ESSAYS ON INFLATION AND OUTPUT: A SEARCH-THEORETIC APPROACH

by

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Abstract

This dissertation examines the welfare effects of inflation on employment and output in three different market settings. The theoretical frameworks build on recent studies in the monetary search literature that explicitly models the microfoundations of money and study how monetary policy interacts with real variables.

The first essay studies the relationship between inflation and unemployment in a general equilibrium framework where inflation has differential effects on employed and unemployed workers. The model finds that inflation can either increase or decrease employment and output, depending on goods and labor market institutions. Sales taxes, the degree of competitiveness in the goods market and imperfect indexation of unemployment insurance benefits are the major factors determining the direction of this relationship. Through a comparison of these parameters, the model predicts an inflation-unemployment relation that is qualitatively consistent with the empirical evidences.

The second essay, co-authored with Liang Wang and Randall Wright, investigates the effect of inflation on people’s trading behavior in the goods market. By focusing on buyers’ search intensity on the extensive margin, the model unambiguously predicts a rise in inflation leads to an increase in the speed with which agents spend their money and velocity. This is consistent with the phenomenon described by the conventional "hot potato" effect of inflation. We also discuss the welfare implications of different monetary policy. In some
circumstances inflating above the Friedman rule may be optimal, but the effect of inflation on output is always negative.

The third essay, co-authored with Allen Head, Guido Menzio and Randall Wright, examines the effect of monetary growth on output in a general equilibrium model where price stickiness arises as an equilibrium outcome. The model makes several predictions about individual firms’ price adjustment behavior that are consistent with micro data. For instance, the frequency (duration) of price changes increases (decreases) with inflation and the price change hazard declines over time. In contrast to the New Keynesian literature, price rigidities in our model does not generate monetary non-neutrality. Higher inflation reduces real output in the long run, but changes in the aggregate price level has no effect on real allocations.
Co-Authorship

Chapter 2 of this thesis was co-authored with Liang Wang and Randall Wright. Chapter 3 was co-authored with Allen Head, Guido Menzio and Randall Wright.
Dedication

To my parents: for your constant encouragement and unconditional love. Without your support, this would not have been possible.
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Chapter 1

General Introduction

The effects of inflation on economic activity, in particular production and output, has been a primary focus of macroeconomic studies for many decades. The existing literature has developed different theories to answer this question. For example, the classical quantity theory of money states that permanent increases in money growth and inflation have no real effects in the long run. Economic activities such as capital accumulation and production are determined by fundamentals that are invariant to anticipated monetary expansions (for example, Sidrauski 1969). The argument of the inflation tax effect, however, emphasizes that a higher growth rate of money stock erodes money’s purchasing power and reduces its real return. Depending on the specific roles of money in the economy, inflation acting as a tax on money holdings could cause different distortions on households decisions.

For instance, in the neoclassical models of Mundell (1963) and Tobin (1965), inflation changes real output through its impact on capital accumulation. A higher rate of inflation lowers the return from holding money balances and causes a substitution towards capital investment. Such an adjustment in agents’ portfolio raises the steady-state capital stock and thus the long-run real output. This is known as the Mundell-Tobin effect. In contrast,
Stockman (1981) developed a model that predicted a negative relationship between inflation and capital accumulation. In his economy, agents face a cash-in-advance constraint on both consumption and investment spending. A rise in inflation reduces the purchases of both consumption goods and capital, and thereby lowers output in the long run. In addition to the effect on capital, changes in inflation may also lead to distortions on labor decisions.

The natural rate hypothesis in Friedman (1968) and Phelps (1967) states that employment and output are independent of inflation variations in the long run, whereas the inflation tax effect in Greenwood and Huffman (1987) and Cooley and Hansen (1989) suggests a negative correlation between inflation and employment. In the latter case, a higher rate of inflation causes agents to substitute away from cash goods - consumption to leisure, which reduces labor supply and the equilibrium output.

Recent papers have studied these effects in models that explicitly consider market frictions. By introducing search and matching frictions into the goods market, the search-theoretic models have provided micro-foundations for the use of money as a medium of exchange (e.g. Shi, 1997; Lagos and Wright, 2005; Rocheteau and Wright, 2005). This has important implications for the welfare analysis of the inflation tax effect both qualitatively and quantitatively. In addition, the consideration of frictions also allows researchers to explore additional channels through which inflation impacts the real economy. For example, the existence of search frictions implies that agents need to invest on time to look for a trading partner. By eroding the real return of money holdings, inflation encourages people to search harder for either a better deal or a higher probability to spend their money. This could lead to an increase in the number of trades and the aggregate output (Shi, 1998; Shi 1999; Head and Kumar, 2005).
The findings in the empirical literature are mixed. While early studies seemed to support the natural rate hypothesis (Lucas, 1973; Alberro, 1981; Kormendi and Meguire, 1984; King and Watson, 1997), Ericsson, Irons, and Tryon (2001) and Rapach (2003) found evidences of a positive long-run response of the real output to a permanent inflation shock. In addition, the empirical tests in Bullard and Keating (1995) suggested that inflation and output might be positively related in low inflation countries, but the correlation was close to zero or negative in high inflation countries.

An important message from previous studies is that the effects of inflation depend on the role of money in the economy and how money interacts with real variables. As such, models that are used to examine the inflation-output relation should be explicit about fundamentals that give rise to the specific roles of money. This dissertation builds on the recent developments in the micro-founded monetary literature, and studies the real effects of inflation in three different settings, where market frictions are explicitly modeled to introduce the roles of money in the economy.

Chapter two investigates the inflationary effects of monetary policy on labor market decisions. The model extends search-theoretic models of labor and money in Berentsen, Menzio and Wright (2008), by introducing the differential effects of inflation on employed and unemployed workers into their economy. Due to either imperfect indexation of the unemployment insurance (UI) benefit or heterogeneous money holdings, the unemployed are exposed to a larger cost of inflation than the employed. A higher rate of inflation reduces the wealth of unemployed workers and raises their incentives to work. This generates a positive effect on employment. On the other hand, inflation reduces the profits of sales that involve cash transactions, and lowers firms’ return from hiring workers, thereby raising unemployment. In steady state the effects of inflation on employment and output are either
positive or negative, depending on goods and labor markets institutions.

The theoretical model is solved numerically to demonstrate how different institutions affect the long-run inflation-unemployment relationship. Quantitative examples suggest that sales taxes, the degree of competitiveness in the goods market and imperfect indexation of unemployment insurance benefits play major roles in determining the direction of this relationship. Through a comparison of these parameters, the model is applied to explain the different long-run Phillips curves observed across different countries. For instance, many empirical studies have demonstrated that the inflation-unemployment relationship tends to be positive in the US, while negative in Europe. One consensus in the comparison of market institutions is that the sales tax is higher and UI benefits are more generous and imperfectly indexed in most European countries than those in the US. Both of these features favor a negative relationship between inflation and unemployment in the model. Due to the interactions between labor and goods market frictions, the welfare implications of this model are not straightforward. The optimal monetary policy depends on both labor market efficiency and the specific relationship between inflation and unemployment. In the presence of distortions in the labor market, deviating from the Friedman rule may be optimal.

Chapter three, co-authored with Liang Wang and Randall Wright, examines how inflation changes people’s spending behavior and thus the velocity of money in the long run. Conventional wisdom tells that high inflation or nominal interest rates make people spend their money faster, trying to get rid of it like a "hot potato", and this is a channel through which inflation changes velocity and output. Monetary theory with endogenous search intensity seems ideal for studying this phenomenon, since it explicitly models people’s search and spending behavior in the goods market. However, in the standard environment,
the inflation tax effect always induces a lower surplus from monetary exchange and hence discourages buyers to trade. This means that in these models buyers search less and end up spending their money more slowly. The approach adopted in our model is to consider buyers’ decision to participate in monetary exchange in the first place rather than their search intensity, i.e., it focuses on the extensive rather than the intensive margin. The model robustly predicts a positive effect of inflation on the speed with which buyers spend their money, as well as velocity.

We also analyze the welfare implications of monetary policy. In the standard monetary model with bargaining, two conditions are necessary to eliminate inefficiency in the goods market: zero nominal interest rate plus buyers having all the bargaining power. In our model, however, these two conditions only achieve efficiency along the intensive margin. With search and matching externalities, low inflation could potentially generate too many entries by buyers. Hence, inflating above the Friedman rule may be optimal. Nevertheless, the effect of inflation on aggregate output is always negative due to the reduction in the number of buyers trading in the market.

The fourth chapter, co-authored with Allen Head, Guido Menzio and Randall Wright, examines the implications of monetary policy on output in a model where price stickiness is a feature of the equilibrium outcome. The model presents a general equilibrium framework in which sellers post nominal prices that may not respond to changes in the aggregate price level. Due to search frictions, monetary equilibrium is characterized by a non-degenerate price distribution, with low price sellers earning less per unit but selling a larger quantity. This implies that when the money supply increases, it is not necessary for all sellers to raise prices, as long as the distribution of real prices remains invariant. Profit maximization is consistent with some sellers resetting prices infrequently, even though they are allowed to
change prices for free whenever they like.

The model has many testable predictions. We derive closed-form solutions for the price density and determine how it depends on inflation. In terms of the pattern of price changes, we find that sellers who change at a point in time need not choose the same new price – they may raise or lower their prices, despite positive inflation. The model also makes several other predictions that can be compared with the data. For example, the model suggests that the frequency (duration) of price changes is increasing (decreasing) with inflation and the price change hazard is declining over time. All of these are consistent with the evidence of micro price data in Nakamura and Steinsson (2008) or Klenow and Kryvtsov (2007).

Although our theory provides microfoundations for a key feature of Keynesian economics – price stickiness, the policy implications are very different. Our model implies classical neutrality: individual prices appear to be sticky, but shocks to money have no effect on real allocations. Nevertheless, money is not super – neutral in the long run: as in standard monetary models, inflation imposes a tax on monetary exchange and reduces real output.
Chapter 2

Inflation and Unemployment: The Role of Goods and Labor Market Institutions

2.1 Introduction

This chapter develops a general equilibrium model to study the effects of inflation on unemployment and output. The main framework extends the environment in Berentsen, Menzio and Wright (2008) by allowing inflation to have differential effects on employed and unemployed workers. As in their economy, a higher rate of inflation reduces firms’ profits in the goods market and discourages them from hiring workers. But at the same time, the differential effect implies that unemployed workers are exposed to a larger inflation cost comparing to the employed, so they are more willing to accept a job offer as inflation rises. In equilibrium, the implied inflation-unemployment relationship is either positive or negative, depending on goods and labor market institutions.

The link between inflation and employment has been a major focus in macroeconomic literature for many decades. Empirical studies employ various methodologies to estimate
this relationship, and find different slopes of the long-run Phillips curve across countries\(^1\). For example, based on U.S. low frequency data, Beyer and Farmer (2007) and Berentsen, Menzio and Wright (2008) both document a positive correlation between inflation and unemployment. Karanassou, Sala and Snower (2003), on the other hand, estimate a panel regression for 22 European countries and find a long-run trade-off in the two variables. The similar negative relationship is also presented in Franz (2005) and Schreiber and Wolters (2007), with both focusing on the time series for Germany.

In spite of the divergent empirical evidence, most theoretical literature predicts inflation and unemployment to be positively correlated. Early paper by Cooley and Hansen (1989) introduces a cash-in-advance constraint into Rogerson’s (1988) indivisible labor model. In their economy, a rise in anticipated inflation reduces labor supply due to a substitution between consumption and leisure, and increases unemployment\(^2\). Recent studies analyze this relationship in models that explicitly incorporate search frictions in labor and goods markets, for example, Lehmann (2006), Berentsen, Menzio and Wright (2008) and Kumar (2008). Inflation in these environments acts as a tax on activities that involve cash transactions. A higher rate of inflation reduces the demand for money balances, and lowers the profit of selling in the goods market. This discourages firm from hiring workers, thereby raising unemployment.

The micro-founded models of labor and monetary exchange provide a natural environment to study the mechanism through which inflation affects employment and output. However, despite the large heterogeneity between employed and unemployed workers, little attention has been paid to the differential effects of inflation. In particular, with different

\(^{1}\)In the literature, the "long-run Phillips curve" is often used to describe the NAIRU (non-accelerating inflation rate of unemployment). Here it simply refers to the long-run relationship between inflation and unemployment.

\(^{2}\)A separate strand of literature studies this issue in environments with nominal or real rigidities, for example, Blanchard and Gali (2008), Gertler and Trigari (2007), and Faia (2008).
income resources and portfolio choices, employed and unemployed workers are subject to different degrees of the inflation cost. Since agents’ incentives to work depend on the relative income levels of different employment status, the heterogeneous effects potentially present an additional channel through which inflation influences wage determination and employment.

This chapter develops a general equilibrium model to investigate the implications of these asymmetric effects. The main framework combines the random matching model of labor in Mortensen and Pissarides (1994) with the monetary search economy of Lagos and Wright (2005). The model gives rise to an inflation-unemployment relation that is either positive or negative, depending on goods and labor market institutions. The positive effect is generated through the same labor demand mechanism as discussed in the previous literature. The negative effect, however, arises from the asymmetric effects of inflation on unemployed and employed workers. In an economy where the unemployed are more vulnerable to the inflation tax, a higher rate of inflation reduces the value of outside option for workers and lowers their bargained wages. This encourages firms to create more job vacancies and lowers unemployment. At the same time, the reduction in the outside value also raises workers’ incentives to work and further pushes down the unemployment rate.

The crucial element in this model is the asymmetric effects of inflation. The model presents two cases in which inflation affects employed and unemployed workers differently. The first case is to consider the imperfect indexation of government policy. In countries where the unemployment insurance (UI) benefit is not perfectly indexed, a higher rate of inflation directly lowers the real income of unemployed workers. Comparing with the employed ones who receive constant real wages, imperfect indexation induces a larger
welfare loss for the unemployed. The second case is based on the observation of heterogeneous money holdings across households. The empirical evidence described in previous studies, for example, Avery et al. (1987), Mulligan and Sala-i-Martin (2000), Attanasio et al. (2002) and Erosa and Ventura (2002), strongly suggests that low income households tend to hold more cash in their portfolios and also use cash for a larger fraction of their expenditure. These differences in cash balances imply that low income households (the unemployed) will suffer a larger welfare loss in an inflationary economy, as their assets are more exposed to the inflation tax. Overall, the two types of heterogeneous effects offer the same implication: as long as unemployed workers are hurt by inflation to a larger extent, the value of their outside option is going to decrease with inflation, and a negative effect of inflation on unemployment arises\(^3\).

Given the two opposite effects of inflation on unemployment, which one dominates depends on the source of heterogeneity and the corresponding market institutions. In an economy with imperfect policy indexation, the level of UI benefit directly determines the sign of the net effect. More generous UI benefit implies the unemployed suffer a larger cost when inflation rises, and therefore favors a negative inflation-unemployment relation. With perfect indexation, however, when the differential effect comes from heterogeneous money holdings only, labor market institutions such as UI benefit and worker’s wage bargaining power only generate a second-order general equilibrium effect. In this case, goods market institutions become the major determinants. For instance, a higher sales tax rate and a more competitive goods market (proxied by firms’ bargaining power) both reduce firms’ profits and weaken the negative effect of inflation on job creation, whereas a larger difference in

\(^3\)Other studies that emphasize the distributional effects of inflation associated with the negative correlation between labor income and money holdings include Albanesi (2007a), da Costa and Werning (2007), and Albanesi (2007b). The first one focuses on the correlation between inflation and income inequality, and the latter two study the optimal fiscal and monetary policy in a heterogeneous agents economy.
cash holdings boosts the heterogeneous effect and favors an inflation-unemployment trade-off. Through a comparison of different market institutions, the model enables us to explain the different sloped long-run Phillips curves observed in different countries. For instance, by comparing the sales tax and UI benefit in U.S. and Europe, the model predicts inflation-unemployment relationships that are qualitatively consistent with the empirical evidences reviewed above.

The results in this chapter to some extent are consistent with those of Rocheteau, Rupert and Wright (2008) and Dong (2008). Both these studies integrate Rogerson’s indivisible labor into the monetary search environment of Lagos and Wright (2005). The long-run Phillips curves in their economies are also either positive or negative, but the net effect depends on specifications of preferences. In contrast, my model suggests that institutions in goods and labor markets are the determinant factors.

Shi (1998) studies this relationship in a model where agents choose their search intensities in both goods and labor markets. Inflation in his economy has two opposing effects, but the enhancing effect hinges on the positive correlation between inflation and search intensity. Reed and Schreft (2007) introduce financial market frictions into a labor search economy. In their model, an increase in inflation lowers the return on financial assets. Depending on workers’ risk aversion parameter, the negative effect on asset values may induce workers to exert more or less effort to work and hence generates different sloped long-run Phillips curve.

In terms of the mechanism through which inflation encourages employment, the paper by Heer (2003) is closely related. In his model, the reservation wage of household depends on the consumption of cash goods. An increase in inflation lowers consumption and the bargained wage, thereby encouraging firms to hire more workers. Money in his economy
is introduced through a cash-in-advance constraint and the cost of inflation is reflected by the capital-money substitution effect. In contrast, my model explicitly considers frictions in the goods market, which allows one to directly examine the effect of inflation on the cost of holding money, and investigate the interaction between market institutions and monetary policy.

The remainder of this chapter is organized as follows. The next section describes the basic economic environment. The third section defines the stationary monetary equilibrium, as well as characterizes the conditions under which a unique equilibrium exits. Section four compares the inflation-unemployment relationship in economies with and without asymmetric effects of inflation. Section five presents the numerical results that demonstrate how different goods and labor market institutions affect the long-run inflation-unemployment relationship. Finally, conclusions are given in Section six.

### 2.2 The Economy

Time is discrete and continues forever. Each period consists of two subperiods. In the first sub-period, a labor market and a decentralized goods market open at the same time. The labor market, or LM, follows the structure of the search economy in Mortensen and Pissarides (1994, 1999), with match specific productivity. The decentralized goods market, or DM, is characterized by a random matching environment, where money is essential as a medium of exchange due to frictions such as anonymity, double coincidence of wants, and limited commitment. To capture the idea of heterogeneous cash expenditure in consumption, we allow part of the trades to be carried out using credit. Nevertheless, as long as some bilateral meetings are anonymous, money is still essential in this economy. In the second sub-period, there is a centralized goods market, or CM, where frictionless Walrasian
trades can take place with or without money. The CM/DM structure follows from Lagos and Wright (2005).

The economy is populated by two types of private agents: workers and firms, indexed by $w$ and $f$, respectively. Workers work in the LM and enjoy utility from consuming in both the DM and CM. The measure of workers is normalized to 1. Firms hire workers to produce. They sell their outputs in the goods market. Assume the measure of firms is arbitrarily large. As is standard in search theory of labor, a worker and a firm are brought together through some matching technology. When they meet, a match-specific productivity $y$ is drawn from a distribution $F$ with support $[y_l, y_u]$, and both the worker and the firm can observe this productivity. In general, there exists a reservation value, $y_R$, which is characterized below, such that firms and workers agree to match if and only if $y \geq y_R$. For simplicity, assume that a successful job match with productivity $y$ produces $y$ units of output each period.

Let subscript $e$ indicate employment status: $e = y$ if a worker is successfully matched with a firm and together they produce with productivity $y$; and $e = 0$ otherwise. The utility of a worker has a quasi-linear form: $v(q) + x$, where $q \in \mathbb{R}_+$ is the amount of goods consumed in the DM, and $x \in \mathbb{R}_+$ is the consumption in the CM. Assume $v$ is trice continuously differentiable with $v' > 0$, $v'(\infty) = 0$ and $v'(0) = \infty$. In the DM with probability $\alpha_e$, a worker is matched with a firm to trade. Let $\rho_e$ denote the probability that this match is anonymous, in which case firms only take money in exchange for goods. Then with probability $1 - \rho_e$, workers can use credit. In the baseline economy, employed and unemployed workers may have different $\alpha_e$ or $\rho_e$, which are assumed to be exogenous. But all workers that have a job face the same probabilities, i.e., $\alpha_y = \alpha_1$ and $\rho_y = \rho_1$, for all $y$.

\[4\] These assumptions are adopted to keep the focus on the effect of heterogeneity between employed and
In addition to the private agents, there exists a government who consumes $G$ in each period. To examine how different policies affect the equilibrium outcomes, the model considers two types of fiscal policies: a sales tax on each dollar transaction in the DM and an unemployment insurance (UI) benefit paid to unemployed workers in the CM. The government also conducts monetary policy to increase the stock of money at a gross rate $\gamma$.

To understand how consumption and production take place in each market, it is useful to first describe the timing of events within each period $t$. As demonstrated in Figure 2.1, agents enter into the first sub-period with different employment status. Employed workers join firms to produce output $y$, while the unemployed start to search for jobs. At the same time, both types of workers can participate in the DM and purchase goods for consumption. New jobs are created and some existing ones are destroyed at the end of Sub-period 1. In Sub-period 2, workers first receive their incomes, either in forms of wages (for the employed) or UI benefits (for the unemployed). Firms then distribute their profits to workers as dividend payments. Through exchanges in a Walrasian market, workers acquire consumption, as well as currency to carry into period $t + 1$.

Now consider agents’ optimal choices in a representative period $t$. Let $V_{e}^{j}$ and $W_{e}^{j}$ denote the expected value of a type $j \in \{w, f\}$ agent during Sub-period 1 and Sub-period 2, respectively, conditional on employment status $e$. As the following analysis focuses on steady state only, the time subscript $t$ is dropped for ease of notation. Denote $\hat{X}$ as the value of any variable $X$ in the next period.

unemployed workers, and at the same time maintain simplicity. One can easily endogenize $\alpha_e$ by introducing a matching function in the DM or letting buyers to choose their search intensity. $\rho_e$ can be justified through worker’s choice of transaction patterns, or a credit limit imposed on the DM trade, which depends on individual’s income in the CM. In the later case, the credit extended to the unemployed is lower than the employed, as they have less incomes. In any of these contexts, all major analytical results remain unchanged.
2.2.1 Workers

We now examine workers’ value functions in each sub-period, starting with the second one. A worker that enters the CM with employment status \(e\) and money balances \(m_e\), chooses consumption \(x\) and money holdings for next period \(\hat{m}_e\), to solve

\[
W^w_e(m_e) = \max_{x, \hat{m}_e} \left[ x + \beta \hat{V}_e^w(\hat{m}_e) \right] \tag{2.1}
\]

s.t. \(x = \phi(m_e - \hat{m}_e) - T + \Psi\),

where \(\phi\) is the real price of money in the CM, \(T\) is a lump-sum tax, and \(\Psi\) is the real dividend income. The assumption of linear preference follows Mortensen and Pissarides (1994), and also keeps the model analytically tractable as in Lagos and Wright (2005).

Using the budget constraint to eliminate \(x\) in (2.1), we obtain

\[
W^w_e(m_e) = \phi m_e - T + \Psi + \max_{\hat{m}_e} \left[ -\phi \hat{m}_e + \beta \hat{V}_e^w(\hat{m}_e) \right]. \tag{2.2}
\]

The first order condition with respect to \(\hat{m}_e\) gives

\[
\phi = \beta \hat{V}_e^w(\hat{m}_e), \tag{2.3}
\]
and the envelope condition implies $W_e^w$ is linear in $m_e$: $W_e^w(m_e) = \phi$. As in Lagos and Wright, the linear preference eliminates the wealth effect and makes the choice of $\hat{m}_e$ independent from agent’s current money balances. It may depend, however, on workers’ employment status $e$, if employed and unemployed workers expect different consumption levels over the next period.

Moving back to the first sub-period, workers with different $e$ are engaged in different activities. The employed ones work in the LM and consume in the DM with probability $\alpha_1$. At the end of Sub-period 1, they lose their jobs at an exogenous rate $\delta$. Separated workers stay unemployed for one period. Those who survive the separation shock continue to earn the same wage next period. The nominal wage $w^n(y)$, which is a function of productivity $y$, is distributed at the beginning of the CM\(^5\). Using the linearity property of $W_e^w$, the expected value for an employed worker can be written as

\[
V_{w}^y(m_y) = \alpha_1 \left\{ \rho_1 [v(q_y) - \phi d_y] + (1 - \rho_1) [v(q^c_y) - \phi d^c_y] \right\} \\
+ \delta W^w_0(m_y + w^n(y)) + (1 - \delta) W^w_y(m_y + w^n(y)),
\]

where $q_y$ is the amount of goods purchased using cash $d_y$, and $q^c_y$ is the amount obtained through credits. The crucial difference is that cash transactions are subject to the constraint $d_y \leq m_y$, while in credit trades workers consume $q^c_y$ first and pay $d^c_y$ in the CM\(^6\). The amount purchased in the credit transaction is unconstrained \(^7\). The terms of trade $(q_y, d_y)$ and $(q^c_y, d^c_y)$ are both determined in Section three. Note that the value function in (2.4) consists of two parts. The first item on the right hand side is the net payoff of consuming

\(^5\)Note that firms pay wages in the CM, so it makes no difference if they are paid in nominal or real terms.

\(^6\)Given the quasi-linear utility in the CM, one can easily show that the credit repayment $d^c_y$ enters into the value function $W^w_e$ linearly. Hence, for simplicity I skip the steps of solving credit repayment problem in the CM, and express the value function $V_{w}^y$ in terms of net payoff $v(q_y) - \phi d_y$ directly.

\(^7\)Here we are abstract from issues like default. Since credit trade meetings are not anonymous, I assume workers commit to pay credit back in the CM.
in the DM, which depends on the probability of trade, \( \alpha_1 \), and the expected utility from cash and credit purchases. The terms in the second line represent the expected continuation values of carrying \( m_y \) units of money into the CM. As wages are paid at the beginning of sub-period 2, workers’ nominal balances are independent of their employment status.

Unemployed workers, on the other hand, enjoy leisure \( l \) in Sub-period 1 and receive an UI benefit \( B \) once entering the second sub-period. In addition, they search for jobs in the LM and consume with probability \( \alpha_0 \) in the DM. A job offer arrives at the rate \( \lambda_w \). Workers accept an offer if and only if \( y \geq y_R \), but they do not start working until the next period. Overall, the expected value for an unemployed worker entering sub-period 1 with \( m_0 \) dollars is

\[
V_w^u(m_0) = l + \alpha_0 \{ \rho_0 [v(q_0) - \phi d_0] + (1 - \rho_0) [v(q_0^c) - \phi d_0^c] \}
+ \lambda_w \int_{y_R} W^w_y(m_0 + B) dF(y) + [1 - \lambda_w + \lambda_w F(y_R)] W^w_0(m_0 + B)
\]

(2.5)

where \([1 - \lambda_w + \lambda_w F(y_R)]\) is the probability that the worker receives no job offer, or he gets one but turns it down. The job arrival rate \( \lambda_w \) is endogenously determined by the matching technology, \( \lambda_w = M(u, v)/u \), where \( u \) is the unemployment rate and \( v \) is the number of vacancies created by firms. Following the usual assumptions, \( M \) is nonnegative, increasing in both arguments and concave. Moreover, it displays constant returns to scale, so the matching function can be rewritten as \( \lambda_w = M(1, \sigma) \), where \( \sigma = v/u \) is the measure of labor market tightness.

### 2.2.2 Firms

In Sub-period 1, firms with \( e = 0 \) have no production, so \( V^f_0 = 0 \). Matched firms produce output \( y \) in the LM and with probability \( \alpha_f \), they sell their products in the DM immediately.
In general, $\alpha_f$ depends on $\alpha_0$, $\alpha_1$ and the total number of matched firms. Once a firm sells $q_f$ units of output, the rest $y - q_f$ is carried into the CM as inventory. Without loss of generality, assume that each unit of unsold product can be transformed directly into one unit of consumption goods in the CM. In other words, the opportunity cost of selling in the DM is linear: $c(q_f) = q_f$. Putting together, the expected value of a firm with $e = y$ is expressed as

$$V_f^y = \alpha_f \left[ \delta W_{0}^f(y - q_f, (1 - s)d_f - w^n(y)) + (1 - \delta) W_{y}^f(y - q_f, (1 - s)d_f - w^n(y)) \right]$$

$$+ (1 - \alpha_f) \left[ \delta W_{0}^f(y, -w^n(y)) + (1 - \delta) W_{y}^f(y, -w^n(y)) \right], \quad (2.6)$$

where $(1 - s)d_f$ is the after-tax revenue received from selling $q_f$ units of output, including both money and credit trades with the two types of workers. The equilibrium conditions for $(q_f, d_f)$ are specified below. Note that the firms’ valuation in this sub-period is a convex combination of continuation values in the subsequent CM. With different matching status, $e$, $W_c^f$ are functions of total inventory brought into the second sub-period, as well as their cash balances after wage payment. Here it is assumed firms commit to pay wages $w^n(y)$ to their previous employees.

Moving to the second sub-period, it should be obvious that no firm acquires positive $\hat{m}$, as they do not need cash during Sub-period 1. The expected value for a matched firm with $x$ units of inventory and $d_f$ dollars of sale revenue is simply

$$W_{y}^f(x, d_f) = x + \phi d_f + \beta V_f^y. \quad (2.7)$$

The $e = 0$ type of firm decides whether to open a vacancy in the next LM by paying a fix cost $k$. With $\lambda_f$, they will be matched with a worker, but production takes place if and only
if the realization of $y$ is higher than $y_R$. Hence, the value of a firm with $e = 0$ is

$$W_f^f(x, d_f) = x + \phi d_f + \max \left\{ 0, -k + \beta \lambda_f \int_{y_R} W_{y}^f(0, 0) dF(y) + \beta [1 - \lambda_f + \lambda_f F(y_R)] W_0^f(0, 0) \right\}. \tag{2.8}$$

By constant returns, $\lambda_f = \mathcal{M}(u, v)/v = \mathcal{M}(1/\sigma, 1)$. With free entry, firms continue to post vacancies until the net surplus equals 0. So in equilibrium, $W_0^f(x, m) = x + \phi m$ and

$$k = \beta \lambda_f \int_{y_R} W_{y}^f(0, 0) dF(y) + \beta [1 - \lambda_f + \lambda_f F(y_R)] W_0^f(0, 0). \tag{2.9}$$

Combining (2.6) and (2.7) together and imposing the steady-state condition $V_f^f = \hat{V}_f^f$, we then obtain a steady-state version of (2.9)

$$k = \frac{\beta^2 \lambda_f}{1 - \beta (1 - \delta)} \int_{y_R} [y - w(y) + \pi] dF(y), \tag{2.10}$$

where $\pi = \alpha_f [(1 - s) \phi d_f - q_f]$ denotes the expected after-tax profits from selling in the DM.

At the aggregate level, the average profit across those firms that are in production is

$$(1 - u) \int_{y_R} [y - w(y) + \pi] dF(y)/[1 - F(y_R)] - vk.$$ Assume that all workers, independent of their types, hold the same portfolio of shares. Then the real dividend $\Psi$, allocated to each worker in the CM is simply equal to the average profits.\footnote{Note that, with quasi-linear utility, how much dividend workers receive has no effect on their optimal decisions of $x$ and $\hat{m}_e$. This is true even if we allow workers to choose the amounts of shares in the firm.}

### 2.3 Equilibrium

To characterize the equilibrium, it remains to specify the pricing mechanisms in each of the three markets. Following standard specifications in the related literature, the CM is assumed to be perfectly competitive and agents are price taker, while the LM is characterized
by Nash wage bargaining. The terms of trade in the DM are also determined by pairwise bargaining, although one can easily modify this to allow for other pricing protocols. One advantage of the bargaining mechanism is that it allows us to use seller’s bargaining power as a proxy for the degree of competition in the goods market, and examine how different market structures affect the implied inflation-unemployment relation.

2.3.1 Optimal Decisions in the DM

We now examine the equilibrium conditions in the DM. Let \( \eta \) denote the seller’s bargaining power. In any monetary trade, the surplus for a worker from trading \( q_e \) units of goods is \( v(q_e) - \phi d_e \), and the surplus for the firm is \((1 - s)\phi d_e - q_e\), as the government collects \( s\phi d_e \) dollars of sales tax from each transaction. Workers cannot spend more than the amount of money they have, so the generalized Nash bargaining problem is to

\[
\max_{q_e,d_e} [v(q_e) - \phi d_e]^{1-\eta} \cdot [(1 - s)\phi d_e - q_e]^\eta \quad \text{s.t.} \quad d_e \leq m_e \quad \text{and} \quad q_e \leq y.
\]

The standard results in Lagos and Wright (2005) and related models apply here. First, \( d_e = m_e \) for any \( q_e \leq q_e^* \), where \( q_e^* \) is the first best allocation satisfying \( v'(q_e^*) = 1 \). Secondly, the first order condition for \( q_e \) yields

\[
\phi m_e = g(q_e) = \frac{\eta v(q_e) + (1 - \eta) v'(q_e) q_e}{(1 - s)(1 - \eta)v'(q_e) + \eta} \quad (2.11)
\]

\(^9\)In the working paper, Berentsen, Menzio and Wright (2008) present alternative versions with price taking and price posting with directed search. In general the qualitative results under bargaining and price taking are consistent, as long as firm’s profits are positive. The modification to directed search, however, requires some additional changes in this environment, as it considers endogenous trading probabilities in the DM.

\(^10\)An alternative way to model the sale tax is to let workers (buyers) pay \((1 + s)\phi d_e\) when they make the purchase. Given the cash constraint, in equilibrium workers pay \( \phi m_e \) and firms receive \( \phi m_e / (1 + s) \), which leads to similar results as in the above setup.

\(^11\)It is assumed that the distribution of \( y \) is high enough such that firms always have the capacity to satisfy the demand in the DM.
The bargaining problem for credit trades has the similar format, except there is no cash constraint. In this case, Nash bargaining yields

\[ v'(q^c) = \frac{1}{1 - s} \quad \text{and} \quad \phi d^c = \eta v(q^c) + (1 - \eta)v'(q^c)q^c. \] (2.12)

Notice that \( q^c \) and \( d^c \) are independent of the amount of money workers bring into the DM, so the units of goods consumed in the credit trade are the same for different types of workers. The consumption in the monetary trade, however, is related to the choice of \( \hat{m}_e \). Inserting \( d_e = m_e \) into (2.4) and (2.5) and using (2.11), we obtain the marginal benefit of carrying money into the DM

\[ V^{w'}_e(m_e) = \alpha_e \rho_e \left[ \frac{\phi v'(q_e)}{g'(q_e)} - \phi \right] + \phi. \] (2.13)

In a stationary monetary equilibrium, with \( q_e \) constant over time and \( \phi/\hat{\phi} = \gamma \), substituting (2.13) into (2.3) gives the following equilibrium conditions,

\[ \frac{v'(q_e)}{g'(q_e)} = \frac{i}{\alpha_e \rho_e} + 1, \] (2.14)

where we have replaced \( (\gamma/\beta - 1) \) by the nominal interest rate, \( i \), using the Fisher equation \( 1 + i = \gamma/\beta \).

Under the assumptions that \( \alpha_y = \alpha_1 \) and \( \rho_y = \rho_1 \), (2.14) implies all employed workers that trade in the DM have the same consumption, i.e., \( q_y = q_1, \forall y \). By virtue of (2.11), the amounts of money they bring in the DM are also the same: \( m_y = m_1, \forall y \). Unemployed workers, on the other hand, may consume different amounts given \( \alpha_0 \neq \alpha_1 \) and \( \rho_0 \neq \rho_1 \). In general, the difference between \( q_0 \) and \( q_1 \) depends on the properties of the LHS of (2.14).

Wright (2009) recently proves that there is generically a unique steady state of \( q \) even if the LHS of (2.14) is not monotone, and he shows that \( q \) is strictly increasing with the trading probabilities. In this case, using \( m_0 \) and \( m_1 \) as the optimal money holdings for the employed and the unemployed, respectively, we get
**Proposition 2.1.** For all $i > 0$, $\alpha_0 > \alpha_1$ or $\rho_0 > \rho_1$ implies that $m_0 > m_1$ and $q_0 > q_1$. Both $q_0$ and $q_1$ are decreasing in $i$, but the difference between them is increasing in $i$. In the limit as $i \to 0$, $q_0$ and $q_1$ converge to the same level, $\tilde{q}$ where $\tilde{q} \leq q^*$ with $\tilde{q} = q^*$ iff $\eta = 0$ and $s = 0$.

**Proof:** See Appendix A.1.a

The differences between employed and unemployed people’s trading probabilities induce them to choose different money balances, and thereby consume different amounts of cash goods in the DM. Inflation, as a tax on cash consumption, reduces both $q_0$ and $q_1$, but the difference between them diminishes as interest rates approach to zero. At the Friedman rule $i = 0$, so there is no cost of holding nominal balances, and the two types of workers consume the same amount of $q$. Consumptions in the credit trades are not subject to the cash constraint, so according to (2.12) $q^c$ is efficient as long as there is no sales tax in the DM.

On the firm’s side, the expected sales in the DM depend on transactions in both monetary and credit trades, as well as trading probabilities $\alpha_e$ and $\rho_e$. Suppose that all the firms that produce in the LM have an equal probability of meeting a worker in the DM. Given that the total measure of potential buyers is $\alpha_0u$ unemployed workers plus $\alpha_1(1-u)$ employed workers, where $u$ denotes the unemployment rate, the expected sale of each firm is

$$
\alpha_f q_f = \{u\alpha_0[\rho_0 q_0 + (1 - \rho_0)q^c] + (1 - u)\alpha_1[\rho_1 q_1 + (1 - \rho_1)q^c]\} / (1 - u).
$$

Similarly, the expected revenues are

$$
\alpha_f d_f = (1 - s) \{u\alpha_0[\rho_0 m_0 + (1 - \rho_0)d^c] + (1 - u)\alpha_1[\rho_1 m_1 + (1 - \rho_1)d^c]\} / (1 - u),
$$

where $d_e = m_e$ has been inserted. Using the bargaining solution in (2.11) and (2.12), one
can rewrite the firm’s expected profits

\[
\pi = \frac{u\alpha_0 \rho_0}{1-u} [(1-s)g(q_0) - q_0] + \alpha_1 \rho_1 [(1-s)g(q_1) - q_1] \\
+ \left[ \frac{u\alpha_0 (1-\rho_0)}{1-u} + \alpha_1 (1-\rho_1) \right] \eta [(1-s)v(q^c) - q^c].
\] (2.15)

Finally, taking as given the equilibrium quantities exchanged in the DM the government meets its budget constraint at each period:

\[
G + bu = s\alpha_0 [\rho_0 g(q_0) + (1-\rho_0)\phi d^c] + s(1-u)\alpha_1 [\rho_1 g(q_1) + (1-\rho_1)\phi d^c] + (\gamma - 1)\phi M,
\]

where \(\phi d^c\) is given in (2.12).

### 2.3.2 Optimal Decisions in the LM

In the LM, wages are determined by the generalized Nash bargaining between a worker and a firm. Let \(w(y) = \phi w^n(y)\) denote the real wage and \(b = \phi B\) represent the UI benefits in real terms. The surplus for a worker from a successful negotiation is equal to

\[
S^w_y = W^w_y(m) - W^w_0(m) = W^w_y(0) - W^w_0(0),
\]

by linearity of \(W^w_e\). Using (2.2), (2.4) and (2.5), one can derive the following expression for \(S^w_y\):

\[
\frac{1}{\beta} S^w_y = w(y) - b - l + i [g(q_0) - g(q_1)] + D + (1-\delta) \hat{S}^w_y - \lambda_w \int_{yR} \hat{S}^w_y dF(y),
\] (2.16)

where

\[
D = \alpha_1 \rho_1 [v(q_1) - g(q_1)] - \alpha_0 \rho_0 [v(q_0) - g(q_0)] \\
+ [\alpha_1 (1-\rho_1) - \alpha_0 (1-\rho_0)] (1-\eta) [v(q^c) - v'(q^c)q^c]
\]

is the difference of trade surplus in the DM between employed and unemployed workers.

The payoff for a firm from a successful match is \(S^f_y = W^f_y(x, d_f) - W^f_0(x, d_f)\). Applying the optimal decisions in (2.6) to (2.8), \(S^f_y = \beta \left[ y - w(y) + \pi + (1-\delta) \hat{S}^f_y \right]\). Let
θ be the worker’s bargaining power, and the equilibrium wage solves the following Nash product

$$\max_w [S^w_y(w)]^\theta [S^f_y(w)]^{1-\theta}. $$

Taking as given future surpluses \( \hat{S}_y \), the solution to the maximization problem satisfies

$$\theta S^f_y = (1 - \theta) S^w_y. \quad (2.17)$$

We now proceed to solve for the equilibrium wage \( w(y) \). First consider the reservation value for an unemployed worker. By definition, \( y_R \) is the cut-off level of productivity such that workers and firms agree to match if and only if \( y \geq y_R \). At \( y_R \), the total surplus from a job match satisfies \( S^w_{yR} + S^f_{yR} = 0 \). By virtue of (2.17), this is equivalent to saying that \( S^w_{yR} = 0 \). Replacing \( y \) by \( y_R \) in (2.16), one can obtain an expression for \( w(y_R) \). Substituting it back into (2.16) and imposing the steady-state condition, we get

$$S^w_y = \frac{\beta [w(y) - w(y_R)]}{1 - \beta (1 - \delta)}. \quad (2.18)$$

Together with the steady-state equation for firm’s surplus, one can solve (2.17) for the equilibrium bargaining wage

$$w(y) = \theta y + (1 - \theta) y_R + \pi. \quad (2.19)$$

Inserting (2.19) into (2.10), firm’s free entry condition becomes

$$k = \frac{\beta^2 (1 - \theta) M(1/\sigma, 1)}{1 - \beta (1 - \delta)} \int_{y_R} (y - y_R) dF(y), \quad (2.20)$$

where the arrival rate \( \lambda_f \) has been replaced using the matching function. Note that (2.20) defines one equation for the equilibrium \( y_R \) and \( \sigma \).

To obtain the second equation, we first characterize the bargaining wage at the reservation level using (2.16) and (2.20)

$$w(y_R) = b + l + i [g(q_1) - g(q_0)] - \mathcal{D} + \frac{\sigma \theta k}{\beta (1 - \theta)}. \quad (2.21)$$
Replacing $y$ by $y_R$ in (2.19) and plugging in $w(y_R)$, one can get the reservation productivity

$$y_R = b + l + i [g(q_1) - g(q_0)] - D + \frac{\sigma k}{\beta (1 - \theta)} - \pi,$$

(2.22)

where the DM profits $\pi$ is defined in (4.60). Note that $\pi$ is a function of the equilibrium unemployment rate, $u$, which in the steady state satisfies

$$u = \frac{\delta}{\delta + M(1, \sigma)[1 - F(y_R)]}.$$

(2.23)

Hence the expression in (2.22) defines the second equilibrium condition for $(y_R, \sigma)$.

A nice property of this model is that the optimal choices in the DM can be solved separately from the LM equilibrium conditions. As demonstrated above, (2.14) determines unique solutions for $q_0$ and $q_1$, while $q^c$ is defined in (2.12). Hence, the DM equilibrium results enter into expressions (2.20) and (2.22) only as constants. Overall, the characterization of the solution to the worker’s and firm’s problems leads to the following definition of equilibrium.

**Definition 4.1:** A Stationary Monetary Equilibrium is a collection of the consumption levels in the DM, $(q_0, q_1, q^c)$; a reservation productivity, $y_R$; and a measure of labor market tightness, $\sigma$ satisfying

1. workers’ consumption in the credit trade, $q^c$, is given by (2.12);
2. given the bargaining solutions of (2.11), workers’ consumption in the cash trade, $q_0$ and $q_1$, solve the money demand problem (2.14);
3. given $(q_0, q_1, q^c)$, $y_R$ is constructed as in (2.22);
4. given $y_R$, $\sigma$ satisfies the free entry condition (2.20).

The free entry condition in (2.20) describes a downward sloping curve called $FE$ over $(y_R, \sigma)$ space, as shown in Figure 2.2. Intuitively, higher $y_R$ makes job matches less profitable for firms, so they reduce vacancy postings, which lowers the labor market tightness.
The expression in (2.22), on the other hand, presents an upward sloping reservation productivity curve \( (RP) \). When \( \sigma \) is larger, it is easier for workers to find a job, so they are more willing to turn down any potential match with low productivity. The curve \( RP \) deviates from the standard literature in two aspects. First, the reservation wage takes account of the different surplus workers receive from the DM trade under different employment status. This is a result of heterogeneous cash consumptions between being employed and unemployed. Second, from a firm’s point of view, a potential job match not only enables it to produce but also generates a positive surplus from the DM trade. Consequently, the expression for \( y_R \) includes an additional term \( \pi \). The \( FE \) and \( RP \) curves together define unique solutions for \( y_R \) and \( \sigma \) as long as \( RP \) intersects the horizontal line at a point lower than \( \bar{y} \). Proposition 2.2 summarizes the existence and properties of the equilibrium.

**Proposition 2.2.** If the highest productivity level \( \bar{y} \) satisfies \( \bar{y} > b + l \), there exists a unique stationary monetary equilibrium with \( (q_0, q_1, \bar{q}, y_R, \sigma) \) characterized by equations (2.14), (2.12) (2.20) and (2.22).

**Proof:** See Appendix A.1.2.

Before turning to examine the relationship between inflation and unemployment, it is important to note that even though we allow the unemployed to consume a larger amount of cash goods in the DM, their overall consumption and welfare are still lower than those of the employed. The intuition is as follows. First, workers who have a job earn higher income than those who do not, and this is guaranteed by the Nash bargaining solutions. Unemployed workers consume more in the cash transaction, but less in the credit purchase and the subsequent CM. With positive nominal interest rates, higher money balances impose a larger cost of inflation to the unemployed, so their overall consumption and welfare are necessarily lower than the employed. The fact that unemployed workers endure a higher
burden of the inflation tax has important implications for the inflation-unemployment relation, which is explored in the next section.

\[ \text{Figure 2: Effects of increase in } i \text{ on } \sigma \text{ and } y_R \]

2.4 Inflation and Unemployment

We now examine how changes in monetary policy, in particular inflation or interest rates, affect the steady-state unemployment rate. To illustrate the importance of asymmetric effects of inflation on labor market decisions, we first consider a benchmark economy in which employed and unemployed workers are exposed to the same inflation tax.
2.4.1 Economy with Symmetric Effects of Inflation

Suppose now all workers face the same probability of monetary trade in the DM. With $\alpha_0 = \alpha_1 = \alpha$ and $\rho_0 = \rho_1 = \rho$, the DM equilibrium conditions in (2.11) and (2.14) imply that $q_0 = q_1 = q$ and $m_0 = m_1 = m$. Substituting these results into (2.22), the steady-state reservation productivity is then simplified to

$$y_R = b + l + \frac{\sigma \theta k}{\beta (1 - \theta)} - \frac{\alpha}{1 - u} \left\{ \rho [(1 - s)g(q) - q] + (1 - \rho) \eta [(1 - s)v(q^c) - q^c] \right\}.$$  

(2.24)

The LM equilibrium now is given by the intersection of the FE curve in (2.20) and the new RP curve in (2.24). To see the effect of inflation on unemployment, we first examine how changes in $i$ shift the FE and RP curves. Note that $i$ does not enter directly into equation (2.20), so changes in nominal interest rates only shifts RP. Keeping $\sigma$ constant, totally differentiating (2.24) yields the sign of $\partial y_R / \partial i$ as

$$- \left\{ \frac{\alpha \rho}{1 - u} [(1 - s)g'(q) - 1] \frac{\partial q}{\partial i} \right\}.$$ 

According to Proposition 2.1, $\partial q / \partial i < 0$ and $(1 - s)g'(q) - 1 > 0$, so an increase in $i$ shifts RP to the right, which raises $y_R$ and lowers $\sigma$. Given (2.23), this leads to a higher unemployment rate in the new steady state. To summarize, we have the following proposition.

**Proposition 2.3.** With $\alpha_0 = \alpha_1$ and $\rho_0 = \rho_1$, $q_0 = q_1 = q$. In this case, a rise in $i$ decreases $q$ and increases both $y_R$ and $u$.

In the economy that is absent from the asymmetric effects, inflation impacts labor market only through changes in firms’ profits. A higher rate of inflation, acting as a tax on transactions in the DM, reduces firms’ return from hiring workers, and thereby encourages unemployment. This results is the same as that in Berentsen, Menzio and Wright (2008), except the arrival rate $\alpha$ is endogenous in their model and productivity is the same across
all matches. Now to study the asymmetric effect channel, we turn to environments where employed and unemployed workers experience different costs of inflation.

### 2.4.2 Economies with Asymmetric Effects of Inflation

This section presents two scenarios that describe the heterogeneous effects of inflation. The first sub-section highlights the differences in workers’ income resources. More specifically, wages for the employed are determined by bargaining, and in the long run nominal wages grow at the same rate as inflation. UI benefits, on the other hand, reply on government policy settings and may not be perfectly indexed to the inflation. The second sub-section considers heterogeneous transaction patterns across population. Consistent with empirical evidences, workers who do not have a job tend to use cash more often in their expenditures and as a result, are more exposed to the inflation tax. The two types of differential effects provide the same implication: as long as the unemployed endure a larger cost, their incentives to work are increasing with inflation.

**Imperfect Indexation of UI Benefit**

In many countries, unemployment benefits are either indexed to workers’ previous nominal earnings - e.g., Canada or U.S. - or paid up to a certain specified level - e.g., UK. In the former case, positive inflation erodes the real value of UI benefits that are indexed to the nominal wages. Hence, a higher rate of inflation implies a lower real income for those unemployed\(^{12}\). In the second scenario, most governments adjust the amount of benefits

\(^{12}\)This is reminiscent of the literature that analyzes the impact of inflation on the distortionary nominal tax system. For example, Feldstein (1982) first discussed how inflation increases the effective tax rate on real capital income. Heer and Sussmuth (2007) extended this idea and examined how inflation increases the real tax burden on saving incomes and reduces the return on savings. Note that this type of "rigidities" in government policy does not hinge on any restriction on agent’s optimal choices. It describes the existing fiscal structure, which in many studies are taken exogenously.
only periodically (e.g., the Jobseeker’s Allowance Rate in UK is revised on a yearly base). With imperfect adjustment, unemployed workers’ welfare is also subject to the risk of inflation.

To investigate the effect of imperfect indexation, I modify the previous economy by indexing UI benefit to the average nominal earnings in the previous period. With inflation rate $\gamma$, the real value of $B$ in the steady state now becomes $\tau \bar{w} / \gamma$, where $\tau$ is the replacement ratio and $\bar{w}$ is the average real wage. Accordingly, the first term on the RHS of (2.24) changes to $\tau \bar{w} / \beta (1 + i)$. To see the effect of a rise in inflation, the sign of $\partial y_R / \partial i$ now is

$$-rac{\tau \bar{w}}{\beta (1 + i)^2} - \left\{ \frac{\alpha_p}{1 - u} [(1 - s)g'(q) - 1] \frac{\partial q}{\partial i} \right\}.$$  

(2.25)

The first term in (2.25) introduces a negative effect of inflation on $y_R$ and $u$. For convenience, we name it the “worker” effect. Intuitively, a rise in nominal interest rates reduces the wealth for the unemployed, thereby decreasing worker’s reservation wage. Combining with the second effect on firm’s vacancy posting - call it the “firm” effect, inflation in this economy either increases or decreases unemployment.

The two conflicting effects raise the question of what factors ultimately determine the slope of the long-run Phillips curve. Heterogeneity in this economy arises from imperfect indexation of social security system, so the level of UI benefits naturally determines the magnitude of the “worker” effect. A more generous UI scheme exposes the unemployed to a larger risk of inflation and amplifies its positive effect on employment. In general, there exists a cut-off level $\hat{\tau}$, such that, with everything else constant, $\tau < \hat{\tau}$ implies a positive inflation-unemployment relationship, while $\tau > \hat{\tau}$ gives a negative one.

The second set of the relevant factors is the parameters that characterize the goods market structure. The size of the “firm” effect in (2.25) depends on firm’s surplus from the DM trade. With Nash bargaining, the bargaining power of the firm, $\eta$, determines
how much surplus is allocated to the firm. Intuitively, one can view $\eta$ as the proxy for the degree of competition in the goods market. A higher $\eta$ implies that the DM is less competitive, so each firm earns more profits and a positively sloped Phillips curve tends to arise. On the other hand, the sales tax reduces the quantity exchanged in the DM, as well as the firm’s surplus from each sale. This tends to mitigate the “firm” effect and favor an inflation-unemployment trade-off.

Finally, labor market institutions such as worker’s wage bargaining power, may also play a role in determining the sign of $\partial u/\partial i$. Note that the second term in (2.25) is a function of the equilibrium unemployment rate. This is because sellers in this economy are the firms that are matched with workers and produce in the LM, so a higher $u$ implies less firms selling in the DM and each existing firm earns a higher profit. Nevertheless, as this only works through a second order general equilibrium channel, the impact of $u$ is hardly significant.

**Heterogeneous Money Balances**

As reviewed previously, a large number of studies have provided compelling evidences on heterogeneous transaction patterns and currency holdings across households with different income levels. This section first presents some estimate results to illustrate the different cash holding and spending behavior between employed and unemployed workers. The estimation is based on the Italian Household Survey Data of Income and Wealth 2004\textsuperscript{13}, and results are reported in Table A.1 in Appendix A.2. The main message from this exercise is that unemployed households on average hold a larger proportion of their wealth in cash, and also use money for a greater fraction of their monthly expenditure.

There are various ways to model heterogeneous money holdings between employed

\textsuperscript{13}See Appendix A.3 for a detailed description of the dataset.
and unemployed households. Following the general environment developed in Section 3, I examine the effects of heterogeneity arising from different probabilities of monetary trade. Given $\alpha_0 > \alpha_1$ or $\rho_0 > \rho_1$, unemployed workers face a higher probability of using cash, so they choose to carry more money into the DM. As a result, their cash expenditure as a percentage of total consumption is also larger than the employed.

To understand the impact of monetary policy on the labor decisions, we again examine how changes in $i$ shift the $RP$ curve (the $FE$ curve still remains unchanged). Keeping $\sigma$ constant, totally differentiating (2.22) yields the sign of $\frac{\partial y_R}{\partial i}$ as\footnote{This assumption is consistent with the estimation results in Table A.1, which demonstrates unemployed workers use cash more often in their total expenditure.}

\[ [g(q_1) - g(q_0)] - \left\{ \frac{u\alpha_0 \rho_0}{1 - u} [(1 - s)g'(q_0) - 1] \frac{\partial q_0}{\partial i} + \alpha_1 \rho_1 [(1 - s)g'(q_1) - 1] \frac{\partial q_1}{\partial i} \right\}. \]

(2.26)

The term $[g(q_1) - g(q_0)]$ captures the differential effects of inflation. By Proposition 2.1, $q_0 > q_1$, so $g(q_1) - g(q_0) < 0$. Intuitively, higher consumption of cash goods makes the unemployed more exposed to the inflation tax, so an increase in inflation or nominal interest rates reduces the value of outside options for workers, and lowers the reservation productivity at which firms and workers agree to match. This shifts the $RP$ curve in Figure 2.2 to the left. The second term in (2.26) corresponds to the previous “firm” effect, although the DM surplus now covers the heterogeneous sales to both employed and unemployed workers. With $\frac{\partial q}{\partial i} < 0$ and $(1 - s)g'(q) - 1 > 0$, the “firm” effect shifts $RP$ to the right.

As illustrated in Figure 2.2, the two opposite effects move the $RP$ curve into different directions. When the “firm” effect dominates, the $RP$ curve shifts to the right, which increases $y_R$ and decreases $\sigma$ as the $FE$ curve is downward sloping. By virtue of (2.23), this leads to a rise in the steady-state unemployment rate. On the other hand, if the “worker”
effect dominates, $y_R$ goes down and $\sigma$ goes up. The equilibrium unemployment rate falls. Summarizing these results, we establish the following proposition

**Proposition 2.4.** Define

$$u_\alpha \left[ \frac{(1-s)}{1-s} g'(q_0) - 1 \right] \frac{\partial q_0}{\partial i} + \alpha_1 \rho_1 \left[ (1-s) g'(q_1) - 1 \right] \frac{\partial q_1}{\partial i}$$

as the “firm” effect and $g(q_1) - g(q_0)$ as the “worker” effect. In the steady state, $\partial u/\partial i > 0$ iff the “firm” effect dominates the “worker” effect; otherwise, $\partial u/\partial i < 0$.

A formal proof of this proposition is omitted as the results are intuitive. Again we are interested in how different market institutions shape the equilibrium inflation-unemployment relationship. First, the asymmetric effects of inflation are generated through heterogeneous cash holdings. Naturally, a more dispersed distribution of money balances amplifies the “worker” effect, which tends to generate an inflation-unemployment trade-off if other variables remain the same.

Goods market institutions play a similar role as in the previous economy, except that $\eta$ and $s$ affect the sizes of both effects now. In an economy where firms have less market power, workers obtain a larger share of the trading surplus. Consequently, the magnitude of the “worker” effect relative to the “firm” effect becomes bigger, which leads to a negatively sloped long-run Phillips curve. In the extreme case with $\eta = 0$, firms earn zero surplus from the DM trade, so the inflation-unemployment relationship is channeled through the negative effect on workers’ outside option only. Sales tax, however, reduces both terms in (2.26), so it is not clear which effect is more likely to dominate. The roles of labor market institutions are still insignificant, as in the previous case. With UI benefit perfectly indexed, $b$ and $\theta$ change the equilibrium unemployment rate only, which strengthens the “firm” effect in a second order.
Due to the complex functional form of $g(q)$, it is difficult to study comparative statics analytically in this model. This section presents some numerical examples to demonstrate how different factors in goods and labor markets affect the inflation-unemployment relationship. The parameters are chosen to be consistent with those in the standard literature, although they are mainly for illustrative purpose only. The length of each period is considered to be one quarter. Utility in the DM has the form of $v(q) = Aq^{1-\nu}/(1 - \nu)$. The Matching function in the LM is assumed to be $\mathcal{M}(u, v) = Zu^{1-\xi}v^\xi$.

### Table 2.1: Parameter Values in the Benchmark Economy

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor</td>
<td>$\beta = 0.992$</td>
</tr>
<tr>
<td>DM utility function</td>
<td>$A = 1.013$</td>
</tr>
<tr>
<td></td>
<td>$\nu = 0.2$</td>
</tr>
<tr>
<td>LM matching function</td>
<td>$Z = 0.364$</td>
</tr>
<tr>
<td></td>
<td>$\xi = 0.28$</td>
</tr>
<tr>
<td>Worker’s wage bargaining power</td>
<td>$\theta = 0.72$</td>
</tr>
<tr>
<td>Job separation rate</td>
<td>$\delta = 0.05$</td>
</tr>
<tr>
<td>Value of nonmarket activity</td>
<td>$l = 0.417$</td>
</tr>
<tr>
<td>Vacancy posting cost</td>
<td>$k = 1.25 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

In the general version of the model, the productivity, $y$, is assumed to be match-specific. This is to capture the effects of inflation not only on the bargaining between workers and firms, but also on the job acceptance decision. Nevertheless, since the comparison of the two conflicting effects in (2.26) remain consistent even with homogenous productivity, I normalize $y$ to one for simplicity. This means the negative effect of inflation on unemployment works only through the bargained wage, but not the reservation productivity. Unemployment insurance benefit, $b$, is chosen to meet a replacement ratio of about 5% in the benchmark economy. I start with firms’ bargaining power in the DM, $\eta = 0.35$, and then
compare the equilibria under different values. Worker’s wage bargaining power, $\theta$, is set to equal the elasticity of the matching function, which satisfies the Hosios (1990) condition. The sales tax is set to zero in the benchmark case. Finally, the probabilities of monetary trade are chosen to generate a 10% cash spending difference between the employed and unemployed, roughly consistent with the number in Table A.1. From (2.14) and (2.26), worker’s optimal decisions depend on the combination of $\alpha$ and $\rho$ together. To simplify the computation, I normalize $\rho = 1$, i.e., assuming all trades in the DM are anonymous, and set $\alpha_0 = 0.9$ and $\alpha_1 = 0.5$. The values of the rest parameters are reported in Table 1, in accordance with the calibration exercises in Berentsen, Menzio and Wright (2008) except a few variations\textsuperscript{16}. Figure 2.3 displays the benchmark economy. Following a rise in inflation from 0 to 10%, the unemployment rate first increases by approximately 0.19 percentage

\textsuperscript{16}Adjustments have been made for $k$, $l$ and $\nu$. This is to guarantee the existence of equilibrium for a large set of parameters.
point up to 9% inflation, and then starts to decline\textsuperscript{17}.

The first exercise is to reduce the value of $\alpha_1$ from 0.5 to 0.3. Such a change in the trading probability leads to a larger difference in unemployed and employed workers’ real balances (a rise from 10% to 25%). As a result, the “worker” effect becomes stronger and the inflation-unemployment relationship starts to slope downward as shown in part (a) of Figure 2.4.

Part (b) of Figure 2.4 compares the equilibria with and without the sales tax. Although a positive $s$ reduces the trading surplus for both workers and firms, the numerical result suggests a larger impact on the firm’s side. One reason is that the magnitude of the “worker” effect is mainly determined by the difference between $q_0$ and $q_1$. The sales tax in the DM lowers the quantity exchanged in each transaction, but the gap between $q_0$ and $q_1$ generally does not change much. Firms’ profits, on the other hand, drop directly after paying the sales tax. Therefore, the reduction in the “firm” effect tends to outweigh the “worker” effect, which leads to an equilibrium that favors a negatively sloped Philips curve.

Part (c) of Figure 2.4 illustrates the effect of firm’s bargaining power. The four graphs correspond to different values of $\eta$. In the upper left panel with $\eta = 0.1$, the inflation-unemployment relationship is downward sloping. Following a rise in inflation from 0 to 10 percent, the unemployment rate drops from 11.6 to 8.7 percent. Moving to the upper right panel, with $\eta$ rising back to the benchmark level, unemployment starts to increase with inflation. A further rise in $\eta$ only raises the slope of the curve, and overall inflation and unemployment tend to be negatively related in a more competitive market (lower $\eta$).

\textsuperscript{17}Notice that the change in the unemployment rate is quite small in the benchmark economy, but one can obtain a large variation simply by adjusting the value of non-market activity $l$. See Berentsen, Menzio and Wright (2008) for details. Here the value of $l$ is chosen to ensure the existence of equilibrium in a large range of parameter values, for the purpose of later sensitivity analysis.
Figure 2.4: Goods Market Institutions

(a) Effects of Heterogeneity in Money Holdings

(b) Effects of Sales Tax

(c) Effects of Firm’s Bargaining Power
Figure 2.5: Labor Market Institutions

(a) Effects of Wage Bargaining Power

(b) Effects of UI Benefit with Perfect Indexation

(c) Effects of UI Benefit under Imperfect Indexation with $\alpha_0 = \alpha_1 = 0.9$ and $\eta = 0.5$
The sensitivity analysis for $\theta$ and $b$ is presented in parts (a) and (b) of Figure 2.5. Following a rise in worker’s wage bargaining power or UI benefit, the unemployment rate goes up at every level of inflation. In the new steady state, the slope of the inflation-unemployment curve becomes steeper, but the direction of the curve remains unchanged. These modest effects are consistent with the previous conjecture that labor market institutions only influence the equilibrium outcome through a second order channel. Nevertheless, when the asymmetric effects are introduced through imperfect policy indexation, the role of UI benefits turns to be significant. As shown in (2.25), the “worker” effect in this case is a function of UI replacement ratio. A generous UI benefit strongly favors the “worker effect”, and leads to a downward sloping inflation-unemployment relationship. This is illustrated in part (c) of Figure 2.5, where $\alpha_0$ is set to be the same as $\alpha_1$, and the bargaining powers is adjusted to be equal between workers and firms. In this case, the slope of the curve switches from positive to negative when $\tau$ rises from 0.44 to 0.6.

Before concluding the roles of market institutions, one may notice that most curves in the above numerical examples exhibit a hump shape, which suggests the inflation-unemployment relationship also varies across different monetary regimes. One explanation is that the disparity between employed and unemployed workers’ money balances is increasing with inflation. At the low inflation level, the distribution of the inflation burden is narrower, so the positive effect tends to dominate. In the limit, as nominal interest rate approaches zero, there is no cost of holding money and the asymmetric effects become negligible.

Given the above sensitivity analysis, one important question is whether the model can account for the cross-country difference in the long-run Phillips curve. As reviewed earlier, empirical literature seems to suggest that inflation and unemployment tend to be positively
correlated in the U.S., while negatively in Europe. In general, the goods and labor markets in the two areas differ in many aspects. To be consistent with the current model environment, I focus on two parameters: the sales tax and the UI benefit. In the OECD working paper, Carey and Tchilinguirian (2000) estimate the average effective tax rates for different countries and areas. According to their estimation, between 1980 and 1997, the consumption tax in U.S. was around 6%, while the average of European Union was close to 18.6%. A higher sales tax in the present model suggests a downward sloping long-run Phillips curve, if other factors remain the same. This is qualitatively consistent with the empirical evidence observed in Europe. Secondly, when we compare labor market institutions, one consensus is that most European countries have a more generous UI benefit scheme than the U.S.. In the model with imperfect policy indexation, higher UI benefits tend to give rise to a negatively sloped inflation-unemployment relationship.

Finally the model also provides some implications for the optimal monetary policy in the long run. In general, it is difficult to draw one conclusion from the above analysis, since the interaction between monetary policy and labor decision implies that the optimal rate of inflation depends on both labor market efficiency and the underlying inflation-unemployment relation. To consider a simple scenario, suppose now the Hosios condition is satisfied in the labor market, with worker’s wage bargaining power $\theta$ equal the elasticity of the matching function $\xi$. This may be achieved through government fiscal policy, or simply efficient market outcomes. In this case, the optimal monetary policy is to set $i = 0$, no matter inflation and unemployment are positively or negatively related. Intuitively, a positive interest rate imposes a tax on the consumption of cash-intensive goods, which reduces the welfare for both employed and unemployed workers. Even under the case where inflation encourages employment and output in the long-run, the first order welfare loss
always dominates the second order employment effect, and the Friedman rule (with \( i = 0 \)) is optimal.

In an alternative scenario, suppose now the Hosios condition is violated and the labor market is inefficient. With certain restrictions on the fiscal policy (UI benefit or income tax), the equilibrium unemployment rate may be above or below the efficient level. Under these circumstances, it may be optimal to deviate from the Friedman rule. Nevertheless, the specific level of inflation rate depends on labor market outcomes and the underlying inflation-unemployment relation. To infer a detailed optimal policy rule, we need additional welfare analysis and a close examination of the market characteristics. This is left for future work.

2.6 Conclusion

This chapter has presented a general equilibrium model to study the long-run relationship between inflation and unemployment. The model takes seriously the observation that unemployed people are affected more heavily by the inflation tax than the employed ones, and studies its implication for the inflation-unemployment relation. Overall the model predicts that changes in monetary conditions have two opposing effects on the labor market. Following a rise in inflation or nominal interest rate, the asymmetric effect of inflation reduces the value of outside options for workers, which lowers both the bargained wage and reservation productivity, thereby decreasing unemployment. At the same time, inflation as a tax on the cash-intensive activities, reduces firm’s return from job creation and raises unemployment. In the steady state, inflation either increases or decreases output.

The main message from the quantitative examples is that goods market factors, such as
the degree of heterogeneity, the level of sales tax and the overall competitiveness, play major roles in determining which of the two effects dominates. In contrast, labor market institutions, like UI benefit or wage bargaining power, only affects the inflation-unemployment relationship modestly. The exception is when UI benefit is not perfectly indexed to inflation, in which case unemployed workers suffer an additional cost of inflation due to the real value depreciation of the nominal benefit. Overall, through a comparison of the sales tax and UI benefit, the model generates an inflation-unemployment relation qualitatively consistent with the empirical evidence.

The implications for the optimal policy in this model are complicated. In the absence of any restriction on policy instruments, the optimum is to achieve labor market efficiency using fiscal policy, and set the monetary policy at the Friedman rule. This is welfare improving even when inflation encourages employment. When distortions exist in the labor market, deviating from the Friedman rule may be optimal. The implementation, however, requires detailed analysis of the goods market structure and labor market institutions.

Several extensions can be made for future research. First, the numerical examples in Section 5 only demonstrate that the model implications are qualitatively consistent with the empirical evidence. One extension is to calibrate the model and quantitatively examine the cross-country difference in the relationship between inflation and unemployment. Secondly, the probabilities of monetary trade in the DM are taken exogenously in the present economy. One can endogenize these probabilities by allowing workers or firms to choose how much of the DM trade can be carried out using credit. For example, a firm may extend its credit to a worker subject to a borrowing limit that is proportional to the worker’s earnings. Lastly, the present paper focuses on the steady state analysis. To understand how labor and goods markets frictions affect the conduct of monetary policy, one can introduce
short-run stochastic shocks into the model and study the dynamics effect of monetary.
Chapter 3

On The "Hot Potato" Effect of Inflation:
Intensive versus Extensive Margins

This chapter was co-authored with Liang Wang and Randall Wright\(^1\).

3.1 Introduction

Conventional wisdom has it that when inflation or nominal interest rates rise people try to spend their money more quickly – like a “hot potato” they want to get rid of sooner rather than later – and this is a channel through which inflation potentially affects velocity, output and welfare. For the purpose of this chapter, this is our definition of the “hot potato” effect: when inflation increases, people spend their money faster. Search-based monetary theory seems ideal for studying this phenomenon, once we introduce endogenous search intensity, as in standard job-search theory (Mortensen 1987). This is done by Li (1994, 1995), assuming buyers search with endogenous intensity, in a first-generation model of money with

\(^1\)© Macroeconomic Dynamics (forthcoming in January 2011)
indivisible goods and indivisible money along the lines of Kiyotaki and Wright (1993). One cannot study inflation directly in this model, of course, but Li proxies for it with taxation. Among other results, his model predicts that increasing the inflation-like tax unambiguously makes buyers search harder and spend their money faster, thus increasing velocity, and actually improving welfare.

His results may appear natural, but they do not easily generalize to relaxing the assumption of indivisible goods and money, which were made for convenience and not meant to drive substantive conclusions. Why? People cannot in general avoid the inflation tax by spending money more quickly – again like a “hot potato” buyers can only pass it on to sellers. Sellers are not inclined to absorb the incidence of this tax for free. Once we relax the restriction of indivisible goods and money, the terms of trade adjust with inflation, and the net outcome is that buyers reduce rather than increase their search effort. Intuitively, as a tax on monetary exchange, inflation reduces the return to this activity; when the return falls, agents invest less; and this means in the models that buyers search less and end up spending their money more slowly. The prediction that search effort increases with inflation depends on the terms of trade not being allowed to adjust.²

Lagos and Rocheteau (2005) prove these results using the search-based model in Lagos and Wright (2005) with divisible goods and money, which allows the terms of trade to be determined by bargaining, and allows one to introduce inflation directly rather than proxy for it by taxation. They show an increase in inflation reduces the surplus from monetary trade and hence buyers’ incentive to search, so they spend their money less, not more, quickly. Lagos and Rocheteau go on to show one can get buyers to search more with inflation in a model with price posting as in Rocheteau and Wright (2005), rather than

²This is reminiscent of Gresham’s law: good money drives out bad money when prices are fixed, but not necessarily when they are flexible. See Friedman and Schwartz (1963, fn. 27) for a discussion and Burdett et. al. (2001, Sec. 5) for a theoretical analysis of this idea.
bargaining, for some parameter values. The trick is this: even though the total surplus falls with inflation, if buyers’ share of the surplus goes up enough, which is possible under posting if parameters are just right, they may get a higher net surplus and hence increase search effort. This is clever, but not especially robust, in that one might think the “hot potato” effect is so natural it ought not depend on extreme parameter values or on the pricing mechanism (posting vs. bargaining).

There is much additional work on the problem. Ennis (2008) assumes sellers have an advantage over buyers in terms of the frequency with which they can access a centralized market where they can off load cash (like Keynes’ cheese merchant in the epigram). Thus, inflation increases buyers’ incentive to find sellers, because sellers can get money to the centralized market faster.\(^3\) Dong and Jiang (2009) present a similar analysis in a model based on private information. Nosal (2008) assumes buyers meet sellers with different goods and have to decide when to make a purchase. They use reservation strategies, and as inflation rises their reservation values fall, increasing the speed at which they trade. Previously, Shi (1998) endogenized search intensity in the Shi (1997) model, and showed it can increase with inflation, due to general equilibrium effects, for some parameter values.

We propose a new approach to study the "hot potato" effect of inflation. Our idea is to focus on the extensive rather than the intensive margin – i.e. on how many buyers are searching, rather than on what any particular buyer does. The idea is obvious, once one sees it, but we think it is nonetheless interesting. If one will allow us to indulge in the Socratic method, for moment, consider this. The goal is to get buyers to trade more quickly when the gains from trade are reduced by inflation. What kind of theory of the goods market would predict that buyers spend their money faster when the gains from trade are lower?\(^3\)

\(^3\)This is reminiscent of the model of middlemen by Rubinstein and Wolinsky (1995), where there are gains from trade between sellers and middlemen because the latter meet buyers more quickly than the former meet buyers.
That would be like a theory of the labor market that predicts firms hire more quickly when we tax recruiting. What kind of model of the labor market could generate that?

The answer is, the textbook model of search and recruiting in Pissarides (2000). It does so because it focuses on the extensive margin – a free-entry or participation decision by firms. When recruiting is more costly, and thus less profitable, in that model, some firms drop out, increasing the hiring rate for those remaining through a standard matching technology. Of course firms hire faster when we tax them, since that is the only way to keep profit constant! The same logic works for the goods market. Of course people spend their money faster when inflation rises, since that is the only way to satisfy the analogous participation condition for consumers. This corresponds well to the casual observation that people are less likely to participate in monetary exchange when inflation is high, perhaps reverting to barter, home production, etc. Our results are also robust, in the sense that they do not depend much on parameters or pricing mechanisms.

There are at least two reasons for being interested in search behavior, along either the intensive or extensive margin. One concerns welfare and optimal policy: we want to know if there is too little or too much search, and how policy might correct any inefficiency. The other concerns positive economics. As mentioned, if buyers spend their money faster when inflation rises, this is one (if not the only) channel through which velocity depends on inflation and nominal interest rates. Understanding how velocity depends on monetary policy is important, since this is basically the same as understanding how money demand, or welfare, depends on monetary policy, as discussed e.g. by Lucas (2000). In the simplest models, velocity and search intensity are identically equal. In more complicated models, velocity depends on several effects, but the speed with which agents spend their money is still one of the relevant effects.
This chapter is organized as follows. In the next section we begin by presenting the data to confirm the conventional wisdom that velocity is increasing in inflation or nominal interest rates.\footnote{While it would be nice to have direct evidence on the speed with which agents spend their money, we do not; hence we look at velocity.} We then move to theory. Section three considers models with indivisible money in order to introduce some assumptions and notation, and to review the results in Li (1994, 1995). In Section four we consider models with divisible money, and show the following: with an endogenous search intensity decision (the intensive margin), the speed with which agents spend their money falls with inflation, as in Lagos and Rocheteau (2005); but with a participation decision (the extensive margin), the speed with which agents spend their money, and also velocity, always increase with inflation. We also discuss welfare implications, and show that with an endogenous participation decision for buyers, the Friedman rule might not be optimal – positive inflation or nominal rates can be desirable. Section five concludes.

3.2 Evidence

We use quarterly US data between 1955 and 2008 and Canada data between 1968 and 2006. Figure B.1a shows for the US the behavior of inflation $\pi$, and two measures of the nominal rate $i$, the government bond (T-Bill) rate and the Aaa corporate bond rate. Figure B.2a shows similar series for Canada.\footnote{Except instead of the Aaa corporate rate for Canada we use the Prime Corporate Paper, which is a weighted average of rates posted for 90-day paper by major participants in the Canadian market.} Dotted lines are raw data and solid lines are HP trends. The models below satisfy the Fisher equation, $1 + i = (1 + \pi)/\beta$ where $\beta$ is the discount factor. As one can see, this relationship is not literally true but not a bad approximation to the data. Figures 1b and 2b show velocity $v = PY/M$ for the US and Canada, where $P$ is...
the price level, \( Y \) real output, and \( M \) the money supply, for three measures of money, \( M_0 \), \( M_1 \) and \( M_2 \). We call the three velocity measures \( v_0 \), \( v_1 \) and \( v_2 \). Obviously, \( v \) is lower for broader definitions of \( M \). Also, although \( v \) has relatively small deviations between raw data and trend, there are interesting trend movements in \( v_0 \) and \( v_1 \).

Figure B.3 shows scatter plots for the US raw data on all three measures of \( v \) versus \( \pi \) and \( v \) versus \( i \) (we show only T-bill rates, but the picture looks similar for Aaa rates). Figure B.4 shows scatter plots after filtering out higher frequency movements in the series, i.e. scatter plots of the HP trends; Figure B.5 shows something similar after filtering out the low frequency movements, i.e. scatter plots of the deviations between the data and trends.\(^6\) Table B.1 gives the correlations. From the figures or the table, one can see that for the US data \( v_1 \) and especially \( v_0 \) move together with \( \pi \) or \( i \) in the raw data, while \( v_2 \) does not. However, \( v_2 \) is strongly positively correlated with \( \pi \) or \( i \) at high frequencies, while the correlations for \( v_0 \) are driven mainly by the low frequency observations, and the correlations for \( v_1 \) are positive at both high and low frequencies. Similar observations prevail in the Canadian data, with some interesting differences that we do not have time to dwell on.

There appears to be a structural break in velocity in the US data, especially \( v_1 \). Informally, looking at the charts, one might say that sometime in the early 1980s interest rates began to drop while \( v_1 \) stayed flat. Or one might argue that the big change was in the mid 1990s when \( \pi \) and \( i \) continued to fall but \( v_1 \) started upward. To control for this in a simple way, Table B.1 also reports the correlations for the US when we stop the sample in 1982 (results are similar when we stop in 1995). We find that \( v_0 \) moves about as strongly with \( \pi \) or \( i \), but now both \( v_1 \) and \( v_2 \) move much more with \( \pi \) or \( i \), at both high and low frequency. We conclude from all of this that the preponderance of evidence indicates all

\(^6\) Scatter plots for the Canadian data look similar and are omitted.
measures of $v$ move positively with $\pi$ or $i$, although for some measures this is mainly in the high frequency and for others in the low frequency.

We want it to be clear we are not suggesting that these observations constitute a puzzle – i.e. that they are inconsistent with existing theories. Many models, including those with some but not all goods subject to a cash-in-advance constraint, as well as most recent search models, and many other models, can in principle match these data. In fact, since $v$ is the inverse of $M/PY$, and it is common to take $M/PY$ as a measure money demand, any model where money demand decreases with $i$ should be at least roughly consistent with the evidence on $v$. The purpose of this empirical digression is this: one reason for being interested in search behavior is that it contributes to the relationship between inflation and velocity, and we simply want to document what this relationship is. We now move to theory.

3.3 Indivisible Money and Goods

A $[0, 1]$ continuum of agents meet bilaterally and at random in discrete time. They consume and produce differentiated nonstorable goods, leading to a standard double coincidence problem: $x$ is the probability a representative agent wants to consume what a random partner can produce. As agents are anonymous, credit is impossible, and money is essential. So that we can review earlier results, for now goods and money are indivisible, and there is a unit upper bound on money holdings. Given $M$ total units of money, at any point in time there are $M$ agents each with $m = 1$ unit, called buyers, and $1 - M$ with $m = 0$, called sellers. Only sellers can produce, so if two buyers meet they cannot trade (one interpretation is that, after producing, agents need to consume before they produce again). Only buyers can search, so sellers never meet (one interpretation is that they must produce
at fixed locations). Hence, all trade has a buyer giving 1 unit of money to a seller for \( q = 1 \) units of some good; there is no direct barter.

Each period, a buyer meets someone with a probability \( \alpha \). The probability he meets a seller that produces what he wants, a so-called trade meeting, is \( \alpha_b = \alpha (1 - M) x \). This is also velocity: \( \alpha_b = v = PY/M \) since \( PY = M \alpha_b \). The probability of such a meeting for a seller is \( \alpha_s = \alpha_b M/(1 - M) = \alpha M x \). Buyers choose search intensity. Given \( M \) and \( x \), they can choose either \( \alpha \) or \( \alpha_b \). We adopt the convention that they choose \( \alpha_b \), and we write search cost as \( k(\alpha_b) \), where \( k(0) = k'(0) = 0 \), \( k'(\alpha_b) > 0 \) and \( k''(\alpha_b) > 0 \) for \( \alpha_b > 0 \).\(^7\) Policy is modeled as a tax on money holdings, but since it is indivisible, rather than taking away a fraction of your cash we take it all with probability \( \tau \) each period (one interpretation is that buyers, in addition to meeting sellers, also meet government agents with confiscatory power). To focus on steady states we keep \( M \) constant by giving money to a seller each period with probability \( \tau M/(1 - M) \). This tax proxies for inflation.

Although for now \( q \) is indivisible, in general \( u(q) \) and \( c(q) \) are utility from consumption and disutility from production, where \( u(0) = c(0) = 0 \), \( u'(q) > 0 \), \( c'(q) > 0 \), \( u''(q) < 0 \), \( c''(q) \geq 0 \), \( u'(0)/c'(0) = \infty \), and \( q^* \) solves \( u'(q^*) = c'(q^*) \). Let \( \beta = 1/(1 + r) \) be the discount rate. Let \( V_b \) and \( V_s \) be the value functions for buyers and sellers. Given that sellers are willing to trade goods for money, which we check below, these satisfy the Bellman

\(^7\)This is how search is assumed to operate in Li (1994, 1995). Here is a physical environment consistent with the specification. There is some number of agents \( N_A \) and locations \( N_L > N_A \). Each period a seller occupies a location. Then each buyer samples a location, in a coordinated manner say, they sample sequentially, and no one samples the same location as a previous buyer (to avoid the coordination friction emphasized in the directed search literature). The number of sellers is \( N_A (1 - M) \), and each produces your desired good with probability \( x \). Hence, your probability of a trade meeting is \( \alpha_b = \alpha (1 - M) x \) with \( \alpha = N_A/N_L \). The key to this specification is this: when you choose your search effort, it affects your probability \( \alpha_b \), but not that of other buyers, although it does affect \( \alpha_s \) for sellers. Lagos and Rocheteau (2005) use a different setup, starting with an underlying matching technology giving the number of meetings as a function of total search effort by buyers and the number of sellers, \( n(M \bar{e}, 1 - M) \), where \( \bar{e} \) is average buyer effort. The probability a given buyer meets a seller is \( \epsilon n(M \bar{e}, 1 - M)/e M \), where \( e \) is his own effort. In this setup your search effort affects this probability for other buyers. This complicates the analysis but does not affect the results. In any case, we return to general matching functions below.
equations:

\[(1 + r) V_b = -k(\alpha_b) + \tau V_s + \alpha_b[u(q) + V_s] + (1 - \tau - \alpha_b) V_b \quad (3.1)\]
\[(1 + r) V_s = \frac{\tau M}{1 - M} V_b + \alpha_s[-c(q) + V_b] + \left(1 - \frac{\tau M}{1 - M} - \alpha_s\right) V_s \quad (3.2)\]

In (3.1), e.g., $\tau$ is the probability of having your money taxed away, $\alpha_b$ is the probability of a trade meeting, and $1 - \tau - \alpha_b$ is the probability of neither event.$^8$

As we said, for now we take $q = 1$ as fixed, as in first-generation money-search models, and write $u = u(1)$ and $c = c(1)$, assuming $c < u$. Also, for now we ignore the constraint $\alpha_b \leq 1 - \tau$, and return to it later. Then the necessary and sufficient FOC for $\alpha_b$ is

\[k'(\alpha_b) = u + V_s - V_b. \quad (3.3)\]

Solving (3.1) and (3.2) for $V_s$ and $V_b$, and inserting these plus $\alpha_s = \alpha_b M/(1 - M)$ into (3.3), we can reduce this to

\[T(\alpha_b) = [r(1 - M) + \tau + M \alpha_b] u - M \alpha_b c + (1 - M)k(\alpha_b) \quad (3.4)\]
\[- [r(1 - M) + \tau + \alpha_b] k'(\alpha_b) = 0.\]

It is easy to show $T(0) > 0$ and $T(\bar{\alpha}_b) < 0$, where $\bar{\alpha}_b = (1 - M)x$ is the natural upper bound, assuming $k'(\bar{\alpha}_b) = \infty$. Hence, there exists $\alpha_b^e \in (0, \bar{\alpha}_b)$ with $T(\alpha_b^e) = 0$. Although $T$ is not monotone, in general, a sufficient condition for uniqueness is $k'''' > 0$, since this makes $T$ concave. To show $\alpha_b^e$ is an equilibrium, we have only to check sellers want to trade, $c \leq V_b - V_s$, which holds iff

\[(1 - M)\alpha_b u - [(r + \alpha_b)(1 - M) + \tau] c - (1 - M)k(\alpha_b) \geq 0. \quad (3.5)\]

$^8$We assume payoffs $-k(\alpha_b), u(q)$ and $c(q)$ are all received next period, which is why the value functions $V_b$ and $V_s$ discount everything on the right; this affects nothing of substance, but makes for an easier comparison to models with divisible money. Also, as we only consider steady states, value functions are always time invariant.
Assuming this holds with strict inequality at \( \tau = 0 \) (see below), monetary equilibrium exists for all \( \tau \leq \tilde{\tau} \) where \( \tilde{\tau} > 0 \) satisfies (3.5) at equality. In terms of the effects of policy, given that equilibrium is unique, the key result in Li follows immediately: \( \partial \alpha^e_b / \partial \tau > 0 \). Thus, a higher tax rate (read higher inflation) increases search intensity \( \alpha^e_b \), and hence velocity \( v \).

In terms of optimality, in this model, average welfare \( MV_b + (1 - M)V_s \) is proportional to \( \alpha_b(u - c) - k(\alpha_b) \). Hence the optimal \( \alpha^*_b \) satisfies \( k'(\alpha^*_b) = u - c \). Comparing this with equilibrium condition (3.3), \( \alpha^*_b = \alpha^*_b \) iff \( c = V_b - V_s \). Hence, the optimal \( \tau \) is the maximum feasible \( \tilde{\tau} \), which implies that sellers get no gains from trade. This is a version of the standard Hosios (1990) condition saying, in this case, that buyers should get all surplus since they make all the investment in search effort. To put it another way, buyers equate the marginal cost of search to their private benefit, but unless they get all the gains from trade, sellers also get some benefit that is not internalized.

Also, given \( \tau = \tilde{\tau} \) implies \( \alpha^*_b = \alpha^*_b \), we can rearrange (3.5) at equality for

\[
\tilde{\tau} = \frac{1 - M}{c} \left[ \alpha^*_b(u - c) - k(\alpha^*_b) - rc \right],
\]

(3.6)

where \( \alpha^*_b \) is given by \( k'(\alpha^*_b) = u - c \). Hence, \( \tilde{\tau} > 0 \) iff \( rc < \alpha^*_b(u - c) - k(\alpha^*_b) \). Finally, up to now we ignored the constraint \( \alpha_b \leq 1 - \tau \). Since \( \alpha^*_b \) is increasing in \( \tau \), with \( \alpha^*_b = \alpha^*_b \) at \( \tau = \tilde{\tau} \), this will be valid in all equilibria as long as \( \alpha^*_b \leq 1 - \tilde{\tau} \). In conclusion, in this model, monetary equilibrium exists iff \( \tau \leq \tilde{\tau} ; \alpha^*_b \) and \( v \) are increasing in \( \tau \); and the optimal policy \( \tau = \tilde{\tau} \) maximizes \( \alpha^*_b \) and \( v \). The main substantive result is that buyers spend their money faster when the inflation-like tax increases.\(^9\)

We want to know if these substantive results are robust, and how they generalize. There are several directions one could go in this endeavor, and obviously allowing the terms

\(^9\)In terms of technical details, notice that for \( \tau \) near the optimum \( \tilde{\tau} \) we have \( T'' < 0 \), and hence uniqueness follows even without the restriction \( k''' > 0 \). And of course we know \( \alpha^*_b < \alpha^*_b \) for all \( \tau < \tilde{\tau} \) in any equilibrium even if we have multiple equilibria. Finally, all this takes \( M \) as given, but it is a simple exercise to optimize over \( M \) as well as \( \tau \).
of trade to be something other than a one-for-one swap of money for goods is desirable. Ultimately we want to consider the most recent search-based models where goods and money are divisible. There are several versions one could use, including those that build on Shi (1997), Green and Zhou (1998), or Molico (2006). We will use the model in Rocheteau and Wright (2005), which has the convenient feature that there are always two types of agents in the economy, buyers and sellers, that correspond well to the two types in the Li model. Before we go to the case where goods and money are divisible, however, it is useful to consider the case where they are indivisible but we incorporate some other elements of the setup to be analyzed below.\footnote{A different approach is to keep \( m \in \{0, 1\} \), but make \( q \) divisible, determined using bargaining as in Shi (1995), Trejos and Wright (1995) or Rupert et al. (2001). Assuming buyers make take-it-or-leave-it offers, to reduce the algebra, bargaining implies}

An important element of the models below is an alternating market structure: each period there convenes a decentralized market, DM, like the one analyzed above, as well as a centralized market, CM, without frictions. The population is again \([0, 1]\), but it now consists of two permanently different types, called buyers and sellers. Assume the measure of buyers is \( N \), with \( N > M \), so that money is scarce. Types are defined as follows: buyers always want to consume but cannot produce in the DM; sellers can always produce but do not want to consume in the DM. One cannot have two such types in models with only a

\[
c(q) = \frac{\alpha_b u(q) - k(\alpha_b)}{r + \alpha_b + \tau/(1 - M)}.
\]

Equilibrium is a pair \((q, \alpha_b)\) solving this plus the FOC for effort, \( k'(\alpha_b) = u(q) - c(q) \). The first relation defines a curve in \((q, \alpha_b)\) space we call BS; it looks like a loop starting at \((0, 0)\) since for small \( \alpha_b \) there are two solutions, say \( q^H \) and \( q^L \), and for large \( \alpha \) there are none. The FOC defines a curve we call SE; it is strictly concave, goes through \((0, 0)\), and is maximized at \( q = q^* \) where \( u'(q^*) = c'(q^*) \). One can show BS and SE cross where BS is vertical in \((q, \alpha_b)\) space, and an increase in \( \tau \) shifts BS left along the SE curve. Since it is not clear there is a unique intersection (although this seems to be true in examples), consider the equilibrium with the highest \( q \). Then an increase in \( \tau \) reduces \( q \), but whether or not this reduces \( \alpha_b \) depends on whether \( q \) is above or below \( q^* \), so the results are ambiguous in general. As we show below, this is an artifact of indivisible \( m \). And even with indivisible \( m \), one could say it is an artifact of not allowing lotteries, as in Berentsen et al. (2002), since in that model we never get \( q > q^* \).
DM, since sellers will not produce for money if they never get to be buyers in some future
DM. But in this model, sellers may value money even if they never get to be buyers in a
future DM because they can spend it in the CM.

Let $W^b_m$ and $V^b_m$ be the value functions for buyers in the CM and DM, respectively,
where $m \in \{0, 1\}$ indicates whether they have money or not. For sellers, replace the
superscript $b$ by $s$. In the CM, all agents trade money, labor, and a consumption good $X$
different from the goods traded in the DM. Given a production function $x = H$, the real
wage is 1, and we denote by $\phi$ the price of $m$ in terms of $X$. Assuming there is discounting
between the CM and DM, but not between the DM and CM, for an agent of type $j \in \{b, s\}$
we have

$$W^j_m = \max_{X,H,\hat{m}} \left\{ U(X) - H + \beta V^j_m \right\}$$

s.t. $X = H + \phi(m - \hat{m})$, $\hat{m} \in \{0, 1\}$

where $U(X)$ is a utility function satisfying the usual assumptions, and utility over $H$ is
linear.\footnote{Quasi-linear utility is necessary to keep the model tractable once we allow divisible money, but it is easy
to generalize many other elements of the model ($b$ and $s$ can have different CM utility functions $U^b$ and $U^s$,
we can have firms in the CM with nonlinear production functions, and so on).}

As is standard, it is easy to see that the choices of $X$ and $\hat{m}$ are independent of $m$, that
$X = X^*$ where $U'(X^*) = 1$, and that $W^b_1 - W^b_0 = \phi$. In terms of $\hat{m}$, it should be obvious
that sellers have no incentive to take money out of the CM, so they set $\hat{m} = 0$. Indeed,
sellers are somewhat passive in this model, and the only thing we have to check (as in the
previous model) is that they are actually willing to produce the indivisible DM good for a
unit of money; this requires $c \leq \phi$. For buyers, since $M < N$, in equilibrium some take
$\hat{m} = 1$ out of the CM and others take $\hat{m} = 0$; this requires they are indifferent between the
two options,
\[ \phi = \beta (V_1^b - V_0^b). \quad (3.7) \]

Given this, and continuing for now to use taxation as in Li’s model, the DM value functions for buyers are
\[ V_1^b = \tau W_0^b + \alpha_b (u + W_0^b) + (1 - \alpha_b - \tau) W_1^b - k(\alpha_b) \quad (3.8) \]
\[ V_0^b = W_0^b \quad (3.9) \]

Subtracting these and using (3.7), we have
\[ \phi = \frac{\beta [\alpha_b u - k(\alpha_b)]}{1 - \beta (1 - \alpha_b - \tau)}. \quad (3.10) \]

Furthermore, any buyer with money in the DM chooses \( \alpha_b \) to solve
\[ k'(\alpha_b) = u - \phi. \quad (3.11) \]

Combining (3.10) and (3.11), we get the analog of (3.4) from the previous model:
\[ T(\alpha_b) = (1 - \beta + \beta \tau) u + \beta k(\alpha_b) - [1 - \beta + \beta (\alpha_b + \tau)] k'(\alpha_b) = 0. \]

Again, \( T(0) > 0 \) and \( T(\bar{\alpha}_b) < 0 \), where \( \bar{\alpha}_b \) is a natural upper bound. Moreover, \( T'(\alpha_b) = -[1 - \beta + \beta (\alpha_b + \tau)] k''(\alpha_b) < 0 \). Hence, there exists a unique \( \alpha_b^* \in (0, \bar{\alpha}_b) \) such that \( T(\alpha_b^*) = 0 \). It only remains to check the participation condition \( c \leq \phi \) for sellers at the equilibrium value of \( \phi \), which holds iff
\[ \beta [\alpha_b u - k(\alpha_b)] - [1 - \beta (1 - \alpha_b - \tau)] c \geq 0. \quad (3.12) \]

Monetary equilibrium exists for all \( \tau \leq \bar{\tau} \) where \( \bar{\tau} > 0 \) satisfies (3.12) at equality.\(^{12}\)

\(^{12}\)As in the previous section, this assumes \( \alpha_b < 1 - \bar{\tau} \), which is valid if \( c \geq \beta [\alpha_b u - k(\alpha_b)] \).
We can easily differentiate to get $\partial\alpha^e_b/\partial\tau > 0$, so that a higher tax rate (read higher inflation) increases search intensity. Velocity is slightly more complicated here, because of the two-sector structure. Nominal spending is $PY = PX^* + M\alpha_b$, where $P = 1/\phi$ is the nominal price level, the first term is CM spending, and the second is DM spending. Hence,

$$v = \frac{X^*}{M\phi} + \alpha_b = \frac{X^*}{M[u - k'(\alpha_b)]} + \alpha_b,$$

(3.13)

by virtue of (3.11). Therefore $\partial v/\partial\tau > 0$ iff $\partial\alpha_b/\partial\tau > 0$, and so $v$ also increases with $\tau$. In terms of optimality, as before, $\alpha^e_b = \alpha^*_b$ iff sellers get no surplus, which here means $c = \phi$. Again, the optimal policy is the maximum feasible tax $\bar{\tau}$. This model with an alternating CM-DM structure therefore delivers the same basic results as the Li model. Hence, it is a good framework to use when we relax the assumption of indivisible money.

### 3.4 Divisible Money

It is desirable to allow $m \in \mathbb{R}_+$, not only because $m \in \{0, 1\}$ is restrictive in a descriptive sense, but because we can then determine the terms of trade in a nontrivial way, and we can analyze inflation directly instead of proxying for it with taxation. Although we ultimately allow both goods and money to be divisible, it facilitates the presentation to start with the case where the DM $q = 1$ is still indivisible but $m \in \mathbb{R}_+$. An additional virtue of divisible money is that now we can endogenize search on the extensive margin – determining the number of buyers who go to the DM – while with $m \in \{0, 1\}$ this was pinned down by the exogenous $M$.

Now assume that aggregate money supply grows as $\hat{M} = (1 + \gamma)M$. In steady state with real balances $\phi M$ constant, the gross inflation rate is $\hat{P}/P = \phi/\hat{\phi} = \hat{M}/M = 1 + \gamma$. Let $z = \phi m$ denote the real balances that an agent brings to the CM, and $\hat{z} = \hat{\phi} \hat{m}$ is the
amount he takes out of this market and into next period’s DM. Let \( W^j(z) \) and \( V^j(z) \) be the CM and DM (time-invariant) value functions, for any \( z \in \mathbb{R}_+ \). The CM problem becomes

\[
W^j(z) = \max_{X,H,\hat{z}} \left\{ U(X) - H + \beta V^j(\hat{z}) \right\}
\]

s.t. \( X = H + z - (1 + \gamma)\hat{z} + \phi \gamma M, \hat{z} \in \mathbb{R}_+ \)

where \( \phi \gamma M \) is a lump-sum money transfer. Again \( X = X^* \), and now \( \partial W^j / \partial z = 1 \). Sellers still choose \( \hat{z} = 0 \), but here the choice of \( \hat{z} \) for buyers is slightly more complicated, since we cannot just take the FOC due to the fact that \( V^b(\hat{z}) \) may not be differentiable. In particular, a seller is willing to trade in the DM iff a buyer’s real balances are enough to cover his cost \( c \). Hence, there exists a cut-off \( z^* \), characterized below, such that trade occurs iff \( z \geq z^* \).

The DM value function for a buyer is the following: first, if \( z \geq z^* \) then

\[
V^b(z) = -k(\alpha_b) + W^b(z) + \alpha_b \left[ u + W^b(z - d) - W^b(z) \right],
\]

where \( d \) denotes the amount of real balances exchanged, and the term in brackets is his surplus from a DM trade, which reduces to \( u - d \) using \( \partial W^b / \partial z = 1 \). Second, if \( z < z^* \) then \( V^b(z) = W^b(z) \). The seller’s surplus is \(-c + d\), and so \( z^* = c \). Although there are several ways to determine the terms of trade, in much of this paper we use bargaining. However, in this version of the model, with indivisible goods, since a buyer in the DM cannot pay more than he has he can effectively commit to not pay more than \( z^* \) by not bringing more, and in this way he can capture the entire surplus.\(^{13}\)

Of course, as always, for this to be an equilibrium a seller has to be willing to trade, but this is true by definition of \( z^* \). Additionally we now have to check buyers are willing to participate and bring \( \hat{z} = z^* \), rather than \( \hat{z} = 0 \), since there is an ex ante cost to participating.

\(^{13}\)We do not dwell on this issue since we soon move to models with divisible goods and money. See Jean et al. (2009) for an extended discussion.
in the DM, which is the cost of acquiring the real balances in the previous CM. It is a matter of algebra to check they are willing to bring $\hat{z}^*$ iff $\alpha_b(u - c) - ic - k(\alpha_b) \geq 0$, where $i$ (the nominal interest rate) satisfies $1 + i = (1 + \gamma)/\beta$, and $\alpha_b$ is the choice of search intensity given by $k'(\alpha_b) = u - c$. Note that $\alpha_b = \alpha_b^*$ does not depend on $i$, since the cost of bringing money to the DM in the first place is sunk when buyers choose $\alpha_b$. In any case, we have the result that a monetary equilibrium exists iff $i \leq \bar{i}$, where $\bar{i} = [\alpha_b^*(u - c) - k(\alpha_b^*)] / c$.

Interestingly, in this model, from $k'(\alpha_b) = u - c$ we immediately get $\partial \alpha_b^* / \partial i = 0$; inflation has no effect on equilibrium search. It also has no effect on velocity, which turns out to be $v = X^*/Nc + \alpha_b^*$, where $N$ is the measure of buyers. This is an artifact of indivisible goods, however, as we will soon see. Before getting into that analysis, we can preview the results to come along the extensive margin. Suppose instead of a search cost $k(a_b)$ we assume buyers have to pay a fixed cost $k_b$ to participate in the DM, but once they are in $\alpha_b$ is not their choice. Let $\sigma_b$ be the fraction of buyers that choose to participate (we assume the number of participants is less than the total number of buyers $N$). Since sellers get in for free, all $1 - N$ of them participate. In equilibrium, assuming again an interior solution, the buyers’ participation decision implies $\alpha_b(u - c) - ic = k_b$. Hence, $\partial \alpha_b^* / \partial i > 0$, so more inflation increases the probability of a trade meeting for buyers. Velocity in this case is given by $v = X^*/\sigma_b c + \alpha_b^*$. As $\sigma_b$ is decreasing in $\alpha_b$, more inflation also increases $v$.

So the extensive margin looks promising, and to analyze this in detail, we now move to the case of divisible DM goods. The CM problem is still given by (3.14), except now $V^j(\hat{z})$ is differentiable, given the way we determine the terms of trade using bargaining. That is, in the DM, the pair $(q, d)$ is now determined by generalized Nash bargaining, with threat points equal to continuation values and bargaining power for the buyer denoted $\theta$. The key

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14This is explained in more detail below for the model with divisible goods (as well as divisible money).
difference is that now by bring more \( \hat{z} \) the buyer can get more \( q \). One can show, exactly as in Lagos and Wright (2005), that in any equilibrium, if buyers bring \( \hat{z} \) then \( d = \hat{z} \) and \( q \) solves \( g(q) = \hat{z} \), where\(^{15}\)

\[
g(q) = \frac{\theta c(q)u'(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)}.
\] (3.16)

This implies \( \partial q / \partial \hat{z} = 1 / g'(q) > 0 \), again, bring more money implies you get more stuff, unlike the case of indivisible goods.

Given all of this, we have

\[
V^b(\hat{z}) = -k(\alpha_b) + W^b(\hat{z}) + \alpha_b[u(q) - \hat{z}],
\]

where for now we return to the intensive margin, with \( \alpha_b \) an individual choice. Thus, it satisfies

\[
k'(\alpha_b) = u(q) - g(q),
\] (3.17)

after inserting the bargaining solution \( \hat{z} = g(q) \). To determine \( \hat{z} \) and hence \( q = g^{-1}(\hat{z}) \), consider the FOC for \( \hat{z} \) for buyers in the CM:

\[
1 + \gamma = \beta \frac{\partial V^b}{\partial \hat{z}} = \beta \left[ \alpha_b \frac{u'(q)}{g'(q)} + 1 - \alpha_b \right].
\]

Using \( 1 + i = (1 + \gamma) / \beta \), we reduce this to

\[
\frac{i}{\alpha_b} = \frac{u'(q)}{g'(q)} - 1.
\] (3.18)

Equilibrium now is a pair \((q, \alpha_b)\) solving (3.17) and (3.18).

Several remarks can be made about this model. For example, setting \( \theta = 1 \) implies \( g(q) = c(q) \) and then (3.17) guarantees search effort is efficient; this is again the Hosios

\(^{15}\)The surplus for a buyer is \( \Sigma^b = u(q) - \hat{z} \), and the surplus for a seller is \( \Sigma^s = \hat{z} - c(q) \). Given buyers do not bring more money than they spend, insert \( d = \hat{z} \) into the generalized Nash product \( \Sigma^b_\theta \Sigma^{1-\theta}_s \), take the first-order condition with respect to \( q \), and rearrange to get (3.16).
condition. Given $\theta = 1$, one can show $q < q^*$ for all $i > 0$, where $q^*$ is the efficient $q$, and $q = q^*$ iff we follow the Friedman rule $i = 0$. Hence the Hosios condition and the Friedman rule in combination define the efficient $\alpha_b^*$ and $q^*$. In any case, (3.17) and (3.18) define two curves in $(q, \alpha_b)$ space we call the SE and BS (for search effort and bargaining solution). As shown in Figure 3.1, both curves start at the $(0, 0)$; SE increases as $q$ increases from 0 to $q^*$ and then decreases to $\alpha_b = 0$ when $q = \hat{q}$, where $\hat{q} > 0$ solves $u(q) = c(q)$; BS increases to $(\tilde{q}, 1)$ where $\tilde{q} \in (0, q^*]$ solves $u'(\tilde{q}) = (1 + i)g'(\tilde{q})$. They could potentially intersect at multiple points, but it is easy to check that the SOC for the buyer’s choice of $q$ and $\alpha$ only holds when BS intersects SE from below.

![Figure 3.1: Equilibrium under Intensive Search Margin](image)

When we increase the inflation rate $\gamma$ or equivalently the nominal interest rate $i$, BS rotates up, which means at any point where BS intersects SE from below $q$ and $\alpha_b$ both fall.
More formally, differentiate (3.17) and (3.18) to get

\[ \frac{\partial q}{\partial i} = -k''D \quad \text{and} \quad \frac{\partial \alpha_b}{\partial i} = -\frac{u' - g'}{D}, \]

where \( D = -\alpha_b \ell' k'' - (u' - g')(\ell - 1), \) with \( \ell = \ell(q) \equiv u'(q)/g'(q). \) The SOC is \( D > 0, \)
and since \( u' > g' \) for all \( i > 0 \) by (3.18), we conclude that \( q \) and \( \alpha_b \) fall with \( i. \) This is the
result in Lagos and Rocheteau (2005): inflation makes buyers spend their money less and not more quickly, because it reduces the buyers’ surplus, which makes them less willing to invest in costly search.

At this point we move to study the extensive rather than the intensive margin of search – i.e. instead of search intensity we return to a free entry decision by buyers.\(^{16}\) To this end we now assume a standard matching function \( n = n(\sigma_b, \sigma_s), \) where \( n \) is the number
of trade meetings and now we interpret \( \sigma_b \) and \( \sigma_s \) as the measures of buyers and sellers in
the DM (and not the measures in the total population, as some may not go to the DM). An
individual’s probability of a trade meeting is \( \alpha_j = n(\sigma_b, \sigma_s)/\sigma_j, \) for \( j = b, s. \) Assume \( n \)
is twice continuously differentiable, homogeneous of degree one, strictly increasing, and
strictly concave. Also \( n(\sigma_b, \sigma_s) \leq \min(\sigma_b, \sigma_s), \) and \( n(0, \sigma_s) = n(\sigma_b, 0) = 0. \) Define the
buyer-seller ratio, or market tightness, by \( \delta = \sigma_b/\sigma_s. \) Then \( \alpha_b = n(1, 1/\delta), \) \( \alpha_s = n(\delta, 1), \)
and \( \alpha_s = \delta \alpha_b. \) Also, \( \lim_{\delta \to \infty} \alpha_b = 0 \) and \( \lim_{\delta \to 0} \alpha_b = 1. \)

Participation decisions are made by buyers, who have to pay a fixed cost \( k_b \) to enter,
while sellers get in for free and so all of them participate. We focus on the situation where
the total measure of buyers \( N \) is sufficiently big that some but not all go to the DM, which
means that in equilibrium they are indifferent. Of course this means buyers get zero ex-
pected surplus from participating in the DM, although those who actually trade do realize

\(^{16}\)This is in a sense opposite to the approach in the literature on limited participation in both reduced-form
models (e.g. Alvarez et al. 2008 or Khan and Thomas 2007) and search models (Chiu and Molico 2007) of
money. Those models assume agents have to pay a cost to access something analogous to our CM, sometimes
interpreted as a financial sector.
a positive surplus (just like the firms in Pissarides 2000). If one does not like this, it is easy enough to assume all buyers draw a participation cost at random from some distribution $F(k)$ each period. Then instead of all buyers being indifferent, there will be a marginal buyer with cost $k^*$ that is indifferent about going to the DM, while all buyers with $k < k^*$ strictly prefer to go since they get a strictly positive expected surplus. Given this is understood, for ease of presentation we focus on the case where $k$ is the same for all buyers.

For a buyer who does not go to the DM, $X = X^*$ and $\hat{z} = 0$. For one who does, he pays cost $k_b$ next period, but he has to acquire $\hat{z}$ in the current CM. Algebra implies he wants to go iff $-(1 + \gamma)\hat{z} + \beta [ -k_b + \alpha_b u(q) + (1 - \alpha_b)\hat{z} ] \geq 0$. Using (3.16) and inserting the nominal rate $i$, this can be written

$$-ig(q) - k_b + \alpha_b [u(q) - g(q)] \geq 0.$$  

(3.19)

There are two costs to participating in the DM: the entry cost $k_b$; and the cost of bringing real balances $ig(q)$. The benefit is $\alpha_b$ times the surplus. In equilibrium, (3.19) holds at equality:

$$\alpha_b = \frac{ig(q) + k_b}{u(q) - g(q)}.$$  

(3.20)

Given $q$, this determines $\alpha_b$. Then one gets the measure of buyers $\sigma_b$ from $\alpha_b = n(1, \sigma_s/\sigma_b)$, with $\sigma_n = 1 - N$. A monetary equilibrium is a solution $(q, \alpha_b)$ to the free entry and bargaining conditions (3.20) and (3.18), defining the FE and BS curves in Figure 3.2.

Restricting attention to the relevant region of $(q, \alpha_b)$ space, $(0, \bar{q}) \times [0, 1]$, it is routine to verify the following: the curves are continuous, BS is upward sloping and goes through $(0, 0)$, while FE is downward (upward) sloping to the left (right) of the BS curve, hitting a minimum where the curves cross.\(^{17}\) Hence, there is a unique equilibrium, and in equilibrium we have $\partial \alpha_b / \partial i > 0$ and $\partial q / \partial i < 0$. To see this, note that as $i$ increases the BS

\(^{17}\)Proof: The properties of BS are obvious. The slope of FE is given by $\partial \alpha_b / \partial q \approx (u-q)ig' - (ig + k_b)(u' - g')$ where $\approx$ means “equal in sign.” Eliminating $k$ using (3.20) and simplifying, $\partial \alpha_b / \partial q \approx i + \alpha_b - \alpha_b u'/g'$. 

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and FE curves both shift up, so $\alpha_b$ increases. To see what happens to $q$, rewrite the model as two equations in $q$ and $\alpha_b/i$ by dividing (3.20) by $i$. This new version of FE satisfies the same properties as before: it is downward (upward) sloping to the left (right) of the BS curve. But now as $i$ increases the FE curve shifts down while the BS curve does not shift ($q$ as a function of $\alpha_b/i$ does not change when $i$ changes). Hence $q$ falls.

Now consider $v = Y/\phi M$. Total real output is $Y = Y_C + Y_D$. Real CM output is $Y_C = X^*$ as always, and real DM output is $Y_D = n(\sigma_b, \sigma_s)\phi M/\sigma_b = \alpha_b\phi M$, since $M/\sigma_b$ is total cash per buyer participating in this market. Thus,

$$v = \frac{X^* + \alpha_b\phi M}{\phi M} = \frac{X^*}{\sigma_b g(q)} + \alpha_b,$$

From the CM problem, the derivative of the objective function $1 + \gamma = \beta \partial V^b/\partial \hat{z}$ can be rewritten in terms of $q$ as $-(i + \alpha_b) + \alpha_b u'(q)/g'(q) = -\partial \alpha_b/\partial q$. There is a unique solution to this maximization problem, $\partial \alpha_b/\partial q$ is positive (negative) as $q$ is less (greater) than the solution which is given by (3.18). Hence the FE curve is decreasing (increasing) to the left (right) of the BS curve.

Figure 3.2: Equilibrium under Free Entry by Buyers
using \( M = \sigma_b g(q)/\phi \). Since \( \partial \alpha_b / \partial i > 0 \), we have \( \partial \sigma_b / \partial i < 0 \), and we already know \( \partial q / \partial i < 0 \). Therefore we conclude that \( \partial v / \partial i > 0 \). Hence, this model unambiguously predicts that velocity increases with \( i \). And, again, it predicts the “hot potato” effect \( \partial \alpha_b / \partial i > 0 \), for the following intuitively plausible reason: an increase in inflation or rates must lead to buyers spending their money more quickly, since this is the only way to satisfy the free entry condition.

These results are natural, and they are quite robust, at least as long we maintain the assumption that buyers are the ones that face a DM participation choice.\(^{18}\) They are robust in the sense that in our baseline model the results do not depend on parameter values. They also do not depend much on the pricing mechanism. The same qualitative results hold with proportional rather than Nash bargaining (as used in money models by Aruoba et al. 2007), and with Walrasian price taking (as used by Rocheteau and Wright 2005). We also tried price posting with directed search. Recall that Lagos and Rocheteau (2005) could get agents to spend their money faster under this pricing mechanism for some parameter values in their intensive-margin model. In our extensive-margin model, under price posting and directed search, agents might or might not spend their money faster with inflation depending on parameters. So in both models, the results are ambiguous under price posting and directed search. But Lagos and Rocheteau can only get the desired “hot potato” effect for very low inflation; we get it for all parameters except possibly very low inflation.

Finally, we analyze welfare, which was part of our original motivation for this study. As in most related models, the Friedman rule \( i = 0 \) plus the Hosios condition \( \theta = 1 \) are necessary and sufficient for \( q = q^* \). But given \( q^* \), we may not get efficiency in terms of entry, since there is a search externality at work: participation by buyers increases the

\(^{18}\)For the record, we also studied the model where all buyers enter but sellers have to pay \( k_s > 0 \), and the model where each agent can choose to be a buyer or seller. In those models, the results are ambiguous, and \( v \) can increase or decrease with \( i \) in examples.
arrival rate for sellers and decreases it for other buyers. As is often the case in models with entry, there is a Hosios condition for efficient participation, which sets the elasticity of the matching function with respect to the number of buyers equal to their bargaining power \( \theta \).

But this conflicts in general with the condition \( \theta = 1 \) required for \( q = q^* \).

Formally, the planner’s problem is to choose sequences for \( \{\sigma_{bt}, q^b_t, q^s_t, X_t, H_t\} \) to maximize

\[
\sum_{t=0}^{\infty} \beta^t \left\{ n(\sigma_{bt}, \sigma_s) \left[ u(q^b_t) - c(q^s_t) \right] - \sigma_{bt}k_b + U(X_t) - H_t \right\},
\]

subject to \( q^b_t \leq q^s_t \) and \( X_t \leq H_t \). Optimality requires for all \( t \)

\[
\frac{u'(q)}{c'(q)} = 1 \quad (3.21)
\]

\[
n_1(\sigma_b, \sigma_s) \left[ u(q) - c(q) \right] = k_b. \quad (3.22)
\]

We want to compare this with the equilibrium conditions under Nash bargaining, which we repeat here for convenience:

\[
\frac{u'(q)}{g'(q)} = 1 + \frac{i}{\alpha_b} \quad (3.23)
\]

\[
\alpha_b \left[ u(q) - g(q) \right] = ig(q) + k_b. \quad (3.24)
\]

Clearly \( i = 0 \) and \( \theta = 1 \) achieve \( q = q^* \), and given this, entry is efficient iff \( n_1(\sigma_b, \sigma_s) = \alpha_b \), which is equivalent to saying the elasticity of the matching function with respect to \( \sigma_b \) is 1.

In general, we do not get efficiency if this elasticity is not 1. For instance, if the matching function is Cobb-Douglas we can assign whatever value \( \eta \in (0, 1) \) to this elasticity. With \( \eta < 1 \) and \( \theta = 1 \), the number of buyers in the DM is necessarily too high. An important implication is that for a given \( \theta \), and especially for a relatively high \( \theta \), the Friedman rule \( i = 0 \) may not be optimal. When the number of buyers is too high, a small increase in the nominal rate from \( i = 0 \) entails a welfare cost, because it reduces \( q \), but it also brings
the number of buyers closer to the efficient level. When $\theta = 1$, for $i$ near 0, the welfare consequence of the effect on $q$ is of second order because $q$ is near $q^\ast$, and therefore the net gain is positive and the optimal policy is $i > 0$. It is not hard to construct explicit examples to this effect.\footnote{Similar results can be found in Nosal and Rocheteau (2009), although there it is slightly easier, because they assume that buyers who do not participate become sellers in the DM while we assume they simply sit out. Hence, in their model, when the number of buyers is too high, inflation can, by reducing the number of buyers and increasing the number of sellers actually increase the number of DM trades. For us inflation always reduces the number of DM trades because it decreases the number of buyers and the number of sellers is fixed. But it can still increase welfare.}

To illustrate the above analysis, consider the following quantitative exercise. Assume the relatively standard functional forms:

**DM utility:**

$$u(q) = \frac{q^\ast}{1-\mu} - b(1-\mu)$$

**DM cost of production:**

$$c(q) = q$$

**CM utility:**

$$U(X) = AlnX - H$$

**Matching function (Kiyotaki-Wright):**

$$n(\sigma_b, \sigma_s) = \frac{\sigma_b\sigma_s}{\sigma_b+\sigma_s}$$

Set $\beta = 1/1.03$ and $b = 0.0001$, and normalize $\sigma_s = 1$. Then calibrate the remaining parameters as follows. Set $A$ to match average $M/PY$, and $\mu$ to match the interest elasticity of $M/PY$, in the annual U.S. data (1948-2005), as shown by the model’s implied “money demand” curve shown in Figure 3.3. Finally, set entry cost $k$ so the DM contributes 10\% to aggregate output, and $\theta$ so that the DM markup is 30\%.\footnote{The parameter $b$ is here for purely technical reasons, so that $u(0) = 0$, but is set close to 0 so that DM utility displays approximately constant relative risk aversion. The 10\% DM share is targeted so that the results are easily comparable to Lagos and Wright (2005). The 30\% DM markup target is discussed in Aruoba et al. (2009). The results of the calibration are $(A, \mu, k, \theta) = (2.709, 0.373, 0.147, 0.671)$.}

The welfare cost of inflation is measured using the standard method: we ask how much total consumption agents would be willing to give up to reduce inflation to the Friedman rule. In general, bargaining power $\theta$ plays a key role in these calculations. Figure 3.4 shows
the cost of inflation as it ranges up to 20%, under different values of $\theta$. For our benchmark value of $\theta = 0.671$, optimal inflation is above the Friedman rule, but still negative. At the optimal policy, with $\theta = 0.671$, welfare in consumption units is 0.2% above what it would be at the Friedman rule. Also shown is the case $\theta = 1$, where optimal inflation is well over 5%, and at the optimal policy welfare is nearly 2% above what it would be at the Friedman rule. And finally, for $\theta = 1/2$, the Friedman rule is optimal – positive nominal interest rates are not always efficient, but they could be. Again, these results are not meant to be definitive, and are certainly sensitive to parameter values, but they clearly indicate to us that extensive-margin models are worth further study.

## 3.5 Conclusion

This chapter studied the relationship between inflation and nominal interest rates, on the one hand, and the speed with which agents spend their money, on the other. We are mainly
interesting in what we call the “hot potato” effect: when inflation increases, people spend their money faster. We also discussed the effects of inflation or interest rates on velocity and on welfare. We first presented some evidence on velocity, showing that it tends to increase with inflation and nominal interest rates. We then discussed some simple models with indivisible money and goods. While it is possible to generate a positive relationship between the variables in question, in a sense this is an artifact of indivisibilities: a model with divisible money and endogenous search intensity unambiguously predicts a negative relationship.

Then we changed the framework by focusing on the extensive rather than intensive search margin – i.e., on how many buyers are searching, rather than on what any given buyer is doing. This is the main contribution of the paper. Now the model unambiguously predicts a rise in inflation leads to an increase in the speed with which agents spend their
money, which is the “hot potato” effect we set out to capture. While undoubtedly both margins can be relevant, in reality, we think focusing on the extensive margin is interesting for a variety of reasons, including the implications for welfare. In particular, it is not hard to get inflation above the Friedman rule to be the optimal monetary policy in this framework. Also, although we do not have direct evidence on this, the predictions of the model are consistent with the casual empirical observation that people are less inclined to participate in cash-intensive market activity during periods of higher inflation.

In terms of methodology, the exercise in this chapter also suggests that search-theoretic monetary model provides a right tool for us to study certain questions we are interested in. The issues raised in this model are all about search and matching. We are interested in the speed with which buyers spend their money. The relevant arrival rates are either determined on the intensive margin using search intensity, or on the extensive margin using a matching function and endogenous participation. By explicitly modeling the frictions in the goods market, search-theoretic models of money provide useful insights for us to understand the interaction between monetary policy and trading behavior in the goods market.
Chapter 4

Equilibrium Price Dispersion and Rigidity

This chapter was co-authored with Allen Head, Guido Menzio and Randall Wright.

4.1 Introduction

This chapter develops a framework in which sellers post prices in nominal terms, and in equilibrium some fraction of individual prices may not respond to changes in the aggregate price level. As in the model of Lagos and Wright (2005), money is essential as a medium of exchange, and hence it is natural that prices are posted in dollars. As in the model of Burdett and Judd (1983), there is a non-degenerate distribution of prices, even if both sellers and buyers are homogeneous. Due to search frictions, sellers can post high or low prices and have the same expected profit, since the latter yields more sales. As the money supply and the aggregate price level increase, profit maximization does not require that all sellers change their nominal prices, only that the aggregate distribution of real prices
remain the same. Hence profit maximization is consistent with many sellers changing prices infrequently, in spite of the fact that sellers may adjust their prices whenever they like—we do not restrict the timing of price changes as does Calvo (1983)—and without cost—we do not impose menu costs, as do, for example, Caplin and Spulber (1987).

Although our theory implies that individual sellers may not need to change their nominal prices frequently—and so one could say that they can rationally afford to be somewhat inattentive to their own price, to the aggregate price level, and to monetary policy—it nonetheless has many testable predictions. The distribution of relative prices is determinate, and as in some other search-based models, we may derive a closed-form solution for the density. Also, while the empirical distribution may or may not look like that predicted by a simple version of the model, it is possible in extended versions to generate price densities with a wide range of properties.¹ Moreover, although the real distribution is invariant to the nominal price level, its shape does depend on the inflation rate in a precise way that is in principle testable against macro data. Also, the price distribution has non-degenerate support. When the aggregate price level increases some sellers drop out of the support and have to adjust. Thus, the frequency of price changes tends to increase with inflation, as observed by Nakamura and Steinsson (2008), Gagnon (2009), and others.

In addition to delivering as an equilibrium outcome that individual sellers may change their prices infrequently, even when inflation is nonzero, the theory makes several other predictions we can compare with the data. For instance, when sellers change their prices in our model they do not necessarily all set them to the same level — and in particular they do not set them as in the standard (s, S) theory. After a change, a given seller may increase or decrease its price, despite positive inflation. Also, sellers who have not changed their price

¹The baseline model of Burdett and Mortensen (1998), for example, predicts a wage distribution that does not match the wage data very well, and so empirical work typically uses generalized versions with heterogeneous firms and workers (e.g. see Bontemps, Robin and van den Burg, 2000).
for a long time are more likely to change in any given period, since they are more likely to fall out of the equilibrium price support, especially in high inflation environments. All of these predictions can be taken to the data.

Because we have a genuine monetary theory, as we said, as a natural outcome we can generate nominal price setting, and these prices tend to be sticky. This is in sharp contrast to simply assuming that prices are set in dollars, often in models that do not even have money, and then assuming that they can be adjusted only infrequently or at a cost. In this sense our theory provides a micro-foundation for a key feature in many Keynesian models that was never derived endogenously. But it is critical to understand that the policy implications of our theory are very different. Our model exhibits classical neutrality: although many individual nominal prices can stay fixed when there is inflation, since the real price distribution is invariant, if there were to be a one-time shock to money and hence to the nominal price level, the nominal distribution would shift immediately and there would be no real effect. That is, price stickiness in our theory does not generate monetary non-neutrality.

Other authors have developed models in which in spite of exogenously imposed price stickiness, money is either neutral or nearly so (see, for examples, Caplin and Spulber, 1987 and Golosov and Lucas, 2007 ). Our theory differs in a number of respects. First, as stated above, monetary neutrality is a robust feature of our economy, in spite of the fact that money plays an essential role in exchange and inflation may have significant welfare costs. Second, price stickiness in our model arises entirely endogenously.

The rest of this chapter is organized as follows. The next section proceeds to describe the basic environment. Section three defines a stationary monetary equilibrium and introduces a refinement such that there exists a unique equilibrium of this type. Section four describes the behavior of nominal prices in the equilibrium, and introduces a parametric
representation of the extent of nominal price stickiness. This section also contains a cal-
ibration and discusses the ability of the theory to account for the facts discussed above.
Section five considers the robustness of our main findings to a variety of modifications. We
work out explicitly versions of the economy with divisible goods in which sellers are and
are not restricted to the use of linear prices. The last section concludes.

4.2 The Model

4.2.1 The Environment

Time is discrete and indexed by \( t \). Each time period is divided into two consecutive sub-
periods. In the first sub-period, exchange of a produced and indivisible good takes place
in an anonymous market with search frictions. We will refer to this market as the Burdett-
Judd market (BJ market). In the second sub-period, exchange of an endowed good takes
place in a Walrasian market. We will refer to this market as the Arrow-Debreu market (AD
market). It is worth stressing that there is nothing special about this particular ordering of
the two markets, nor about the assumption that the good exchanged in the BJ market is
indivisible, as we will show later.

In the economy there are a large number of households with measure 1. Each household
maximizes the utility function

\[
\sum_{t=0}^{\infty} \beta^t u 1[q_t = 1] + x_t, \tag{4.1}
\]

Here \( q_t \in \{0, 1\} \) is the household’s consumption of the good exchanged in the BJ market
(we will refer to this as the BJ good), \( x_t \in \mathbb{R} \) is the consumption of the good exchanged
in the AD market (we will refer to this as the AD good), \( u > 0 \) is the utility derived from
consuming a unit of the BJ good, and $\beta \in (0, 1)$ is the discounting factor. Each household receives an endowment of $\pi > 0$ units of the AD good in every period.

There is also a positive measure of firms, $s$. Each of these firms can produce unlimited units of the BJ good at a constant cost, $c \in (0, u)$, per unit. They derive no utility from the consumption of the BJ good, but have linear utility in the AD good, of which they have no endowment.

We assume that in the BJ market, households are anonymous, so exchange is facilitated by the existence of fiat money. The government controls the supply of this money. In the AD market of period $t - 1$, the quantity of fiat money is $M_{t-1}$. At the beginning of the AD market of period $t$, the government injects $(\mu - 1)M_{t-1}$ units of fiat money as uniform lump-sum transfers to the households, i.e. $T_t = (\mu - 1)M_{t-1}$. Therefore, during the AD market of period $t$, the quantity of fiat money in circulation is $M_t = \mu M_{t-1}$. For the sake of simplicity, we assume that the growth rate of money, $\mu$, is constant over time.\(^2\)

In the BJ market of the current period, each firm posts a nominal price $p \geq 0$, taking as given the amount of money, $\hat{m}$, brought into the market by prospective buyers (households), and the cumulative distribution function, $F(p)$, of nominal prices posted by all other firms. In the BJ market, there is a measure $b \in [0, 1]$ of households, who have entered the market at a cost $k \geq 0$. Following Burdett and Judd (1983), we assume that households observe the entire price distribution, but can only purchase the good from a random sample of firms. In particular, a household can purchase the good from two firms with probability $\alpha_2(b) = (\lambda(b))^2$ and it can purchase the good from just one firm with probability $\alpha_1(b) = 2\lambda(b)(1 - \lambda(b))$. Also, with probability $\alpha_0(b) = (1 - \lambda(b))^2$ a household, having entered the BJ market, has no opportunity to purchase any good. Here $\lambda : \mathbb{R}_+ \rightarrow [0, 1]$ is a twice

\(^2\)From this point, we suppress time subscripts and use $x_{t+1}$ and $x_{t-1}$ to distinguish quantities and prices in the next and previous periods respectively.
differentiable function of \( b \) such that \( \lambda(0) = 1, \lambda(1) = 0, \) and \( \lambda'(b) < 0. \) Note that firms’ sales revenues are collected in fiat money.

At the beginning of the AD market, each household receives its transfer \( T \), chooses how many units of the AD good to purchase or sell, whether or not to enter the subsequent BJ market, and how much money, \( \hat{m} \) to carry into the next period. Since households are recognizable in the AD market, exchange may take place with either money or credit. Let \( \phi \) denote the unit of the AD good that can be purchased with one unit of money. If a firm makes a sale in the previous BJ market, it carries the cash receipts into the AD market, and uses them to purchase the AD good.

### 4.2.2 Firm Optimization

Consider a firm that posts a nominal price \( p \) in the BJ market. Following Burdett and Judd, no firm would post a price higher than the buyer’s reservation value, \( u/\phi \), as this only results in no sale. At the same time, no buyer can purchase from a seller who posts \( p > m^* \), where \( m^* \) is the equilibrium money holdings of a representative household. Thus, \( p \leq \min\{m^*, u/\phi\} \). Given the posting price \( p \), a firm meets \( \alpha_1(b)b \) buyers who cannot purchase the good from any other seller because they observe no other price. These buyers purchase the good from the firm with probability one. The firm also meets \( 2\alpha_2(b)b \) buyers who have the opportunity to purchase from another seller. The firm sells to these buyers only if the other seller’s price exceeds that posted by the firm, which occurs with probability \( 1 - F(p) \). The firm’s expected profits, \( \Pi(p) \), are then equal to

\[
\Pi(p) = (\phi p - c) b \left[ \alpha_1(b) + 2\alpha_2(b)(1 - F(p)) \right].
\]

Now suppose one firm posts a price \( p > \min\{m^*, u/\phi\} \). The firm meets \( \frac{b}{s} \left[ \frac{\alpha_1(b) + 2\alpha_2(b)}{\alpha_1(b) + \alpha_2(b)} \right] \) buyers, but it does not sell the good to any of them, since they all have a better option.
Therefore, the firm’s expected profits, $\Pi(p)$, are equal to zero.

In equilibrium, every price on the support, $\mathcal{F}$, of the cumulative distribution of nominal prices, $F(p)$, maximizes firms’ profits. As established in Burdett and Judd, this condition implies that the cumulative distribution function $F(p)$ is continuous over a compact support $\mathcal{F} = [p, \bar{p}]$. Since $\Pi(p) = \Pi(\bar{p})$ for every $p \in \mathcal{F}$, we may derive an expression for $F(p)$ analytically:

$$F(p) = 1 - \frac{\alpha_1(b)}{2\alpha_2(b)} \left( \frac{\phi\bar{p} - \phi p}{\phi p - c} \right). \quad (4.3)$$

Again following the logic of Burdett and Judd, the upper support of the price distribution is given by the monopoly price, $\bar{p} = \min\{m^*, u/\phi\}$, which a firm would post if it knew with certainty that any buyer observing its price would not receive any other price quote. Since $F(p) = 0$, the lower support is given by

$$p = \frac{\alpha_1(b)\phi \bar{p} + 2\alpha_2(b)c}{(\alpha_1(b) + 2\alpha_2(b))\phi}. \quad (4.4)$$

It is useful to measure prices in units of the AD good rather than in units of money. We refer to such prices as real prices. In order to compute a real price, it is sufficient to notice that $p$ units of money are equivalent to $p\phi$ units of the AD good. Therefore, we define the cumulative distribution function, $H(z)$, of real prices in the BJ market and note that given $F(p)$ it satisfies:

$$H(z) = H(\phi p) = F(p) = 1 - \frac{\alpha_1(b)}{2\alpha_2(b)} \left( \frac{z - \bar{z}}{z - c} \right). \quad (4.5)$$

Denote the support of $H(z)$, $\mathcal{H}$. The highest real price, $\bar{z}$, is equal to $\phi \bar{p} = \min\{\phi m^*, u\}$. And the lowest real price, $\underline{z}$, is equal to

$$\underline{z} = \phi p = \frac{\alpha_1(b)\bar{z} + 2\alpha_2(b)c}{[\alpha_1(b) + 2\alpha_2(b)]}. \quad (4.6)$$
4.2.3 Household Optimization

Let $V(m)$ and $W(m)$ denote the lifetime utility of households with $m$ units of money at the beginning of the current BJ and AD markets, respectively.

Arrow-Debreu Market

Consider first a household who enters the AD market with $m$ units of money, and has decided to participate in the subsequent BJ market. The income of this household is $\bar{x} + \phi(m + T)$, which is allocated between consumption of the AD good, $x_1$, and money holdings for the BJ market, $\hat{m}_1$. The lifetime utility, $W_1(m)$, then is given by

$$W_1(m) = \max_{x_1, \hat{m}_1} -k + x_1 + \beta V_{+1}(\hat{m}_1),$$

s.t.

$$x_1 + \phi \hat{m}_1 = \bar{x} + \phi(m + T),$$

$$\hat{m}_1 \geq 0.$$

It is immediate to verify that the necessary condition for an interior choice of $\hat{m}_1$ is

$$\beta V_{+1}^{-}(\hat{m}_1) - \phi \geq 0, \quad \beta V_{+1}^{+}(\hat{m}_1) - \phi \leq 0,$$

(4.8)

where $V_{+1}^{-}$ and $V_{+1}^{+}$ denote, respectively, the left and the right derivatives of the value function $V_{+1}$ with respect to $\hat{m}$. As we will show later, the value function $V_{+1}(\hat{m}_1)$ may not be differentiable at $\hat{m}$. Moreover, with linear preference the value function $W_1$ is linear with respect to $m$, i.e. $W_1(m) = \phi m + W_1(0)$.

Next, consider a household who enters the AD market with $m$ units of money and has decided to skip the subsequent BJ market. This household’s lifetime value differs from that given by (4.7) in that they do not pay the BJ market entry cost, and expect to do nothing
between the current and next period AD market:

\[ W_0(m) = \max_{x_0, \hat{m}_0} x_0 + \beta W_{+1}(\hat{m}_0), \]

s.t. \[ x_0 + \phi \hat{m}_0 = \bar{x} + \phi (m + T), \]

\[ \hat{m}_0 \geq 0. \] \tag{4.9}

In this case it is immediate to verify that the optimal choice of money holdings is \( \hat{m}_0 = 0 \). Also, the value function \( W_0 \), like \( W_1 \), is linear with respect to \( m \), i.e. \( W_0(m) = \phi m + W_0(0) \).

Finally, a household who enters the AD market and has yet to decide whether to participate in the next BJ market has lifetime utility,

\[ W(m) = \max \{ W_0(m), W_1(m) \} = \phi m + \max \{ W_0(0), W_1(0) \}, \] \tag{4.10}

where the second line makes use of the fact that \( W_i(m) \) is equal to \( \phi m + W_i(0) \), for \( i = 1, 2 \).

Therefore, the derivative of \( W \) with respect to \( m \) exists and is given by

\[ W'(m) = \phi. \] \tag{4.11}

Equation (4.11) verifies our initial conjecture that the marginal utility of a unit of money in the AD market is equal to \( \phi \).

**Burdett-Judd Market**

Next, consider a household who enters the BJ market with \( m \) units of money, \( m \in [0, u/\phi] \).\footnote{In this environment, households never choose to enter the BJ market with more than \( u/\phi \) units of money.}

With probability \( \alpha_0(b) \), the household does not meet a firm and he enters the AD market with \( m \) units of money. With probability \( \alpha_1(b) \), the household meets exactly one firm.
If the firm’s price \( p \) is smaller than \( m \), the household purchases the good and transits to the AD market with \( m - p \) units of money. If the price is greater than \( m \), the household cannot afford the good and he enters the AD market with \( m \) units of money. Finally, with probability \( \alpha_2(b) \), the household meets two firms, and if the lower of the two prices is smaller than \( m \), the household purchases at the lower price and enters the AD market with their remaining money, \( m - p \).

Overall, the household’s lifetime utility \( V(m) \) is equal to

\[
V(m) = W(0) + \phi m + \left[ \alpha_1(b) + \alpha_2(b) \right] \int_{\bar{p}}^{p} [p \leq m] (u - \phi p) dJ(p), \tag{4.12}
\]

where

\[
J(p) = \frac{\alpha_1(b) F(p) + \alpha_2(b) [1 - [1 - F(p)]^2]}{\alpha_1(b) + \alpha_2(b)}, \quad \forall p \in \mathcal{F}, \tag{4.13}
\]

is the distribution of the lowest price observed (i.e. of transaction prices), and \( W_t(m) = W_t(0) + \phi_t m \).

The derivative of the value function \( V \) with respect to \( m \) exists for all \( m \) not equal to either \( p \) or \( \bar{p} \). For all \( m \) strictly smaller than \( p \) and all \( m \) greater than \( \bar{p} \), \( V'(m) = \phi \). For all \( m \in (\underline{p}, \bar{p}) \) we have

\[
V'(m) = \phi + (u - \phi m) J'(m). \tag{4.14}
\]

Clearly, \( V'(m) > 0 \) for all \( m \in (\underline{p}, \bar{p}) \). It is also straightforward to show that \( V''(m) \leq 0 \).

### 4.3 Equilibrium

#### 4.3.1 Definition

In this paper, we restrict attention to stationary equilibria. That is, we consider equilibria in which all the real variables are constant over time, i.e. \( z_t = z, H_t = H, b_t = b \) for
\(t = 0, 1, \ldots\), and all the nominal variables grow at the same rate, \(\mu\), as the supply of money. That is, \(F_{t+1}(\mu p) = F_t(p_t)\) for all \(p_t, p_{t+1}\) in \(\mathcal{F}_t\) and \(\mathcal{F}_{t+1}\) respectively, \(m_{t+1} = \mu m_t\), and \(\phi_{t+1} = \phi_t/\mu\).

To solve the household’s problem, we first substitute (4.14) into (4.8). Imposing the stationarity condition, one can derive the equilibrium conditions for the optimal real balance the household carries into the BJ market, \(\bar{z}\),

\[
\mu \geq \beta, \quad \mu \leq \beta \left[ 1 + \alpha_1(b) H'(\bar{z})(u - \bar{z}) \right]. \tag{4.15}
\]

Given \(\bar{z}\), we then consider households’ participation in the BJ market. Using (4.10), the difference between the lifetime utility of a household who chooses to participate and that of one who chooses to skip it is given by

\[
\Psi(b, \bar{z}) = W_1(0) - W_0(0) = [\alpha_1(b) + \alpha_2(b)] \int_{\bar{z}}^{\bar{z}} (u - z) dG(z) - \frac{k}{\beta} - \left( \frac{\mu - \beta}{\beta} \right) \bar{z}, \tag{4.16}
\]

where \(G(z)\) is the distribution of the lowest real price observed, derived from \(H(z)\) just as \(J(p)\) is from \(F(p)\) at (4.13). Putting together, we have the following definition of equilibrium.

**Definition 4.1:** A Stationary Monetary Equilibrium is a distribution of real prices in the BJ market, \(H\); a level of real monetary balances, \(\bar{z} > 0\); and a measure of households who participate in the BJ market, \(b > 0\), satisfying

1. given \(\phi = \bar{z}/M\), the real price distribution \(H\), is constructed as in (4.5);
2. given \(H, \bar{z}\) solves the money demand problem as in (4.15);
3. free entry implies that \(b\) satisfies

\[
\begin{align*}
\text{if } b = 0, \text{ then } &\Psi(0, \bar{z}) \leq 0, \\
\text{if } b = 1, \text{ then } &\Psi(1, \bar{z}) \geq 0, \\
\text{if } b \in (0, 1), \text{ then } &\Psi(b, \bar{z}) = 0. 
\end{align*}
\tag{4.17}
\]
Condition (1) derives the equilibrium distribution of posted real prices, using firms equal profit condition. Given the price distribution, condition (2) guarantees that the real balances carried by the buyers into the BJ market solve the problem of the household. In the third condition, free entry requires that the payoff from participating in the BJ market is high enough to compensate the entry cost \( k \).

Now considering equations (4.13) and (4.15), it is clear that there is a strategic complementarity between households’ choice of money holdings and firms’ choice of prices. On the one hand, households have no incentive to hold more money than the highest price posted by any firm. On the other hand, firms have no incentive to set a price that is higher than the quantity of money carried by households. As a result, our model admits a continuum of stationary equilibria indexed by the household’s money holdings. All these equilibria offer similar implications to price stickiness. For the purpose of this chapter, however, we focus on a particular equilibrium which yields the highest household’s money balance.

**Definition 4.2:** A Refined Stationary Monetary Equilibrium is stationary monetary equilibrium \((H, \bar{z}, b)\) such that:

\[
\mu = \beta \left[ 1 + \alpha_1(b) H'(\bar{z})(u - \bar{z}) \right].
\]  

(4.18)

### 4.3.2 Existence and Uniqueness

First, note that (4.18) guarantees that in equilibrium the household’s choice of real balances, \( \bar{z} \), is optimal given the real price distribution, \( H \), and the fraction of households who participate in the BJ market, \( b \). Using the equilibrium condition (4.5) to substitute for \( H \), we may rewrite (4.18) as

\[
i = \frac{\alpha_1(b)^2}{2 \alpha_2(b)} \frac{u - \bar{z}}{\bar{z} - c},
\]

(4.19)
where \( i \geq 0 \) denotes the nominal interest rate, \((\mu - \beta)/\beta\). Since by assumption the growth rate of money, \( \mu \), is greater than the discount factor, \( \beta \), \( i \) is always positive. Equation (4.19) holds if real balances are equal to

\[
\bar{z}_{RB}(b) = \frac{2 (1 - \lambda(b))^2 u + ic}{2 (1 - \lambda(b))^2 + i}.
\]  

(4.20)

From (4.20), it follows that, for \( b = 0 \), the household’s optimal real balances are equal to the firm’s cost of production, \( c \). For \( b = 1 \), the household’s optimal real balances are equal to \((2u + ic)/(2 + i)\). Moreover, for all \( b \) between 0 and 1, the household’s optimal real balances are strictly increasing with respect to \( b \). Intuitively, the higher is the measure of buyers in the BJ market, the lower the probability that two firms compete for the same buyer, the higher the distribution of prices (in the sense of first order stochastic dominance), and the higher are the household’s optimal real balances. The green line in Figure 1 depicts (4.20), optimal real balances as a function of the measure of households who participate in the BJ market.

Next, consider the free entry condition, (4.17). This condition guarantees that the measure of households who participate in the BJ market, \( b \), is consistent with the household’s optimal entry decision given the firms’ real price distribution, \( H \), and the household’s real balances, \( \bar{z} \). Using (4.5) to substitute for the real price distribution, we rewrite (4.16) as

\[
\Psi(b, \bar{z}) = \alpha_1 \int_{\bar{z}}^{\Psi} \frac{\alpha_1 (z-c)(u-z)}{(z-c)^2} \, dz + \alpha_2 \int_{\bar{z}}^{\Psi} \left( \frac{\alpha_1}{2\alpha_2} \right)^2 \frac{(z-c)(z-u-z)(u-z)}{(z-c)^3} \, dz - \frac{k}{\beta} - i\bar{z}
\]

(4.21)

where the second line of (4.21) uses the following facts:

\[
\int_{\bar{z}}^{\Psi} \frac{\alpha_1 (z-c)(u-z)}{(z-c)^2} \, dz = u - c - \frac{\alpha_1 (z-c)}{\alpha_2} \ln \left( \frac{\alpha_1 + 2\alpha_2}{\alpha_1} \right)
\]  

(4.22)

\[
\int_{\bar{z}}^{\Psi} \left( \frac{\alpha_1}{2\alpha_2} \right)^2 \frac{(z-c)(z-u-z)(u-z)}{(z-c)^3} \, dz = u - c - \frac{\alpha_1 (z-c)}{2\alpha_2} + \frac{\alpha_1^2 (z-c)}{2\alpha_2^2} \ln \left( \frac{\alpha_1 + 2\alpha_2}{\alpha_1} \right)
\]  

(4.23)
From \((4.21)\), it follows that the household’s surplus from entering the BJ market is strictly positive for all \(\bar{z} < \bar{z}_{FE}(b)\); it is strictly negative for all \(\bar{z} > \bar{z}_{FE}(b)\); and it is equal to zero for \(\bar{z} = \bar{z}_{FE}(b)\), where \(\bar{z}_{FE}(b)\) is given by

\[
\bar{z}_{FE}(b) = \frac{2\lambda(b)(1 - \lambda(b))u + \lambda(b)^2(u - c) - k/\beta}{i + 2\lambda(b)(1 - \lambda(b))}.
\] (4.24)

From \((4.24)\) and the equilibrium condition \((4.17)\), it follows that the measure of households who participate in the BJ market is \(b = 0\) if \(\bar{z} \geq \bar{z}_{FE}(0)\), where \(\bar{z}_{FE}(0)\) is equal to \((u - c - k/\beta)/i\). The equilibrium measure of buyers who participate in the BJ market is \(b = 1\) if \(\bar{z} \leq \bar{z}_{FE}(1)\), where \(\bar{z}_{FE}(1)\) is equal to \(-(k/\beta)/i\). And the equilibrium measure of buyers in the BJ market is \(b \in (0, 1)\) if \(\bar{z} = \bar{z}_{FE}(b)\), where \(\bar{z}_{FE}(b)\) is strictly decreasing with respect to \(b\). Intuitively, the higher are the households’ real balances, the higher the price distribution (in the sense of first order stochastic dominance), the lower the household’s surplus from entering the BJ market, and the lower the measure of buyers who choose to enter. The red line in Figure 4.1 is the plot of the real balances, \(\bar{z}\), which induce \(b\) households to participate in the BJ market. That is, the red line in Figure 1 is the plot of \(\bar{z}_{FE}(b)\).

Given the properties of \(\bar{z}_{RB}(b)\) and \(\bar{z}_{FE}(b)\), it is immediate to prove that there exists one and only one \(b\) such that \(\bar{z}_{RB}(b) = \bar{z}_{FE}(b)\) for all nominal interest rates \(i\) smaller than \(\tilde{i}\), where \(\tilde{i}\) is equal to \((u - c - k/\beta)/c\). Moreover, it is immediate to prove that there exists no \(b\) such that \(\bar{z}_{RB}(b) = \bar{z}_{FE}(b)\) for any nominal interest \(i\) strictly greater than \(\tilde{i}\). These findings lead to the following theorem.

**Proposition 4.1.** (Existence and Uniqueness) For all \(i \in [0, \tilde{i}]\), there exists a unique refined monetary stationary equilibrium. For all \(i > \tilde{i}\), a refined monetary stationary equilibrium does not exist.
4.4 The Behavior of Nominal Prices

One of the properties of our stationary monetary equilibrium is that firms are indifferent between posting any real price that falls in the interval between $z$ and $\bar{z}$, as all these prices yield the same profit. A firm that posts a low price earns less per unit, but makes it up on the volume. In an inflationary environment, nominal distribution shifts up over time but real distribution remains the same. For each individual firm, the real price $z$ falls as the aggregate price level goes up, but many firms do not need to change their prices as long as $z \in \mathcal{F}$. For this reason, our model has the potential to support a pricing policy in which firms do not change their nominal prices in every period, and price rigidities arise as an
equilibrium outcome.

In this section, we investigate the possibility that the price stickiness implied by our model is consistent with the observed behavior of price adjustment. To this aim, we first specify a parametric (and relatively flexible) family of pricing policies. The pricing policies in this family combine some of the elements of time-dependent pricing models and some of the elements of state-dependent pricing models. Second, we identify which of the pricing policies in this family can be supported in the refined stationary equilibrium of our model. Third, for every policy that is supported by the equilibrium, we compute some of the statistics that are used in the empirical literature to measure the extent of nominal rigidities (e.g., the average duration of a price, the frequency of a price change, etc...). Finally, we calibrate the model and consider whether there exists a pricing policy that is supported by the equilibrium and that can match the statistical properties of the US micro-data on prices. We carry out this analysis for the case of an inflationary economy, i.e. $\mu > 1$, as this is empirically relevant.

### 4.4.1 Supportable Pricing Policies

Suppose that firms adjust their prices according to the policy

$$p_{t+1} = \begin{cases} 
  p, & \text{w/prob } \rho; \\
  \bar{z}/\phi_{t+1}, & \text{w/prob } 1 - \rho, \\
  \bar{z}/\phi_{t+1}, & \text{otherwise.}
\end{cases}$$

(4.25)

where $\bar{z}$ is a random variable drawn from a cumulative distribution function $L(z)$. According to the policy (4.25), a firm changes its price with certainty if the price it posted in the previous period was lower (in real terms) than the lower support of the equilibrium price.
distribution $H$. Otherwise, a firm keeps its price unchanged with probability $\rho$, and it readjusts it with probability $1 - \rho$. The two parameters that characterize the pricing policy (4.25) are the probability $\rho \in [0, 1]$, and the cumulative distribution function $L : [\underline{z}, \bar{z}] \rightarrow [0, 1]$.

The pricing policy (4.25) is consistent with the stationary equilibrium $(H, \bar{z}, b)$ if the distribution of real prices in period $t + 1$ is equal to $H$ given that all firms follow (4.25) and the distribution of prices in period $t$ is $H$. To see that (4.25) is consistent with the stationary monetary equilibrium, note that for any real price, $z$, the fraction of prices (and associated firms) that exits the interval $[\underline{z}, z]$ is equal to

\begin{equation}
X(z) = \begin{cases} 
H(z), & \text{if } z < \mu\underline{z}, \\
H(\mu\underline{z}) + (1 - \rho) [H(z) - H(\mu\underline{z})], & \text{if } z \geq \mu\underline{z}.
\end{cases}
\end{equation}

(4.26)

Conversely, the fraction of real prices (firms) that enters the interval $[\underline{z}, z]$ is equal to

\begin{equation}
I(z) = \begin{cases} 
\rho [H(\mu z) - H(\mu\underline{z})] + M(\rho)G(z), & \text{if } z < \mu\underline{z}, \\
\rho [H(\mu z) - H(z)] + M(\rho)G(z), & \text{if } z \geq \mu\underline{z}.
\end{cases}
\end{equation}

(4.27)

where $M(\rho)$ is defined as

\begin{equation}
M(\rho) = [H(\mu\underline{z}) + (1 - \rho)(1 - H(\mu\underline{z}))],
\end{equation}

(4.28)

and represents the fraction of firms that change their price under the policy, (4.25).

Stationarity requires the flows of real prices in and out of the interval $[\underline{z}, z]$ to be equal. This is true if $L$ satisfies:

\begin{equation}
L(z) = \frac{[H(z) + \rho (H(\mu\underline{z}) - H(\mu z))]}{M(\rho)}.
\end{equation}

(4.29)

To verify that $L(\cdot)$ as defined by (4.29) is a legitimate cumulative distribution function, note first that given the properties of $H(z)$, $L(\underline{z}) = 0$ and $L(\bar{z}) = 1$. Differentiating the
equilibrium $H(z)$ with respect to $z$, we may write $L'(z)$ as

$$L'(z) = \begin{cases} 
\frac{1}{M(\rho)} \frac{\alpha_1(b)}{2\alpha_2(b)} \left[ \frac{(z-c)}{(z-c)^2} - \frac{\rho(z-c)}{(\rho z-c)^2} \right], & \text{if } z \leq \overline{z}/\mu, \\
\frac{1}{M(\rho)} \frac{\alpha_1(b)}{2\alpha_2(b)} \frac{(z-c)}{(z-c)^2}, & \text{if } z > \overline{z}/\mu.
\end{cases} \tag{4.30}$$

From (4.30), it is immediate to verify that $L'(z) \geq 0$ for all $z \in [\overline{z}, \overline{z}]$. Thus, $L(z)$ is a cumulative distribution function. We then have the following proposition.

**Proposition 4.2:** (Supportable Pricing Policies). The pricing policy (4.25) supports the stationary equilibrium $(H, \overline{z}, b)$ if and only if $\rho \in [0, 1]$ and $L(z)$ is given by (4.29).

For the pricing policies that are supported by the equilibrium, we now compute statistics similar to those used in the empirical literature to measure the extent of nominal rigidities. First, we compute the average price duration. Let $N$ denote the largest integer such that $\mu^N \overline{z} \leq \overline{z}$. In period $t$, there are $H(\mu^n \overline{z}) - H(\mu^{n-1} \overline{z})$ prices in the interval $[\mu^{n-1} \overline{z}, \mu^n \overline{z}]$ for $n = 1, 2, \ldots N$. There are also $1 - H(\mu^N \overline{z})$ prices in the interval $[\mu^N \overline{z}, \overline{z}]$. If a price belongs to the interval $[\mu^{n-1} \overline{z}, \mu^n \overline{z}]$ in period $t$, it will change in period $t + i$ (and not until then) with probability $\rho^{i-1}(1 - \rho)$ for $i = 1, 2, \ldots n - 1$, and it will change in period $t + n$ with probability $\rho^{n-1}$. If a price belongs to the interval $[\mu^N \overline{z}, \overline{z}]$ in period $t$, it will change in period $t + i$ with probability $\rho^{i-1}(1 - \rho)$ for $i = 1, 2, \ldots N$, and it will change in period $t + N + 1$ with probability $\rho^N$.

The average price duration, $A(\rho)$, is then equal to

$$A(\rho) = \left\{ \sum_{n=1}^{N} \left[ H(\mu^n \overline{z}) - H(\mu^{n-1} \overline{z}) \right] \frac{1 - \rho^n}{1 - \rho} \right\} + \left[ H(\overline{z}) - H(\mu^N \overline{z}) \right] \frac{1 - \rho^{N+1}}{1 - \rho}. \tag{4.31}$$

Since the stationary equilibrium is consistent with the pricing policy (4.25) for any $\rho \in [0, 1]$ and $A(\rho)$ is continuous and increasing in $\rho$, the stationary equilibrium is consistent with any average price duration between a minimum of $A(0)$ and a maximum of $A(1)$.

Next, consider the frequency with which prices change. The probability that a given firm changes its price from one period to the next is of course equal to the fraction of firms
that change their price. We defined this as $M(\rho)$ above, at (4.28). Thus $M(\rho)$ represents the average frequency of price changes as a function of $\rho$. Since the stationary equilibrium is consistent with the pricing policy (4.25) for any $\rho \in [0, 1]$ and $M(\rho)$ is continuous and decreasing with respect to $\rho$, the stationary equilibrium is consistent the frequency of price changes can take on any value between a minimum of $M(1)$ and a maximum of $M(0)$. This leads to the following proposition.

**Proposition 4.3. (Nominal Price Rigidity)** The stationary equilibrium $(H, z, b)$ is consistent with any average price duration in the interval $[A(0), A(1)]$, and any average frequency of price adjustment in the interval $[M(1), M(0)]$.

Finally, we consider the price change hazard; that is, the probability with which a firm changes its price conditional on the number of periods since its last price change. Models of micro price adjustment typically have strong predictions for these probabilities. For example, those based on Calvo pricing typically have a flat hazard. That is, the probability of the next price change is constant and independent of the number of periods since the last one. Moreover, in these models this probability is an exogenous parameter. Models based on menu costs which imply a pricing policy of the $(s, S)$ variety imply a hazard of 0 for the $s - 1$ periods following a price change and a price change with certainty in the $s$th period. Neither of these theories fares well empirically, as hazards which decline over time have been documented in a number of studies (see e.g. Nakamura and Steinsson, 2008 or Klenow and Kryvtsov, 2008).

In our economy, the real prices of firms that have just reset are distributed according to $L(z)$ on support $\mathcal{H} = [\bar{z}, \tilde{z}]$. With $N$ defined (as above) to be the largest integer such that $\mu^N \bar{z} < \tilde{z}$, we may write the price change hazard as a function of the number of periods
since a firm’s last price change, \( i \leq N + 1 \), as follows:

\[
\chi(i) = \begin{cases} 
1 - \frac{\rho[1-L(\mu z)]}{[1-L(\mu^{i-1} z)]}, & i = 1, \ldots, N; \\
1, & i = N + 1.
\end{cases}
\] (4.32)

In general, it is difficult to characterize the hazard, as its properties depend on the money creation rate, \( \mu \), and the price policy parameter, \( \rho \). For the case of \( \rho = 1 \) (i.e. minimum price change frequency or maximum price duration), we have the following proposition:

**Proposition 4.4:** (A Decreasing Price Change Hazard). *In a stationary monetary equilibrium with \( \rho = 1 \) in the pricing policy (4.25), \( \chi(i) - \chi(i - 1) < 0 \) for \( i \in \{2, N - 1\} \).*

Proof: See Appendix C.1.1.

The case of \( \rho = 1 \) implies the maximum price stickiness. According to (4.25), a firm adjusts its price only if its real price falls out of the price support. In this case, as the number of periods since the firm’s last change increases, the probably with which the firm changes its price becomes lower, since those firms who survive without price changes for a longer period are the ones that have relatively higher prices. Eventually, as \( i \) approaches \( N \), the firm changes its price with probability one. Thus, our model naturally generates a price change hazard that is at least initially decreasing.\(^4\) This characteristic of our equilibrium is broadly consistent with the empirical evidence and stands in marked contrast to the predictions of models including menu costs or temporal restrictions on price changes.

### 4.4.2 Confronting the Data

The parameters of the model are the inflation rate, \( \mu \) (which we allow to vary rather than calibrate), the discount factor, \( \beta \), the meeting-probability function, \( \lambda(b) \), the utility cost of

\(^4\)In our analysis of a calibrated economy below, we show that \( N \) may be vary large so that the hazard may be decreasing over a time horizon significantly longer than that calculated in empirical studies. We also illustrate that a decreasing hazard can in principle occur for a pricing policy associated with any \( \rho > 0 \).
entering the BJ market, \( k \), the utility of the BJ good, \( u \), and the cost of producing the BJ good, \( c \). We choose the model time period to be one month and set \( \beta \) equal to 0.996 so that the annual real interest rate in the model is 4.8 percent, which is the average of the US interest rate on triple-A bonds over the period between 1991 and 2002 (the period studied by Bils and Klenow, 2004). We parameterize the function \( \lambda(b) = \min\{(2b)^{-1/2}, 1\} \), and set \( k \) equal to 0.41 so that the average markup in the BJ market is 15 percent when \( \mu = 1.002 \), which corresponds to an annual inflation rate of 2.5%, again matching the U.S. average for 1991-2002. Note that the average markup in this model is calculated as \( \int_{\underline{z}}^{\bar{z}} \frac{z-u}{c} dG(z) \). The 15 percent of markup is within the range of estimates for the US economy (see, e.g., Rotemberg and Woodford, 1991, and Basu and Fernald, 1997). We normalize the utility \( u \) to 1, and we tentatively set the cost \( c \) to 0.5.

Given these parameter values, we compute the stationary equilibrium of the model for inflation rates between 0.1 and 25 percent, per quarter. Firms adjust their prices according to the policy in (4.25). In general, \( \rho \) measures the degree of price stickiness. Since firms who post a price high enough only adjust their nominal prices randomly (with a probability \( 1 - \rho \)), the average price durations and the frequency of price adjustment both vary with \( \rho \) at any given inflation rate.

Figure 4.2 plots the maximum and minimum average price durations (\( A(1) \) and \( A(0) \) respectively) for this range of quarterly inflation rates. Similarly, Figure 4.3 plots the maximum and minimum frequency of price adjustment, (\( M(0) \) and \( M(1) \) respectively). As the inflation rate increases, the maximum average price duration falls and the minimum frequency of price adjustment increases. Intuitively, as the inflation rate rises, a given nominal price moves faster (in real terms) through the interval \([\underline{z}, \bar{z}]\) and, hence, must change sooner. Since the equilibrium is always consistent with a pricing policy in which firms
reset their prices in every period, the minimum average price duration and the maximum frequency of price adjustment do not depend on the inflation rate.

Several predictions of the model are testable. Together with the minimum and the maximum frequency of price adjustment in equilibrium, Figure 4.3 presents a scatter plot of the empirical frequency of price adjustment and of the quarterly inflation rate from several studies covering different countries and time periods (the figure includes the frequency/inflation pair for each of the empirical studies considered by Golosov and Lucas, 2007). From this figure, we can see that the model can account for the frequency at which US prices changed over the low-inflation period between 1991 and 2002 (see Bils and Klenow, 2004).

Since the model is calibrated to the US economy, comparison to the Bils and Klenow evidence is the only rigorous empirical test. We find it encouraging, however, that the
model can also account for the frequency at which Mexican prices changed both during a high-inflation period (1995-1996) and during a low-inflation period (2000-2002) (see Gagnon, 2009). Moreover, the model can account for the frequency of price adjustment observed in the EU over the low-inflation period between 1995 and 2000 (see Dhyne et al., 2005), and for the frequency of price adjustment observed in Israel over the period of moderate inflation between 1991 and 1992 (see Baharad and Eden, 2004). Our model fails only to account for the episodes of hyper-inflation in Israel (1978-1979 and 1981-1982) and Poland (1990-1996).\footnote{Of course, it is possible that if the model were re-calibrated to the Israeli or Polish economies over the time periods considered it could in fact account for these observations.}

Finally, Figure 4.4 depicts the price change hazard for two specific cases. In both inflation is equal to 2.5% annually. We then plot the hazard for two pricing policies supported

![Figure 4.3: Price Change Frequency](image)

**Figure 4.3: Price Change Frequency**

\[\text{Figure 4.3: Price Change Frequency}\]

\[\text{Figure 4.4: Price Change Hazard for Two Specific Cases. In both inflation is equal to 2.5% annually. We then plot the hazard for two pricing policies supported}\]

\[\text{Figure 4.4: Price Change Hazard for Two Specific Cases. In both inflation is equal to 2.5% annually. We then plot the hazard for two pricing policies supported}\]

\[\text{Figure 4.4: Price Change Hazard for Two Specific Cases. In both inflation is equal to 2.5% annually. We then plot the hazard for two pricing policies supported}\]
by the stationary equilibrium, \( \rho \in \{.5, 1\} \). For this level of inflation, \( N = 200 \). That is, it takes 201 months for a seller who has posted the lower support of the distribution \( L \) to be required to make a price change. With \( \rho = 1 \), Proposition 4.3 applies and the hazard decreases for the first \( 199(N - 1) \) periods following a price change. With \( \rho = .5 \), the price change hazard is bounded below by \( .5 \), and decreases for the first periods following a price change. Of course, \( \rho = 0 \), all sellers change prices every period, and the price change hazard is constant and equal to one.

![Figure 4.4: Hazard Rate at 2.5% Annual Inflation Rate](image)

**Figure 4.4:** Hazard Rate at 2.5% Annual Inflation Rate

### 4.5 Extensions

The economy described above has a number of particular characteristics which may be viewed as representing restrictive assumptions (*e.g.* the indivisibility of the BJ good). In
fact our main qualitative findings are robust to the relaxation of many of these assumptions. In general, any modification to the environment that preserves price dispersion and indifference of sellers across price levels at a point in time will leave these findings unaffected.

In this section, we describe briefly two environments, both with a divisible BJ good. In the first, sellers are restricted, as in our base model, to post linear prices, and in the second we consider a general environment where sellers can post a quantity-price combination. In both of these extensions, we fix the number of buyers present in the BJ market. Entry could, however, be handled exactly as in our main economy presented above.

### 4.5.1 Divisibility of the BJ Good

In this extension the exchange and production in the BJ market proceed much as they do in the model of Head and Kumar (2005). We show that the divisibility of the BJ good does not substantially affect the distributions of real and nominal prices in and so the results for our main model are robust to divisibility of the BJ good.

**The Environment**

In this extension, the AD market functions just as that in the main model described above. In the BJ market there is a fixed measure, \( b \), of identical buyers, and a unit measure of identical firms. Firms derive no utility from consuming the BJ good, but can produce any quantity of it at constant marginal disutility, \( c \). Buyers have preferences over consumption of the BJ and AD goods represented by:

\[
\sum_{t=0}^{\infty} [u(q_t) + x_t],
\]

(4.33)

Their environment, however, does not have an AD market—all trade effectively takes place in the BJ market. As a result the economy’s monetary equilibrium may be significantly different from that described both above for the main model and here for the extension to divisibility of the BJ good.
where \( q_t \in \mathbb{R} \) is the quantity of BJ goods purchased by the buyer. Let \( u \) be three times continuously differentiable, with \( u(0) = 0, u' > 0 \), and \( u'' < 0 \). Also, assume there exists \( q^* > 0 \) such that \( u'(q^*) = c \). For clarity, we effectively restrict attention to the utility function considered by Lagos and Wright (2005);

\[
u(q) = \frac{(q + \nu)^{1-\eta} - \nu^{1-\eta}}{1-\eta} \quad \nu > 0, \quad \eta \geq 0,
\]

although our results typically generalize. As above, in every period, each buyer is endowed with \( \bar{x} > 0 \) units of the AD good.

Exchange in the BJ market follows the pattern described by Head and Kumar (2005). At the opening of the market, each firm posts nominal price \( P \), and commits to satisfy demand at that price. Households again observe \( k \) prices with probability \( \alpha_k \) for \( k = 1, 2 \). Let \( F(P) \) denote the distribution of nominal prices in the current period, and let \( H(p) \) denote the distribution of real prices, where as above we will restrict attention to equilibria in which the all real quantities are constant and nominal variables grow at the rate of money creation, \( \mu \).

Given the price information they receive, buyers decide whether and how much to purchase. Clearly, if a buyer purchases it will do so exclusively at the lowest price it observes. Let \( J(p) \) denote the distribution of the lowest (real) price observed by a buyer:

\[
J(p) = \frac{\alpha_1 H(p) + \alpha_2 [1 - [1 - H(p)]^2]}{\alpha_1 + \alpha_2}, \quad \forall p \in \mathcal{H},
\]

where \( \mathcal{H} \) is the support of \( H(\cdot) \).

**Buyers’ Optimization**

Let \( W(z) \) denote the expected value for a buyer who enters the AD market of the current period with real balances \( z \) and \( V_{+1}(\hat{z}) \) the expected value of entering the subsequent BJ
market (in period \( t + 1 \)) with balance \( \hat{z} \).

1). The AD market:

A buyer chooses \( x \) and future real balances, \( \hat{z} \) to solve

\[
W(z) = \max_{x,\hat{z}} \{ x + \beta V_{+1}(\hat{z}) \}
\]

s.t. \( x = \bar{x} + z - \hat{z} \mu + \phi T \). \hspace{1cm} (4.36)

The necessary condition for the optimal choice of \( \hat{z} \) is

\[
\beta V'_{+1}(\hat{z}) \geq \mu,
\]

with complementary slackness and the envelope condition is

\[
W'(z) = 1. \hspace{1cm} (4.38)
\]

2). The BJ market:

A buyer who enters the BJ market of the current period with \( m \) units of money solves:

\[
V(z) = W(0) + z + (\alpha_1 + \alpha_2) \int_{\mathcal{H}} \left[ u \left( \frac{d(z, p)}{p} \right) - d(z, p) \right] dJ(p),
\]

where \( d(z, p) \) denotes the buyer’s optimal real expenditure at real price \( p \). Given the lowest price observed, \( p \), a buyer with real balances, \( z \), chooses \( d(z, p) \) to solve

\[
\max_d \left\{ u \left( \frac{d}{p} \right) + W(z - d) \right\}, \hspace{1cm} \forall p
\]

s.t. \( d \leq z \), \hspace{1cm} (4.41)

\[
u \left( \frac{d}{p} \right) - d \geq 0. \hspace{1cm} (4.42)
\]

In general, the buyer’s expenditure constraint, (4.41), may be non-binding for some prices. Moreover, (4.42) reflects the requirement that a buyer’s payoff from trade be non-negative.
Otherwise, the buyer would prefer to carry money into the AD market rather than spend it in the BJ market. This requirement implies a reservation price, $p^r$, such that $u\left(\frac{z}{p^r}\right) = z$. Households trade if and only if the lowest price they observe satisfies $p \leq p^r$.

The following proposition characterizes buyers’ optimal expenditure rule which depends, of course, on $u(q)$:

**Proposition 4.5:** Define the real price $\hat{p}$ such that $u'\left(\frac{z}{\hat{p}}\right) = 1$. Depending on the value of $\eta$, buyers’ optimal expenditure rule, $d(z, p)$, has the following form:

(i) when $\eta \leq 1$,

$$d(z, p) = \begin{cases} 
  z, & \text{if } p \leq \hat{p} \\
  pu'^{-1}(p), & \text{otherwise}
\end{cases} \quad (4.43)$$

(ii) when $\eta > 1$

$$d(z, p) = \begin{cases} 
  pu'^{-1}(p), & \text{if } p < \hat{p} \\
  z, & \text{if } \hat{p} \leq p \leq p^r \\
  0, & \text{otherwise}
\end{cases} \quad (4.44)$$

For a proof of this proposition see Appendix C.1.2. The cutoff price $\hat{p}$ is that at which the buyer’s expenditure constraint binds. At this point the marginal benefit from spending one unit of money in the BJ market is at least as great as its marginal cost, i.e. its value in the subsequent AD market. Given Proposition 6.1, the derivative of $V$ exists for all $z$ and satisfies:

$$V'(z) = \begin{cases} 
  1 + (\alpha_1 + \alpha_2) \int_{p^r}^{\hat{p}} u'\left(\frac{z}{p}\right) \frac{1}{p} - 1 \, dJ(p), & \text{if } \eta \leq 1; \\
  1 + (\alpha_1 + \alpha_2) \int_{p^r}^{\hat{p}} u'\left(\frac{z}{p}\right) \frac{1}{p} - 1 \, dJ(p), & \text{if } \eta > 1.
\end{cases} \quad (4.45)$$

Clearly, given the assumed properties of $u(\cdot)$, $V'(z) > 0$ and $V''(z) < 0$.

We assume (similarly to Lagos and Wright (2005)) that $\nu$ is sufficiently small that the Arrow-Pratt measure of relative risk aversion satisfies, $-\frac{\nu q}{u'(q)} > 1$. 98
Firm Optimization

A firm’s expected profit from posting real price \( p \) the BJ market is given by

\[
\pi(p) = b \left[ \left( 1 - \frac{c}{p} \right) d(\bar{z}, p) \right] [\alpha_1 + 2\alpha_2(1 - H(p))] .
\] (4.46)

Taking as given buyers’ optimal expenditure rule, \( d(\bar{z}, p) \), as well as the common distribution of posted real prices, \( H(p) \), firms choose a price, \( p \), to maximize expected profits. To ensure that the expected profits are non-negative, no firm will post a price lower than \( p^* = c \), which we refer to as the marginal cost price.

We also define the monopoly price as that which a firm would post if it knew with certainty that any buyer observing its price would not receive any other price quote (i.e. would have no other opportunity to trade). Following the logic of Burdett and Judd (1983) it is straightforward to show that if any firm believes that all its competitors are charging prices strictly below the monopoly price, then its optimal choice is to post that price. Thus, the upper support of any distribution of prices posted by profit maximizing firms, \( \bar{p} \) must equal the monopoly price. Clearly, \( \bar{p} \leq p^r \) as posting any price higher results in an expected profit of zero. The following proposition (proved in Appendix C.1.3) characterizes the maximum price posted (i.e. the monopoly price, \( \bar{p} \)) by any profit maximizing firm:

**Proposition 4.6:** Given buyers’ optimal expenditure rule, \( d(z, p) \) (see Proposition 4.5), the optimal price posted by a monopoly is given by:

1. if \( \eta > 1 \), the monopoly price equals the buyers’ reservation price, \( p^r \).

2. if \( \eta \leq 1 \), the monopoly price equals \( \max\{p^m, \hat{p}\} \) where \( p^m \) satisfies

\[
u'' \left[ u'^{-1}(p^m) \right] u'^{-1}(p^m) + p^m = c. \] (4.47)

\( \hat{p} \) is as specified in Proposition 4.5.
Since no profit maximizing firm will ever post a price above the monopoly price, $\bar{p}$, or below the marginal cost price, $p^*$, given distribution $H(p)$, an individual firm is indifferent between posting any price, $p \in [p^*, \bar{p}]$, that satisfies

$$d(\bar{z}, p) \left(1 - \frac{c}{p}\right) \left[\alpha_1 + 2\alpha_2(1 - H(p))\right] = d(\bar{z}, \bar{p}) \left(1 - \frac{c}{\bar{p}}\right) \alpha_1. \quad (4.48)$$

Using (4.48), it then follows that optimization by firms generates the following distribution of real posted prices:

$$H(p) = 1 - \frac{\alpha_1}{2\alpha_2} \left[\frac{d(\bar{z}, \bar{p}) (1 - c/\bar{p})}{d(\bar{z}, p) (1 - c/p)} - 1\right], \quad \forall p \in \mathcal{H} = [p^*, \bar{p}], \quad (4.49)$$

where $\bar{p}$ is equal to the monopoly price given by Proposition 4.6 and the lowest price posted, $p$, satisfies

$$\frac{d(\bar{z}, \bar{p}) (1 - c/\bar{p})}{d(\bar{z}, p) (1 - c/p)} = \frac{\alpha_1 + 2\alpha_2}{\alpha_1}. \quad (4.50)$$

**Stationary Equilibrium**

We now may define a stationary equilibrium as above, except that the number of buyers in the BJ market is now fixed, and each buyer’s expenditure rule must be determined in equilibrium:

**Definition 4.3:** A Stationary Equilibrium is a distribution of real prices in the BJ market, $H$; a real money balance that all buyers carry into the BJ market, $\bar{z}$; and a real expenditure rule, $d(\bar{z}, p)$ used by all buyers in the BJ market such that:

1. Taking $H(p)$ as given, buyers carry $\bar{z}$ and use the expenditure rule, $d(\bar{z}, p)$ to solve their optimization problem

2. Given $H(p)$ and $d(\bar{z}, p)$ all prices in the support of $H$ maximize firms’ expected profits.
Using (4.45) in buyers first order condition, (4.45) and defining the nominal interest rate, 

\[ i = \frac{\phi}{\phi_{+1}} - 1 = \frac{\mu}{\beta} - 1, \]  

(4.51)

we have expressions implicitly characterizing real balances in equilibrium:

1. If \( \eta \leq 1 \):

\[ i = \int_{\bar{p}}^{\bar{p}'} \left[ u' \left( \frac{\bar{z}}{p} \right) \frac{1}{p} - 1 \right] dJ(p). \]  

(4.52)

2. If \( \eta > 1 \)

\[ i = \int_{\bar{p}}^{p'} \left[ u' \left( \frac{\bar{z}}{p} \right) \frac{1}{p} - 1 \right] dJ(p). \]  

(4.53)

Thus, in a stationary equilibrium, the marginal cost of carrying additional currency into next period’s BJ market equals the discounted expected marginal utility from spending it at that time. Then we have the following:

**Proposition 4.7:** If \( i > 0 (\mu > \beta) \), there exists a unique symmetric stationary monetary equilibrium.

For the proof of this proposition, see the Appendix C.1.4. The following proposition, which can be proved by direct calculation, characterizes the equilibrium:

**Proposition 4.8:** Given \( \mu > \beta \), and equilibrium real balances, \( \bar{z} \), the equilibrium distribution of real prices, \( H(p) \) is given by

\[ H(p) = 1 - \frac{\alpha_1}{2\alpha_2} \left[ \frac{d(\bar{z}, \bar{p}) (1 - c/\bar{p})}{d(\bar{z}, p) (1 - c/p)} - 1 \right], \quad \forall p \in \mathcal{F}, \]  

(4.54)

where \( \bar{p} \) is given by

\[ \frac{d(\bar{z}, \bar{p}) (1 - c/\bar{p})}{d(\bar{z}, p) (1 - c/p)} = \frac{\alpha_1 + 2\alpha_2}{\alpha_1}. \]  

(4.55)

and \( d(\bar{z}, p) \) and \( \bar{p} \) are given by Propositions 4.5 and 4.6 respectively, depending on \( \eta \).
To demonstrate the model predictions, we calibrate our divisible good economy using the similar strategy as in the indivisible case, and present a similar exercise as in Figure 4.3. Figure 4.5 depicts the scatter plot of the empirical frequency of price adjustment, together with the minimum and the maximum frequency implied by our extended model. As we can see, the extension allows us to account for even the frequency of price adjustment observed in Israel during the episode of 1978 to 1979.

![Figure 4.5: Price Change Frequency in the Divisible Good Economy](image)

Figure 4.5: Price Change Frequency in the Divisible Good Economy
4.5.2 Non-linear Pricing

The Environment

The environment remains essentially the same as for the divisible goods extension considered above: The numbers of buyers and sellers in the BJ market are fixed, preferences are represented by (4.34) and costs are linear.

Sellers’ optimization

Define $\sigma = u(q) - z$ as a buyer’s surplus from exchanging $z$ real balances for $q$ units of the BJ good. Assume that in the BJ market, firms post contracts delivering a particular surplus, $\sigma$. Conditional on surplus offered, the firm chooses $(q, z)$ to maximize its own profit, $\pi(\sigma)$:

$$\pi(\sigma) = \max_{q, z} [\phi z - cq]$$

subject to:

$$u(q) - z \geq \sigma$$

$$z \leq \bar{z}$$

where we assume that the firm believes that all buyers carry the same real balance into the BJ market, $\bar{z}$. It is straightforward to show that for $\sigma \in [0, q^*]$ the profit-maximizing contract is:

$$(q, z) = \begin{cases} 
(u^{-1}(\sigma + \bar{z}), \bar{z}), & \text{if } \sigma \leq u(q^*) - \bar{z}; \\
(q^*, u(q^*) - \bar{z}), & \text{if } \sigma > u(q^*) - \bar{z}.
\end{cases}$$

and that each executed contract results in the following realized profit to the firm:

$$\pi(\sigma) = \begin{cases} 
\phi M - c(\phi M + \sigma), & \sigma \leq q^* - \phi M; \\
q^* - \sigma - c(q^*), & \sigma > q^* - \phi M.
\end{cases}$$
We can associate with each contract a real price, \( p(\sigma) = z/q \), and a nominal price, \( P(\sigma) = z/\phi q \).

In order to make a sale, a seller’s contract must be observed by a buyer who observes no other contract which offers a higher surplus. Let \( S(\sigma) \) denote the CDF of offered contracts, and assume that a buyer observes one contract with probability \( \alpha_1 \) and two with probability \( \alpha_2 \). Let \( b \) be the ratio of buyers to sellers. Let \( J(\sigma) \) denote the distribution of the highest surplus observed:

\[
J(\sigma) = \frac{\alpha_1 S(\sigma) + \alpha_2 S(\sigma)^2}{\alpha_1 + \alpha_2}.
\]  
(4.61)

The expected number of sales by a firm that posts a contract offering surplus \( \sigma \) equals:

\[
n(\sigma) = b\alpha_1 + 2b\alpha_2 S(\sigma).
\]  
(4.62)

where \( b \) is the measure of buyers in the BJ market. The value of a seller who posts such a contract is then given by:

\[
V^s(\sigma) = \begin{cases} 
\bar{z} - cu^{-1}(\bar{z} + \sigma) n(\sigma), & \sigma \leq u(q^*) - \bar{z}; \\
[u(q^*) - \sigma - cq^*] n(\sigma), & \sigma > u(q^*) - \bar{z}.
\end{cases}
\]  
(4.63)

For all sellers to be optimizing, each posted contract must again generate equal expected profit. With \( V^s(0) = [\bar{z} - cu^{-1}(\bar{z})]b\alpha_1 \), the CDF \( S(\sigma) \) may be derived analytically:

\[
S(\sigma) = \begin{cases} 
\frac{\alpha_1}{2\alpha_2} \left[ \frac{e^{u^{-1}(\sigma + \bar{z}) - u^{-1}(\bar{z})}}{\bar{z} - cu^{-1}(\bar{z} + \sigma)} \right], & \sigma \leq u(q^*) - \bar{z}; \\
\frac{\alpha_1}{2\alpha_2} \left[ \frac{\bar{z} - cu^{-1}(\bar{z})}{u(q^*) - \bar{z} - cq^*} - 1 \right], & \sigma > u(q^*) - \bar{z},
\end{cases}
\]  
(4.64)

In general, the maximal surplus offered, may be either greater or less than \( u(q^*) - \bar{z} \). Given \( \bar{z}, \bar{\sigma} > u(q^*) - \bar{z} \) if

\[
\frac{\alpha_1}{2\alpha_2} < \frac{\bar{z} - cq^*}{cq^* - cu^{-1}(\bar{z})}.
\]  
(4.65)
That is, if the fraction of buyers observing two contracts is large enough, (4.65) holds with equality. In this case \( \bar{\sigma} \) is given by
\[
\bar{\sigma} = \frac{[u(q^*) - cq^*] (\alpha_1 + 2\alpha_2) - [\bar{z} - cu^{-1}(\bar{z})] \alpha_1}{\alpha_1 + 2\alpha_2}.
\] (4.66)

From here on we restrict attention to this case.

**Buyers’ optimization**

Consider now a buyer entering the BJ market with \( z \leq \bar{z} \), where \( \bar{z} \) is the highest expenditure required by a posted contract. Buyers have no incentive to enter with \( z > \bar{z} \). This buyer’s value is
\[
V^b(z) = W(0) + z + \begin{cases} 
[\alpha_1 + 2\alpha_2] \int_{u(q^*)-z}^{\bar{q}} \sigma dG(\sigma), & z < \bar{z}; \\
[\alpha_1 + 2\alpha_2] \int_{0}^{\bar{q}} \sigma dG(\sigma), & z \geq \bar{z}.
\end{cases}
\] (4.67)

That is, a buyer who brings in real balance \( z \) less than \( \bar{z} \), can only accept the (high surplus) contracts that supply \( q^* \) and require \( z \leq \bar{z} \). If they bring in \( \bar{z} \), then they can accept any contract.

Now, it can be shown that \( V' \) exists for all \( z \neq \bar{z} \) and satisfies:
\[
V'(z) = \begin{cases} 
[\alpha_1 + 2\alpha_2] \left[ 1 + (u(q^*) - z) G'(u(q^*) - z) \right], & z < \bar{z}; \\
1, & z \geq \bar{z}.
\end{cases}
\] (4.68)

Thus \( V'(z) > 0 \) for all \( z \geq 0 \). We also have that
\[
V''(z) = \begin{cases} 
[\alpha_1 + 2\alpha_2] \left[ -G''(q^*) - z \right] - u(q^* - z) G'''(q^* - z)), & z < \bar{z}; \\
0, & z \geq \bar{z}.
\end{cases}
\] (4.69)

Clearly \( V'' \leq 0 \) and \( V'' < 0 \) for all \( z < \bar{z} \).

Buyers’ first-order condition for choice of \( \hat{z} \) in the AD market of the current period is similar to (4.8):
\[
\beta V'^{-}_{t+1}(\hat{z}) - 1 \geq 0, \quad \beta V'^{+}_{t+1}(\hat{z}) - 1 \leq 0,
\] (4.70)
or, for the case of $\hat{z} < \bar{z}$

$$
\mu \leq \beta \left[ 1 + (u(q^*) - \hat{z})G'(u(q^*) - \hat{z}) \right].
$$

(4.71)

**Stationary Equilibrium**

As for our main model, we associate a refined stationary equilibrium with a real money balance, $\bar{z}$, and a distribution of contracts, $S(\sigma)$, which satisfies (4.64) given $\bar{z}$. In a stationary equilibrium the first-order condition may then be written:

$$
i = (u(q^*) - \bar{z})G'(u(q^*) - \bar{z}),
$$

(4.72)

where as before $i$ is the nominal interest rate given by (4.51).

Let $\Upsilon(\bar{z}) \equiv [\alpha_1 + 2\alpha_2](u(q^*) - \bar{z})G'(u(q^*) - \bar{z}) - i$. Then, a refined stationary monetary equilibrium is associated with a solution to the equation $\Upsilon(\bar{z}) = 0$.

**Proposition 4.9:** Fix $i \geq 0$. Then there exists a unique refined stationary monetary equilibrium.

**Proof:** See Appendix C.1.5.

Suppose that given $i$ such that $\Upsilon(\bar{z}) = 0$, an individual buyer would like to deviate by carrying real balances $\tilde{z} < \bar{z}$. This buyer will forgo the ability to accept certain contracts, but save the marginal cost of carrying money balances, $i$. For this to be advantageous to the buyer, it must be the case that $V_{+1}'(\hat{z}_{+1}) < i$ for some $\hat{z}_{+1} < \bar{z}$. Then,

$$
(u(q^*) - \tilde{z}) \frac{\alpha_1^2}{2\alpha_2} \left[ \frac{(\bar{z} - cu^{-1}(\bar{z})^2)}{(\bar{z} - cq^*)^3} \right] < i. \tag*{(4.73)}
$$

Since its LHS is monotonically decreasing in $\tilde{z}$, (4.73) contradicts $\Upsilon(\bar{z}) = 0$, and so there can be no such profitable deviation.
4.6 Conclusions

Many empirical studies have investigated firms price adjustment behavior using micro price data, e.g. Klenow and Kryvtsov (2008), Nakamura and Steinsson (2008), Bils and Klenow (2004), Gagnon (2009), and Campbell and Eden (2007). Several stylized facts from this empirical literature include: the average frequency of price changes is around 4-8 months; the frequency (duration) of price changes is decreasing (increasing) with inflation; the price change hazard is declining over time; and firms may reduce their prices despite the positive inflation. Most theoretical models in the New Keynesian literature generate price stickiness either by restricting the timing of price changes or imposing menu costs.

This chapter developed a general equilibrium framework in which sellers post prices in nominal terms, and price rigidity arises as an equilibrium outcome. We derived closed form solutions for price density and determined how it depends on inflation. We also calibrated the model and demonstrated that our model predictions are consistent with the above empirical facts from the micro data. In addition, our model can also account for the frequency at which US price changes over the low-inflation period between 1991 and 2002.

Our theory in a sense provides a microfoundation for a key feature of the New Keynesian literature, but the policy implications are very different. A one time monetary shock shifts the nominal price distribution immediately, but the real distribution remains unchanged, and hence there is no effect on real allocations. In other words, price stickiness neither implies nonneutrality nor provides support for the Keynesian policy.

The model can be easily extended to address other interesting policy questions. For example, one can introduce aggregate shocks to money growth and production cost to study the dynamics of the economy, or allow for multiple BJ goods to capture different shapes of the price distribution. We could also add capital and labor in the production sector and
examine the business cycle properties of our model.
Chapter 5

Summary and Conclusions

The long-run relationship between inflation and output is one of the classic topics in macroeconomic literature. The findings from previous studies suggest that this relationship depends on the role of money in the economy and how money interacts with real variables. This dissertation investigates the real effects of inflation in models that are explicit about microfoundations of money and study their implications for optimal monetary policy.

In the first chapter, I studied the interaction between inflation and labor market outcome in a general equilibrium model where inflation has differential effects on employed and unemployed workers. The model predicts an inflation-unemployment relationship that is either positive or negative, depending on goods and labor market institutions. Through a comparison of these institutions across different countries, I found that sales tax and unemployment insurance benefit may be important factors that lead to the different sloped long-run Phillips curves observed in US and Europe. An important policy implication from this model is that optimal monetary policy depends on both labor market efficiency and the specific inflation-unemployment relation. In the presence of distortions in the labor market, I found that deviating from the Friedman rule may be optimal.
The second chapter developed a search-theoretic model of money to investigate the phenomenon of “hot potato” effect: when inflation rises, people spend their money faster. We first documented some evidence on velocity, showing that it tends to increase with inflation and nominal interest rates. We then demonstrated why standard monetary search models might fail to generate these relationships. The approach taken by our model is to focus on search intensity on the extensive margin rather than intensive margin. Our model robustly predicts a positive effect of inflation on velocity, but the impact on aggregate output is always negative due to the reduction in the number of buyers trading in the market. In terms of policy implications, we found that in some circumstances inflation above the Friedman rule may be optimal.

The third chapter investigated the effect of inflation on output in a general equilibrium model where price rigidity arises as an equilibrium outcome. Different from the New Keynesian models, firms in our economy may choose not to adjust their prices even in the absence of any menu cost or other restrictions. We showed that the our model predictions can match a number of empirical facts from the micro data, but policy implications are different from those in the New Keynesian literature. In particular, we found that price rigidity does not necessarily lead to monetary non-neutrality. Higher inflation reduces real output in the long run, but changes in the aggregate price level have no effects on real allocations, even if individual prices appear to be sticky.

A number of extensions could be considered for future research. The models developed here mainly focus on steady state analysis, and thereby emphasize the long-run effects of anticipated inflation on real economic activity. Previous studies in the literature have shown that unanticipated inflation is also an important channel through which monetary policy affects the real economy. Thus, one extension is to introduce monetary shocks into the
above models and study the short-run dynamics of monetary policy.

Furthermore, the interaction between money growth and capital accumulation is another important channel for addressing the real effects of the inflation tax. In standard monetary search models, production in the decentralized market is mostly modeled through a simple cost function. One can consider to add capital in the production process and study the impact of inflation on capital accumulation and output growth. In addition, introducing capital into the monetary models may raise other interesting questions. For instance, capital as a second type of asset may compete with fiat money as a medium exchange, so models in this direction may be used to address the issue of co-existence and explain liquidity premium, etc. I intend to study these issues in my future research.
Bibliography


Appendix A

Appendix for Chapter 2

A.1 Proofs

A.1.1 Proof of Proposition 2.1

Proof. With $v'(q)/g'(q)$ strictly decreasing in $q$, $q_0 > q_1$ follows directly from (2.14). To prove $m_0 > m_1$, differentiating (2.11) with respect to $q$

$$g'(q) = \frac{(1 - s)(1 - \eta)v'(q)^2 + \eta v'(q) + \eta (1 - \eta)v''(q)[q - (1 - s)v(q)]}{[(1 - s)(1 - \eta)v'(q) + \eta]^2}.$$ 

To see the sign of $g'(q)$, we need to determine the term $[q - (1 - s)v(q)]$. By Nash bargaining, the surplus for the firm is positive as long as $\eta > 0$, that is, $(1 - s)g(q) - q > 0$. Using (2.11), we obtain $(1 - s)v(q) - q > 0$. With $v'(q) > 0$ and $v''(q) < 0$, this gives $g'(q) > 0$. Hence, $m_0 > m_1$ when $q_0 > q_1$.

The results that $\partial q_0/\partial i < 0$ and $\partial q_1/\partial i < 0$ again follow directly from (2.14). To see the effect of change in $i$ on $q_0 - q_1$, we can rewrite (2.14) in a general function form

$$h(q) = \frac{i}{\alpha \rho} + 1.$$
where \( h(q) = \frac{v'(q)}{g'(q)} \) and \( \alpha \) and \( \rho \) represent any arbitrary number of trading probabilities. Differentiating \( q \) with respect to \( \alpha \) yields \( \frac{dq}{d\alpha} = -\frac{i}{\alpha^2 \rho h'(q)} \). With \( \frac{v'(q)}{g'(q)} \) strictly decreasing in \( q \), \( \frac{dq}{d\alpha} \) is positive and increasing in \( i \). That is, the size of the difference in \( q \) resulting from changes in \( \alpha \) is increasing in \( i \). The proof for the effect of different values of \( \rho \) follows the same steps.

Finally, the results in the limiting case, \( i \to 0 \), follow the same arguments as in the standard model of Lagos and Wright (2005), except here the efficient allocation \( q^* \) requires both \( \eta = 0 \) and \( s = 0 \).

### A.1.2 Proof of Proposition 2.2

**Proof.** The uniqueness of \( q_0 \) and \( q_1 \) has been established in Proposition 1. To examine the properties of equilibrium \( y_R \) and \( \sigma \), we first totally differentiate (3.20). The sign of \( \frac{\partial \sigma}{\partial y_R} \) is the same as

\[
-\left[ \mathcal{M}(1, \sigma) \right]^2 \frac{[1 - F(y_R)]}{\mathcal{M}_1},
\]

where \( \mathcal{M}_1 = \frac{\partial \mathcal{M}(u, v)}{\partial u} \). Given the standard assumptions on \( \mathcal{M}(u, v) \), \( \frac{\partial \sigma}{\partial y_R} \) is always positive. Similarly, totally differentiating (2.22) yields

\[
\frac{\partial \sigma}{\partial y_R} = \frac{1 + \frac{\Delta f(y_R)}{\mathcal{M}[1 - F(y_R)]^2}}{\frac{\partial_\theta}{\beta (1 - \theta)} + \frac{\Delta \mathcal{M}_2}{\mathcal{M}^2 [1 - F(y_R)]}} > 0,
\]

where \( \mathcal{M}_2 = \frac{\partial \mathcal{M}(u, v)}{\partial v} \), and \( \Delta = \alpha_0 \delta \{ \rho_0 [(1 - s)g(q_0) - q_0] + (1 - \rho_0) \eta [(1 - s)v(q^c) - q^c] \} \).

The \( FE \) curve in (3.20) intersects the horizontal line at \( \bar{y} \), while the \( RP \) curve in (2.22) crosses at \( b + l \). So the existence of an equilibrium requires that \( \bar{y} > b + 1 \). Under this condition, the \( FE \) and \( RP \) curves intersect only once and the monetary equilibrium is unique.
A.2 Evidence of the Asymmetric Effects of Inflation

Table A.1: Results of Linear Regression for cash holding and spending

<table>
<thead>
<tr>
<th></th>
<th>Cash Holdings (as a percentage of total annual income)</th>
<th>Monthly Consumption in Cash (as a percentage of total expenditure)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment</td>
<td>1.6004</td>
<td>(0.2942)***</td>
</tr>
<tr>
<td></td>
<td>(0.1601)**</td>
<td>(1.5006)***</td>
</tr>
<tr>
<td>Male</td>
<td>0.3232</td>
<td>6.0200</td>
</tr>
<tr>
<td></td>
<td>(0.1585)**</td>
<td>(1.4851)***</td>
</tr>
<tr>
<td>Married</td>
<td>-0.3054</td>
<td>-3.8708</td>
</tr>
<tr>
<td></td>
<td>(0.1585)**</td>
<td>(1.8030)***</td>
</tr>
<tr>
<td>University degree</td>
<td>-0.3292</td>
<td>-15.1348</td>
</tr>
<tr>
<td></td>
<td>(0.1924)*</td>
<td>(1.8030)***</td>
</tr>
<tr>
<td>Age31-40</td>
<td>0.0471</td>
<td>-4.3032</td>
</tr>
<tr>
<td></td>
<td>(0.2818)</td>
<td>(2.6407)*</td>
</tr>
<tr>
<td>Age41-50</td>
<td>-0.0241</td>
<td>-5.3244</td>
</tr>
<tr>
<td></td>
<td>(0.2755)</td>
<td>(2.5816)**</td>
</tr>
<tr>
<td>Age51-65</td>
<td>0.2730</td>
<td>-3.3632</td>
</tr>
<tr>
<td></td>
<td>(0.2795)</td>
<td>(2.6190)</td>
</tr>
<tr>
<td>Age65</td>
<td>0.2203</td>
<td>-3.1902</td>
</tr>
<tr>
<td></td>
<td>(0.6783)</td>
<td>(6.3564)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.5658</td>
<td>81.5602</td>
</tr>
<tr>
<td></td>
<td>(0.2688)***</td>
<td>(2.5191)***</td>
</tr>
</tbody>
</table>

Summary statistics:
Mean of dependent variables:

<table>
<thead>
<tr>
<th></th>
<th>Employed</th>
<th>Unemployed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.6226%</td>
<td>3.2789%</td>
</tr>
<tr>
<td></td>
<td>76.8952%</td>
<td>89.1514%</td>
</tr>
</tbody>
</table>

# of Observations: 3730

Standard errors in parentheses.

* significant at 10%; ** significant at 5%; *** significant at 1%

Data: Italian Survey of Household Income and Wealth 2004 (excluding Not in Labor Force)

A.3 Data

The data is from Italian Survey of Household Income and Wealth 2004, excluding households who are not in labor force. The total number of observation is 3730. The cash holdings as a percentage of total annual income is calculated as the average money holdings in the house divided by net annual disposable income. The monthly consumption in cash as a percentage of total expenditure is calculated as the ratio of monthly cash spending and monthly average spending on all consumption.
Appendix B

Appendix for Chapter 3

B.1 Empirical evidences on Inflation and Velocity

Table B.1: Correlation

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>V0</td>
<td>0.6205</td>
<td>0.7866</td>
<td>0.0851</td>
</tr>
<tr>
<td>V1</td>
<td>0.1722</td>
<td>0.8473</td>
<td>0.5158</td>
</tr>
<tr>
<td>V2</td>
<td>-0.0736</td>
<td>0.5805</td>
<td>0.1884</td>
</tr>
<tr>
<td>Inflation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA</td>
<td>0.8390</td>
<td>0.8718</td>
<td>0.5251</td>
</tr>
<tr>
<td>T-Bill</td>
<td>0.7880</td>
<td>0.7980</td>
<td>0.6049</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1971</td>
<td>0.5864</td>
</tr>
<tr>
<td></td>
<td>0.0075</td>
<td>0.8596</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.6296</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Trend (Low Freq)</th>
<th>V0</th>
<th>V1</th>
<th>V2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation</td>
<td>0.7403</td>
<td>0.1971</td>
<td>-0.1840</td>
</tr>
<tr>
<td>AAA</td>
<td>0.8854</td>
<td>0.416</td>
<td>0.0805</td>
</tr>
<tr>
<td>T-Bill</td>
<td>0.9128</td>
<td>0.1933</td>
<td>-0.1266</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Deviation (High Freq)</th>
<th>V0</th>
<th>V1</th>
<th>V2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation</td>
<td>0.0603</td>
<td>0.1674</td>
<td>0.5016</td>
</tr>
<tr>
<td>AAA</td>
<td>0.0128</td>
<td>0.3235</td>
<td>0.4292</td>
</tr>
<tr>
<td>T-Bill</td>
<td>0.3425</td>
<td>0.3933</td>
<td>0.6623</td>
</tr>
</tbody>
</table>

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Figure B.1a: Inflation and Nominal Interest Rates, US

Figure B.1b: Velocity, US
Figure B.3: Scatter Plots of Raw Data, US

\begin{align*}
\text{Inf}_{V_0} &= 0.7122 \times x + 18.842 \\
R^2 &= 0.3823
\end{align*}

\begin{align*}
\text{Inf}_{V_1} &= 0.1111 \times x + 5.0552 \\
R^2 &= 0.0269
\end{align*}

\begin{align*}
\text{Inf}_{V_2} &= -0.0054 \times x + 1.4393 \\
R^2 &= 0.0054
\end{align*}

\begin{align*}
\text{Inf}_{T-Bill} &= 0.0446 \times x + 16.734 \\
R^2 &= 0.621
\end{align*}

\begin{align*}
\text{Inf}_{T-Bill} &= 0.1327 \times x + 5.7021 \\
R^2 &= 0.0389
\end{align*}

\begin{align*}
\text{Inf}_{T-Bill} &= 0.0004 \times x + 1.8237 \\
R^2 &= 6 \times 10^{-5}
\end{align*}
Figure B.4: Scatter Plots of HP Trend, US

- **V0 vs Inflation HP 1600**
  
  \[ y = 1.0149x + 17.63 \]
  
  \[ R^2 = 0.548 \]
  
- **V0 vs T-bill Rate HP 1600**
  
  \[ y = 1.2846x + 15.063 \]
  
  \[ R^2 = 0.8332 \]

- **V1 vs Inflation HP 1600**
  
  \[ y = 0.3512x + 5.795 \]
  
  \[ R^2 = 0.0389 \]

- **V1 vs T-bill Rate HP 1600**
  
  \[ y = 0.1523x + 3.6099 \]
  
  \[ R^2 = 0.0174 \]

- **V2 vs Inflation HP 1600**
  
  \[ y = 0.0099x + 1.8645 \]
  
  \[ R^2 = 0.0139 \]

- **V2 vs T-bill Rate HP 1600**
  
  \[ y = 0.0060x + 1.8614 \]
  
  \[ R^2 = 0.016 \]
Figure B.5: Scatter Plots of Deviations, US
Appendix C

Appendix for Chapter 4

C.1 Proofs

C.1.1 Proof of Proposition 4.4

Proof. In the case of $\rho = 1$,

$$
\chi(i) = 1 - \frac{1 - L(\mu^i \bar{z})}{1 - L(\mu^{i-1} \bar{z})}, \quad i = 1, 2, \ldots, N, \quad \text{and} \quad L(z) = 1 - \frac{H(\mu z) - H(z)}{H(\bar{z})}.
$$

Putting the two equations together, we get

$$
\chi(i) = \begin{cases} 
1 - \frac{H(\mu^{i+1} \bar{z}) - H(\mu^i \bar{z})}{H(\mu^{i+1} \bar{z}) - H(\mu^i \bar{z})}, & i = 1, \ldots, N; \\
1 - \frac{1 - H(\mu^N \bar{z})}{H(\mu^N \bar{z}) - H(\mu^{N-1} \bar{z})}, & i = N; \\
1, & i = N + 1.
\end{cases} \quad (C.1)
$$

Given the expression of $H$ in (4.5), one can derive

$$
H(\mu^{i+1} \bar{z}) - H(\mu^i \bar{z}) = \frac{\alpha_1}{2\alpha_2} \frac{\bar{z}(\bar{z} - c)(\mu^{i+1} - \mu^i)}{(\mu^{i+1} \bar{z} - c)(\mu^i \bar{z} - c)}.
$$

Plugging this into (C.1), we get

$$
\chi(i) - \chi(i - 1) = \frac{\mu^{i-1} \bar{z} - \mu c}{\mu^i \bar{z} - c} - \frac{\mu^i \bar{z} - \mu c}{\mu^{i+1} \bar{z} - c}, \quad \text{for} \quad i = 2, \ldots, N - 1
$$
One can easily verify that \( \chi(i) - \chi(i - 1) < 0 \) for any \( i = 2, \ldots, N - 1 \).

### C.1.2 Proof of Proposition 4.5

**Proof.** The maximization problem in (4.40) yields the following F.O.C:

\[
u'(d) \left( \frac{d}{p} \right) \frac{1}{p} - 1 - \lambda(p) = 0, \quad \forall p,
\]

where \( \lambda(p) \) is the multiplier of the household’s cash constraint (4.41). If \( \lambda(p) = 0 \), the optimal \( d \) satisfies \( u'(d) \left( \frac{d}{p} \right) \frac{1}{p} = 1 \). And if \( \lambda(p) > 0 \), the cash constraint is binding and \( d = z \). In this case, \( \lambda(p) = u'(z) \left( \frac{z}{p} \right) \frac{1}{p} - 1 \). At the cut-off level, \( \hat{p} \) satisfies \( \lambda(\hat{p}) = 0 \) and \( u'(\hat{z}) \left( \frac{z}{\hat{p}} \right) \frac{1}{\hat{p}} = 1 \).

Differentiating \( \lambda(p) \) with respect to \( p \) yields:

\[
\lambda'(p) = -\frac{1}{p^2} \left[ u''(z) \left( \frac{z}{p} \right) \frac{z}{p} + u'(z) \left( \frac{z}{p} \right) \right].
\]

In the case of \( \eta > 1 \), \( u''(z) \left( \frac{z}{p} \right) \frac{z}{p} + u'(z) \left( \frac{z}{p} \right) < 0 \) and \( \lambda'(p) > 0 \). Hence, \( \lambda(p) > 0 \) if and only if \( p > \hat{p} \). Together with the “participation constraint”, this implies \( d = z \) if \( \hat{p} \leq p \leq p^r \). When \( p < \hat{p} \), \( u'(\hat{z}) \left( \frac{z}{p} \right) \frac{1}{p} < 1 \). The marginal benefit of spending one unit of money is lower than its marginal cost, so the household is better off by decreasing his/her money expenditure. In equilibrium, the household spends \( d < z \) to achieve \( u'(d) \left( \frac{d}{p} \right) \frac{1}{p} = 1 \).

When \( \eta \leq 1 \), \( u''(\hat{z}) \left( \frac{z}{p} \right) \frac{z}{p} + u'(\hat{z}) \left( \frac{z}{p} \right) > 0 \), and \( \lambda'(p) < 0 \). In this case, \( \lambda(p) > 0 \) if and only if \( p < \hat{p} \). The remaining proof follows the same arguments as above.

### C.1.3 Proof of Proposition 4.6

**Proof.** We first consider the case where the household’s cash constraint is binding, i.e. \( d = z \). In this situation, the expected return from posting \( p \) is

\[
\pi(p) = \left( 1 - \frac{c}{p} \right) z. \tag{C.2}
\]
It follows that \( \pi(p) \) is strictly increasing in \( p \), so the monopoly would like to post a price as high as possible.

When \( d < z \), the household’s optimal expenditure satisfies

\[
u' \left( \frac{d(p)}{p} \right) = p \quad \text{or} \quad u'[q(p)] = p, \tag{C.3}\]

where \( q(p) = d(p)/p \) and \( q'(p) = u''(q) \). Taking the household’s expenditure rule as given, the monopolist chooses \( p \) to maximize the following expected return:

\[
\pi(p) = pq(p) - cq(p).
\]

The optimality requires that

\[
\pi'(p) = q + (p - c)q'(p).
\]

Plugging the results of (C.3) and \( q'(p) \) into the above expression, we then have

\[
\pi'(p) = \frac{1}{u''(q)} [u''(q)q + u'(q) - c]. \tag{C.4}
\]

(i). When \( \eta \geq 1 \), \( u''(q)q + u'(q) \leq 0 \) and \( \pi'(p) > 0 \). Together with the result in (C.2), this implies that the expected return for a monopoly is strictly increasing in its prices. In this case, no household purchase in the BJ market if \( p > p^r \), the highest price the monopoly will charge is the household’s reservation price, \( p^r \).

(ii). When \( \eta < 1 \), \( u''(q)q + u'(q) > 0 \). So there exists an \( p \) such that \( \pi'(p) = 0 \). Given the solution of \( q(p) \) in (C.3), the optimal price \( p^m \) satisfies

\[
u''[u'^{-1}(p^m)]u'^{-1}(p^m) + p^m = c. \tag{C.5}\]

Footnote: To ensure a unique maximum point, we also need the second order condition: \( \pi''(p) = q'(p) \left[ 1 + \frac{cu'''}{uu''} + \frac{(u'')^2 - u''''}{uu''} \right] < 0 \). A sufficient condition is \( u'u'''' \leq (u'')^2 \), which holds if \( u' \) is log concave. This is the same condition in Lagos and Wright.
Combining with the case where households spend all their money balances, the monopolist’s price could be either \( \hat{p} \) or \( p^m \), depending on the marginal value of money in the AD market. In general, it can be proved that when the marginal valuation of money is sufficiently low (or inflation is sufficiently high), \( p^m < \hat{p} \) and the household’s cash constraint is always binding. In this case, the optimal monopoly price equals \( \hat{p} \).

C.1.4 Proof of Proposition 4.7

Proof. We first rewrite the equilibrium conditions in (4.52) and (4.53) as

\[
\Psi_1(\bar{z}) = \int_{\hat{p}}^{\bar{p}} \left[ u' \left( \frac{\bar{z}}{p} \right) \frac{1}{p} - \frac{1}{p} \right] dJ(p) - i, \quad \text{and} \quad \Psi_2(\bar{z}) = \int_{\hat{p}}^{p^r} \left[ u' \left( \frac{\bar{z}}{p} \right) \frac{1}{p} - \frac{1}{p} \right] dJ(p) - i. 
\]

(C.6)

Given the assumed properties of \( u(,.) \), both \( \Psi'_1(\bar{z}) < 0 \) and \( \Psi'_2(\bar{z}) < 0 \). Moreover, using the definition of \( \hat{p} \) one can prove that \( \hat{p}'(z) < 0 \) for \( \eta \leq 1 \) and \( \hat{p}'(z) > 0 \) for \( \eta > 1 \). Thus, in each case, there exists a highest value \( \bar{z}^{max} \) such that, at \( \bar{z}^{max} \), \( \hat{p} = p \) (for \( \eta \leq 1 \)) or \( \hat{p} = p^r \) (for \( \eta > 1 \)), and \( \Psi_1(\bar{z}) = \Psi_2(\bar{z}) = -i \). Putting these together, we have

- \( \Psi_1 \) and \( \Psi_2 \) are both continuous and differentiable for \( \bar{z} \in (0, \bar{z}^{max}) \);

- \( \Psi'_1(\bar{z}) < 0 \) and \( \Psi'_2(\bar{z}) < 0 \);

- \( \lim_{\bar{z} \to 0} \Psi_1(\bar{z}) = \lim_{\bar{z} \to 0} \Psi_2(\bar{z}) = \infty \) and \( \Psi_1(\bar{z}^{max}) = \Psi_2(\bar{z}^{max}) = -i \).

Therefore, there exists a unique \( \bar{z} \) such that \( \Psi_1(\bar{z}) = 0 \) or \( \Psi_2(\bar{z}) = 0 \).

C.1.5 Proof of Proposition 4.9

Proof. Note first that

\[
G'(\sigma) = \frac{[\alpha_1 + 2\alpha_2 S(\sigma)]S'(\sigma)}{\alpha_1 + 2\alpha_2} \quad \text{(C.7)}
\]

\footnote{There is no reservation price for the household in this case.}

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and that

\[ S'(\sigma) = \frac{\alpha_1}{2\alpha_2} \left[ \frac{\bar{z} - cu^{-1}(\bar{z})}{[u(q^*) - \sigma - cq^*]^2} \right]. \]  

(C.8)

Thus, we may write the equilibrium condition as:

\[ \Upsilon(\bar{z}) = \frac{\alpha_1}{2\alpha_2} [u(q^*) - \bar{z}] \left[ \frac{(\bar{z} - cu^{-1}(\bar{z})^2)}{(\bar{z} - cq^*)^3} \right] - i. \]  

(C.9)

Given the assumed properties of \( u(\cdot) \),

- \( \Upsilon \) is continuous and differentiable for \( \bar{z} \in (cq^*, q^*) \)
- \( \Upsilon'(\bar{z}) < 0 \)
- \( \lim_{\bar{z} \to cq^*} \Upsilon(\bar{z}) = \infty \) and \( \Upsilon(q^*) = -i \).

As a result, there exists a unique \( \bar{z} \in (cq^*, q^*) \) such that \( \Upsilon(\bar{z}) = 0 \).