively.
There have been numerous studies of the cosmic ray flux from annihilating dark matter and the number has recently increased with the detection of an anomalous cosmic ray excess (e.g. Chang et al. 2008; Adriani et al. 2009). For example, Lavalle et al. (2008a) used an N-body simulation of a Galactic mass halo with a mass resolution of $7 \times 10^5$ M$_\odot$ to determine the density and Green’s functions to propagate cosmic rays produced by annihilating dark matter. They assumed a thermal WIMP with $m_\chi = 200$ GeV and $\langle \sigma v \rangle = 3 \times 10^{-26}$ cm$^3$/s and examined two decay chains, $\chi \chi \rightarrow e^+ e^-$ and $\chi \chi \rightarrow \bar{b}b$. It is important to note that they did not include increases in density from subhalos below resolution threshold. With these assumptions, they found the resulting flux of high energy electrons and the positron fraction was substantially smaller than that observed.

Lavalle et al. (2008b) calculated the effective boost due to Galactic subhalos, however, they did not include boosts in luminosity of Galactic subhalos from deeper levels in the subhalo hierarchy, that is they set $B(m)$ to zero. They examined the effective boost for subhalos with NFW and Moore profiles, several different indices of the subhalo mass function, ($\alpha = -1, -0.9, -0.8$), and two different models for $c(M)$, (B01 & ENS). In this case, they found that the large $B_{\text{eff}}$ of $\sim 100 - 1000$ required to explain the observed excess are strongly disfavoured.

A study by Cholis et al. (2008) focused on the positron fraction measured by PAMELA. They assumed a smooth Galactic halo which followed an NFW profile with a solar density of $\rho_\odot = 0.35$ GeV/cm$^3$ and used the GALPROP code to propagate the cosmic rays. By examining several different decay chains ($\chi \chi \rightarrow e^+ e^-, \mu^+ \mu^-, W^+ W^-, \bar{b}b$, etc) for a thermal WIMP with $m_\chi = 100, 300, 1000$ GeV, they found that direct injection of $e^+ e^-$ pairs from a $m_\chi \geq 300$ dark matter particle reproduced the observed positron fraction. However, again the flux needed to be boosted by a factor of $\sim 5$.

Bergstrom et al. (2009) found that, although it is possible to reproduce the spectral features in the cosmic ray flux with a thermal WIMP, which had $m_\chi \approx 1$ TeV and annihilated primarily to $\mu^+ \mu^-$ pairs, large effective boosts relative to the flux from a smooth halo, ($B_{\text{eff}} \sim 100 - 1000$) are required.

A study by Hooper et al. (2009) found that the observed flux could be reproduced by a single
nearby subhalo. The required annihilation rate for a subhalo at $\sim 1$ kpc was $\sim 10^{37}$ s$^{-1}$. A quick estimate of the flux from Draco ($10^7 - 10^8 M_\odot$) is

$$\sim 10^{35} \text{ s}^{-1} \left( \frac{600 \text{ GeV}}{m_\chi} \right)^{-2} \left( \frac{\langle \sigma v \rangle}{3 \times 10^{-26} \text{ cm}^3/\text{s}} \right) \left( \frac{M}{10^7 M_\odot} \right) [1 + B(M)].$$

Neglecting the boost factor from deeper levels in the subhalo hierarchy, a subhalo with a mass of $10^9 M_\odot$ is required. But study by Brun et al. (2009) found, based on the VL-II simulation, that the probability of a large subhalo ($\gtrsim 10^8 M_\odot$) within $\sim 8$ kpc of the Galactic center is exceedingly low, $p \lesssim 10^{-5}$. 
Chapter 4

Paper I: Subhaloes in Scale-Free Cosmologies

Here I present a paper co-authored with Robert J. Thacker, Lawrence M. Widrow, and Evan Scannapieco as it was published in *Monthly Notices of the Royal Astronomical Society* (MNRAS). The focus of this paper is the properties of subhalos, particularly how they might change as one approaches the bottom of the CDM hierarchy. Note that I have made a minor alterations to the manuscript. The convention in MNRAS is to use “haloes” and “subhaloes” instead of “halos” and “subhalos”. For clarity I have altered the manuscript and continue to use “halos” and “subhalos”. The focus of this manuscript is to explore the dependence of the subhalo mass function on the spectral index $n$ of the linear matter power spectrum using scale-free Einstein-de Sitter simulations with $n = -1$ and $n = -2.5$. By checking these different indices I am effectively exploring the largest scales and the smallest scales CDM hierarchy. A final note, in this paper the subhalo mass function index is defined via $dN/d\ln m \propto m^\alpha$ whereas in Section 3.3.2 and Chap. 5 I use $dN/d\ln m \propto m^{-\alpha}$, a simple change of sign.

I am the primary author of this paper. I helped run the simulations discussed in this chapter. I also wrote the subhalo group finder based on a previously written halo finder. I analyzed the halos...
4.1 Introduction

In the current Cold Dark Matter (CDM) paradigm, structure forms hierarchically; small-scale density fluctuations collapse to form an early generation of halos, which are the progenitors of larger-scale systems. While some progenitors survive as substructure, others are tidally disrupted and become the smooth component of the new system. The distribution of substructure within galaxy-size halos is of great interest as galaxies and satellites, acting as baryonic tracers of the underlying mass distribution, can provide observational checks of the current CDM paradigm. The subhalo distribution at much smaller scales, likely devoid of baryonic tracers, is of practical interest for DM detection experiments.

In this paper, we examine the properties of subhalos in a pair of scale-free cosmologies meant to mimic two different scales in the Universe. Objects in a scale-free simulation can be related to objects at different scales in the CDM hierarchy via the scale dependence of effective spectral index $n_{\text{eff}} \equiv d \ln P(k)/d \ln k$ of the $\Lambda$CDM concordance model. We use scale-free simulations for their simplicity as there is only one physical scale in the simulation, the nonlinear scale. This feature allows us to examine whether halo substructure remembers initial conditions, namely the index of the initial power spectrum.

Dark matter halos in $\Lambda$CDM simulations exhibit properties of self-similar and fractal systems. For example galactic halos appear to be rescaled versions of cluster halos (Moore et al., 1999). Moreover halos contain subhalos whose properties such as the density profile are similar to those of halos (e.g. Diemand et al., 2008 and Springel et al., 2008). The subhalo mass and circular velocity distributions show little scale dependence if normalized in terms of the host halo mass or maximum velocity (e.g. Kravtsov et al., 2004, Gao et al., 2004, and Reed et al., 2005a). Furthermore, the subhalo mass distribution is characterized by a power-law, $dN/d \ln M_{\text{sub}} \propto M_{\text{sub}}^{\alpha}$.
\( \alpha = -0.8 \) to \(-1.0 \) (e.g. [Stoehr et al., 2003; Gao et al., 2004; Diemand et al., 2007; Madau et al., 2008] and [Springel et al., 2008]). Note that a power-law index of \(-1\) implies a scale-free distribution with a constant mass in substructure per logarithmic mass interval. The most recent studies seem to favour \( \alpha = -0.9 \) for galactic halos down to a subhalo mass of \( \sim 10^5 \) M\(_\odot\) ([Madau et al., 2008 and Springel et al., 2008]). Substructure is not entirely independent of environment as the amount of it in a given halo depends on the halo’s formation time or peak height and concentration; halos that form earlier from high-\(\sigma\) peaks have less substructure than low-\(\sigma\) halos of similar mass while the less concentrated the halo’s density profile the more substructure it contains (e.g. [Bullock et al., 2001; Gao et al., 2004] and [Zentner et al., 2005]).

These ideas suggest a simple picture of substructure: halos contain a scale-free distribution of subhalos, which in turn contain a similar distribution of subsubhalos, all the way down the CDM hierarchy. The scale of the bottom of the hierarchy is set by fundamental physics, namely the free-streaming and collisional dampening scales of dark matter. If, for example, dark matter is the neutralino, a Weakly Interacting Massive Particle (WIMP) predicted by supersymmetric extensions to the Standard Model (SUSY), the first objects form at a redshift of \( z \gtrsim 60 \) and have masses \( M \lesssim 10^{-6} \) M\(_\odot\) ([Green et al., 2004, 2005]. [Diemand et al., 2005] investigated the formation of these objects and postulated that there would be \( \sim 10^{15} \) of them in the Milky Way (MW) halo today. The formation of a rare, high redshift 0.014 M\(_\odot\) object was simulated by [Diemand et al., 2006]. They found that the subhalo mass function is very similar to that of cluster halos. Again the suggestion is that self-similarity in halo’s substructure extends all the way down the hierarchy.

Self-similarity in the distribution of substructure may have important implications for a variety of experiments such as GLAST, which will search for \( \gamma\)-rays from DM self-annihilation. The annihilation signal is very sensitive to the slope of the subhalo mass distribution and the number of WIMP-scale subhalos since substructure can boost the flux by factors of a 3-100 (e.g. [Diemand et al., 2007; Kuhlen et al., 2008; Pieri et al., 2008; and Strigari et al., 2008]). The amount of dark matter locked up in substructure also has ramifications for direct DM detection as it reduces the density of the smooth background component of the Galactic halo but also increases the local density
inside subhalos (Kamionkowski and Koushiappas, 2008).

The extrapolation used to make predictions of the $\gamma$-ray flux are non-trivial since the MW halo is a factor of $10^{18}$ times more massive than the smallest subhalos. The estimate by Diemand et al. (2005) of $\sim 10^{15}$ small subhalos neglects mergers and tidal interactions inherent in the hierarchical structure formation scenario. Halos must survive similar-mass mergers and accretion, along with other dynamical processes that exist in galaxies (Zhao et al., 2007). Furthermore, the choice of $\alpha$ is crucial as small changes in $\alpha$ by as little as 0.1 can change the number of subhalos at the bottom of the hierarchy by an order of magnitude or more.

These results suggest that the internal properties of individual halos (e.g. the subhalo mass function) have no memory of the initial power spectrum. However, this would be suprising considering other internal properties of dark matter halos, such as concentration of the density profile, depend on properties of the primordial power spectrum (e.g. Reed et al. 2005b). For example, scale-free simulations by Knollmann et al. (2008) show that halos have less concentrated density profiles as $n \to -3$.

There are other theoretical reasons to suspect that substructure is qualitatively different at small scales. As one approaches the bottom of the CDM hierarchy, $n_{\text{eff}}$ monotonically decreases to $-3$ down to the cutoff at the WIMP free-streaming scale. The dimensionless power spectrum, $\Delta^2(k) \propto k^{n_{\text{eff}}+3}$, becomes scale independent as $n \to -3$ and objects collapse simultaneously over a wide range in scales above the free-streaming scale. Halos at these scales do not form in a clean hierarchical fashion and may not virialize before merging with or being accreted by other halos, which may influence the distribution of substructure. In short, structure formation at the smallest scales in the CDM hierarchy may be qualitatively different from structure formation via hierarchical clustering that occurs at galactic scales.

There are hints in previous studies that substructure does have a spectral dependence. Diemand et al. (2006) showed substructure in a high redshift subsolar mass object was more susceptible to tidal disruption, with as much as $20 - 40\%$ of it being disrupted within an expansion factor of 1.3 as compared to $1\%$ in a low redshift cluster halo. Springel et al. (2008) found that subgalactic subhalos
have less substructure than galactic halos when regions of the same overdensity are compared. They also presented evidence that suggests the slope of the subsubhalo mass function flattens as the subhalo host mass decreases. The scale-free simulations of Reed et al. (2005a) appear to show that the subhalo velocity distribution flattens when $n < -2$.

However, simulations of $n < -2$ cosmologies are notoriously difficult to perform due to finite volume effects (Smith et al., 2003). Missing power from modes larger than the simulation box affect statistical quantities such as the two-point correlation function and halo mass function (Bagla and Ray, 2005; Power and Knebe, 2006; and Bagla and Prasad, 2006). Provided large-scale modes have negligible amplitudes and the scale of interest is a small fraction of the box size, the internal properties of halos appear unaffected. One can also correct for deviations in statistical quantities, though these corrections become increasingly large as $n \rightarrow -3$. Generally, it is preferable to simply halt a simulation before finite volume effects become an issue. We revisit the criterion presented by Smith et al. (2003) that ensures the finite volume effects are negligible and show that it has not been met in the prior simulations of scale-free power spectra with $n < -2$.

We run a pair of scale-free Einstein-de Sitter simulations in order to quantify the spectral dependence of the subhalo mass function. As most studies focus on clusters and galaxies, where $n_{\text{eff}} \approx -1.8$ and $-2.1$ respectively, we choose spectral indices of $n = -1$ and $n = -2.5$ to bracket these scales in the CDM hierarchy. Finite volume effects are carefully considered in choosing the end point of our simulations. We search all large halos for subhalos and vary the parameters of the group finding algorithm in order to quantify systematics. Our results show that subhalo mass function does depend on $n$, in contrast to previous results, and is sensitive to the parameters used to find subhalos.

Our paper is organized as follows: In section Section 4.2, we discuss particle number requirements and the difficulties with simulating $n \rightarrow -3$ cosmologies. The initial conditions and cosmological parameters used along with the technical details of the simulations are presented in Section 4.3. In Section 4.4, we compare the halo mass distribution from our simulations with theoretical models. In Section 4.5, we present our results for the properties of subhalos and the dependence of
subhalos on the spectral index. The paper concludes in Section 4.6 with a summary and discussion.

### 4.2 Particle requirements

Simulations are a compromise between two competing goals: good statistics, which favours the use of a physically large box, and high resolution, which promotes the use of a small box to achieve high resolution of individual objects. Since substructure is sensitive to the softening length used in simulations the second of these issues tends to dominate the choice of box size. Indeed, substructure in halos composed of $\lesssim 10^5$ particles tends to evaporate due to numerical softening effects (e.g. [Moore et al., 1999] and [Klypin et al., 1999]).

Choosing between a small box with a resolution high enough to resolve small-scale structure while still keeping large-scale modes in the linear regime is problematic. Simulations must be halted before the large-scale modes are nonlinear so as to ensure that mode-mode coupling is correctly represented, and that the amount of power due to missing large-scale modes is negligible. The dimensionless power spectrum is

$$\Delta^2(k) = \frac{V}{(2\pi)^3} 4\pi k^3 P(k), \quad (4.1)$$

where $V$ is the normalization volume and $P(k)$ is the power spectrum. In what follows, $P(k)$ and $\Delta^2(k)$ refer to the power spectrum evolved according to linear theory. We assume a scale-free initial power spectrum so that

$$P(k) = a^2 A k^n, \quad (4.2)$$

where $A$ is the amplitude and $a$ is the cosmological scale factor. Modes are considered to be linear if $\Delta^2(k) \ll 1$ and nonlinear if $\Delta^2(k) \geq 1$ with the nonlinear scale defined by the relation $\Delta^2(k_{NL}) = 1$. The effective index of the power spectrum is

$$n_{\text{eff}}(k) \equiv \frac{d \ln \Delta^2}{d \ln k} - 3. \quad (4.3)$$

For a simulation of size $L$ with $N^3$ particles, the power at a given $k$ is related to the power at the
Figure 4.1 Number of particles required, $N_{\text{req}}^3$ versus spectral index $n$ based on Eq. (4.7) (top) and Eq. (4.10) (bottom) for a variety of constraints on the linearity large-scale modes and the nonlinearity of modes at the Nyquist scale. The constraints for the top panel are $(\Delta^2(k_{\text{Ny}}), \sigma_{\text{miss}}^2) = [(2, 0.04), (1.0, 0.1), (0.5, 0.2)]$ in solid, dashed and dotted curves respectively. The bottom panel indicates the requirements for halos of $10^5$ particles to be $1\sigma$, $2\sigma$ and $3\sigma$ peaks in solid, dashed and dotted curves respectively with the box mass being a $5\sigma$ peak. Filled stars indicate the resolution and spectral index of the two simulations we ran. For comparison, also shown are scale-free simulations from Knollmann et al. (2008) in open crosses, Reed et al. (2005a) in open squares, and Jain and Bertschinger (1998) in open triangles. The effective resolution and effective spectral index of the CDM power spectrum from the Nyquist scale up to $k_{\text{halo}} = (4\pi / 3 M)^{1/3}$ for Diemand et al. (2006, 2007) are shown in thin and thick horizontal lines respectively.
Nyquist wavenumber, $k_{Ny} = \pi N/L$, by

$$\Delta^2(k) = \Delta^2(k_{Ny}) \left( \frac{k}{k_{Ny}} \right)^{3+n}. \quad (4.4)$$

Thus, the power at the box scale $k_b = 2\pi/L$ is

$$\Delta^2(k_b) = \Delta^2(k_{Ny}) \left( \frac{2}{N} \right)^{3+n}. \quad (4.5)$$

We can use this definition to define an end point for a given simulation by requiring that the mode at the box scale is still linear.

The impact of modes larger than the box can be quantified by considering missing variance, that is the difference between the variance integral for an infinite universe and the finite sum over modes within the box. Smith et al. (2003) show that the missing variance for scale-free power spectra is well approximated by

$$\sigma_{\text{miss}}^2 = \Delta^2(k_b) \frac{\sigma_{\text{miss}}^2}{F(3+n)}, \quad (4.6)$$

where $F(x) = 1 - 0.31x + 0.015x^2 + 0.00133x^3$ for $-3 \leq n \leq 1$. As $n \to -3$, $\sigma_{\text{miss}}^2 \to \infty$, that is, missing power plays an increasingly important role. Ideally, one wants $\sigma_{\text{miss}}^2 \to 0$, though so long as $\sigma_{\text{miss}}^2 \ll 1$ the simulation will not suffer significantly from finite volume effects and the large-scale modes will still be linear. Smith et al. (2003) use $\sigma_{\text{miss}}^2 < 0.04$ in their study of the nonlinear evolution of the power spectrum, which we also use as our strictest limit. However, there is not a precise value of $\sigma_{\text{miss}}^2$ that guarantees some level of accuracy will be achieved for any given realization.

We can combine a desired nonlinearity at a given scale and our constraint on $\sigma_{\text{miss}}^2$ to obtain a minimum requirement on the number of particles used in a simulation. The minimum number of particles required to achieve a level of nonlinearity at the Nyquist scale, $\Delta^2(k_{Ny})$ is

$$N_{\text{req}}^3 = 2^3 \left[ \frac{\Delta^2(k_{Ny})}{\sigma_{\text{miss}}^2} \frac{F(3+n)}{3+n} \right]^{3/(3+n)}. \quad (4.7)$$

A useful choice is $\Delta^2(k_{Ny}) > 1$ since this ensures that the simulation is in the nonlinear regime. It is evident from this equation that for $\Delta^2(k_{Ny})/\sigma_{\text{miss}}^2 > 1$, the minimum number of particles rises super-exponentially as $n \to -3$. 


Another measure of nonlinearity is the mass variance, $\sigma^2(M)$, given by

$$
\sigma^2(M) = \frac{V}{(2\pi)^3} \int P(k)|\hat{W}(kR)|d^3k,
$$

where $\hat{W}(x) = (3/x^3)(\sin x - x \cos x)$ is the Fourier transform of the top-hat window function, $R = (3M/4\pi\rho_{bg})^{1/3}$ and $\rho_{bg}$ is the background mass density. A mass scale is linear if $\sigma(M) < \delta_{sc}$, where $\delta_{sc}$ is the critical density for spherical collapse, and nonlinear if $\sigma(M) > \delta_{sc}$. The characteristic mass scale $M_*$ is defined by the relation $\sigma(M_*) = \delta_{sc}$. We require $\sigma(M_{\text{box}}) \ll \delta_{sc}$, where $M_{\text{box}} = \pi N^3/6$ is the mass enclosed in the largest sphere contained in the simulation volume.

As $\sigma(M)$ is a statistical quantity, the criterion that $\sigma(M_{\text{box}}) = \delta_{sc}/\nu$, where $\nu$ is the height of density field in rms units, is a probabilistic one. For example, with $\nu = 2$ and Gaussian initial conditions, 4.5% of an ensemble of simulations would be nonlinear at the box scale. We choose $\nu_{\text{box}} \sim 5$, ensuring that the probability that the box mode is nonlinear is $\sim 10^{-6}$. Note that the effective index at a given mass scale is

$$
n_{\text{eff}}(M) = -3 \frac{d\ln \sigma^2(M)}{d\ln M} - 3.
$$

Using the mass variance, one can impose evolutionary criteria to calculate the minimum number of particles required. Generally, studies focus on halos of a particular mass scale, $M$, which correspond to density peaks above some threshold $\nu$. The minimum number of particles required to investigate halos originating from $\nu_{h}\sigma$ peaks of composed of $N_h$ particles is:

$$
N_{\text{req}}^3 = \frac{6N_h}{\pi} \left( \frac{\nu_{\text{box}}}{\nu_{h}} \right)^{6/(3+n)}.
$$

As $n \to -3$, the mass variance becomes independent of scale and the minimum number of particles again rises super-exponentially.

We demonstrate the relative importance of these constraints in Figure[4.1] on page[78] by plotting $N_{\text{req}}^3$ as a function of $n$. The curves indicate the minimum number required for a given set of constraints, with the constraints becoming relaxed as one goes from solid to dashed to dotted. To be representative of a given cosmology and spectral index, simulations must lie above these curves. Note that the number of particles needed to attain a highly evolved simulation $\Delta^2(k_{N_{\nu}}) \gtrsim 1$ while
still limiting $\sigma_{\text{miss}}^2 \leq 0.04$ for $n < -2$ becomes increasingly impractical. For very negative indices, a more reasonable goal is evolving the simulation till the Nyquist scale is just nonlinear while $\sigma_{\text{miss}}^2 \sim 0.10$. Again, we stress that though our choices of $\sigma_{\text{miss}}^2$ and $\nu_{\text{box}}$ are meant to be conservative, these criteria do not guarantee that our study is free of finite volume effects. Simulations are always missing power and this missing power, even if small, can subtly affect the evolution of the system as shown in Widrow et al. (2009). A rigorous examination of finite volume effects on the subhalo population will require further study.

The missing variance is not as large in $\Lambda$CDM as it is for scale-free power spectra due to the turnover in the power spectrum but still goes as $\Delta^2(k_b)$. This figure clearly shows the difficulties with modelling the bottom of the CDM hierarchy where $n_{\text{eff}} \approx -2.8$. The highest resolution simulation of SUSY halos by Diemand et al. (2006) has $n_{\text{eff}} \approx -2.85 \pm 2.75$. As they examine the substructure of a very rare $3.5\sigma$ peak and halt their simulation early enough, the missing variance does not exceed $\approx 0.05$. However, any extrapolations based on a $3.5\sigma$ peak should be treated with caution.

Equations (4.7) and (4.10) can also be rearranged to determine when to halt a simulation for a given $\sigma_{\text{miss}}^2$ or $\sigma^2(M_{\text{box}})$. Since we are interested in substructure, we focus on $\gtrsim 10^5$ particles for $n = -2.5$ and $N^3 = 720^3$, halting the simulation when $\sigma(M_{\text{box}}) \approx \delta_{\text{sc}}/4.9$ means that these halos originate from rare $\sim 2.5\sigma$ peaks. If our $n = -1$ simulation is evolved to the same $\sigma(M_{\text{box}})$, the halos composed of $10^5$ particles would originate from common $\sim 0.4\sigma$ peaks. Using $\sigma^2(M) \propto a^2$, it is a simple matter to calculate how many e-foldings the simulation must be evolved to achieve this desired end state.

### 4.3 Numerical methods

#### 4.3.1 How to interpret scale-free simulations

The varying slope of the $\Lambda$CDM power spectrum is key to interpreting scale-free simulations in the context of the CDM hierarchy. In Figure 4.2 on the next page, we show the effective spectral
Figure 4.2 Effective spectral index of the CDM power spectrum versus mass (solid) and wavenumber (dashed). Highlighted in thick lines are the wavenumbers and mass scales sampled in simulations such as the Via Lactea simulation and the Aquarius Project. Various mass scales in the CDM hierarchy are indicated by filled circles, with the largest mass corresponds to the an MW halo \( (10^{12} \, \text{M}_\odot) \) and the other two points indicating the largest and smallest galactic subhalo masses examined in previous studies, \( 10^{10} \, \text{M}_\odot \) and \( 10^6 \, \text{M}_\odot \) respectively. Also shown are the indices of our \( n = -1 \) and \( n = -2.5 \) simulations (dotted) with the filled circles indicating masses of \( 5 \times 10^5 \), \( 10^5 \) and 100 particles corresponding to the mass of halos, largest subhalos and smallest subhalos examined.
index as a function of wavenumber and mass based on the ΛCDM power spectrum of \cite{Hu and Eisenstein 1998}. Highlighted are scales sampled in the “Via Lactea” (VL) simulation along with certain mass scales. Most numerical studies focus on clusters and galaxies, corresponding to \( n_{\text{eff}} \approx -1.8 \) to \(-2.2\). Also shown are the indices of our simulations, where the scale of the box size is set to match the scale at which the CDM power spectrum has the same index. Our choice of indices is meant to bracket galaxy and cluster scales. In essence, halos found in an \( n = -2.5 \) simulation are representative of high redshift halos surrounding the first protogalaxies. Objects at this scale may also become the larger subhalos found in galaxies. Though the actual power spectrum is steeper at these scales with \( n_{\text{eff}} \approx -2.7 \), we satisfy ourselves with \( n = -2.5 \) as it is the steepest spectra we can reasonably simulate, though even at this index we are pushing the limits outlined in Section 4.2.

### 4.3.2 Simulations

We run scale-free Einstein-de Sitter (\( \Omega_m = 1 \)) cosmological simulations with power-law power spectra as shown in Eq. (4.2). We choose an amplitude and initial scale factor so that the maximum initial displacement for any given particle is less than 1/2 the initial inter-particle spacing. Due to the random nature of the initial density field this choice does not correspond precisely to a specific amplitude requirement across all power spectra, but ensures that the simulation starts in the linear regime. We draw both simulations from the same random realization and normalize the amplitudes so that \( \sigma(M = 15 \times 10^6 \text{ particles}, a = 1.0) = 0.9 \), which is equivalent to normalizing using \( \sigma_8 \). A final issue concerns transients in the density and velocity field generated by initial conditions generator. To minimize the effect of transients, we use initial conditions generated by
second-order Lagrangian perturbation theory (2LPT) rather than the standard Zeldovich approximation (ZA) \cite{Crocce2006}. The simulations are run using the parallel N-body tree-PM code GADGET-2 \cite{Springel2005}. We halt the $n = -2.5$ simulation when halos composed of $\gtrsim 10^5$ particles originate from $\gtrsim 2.5\sigma$ peaks and the box scale corresponds to a rare $4.9\sigma$. Evolving farther in order to form larger halos and further reduce numerical softening effects is not possible since, even at our chosen end point, we are pushing the limits with $\sigma_{\text{miss}}^2 \approx 0.10$. We also analyze the $n = -1$ simulation when $\gtrsim 2.5\sigma$ halos are composed of $\gtrsim 10^5$ particles for the sake of consistency, though $\sigma_{\text{miss}}^2 \approx 0.001$ at this point. A summary of the simulations is given in Table 4.1 on the preceding page.

### 4.4 Halos

We identify halos at each time step using a friends-of-friends (FOF) group finder with a linking length $\ell_h = 0.2$ times the comoving interparticle spacing $\Delta x = L/N$ \cite{Davis1985}. Only FOF groups with more than 32 particles are kept in the halo catalogue. The comoving halo number density is shown in Figure 4.3 on the next page at three different redshifts along with theoretical predictions from Sheth and Tormen \cite{Sheth1999} (ST) and Press and Schechter \cite{Press1974} (PS). The PS mass function is based on the spherical collapse model while the ST incorporates ellipsoidal collapse with additional mass function parameters calibrated using large-scale $\Lambda$CDM simulations. For both simulations, the ST mass distribution is in better agreement than PS for all redshifts.

### 4.5 Substructure

#### 4.5.1 Searching for subhalos

Identifying subhalos is more difficult than identifying halos. The key issue with defining the outer boundary of a subhalo embedded in a gravitationally bound object. Several methods have been developed to search for substructure, including SKID \cite{Stadel2001} and SUBFIND \cite{Springel2005}.
Figure 4.3 The co-moving halo number density of mass $M$, $dn(M, z)/d\ln M$ denoted by filled points with 1σ error bars at $n = -1$ (top) and $n = -2.5$ (bottom). Curves are predictions from analytic fitting functions proposed by ST (solid) and PS (dashed). Going from right to left corresponds to going to progressively lower redshifts with right panel corresponding to the final redshift analyzed.
In this paper, we use a phase-space FOF (6DFOF) algorithm, which is similar to the one described in Diemand et al. (2006). We limit our subhalo search to halos originating from $\gtrsim 2.5\sigma$ peaks, constraining our analysis to 22 halos composed of $\gtrsim 3.2 \times 10^5$ particles and 29 halos composed of $\gtrsim 1.2 \times 10^5$ particles for the $n = -1$ and $n = -2.5$ simulation respectively. Each halo is associated with a velocity scale $\Delta v = (GM_h/R)^{1/2}$ where $R^3 = M_h/(4\pi \rho_{bg} \ell_h^{-3}/3)$, $M_h$ is the mass of the halo found using a linking length of $\ell_h$ and $\rho_{bg}$ is the background density. Two particles with phase-space coordinates $(x_1, v_1)$ and $(x_2, v_2)$ are linked if

$$\frac{(x_1 - x_2)^2}{(\ell_s \Delta x)^2} + \frac{(v_1 - v_2)^2}{(b_v \Delta v)^2} < 1,$$

where $\ell_s$ and $b_v$ are the dimensionless parameters defining the physical and velocity linking lengths respectively. We pass candidate subhalos through an unbinding routine that checks whether an object is self-bound and removes any unbound particles (Springel et al. 2008). Only subhalos with 20 particles or more are kept in the catalogue. This unbinding routine is a necessary but time consuming process that removes unbound particles from subhalo candidates and also eliminates spuriously linked particles. The fraction of unbound candidates in the initial 6DFOF catalogue varies slightly with $b_v$ and is $\approx 0.05$ and $\approx 0.70$ for the $n = -1$ and $n = -2.5$ simulation respectively.

We set $\ell_s = 0.10$, thus limiting our search to regions within halos with physical overdensities of $\rho/\rho_{bg} \gtrsim 1000$, conversely we try several velocity linking lengths, $b_v = [0.0125, 0.025, 0.05, 0.10]$. For an isolated spherical overdensity, increasing $b_v$ amounts to placing a higher circular velocity cutoff in defining the boundary of a candidate subhalo. The mass associated with a phase-space peak could continuously increase with increasing $b_v$ if not for our unbinding routine, which effectively imposes a tidal limit on candidates. In such a case, our technique is analogous to that used by Diemand et al. (2007). They found phase-space peaks using the 6DFOF algorithm and then estimated a tidal mass by assuming spherical symmetry and fitting an NFW profile (Navarro et al. 1997) plus background to circular velocity profiles of the peaks. Since our method does not impose a particular density profile, we can indirectly examine the validity of spherical symmetry as $n \to -3$ and any systematic bias this assumption introduces in the resulting subhalo mass function by varying $b_v$. 
In Figure 4.4 on the following page, we show different representations of two large halos, one from each of our simulations. The two halos originated from $\sim 3\sigma$ peaks and are composed of $\sim 6 \times 10^5$ particles. For each halo we show the phase-space density calculated using EnBiD (Sharma and Steinmetz 2006) and the subhalo candidates found using two different velocity linking lengths. This figure demonstrates that there are significant substructure differences between the simulations. Substructure at $n = -2.5$ appears more triaxial and less dense than at $n = -1$ and, overall, the $n = -1$ halo looks very similar to a cluster or galaxy halo. At $n = -1$, substructure consists of distinct spherical overdensities with well defined boundaries, whereas substructure is less evident at more negative indices, (see Diemand et al. 2006). Of the 525 subhalo candidates found using $b_v = 0.05$, 464 are bound and contain a total of $\approx 20\%$ of the host halo’s mass, though no single subhalo contains more than $5\%$ of the halo’s mass. In contrast, the $n = -2.5$ halo has 285 candidates, of which only 21 are bound. These bound subhalos contain $2\%$ of the halo’s mass.

Comparing the impact of different velocity linking lengths $b_v$, we find that at $n = -2.5$, using $b_v \geq 0.05$ appears to spuriously link phase-space peaks into long, unbound, filamentary structures, which are subsequently eliminated from the catalogue by our unbinding routine. This is the case in the central region outlined by a solid black circle. However, not all peaks are linked into artificial filaments. Occasionally, peaks are close enough to one another in phase-space and are linked by more than a few particles. Two candidate subhalos, which are split at small $b_v$, are linked at larger $b_v$ into a bound triaxial object. An example of such is circled in a black dashed circle in Figure 4.4 on the next page. The dotted circle indicates a subhalo that is non-spherical regardless of $b_v$ used. These results visually demonstrate that the boundary of subhalos appears less well defined as $n \rightarrow -3$.

### 4.5.2 Phase-space structure of halos

The upper panels of Figure 4.4 on the following page provide the phase-space density as a function of position. Of course the phase-space distribution $f(x, v)$, which provides a complete description of a dynamical system, is a function of six variables, making it cumbersome to deal with. As an alternative, we examine halo phase-space structure using the volume distribution function $v(f)$,
Figure 4.4 Project in x-y plane of two halos originating from $\approx 3\sigma$ peaks in the $n = -1$ (left) and $n = -2.5$ simulation (right) composed of $6.2 \times 10^5$ and $5.5 \times 10^5$ particles respectively. The first row shows the phase-space density of the halos using a logarithmic colour scale, where dark blue regions indicate high phase-space density. In the next two rows, particles are colour coded according to group number found with 6DFOF. Shown are subhalo candidates found with $b_v = 0.05$ and $0.0125$ in the middle and bottom rows, respectively. Three regions are circled in the $n = -2.5$ simulation to highlight the variation in particles linked by the 6DFOF algorithm with different $b_v$ (see discussion in text).
which is the volume of phase-space occupied by phase-space elements of density $f$ in an interval $df$ \cite{Arad2004}. In Figure 4.5 on the next page, we plot $v(f)$ (normalized so that $\int v(f) df = 1$) along with the logarithmic slope, $\gamma_v \equiv d \ln v(f)/d \ln f$ for four halos with $\gtrsim 3 \times 10^5$ particles. Halos from the $n = -1$ simulation have larger volumes with high phase-space density and span a greater range in $f$ than the halos from the $n = -2.5$ simulation. At $n = -1$, $\gamma_v$ decreases to $\approx -2.5$ with increasing $f$ followed by a bump, and has a very similar form to that of $\Lambda$CDM galactic and cluster halos examined by \cite{Sharma2006}. The slope of halos at $n = -2.5$ is shallower at low $f$ and plateaus around a slope of $-2.5$. Several authors have attempted to explain the form of $\gamma_v$ for $\Lambda$CDM halos using a toy model where halos are modelled by a Hernquist sphere that contains smaller Hernquist spheres representing subhalos \cite[e.g.][]{Arad2004, Ascasibar2005, Sharma2006}. \cite{Sharma2006} found that the general shape of a $\Lambda$CDM galactic halo with substructure can be reproduced with these toy models and that the size of the bump appeared to be related to the amount of substructure.

To examine whether the size of the bump in the slope is related to the phase-space density contrast between substructure and the background, we smooth out substructure in the $n = -1$ halo shown in Figure 4.4 on the preceding page. To smooth the halo we first calculate its morphology via the inertia tensor (see below) at different radii. Particles are then moved in a random direction through a small random angle on the surface of the ellipsoid determined by the mass distribution interior to their radii. This simple process leaves the radial density profile and overall morphology unchanged to within $\lesssim 1\%$ while reducing the density contrast of substructure, thereby decreasing the number of candidates and bound subhalos by a factor of $2-3$ and $8-60$ respectively. The result is shown in the right column of Figure 4.5 on the next page. We find that the size of the bump is influenced by both the amount and phase-space density contrast of substructure. The lack of a bump in the halos from the $n = -2.5$ simulation shows that the contrast between substructure and the smooth background of a halo decreases as $n \to -3$. The differences in $\gamma_v$ do not arise solely due to differences in halo morphology as the halos in either simulation have similar morphologies. Thus, the logarithmic slope of $v(f)$ depends on the amount of substructure and hence the differences
Figure 4.5 The normalized volume distribution function \( v(f) \) (top) and logarithmic slope \( \gamma_v \) (bottom) for 4 halos consisting of \( \gtrsim 3 \times 10^5 \) particles. The left and middle columns correspond to \( n = -1 \) and \( n = -2.5 \) simulation respectively. The solid lines corresponds to the halos shown in Figure 4.4 on page 88, the other line types are ordered in decreasing halo mass going from dashed, dotted to dashed-dotted. The right column correspond to a \( n = -1 \) halo where the substructure has been smoothed with increasing smoothing going from dashed to dotted to dashed-dotted and the reference halo denoted by the solid curve.

The seen in Figure 4.5 are suggestive that there may be a spectral dependence in both the amplitude and the slope of subhalo mass function.

### 4.5.3 Morphology

To determine the morphology of a subhalo we follow Dubinski and Carlberg (1991) and Allgood et al. (2006) and diagonalize the weighted moments of inertia tensor

\[
\tilde{I}_{i,j} = \sum_n \frac{x_{i,n}x_{j,n}}{r_n^2}.
\]
The ellipsoidal distance between the subhalo’s centre of mass and the \( n \)th particle is

\[
    r_n^2 = x_n^2 + (y_n/q)^2 + (z_n/s)^2,
\]

where \( q \) and \( s \) are the intermediate-to-major and minor-to-major axis ratios respectively. The axis ratios are calculated after unbound particles are removed for subhalos composed of \( > 100 \) particles.

Figures 4.6-4.7 show the distribution of subhalos in terms of \( q \) and \( s \). We see that subhalos are generally more triaxial at \( n = -2.5 \) compared to \( n = -1 \) and the distribution of axis ratios is substantially broader. At both indices, the distribution remains relatively unchanged as \( b_v \) is varied, save for the changes in the number of filamentary objects with \( q, s \lesssim 0.1 \). These filamentary objects are elongated subhalos in the process of being tidally disrupted with loosely bound tidal tails. These filaments comprise \( \lesssim 1\% \) of the subhalo population for \( b_v \leq 0.05 \) at \( n = -1 \) and for \( b_v \leq 0.025 \) at \( n = -2.5 \), and increase to 20\% for \( b_v = 0.10 \) at both indices. This filamentary population decreases by roughly a factor of \( \approx 10 \) and \( \approx 5 \) as \( b_v \) is halved at \( n = -1 \) and \( n = -2.5 \) respectively, though the trend is not as clear cut at \( n = -2.5 \). The main reason for this dependence on \( b_v \) is that tidal tails are more likely to be linked with the parent subhalo when using large \( b_v \). The decrease in the number of subhalos as \( b_v \) is increased at \( n = -2.5 \) indicates that, as \( n \rightarrow -3 \), local phase-space peaks become less well separated in phase-space and more likely to be embedded in a triaxial overdensity. The increasingly triaxial or filamentary nature makes assigning a mass to a phase-space peak non-trivial. Estimates of a tidal mass assuming spherical symmetry will become increasingly incorrect as the distribution of mass around a phase-space peak becomes increasingly triaxial. These results are insensitive to the minimum subhalo particle number cut applied. Again, the differences in morphology may be indicative of a break in the “universality” of the subhalo mass function.

### 4.5.4 Mass function

We define the dimensionless subhalo mass ratio \( M_f \equiv M_{\text{subhalo}}/M_{\text{halo}} \) and show in Figure 4.8 on page 95 the cumulative subhalo mass function \( N(> M_f) \) summed over all halos in our analysis.
Figure 4.6 Scatter plot of major and minor axis ratios, $q$ and $s$, for subhalos found in the $n = -1$ simulation. Also shown are the projections of these distributions. The grey open circles, squares, triangles and upside down triangles and solid, dashed, dotted, and dashed-dotted lines correspond to $b_v = 0.0125$, $0.025$, $0.05$ and $0.10$ respectively. For clarity we only show a random subsample (10%) and plot contours for $b_v = 0.0125$ and $b_v = 0.05$. Filled points indicate the peaks of the histograms and follow the same marker scheme as the scatter plot. The thin dotted line corresponds to $q = s$. 
Figure 4.7  Same as Figure 4.6 on the preceding page but for the $n = -2.5$ simulation.
We also show the logarithmic slope $\alpha(M_f) \equiv \frac{d \ln (dN/d \ln M_f)}{d \ln M_f}$. Since $dN/d \ln M_f$ is quite noisy, we calculate several logarithmic slopes at a given $M_f$ by varying the spacing used in the five point central difference estimate of the slope. Shown in Figure 4.8 on the following page is the average slope along with the standard deviation. In both simulations, $N(> M_f)$ is linear in the log-log plot provided we exclude low and high mass regions dominated by numerical effects. The flattening in the low mass region corresponds to subhalos composed of fewer than 100 particles and is due to numerical softening effects. The high mass scale, where the distributions begins to steepen, scales roughly as $b_v^3$. This region results from excluding the outer high velocity volumes of a candidate subhalo and the scaling can be understood as follows: $v^2 \propto M/R$, $R \propto M^{1/3}$, thus $v^3 \propto M$. Hence, the more massive the subhalo the larger $b_v$ must be to group particles out to the subhalo’s tidal radius. Since we pass subhalo candidates through an unbinding routine, which does not take into account the background, the loosely unbound central regions of large subhalos will be removed.

In the $n = -1$ simulation, the number of subhalos is very weakly dependent on $b_v$. The 6DFOF algorithm using $b_v = 0.10$ cannot separate subhalos in the central region from the core and underestimates the number of subhalos. The small changes in the number found with $b_v \leq 0.05$ are due to the fact that the 6DFOF algorithm using these velocity linking lengths tends to only link the central regions of subhalos with mass ratios of $M_f \gtrsim 0.01$, underestimating their mass. In some cases, the algorithm only groups the loosely unbound centres of these subhalos and they are subsequently removed by our unbinding routine. However, this does not greatly affect the number since there are few subhalos above the high mass limit imposed by $b_v$. The fraction of mass bound in subhalos with $10^{-4} \leq M_f \leq 5 \times 10^{-3}$ is weakly dependent on $b_v$, decreasing from 0.127 to 0.085 when going from $b_v = 0.0125$ to $b_v = 0.10$. This fraction varies from halo-to-halo by an average of $\approx 10\%$ for $b_v \leq 0.05$ and increases to $\approx 20\%$ for $b_v = 0.10$ with no strong dependence on peak height of the halo. The slope of the subhalo mass function is also unaffected by changes in $b_v$ and, neglecting the regions dominated by numerical effects, is consistent with previous results, as indicated by the grey region in Figure 4.8 on the following page. Even between the catalogues found using $b_v = 0.10$
Figure 4.8 The cumulative subhalo mass function $N(> M_f)$ (top) and logarithmic slope $\alpha$ (bottom) from all halos used in our analysis. The line styles and marker scheme are the same as in Figure 4.6 on page 92. Also shown in the top panel by the solid thin black line is $N(> M) \propto M^{-0.9}$. The solid grey region outlined by black dashed lines in the bottom panel indicates the various slopes found in other studies with the solid horizontal line denoting $\alpha = -0.9$. 
and $b_v \leq 0.05$, where half as many subhalos are found and the fraction of filaments goes from 0.2 to $\lesssim 0.01$, the slope is only shallower for mass ratios of $M_f \lesssim 2 \times 10^{-4}$.

The subhalo distribution at $n = -2.5$ is dependent on $b_v$. The number of subhalos increases as $b_v$ decreases and appears to converge, though we note that few if any bound subhalos are found using $b_v < 0.0125$. The increase in the number found is due to bound filamentary objects found at a given $b_v$ being broken into several smaller triaxial subhalos at smaller $b_v$, as shown in Fig. 4.4 & 4.6-4.7. For example, comparing the subhalos found using $b_v = 0.025$ to those found using $b_v = 0.0125$, the number of filaments decreases from 1.7% to 0.3% of the population but the number of subhalos increases by a factor of 1.6. The apparent convergence is due to there being a finite number of phase-space peaks with local physical overdensities above 1000. The fraction of mass in subhalos between $10^{-4} \leq M_f \leq 5 \times 10^{-3}$ shows a strong dependence on $b_v$, decreasing from 0.035 to 0.0052 when going from $b_v = 0.0125$ to $b_v = 0.10$. In general, the fraction of mass bound in subhalos is a factor of $\sim 3 - 20$ smaller than at $n = -1$. The relative halo-to-halo variation is also greater, with the fraction varying by 26% and 96% using $b_v = 0.0125$ and $b_v = 0.10$ respectively. The slope increases as $b_v$ increases and exhibits fluctuations as a function of $M_f$ that are greater than those at $n = -1$. Generally, $\alpha \gtrsim -0.9$ within the region free of numerical effects. Given the changes in the number of subhalos found, the changes in $\alpha$ could be dismissed as being purely numerical. However, similar changes in the subhalo population are observed at $n = -1$ going from $b_v = 0.10$ to $b_v = 0.05$ yet $\alpha$ is unaffected, whereas at $n = -2.5$ $\alpha$ increases by $\approx 20\%$ when going from $b_v = 0.05$ to $b_v = 0.025$ or $b_v = 0.025$ to $b_v = 0.0125$. These changes in $\alpha$, while being partly numerical, are nonetheless indicative of the underlying physical issue of defining the boundary of a subhalo.

To compare with previous work, we fit a power-law,

$$N(> M_f) = A M_f^\alpha,$$

(4.14)

to the cumulative subhalo mass function, neglecting the low and high mass regions which are dominated by numerical effects. The mass scale above which subhalos are artificially truncated due to
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$b_v$ is very clear at $n = -1$ but not as clear at $n = -2.5$, so we use the $n = -1$ simulation for guidance in defining the range in $M_f$ fitted. We plot the mean $\alpha$ as a function of the spectral index for different $b_v$ in Figure 4.9 on the next page. At $n = -1$, $\alpha \approx -0.9$ independent of the choice of velocity parameter in agreement with previous results from $\Lambda$CDM simulations. Examining the four largest halos in the $n = -1$ simulation, we find the halo-to-halo variation in the fitted index is 0.07. Unlike at $n = -1$, the index at $n = -2.5$ exhibits a strong dependence on $b_v$, though it has similar a halo-to-halo variation of 0.06. We can reproduce $\alpha \approx -0.9$ by choosing $b_v \approx 0.0125$, though in general $\alpha \gtrsim -0.9$. Increasing $b_v$ from 0.0125 to 0.025, that is applying higher velocity cutoffs and then imposing a tidal limit, does flatten the slope by 22%. These results indicate that the slope of the subhalo mass function depends on $n$, becoming shallower but more sensitive to systematics as $n \rightarrow -3$.

To check for any redshift dependence, we examine the evolution of a subset of our halos over 0.8 and 0.5 $e$-foldings in expansion factors for $n = -1$ and $n = -2.5$ respectively. We also compare halos between the two simulations at similar redshifts. However, due to the redshift and mass dependence of $\nu$, we cannot at the same time continue to compare similar mass halos originating from similar $\sigma$ peaks. We find that subhalos do not have a strong systematic redshift dependence in either cosmology, thus the differences between the simulations remain unchanged. At $n = -1$, $\alpha \approx -0.9$ regardless of redshift and at $n = -2.5$ the value of $\alpha$ and its dependence on $b_v$ remains unchanged. It is interesting to note that $\alpha$ does vary randomly with redshift by $\approx 0.05$ and $\approx 0.06$ for $n = -1$ and $n = -2.5$ respectively. This variation is of similar size to the halo-to-halo variation seen, which for the subset studied spans a peak height range of $1 \lesssim \nu \lesssim 4.5$ and $2.2 \lesssim \nu \lesssim 3.3$ for $n = -1$ and $n = -2.5$ respectively.

4.6 Discussion and Conclusion

One of the most intriguing results from the Moore et al. (1999) study of CDM structure is similarity of substructure in galactic and cluster halos. This result suggests that the subhalo mass distribution
Figure 4.9  Spectral dependence of $\alpha$. Filled stars, circles, squares and triangles indicate slopes from subhalos found using $b_v = 0.0125$, 0.025, 0.05 and 0.10 respectively. Marker scheme is the same as in Figure 4.6 on page 92. The grey region and black lines indicate the same slopes as in Figure 4.8 on page 95 at $n_{\text{eff}} = -1.8$ to $-2.2$, which correspond to cluster scales down to galaxy scales in the CDM hierarchy. Error bars correspond to the average variation in the logarithmic slope seen in bottom panel of Figure 4.8 on page 95 within the region fitted by the power-law.
is independent of parent halo mass, and possibly independent of the shape of the primordial power spectrum. At first glance, this similarity may seem surprising since galaxy and cluster halos differ by two orders of magnitude in mass. However, galaxies and clusters probe similar effective spectral indices, \( n_{\text{eff}} \approx -2.2 \) as compared to \( n_{\text{eff}} \approx -1.8 \), and it is the effective index that governs the structure formation process via the relative formation times and merger rates of progenitors.

To explore this issue, we have analyzed substructure in Einstein-de Sitter cosmologies at two widely separated indices, \( n = -1 \) and \(-2.5\), which bracket the indices most often examined in \( \Lambda \)CDM simulations. In the spirit of Smith et al. (2003) we have outlined a set of criteria to minimize the numerical effects which plague simulations of \( n < -2 \) cosmologies. We chose \( n = -2.5 \) as it is the most negative index we can simulate that satisfies our criteria, albeit just barely. Another reason for choosing \( n = -2.5 \) is that it represents a transitional index between hierarchical formation and the singular, perhaps pathological, case of \( n = -3 \) where structures collapse simultaneously. We evolved our simulations until there were halos large enough to analyze for substructure, while at the same time not being too significantly affected by finite volume effects. Substructure was analyzed with a number of tools: we examined the phase-space structure of halos using EnBiD, searched for candidate subhalos using a 6DFOF algorithm, and examined the morphology and mass distribution of bound subhalos. All of these methods showed that substructure in an \( n = -2.5 \) cosmology is different from substructure in an \( n = -1 \) cosmology. For example, substructure leaves a feature in the volume distribution function whose strength depends on the number of subhalos and the density contrast of substructure (Arad et al., 2004; Ascasibar and Binney, 2005; and Sharma and Steinmetz, 2006). This feature was present in our \( n = -1 \) simulation but was greatly reduced in our \( n = -2.5 \) simulation.

The differences in the morphology of bound subhalos between the two cosmologies are striking and appear to be independent of redshift. At \( n = -1 \), subhalos were typically well-defined, spherical overdensities that would lend themselves to the type of analysis outlined by Diemand et al. (2007), where the tidal mass is determined by fitting spherically averaged density profiles to phase-space peaks. At \( n = -2.5 \), subhalos were more triaxial and phase-space peaks were more irregular,
making it difficult to define a boundary. The triaxial or filamentary nature of subhalos, along with the large fraction of unbound candidates, indicates that substructure in an \( n = -2.5 \) cosmology is more susceptible to tidal disruption. These observations suggest that the methods that assume a spherical profile in order to estimate subhalo circular velocities and masses, such as those outlined in \([\text{Diemand et al.}]\,(2006, 2007)\), will become increasingly inaccurate as \( n \to -3 \) and might give misleading results.

The logarithmic slope of the subhalo mass distribution also showed a spectral dependence. At \( n = -1 \), we found \( \alpha \approx -0.9 \) independent of the velocity linking length \( b_v \) and is consistent with slopes from previous studies. At \( n = -2.5 \), we found a range of values \(-0.9 < \alpha < -0.5\) depending on our choice of \( b_v \) with \( \alpha \) tending towards less negative values with larger values of \( b_v \). The sensitivity of \( \alpha \) to \( b_v \) at \( n = -2.5 \) is due to the underlying uncertainty in determining the boundary of a subhalo. Our preferred value is \( \alpha = -0.76 \pm 0.07 \), found using \( b_v = 0.025 \), for a number of reasons: subhalos were found over several decades in \( M_f \) below the numerical mass limit imposed by \( b_v \) and above the mass scales dominated by numerical softening effects; highly filamentary subhalos made up \( \approx 1\% \) of the population; and the number of subhalos appeared to be converging. We also find that the logarithmic slope of the mass function shows no dependence on redshift, though \( \alpha \) does vary by \( \approx 5 - 10\% \) in time for a given halo. This random variation in \( \alpha \) with redshift is of similar size as the halo-to-halo variation, which shows little dependence on the peak height of the halo.

Our preferred slope at \( n = -2.5 \) appears to agree with that of the subsubhalo mass function in \( \sim 10^9 \, M_\odot \) subhalos studied by \([\text{Springel et al.}]\,(2008)\), where \( n_{\text{eff}} \approx -2.4 \). However, we note that the subsubhalo mass functions in \([\text{Springel et al.}]\,(2008)\) are quite noisy with a great deal of scatter in the observed logarithmic slope, so this agreement should be treated with caution. This value is not entirely consistent with that found in \([\text{Gao et al.}]\,(2005)\) and \([\text{Diemand et al.}]\,(2006)\), who both found results consistent with previous studies of much larger scales. Using \textsc{Subfind}, \([\text{Gao et al.}]\,(2005)\) examined substructure in a single high redshift subgalactic halo at \( n_{\text{eff}} \lesssim -2.5 \) originating from an extremely rare \( \approx 5\sigma \) peak, whereas we examined the logarithmic slope averaged over a
number of halos, albeit they examined their halo over a larger redshift range than we did. At their earliest epoch, this rare peak corresponded to a $2 \times 10^5 \, M_\odot$ halo composed of $\approx 2 \times 10^5$ particles with $n_{\text{eff}} \approx -2.7$. Though their halo sampled a more negative index then our simulation, it was an extremely rare peak that only had $\approx 35$ subhalos over a single decade in $M_f$ from $10^{-4} \lesssim M_f \lesssim 10^{-3}$. We also note that the logarithmic slope of this rare peak appears to vary by $\sim 0.1$ with redshift, though there is no systematic dependence on redshift, similar to the variation we observed. Diemand et al. (2006) also examined a single rare halo, though at an effective index slightly closer to $-3$ and at a much higher resolution than Gao et al. (2005) study. They noted that the subhalo mass function was sensitive to the parameters used to find candidates, unlike the mass function of larger halos where $n_{\text{eff}} \gtrsim -2$, in agreement with our observations, although they used SKID and we used a 6DFOF algorithm.

Our results, when combined with those of previous studies (e.g. Moore et al. 1999; Gao et al. 2004; Reed et al. 2005a; Diemand et al. 2007 and Springel et al. 2008), imply that the logarithmic slope, though relatively constant for a given host, is dependent on the host’s mass and its associated spectral index. This slope is constant with a value of $\approx -0.9$ so long as $n \gtrsim -2$ but tends to larger values and possibly with increasing scatter for $n \lesssim -2$. This transition in behaviour might be due to the structure formation process. Provided that this process is “sufficiently” hierarchical, that is halos virialize before being accreted or merging, the subhalo mass function is independent of $n$. As $n \to -3$, halos do not fully virialize before being accreted and consequently the subhalo distribution is influenced by the properties of the power spectrum. This dependence manifests itself in the boundary of a subhalo becoming increasingly ill defined as $n \to -3$; phase-space peaks become more indistinct and irregular, resulting in greater variation in the subhalo mass function while also tending to flatten it. Thus, extrapolating the subhalo mass function at galactic scales to predict the number of subhalos at the bottom of the CDM hierarchy is questionable. It is likely that at $n = -3$ subhalos become generally impossible to distinguish and, in essence, substructure becomes negligible.

The distribution and internal properties of subhalos have important ramifications for dark matter
detectors. Because the $\gamma$-ray flux varies as the local dark matter density squared, indirect detectors, such as GLAST, will see a stronger signal if there is a great deal of dense substructure. Numerous groups have calculated the boost in flux due to substructure (e.g. Strigari et al. [2007], Pieri et al. [2008] and Kuhlen et al. [2008]) but there are two issues with these estimates. The primary issue is the scale dependence of the subhalo mass function. Estimates assume the subhalo mass function observed in $10^{12} \, M_\odot$ halos at subhalo masses of $\gtrsim 10^5 \, M_\odot$ applies to the subsubhalo population and extends to the bottom of the CDM hierarchy. However, it is likely that the subsubhalo mass function of $\sim 10^8 \, M_\odot$ subhalos will have a form similar to that observed at $n = -2.5$ for $b_v = 0.025$ and that the distribution will continue to flatten as one proceeds down the CDM hierarchy. The secondary issue is the use of field halo mass-concentration relations to convert subhalo masses to densities. At small scales, their use is questionable since subhalos become increasingly triaxial and less distinct as $n \rightarrow -3$. However, this could be accounted for in the concentration-mass relation such as the one proposed by Eke et al. [2001] which attempts to account for the possible effects of $n_{\text{eff}}$. Ultimately, as a consequence of these issues, previous results should generally be considered optimistic upper limits. In light of these results we do not attempt to predict the $\gamma$-ray background here and will address this critical issue in a paper in preparation.
Chapter 5

Paper II: Can substructure in the Galactic Halo explain the ATIC and PAMELA results?

Here I present a paper co-authored with Robert J. Thacker, and Lawrence M. Widrow as it was submitted to Physical Review Letters (PRL) with a few corrections based on the referees’ reports as they add clarity. The focus of this paper is how much substructure can enhance the flux of secondaries for annihilating dark matter as outlined in Section 3.4. I use a Monte Carlo calculation technique that is constrained by previous simulation work on halo substructure to investigate the feasibility of large boost factors.

I am the primary author of this paper. I did the extensive literature review necessary to compare and combine the results from numerous other studies with my own regarding subhalos presented in Chap. 4. I also evaluated and analyzed the boost factors using code I wrote and calculated the resulting cosmic ray flux for the dark matter annihilation model discussed in this chapter.
5.1 Main Body

Recent measurements by ATIC (Chang et al., 2008) and PAMELA (Adriani et al., 2009) offer the tantalizing prospect that dark matter has been discovered, albeit indirectly. Both experiments have reported an anonymously large flux in leptonic cosmic rays at energies above 10 GeV. Dark matter candidates, such as WIMPs (Weakly Interacting Massive Particles which arise in theories of supersymmetry), or Kaluza-Klein particles, can annihilate and produce cosmic rays (Bertone et al., 2005). The flux excess might comprise secondary particles from annihilation events in the Galactic halo, a possibility that is now attracting considerable attention.

Though the observed excess flux is 100-1000 times larger than the flux predicted from a smooth Milky Way halo with a standard thermal WIMP particle, the presence of subhalos can boost the signal since the annihilation rate is proportional to the square of the density. But while there is little doubt that dark matter halos are clumpy, the actual boost factor is a matter of some debate. In this paper, we take a critical look at the underlying assumptions in boost factor calculations and discuss the implications of our analysis for the ATIC and PAMELA results as well as for future observations by GLAST (Strigari et al., 2007; Pieri et al., 2008).

Halos in cosmological N-body simulations host numerous subhalos (Moore et al., 1999), which in turn host their own subhalos (Diemand et al., 2007). The distribution of subhalos can be summarized by the mass function, $dN/d\ln f$, where $f \equiv m/M_h$ is the mass fraction of a subhalo of mass $m$ in a host of mass $M_h$. Note that in this paper, a host can refer to a virialized halo or a subhalo. The mass function appears to be well-characterized by a power-law (Gao et al., 2004; Diemand et al., 2004),

$$dN/d\ln f = Af^{-\alpha},$$

for $f < 10^{-2}$. While current simulations probe $dN/d\ln f$ for $f \gtrsim 10^{-6}$, a Galactic-mass WIMP halo should have subhalos down to $f \sim 10^{-18}$ (Martinez et al., 2009), with the possibility of up to nine nested levels of substructure.

The luminosity of annihilation secondaries (electrons, positrons, or photons) from a halo or
subhalo of mass $M_h$ can be written in the form

$$L_{e^\pm,\gamma} = \mathcal{P}_{e^\pm,\gamma} \frac{\langle \sigma v \rangle}{m_\chi} \mathcal{L}(M_h),$$

(5.2)

where $\mathcal{P}_{e^\pm,\gamma}$ are the branching ratios for electron-positron or photon secondaries, $\sigma$ is the total annihilation cross section, $v$ is the relative speed of the dark matter particles, and $m_\chi$, their mass. The quantity $\mathcal{L}(M_h)$ is the volume integral of $\rho^2$ where $\rho$ is the dark matter density. The boost factor due to subhalos is defined implicitly by the relation

$$\mathcal{L}(M_h) = [1 + B(M_h)] \tilde{\mathcal{L}}(M_h).$$

(5.3)

Here $\tilde{\mathcal{L}}(M_h)$ is the volume integral of $\tilde{\rho}^2$ where $\tilde{\rho}$ is the dark matter density for a smooth halo of equivalent mass (i.e., density if one ignores halo substructure). The boost factor is calculated recursively through the expression

$$B(M_h) = \frac{1}{\mathcal{L}(M_h)} \int dN \frac{d}{d\ln f} \frac{[1 + B(m)] \tilde{\mathcal{L}}(m) d \ln f}{\mathcal{L}(M_h)}d \ln f,$$

(5.4)

where $m = f M_h$.

Gamma-ray secondaries travel directly from the source to the observer and therefore the flux measured at Earth due to a source located at a distance $d$ is $L_\gamma/4 \pi d^2$. By contrast, charged particles are scattered by the Galactic magnetic field and lose energy. These processes depend on the initial energy of the secondary particle and cause the arrival direction of the secondary to be essentially independent of the direction to the source. It is therefore useful to define an effective (energy-dependent) global boost due to all subhalos (see Lavalle et al., 2008b for details):

$$B_{\text{eff}}(E; M_h, E_i) = (1 - f_t)^2$$

$$+ \frac{\left( \int G(E) \frac{\tilde{\rho}(x_i)}{M_h} d^3 x_i \right) \left( \int \frac{dN}{d\ln f} \mathcal{L}(m) d \ln f \right)}{\int G(E) \tilde{\rho}^2(x_i) d^3 x_i},$$

(5.5)

where $f_t$ is the total mass fraction in subhalos.

The function $G(E)$ accounts for the propagation of leptons from an initial position $x_i$ with initial energy $E_i$ to the Earth with final energy $E$. This propagation is a diffusive process and consequently
\[ G(E) \propto \exp \left( -\frac{(x_i - x_\odot)^2}{\lambda_D^2(D; E_i)} \right) \]

where \( \lambda_D \) is the diffusion length. This length depends both on the diffusion parameters, which describe the scattering and energy loss processes, and on the initial and final energy of the particles. For monoenergetic electron-positron pairs produced by a 100–1000 GeV WIMP the diffusion length is a few kpc for leptonic cosmic rays with \( E \gtrsim 100 \) GeV and decreases monotonically as \( E \rightarrow E_i \) (Lavalle et al. 2008b).

The integral involving \( G(E) \) in the denominator of Eq. (5.5) accounts for the propagation of cosmic rays originating from the host halo. While the integral in the numerator accounts for propagation of cosmic rays originating from subhalos. Effectively we are treating subhalos as point sources. This is a reasonable assumption as \( \approx 90\% \) of the flux originates from the central region of a (sub)halo as defined by the characteristic radius \( r_s \). As we are interested in subhalos that are numerous enough to enhance the diffusive background, we are generally concerned with \( \lesssim 10^8 \) M\( \odot \) subhalos which have characteristic radii of \( r_s \lesssim 1 \) kpc, less than the diffusion length of a few kpc. We also assume that subhalos trace the host’s dark matter distribution and that this volume distribution is independent of subhalo mass.

The integral involving \( \mathcal{L}(m) \) is simply the total contribution of subhalos to the annihilation flux. The main difference between our work and previous studies (Lavalle et al. 2008b; Pato et al. 2009; Kuhlen and Malyshev 2009) is that we examine the enhancement that arises not only due to subhalos but explicitly due to the entire subhalo hierarchy, i.e., subhalos, subsubhalos, etc., rather than the enhancement from smooth subhalos alone.

The boost factor depends sensitively on the subhalo mass function. Using the often quoted values \( \alpha = 1 \) and \( A = 0.033 \) (Kuhlen et al. 2008), one finds \( B \approx 30 \) for \( \gamma \)-rays and a Galactic-mass halo. The key assumption in obtaining this result is that \( \alpha \) and \( A \) are independent of the host mass and apply to all scales and levels in the subhalo hierarchy. Our goal is to examine the validity of this assumption, in short, to test whether \( \alpha \) and \( A \) depend on the mass of the host. To do so, we compare estimates for \( \alpha \) and \( A \) over a wide range of published simulations.

Though most of the results in our study are based on simulations which assume a standard
ΛCDM cosmology, results from our own simulations of structure formation in scale-free cosmologies ([Elahi et al.] 2009a) as well as results from simulations of warm dark matter cosmologies ([Knebe et al.] 2008) are also included. To assign an effective mass to hosts in these non-standard simulations we appeal to the halo formation process. In the hierarchical clustering paradigm halos arise from primordial density fluctuations with small-scale structures forming first and building up into larger and larger objects. This process is governed by the power spectrum of the perturbations, $P(k)$, where $k$ is the wavenumber of a perturbation mode. Alternatively, one can describe the perturbation spectrum by the mass variance, $\sigma^2(M) \propto \int k^2 dk P(k) W^2(k, M)$, where $W(k, M)$ is the window function enclosing a mass $M \propto k^{-3}$ within a comoving volume of radius $r = 2\pi/k$.

In scale-free cosmological models the primordial power spectrum is a power-law function of wavenumber $k$, that is, $P \propto k^n$ where $n$ is referred to as the spectral index. In terms of the mass variance, $d\ln \sigma^2/d\ln M = -(n + 3)/3$. Though the power-spectrum in a ΛCDM cosmology is more complicated, we can define an effective spectral index, $n_{\text{eff}}(M) = -3(d\ln \sigma^2(M)/d\ln M + 1)$. Thus, the spectral index in a scale-free cosmology may be used as a proxy for (ΛCDM) mass scale ([Elahi et al.] 2009a) by setting $n = n_{\text{eff}}(M)$ and solving for $M$. In this work we take $\Omega_m = 0.25$, $\Omega_\Lambda = 0.75$, $h = 0.73$, $\sigma_8 = 0.9$ and $n_s = 1$ as our reference cosmology.

The use of different algorithms to identify subhalos introduces a further complication. These algorithms range from SUBFIND ([Springel et al.] 2001), which identifies subhalos with local density peaks, to 6DFOF ([Diemand et al.] 2006; [Elahi et al.] 2009a), which searches for clustering in 6D phase space. In addition, some (but not all) researchers apply an unbinding criterion which removes unbound particles when estimating the mass of a subhalo. One reason for such a variety of methods is that subhalo identification is an ill-defined problem.

To characterize the amount of substructure, we introduce $f_{42}$ in place of $A$, where

$$f_{42} = \int_{-4\ln 10}^{-2\ln 10} f \frac{dN}{d\ln f} d\ln f,$$

(5.6)

is the fraction of mass in subhalos in $10^{-2} < f < 10^{-4}$. In Figure 5.1 on page 110 we plot estimates of $\alpha$ and $f_{42}$ from a large sample of studies. When available we plot published values of
\( \alpha \). Otherwise, we fit-by-eye for the mean \( \alpha \) and estimate the uncertainty. The mean \( \alpha \) is then used to determine \( f_{42} \). Error bars for \( f_{42} \) are shown when a study examines several similar mass halos and reports the variation in the subhalo number or mass fraction.

The figure displays a large scatter in \( \alpha \) and \( f_{42} \). For galactic to cluster masses, estimates of \( \alpha \) range from 0.7 and 1.1, which is generally within the estimated uncertainty in \( \alpha \) for a single measurement and the variation in \( \alpha \) with redshift for an individual halo (Gao et al. 2005, Elahi et al. 2009a). The scatter in log \( f_{42} \) is roughly constant with \( M_h \) with most points lying within 0.15 dex of the mean. The scatter may be due to variations in the mass accretion histories of the halos. Accretion events are the dominant source of subhalos. Moreover, a halo’s history affects its concentration parameter (Tasitsiomi et al. 2004) which, in turn, determines how efficiently subhalos are tidally disrupted.

Figure 5.1 also clearly shows that \( \alpha \) and \( f_{42} \) decrease with decreasing \( M_h \). The observed \( M_h \)-dependence should be treated with some caution. For example, a linear fit to \( \alpha(M_h) \) for \( M_h \) between galaxy and cluster masses is consistent at the 1\( \sigma \)-level with \( \alpha = \text{constant} \). Furthermore, at smaller host halo masses (\( n_{\text{eff}} \) closer to \( -3 \)) the subhalo mass function is more sensitive to the type of subhalo-finding algorithm used (Elahi et al. 2009a) and in particular, whether a binding criterion is applied. For smaller \( M_h \), subhalos tend to be less gravitationally bound. Studies that do not correct for unbound particles, such as (Diemand et al. 2006), overestimate subhalo masses and therefore overestimate \( \alpha \) and \( f_{42} \). Even for larger \( M_h \), the application of a correction for unbound particles decreases \( \alpha \) by 0.1 – 0.2 (see, for example, Athanassoula et al. 2009 and Maciejewski et al. 2009).

It should also be noted that the data points from Springel et al. (2008) at subgalactic masses are for subhalo hosts. This study found that the mass fraction of subsubhalos in subhalos tends to be smaller than the mass fraction of subhalos in halos. It may well be that \( \alpha \) depends on the host’s level in the subhalo hierarchy, i.e., that \( \alpha \) of a field halo host differs from that of a similar mass host that sits within the virial radius of a larger halo.

Most boost factor calculations assume \( \alpha \) and \( f_{42} \) are independent of mass, that is, \( \alpha(M_h) = \alpha_o \), \( f_{42}(M_h) = f_o \). However, this assumption does not capture the trends in Figure 5.1 on page 110.
especially at lower $M_h$, and we therefore propose the following phenomenological model:

$$\alpha(M_h) = \alpha_o + \alpha_n \log[n_{\text{eff}}(M_h) + 3],$$

$$\log f_{42}(M_h) = \log f_o + \log f_n \log[n_{\text{eff}}(M_h) + 3].$$

We account for the scatter in $\alpha$ by considering a mean and upper and lower envelop values for $\alpha_o$ and $\alpha_n$ as given in Table 5.1. For $f_{42}$ we fit the mean by eye and assume that it follows a lognormal distribution. This choice is motivated by the fact that other bulk halo parameters, such as the concentration parameter, follow lognormal distributions (Macciò et al., 2008).

We calculate the boost factor using Eq. (5.4) under the assumption that the subhalo mass function is a power-law with an index $\alpha(M_h)$, where the amplitude is normalized using $f_{42}(M_h)$ and $\alpha(M_h)$ subject to the condition that the total fraction of mass in subhalos is less than one. The mass function extends down to a mass $m_o$, the minimum CDM halo mass, which depends on the properties of the dark matter particle. Typical values for a neutralino in the Constrained Minimal Supersymmetric Standard Model are $10^{-9} - 10^{-6} M_\odot$ (Martinez et al., 2009) though studies show that the minimum CDM halo mass can vary between $10^{-12} - 10^{-4} M_\odot$ (Profumo et al., 2006). We use the more probable value of $m_o = 10^{-6} M_\odot$. We also assume that a host cannot contain subhalos with $f > 10^{-2}$ as suggested by most studies. As a consequence, the smallest host mass that contains subhalos is $100m_o$. Finally, we only go to three levels in the subhalo hierarchy as

<table>
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<th>Model #</th>
<th>Figure 5.2 on page 113</th>
<th>color</th>
<th>$\alpha_o$</th>
<th>$\alpha_n$</th>
<th>$f_o$</th>
<th>$f_n$</th>
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<td>0.04</td>
<td>0</td>
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Figure 5.1 Compilation of $\alpha$ (top) and $f_{42}$ (bottom) of studies indicated in the right-hand column. For both left panels: marker colors (types) indicate different studies (algorithms), studies of subhalo hosts are indicated by internal black filled star, colored horizontal error bars indicate mass range examined in study and black dashed vertical lines going from left to right indicate mass dwarf spheroidal galaxies, galaxies, and galaxy clusters. Also in the right panels are the curves of our phenomenological models of $\alpha$ and $f_{42}$ with the thick (thin) lines for model 1 (0). For $\alpha$, the mean, upper and lower values are given by the solid, dashed and dotted lines respectively. For $f_{42}$, the mean of the lognormal distribution is given by the solid line and $+2\sigma$ by the dashed, $-2\sigma$ by the dotted lines. Top panels: vertical black (gray) error bars indicate uncertainty in the published (fit-by-eye) values. Bottom panels: black vertical error bars indicate variation between similar mass hosts.
deeper levels change the boost factor by \( \lesssim 1\% \).

The volume integral of \( \tilde{\rho}^2 \) in Eq. (5.3) is given by

\[
\tilde{L}(M_h) = \frac{\rho_v}{3} M_h c^3(M_h) g(c) ,
\]

where \( \rho_v \) is the virial density of a halo, \( c \equiv r_v/r_s \) is the concentration parameter, \( r_s \) is the profile’s effective radius, and the dimensionless function \( g(c) \) depends on the form of the density profile. We consider the commonly used NFW (Navarro et al., 1997) profile, which is described by two parameters, the characteristic radius \( r_s \) and density \( \rho_s \). We also examine the Einasto profile, which appears to be a better fit to halos from cosmological simulations (Springel et al., 2008). The Einasto profile has an extra parameter, \( \alpha_E \) describing the logarithmic shape which appears to have a mass dependence (Gao et al., 2008). To determine the concentration parameter we use the two-parameter model presented in Macciò et al. (2008) for \( c(M) \) as this model provides a better match to simulation data than the often-used model of Bullock et al. (2001) (B01). We find that an Einasto profile increases the boost factor for subgalactic masses by \( \lesssim 50\% \) and using the Macciò et al. (2008) model for \( c(M) \) instead of the B01 model reduces \( B \) by \( \lesssim 10\% \).

We note that the boost factors saturate for large \( f_{42} \) and small \( m_o \) due to the condition that \( f_t \leq 1 \). For \( \alpha = 1 \) and \( m_o = 10^{-4} \, M_\odot \) the boost factor begins to saturate at \( f_{42} \gtrsim 0.25 \). For \( \alpha = 0.9 \) a similar saturation occurs at \( f_{42} \gtrsim 0.25 \) when \( m_o = 10^{-12} \, M_\odot \). For our choice of \( m_o = 10^{-6} \, M_\odot \) and our subhalo mass function models, the boost factor to begins to saturate at values of \( f_{42} \) that are \( \gtrsim 4\sigma \) away from the mean.

We calculate \( 10^4 \) random realizations of \( B \) for a given \( M_h \) where \( c \) and \( f_{42} \) are sampled from lognormal distributions with \( \sigma_{\log f_{42}} = 0.15 \) dex and \( \sigma_{\log c} = 0.10 \) dex (Neto et al. 2007). Due to the recursive nature of Eq. (5.4), variation in \( c \) not only affects \( \bar{L} \) of the host but that of the host’s subhalos. The distribution of \( B(M_h) \) across the \( 10^4 \) realizations is characterized by a lognormal distribution and is fit for the mean and the variance.

The results are plotted in Figure 5.2 on page 113. We highlight the mass scales of the Galaxy’s dwarf spheroidal satellites as these may be the best candidates for searches of \( \gamma \)-rays secondaries.
The left panel shows how a few key parameters individually affect $B(M_h)$. For reference, we use $m_\phi = 10^{-6} \, M_\odot$, $\alpha = 1.0$, $f_{42} = 0.067$, an NFW density profile with the Bullock prescription for $c(M)$, as well as no variation in $c$ or $f_{42}$ (Kuhlen et al., 2008). (Note that in Kuhlen et al., 2008, the boost factor is calculated for subhalos below the mass resolution of their Via Lactea II simulation, which is $\sim 10^5 \, M_\odot$. This amounts to calculating Eq. (5.4) with the upper limit given by $10^5 \, M_\odot/M_h$.) Decreasing $\alpha$, here for example from 1.0 to 0.9, or using the new model for $f_{42}$, decreases $B(M_{dSph})$ by $\sim 4$. The introduction of scatter in $c$ and $f_{42}$ increases the mean boost factor slightly, though the amount depends on the exact form and width of the distributions. For our choices, $B(M_{dSph})$ increases by $\approx 30\%$. Decreasing $m_\phi$ to $10^{-9} \, M_\odot$ also increases the boost factor by $\approx 30\%$. Using an Einasto profile increases the boost factor by $\lesssim 20\%$ in this case.

The other two panels of Figure 5.2 on the following page show the peak of the distribution and the $2\sigma$ contours from the models listed in Table 5.1 on page 109. We find increasing $\alpha$ increases the boost factor by $\approx 2$ for dSph galaxies, though the differences are generally within the variation caused by $c$ and $f_{42}$. For model 0, the mean $\alpha$ gives $B(M_{dSph}) \sim 0.6^{+1.4}_{-0.4}$, though for $\alpha = 1.0$, $B \sim 8$ are within the $2\sigma$ envelop. Model 1 reduces the boost factors such that $B(M_{dSph}) \gtrsim 2$ lie outside the $2\sigma$ envelop with the mean $\alpha$ giving $B(M_{dSph}) \approx 0.2$.

Thus, subhalos in the Galaxy’s satellites are unlikely to greatly enhance the $\gamma$-ray flux. Previous estimates of the number of satellites that could be detected via their $\gamma$-ray flux by GLAST, such as those by Strigari et al. (2007) and Kuhlen et al. (2008), are probably overly optimistic. Even our revised calculations may overestimate the boost factor as it appears that subhalos have smaller $f_{42}$ than similar mass halos. Consequently, satellites might have their boost factors reduced by a factor of $\gtrsim 2$. So far, GLAST has detected numerous $\gamma$-ray sources, none of which are convincing dark matter annihilation signals (Abdo, 2009).

We now examine the consequences of such boost factors for cosmic rays. To determine the energy dependence in Eq. (5.5), we use the mean propagation parameters listed in Lavalle et al. (2008b) and assume for simplicity that the dark matter particle annihilates directly to monoenergetic $e^\pm$. We assume the Milky Way halo has an NFW density profile with a characteristic radius $r_s =$
Figure 5.2 Boost factors. Left panel shows how $B(M_h)$ changes when a single parameter is changed. Middle (right) panel shows model 0 (1) with thick red, green and blue lines indicating mean boost factor using the mean, upper, and lower values of $\alpha$ respectively. Colored hatched contours indicate the 2σ region arising from the variation in $\log c \& \log f_{12}$. The gray hatched region outlines the mass range for dwarf spheroidal satellite galaxies of MW.

20 kpc and a solar density of $\rho_\odot = 0.43$ GeV cm$^{-3}$. An Einasto profile with a shape parameter of $\alpha_E = 0.17$ ([Springel et al.], 2008) does not drastically alter our results.

Recall that in Eq. (5.5) we assume that the subhalo volume distribution is the same as the halo’s density profile. However, simulations appear to show that subhalos do not trace the host halo’s radial density profile ([Springel et al.], 2008; Diemand and Moore, 2009). Instead, down to the numerical resolution of $\sim 10^5 M_\odot$, tidal stripping destroys subhalos in the center, resulting in a radially anti-biased distribution. There appear to be few subhalos within $\sim 20$ kpc of the halo center. This would imply that subhalos cannot contribute to the observed flux as they are too far away. We argue that this anti-biased radial distribution is unlikely to continue down to smaller mass subhalos since these subhalos should become increasingly less susceptible to tidal disruption as their mass decreases and therefore be able to survive to smaller radii. Consequently, the observed radial distribution is partly due to the numerical resolution of the simulations. It is also difficult to identify substructure in the central, high density, dynamically cold regions of halos with current subhalo finder algorithms. Furthermore, [Diemand and Moore, 2009] found that though tidal fields affect the mass of subhalos,
their annihilation luminosity is less affected and more closely follows the host halo’s radial density profile. Thus, we compromise and only include contributions from subhalos with $m \lesssim 10^4 \text{ M}_\odot$, which still may be an optimistic assumption. Considering only low mass subhalo hosts reduces the contribution of substructure by $\approx 60\%$ for both models 0 and 1.

For a 700 GeV thermal dark matter particle with $\langle \sigma v \rangle = 3 \times 10^{-26} \text{ cm}^3/\text{s}$, the diffusion lengths are 1.3 kpc at $E = 300$ GeV which monotonically decreases to 0.1 kpc at $E = 690$ GeV. We generally find $B_{\text{eff}}(E \geq 300 \text{ GeV}) \sim 1$. Even the most optimistic model, 0-u, gives $B_{\text{eff}}(E \geq 300 \text{ GeV}) = 1.3^{+0.6}_{-0.4}$. This model excludes the $B_{\text{eff}} \sim 200$ required to explain the ATIC signal at the $\sim 14\sigma$ level. These results are in agreement with similar studies \cite{Lavalle08b, Pato09, Kuhlen09}, though these studies neglected deeper levels in the subhalo hierarchy.

The High Energy Stereoscopic System (HESS) \cite{Hess09} and Fermi GLAST \cite{Fermi09} found different though not necessarily contradictory results. Both instruments detect an excess, though not as large as that observed by ATIC, and neither instrument observes the pronounced peak at $\sim 700$ GeV seen by ATIC. The Fermi GLAST measurements require a boost of $\sim 150$ at $E = 300$ which monotonically decreases to $\approx 70$ at $E = 600$. This is still rejected at the $\approx 10 - 12\sigma$ level.

We note that studies show it is possible to reproduce the observed flux, particularly the ATIC bump in the energy spectrum, by a single nearby subhalo \cite{Hooper09, Kuhlen09}. The required annihilation rate is $\sim 10^{37} \text{ s}^{-1}$ for a thermal WIMP with $m_\chi \sim 100 - 1000$ GeV. Neglecting the boost factor from deeper levels in the subhalo hierarchy, a subhalo with a mass of $\gtrsim 10^8 \text{ M}_\odot$ at $\sim 1$ kpc is required. However, \cite{Brun09} find based on a numerical simulation that the probability of such a large subhalo within $\sim 8$ kpc of the Galactic center is exceedingly low, $p \lesssim 10^{-5}$. Incorporating the boost factors does not drastically reduce the minimum mass required to explain the ATIC peak and thereby the probability of a nearby clump.

We conclude that subhalos alone are unlikely to account for the cosmic ray flux anomaly. Thus, our work favors either more mundane sources, such as nearby pulsars, or exotic theories of dark
matter, such as that proposed by Arkani-Hamed et al. (2009) which Kuhlen et al. (2009) showed could account for the high observed flux.
5.2 Paper II: Errata

This section contains supplementary information not included in the submitted paper. First we present the boost factor distribution for several different host masses. This is followed by a brief discussion of how the required effective boost factor depends on the density profile of the Galactic Halo.

Figures 5.3 & 5.4 on the following pages show the distribution of boost factors for a sample of host masses. These histograms show that the boost factors are generally well described by lognormal distributions. The lognormal variance is $\log B \approx 0.27$ and has a slight dependence on mass, decreasing from $\approx 0.28$ at $M_h \sim 10^{-3} M_\odot$ to $\approx 0.26$ at $M_h \sim 10^{10} M_\odot$.

The effective boost for $e^+e^-$ cosmic rays required to reproduce the high energy signal of ATIC depends on $\rho_s^{-2}$. This density can be determined by either specifying the solar density or the concentration of the Milky Way halo and the mass of the halo. Using the concentration parameter and mass yields

$$\rho_s = \frac{\Delta_{\text{vir}} c^3 \rho_c}{3 f_1(c)}, \quad (5.10)$$

where $f_1$ is given by Eq. (3.84). An NFW halo has

$$\rho_s = \frac{\Delta_{\text{vir}} \rho_c c^3}{3} \left[ \ln(1 + c) - \frac{c}{1 + c} \right]^{-1}. \quad (5.11)$$

Determining $\rho_s$ using $\rho_\odot$ is simply a matter of using the virial radius $r_{\text{vir}} = [M/(4\pi \Delta_{\text{vir}} \rho_c/3)]^{1/3}$ and the density profile. For an NFW halo

$$\rho_s = \rho_\odot \left[ \frac{r_\odot}{r_{\text{vir}}} \left( 1 + \frac{r_\odot}{r_{\text{vir}}} \right)^2 \right]. \quad (5.12)$$

For example a $2 \times 10^{12} M_\odot$ halo with $c = 14.7$, the solar density would be $\rho_\odot = 0.43 \text{ GeV/cm}^3$, which is an often used value for the solar dark matter density (see for example Chang et al. 2008). Another often quoted value for the solar density is $\rho_\odot = 0.30 \text{ GeV/cm}^3$, which for a $10^{12} M_\odot$ halo would require a concentration of 10. In this case the required boost factor for energies from
Figure 5.3 Distribution of boost factors $B(M)$ at various masses for subhalo model 0 (solid black curve) and lognormal fits (dashed red curve) along with fit parameters.
Figure 5.4 Same as Figure 5.3 on the preceding page but for subhalo model 1.
∼ 300 − 700 GeV is $B_{\text{eff,req}} \approx 440 − 400$. However, assuming the subhalos still follow the halo’s density profile, the effective boosts actually tend to be unity.

The variation in the effective boost is estimated by

$$
\sigma_{\log B_{\text{eff}}} = \frac{1}{B_{\text{eff}}} \left[ \left( \frac{\partial B_{\text{eff}}}{\partial \log f_{42}} \right)^2 \sigma_{\log f_{42}}^2 + \left( \frac{\partial B_{\text{eff}}}{\partial \log c} \right)^2 \sigma_{\log c}^2 + \left( \frac{\partial B_{\text{eff}}}{\partial \log B} \right)^2 \sigma_{\log B}^2 \right]^{1/2}.
$$

(5.13)

Writing $C_{\text{sub}}$ in Eq. (3.99) as

$$
C_{\text{sub}} = \sum N(M_i)(1 + B(M_i))M_i c^3(M_i)g[c(M_i)],
$$

(5.14)

the derivatives in Eq. (5.13) can be written as

$$
\frac{\partial B_{\text{eff}}}{\partial \log f_{42}} = -2 f_t (1 - f_t) + \frac{B(E)}{A(E)} C_{\text{sub}},
$$

$$
\frac{\partial B_{\text{eff}}}{\partial \log c} = \frac{B(E)}{A(E)} \sum N(M_i)(1 + B(M_i))M_i c(M_i) \left( 3c^2(M_i)g[c(M_i)] + c^3(M_i) \frac{\partial g}{\partial c} \bigg|_{M_i} \right)
$$

$$
\frac{\partial B_{\text{eff}}}{\partial \log B} = \frac{B(E)}{A(E)} \sum N(M_i)B(M_i)M_i c^3(M_i)g[c(M_i)].
$$

We find the lognormal variation $\sigma_{\log B_{\text{eff}}} \approx 0.10 − 0.15$. Thus, the required boost factors are in general ruled out at $\gtrsim 10\sigma$ given our assumptions.
Chapter 6

Paper III: Power spectrum for the small-scale Universe

This chapter is based on Paper III: “Power spectrum for the small-scale Universe”, co-authored with Lawrence M. Widrow, Robert J. Thacker, Mark Richardson, and Evan Scannapieco, which was published in *Monthly Notices of the Royal Astronomical Society* (MNRAS). In this work, I am again looking at the nonlinear gravitational evolution of scale-free cosmologies but here the focus is on the power spectrum, particularly spectra with spectral indices near those of the small-scale CDM power-spectrum (see Section 3.1 for relevant background information). I examine the ability of one-loop PT, HKLM scaling and the Halo Model (see Sections 3.1.3, 3.1.4 & 3.1.4) to describe the evolution of the nonlinear power spectrum.

My contributions to this work are that I ran some of the smaller simulations and analyzed the power spectrum for all but the largest $n = -2$ and $n = -2.25$ simulations. I contributed to the interpretation of the results and the scientific content of the paper as well as helped in the editing process.
CHAPTER 6. PAPER III: POWER SPECTRUM

6.1 Introduction

In this chapter, we provide insight into the small-scale limit of CDM by focusing on scale-free simulations as these cosmologies have only one preferred length scale, the nonlinear scale. As a consequence the power spectrum should evolve according to the self-similar scaling ansatz

\[ \Delta^2(k, a) = \hat{\Delta}^2 \left( \frac{k}{k_{NL}} \right). \]  

(6.1)

Self-similar scaling provides a diagnostic test of whether a simulation has sufficient dynamic range (Jain and Bertschinger, 1998). Contact with the standard ΛCDM cosmology is made by treating parameter \( n \) as a proxy for scale: recall that \( n_{\text{eff}} \approx -1.8 \) corresponds to cluster scales and \( n_{\text{eff}} \approx -2.2 \) to galactic scales (see Figure 2.1 on page 10). The limit \( n \to -3 \) corresponds to the bottom of the CDM hierarchy. We compare our results with the predictions for the power spectrum from (one-loop) perturbation theory and a renormalization group approach suggested by McDonald (2007). We also examine the two widely used fitting formulae for the nonlinear power spectrum, that of PD96 and Smith et al. (2003), and investigate the stable clustering hypothesis versus the halo model.

6.2 Simulations

N-body simulations are carried out with scale-free initial conditions and \( n = -2, -2.25 \) using the parallel tree-PM code GADGET-2 (Springel, 2005) in addition to the two simulations presented in Chap. 4 which have indices of \( n = -1 \) and \( -2.5 \). Like the previous simulations, these simulations are started such that the maximum initial displacement for any given particle is less than 1/2 the initial inter-particle spacing and the initial conditions are generated on a regular grid using 2LPT code (Crocce et al., 2006; Thacker and Couchman, 2006). As mention in Section 3.1.2 the primary benefit of 2LPT is to reduce the impact of spurious transient modes which arise from the truncation of the perturbative expansion. The simulations are run with a softening length of 1/30 of the initial interparticle spacing. GADGET-2 parameters such as the opening angle used in constructing the particle tree and maximum time step criterion were set to their default value. These simulations
were carried out on Saguaro, a 4560 node system with 10 TB of memory located at Arizona State University’s Fulton High Performance Computing Initiative. The \( n = -2.25 \), our largest simulation ran for six days on 1024 processors. The other runs used 256 processors, and typically ran for 3-4 days.

Table 6.1 summarizes key features of the simulations used in this study such as the number of simulation particles and the epochs at which the power spectra are measured. The latter are expressed in terms of the ratio \( a/a_* \) where \( a_* \) is defined as the scale factor at the epoch when the mode on the scale of the box is equal to the nonlinear scale, that is, \( a/a_* = (k_B/k_{NL})^{(n+3)}/2 \). With this definition, \((a/a_*)^2 = \Delta_2(k_B)\).

As \( n \) approaches \(-3\), the absence of modes beyond the box scale induces an error in the nonlinear power spectrum. As in Chap. 4, we examine Smith et al. (2003) ad hoc criterion for the missing variance, \( \sigma_{\text{miss}} \leq 0.04 \). Recall that \( \sigma_{\text{miss}} = \Delta_2^2(k_B) F(3 + n) \), where \( F(x) = (1 - 0.31x + 0.015x^2 + 0.00133x^3)/x \). Figure 6.1 on the following page shows \( \sigma_{\text{miss}} \) for the outputs of our six simulations. We see that the final two outputs in our high-resolution \( n = -2 \) and \( n = -2.25 \) simulations and all but the first few outputs in our \( n = -2.5 \) simulation fail to meet the Smith et al. (2003) criterion. We return to this point below.

Power spectra are calculated using a cubic mesh with \( N_g^3 \) cells (Table 1). We set \( N_g = 2N \) so long as \( N \) is a power of 2, where \( N^3 \) is the number of particles. When \( N \) is not a power of 2 we set \( N_g \) equal to the first power of 2 larger than \( N \). The power spectra are calculated using piecewise quadratic spline interpolation (Hockney and Eastwood, 1981) and adjusted to account for the strong filtering of this mass-assignment scheme. No correction is made for shot noise.

The ratio of the dimensionless power spectrum at the Nyquist frequency, \( k_{Ny} \), to that at the box scale, \( k_B \), provides a measure of a simulation’s dynamic range. For a scale-free power spectrum

\[
\frac{\Delta^2(k_{Ny})}{\Delta^2(k_B)} = \left( \frac{k_{Ny}}{k_B} \right)^{n+3} \propto N^{n+3}.
\] (6.2)

Thus, if \( N^3 = 256^3 \) particles are required to achieve a scaling solution over a reasonable range in \( \Delta^2 \) when \( n = -2 \), Jain and Bertschinger (1998), \( 256^4 \approx 1625^3 \) particles are required at \( n = -2.25 \),
Figure 6.1 Missing variance, $\sigma_{\text{miss}}$, for the six simulations described in the text with line types/colours as follows: solid/blue — $n = -1$; dotted/red — $n = -2$, $N = 32^3$; short-dashed/red — $n = -2$, $N = 256^3$; long-dashed/red — $n = -2$, $N = 1024^3$; short-dashed-dot/green — $n = -2.25$; long-dashed-dot/cyan — $n = -2.5$. The output number, $N_{\text{output}}$ corresponds to the values listed in Table 1. For $n = -2.5$, we show $\sigma_{\text{miss}}$ for the last six outputs. The horizontal black curve corresponds to the criterion adopted by Smith et al. (2003).
Table 6.1 Summary of Simulations

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N^3$</th>
<th>$N^3_g$</th>
<th>Initial scale factor $(a_i/a_*)$</th>
<th>Output scale factors $(a/a_*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$720^3$</td>
<td>1024</td>
<td>0.0014</td>
<td>0.026, 0.11, 0.19, 0.21, 0.24, 0.30</td>
</tr>
<tr>
<td>$-2$</td>
<td>$32^3$</td>
<td>64</td>
<td>0.028</td>
<td>0.052, 0.13, 0.23, 0.31, 0.42, 0.49</td>
</tr>
<tr>
<td></td>
<td>$256^3$</td>
<td>512</td>
<td>0.010</td>
<td>0.031, 0.054, 0.17, 0.29, 0.51, 0.67</td>
</tr>
<tr>
<td></td>
<td>$1024^3$</td>
<td>2048</td>
<td>0.006</td>
<td>0.024, 0.043, 0.063, 0.17, 0.30, 0.36</td>
</tr>
<tr>
<td>$-2.25$</td>
<td>$1584^3$</td>
<td>2048</td>
<td>0.009</td>
<td>0.018, 0.021, 0.039, 0.084, 0.221, 0.394</td>
</tr>
<tr>
<td>$-2.5$</td>
<td>$720^3$</td>
<td>1024</td>
<td>0.015</td>
<td>0.033, 0.074, 0.15, 0.23, 0.26, 0.29, 0.33</td>
</tr>
</tbody>
</table>

$\dagger$ Simulations presented in Chap. 4.

256$^6$ particles are required for $n = -2.5$, and 256$^{12}$ particles are required for $n = -2.75$. We set the particle number for the $n = -2.25$ simulation on the basis of these arguments.

We fully anticipated that self-similar scaling would not be achieved in our $n = -2.5$ simulation. Our run, carried out with $N^3 = 720^3$ illustrates the difficulties that arise as one attempts to simulate highly negative spectral indices. In practice $N^3 = 1584^3$ is the largest simulation we can perform within word-addressing limits in GADGET-2. It is also clear from this discussion that running an $n = -2.5$ simulation at $1584^3$ will yield little improvement over our $720^3$ simulation since, apparently, one requires $N^3 = 65536^3$. Further discussions of the difficulties in simulating scale-free models with $n \to -3$ can be found in Smith et al. (2003) and in Chap. 4.

### 6.3 Results

In Figure 6.2 on page 126 we plot the power spectrum from our $N^3 = 720^3$ $n = -1$ simulation at the six epochs listed in Table 1. The power at a given wavenumber increases with time while $k_{NL}$ (in this figure, the wavenumber where the power spectrum deviates from the linear form, $P(k) \propto k^{-1}$), decreases. These results are in close agreement with previous simulations. The cumulative halo
distribution is also in good agreement with the expected results (see Figure 4.3 on page 85). In Figure 6.3 on page 127 we test the self-similar scaling ansatz by plotting the dimensionless power spectra as a function of $k/k_{NL}$ for each of our four high-resolution simulations. Taken together, the spectra from different epochs yield a composite power spectrum. The $n = -1$ power spectrum (upper left panel) covers 7 orders of magnitude in $\Delta^2$ or, equivalently, 3-4 orders of magnitude in $k$. The fact that the composite power spectrum is very nearly a single-valued function of $k/k_{NL}$ indicates that self-similar scaling is essentially achieved. Note also that $\Delta^2/\Delta^2_L < 1$ for most values of $k$, which is consistent with PT (see Eq. (3.32) and note that $\lambda(n = -1) < 0$).

As with the $n = -1$ run, the composite spectra for $n = -2$ and $n = -2.25$ show excellent consistency with the scaling hypothesis. However, a departure from self-similar scaling is observed at large $a$ in the $n = -2.5$ simulation highlighting the difficulty of simulating $n \rightarrow -3$ spectral indices. Note that the high-$k$ feature in some of the early timesteps of our $n = -2.5$ simulation is a remnant of the grid used in setting up the initial conditions. The feature is subdominant to the physical small-scale power at later times in the $n = -2.5$ simulation.

The effect of resolution in achieving self-similar scaling is illustrated in Figure 6.4 where we compare the spectra from the three $n = -2$ simulations. A departure from self-similar scaling is apparent in the $N = 256^3$ simulation and quite severe for $N = 32^3$. It may be that these simulations are over-evolved, as suggested by Figure 6.1 on page 123. In any case, based upon the scaling arguments presented in the previous section, an $N = 32^3$, $n = -2$ simulation should be comparable to an $N \simeq 32^6 = 1024^3$, $n = -2.5$ simulation. Hence, it is not surprising that the departures from self-similar scaling seen in the right-hand panels of Figure 6.4 on page 128 are comparable to those seen in our $n = -2.5$ simulation (lower-right panel of Figure 6.3 on page 127). The departure manifests itself in a suppression of power at small $k$ or large scales, as expected since power is missing due to the finite size of the simulation volume.
Figure 6.2  Power spectrum, \( P(k) \) as a function of wavenumber \( k \) for the \( n = -1 \) simulation. Different colours and symbols correspond to different output times as given in Table 6.1. The sequence, from early to late times is magenta-blue-cyan-green-brown-red, or, alternatively, open square-filled square-open triangle-filled triangle-open circle-filled circle.
Figure 6.3  Dimensionless power spectrum, $\Delta^2$ as a function of wavenumber $k$. Colours and symbols are the same as in Figure 6.2. Bottom panel in each quadrant shows the ratio $\Delta^2/\Delta^2_L$. Upper left — $n = -1$; Upper right — $n = -2$; Lower left — $n = -2.25$; Lower right — $n = -2.5$. 
Figure 6.4 Dimensionless power spectrum, $\Delta^2$, and the ratio $\Delta^2/\Delta^2_L$ for $n = -2$ from simulations with different numbers of particles. Black points are power spectra at different timesteps measured in our highest resolution ($N = 1024^3$) simulation. Superimposed in colour are measurements from the $N = 256^3$ (left) and $N = 32^3$ (right) simulations. As in Figure 6.2, the sequence, from early to late times is magenta-blue-cyan-green-brown-red or, alternatively, open square-filled square, open triangle-filled triangle-open circle-filled circle.
6.3.1 Perturbative or Quasilinear Regime

We now focus on the quasilinear regime in order to illustrate the improvement renormalization group methods brings to perturbation theory. We implement the RG approach by solving Eq. (3.39), which for scale-free cosmologies where the growth function \( D(a) = a^2 \), can be written as

\[
\frac{d\hat{P}}{d(a^2)} = \hat{P}_{13} + \hat{P}_{22},
\]

assuming an initial scale-free power spectrum with \( n = -1, -2, -2.25 \) or \(-2.5\), where the \( \hat{\cdot} \) indicates the power spectrum has been divided by \( a^2 \). The initial spectrum is “evolved” forward in time (or equivalently, scale factor \( a \)) using a 4th-order Runge Kutta scheme with an adaptive stepsize (see, for example, [Press et al., 1992]). Each Runge Kutta step requires an evaluation of the \( P_{13} \) and \( P_{22} \) integrals. As with N-body simulations, we must truncate the initial power spectrum at both high and low wavenumbers. Otherwise, the integrals would diverge. Scoccimarro and Frieman (1996b) use sharp cutoffs which are convenient for power-law spectra with integer \( n \) since analytic expressions can be derived. Smooth cutoffs are more manageable for the RG analysis where \( \hat{P}_{13} \) and \( \hat{P}_{22} \) must be evaluated numerically. The IR and UV cutoffs as \( k_{IR} \) and \( k_{UV} \) used are

\[
f_{IR} (k, k_{IR}, \Delta k) = \frac{1}{2} \left[ \text{erf} \left( \frac{\ln (k/k_{IR})}{\Delta_k k} \right) + 1 \right],
\]

\[
f_{UV} (k, k_{UV}, \Delta k) = \frac{1}{2} \text{erfc} \left( \frac{\ln (k/k_{UV})}{\Delta_k k} \right).
\]

The ratio \( k_{UV}/k_{IR} \), which corresponds to the dynamic range of the calculation, is set to \( 10^6 \) while \( \Delta_k \), which determines the sharpness of the k-space cutoffs, is set equal to 0.5. For \( n < -1 \), \( P_{13} \) and \( P_{22} \) have terms of order \((k_{IR}/k_{UV})^{n+1}\) which cancel, leaving behind a residual term of order \( k^{2n+3} \). As mentioned the numerical challenge is determining the surviving terms which can be much smaller than the terms that cancel, especially for large \( k \) and small \( n \). Widrow developed a technique capable of dealing with the cancellations and accurately determine the surviving term using the Romberg integration routine from [Press et al.] (1992).

The solution to Eq. 6.3 yields an evolutionary sequence for the power spectrum, \( \hat{P}(k, a) \). Departures from the linear power spectrum increase with \( a \) beginning at high wavenumber. We evolve
\( \hat{P} \) until the \( k_{NL} \) is roughly equal to the geometric mean of \( k_{UV} \) and \( k_{IR} \). Since our dynamic range is a full three orders of magnitude greater than is found in our N-body simulations, finite “box” effects are much less a concern here. Moreover, our results are insensitive to the form of the cutoff functions, \( f_{UV} \) and \( f_{IR} \), since they are derived in a region well inside the computation box.

In Figure 6.5 on the next page we show the measured \( \Delta^2 \) in the mildly nonlinear regime together with predictions from PT, RGPT, and the Zel’’dovich approximation. The latter is discussed in Taylor and Hamilton (1996). Also shown are the fitting formulae of PD96 and Smith et al. (2003). Note that in the \( n = -1 \) case, \( \Delta^2 / \Delta^2_{NL} \) slowly decreases with increasing \( k \) for \( k \lesssim k_{NL} \) and it is difficult to discern the true self-similar evolution of the power spectrum given that the actual initial power spectrum has a large-\( k \) cutoff. On the other hand, for \( n = -2 \), and \( -2.25 \), it is clear that RG does the best job of tracking the power spectrum in the quasilinear regime the RG power spectra does not quite capture the rapid evolution of the measured power spectra. The situation is less clear for \( n = -2.5 \) where the validity of the simulation is in doubt, although RG does push the PT prediction in the right direction. The question remains as to whether agreement between RG and the simulations might be improved by refinements in the RG analysis, such as those suggested by McDonald (2007).

### 6.3.2 Nonlinear Regime

In the top left corner of Figure 6.6 on page 134 we plot the complete power spectrum for our high-resolution, \( n = -1 \) simulation together with the predictions of PD96 and Smith et al. (2003). Also shown is our own fitting formula given by

\[
\Delta^2(k) = \Delta^2_L(k) g(k/k_{NL}),
\]

(6.6)

where

\[
g(x) = \left( \frac{1 + A x + B x^\alpha}{1 + C x^\gamma} \right)^\beta,
\]

(6.7)

and \( k_{NL} = (a^2 A/2\pi^2)^{-1/(n+3)} \). The form of this formula is motivated by that of PD96 but has one additional parameter to allow for a more general behaviour in the \( k \to \infty \) limit. The parameters,
Figure 6.5 The ratio $\Delta^2/\Delta_L$ as a function of $k/k_{NL}$ from our four high-resolution simulations as labeled in each panel. Black points are from the simulation with the different symbols representing results from different outputs (from early to late outputs: solid squares, open triangles, solid triangles, open circles, solid circles). Line colours/types are: blue/solid — RG; green/dotted — one-loop; cyan/dot-dashed — Zel’dovich approximation; red/short-dashed — Smith et al. (2003) fitting formula; magenta/long-dashed — PD96 fitting formula.
Table 6.2 Parameters for fitting formula Eq. (6.6)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>α</th>
<th>γ</th>
<th>β</th>
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</thead>
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<td>1.571</td>
<td>1.845</td>
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<td>1.243</td>
<td>1.266</td>
<td>8.647</td>
</tr>
<tr>
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<td>4.038</td>
<td>0.348</td>
<td>1.413</td>
<td>1.372</td>
<td>0.659</td>
</tr>
<tr>
<td>-2.5</td>
<td>174.0</td>
<td>110.2</td>
<td>4.532</td>
<td>1.492</td>
<td>1.231</td>
<td>0.879</td>
</tr>
</tbody>
</table>

derived by performing a nonlinear least-squares fit (see, for example, Press et al., 1992), are given in Table 6.2. The lower panel in Figure 6.6 shows the logarithmic slope $\mu \equiv d \ln P/d \ln k = d \ln \Delta^2/d \ln k - 3$. Note that $\mu$ monotonically decreases with $k$ near the Nyquist wavenumber.

In the top right and lower left panels of Figure 6.6, we show $\Delta^2$ and $\mu$ for our high-resolution, $n = -2$ and $-2.25$ simulations. Again, $\mu$ decreases monotonically in the large-$k$ limit. Nevertheless, by design, virtually all fitting formulae (including our own) have a power-law form in the high-$k$ limit, that is, $P(k) \propto k^{\mu}$ as $k \to \infty$. The lesson is that fitting formula should not be extrapolated to scales below the smallest scales probed by the simulation used in their construction.

Neither the PD96 nor the Smith et al. (2003) fitting formulae do a particularly good job of fitting the power spectra from our simulations. For $n = -2$, the Smith et al. (2003) formula provides a reasonable fit up to $k/k_{NL} \approx 5$ but decreases too rapidly beyond this point. Conversely, PD96 predict that $\Delta^2/\Delta_L^2$ is constant in the large-$k$ limit whereas the measured power spectrum shows a clear decline. The discrepancies between predicted and measured power spectra for $n = -2.25$ are equally severe. By contrast, Eq. (6.6) provides an excellent fit to the nonlinear power spectra from our high-resolution simulations. And while it has one more parameter than the fitting formula of PD96, it has two fewer than that of Smith et al. (2003).

The $n = -2.5$ case, shown in the bottom right corner of Figure 6.6, is difficult to analyse because of the departure from self-similar scaling. In this plot, we truncate the power spectra from different timesteps at large $k$, where the effects of aliasing is apparent, and at small $k$, where the
effects of missing large-scale power is apparent. We contend that this procedure yields a roughly continuous curve, which provides a reasonable facsimile of the true power spectrum. The plausibility of this procedure is illustrated in the left-hand panels of Figure 6.4 on page 128 where one can imagine carrying out a similar procedure with our \( n = -2, N = 256^3 \) results to yield an approximate form for the power spectrum from our high-resolution simulation.

### 6.3.3 Asymptotic Slope and the Halo Model

Here we examine the high-\( k \) limit of our simulations in the context of the halo model. Following Seljak (2000), we recast Eq. (3.49) using \( \nu \equiv (\delta_{sc} / \sigma(M))^2 \) as the integration variable where \( \delta_{sc} \) is the critical overdensity for spherical collapse (\( \delta_c \simeq 1.68 \) in an Einstein-de Sitter universe) and \( \sigma(M) \) is the mass variance. For scale-free models, \( M \propto \nu^{3/(n+3)} \). One finds

\[
P_{1h}(k) = \frac{1}{(2\pi)^3} \int h(\nu) \left( \frac{M}{\rho_{bg}} \right) \left[ \frac{\tilde{\rho}(k, M)}{M} \right]^2 d\nu,
\]

where

\[
h(\nu)d\nu = \frac{M}{\rho_{bg}} \frac{dn}{dM} \frac{dM}{d\nu} d\nu,
\]

is a dimensionless form for the halo mass function.

Recall that halos at small \( r \) have density profiles that are approximately \( \rho_s / (r/r_s)^{\gamma} \), where for NFW \( \gamma = 1 \) (see Section 3.3.1). The characteristic halo scale length, \( r_s \), depends on the halo mass through the relation \( r_s = r_{vir} / c \). For simplicity we assume that the concentration parameter is a power-law function of mass, \( c \propto M^{-\beta} \) so that \( r_s \propto M^{(1+3\beta)/3} \).

Our focus is on the power spectrum in the high-\( k \) limit where the Fourier transform of \( \rho(r) \) may be approximated by a step function,

\[
\tilde{\rho}(k, M) \simeq M \Theta \left( 1 - kr_s(M) \right).
\]

(At wavenumbers above \( k = r_s^{-1}, \tilde{\rho} \) decreases as \( (kr_s)^{3-\gamma} \) but the Heaviside function provides a suitable form for our discussion.) Press and Schechter (1974) and Sheth and Tormen (1999) both provided analytic expressions for \( h(\nu) \) (see Eq. (3.52) & Eq. (3.54) in Section 3.1.4). In the small-\( \nu \)
Figure 6.6 The ratio $\Delta^2 / \Delta_L^2$ as a function of $k / k_{NL}$ for the high resolution simulations with $n = -1$ (top left), $n = -2$ (top right), $n = -2.25$ (bottom left), $n = -2.5$ (bottom right). Shown is the full range in $k$ probed by the simulation is shown. The black points are from the simulation. The red curve is the Smith et al. (2003) fitting formula. The magenta curve is PD96 fitting formula. The blue curve is our own fitting formula, Eq. 6.6. Plotted in the lower panel is the logarithmic slope, $\mu$ of the power spectrum (see text).
limit (i.e., common peaks corresponding to small $M$ and large $k$), one finds $v_h \propto v^p$ where $p = 0.5$ ($p = 0.2$) for $\text{PS}$ ($\text{ST}$). Putting all this together and making the substitution $u = k^{3+n/3} v$ it can be shown that

$$P_{1h} \propto \int \left( k^{3+n/3} u \right)^p \left( k^{-3+n/3} u \right)^{n/3} \Theta^2(1-u) du$$  \hspace{1cm} (6.11)

This gives that in the large-$k$ limit $\Delta_{1h}^2 \propto k^{\bar{\mu}_{1h}+3}$ where

$$\bar{\mu}_{1h} = \frac{-3 - p(n + 3)}{1 + 3\beta}.$$  \hspace{1cm} (6.12)

In Figure 6.7 on the next page we plot our results for the asymptotic slope of the nonlinear power spectrum, $\bar{\mu}$, together with those from the Smith et al. (2003) simulations. We make the point of displaying our results as upper bounds on $\bar{\mu}$ since the slope of $\mu$ appears to be a decreasing function of $k$ as $k \rightarrow k_{Ny}$ (see Figure 6.6 on the preceding page). For $n = -1$ and $-2$, these bounds agree with the quoted values from the Smith et al. (2003) simulations. Furthermore, the functional dependence of $\bar{\mu}$ on $n$ from their fitting formula appears to be consistent with our $n = -2.25$ and $n = -2.5$ results.

Our results suggest that $\mu$ increases with increasing $n$ and that $\bar{\mu}(n \rightarrow -3) = -3$. In other words, as $n \rightarrow -3$, the nonlinear dimensionless power spectrum becomes independent of $k$ (i.e., equal power per logarithmic wavenumber bin) just as with the linear power spectrum. Formulations of the halo model with constant, nonzero $\beta$ cannot reproduce this behaviour. To illustrate this point, we plot these predictions for $\bar{\mu}_{1h}$ assuming $\beta = 0.15$, as in Seljak (2000), and either the $\text{PS}$ or $\text{ST}$ values for $\alpha$. These predictions are inconsistent with our simulation results and those of Smith et al. (2003).

Clearly, the dependence of the concentration parameter on halo mass is central to the development of the halo model. For example, using the B01 model, which for scale-free cosmologies predicts $\beta = (n + 3)/6$ seems to yield the desired behaviour for $\bar{\mu}_{hh}$ in the $n \rightarrow -3$ limit. The lower panel in Figure 6.7 shows this prediction. While it does better than constant-$\beta$ versions of the halo model, the predicted $\bar{\mu}$ tends to lie above the values obtained in the simulation. Note that the Macciò et al. (2008) model is equivalent to the B01 model for scale-free cosmologies.
Figure 6.7 Asymptotic slope of the power spectrum, $\bar{\mu}$, as a function of the slope, $n$, of the linear power spectrum. The black points are from our simulations while the green points are from the Smith et al. (2003) simulations. In the upper left panel we show the predictions of the halo model (Eq. 6.12) for $\beta = 0.15$ and $p = 0.5$ (Press & Schechter). The upper right panel shows the prediction of the halo model assuming $\beta = 0.15$ and $p = 0.2$ (Sheth & Tormen). The lower left panel shows the predictions from PD96 (magenta curve) and Smith et al. (2003) (red curve). The lower right panel, shows the predictions of the halo model using B01 model for the $c-M$ relation.
6.4 Summary

The evolution of the power spectrum in scale-free cosmologies is remarkably simple — the di-
mensionless power spectrum, when written as a function of the ratio \( k/k_{NL} \), is time-independent. Obvi-
osely the finite computation volume of simulations breaks the scale-free nature of the problem
and leads to departures from the scaling solution. The dimensionless power spectrum provides a
simple test of whether finite volume effects have corrupted the simulation (Jain and Bertschinger
1998).

Our high-resolution \( n = -1, -2, -2.25 \) simulations demonstrate the scaling solution across
the simulation volume while showing clear differences with simulations performed at lower reso-
lution. Moreover, our results differ markedly from the the fitting formula provided by PD96 and
Smith et al. (2003). A plausible power spectrum for \( n = -2.5 \) was constructed by stitching together
outputs from different timesteps. Though it shows significant, and entirely expected departures from
the scaling solution, it represents our best estimate of the power spectrum for models with \( n \) this
small. We summarize our results for our four high-resolution simulations by means of a simple
empirical fitting formula. This formula has one more parameter than PD96 but two fewer than that of Smith et al. (2003).

The renormalization group improvements to perturbation theory developed in McDonald (2007)
represent a promising avenue for studying scale-free models with \( n < -2 \) or the low-mass limit of
the CDM hierarchy. Not surprisingly, the calculations become more difficult as \( n \to -3 \). Our
analysis of the \( n = -2 \) and \( -2.25 \) cases confirms McDonald’s claim that RG improves the power
spectrum predictions of perturbation theory when compared to simulations with the caveat that, as
\( n \) becomes more negative, the RG-prediction fails to capture the rapid rise seen in the simulations.
Our analysis for the \( n = -2.5 \) case is less conclusive but departures in the simulations from the
scaling solution suggest that some of the problem may reside there rather than in the PT analysis.
In principle, RG-improved PT can yield a handle on the form of the power spectrum in the mildly
nonlinear regime even as \( n \to -3 \).
The halo model provides a theoretical framework for understanding the two-point correlation function and nonlinear power spectrum. Our results, with respect to this model, are somewhat inconclusive. We agree with Smith et al. (2003) that the stable clustering hypothesis of Hamilton et al. (1991) and PD96 fails. On the other hand, the halo model appears to have difficulty reproducing the relation between the asymptotic slope of the power spectrum and $n$. We must therefore settle for an empirical fitting formula for the nonlinear power spectrum.
Chapter 7

Conclusion

7.1 Summary

I have examined a series of scale-free simulations, the largest yet run, in an effort to examine different scales in the Cold Dark Matter cosmology. Using two high resolution simulations, one with $n = -1$ and the other with $n = -2.5$, I find, contrary to the often made assumption, the subhalo mass function is not scale-free. In fact, subhalos show a dependence on the spectral index. There is less substructure at more negative spectral indices and the logarithmic slope of the mass function decreases with decreasing $n$. At $n = -1$, the index $\alpha \approx 0.9$ and at $n = -2.5 \alpha \approx 0.7 - 0.8$. Furthermore, subhalos are more triaxial and less tightly bound at the more negative index. As a consequence, the mass function appears to be more sensitive to the algorithm used to identify subhalos. It seems that dark matter halos do not have “universal” density profiles and substructure is not scale-free. There is no simple scaling which will map the subhalo mass function of one halo to that of another halo with a different mass.

I present a new phenomenological model for the subhalo mass function based on numerous studies of subhalos, including my own. I still assume that the subhalo mass function of a given halo is a power-law, $dN/d\ln f = A(f_{42})f^{-\alpha}$. In my model, the index $\alpha$ and the amount of substructure, given by $\log f_{42}$, are no longer constant. Instead these quantities depend on the host’s mass through
the effective spectral index $n_{\text{eff}}(M)$.

In the context of annihilating dark matter, I find this new model for the subhalo mass function predicts that substructure is unlikely to greatly enhance the total annihilation luminosity of a dark matter halo. Consequently, the $\gamma$-ray flux boost factor from substructure for satellites of the Galaxy ($\sim 10^7 - 10^8 \, M_\odot$) is $\lesssim 0.5$. Substructure also does not provide a significant global enhancement to the cosmic ray flux for monoenergetically produced of electron-positron pairs, such as would be produced by a KK particle. My study is not the only one to claim substructure does not greatly enhance annihilation flux. However, my work has focused on the cumulative effect of all objects in the subhalo hierarchy, i.e., subhalos, subsubhalos, etc. Other studies have neglected deeper levels in the subhalo hierarchy and focused solely on the effect of Galactic subhalos.

By examining the power spectrum of all the scale-free simulations I find that one-loop Perturbation Theory does not accurately reproduce the observed behaviour at $n \leq -2$ in the quasilinear regime. This discrepancy can be reduced by using a Renormalization Group scheme. Despite the improvement, I note that the discrepancy between RGPT and my simulations, though small, appears to increase as the index becomes increasingly negative.

I also find that previous fitting formulae for the nonlinear power spectrum do not describe the high resolution simulations presented. I present a new phenomenological formula that accurately describes the data. Note though that the coefficients of this fitting function do not appear to display a systematic dependence on the spectral index.

Finally, I agree with previous studies that the asymptotic logarithmic slope of the nonlinear power spectrum does not agree with the stable clustering hypothesis and favours the halo model. Yet, even the halo model does not appear to capture the observed behaviour. The slope lies below the prediction and continuously decreases. It appears that analytic models might not be able to capture the detailed evolution of the power spectrum and we might have to satisfy ourselves with empirical fitting functions.
CHAPTER 7. CONCLUSION

7.2 Discussion and Future Work

7.2.1 Subhalos

My claim that \( \alpha \) and \( f_{42} \) have a spectral dependence rests on the difference observed between the \( n = -1 \) and \( n = -2.5 \) simulations. It is possible that the observed change in these parameters might be, in part, due to the fact that the \( n = -2.5 \) runs suffers from finite volume effects. However, previous work by Elahi et al. (2005) indicated that including large-scale power actually decreases the amount of substructure, which would increase the difference between the two scale-free simulations. This early work and its implications should be treated with caution as the simulations in the study suffered from serious finite volume effects and normalization issues, particularly simulations with more negative spectral indices where the effect is the largest.

This claim might either be confirmed or refuted by the other scale-free simulations, particularly the \( n = -2.25 \) run, presented in Chap. 6, which shows negligible missing power. Unfortunately, I have not yet studied the substructure in all of the scale-free simulations as examining substructure is computationally more intensive than measuring the power spectrum. I hope to address these issues in the future.

A similar trend in \( \alpha \) and \( f_{42} \) is observed in the simulations by Springel et al. (2008) when comparing subhalo hosts to the halo. The subhalos have masses corresponding to more negative spectral indices than the host halo. I claim that the decrease in \( \alpha \) and \( A \) (or \( f_{42} \)) is due to the difference in the effective spectral index between the two mass scales. However, I have not investigated what affect, if any, a host’s level in the subhalo hierarchy has on its substructure. One way of determining if such an affect exists would be to run a scale-free simulation at high enough resolution to resolve subsubhalos. It would also be important to ensure that the results are free of numerical effects by using a spectral index at which the subhalo mass function appears to be insensitive to the subhalo algorithm used and where finite volume effects are easily minimized. For example, I could re-simulate one of the \( n = -1 \) halos. However, the required particle resolution and computation time is prohibitive, making such an endeavour unlikely for an artificial cosmology. Instead, I could run
a nested simulation of a cluster mass object (10^{14} M_\odot) and examine the substructure of its galactic mass subhalos.

There is also the question of halo-to-halo scatter. A number of studies indicate that subhalos distributions show halo-to-halo scatter ([Gao et al., 2004]). This scatter is due in part to the evolution of subhalos as they undergo mass loss and dynamical friction which also causes the subhalo distribution to evolve ([Taylor and Babul, 2004]). This indicates that the details of a halo’s formation history influences the subhalo population. I have not yet examined the halo-to-halo scatter in my own simulations. Instead, I stacked the subhalo population of approximately 20 similar mass halos originating from rare, high-sigma peaks to improve the statistics and examine the mean subhalo mass function. I could investigate the merger history of each halo in detail, which would shed light on whether the halo-to-halo scatter itself has a spectral dependence. I would also be able to examine how, for example, halo age correlates with the subhalo mass function and the parameters $\alpha$ and $f_{42}$.

However, to truly examine the halo-to-halo scatter and determine how either $\alpha$ or $f_{42}$ correlate with quantities such as the halo’s age, it might be necessary to re-simulate these halos at higher resolution. The reason why higher resolution might be required is that the two processes mentioned, dynamical friction and subsequent mass loss, should effect large subhalos. This appears to be clearly demonstrated by [Springel et al., 2008] study, which examined six similar mass halos with different formation redshifts. This study shows that the subhalo mass function shows that most of halo-to-halo scatter appears at $f \gtrsim 10^{-3}$, the high mass fraction end. The results from this study suggests that $\alpha$, if measured for $f \lesssim 10^{-3}$, would not show strong halo-to-halo scatter but $f_{42}$ would. Thus, it might be necessary to examine the subhalo mass function in terms of $f_{64}$, the fraction of mass between $10^{-6} - 10^{-4}$, and $\alpha$ to reduce these variations. The difficulty would be in comparing our results to other studies since the subhalo mass function is generally examined at mass fractions in the range of $10^{-4} - 10^{-1}$.

I also note that using the subhalo mass or velocity function itself may be an issue. These descriptions of the subhalo population are used because they provide a simple means of parameterizing the density distribution of a halo $\rho(x)$ into two components, one corresponding to the average radial
density profile, the other to subhalos. Such a parameterization lends itself to extrapolations of the effects subhalos have on quantities such as the dark matter annihilation rate. However, the subhalo mass and velocity functions appear to become increasingly sensitive to the way in which subhalos are identified as one approaches smaller scales and more negative spectral indices. Perhaps, a different approach is required.

Finally, it is likely that the amount of substructure contained in halos from cosmological simulations is actually underestimated by most algorithms since they are not able to disentangle the central high density, high velocity regions of these halos. Simulations show galactic halos contain numerous tidal streams from disrupted satellites in the outer regions (e.g. Diemand et al., 2008; Maciejewski et al., 2009). The central few kpc might contain numerous dynamically cold streams, remnants of tidally disrupted subhalos. It is unlikely that these streams comprise a large fraction of the host mass. Nevertheless, given the effect such velocity streams would have on the energy signature measured by direct dark matter detection experiments, it would be fruitful to quantify the stream population in dark matter halos.

However, most current algorithms used to find substructure are ill suited to this task. For instance, the two algorithms that use phase-space information, 6DFOF (see Section 4.5.1) and HFS (see Section 3.3.2 and Maciejewski et al., 2009) search for regions of high phase-space density. Streams are kinematically distinct and need not have a high physical density. Thus, searching for them requires a new approach. I plan on exploring various ideas to construct a stream finder. One idea involves determining the local Maxwellian velocity distribution and searching for outliers which are moving in roughly the same direction.

### 7.2.2 Boost Factors

I find large boost factors are very unlikely in the context of my model. However, it should be pointed out that these boosts are very sensitive to the concentration parameter (Martinez et al., 2009). I use the Macciò et al. (2008) model, which is an improvement over the Bullock et al. (2001) and Eke et al. (2001) models, but does not include the effect of baryons.
As baryons fall into the potential well of a dark matter halo, they will pull the dark matter along. Since baryonic matter can cool through radiative emission and collisions, it can become far more dense than dark matter. It is generally assumed that baryons will contract the dark matter leading to cuspier density profiles and higher concentrations. [Blumenthal et al. (1986)] proposed that the halos contracted adiabatically. [Gnedin et al. (2004)] altered this prescription to that it did not assume circular orbits, reducing the amount of pinching.

However, [Abadi et al. (2009)] found both prescriptions overestimate the effect. They compared two series of simulations, one that followed the evolution of dark matter in ΛCDM halos and another which included gas dynamics with just radiative cooling but neglected feedback mechanism. Feedback mechanisms, such as supernovae, re-inject energy into the gas thereby heating it and reducing its density. These gas simulations are unrealistic as gas cools too efficiently without any feedback process. This study also found that the amount of contraction depended not only how much baryonic mass has been deposited at the centre but also the mode of its deposition. It appears the baryonic contraction of a halo depends on the details of its assembly history. It is due to this uncertainty that I did not attempt to include baryonic pinching in my estimates of boost factors. And, although it is a simple matter to include a prescription for contraction into the calculations, the simulations by [Abadi et al. (2009)] suggest that any simple prescription is unrealistic.

Another issue is the density profile. In the calculations of boost factors I assume halos and subhalos have an NFW profile. Simulations in fact show that halos are described by the Einasto profile [Springel et al., 2008]. I find using an Einasto profile increases the boost factor by ≈ 30 – 50% for α = 0.9 and constant f_{12}. However, \( \bar{C} \), the volume integral of the square of the smooth density component, for an Einasto halo with a shape parameter common for (sub)galactic mass hosts (\( \alpha_E = 0.17 \)) is ≈ 50% smaller than a similarly concentrated NFW halo. The net affect is a decrease in the annihilation rate for Einasto halos.

Furthermore, simulations show that subhalos do not follow the density profile of the host. [Springel et al., 2008] found the subhalo volume distribution is characterized by an Einasto profile with a large characteristic radius (≈ 0.8 times the virial radius of the host) and a shape parameter of
\[ \alpha = 0.68. \] I argue that absence of substructure is a numerical effect. If however, there is truly a
dearth of subhalos in the Galactic centre, then for my model of monoenergetic production of \( e^+e^- \) pairs from a \( m_\chi = 700 \text{ GeV} \) particle, subhalos tend to be too far away to significantly contribute to
the cosmic ray flux. The result is \( B_{\text{eff}} \approx (1 - f_t)^2 \).

Another issue with the boost factor calculations done in this work lies in the use of the subhalo
mass function. Most studies of dark matter halos are interested in bound subhalos, that is long-lived
overdensities as opposed to transient ones. However, the annihilation signal scales as the density
squared, so whether an overdense region is bound or not is irrelevant. Using the distribution of
bound mass associated with an overdensity will underestimate the signal. For example, caustics,
which are unbound high density tidal streams, would enhance the indirect signal \cite{Vogelsberger2009}. On the other hand, transient overdensities tend to be small in comparison to long-lived
subhalos, or represent a small fraction of the host halo mass, so the effect of unbound overdensities
is likely negligible.

The effective boost in cosmic ray flux can vary greatly with the choice of annihilating channel. I
assumed the monoenergetic production of leptons since ATIC \cite{Chang2008} and PPB-BETS
\cite{Torii2008} showed a strong peak suggestive of such a source. However, measurements by
HESS \cite{F. Aharonian2009} and GLAST \cite{Abdo2009} do not show
a strong feature. In this case, perhaps a heavy neutralino with a mass of a few TeV might be the
optimal candidate. A heavy neutralino could produce very high energy muons, which subsequently
produce a cascade of high energy electrons. The volume of the dark matter halo which contributes
to the measured excess flux at a \( \sim 100 - 1000 \text{ GeV} \) would increase due to the larger diffusion
lengths, thereby reducing the effective boost factor required.

It is also possible that the annihilation rate is enhanced by the Sommerfeld effect \cite{Sommerfeld1931}. This is a low velocity enhancement to the annihilation cross section that increases \( \langle \sigma v \rangle \) by
\( 1/v \) and occurs due to multiple exchanges of a gauge boson between the two annihilating dark matter
particles. This gauge boson can either be a new force carrier \cite{Arkani-Hamed2009} or bosons
like the \( W^\pm \) \cite{Lattanzi2009}. \cite{Lattanzi2009} showed that the enhancement can
be appreciable if even a small component of the dark matter has velocity dispersions below several km/s. I could investigate this by altering the calculations and using scaling relations between halo mass and velocity dispersion. In this case, the issue might not be whether large effective boosts are possible but whether the cumulative effect of all levels in the subhalo hierarchy produce too large an enhancement.

### 7.2.3 The Nonlinear Power Spectrum

At the moment I have examined the new empirical fitting functions at only a few indices. The coefficients do not appear to have an obvious dependence on the spectral index. However, by filling in the gaps in $n$, say for example about transitions points such as $n \approx -1.4$ where one-loop PT corrections become negligible small, I should be able to determine the spectral dependence of the coefficients. This will enable us to model or interpolate the nonlinear power spectrum for arbitrary $n$ and therefore arbitrary shape.

Furthermore, only one particular RG scheme has been examined so far and only at a few indices. Although this RG scheme improves one-loop PT, there still appears to be a discrepancy which increases and begins earlier, i.e., at smaller values of the dimensionless power spectrum earlier, as $n$ decreases. Thus, filling in the gaps would not only improve the new fitting function but would help determine exactly what condition indicates when PT, or RGPT break down.

I find that the asymptotic slope of the power spectrum does not exactly match with predictions of the halo model. However, if halos truly follow Einasto profiles with ever increasing central logarithmic slopes, perhaps the discrepancy lies not in the model itself, but with the assumed form of the density profile. By re-simulating halos from the current simulations at higher resolution I will be able to measure their central logarithmic slope to very small radii and verify if the flaw is in the model itself.

Determining when various models hold and developing a fitting function that works for arbitrary power spectra would be useful in the estimation of cosmological parameters from surveys of large-scale structure.
Finally, I could investigate a somewhat technical issue, how accurately do re-simulations reproduce the nonlinear power spectrum. A re-simulation is where a high resolution region is nested inside a lower one so as to improve the mass resolution and include power at scales smaller than the parent simulation’s Nyquist scale, while still capturing the large scale tidal field of parent cosmological simulation. These simulations are now in common use (e.g., Diemand et al., 2006, 2007; Springel et al., 2008). An issue with re-simulations is that the density of modes sampled near the fundamental wavelength of the nested region is lower than it would if the entire parent simulation was at resolution of the nest region. This might reduce the size of mode coupling terms and could effect the nonlinear evolution. By using the high resolution simulations as a baseline, I could examine how accurately nest simulations reproduce the nonlinear power spectrum.
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