HALL CONDITIONS FOR EDGE-WEIGHTED BIPARTITE GRAPHS

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Abstract. A weighted variant of Hall's condition for the existence of matchings is shown to be equivalent to the existence of a matching in a lexicographic product. This is used to introduce characterizations of those bipartite graphs whose edges may be replicated so as to yield semiregular multigraphs or, equivalently, semiregular edge-weightings. Such bipartite graphs will be called semiregularizable. For example, Hall's theorem guarantees the existence of a spanning $(1,1)$-semiregularizable forest (i.e., a perfect matching) while a p/q rational variant of Hall's theorem guarantees a spanning (p,q)-semiregularizable forest. Some infinite families of semiregularizable trees are described and all semiregularizable trees on at most 11 vertices are listed. Matrix analogues of some of the results are mentioned and are shown to imply some of the known characterizations of regularizable graphs.

Key words. Hall's theorem, bipartite graphs, regular, semiregular, regularizable, semiregularizable.

AMS subject classifications. 05C50, 05C70, 05C38

1. Introduction. Proofs of Hall's 1935 matching theorem abound in the literature (see, for example, [1, 2, 4, 10] and Hall [6]). The theorem may be presented as the following *employment* result.

In a small town, there are m unemployed workers and n available jobs. It is desired to give all of the workers jobs that they are suited for. Let $|S|$ denote the cardinality of a set S. Hall's theorem asserts that

It is possible to give each worker a suitable job if and only if every set S of workers is collectively suited for $|S|$ or more of the jobs.

By replacing each worker i of the m workers by a set of p_i identical clones, the following *overemployment* result may be deduced from the employment result (See also Exercise 25g, [10, p. 118] and Example 2.3).

Suppose that for each $i = 1, \ldots, m$, worker i wishes to take on p_i suitable jobs. It is possible to give each worker the number of suitable jobs that he/she wants if and only if every set S of workers is collectively suited for $\sum_{i \in S} p_i$ or more of the jobs.

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The employment results may be restated in terms of bipartite graphs. The graph theoretic notation used here is standard (see, for example, Bondy and Murty [2]), except that the term *graph* is always reserved for a graph that may have loops but no multiple edges while the term *multigraph* allows loops and multiple edges. Also, a subset M of the edge-set $E(G)$ of a graph G *covers* a subset S of the vertex set $V(G)$ of G if each vertex in S is incident to some edge of M. Thus, a matching in G is perfect if it covers all of the vertices of G.

Throughout the paper, $G(X,Y)$ always denotes a bipartite graph (necessarily without loops) with vertex parts $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. If each vertex in X is adjacent to each vertex in Y, then $G(X,Y)$ is a *complete* bipartite graph and is denoted by $K_{m,n}$. A *star* is a graph $K_{a,b}$ with $a = 1$ or $b = 1$. Also, if k is a positive integer, $[k]$ denotes the set $\{1, 2, \ldots, k\}$.

For a subset S of the vertex set $V(G)$ of a graph G, let $N(S)$ denote the set of all neighbors in G of vertices in S. In graph theoretic terms, the employment and overemployment results may be restated as the following theorem and corollary.

THEOREM 1.1. *(Hall's theorem) $G(X,Y)$ contains a matching that covers X if and only if*

$$|N(S)| \geq |S| \quad \text{for all subsets} \quad S \subseteq X. \tag{1.1}$$

COROLLARY 1.2. *Let $p_i, i \in [m]$ be positive integers. Then $G(X,Y)$ contains a family $K_{1,p_i}, i \in [m]$ of vertex disjoint stars centered at the vertices $x_i, i \in [m]$ in X if and only if*

$$|N(S)| \geq \sum_{x_i \in S} p_i \quad \text{for all} \quad S \subseteq X. \tag{1.2}$$

The result in Corollary 1.2 is obtained from Hall's theorem by replicating the vertices in X. In the next section, it will be seen that replicating the vertices in Y as well as those in X leads to a condition equivalent to the existence of a matching $\mathcal{M}$ in a lexicographic product graph $\mathcal{G}(\mathcal{X},\mathcal{Y})$ obtained from $G(X,Y)$.

2. Hall conditions for lexicographic products. Let $\mathcal{G}(\mathcal{X},\mathcal{Y})$ be the lexicographic product graph obtained from $G(X,Y)$ by replacing each vertex x_i in X by an empty graph with vertex set $\mathcal{X}_i$ and each vertex y_j in Y by an empty graph with vertex set $\mathcal{Y}_j$. Thus, $\mathcal{X} = \dot{\bigcup}_{i=1}^{m} \mathcal{X}_i$ is the disjoint union of nonempty sets $\mathcal{X}_i$, $i \in [m]$, $\mathcal{Y} = \dot{\bigcup}_{j=1}^{n} \mathcal{Y}_j)$ is the disjoint union of nonempty sets $\mathcal{Y}_j$, $j \in [n]$ and each vertex in $\mathcal{X}_i$ is adjacent (respectively, nonadjacent) to each vertex in $\mathcal{Y}_j$ if x_i is adjacent (respectively, nonadjacent) to y_j in G. That is, the bipartite subgraph $\mathcal{G}(\mathcal{X}_i, \mathcal{Y}_j)$ of $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ induced by $(\mathcal{X}_i, \mathcal{Y}_j)$ is complete if x_i is adjacent to y_j in G and empty if x_i is nonadjacent to y_j in G.

LEMMA 2.1. *Let* $\mathcal{G}(\mathcal{X}, \mathcal{Y}) = \mathcal{G}(\dot{\bigcup}_{i=1}^{m} \mathcal{X}_i, \dot{\bigcup}_{j=1}^{n} \mathcal{Y}_j)$ *be a lexicographic product graph formed from* $G(X, Y)$*. Then there is a matching in* $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ *that covers* $\mathcal{X}$ *if and only if*

$$\sum_{y_j \in N(S)} |\mathcal{Y}_j| \geq \sum_{x_i \in S} |\mathcal{X}_i| \text{ for all subsets } S \subseteq X. \tag{2.1}$$

The graph $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ *has a perfect matching if and only if condition (2.1) holds and* $\sum_{y_j \in Y} |\mathcal{Y}_j| = \sum_{x_i \in X} |\mathcal{X}_i|$.

Proof. If $\mathcal{S} \subseteq \mathcal{X}$, let $\mathcal{N}(\mathcal{S}) \subseteq \mathcal{Y}$ denote the neighbors of $\mathcal{S}$ in $\mathcal{G}(\mathcal{X}, \mathcal{Y})$.

Suppose that there is a matching in $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ that covers $\mathcal{X}$. Let $S \subseteq X$ and let $\mathcal{S}$ denote[1] the set of all $\sum_{x_i \in S} |\mathcal{X}_i|$ copies in $\mathcal{X}$ of members of S. Note that if S is regarded as a subset of $\mathcal{X}$ by taking one copy of each vertex of S in $\mathcal{X}$, then $\mathcal{N}(\mathcal{S}) = \mathcal{N}(S)$. Because the members of $\mathcal{S}$ are matched to $|\mathcal{S}|$ members of $\mathcal{Y}$, $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}| = \sum_{x_i \in S} |\mathcal{X}_i|$. But $|\mathcal{N}(\mathcal{S})| = |\mathcal{N}(S)| = \sum_{y_j \in N(S)} |\mathcal{Y}_j|$, so inequality (2.1) follows.

Suppose now that condition (2.1) holds. By Hall's Theorem, to show that there is a matching in $\mathcal{G}$ that covers $\mathcal{X}$, it is sufficient to show that $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$ for each $\mathcal{S} \subseteq \mathcal{X}$. Given $\mathcal{S} \subseteq \mathcal{X}$, let S be the members of X that have at least one copy in $\mathcal{S}$. Then $|\mathcal{S}| \leq \sum_{x_i \in S} |\mathcal{X}_i|$. Also, $|\mathcal{N}(\mathcal{S})| = \sum_{y_j \in N(S)} |\mathcal{Y}_j|$, so condition (2.1) implies that $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$.

Finally, a matching in $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ that covers $\mathcal{X}$ will be perfect if and only if $|\mathcal{X}| = |\mathcal{Y}|$, that is, if and only if $\sum_{y_j \in Y} |\mathcal{Y}_j| = \sum_{x_i \in X} |\mathcal{X}_i|$. □

More can be said if inequality (2.1) is strict for proper nonempty subsets S and exact when $S = X$.

LEMMA 2.2. *Let* $G(X, Y)$ *and* $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ *be as in Lemma 2.1. Then (2.1) holds with equality for* $S = X$ *and is strict for* $\emptyset \subsetneq S \subsetneq X$*, if and only if* $G(X, Y)$ *is connected and each edge in* $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ *is in a perfect matching of* $\mathcal{G}(\mathcal{X}, \mathcal{Y})$.

Proof. Suppose that (2.1) holds for $S = X$ and is strict for $\emptyset \subsetneq S \subsetneq X$. Then $G(X, Y)$ must be connected, otherwise a strict inequality is obtained whenever S is a subset of X determined by a component and so equality could not hold when $S = X$. Given $\mathcal{S} \subseteq \mathcal{X}$, let S be the members of X that have at least one copy in $\mathcal{S}$. Then $|\mathcal{N}(\mathcal{S})| = \sum_{y_j \in N(S)} |\mathcal{Y}_j| \geq \sum_{x_i \in S} |\mathcal{X}_i| \geq |\mathcal{S}|$. Thus $|\mathcal{N}(\mathcal{S})| > |\mathcal{S}|$ if either S is a

[1] Please note the slight distinction between the letters S and $\mathcal{S}$.

proper nonempty subset of X or if $S = X$ and $|\mathcal{S}| < \sum_{x_i \in S} |\mathcal{X}_i| = |\mathcal{X}|$. It follows that if xy is an edge of $\mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{X} - x$, then $\mathcal{S}$ has at least $|\mathcal{S}|$ neighbours in $\mathcal{Y} - y$. Thus, by Hall's theorem, there is a matching in $\mathcal{G} - x - y$ of $\mathcal{X} - x$ into $\mathcal{Y} - y$ that covers $\mathcal{X} - x$, and so a matching in $\mathcal{G}$ of $\mathcal{X}$ into $\mathcal{Y}$ that contains xy and covers $\mathcal{X}$. Because G is connected, $N(X) = Y$, so $\sum_{y_j \in Y} |\mathcal{Y}_j| = \sum_{y_j \in N(X)} |\mathcal{Y}_j| = \sum_{x_i \in X} |\mathcal{X}_i|$. Thus $|\mathcal{X}| = |\mathcal{Y}|$, so the matching is perfect.

Suppose that $G(X,Y)$ is connected and each edge in $\mathcal{G}(\mathcal{X},\mathcal{Y})$ is in some perfect matching. Then $N(X) = Y$ and equality holds in (2.1) when $S = X$. Given a subset S of X, let $\mathcal{S}$ denote the set of all copies in $\mathcal{X}$ of members of S. Then inequality (2.1) holds, so $|N(\mathcal{S})| = \sum_{y_j \in N(S)} |\mathcal{Y}_j| \geq \sum_{x_i \in S} |\mathcal{X}_i| = |\mathcal{S}|$. The inequality must be strict if S is a proper nonempty subset of X. Suppose not. Then $|\mathcal{N}(\mathcal{S})| = |\mathcal{S}|$, so every perfect matching of $\mathcal{G}(\mathcal{X},\mathcal{Y})$ must match $\mathcal{S}$ to $\mathcal{N}(\mathcal{S})$. But each edge of $\mathcal{G}$ that is incident to $\mathcal{N}(\mathcal{S})$ must be in some perfect matching and so must be incident to $\mathcal{S}$. Thus the subgraph of $\mathcal{G}$ induced by $\mathcal{S}$ and $\mathcal{N}(\mathcal{S})$ must be a proper component of $\mathcal{G}$, a contradiction. □

Example 2.3. Suppose that inequality (2.1) holds in Lemma 2.1 with $|\mathcal{Y}_j| = 1$ for $j \in [n]$. Then $\mathcal{G}(\mathcal{X},\mathcal{Y})$ has a matching $\mathcal{M}$ that covers $\mathcal{X}$. By collapsing the vertex sets $\mathcal{X}_i$ to single vertices x_i, the edges of $\mathcal{M}$ yield a forest $\mathcal{M}(X,Y)$ of vertex disjoint stars K_{1,p_i} in $G(X,Y)$ where $p_i = |\mathcal{X}_i|$ and the stars are centered at the m vertices x_i in X. (A vertex y_i is isolated in $\mathcal{M}(X,Y)$ if no edge of $\mathcal{M}$ is incident to a vertex of $\mathcal{Y}_j$.) Conversely, the existence of such a forest of stars in $G(X,Y)$ implies the existence of a matching in $\mathcal{G}(\mathcal{X},\mathcal{Y})$ that covers $\mathcal{X}$. Thus, Corollary 1.2 is a special case of Lemma 2.1.

Example 2.4. Suppose now that inequality (2.1) holds in Lemma 2.1 with $|\mathcal{X}_i| = 1$ for $i \in [m]$. Then $\mathcal{G}(\mathcal{X},\mathcal{Y})$ has a matching $\mathcal{M}$ that covers $\mathcal{X}$. Collapsing the vertex sets $\mathcal{Y}_j$ to single vertices y_j, the edges of $\mathcal{M}$ yield a forest $\mathcal{M}(X,Y)$ in $G(X,Y)$ of vertex disjoint stars $K_{b_j,1}, b_j \leq |\mathcal{Y}_j|$ centered at the vertices y_j in Y. The edges of the stars cover X and a star at y_j is taken to be the single vertex y_j if no vertex in $\mathcal{Y}_j$ is incident to a vertex of $\mathcal{M}$. Conversely, the existence of such a forest of stars in $G(X,Y)$ implies the existence of a matching in $\mathcal{G}(\mathcal{X},\mathcal{Y})$ that covers $\mathcal{X}$. If $\sum_{j=1}^{n} |\mathcal{Y}_j| = m$, then $\mathcal{M}$ is perfect, the edges of the stars also cover Y, and $b_j = |\mathcal{Y}_j|$ for each $j \in [n]$.

In Example 2.3 the vertex sets $\mathcal{Y}_j$ were singletons and the edges of the matching $\mathcal{M}$ in $\mathcal{G}(\mathcal{X},\mathcal{Y})$ yielded a simple subgraph of $G(X,Y)$ when the vertex sets $\mathcal{X}_i$ were collapsed back to single vertices x_i. Similarly, a simple subgraph also resulted in Example 2.4 where the vertex sets $\mathcal{X}_i$ were singletons. However, in general, when the vertex sets $\mathcal{X}_i$, $\mathcal{Y}_j$ are all collapsed to single vertices, then the edges of a matching $\mathcal{M}$ in $\mathcal{G}(\mathcal{X},\mathcal{Y})$ yield a *multigraph* $\mathcal{M}(X,Y)$ (possibly with isolated vertices) whose underlying simple graph is a spanning subgraph of $G(X,Y)$.

An *edge-weighting* of a graph G with vertex set V is a nonnegative integer valued function $w : V \times V \to \mathbb{N} \cup \{0\}$ such that $w(uv) = w(vu)$ for all $u, v \in V$ and $w(uv) = 0$ whenever u is not adjacent to v in G. Thus an edge-weighting may have value 0 on an edge, but must be 0 on all nonedges. An edge-weighting of G is called *positive* if it is positive on each edge of G. An edge-weighting w is extended to vertices $v \in V$ by taking $w(v)$ to be the sum of the weights on the edges incident to v.

The multigraph $\mathcal{M}(X, Y)$ determined by a matching $\mathcal{M}$ in $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ may be specified by an edge-weighting $w : X \times Y \to \mathbb{N} \cup \{0\}$ of $G(X, Y)$. The edge-weighting will have the following properties for vertices $x_i \in X$ and $y_j \in Y$:

1. $w(x_i y_j)$ is the number of edges of $\mathcal{M}$ between $\mathcal{X}_i$ and $\mathcal{Y}_j$ (possibly 0).
2. $w(x_i)$ is the number of edges of $\mathcal{M}$ incident to vertices in $\mathcal{X}_i$, while $w(y_j)$ is the number of edges of $\mathcal{M}$ incident to vertices in $\mathcal{Y}_j$.
3. $w(x_i) \leq |\mathcal{X}_i|$ for all i and $w(y_j) \leq |\mathcal{Y}_j|$ for all j.
4. $w(x_i) = |X_i|$ for all i if and only if $\mathcal{M}$ covers X, while $w(y_j) = |Y_j|$ for all j if and only if $\mathcal{M}$ covers Y.

Conversely, by expanding vertices x_i, y_j to sets $\mathcal{X}_i$, $\mathcal{Y}_j$, it follows that for every edge-weighting $w : X \times Y \to \mathbb{N} \cup \{0\}$ of $G(X, Y)$ with property 3 and with $w(x_i y_j) > 0$ on some edge, there is at least one matching $\mathcal{M}$ in the lexicographic product graph $\mathcal{G}(\mathcal{X}, \mathcal{Y}) = \mathcal{G}(\dot{\bigcup}_{i=1}^m \mathcal{X}_i, \dot{\bigcup}_{j=1}^n \mathcal{Y}_j)$ that satisfies properties 1-4.

From the above, it follows that $\mathcal{G}(\mathcal{X}, \mathcal{Y}) = \mathcal{G}(\dot{\bigcup}_{i=1}^m \mathcal{X}_i, \dot{\bigcup}_{j=1}^n \mathcal{Y}_j)$ has a perfect matching if and only if $G(X, Y)$ has an edge-weighting w with $w(x_i) = |\mathcal{X}_i|$ for each $i \in [m]$ and $w(y_j) = |\mathcal{Y}_j|$ for each $j \in [n]$. Thus, if $p_i = |\mathcal{X}_i|$ and $q_j = |\mathcal{Y}_j|$, then Lemmas 2.1 and 2.2 imply the following result.

LEMMA 2.5. *Let $p_i, i \in [m]$ and $q_j, j \in [n]$ be positive integers. Then $G(X, Y)$ has an edge-weighting $w : X \times Y \to \mathbb{N} \cup \{0\}$ such that*

$$w(x_i) = p_i \text{ for each } i \in [m] \quad \text{and} \quad w(y_j) = q_j \text{ for each } j \in [n] \tag{2.2}$$

if and only if

$$\sum_{j=1}^{n} q_j = \sum_{i=1}^{m} p_i \quad \text{and} \quad \sum_{y_j \in N(S)} q_j \geq \sum_{x_i \in S} p_i \text{ for all subsets } S \subseteq X. \tag{2.3}$$

Moreover, (2.3) holds and the inequality is strict whenever S is a proper nonempty subset of X if and only if G is connected and, for each edge $x_i y_j$ in G, there is a edge-weighting w of G (depending on $x_i y_j$) that satisfies (2.2) with $w(x_i y_j) > 0$.

The next two sections contain applications of Lemma 2.5 in which the p_i and the q_j are constants p, q.

3. Hall conditions and semiregularizable bipartite graphs. Let r be a positive integer. A graph or multigraph is *r-regular* if each vertex is incident to r edges. A graph is *r-regularizable* if its edges may be replicated to form an r-regular multigraph. Equivalently, a graph G is r-regularizable if it has a positive edge-weighting for which the associated vertex weights, $w(v), v \in V(G)$, are all equal to r. A graph is said to be *regularizable* if it is r-regularizable for some positive integer r.

Characterizations of regularizable graphs are well-known and are discussed in Section 7. It is natural to refine the notion for bipartite graphs. Indeed, it will be seen in Section 7. that results on regularizable graphs often follow from the following bipartite refinement.

Let p, q be positive integers. A bipartite graph or multigraph $G(X, Y)$ is *(p, q)-semiregular* if each edge in X is incident to p edges and each vertex in Y is incident to q edges. A bipartite graph $G(X, Y)$ is *(p, q)-semiregularizable* (briefly, *(p, q)-srz*) if its edges may be replicated to form a (p, q)-semiregular bipartite multigraph. Equivalently, a bipartite graph $G(X, Y)$ is (p, q)-srz if it has a positive edge-weighting that is (p, q)-semiregular, that is, an edge-weighting that is positive on the edges of G and has associated vertex weight p on each vertex of X and vertex weight q on each vertex of Y. A graph $G(X, Y)$ is *semiregularizable* (briefly, *srz*), if it is (p, q)-srz for some positive integers p, q.

Remark 3.1. When searching for a positive semiregular edge-weighting of a bipartite graph $G(X, Y)$, note that it is sufficient to first find a *rational* valued edge-weighting that is positive on edges of G and constant on vertices of X and on vertices of Y. Also, it is helpful to note that the edge-weights may be assumed to be constant on edge-orbits under the action of the automorphism group. For if w is a semiregular edge-weighting of G and Γ is the group of automorphisms of G, then it is straightforward to check that the function $\bar{w} : X \times Y \to \mathbb{Q}$ defined by

$$\bar{w}(xy) = \frac{1}{|\Gamma(G)|} \sum_{\gamma \in \Gamma} w(\gamma(x)\gamma(y))$$

is a rational valued function that is constant on edge-orbits of G and that some integral multiple of $\bar{w}$ is semiregular.

Example 3.2. Positive semiregular edge-weightings are easily found for the odd paths P_5, P_7 and P_9 and are shown in Figure 3.1. The weighting pattern shown there works for all odd paths $P_{2k+1}(X, Y)$ with $|X| = k, |Y| = k + 1,\ k \geq 1$. Also, P_2 is regular and so semiregular. Theorem 3.3 below implies that no even path P_{2k}, $2k \geq 4$ is srz.

If a graph $G(X, Y)$ has connected components $G_i(X_i, Y_i)$ with $X_i \subseteq X$ and

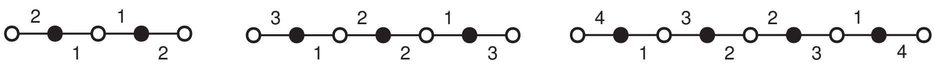

FIG. 3.1. *Positive* $(k+1,k)$*-semiregular edge-weightings of* P_{2k+1} *for* $k = 2,3,4$*. Vertices in* X *are black, vertices in* Y *are white.*

$Y_i \subseteq Y$, for $i \in [k]$, it is clear that $G(X,Y)$ is (p,q)-srz if and only if each of its components $G_i(X_i,Y_i)$ is. Thus, to determine if a graph is srz, it is sufficient to examine its connected components.

THEOREM 3.3. *A bipartite graph* $G(X,Y)$ *is connected and semiregularizable if and only if*

$$|N(X)| = |Y| \quad \text{and} \quad |X||N(S)| > |Y||S| \text{ whenever } \emptyset \subsetneq S \subsetneq X. \tag{3.1}$$

Proof. Suppose that condition (3.1) holds. Then the conditions in the second part of Lemma 2.5 hold with $p_i = |Y|$ and $q_j = |X|$ for all i,j. Thus, G is connected and for each edge xy of $G(X,Y)$, there is a $(|Y|,|X|)$-semiregular edge-weighting of G that is positive on xy (but perhaps 0 on some edges of G). Adding the weight functions over all edges xy gives a positive edge-weighting of G with associated vertex weights $p = |E||Y|$ on the vertices of X and $q = |E||X|$ on the vertices of Y. Thus $G(X,Y)$ is (p,q)-srz.

Suppose now that $G(X,Y)$ is connected and srz. Then for some pair p,q of positive integers, G is (p,q)-srz with some positive edge-weighting w. Let $\emptyset \subsetneq S \subsetneq X$. Because G is connected and w is positive on each edge of G, Lemma 2.5 implies that $q|Y| = p|X|$ and $q|N(S)| > p|S|$. Thus, $|X||N(S)| > |Y||S|$. □

Theorem 3.3 implies that if $G(X,Y)$ is srz, then adding new edges xy with $x \in X$, $y \in Y$ to components of $G(X,Y)$ always yields a srz bipartite graph. A constructive proof of this consequence of Theorem 3.3 is given in the following lemma.

LEMMA 3.4. *Suppose that* $H(X,Y)$ *is a connected spanning bipartite srz subgraph of a bipartite graph* $G(X,Y)$*. Then, by successively adding edges of* G *to* H*, each positive semiregular edge-weighting on* H *can be modified to give a positive semiregular edge-weighting on* G*.*

Proof. Let w be a positive (p,q)-semiregular edge-weighting on H. Suppose that xy is an edge of G that is not an edge of H. Because H is connected, $H + xy$ has a (necessarily even) cycle C that contains xy. Alternately color the edges of C black and white so that xy is black and let h be a positive integer that is less than or equal to the minimum of the weights of the white edges in C. Give xy an initial weight of 0 and each remaining edge of $H + xy$ the same weight as before. Then add $h/2$ to each black edge on C and subtract $h/2$ from each white edge on C. If h is even, this gives a positive (p, q)-semiregular edge-weighting on $H + xy$. If h is odd, then doubling all of the weights gives a positive $(2p, 2q)$-semiregular edge-weighting on $H + xy$.

Repeating the above procedure eventually yields a positive semiregular edge-weighting for G. □

EXAMPLE 3.5. A spanning path in a graph is called a *Hamiltonian path*. It follows from Example 3.2 and Lemma 3.4 that every bipartite graph of odd order that contains a Hamiltonian path is semiregularizable.

Lemma 3.4 leads us to examine connected srz graphs that are edge minimal; that is, it leads us to look for srz trees other than the odd paths.

4. Semiregularizable trees and forests. Other (p, q)-srz trees may sometimes be formed from a given (p, q)-srz tree $T(X, Y)$ by the following *weight-switching procedure*.

If $x \in X$ and $y \in Y$ are nonadjacent in T, let C be the (necessarily even) cycle in $T + xy$. Alternately colour the edges of C black and white, making xy black. Let w be a positive (p, q)-semiregular edge-weighting of T and let h be the minimum weight of the white edges on C. Subtract h from the white edges of C, add h to the weighted black edges of C and give xy a weight of h. Then the modified weight function is positive on the edges of a (p, q)-srz forest. The forest will be a tree if only one white edge on C had the original minimum weight of h (See Figure 4.1). In particular, as observed later in Remark 4.3, a tree will always be obtained if $\gcd(|X|, |Y|) = 1$.

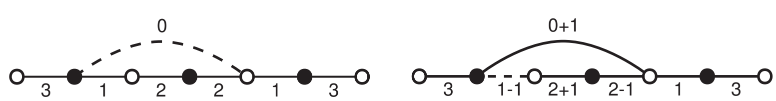

FIG. 4.1. *A* $(4, 3)$*-srz tree formed from the* $(4, 3)$*-srz path* P_7.

For an example where a forest with two trees is obtained by applying the weight-switching procedure to a tree, see Figure 4.2. Also, sometimes an isomorphic tree is obtained. For example, if the $(3, 2)$-srz path P_5 is used in place of P_7 in Figure 4.1, no new $(3, 2)$-srz tree results from the weight-switching procedure.

To gain further insight into srz graphs (and, in particular, trees) it is helpful to employ some linear algebra.

If G is a graph, let $N = N(G)$ denote the $|V| \times |E|$ vertex-edge *incidence matrix* of G; that is, $N_{x,e} = 1$ if vertex x is incident to edge e and $N_{x,e} = 0$ otherwise.

Rational valued functions $w : E(G) \to \mathbb{Q}$ with associated vertex values $w(v) =$

$\sum_{u,uv\in E} w(uv)$ that equal preassigned values $b(v) \in \mathbb{Q}$ for $v \in V$ correspond to solutions w of $Nw = b$ where w is a column $|E|$-vector whose entries are the values $w(uv)$ on the edges uv of G. Thus, if N is the vertex-edge incidence matrix of a bipartite graph $G(X, Y)$, then $G(X, Y)$ is (p, q)-srz if and only if the equation

$$Nw = b \quad \text{where} \quad b(x) = p \text{ for } x \in X \text{ and } b(y) = q \text{ for } y \in Y \tag{4.1}$$

has a positive integral solution w.

Lemma 4.1. *Let $T(X, Y)$ be a tree and let $\frac{|Y|}{|X|} = \frac{p}{q}$ where $\gcd(p, q) = 1$. Then there is a unique integer valued function $w : E(T) \to \mathbb{Z}$ such that $\sum_{y,xy\in E} w(xy) = p$ for each $x \in X$ and $\sum_{x,xy\in E} w(xy) = q$ for each $y \in Y$.*

Thus, if a tree T is srz, then it has a unique positive (p, q)-semiregular edge-weighting where $p|X| = q|Y|$ and $\gcd(p, q) = 1$.

Proof. Because T is a tree, every $(m + n - 1) \times (m + n - 1)$ submatrix of its incidence matrix N has determinant ± 1. Using Cramer's rule, it is easily checked that if $p|X| = q|Y|$, then the equation (4.1) has a unique integral solution when $b(x) = p$ for $x \in X$ and $b(y) = q$ for $y \in Y$. □

In determining the srz trees $T(X, Y)$, it may of course be assumed that $|Y| \geq |X|$. Then the previous lemma implies that a srz tree $T(X, Y)$ is always $(|Y|, |X|)$-srz. This is not the case for all srz graphs. For example, if $G(X, Y)$ is the graph with $|X| = 2$ and $|Y| = 3$ obtained by attaching a vertex of degree 1 to a vertex of a 4-cycle, then $G(X, Y)$ is $(6, 4)$-srz, but not $(3, 2)$-srz.

The following lemma allows us to further restrict our attention to srz trees $T(X, Y)$ with $|Y| \geq |X|$ where $|X|$ is not a divisor of $|Y|$. Unfortunately, for these remaining cases, there may be more than one srz tree with vertex parts X, Y. For, as already seen in Figure 4.1, if $T(X, Y)$ is (p, q)-srz, then the weight-switching procedure may yield different (p, q)-srz trees with the same vertex parts as T.

Lemma 4.2. *Let k be a positive integer. The only semiregularizable forests $F(X, Y)$ for which $|Y| = k|X|$, are those whose trees are all stars $K_{1,k}$. In particular, the only semiregularizable forests with $|Y| = |X|$ are the perfect matchings between X and Y.*

Proof. Suppose $F(X, Y)$ is (p, q)-srz and $T_i(X_i, Y_i)$ is a tree in F with $X_i \subseteq X$ and $Y_i \subseteq Y$. Then T_i is also (p, q)-srz so $\frac{p}{q} = \frac{|Y_i|}{|X_i|} = \frac{|Y|}{|X|} = k$. By Lemma 4.1, $T_i(X_i, Y_i)$ is $(k, 1)$-srz. Then each vertex in Y_i has weight 1 and so is incident to a single edge in T_i. But each edge of T_i is incident to a vertex in Y_i, so all of the edges in T_i have weight 1. Because vertices in X_i have weight k, each vertex in X_i is incident to k edges. But T_i is connected, so $T_i = K_{1,k}$. □

Remark 4.3. If a forest $F(X, Y)$ is semiregularizable and $\gcd(|X|, |Y|) = 1$, then the forest must be a single tree. For, if $T_i(X_i, Y_i)$ is a tree in F then, as in the proof of Lemma 4.2, $\frac{|Y_i|}{|X_i|} = \frac{|Y|}{|X|}$ so $|X_i| = |X|$ and $|Y_i| = |Y|$.

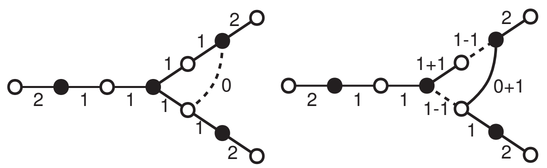

FIG. 4.2. *Adding the dashed edge to the* $(3, 2)$*-srz tree on the left and applying the weight-switching procedure yields a forest with two* $(3, 2)$*-srz* P_5*'s.*

If the weight-switching procedure illustrated in Figure 4.1 is applied to a (p, q)-srz tree $T(X, Y)$ with $\gcd(|X|, |Y|) = 1$, then Remark 4.3 implies that a (p, q)-srz tree will always be obtained. Figure 4.2 shows that this need not be the case if $|X|, |Y|$ have a common factor. This raises the following question.

QUESTION 4.4. *If* $T(X, Y)$ *is a srz tree and* $\gcd(|X|, |Y|) = 1$*, must every srz tree with vertex parts* X, Y *be obtainable from* $T(X, Y)$ *by successive applications of the weight-switching procedure?*

The question has been answered affirmatively for srz trees with 11 or fewer vertices. The srz trees are found by solving the incidence matrix equation (4.1) for each tree of order 11 or less (with p, q relatively prime and $p|X| = q|Y|$). A tree is srz if and only if each entry of the (unique) solution is positive. The srz trees of order 11 or less are provided in the Appendix.

5. Rational Hall conditions and semiregularizable forests. In Theorem 5.1 below, a rational variant of Hall's Theorem 1.1 on perfect matchings in bipartite graphs guarantees the existence of a srz forest. By Corollary 4.2, the only (p, q)-srz forests with $p = q$ are the perfect matchings, so Hall's Theorem 1.1 is a special case. Corollary 1.2 with $p_i = p$ for all $i \in [m]$ also follows from Theorem 5.1.

THEOREM 5.1. *Suppose that* $G(X, Y)$ *is a bipartite graph and that* p, q *are positive integers. Then* $G(X, Y)$ *contains a spanning* (p, q)*-srz forest if and only if*

$$p|X| = q|Y| \quad \textit{and} \quad |N(S)| \geq \frac{p}{q}|S| \quad \textit{for all } S \subseteq X. \tag{5.1}$$

Proof. The necessity is immediate. It also follows from Lemma 2.5.

Suppose that condition (5.1) holds. Then condition (2.3) of Lemma 2.5 holds with $p_i = p,\ i \in [m]$ and $q_j = q, \in j \in [n]$. Thus G has a (p, q)-semiregular edge-weighting

w (possibly with weight 0 on some edges of G). Let $H(X, Y)$ be the subgraph of $G(X, Y)$ induced by the edges of positive weight. Because p, q are positive and w is semiregular, $H(X, Y)$ is a spanning subgraph of $G(X, Y)$. If $H(X, Y)$ is not a forest, then it contains a (necessarily even) cycle C. The cycle may be broken by the following *cycle breaking* procedure.

Alternately color the edges of C black and white. Let c be the minimum weight on, say, the white edges of C. Alternately add c, $-c$ to the weights on the black and white edges of C, respectively, to from a new (p, q)-semiregular edge-weighting on $G(X, Y)$ for which at least one white edge of C has weight 0.

Repeating the cycle breaking procedure eventually yields a (p, q)-semiregular edge-weighting whose edges of positive weight form an acyclic spanning subgraph of $G(X, Y)$, that is a (p, q)-srz spanning forest. □

More can be said if the inequality in (5.1) is strict. Because a union of srz bipartite graphs with common vertex parts is srz, the next theorem implies the main implication in the characterization of srz bipartite graphs given earlier in Theorem 3.3.

THEOREM 5.2. *Suppose that $G(X, Y)$ is a bipartite graph and that $N(X) = Y$. Let p, q be positive integers. If*

$$p|X| = q|Y| \quad \text{and} \quad |N(S)| > \frac{p}{q}|S| \text{ for all nonempty subsets } S \subsetneq X, \tag{5.2}$$

then each edge xy in $G(X, Y)$ is in some spanning (p, q)-srz forest $F_{xy}(X, Y)$ in $G(X, Y)$.

Proof. Suppose that (5.2) holds. Then $|N(X)| = |Y| = \frac{p}{q}|X|$, so (5.1) holds. If xy is an edge of G then, by Theorem 5.1, there is a (p, q)-semiregular edge-weighting w of $G(X, Y)$ that is positive on xy. In the cycle breaking procedure used in the proof of Theorem 5.1, there is always the option of choosing either the black or the white edges of C. Thus the forest that eventually results can always be chosen to contain xy. □

The cycle breaking procedure of Theorem 5.1 also gives the following alternate characterization of srz bipartite graphs.

THEOREM 5.3. *A bipartite graph $G(X, Y)$ is srz if and only if each edge xy of G is in some spanning srz forest $F_{xy}(X, Y)$ in G.*

Proof. If each edge xy is in a srz forest $F_{xy}(X, Y)$, then the forest must be spanning. Adding the associated weight functions w_{xy} over all edges xy in G yields a semiregular weight function that is positive on all edges of G. Thus G is srz.

Suppose now that $G(X, Y)$ has a (p, q)-semiregular edge-weighting w and let xy

be an edge of G. In the cycle breaking procedure used in the proof of Theorem 5.1, there is always the option of choosing either the black or the white edges of C. Thus the (p, q)-srz forest that eventually results can always be chosen to contain xy. □

REMARK 5.4. Note that if the conditions in Theorem 5.1 or 5.2 hold, then they hold when p, q are chosen to be relatively prime. Thus the srz spanning forests in Theorems 5.1 and 5.2 and in Theorem 5.3 can always be chosen to be (p, q)-srz where $p|X| = q|Y|$ and $\gcd(p, q) = 1$. This agrees with Lemma 4.1 and also gives an alternate proof of it if it is first shown that a rational valued semiregular edge-weighting of a tree is uniquely determined by the weights on the twigs.

The characterization of srz bipartite graphs given in Theorem 5.3 becomes much simpler for bipartite graphs whose part sizes are relatively prime.

COROLLARY 5.5. *Let $G(X, Y)$ be a bipartite graph such that* $\gcd(|X|, |Y|) = 1$. *Then G is semiregularizable if and only if it contains a spanning semiregularizable tree.*

Proof. The sufficiency follows from Lemma 3.4.

If G is srz, then G has a srz forest by Theorem 5.3 and that forest must be a tree by Remark 4.3. □

Unfortunately, the assumption that $\gcd(|X|, |Y|) = 1$ cannot be dropped from Corollary 5.5. For example, the graph G in Figure 5.1 has no srz spanning tree. For, by Lemma 4.1, a spanning srz tree in G would have to be $(2, 1)$-srz. But Lemma 4.2 implies that a $(2, 1)$-srz tree is a $K_{1,2}$, and so has only three vertices.

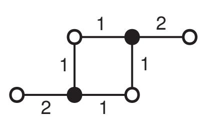

FIG. 5.1. *A connected $(4, 2)$-srz graph with no spanning srz tree.*

Lemma 4.2 immediately implies the following special case of Theorem 5.3.

COROLLARY 5.6. *Let $G(X, Y)$ be a bipartite graph with $|Y| = k|X|$ for some positive integer k. Then $G(X, Y)$ is semiregularizable if and only if each edge of G is in some spanning forest whose components are stars $K_{1,k}$, each centered on a vertex of X.*

In particular, if $k = 1$, Corollary 5.6 gives the following special case.

COROLLARY 5.7. *Let $G(X, Y)$ be a bipartite graph with $|Y| = |X|$. Then $G(X, Y)$ is semiregularizable (and so regularizable) if and only if each edge of G is in a perfect matching of $G(X, Y)$.*

6. Semiregularizable $(0,1)$-matrices. There is a $1-1$ correspondence between $m \times n$ matrices B with nonnegative integer entries and bipartite graphs or multigraphs G with vertex parts $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$. (Take $B_{i,j}$ to be the number of edges between x_i and y_j in $G(X,Y)$.) The matrix $B = B(G)$ is the *biadjacency matrix* of $G(X,Y)$. Using this correspondence, terminology and results on bipartite graphs G can all be interpreted in terms of $(0,1)$-matrices B.

For example, a subset $I \subseteq [m]$ corresponds to a subset $S \subseteq X$, and the set $N(I) = \{j \in [n] : B_{i,j} = 1$ corresponds to $N(S)$. Also, it is appropriate to say that an $m \times n$ $(0,1)$-matrix B is (p,q)-*semiregularizable* if the $1's$ of B may be replaced by positive integers so that the sum of the entries in each row equals p while the sum of the entries in each column equals q. For then, B is (p,q)-srz if and only if its bipartite graph $B(G)$ is (p,q)-srz.

A matching with t edges in a bipartite graph $G(X,Y)$ corresponds to a t-*diagonal* in its $(0,1)$-biadjacency matrix B; that is, a collection of t nonzero entries in B no two of which share the same row or column.

Using this terminology, Corollary 5.7 may be restated as follows.

Lemma 6.1. *An $n \times n$ $(0,1)$-matrix B is semiregularizable (and therefore regularizable) if and only if each 1 of B is in an n-diagonal of B.*

Two diagonals in a matrix are said to be *disjoint* if they have no entry positions in common. Corollary 5.6 can be restated as follows.

Lemma 6.2. *An $m \times n$ $(0,1)$-matrix B with $n = mk$ is semiregularizable if and only if each 1 of B is in a collection of k disjoint m-diagonals in B.*

An $n \times n$ matrix A is *irreducible* if there is no $n \times n$ permutation matrix P such that $P^{\mathrm{T}}AP = \begin{bmatrix} L & M \\ O & N \end{bmatrix}$ where L and N are square matrices of order at least 1.

If M is an $m \times n$ matrix, let $\tilde{M} = \begin{bmatrix} O & M \\ M^{\mathrm{T}} & O \end{bmatrix}$. Then a bipartite graph $G(X,Y)$ with biadjacency matrix B has adjacency matrix $A = \tilde{B}$.

A graph G is connected if and only if its adjacency matrix $A(G)$ is irreducible [3, p.55]. Thus, a bipartite graph with biadjacency matrix B is connected if and only if its adjacency matrix $\tilde{B}$ is irreducible.

Using biadjacency matrices B, the final statement in Lemma 2.5 implies the existence of nonnegative integer matrices with positive entries at preassigned locations and with row and column sums proportional to given positive integers $p_i, i \in [m]$ and $q_j, j \in [n]$.

Theorem 6.3. *Let B be an $m \times n$ $(0,1)$-matrix and let $p_i, i \in [m]$ and $q_j, j \in [n]$ be positive integers. Then*

$$\sum_{j=1}^{n} q_j = \sum_{i=1}^{m} p_i \quad \text{and} \quad \sum_{j \in N(I)} q_j > \sum_{i \in I} p_i \text{ for all proper nonempty subsets } I \subsetneq [m], \tag{6.1}$$

if and only if $\tilde{B}$ is irreducible and, for some $r > 0$, the 1's of B may be replaced by positive integers so that for each $i \in [m]$ and $j \in [n]$, row i has sum rp_i, and column j has sum rq_j.

Proof. By Lemma 2.5, given a positive entry of B, condition (6.1) guarantees the existence of a nonnegative integer matrix that is positive on the chosen entry and that has prescribed row and column sums p_i and q_j. Adding these matrices gives the proportional result. □

In particular, Theorem 6.3 implies the following result on the existence of matrices with constant row sums p and constant column sums q.

Corollary 6.4. *Let B be an $m \times n$ $(0,1)$-matrix. Then B is semiregularizable and $\tilde{B}$ is irreducible if and only if $mp = nq$ and the sum of every k rows of B, $1 \leq k < m$, has more than $\frac{pk}{q}$ nonzero entries.*

Let B be an $m \times n$ $(0,1)$-matrix with $2 \leq m \leq n$. As in Fenner and Loizou [5, p.218], B is called *fully indecomposable* if there are no permutation matrices P and Q, $m \times m$ and $n \times n$, respectively, such that

$$PBQ = \begin{bmatrix} L & M \\ O & N \end{bmatrix} \tag{6.2}$$

where L is a square submatrix, and both L and the bottom left-hand zero submatrix each have at least one row and one column. Thus, B is fully indecomposable if and only if the sum of every k rows of B, $1 \leq k < m$, has more than $k + n - m$ nonzero entries. Equivalently [5, p.223], B is fully indecomposable if and only if its associated bipartite graph $G(X, Y)$ satisfies the condition

$$|N(S)| > |S| + |Y| - |X| \text{ for all proper nonempty subsets } S \subsetneq X. \tag{6.3}$$

The notion of full indecomposability is usually presented for square matrices only. Results for $m \times m$ matrices may be extended to $m \times n$ matrices with $m \leq n$ using the fact that an $m \times n$ $(0,1)$-matrix is fully indecomposable if and only if each of its $m \times m$ submatrices is fully indecomposable

Lemma 6.5. *Let B be an $m \times n$ $(0,1)$-matrix with $2 \leq m \leq n$. If B is fully indecomposable then B is semiregularizable and $\tilde{B}$ is irreducible. The converse statement holds if $m = n$.*

Proof. If $G(X,Y)$ is the bipartite graph with biadjacency matrix B, then for all proper nonempty subsets $S \subsetneq X$,

$$|X||N(S)| > |X||S| + |X|(|Y| - |X|) \geq |Y||S|.$$

By Theorem 3.3, $G(X,Y)$ is connected and srz. Thus, $\tilde{B}$ is irreducible and B is srz.

Conversely, if $\tilde{B}$ is irreducible and B is srz, then the associated bipartite graph $G(X,Y)$ is connected and, by Theorem 3.3, $|X||N(S)| > |Y||S|$ for all proper nonempty subsets $S \subsetneq X$. If $m = n$, then $|X| = |Y|$ and B is fully indecomposable by (6.3). ◻

7. Regularizable graphs. There are a number of characterizations of regularizable graphs in the literature. Many of them follow immediately from the results in the previous section by taking the matrix B there to be the adjacency matrix A of a graph G.

A collection of vertex disjoint cycles and edges in a graph G is called a 2-*matching.* A 2-matching is *perfect* if it covers all of the vertices of G. The following result is well-known (see also [7, p.]). The nonbipartite graphs here are allowed loops.

Theorem 7.1. *A nonempty graph G is regularizable if and only if each edge of G is in a perfect 2-matching of G.*

Proof. If G is connected and regularizable then, by Lemma 6.1 each 1 of its adjacency matrix A is in an n-diagonal. An n-diagonal in A corresponds to a spanning collection of directed cycles in G, including 1-cycles or loops. The 2-cycles may be regarded as edges. Thus each edge of G is in a spanning subgraph whose components are cycles and single edges. Conversely, to each such collection, associate an edge-weighting for which isolated edges have a weight of 2 and edges on cycles have a weight of 1. Adding these edge-weightings over all edges in G yields a positive regular edge-weighting of G. Thus G is regularizable. ◻

Additional characterizations result if the bipartite and nonbipartite cases are distinguished. As observed earlier, a graph is regularizable if and only if each of its connected components is, so it is sufficient to consider connected graphs. The following characterizations of connected regularizable bipartite graphs and square fully indecomposable matrices are well-known, and are listed for completeness. For these and additional results, see Lovász and Plummer [7, p.122] and Brualdi and Ryser [3, p.112-117].

Theorem 7.2. *The following statements are equivalent for a bipartite graph $G(X,Y)$ with $m \times n$ biadjacency matrix B.*

1. *$G(X,Y)$ is connected and regularizable.*

2. *$G(X, Y)$ is connected and each edge of G is in a perfect matching.*
3. *$|X| = |Y|$ and $|N(S)| > |S|$ for all proper nonempty subsets $S \subsetneq X$.*
4. *$G = K_2$ or $|X| = |Y| \geq 2$ and, for each $x \in X$ and $y \in Y$, $G - x - y$ has a perfect matching.*
5. *Either $B = [1]$ or B is square of order $n \geq 2$ and every $(n-1) \times (n-1)$ submatrix of B has an $(n-1)$-diagonal.*
6. *B is $n \times n$ and fully indecomposable.*
7. *The only line cover of B consists of all rows or all columns.*
8. *X and Y are the only minimum point covers of $G(X, Y)$.*

In [7, p.122] a bipartite graph $G(X, Y)$ is termed *elementary* if it has a perfect matching and the set of edges that are in perfect matchings form a connected graph. In addition to the graphical properties stated in Theorem 7.2, it is shown in [7, p.122] that a bipartite graph $G(X, Y)$ is connected and regularizable if and only if it is elementary.

For nonbipartite graphs (possibly with loops), the following characterizations of regularizability are also known. For these and other characterizations, see [7, p.218]. There, graphs G for which $|V| \geq 2$ and $G - v$ has a perfect 2-matching for each $v \in V$ are termed 2-bicritical.

THEOREM 7.3. *The following statements are equivalent for a nonempty nonbipartite graph G of order n with adjacency matrix A.*

1. *G is connected and regularizable.*
2. *$|N(S)| > |S|$ for all proper nonempty sets S of independent vertices in G.*
3. *$|N(S)| > |S|$ for all proper nonempty subsets S of vertices in G.*
4. *Either G is a single loop or, for all $v \in V$, $G - v$ is connected and regularizable.*
5. *Either G is a single loop or, for all $v \in V$, $G - v$ is connected and has a perfect 2-matching.*
6. *Either $A = [1]$ or every principal $(n-1) \times (n-1)$ submatrix of A has an $(n-1)$-diagonal.*
7. *A is fully indecomposable.*

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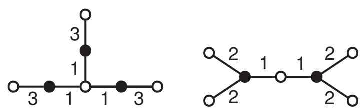
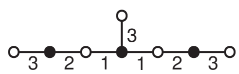
Appendix: Semiregularizable trees of order 11 or less

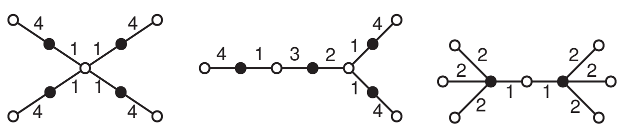
Let $T_{m,n}$ denote a tree on $m+n$ vertices having a proper 2-colouring with m black vertices and n white vertices. If $T_{m,n}$ is (p,q)-srz, it is necessary (but not sufficient) that $mp = nq$. The only srz tree $T_{m,n}$ with $m = n$ is the single edge tree $T_{1,1}$. The only srz trees $T_{m,n}$ on $m+n$ with $m < n$ and $m+n \le 6$ are the stars $K_{1,n}$ and the odd path P_5. Stars and odd paths are always srz. The remaining srz trees with $n > m$ and $7 \le m+n \le 11$ are all shown below. In each case, if $\gcd(p,q) = 1$, then (p,q)-srz trees of the same order can be obtained from one another by a succession of weight-switches as in Figure 4.1.



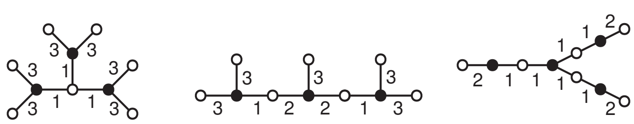
The two srz trees on 7 vertices other than $K_{1,6}$ and P_7.



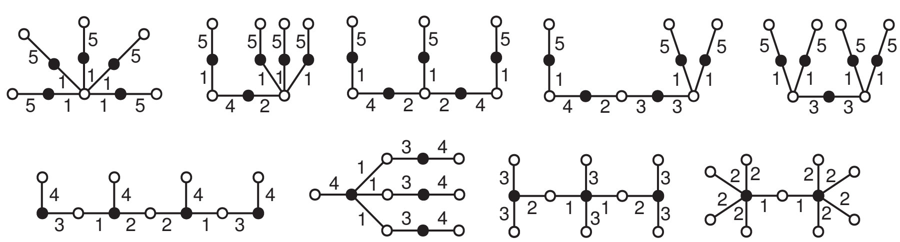
The only srz tree on 8 vertices other than $K_{1,7}$.



The three srz trees on 9 vertices other than $K_{1,8}$ and P_9.



The three srz trees on 10 vertices other than $K_{1,9}$.



The nine srz trees on 11 vertices other than $K_{1,10}$ and P_{11}.

FIG. 7.1. *The srz trees $T_{m,n}$ with $m < n$ and $m+n \le 11$ that are not stars or odd paths.*