Controllability of a hovercraft model
(and two general results)
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Abstract

Modelling and controllability studies of a hovercraft system are undertaken. The system studied is a little more complicated than some in the literature in that the inertial dynamics of the thrust fan are taken into account. The system is shown to be representative of a large class of systems that are controllable only a set described by the zeros of a nontrivial analytic function. Recent results for controllability using vector-valued quadratic forms are useful in arriving at the stated conclusions. As part of the development, two new controllability results of independent interest are proved.

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1. Introduction

In various papers [e.g., Bullo, Leonard, and Lewis 2000, Bullo and Lynch 2001, Lewis and Murray 1997, Manikonda and Krishnaprasad 1996] the control theory for a simplified hovercraft model is undertaken. The model consists in these papers of a planar rigid body moving on a flat frictionless surface propelled by a variable-direction thruster that applies a force to the body at a point distant from its centre of mass. Motion planning algorithms suggested by the work of Bullo and Lynch [2001] have been implemented on a physical hovercraft at Queen’s University. One of the effects not accounted for in the simplified model, but which have a noticeable effect on the physical system, is the inertial dynamics of the fan. Maneouvres that require rapid changes of direction of the thrust fan are seen to cause significant deviations in the trajectory of the hovercraft. In this paper we model these dynamics, and illustrate how they affect the controllability analysis for the system. As we cause significant deviations in the trajectory of the hovercraft.

In this paper we model these inertial dynamics of the thrust fan are taken into account. The system is shown to be representative of a large class of systems that are controllable only a set described by the zeros of a nontrivial analytic function. Recent results for controllability using vector-valued quadratic forms are useful in arriving at the stated conclusions. As part of the development, two new controllability results of independent interest are proved.

However, it interestingly turns out that the controllability of our hovercraft model cannot be decided by using the results of Bullo and Lewis [2003] alone. Therefore, in Section 4, when we analyse the controllability of the hovercraft system, we prove two general results that characterise the controllability of a class of systems, of which the hovercraft is one. With our general results we show that the hovercraft model is only controllable from a subset of configuration space described by the intersection of the zero set of a finite collection of nontrivial analytic functions. That is to say, the system is almost never controllable.

2. Modelling

In this section we present the model for the system we study, and contrast it with the simpler model initially studied by Lewis and Murray [1997]. We adopt a modification of the approach of the approach of Bullo, Leonard, and Lewis [2000] who consider invariant systems on Lie groups. While the simpler model of Lewis and Murray [1997] is an invariant system on a Lie group, the more sophisticated model we present here is not. It is, however, an invariant system on a trivial principal fibre bundle, as considered by Cortés, Martínez, Ostrowski, and Zhang [2002]. A Hamiltonian setting for this sort of problem is considered by Manikonda and Krishnaprasad [1997].

2.1. The physics. The system we study is shown in Figure 1. We have a planar rigid body moving in a plane orthogonal to the direction of gravity. Sitting atop the rigid body is a fan which may be rotated via the torque \(\tau\), and which provides a thrust \(F\). The frame \(\{e_1, e_2\}\) is inertial, and we affix to the centre of mass of the body a frame \(\{f_1, f_2\}\), choosing the \(f_1\)-axis so that along it lies the point of application of the thrust force. The frame \(\{g_1, g_2\}\) is affixed to the centre of mass of the thrust fan, which we assume to coincide with its point of rotation. We assume that the direction of the thrust force is along \(g_1\). Let us describe the configuration space by locating the frames \(\{f_1, f_2\}\) and \(\{g_1, g_2\}\) relative to the inertial frame. The frame \(\{f_1, f_2\}\) is specified relative to \(\{e_1, e_2\}\) by an element of \(SE(2)\). The frame \(\{g_1, g_2\}\) is uniquely determined by its relative orientation to \(\{f_1, f_2\}\), i.e., by an element of \(SO(2)\). Thus \(Q = SE(2) \times SO(2)\). As coordinates we take \((x, y, \theta, \phi)\) where \((x, y)\) is the location of the origin of \(\{f_1, f_2\}\) relative to \(\{e_1, e_2\}\), \(\theta\) is prescribed by demanding that

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

have columns representing the components of \(\{f_1, f_2\}\) in the basis \(\{e_1, e_2\}\).
and where $\phi$ is prescribed by demanding that \[
\begin{pmatrix}
\cos \phi - \sin \phi \\
\sin \phi \\ 
\cos \phi
\end{pmatrix}
\] have columns representing the components of $(g_1, g_2)$ in the basis $\{f_1, f_2\}$.

Now let us define the kinetic energy Lagrangian for the system. We denote the mass of the body and of the fan by $m_{\text{body}}$ and $m_{\text{fan}}$ respectively, and we denote by $J_{\text{body}}$ and $J_{\text{fan}}$ the moment of inertia of the body about its centre of mass and the moment of inertia of the fan about its axis of rotation, respectively. With this notation the kinetic energy in the given coordinates is

$$\text{KE} = \begin{bmatrix} v_x & v_y & v_\theta & v_\phi \end{bmatrix} \begin{bmatrix}
m_{\text{body}} & 0 & m_{\text{fan}} h \sin \theta & 0 \\
0 & m_{\text{body}} + m_{\text{fan}} & -m_{\text{fan}} h \cos \theta & 0 \\
m_{\text{fan}} h \sin \theta & -m_{\text{fan}} h \cos \theta & J_{\text{body}} + J_{\text{fan}} + m_{\text{fan}} h^2 & J_{\text{fan}} \\
0 & 0 & J_{\text{fan}} & J_{\text{fan}} \end{bmatrix} \begin{bmatrix} v_x \\
v_y \\
v_\theta \\
v_\phi \end{bmatrix}.$$  

(2.1)

Now we define in the given set of coordinates the two forces acting on the system. We denote by $F^1$ the thrust force and by $F^2$ the torque rotating the fan. Elementary considerations give $F^1 = (\cos(\theta + \phi) \sin(\theta + \phi), -h \sin \phi)$ and $F^2 = (0, 0, 0, 1)$.

2.2. The mathematics. Now let us put the above physical model into a mathematical framework. First we note that $Q = SE(2) \times SO(2)$ is the total space of a principal fibre bundle with structure group $G = SE(2)$ with the left group action $\Phi: (a_1, (a_2, \phi)) \rightarrow (a_1 a_2, \phi)$. If we denote points in $G$ as $(x, y, \theta, \phi)$ in correspondence to our coordinates above, multiplication takes the form

$$a_1 \cdot a_2 = (x_1 y_2, y_1 \theta_2 - y_2 \sin \theta_1, y_1 + y_2 \cos \theta_1, x_1 y_2 \theta_2) + (x_2, y_2, \theta_2).$$

Below we will interchange $a$ and $(x, y, \theta, \phi)$ as is convenient. We denote by $\gamma: \mathbb{R} \rightarrow SO(2)$ the projection onto the second factor, which corresponds in a natural way to the projection onto the set of orbits for the group action.

Let us now indicate the manner in which the our system is a $G$-invariant system on this principal bundle. Corresponding to the kinetic energy determined above there is a Riemannian metric $g$ with the property that $\text{KE}(v_\alpha) = \frac{1}{2}g(v_\alpha, v_\alpha)$.

2.1 LEMMA: The Riemannian metric $g$ is $G$-invariant.

Proof: To check this in coordinates, one fixes $\tilde{a} = (x, y, \theta) \in SE(2)$, then computes the Jacobian of the map $\Phi_2: (a, \phi) \rightarrow (\tilde{a}, \phi)$. Let us denote this Jacobian evaluated at $(a, \phi)$ by $D\Phi_2(a, \phi)$. Let us also denote the components of $g$ in our coordinates $(a, \phi)$ by $[g(a, \phi)]$. Thus $[g(a, \phi)]$ is the matrix appearing in (2.1). With this notation, $G$-invariance is given by the condition

$$D\Phi_2^T(a, \phi) [g(\tilde{a}, \phi)] D\Phi_2(a, \phi) = [g(a, \phi)].$$

This can now be directly checked.

Recall that a $G$-invariant Riemannian metric gives the principal bundle $\pi: Q \rightarrow SO(2)$ a connection. Indeed, if we let $VQ = ker(\pi^\gamma)$ be the subbundle of tangent spaces of the $G$-orbits, then $HQ = VQ^\perp$ (orthogonal complement taken relative to $g$) is a $G$-invariant subbundle complementary to $VQ$, i.e., a principal connection. This is the mechanical connection [Marsden and Ratiu 1999].

The forces are represented as one-forms on $Q$, and are thus given by

$$F^1 = \cos(\theta + \phi) dx + \sin(\theta + \phi) dy - h \sin \phi d\theta, \quad F^2 = 0.$$ 

These one-forms are also $G$-invariant.

2.2 LEMMA: The one-forms $F^1$ and $F^2$ are $G$-invariant.

Proof: We use the notation $D\Phi_2(a, \phi)$ from the proof of Lemma 2.1. Here, if $F^a(a, \phi), a \in \{1, 2\}$, denotes the vector of components of $F^a$ in our coordinate system, we should check that

$$D\Phi_2(a, \phi)^T [F(\tilde{a}, \phi)] = [F(a, \phi)].$$

This can be directly checked.

Therefore, from the preceding two lemmas we know that if we define vector fields $Y_a = g^T(F^a), a \in \{1, 2\}$, on $Q$, these vector fields will be $G$-invariant. We also know that the Levi-Civita connection $\hat{\nabla}$ for $g$ will be $G$-invariant. The system is then an affine connection control system $\Sigma_{\text{aff}} = (Q, \hat{\nabla}, \mathcal{X}, U)$ corresponding to the system in Figure 1 is a $G$-invariant system on the principal fibre bundle $\pi: Q = SE(2) \times SO(2) \rightarrow SO(2)$.

This structure leads to an interesting consequence concerning the input vector field $Y_2$. Note that the one-form $F^2$ annihilates the vertical bundle $VQ$. Therefore, the vector field $Y_2$ is $g$-orthogonal to $VQ$. This, along with $G$-invariance of $Y_2$, implies that $Y_2$ is the horizontal lift of a vector field $Y_2$ on $SO(2)$. Indeed, one verifies that $Y_2 = \frac{\partial}{\partial \theta}$. This observation turns out to yield the following.

2.4 LEMMA: The vector field $\nabla_{Y_2} Y_2$ is horizontal.

Proof: One can simply check this directly (as we shall do in Section 4). However, let us indicate how this comes up in a more general setting for readers familiar with the symmetry picture in mechanics. The bundle $HQ$ can be realised as the zero level set for the momentum map $J: HQ \rightarrow g^\ast$ defined by

$$J(q, \xi) = g(\nu_q, \xi q),$$

where $\nu_q$ denotes the infinitesimal generator associated with $\xi \in g$. This means that the subbundle $HQ$ is invariant under the unforced dynamics as a consequence of (actually, the definition of) conservation of momentum. Thus $HQ \subset TQ$ is invariant under the geodesic flow. As shown by Lewis [1998], this means that $HQ$ has the property that $\nabla_{Xq} X$ is a section of $HQ$ for any vector field $X$ that is a section of $HQ$. From this the result follows.
We shall see in Section 4 that the lemma has an immediate interpretation in terms of the decoupling vector fields of Bullo and Lynch [2001].

2.5 Remark: The simpler hovercraft model considered in the papers [Bullo, Leonard, and Lewis 2000; Bullo and Lynch 2001, Lewis and Murray 1997] does not have a degree of freedom associated to the fan. The input force for the system is a force $F$ applied directly at the position of the fan. Thus it is a system with a three-dimensional configuration space and two inputs. Bullo, Leonard, and Lewis [2000] show that the system is a left-invariant system on $SE(2)$. \hfill \Box

3. A review of general controllability results

In the next section we investigate the controllability properties of our hovercraft model. As we shall see, the structure is sufficiently complicated that we need to use the approach recently laid out by Bullo and Lewis [2003] (see also [Hirschorn and Lewis 2001, 2002]). We also consider our system in the context of the work of Bullo and Lynch [2001], inasmuch as this is possible. In this section we provide a review of the necessary machinery.

3.1 Definition: Let $\Sigma = (Q, \nabla, \mathcal{F} = \{Y_1, \ldots, Y_m\}, U)$ where $Q$ is the configuration manifold, $\nabla$ is an affine connection on $Q$, $\mathcal{F}$ are vector fields on $Q$, and $U \subseteq \mathbb{R}^n$. The control set $U$ is almost proper if $\text{aff}(U) = \mathbb{R}^n$ and if $0 \not\in \text{conv}(U)$. $U$ is proper if $0 \in \text{conv}(U)$. Here $\text{aff}(U)$ is the affine hull of $U$ and $\text{conv}(U)$ is the convex hull of $U$. Bullo and Lewis [2003] consider a slightly more general setup than this, but what we say here is all that is necessary. The control equations are then

$$\nabla_{\gamma(t)} \gamma(t) = \sum_{a=1}^m u_a(t) Y_a(\gamma(t)). \tag{3.1}$$

A controlled trajectory for $\Sigma$ is a pair $(\gamma, u)$ where $u : [0, T] \to U$ is measurable and $\gamma : [0, T] \to Q$ satisfies (3.1).

For $T > 0$ denote

$$\mathcal{R}_{TQ}(q_0, T) = \{ \gamma(T) \mid (\gamma, u) \text{ is a controlled trajectory on } [0, T] \text{ with } \gamma(0) = q_0 \},$$

where $q_0$ denotes the zero vector in $T_{q_0}Q$, and denote $\mathcal{R}_{TQ}^0(q_0, t) = \bigcup_{0 \leq s \leq T} \mathcal{R}_{TQ}(q_0, t)$. These are therefore reachable states in $TQ$ starting from zero initial velocity at the configuration $q_0$. We also consider the reachable configurations which we denote

$$\mathcal{R}_{TQ}^0(q_0, T) = \tau_Q(\mathcal{R}_{TQ}^0(q_0, T)), \quad \mathcal{R}_{TQ}^0(q_0, t) = \tau_Q(\mathcal{R}_{TQ}^0(q_0, t)),$$

where $\tau_Q : TQ \to Q$ is the tangent bundle projection. We may now state the versions of controllability that are of interest to us in this case. More refined notions are considered by Bullo and Lewis [2003].

3.3. General controllability results. We now provide conditions for accessibility of affine connection control systems as given by Lewis and Murray [1997]. We let $\Sigma = (Q, \nabla, \mathcal{F}, U)$ be an affine connection control system and define the associated symmetric product by

$$(X : Y) = \nabla_X Y + \nabla_Y X.$$
3.3 Definition: $B_{\mathbf{q}_0}$ is
(i) \textit{definite} if there exists $\lambda \in \text{ann}(\mathbf{q}_0)$ so that $\lambda B_{\mathbf{q}_0}$ is positive-definite;
(ii) \textit{semidefinite} if there exists $\lambda \in \text{ann}(\mathbf{q}_0)$ so that $\lambda B_{\mathbf{q}_0}$ is strictly positive-semidefinite (i.e., positive-semidefinite and nonzero);
(iii) \textit{indefinite} if for each $\lambda \in \text{ann}(\mathbf{q}_0)$, $\lambda B_{\mathbf{q}_0}$ is neither positive nor negative-semidefinite.

A simplified version of the results of Bullo and Lewis [2003] are the following.

3.4 Theorem: Let $\Sigma = (Q, \nabla, \mathcal{F}, U)$ be an analytic affine connection control system. The following statements hold:
(i) if
(a) $U$ is proper,
(b) $\Sigma^{(\infty)}(\mathbf{y})_{\mathbf{y}_0} = \Sigma^{(2)}(\mathbf{y})_{\mathbf{y}_0}$, and if
(c) $B_{\mathbf{q}_0}$ is indefinite
then $\Sigma$ is STL from $\mathbf{q}_0$ if it is accessible from $\mathbf{q}_0$ and is STLCC from $\mathbf{q}_0$ if it is configuration accessible from $\mathbf{q}_0$;
(ii) if
(a) $U$ is compact,
(b) $\mathbf{q}_0$ is a regular point for $Y$, and
(c) $B_{\mathbf{q}_0}$ is definite
then $\Sigma$ is not STLCC from $\mathbf{q}_0$.

3.5 Remark: Theorem 3.4 is a “first-order” result, meaning that its hypotheses require knowledge of the first derivatives of the vector fields $\mathcal{F}$. The case where $B_{\mathbf{q}_0}$ is semidefinite will generally be the boundary situation where one will have to use higher-order conditions to determine controllability. However, as we shall see in Theorem 4.3, for some systems one can determine controllability in the semidefinite case without resorting to higher-order conditions. Indeed, our new result stated as Theorem 4.2 is a generalisation of the second part of Theorem 3.4, and allows $B_{\mathbf{q}_0}$ to be semidefinite. \hfill $\Box$

3.4. Kinematic reductions. Let $\Sigma = (Q, \nabla, \mathcal{F}, U) = \mathbb{R}^m)$ be an affine connection control system. Bullo and Lynch [2001] consider a class of vector fields with interesting properties.

For an interval $[0, T]$ a \textit{reparameterisation} is a differentiable map $\tau: [0, \tilde{T}] \rightarrow [0, T]$ with the properties
1. $\tau(0) = 0$,
2. $\tau(\tilde{T}) = T$, and
3. $\tau'(t) > 0$ for $t \in ]0, \tilde{T}[$.

The following definition combines a definition of Bullo and Lynch [2001] and its generalisation by Bullo and Lewis [2003].

3.6 Definition: Let $\Sigma = (Q, \nabla, \mathcal{F}, U) = \mathbb{R}^m$ be an affine connection control system.

(i) A vector field $X$ on $Q$ is a \textit{decoupling vector field} if for every integral curve $\gamma: [0, T] \rightarrow Q$ for $X$ and for every reparameterisation $\tau: [0, \tilde{T}] \rightarrow [0, T]$, there exists a control $u: [0, \tilde{T}] \rightarrow \mathbb{R}^m$ such that $(\gamma \circ \tau, u)$ is a controlled trajectory for $\Sigma$.
(ii) A constant rank distribution $X$ is a \textit{kinematic reduction} of $\Sigma$ if every $X$-valued vector field is a decoupling vector field.
(iii) $\Sigma$ is \textit{kinematically controllable} if there exists kinematic reductions $X_1, \ldots, X_k$ for $\Sigma$ with $\text{Lie}^{(\infty)}(X_1 + \cdots + X_k) = TQ$. \hfill $\Box$

To provide connections between controllability and kinematic controllability, we extend the definition of the vector-valued quadratic form $B_{\mathbf{q}_0}$ to all of $Q$. Thus we define a map $B_{\mathbf{y}}: Y \times Y \rightarrow TQ/Y$ by $B_{\mathbf{y}}(X, Y) = B_{\mathbf{q}_0}(X(q), Y(q))$ for $Y$-valued vector fields $X$ and $Y$. The notation $Q_{\mathbf{y}_0}$ and $\lambda B_{\mathbf{y}_0}$ can then be defined in the obvious manner. Kinematic reductions are then interestingly characterised by the following result of Bullo and Lewis [2003].

3.7 Theorem: Let $\Sigma = (Q, \nabla, \mathcal{F} = \{Y_1, \ldots, Y_n\}, \mathbb{R}^m)$ be an affine connection control system with $Y$ of constant rank and let $X$ be a distribution of constant rank. The following statements are equivalent:
(i) $X$ is a kinematic reduction of $\Sigma$;
(ii) $\Sigma^{(i)}(X) \subset Y$;
(iii) $X \subset Y$ and $Q_{\mathbf{y}_0}|X = 0$.

4. Controllability for the hovercraft model

We now apply the results reviewed in Section 3 to the system of Section 2. We first characterise the accessibility of the system, since this is straightforward. Next we turn to controllability. To give a little context to the controllability of the hovercraft, we first prove two general controllability results for affine connection control systems. One result extends Theorem 3.4 to a situation where $B_{\mathbf{q}_0}$ is semidefinite. The other result is a structural result for two-input systems with one decoupling vector field. This is interesting application of the vector-valued quadratic form technology. Then we look at the controllability of the hovercraft. As we see, the system $\Sigma_{m\tau}$ has quite a complicated structure as concerns its controllability. To summarise, we shall show that the system is accessible from all configurations and that the only configurations from which the system is STLCC are those shown in Figure 2, and any $SE(2)$-translation of the configurations shown.

4.1. Accessibility of the hovercraft model. Let us first provide some explicit expressions for the vector fields $Y_1$, $Y_2$, and some of their symmetric products so that we can see how the analysis might proceed. The $SE(2)$-invariance of the system is useful here since we may without loss of generality evaluate all symmetric products at $(x, y, \theta, \phi) = (0, 0, 0, 0)$, essentially meaning we evaluate at the group identity. Thus in the expressions immediately below this simplification is tacitly made. We introduce the constants

$$C_1 = m_{\text{body}}m_{\text{fan}}h^2 + J_{\text{body}}(m_{\text{body}} + m_{\text{fan}}), \quad M = m_{\text{body}} + m_{\text{fan}}.$$
and compute

\[
Y_1 = \cos \phi \frac{\partial}{\partial x} + \frac{J_{\text{body}} \sin \phi}{C_1} \frac{\partial}{\partial y} + \frac{m_{\text{body}} h \sin \phi}{C_1} \frac{\partial}{\partial \theta} + \frac{m_{\text{body}} h}{C_1} \frac{\partial}{\partial \psi}
\]

\[
Y_2 = -\frac{m_{\text{fan}} h}{C_1} \frac{\partial}{\partial y} - \frac{M}{C_1} \frac{\partial}{\partial \theta} + \frac{J_{\text{fan}} C_1}{C_1} \frac{\partial}{\partial \psi}
\]

\[
(Y_1 : Y_1) = -\frac{m_{\text{body}}^2 m_{\text{fan}} h^4 \sin(2\phi)}{M C_1} \frac{\partial}{\partial y} + \frac{m_{\text{body}} h^2 \sin(2\phi)}{C_1} \frac{\partial}{\partial \theta} + \frac{m_{\text{body}} h^2 \sin(2\phi)}{C_1} \frac{\partial}{\partial \psi}
\]

\[
(Y_1 : Y_2) = -\frac{\sin \phi}{J_{\text{fan}} M C_1} + \frac{(J_{\text{body}} C_1 - J_{\text{fan}} m_{\text{body}} m_{\text{fan}} h^2) \sin \phi}{C_1} \frac{\partial}{\partial y} - \frac{m_{\text{body}} h^2 (C_1 + J_{\text{fan}} M) \cos \phi}{J_{\text{fan}} C_1} \frac{\partial}{\partial \theta} + \frac{m_{\text{body}} h^2 (C_1 + J_{\text{fan}} M) \cos \phi}{J_{\text{fan}} C_1} \frac{\partial}{\partial \psi}
\]

\[
(Y_2 : Y_2) = 0
\]

\[
(Y_1 : Y_2) = -\frac{m_{\text{body}} m_{\text{fan}} h^3 (C_1 + J_{\text{fan}} M) \cos(2\phi)}{J_{\text{fan}} M C_1^2} \frac{\partial}{\partial y} - \frac{m_{\text{body}}^2 h^2 (C_1 + J_{\text{fan}} M) \cos(2\phi)}{J_{\text{fan}} M C_1^2} \frac{\partial}{\partial \theta} + \frac{m_{\text{body}}^2 h^2 (C_1 + J_{\text{fan}} M) \cos(2\phi)}{J_{\text{fan}} M C_1^2} \frac{\partial}{\partial \psi}
\]

\[
(Y_2 : Y_2) = -\frac{\cos \phi}{J_{\text{fan}} M} \frac{\partial}{\partial x}
\]

With these computations we have the following result.

4.1 Proposition: The affine connection control system $\Sigma_{\text{fan}}$ defined in Section 2.2 is inaccessible from every $q_0 \in Q$.

Proof: Provided that $\cos(2\phi) \neq 0$ one can check that the vector fields \{Y_1, Y_2, (Y_1 : Y_2), (Y_1 : Y_1)\} are linearly independent at q_0, and provided that $\sin \phi \neq 0$ one can show that the vector fields \{Y_1, Y_2, (Y_1 : Y_2), (Y_1 : Y_1)\} are linearly independent at q_0. Since the set of points where both $\cos(2\phi)$ and $\sin \phi$ vanish is empty, the result follows.

4.2 Two new controllability results. To investigate the controllability of our hovercraft model it is interesting to state a slightly more general result that characterizes systems like $\Sigma_{\text{fan}}$. Part of this is the following general result which gives a version of the necessary condition of Theorem 3.4 when the quadratic form is only semidefinite.

4.2 Theorem: Let $\Sigma = (Q, \nabla, \mathcal{F} = \{Y_1, \ldots, Y_m\}, U)$ be an affine connection control system and let $q_0 \in Q$ have the following properties:

(i) $q_0$ is a regular point for $Y$;
(ii) the distribution $Y_1$ generated by the vector fields \{Y_1, \ldots, Y_n\} has $q_0$ as a regular point;
(iii) the distribution $Y_2$ generated by the vector fields \{Y_{k+1}, \ldots, Y_m\} is a kinematic reduction for $\Sigma$;
(iv) with $Y_1$ and $Y_2$ as above, $Y = Y_1 \oplus Y_2$;
(v) $U$ is compact;
(vi) $B_{Y_1, Y_2, q_0}$ is definite.

Then $\Sigma$ is not STLCC from $q_0$.

Proof: We work locally. Therefore we may assume that the sets of vector fields \{Y_1, \ldots, Y_n\} and \{Y_{k+1}, \ldots, Y_m\} are linearly independent in a neighbourhood of $q_0$. First we show that the system is not weakly STLCC from $q_0$ using calculations of Hirschorn and Lewis [2002]. We will not provide here a self-contained justification for all of our computations since they take considerable space, but we refer to the paper [Hirschorn and Lewis 2002]. The calculation uses the Chen-Fliss-Sussmann series [Chen 1957, Fliss 1981, Sussmann 1983]. For an analytic control-affine system

\[
\xi(t) = f_0(\xi(t)) + \sum_{a=1}^{m} u_a(t) f_a(\xi(t)), \quad \xi(t) \in M,
\]

on a manifold $M$ with a compact control set, and for an analytic function $\phi$, the Chen-Fliss-Sussmann series gives the following formula for the value of $\phi$ along a controlled

\[
\text{Our use of $\phi$ for something other than the fan angle will be restricted to this proof.}
\]
trajectory $(\xi, u)$:
\[
\phi(t) = \sum (\int_0^t u) f(t) \phi(t)\]
The sum is over multi-indices $f = (a_1, \ldots, a_n)$ in $\{0,1, \ldots, m\}$,
\[
\int_0^t u f = \int_0^t \int_0^t \cdots \int_0^t u a_1(\tau_1) \cdots u a_n(\tau_n) d\tau_1 d\tau_2 \cdots d\tau_n,
\]
and
\[
f(t) \phi = f a_1 \cdots f a_n \phi.
\]
We adopt the convention that $u_0 = 1$. If we regard (3.1) as a control-affine system, this means taking $M = TV, f_0 = Z$ (the geodesic spray for $\nabla$), and $f_a = \text{vfl}(Y_a)$ (the vertical lift). We refer to [Lewis and Murray 1997] for details on how affine connection control systems are control-affine systems.

The function we evaluate is defined as follows. We let $\lambda$ be an analytic one-form defined in a neighbourhood of $q_0$ with the following properties:
1. $\lambda$ annihilates the distribution $Y$;
2. $\lambda(q_0) B_{\lambda q_0} | \Sigma_0 = 0$ is negative-definite.

By a linear input transformation one can ensure that the input vector fields diagonalise $\lambda(q_0) B_{\lambda q_0}$, with the nonzero diagonal entries (i.e., those corresponding to $(a_1, \ldots, a_n)$) being $-1$. We assume this input transformation to have been made. We then define a function $\phi_q$ on $TQ$ by $\phi_q(v_q) = \lambda(q) \cdot v_q$, and we also define
\[
\Phi^\perp_q = \{ v_q \in TQ | \phi_q(v_q) > 0 \}, \quad \Phi^\parallel_q = \{ v_q \in TQ | \phi_q(v_q) < 0 \}.
\]
Note that in any neighbourhood $V$ of $\Sigma_0$ in $Q$ the sets $V \cap \Phi^\perp_q$ and $V \cap \Phi^\parallel_q$ will be nonempty since $\phi_q$ is linear on the fibres of $TQ$. Therefore, we can show that $\Sigma$ is not STLCC from $q_0$ by showing that $\phi_q$ has constant sign along any controlled trajectory of $\Sigma$. One may directly verify that $\phi_q$ has the following properties:
1. $f_a \phi_q$, $a \in \{1, \ldots, m\}$, is zero in a neighbourhood of $q_0$;
2. $\text{ad}_{f_a} f_b \phi_q(0_q) = 0$, $a \in \{1, \ldots, m\}$, $k \in \mathbb{N}$;
3. $[f_a, f_k, f_b] \phi_q(0_q) = -1$, $a \in \{1, \ldots, k\}$ (this and the next two facts use the formula $[f_a, f_b] \phi_q = \text{vfl}(Y_a \cdot Y_b)$);
4. $f_a \phi_q(f_b \phi_q(0_q)) = 0$, $b \in \{k + 1, \ldots, m\}$;
5. $f_a \phi_q(0_q) = 0$, $a, b \in \{1, \ldots, m\}, a \neq b$.

For an input $u : [0, T] \rightarrow U$ let us define
\[
\|u\|_{L^2}^2 = \max_{a \in \{1, \ldots, m\}} \left( \int_0^T |u_a(t)|^2 \right)^{1/2}.
\]

The calculations of Hirschorn and Lewis [2002] now immediately give the following inequality for $\phi_q(\gamma(t))$ along a controlled trajectory $(\gamma, u)$ for an affine connection control system like that under consideration here:
\[
\phi_q(\gamma(t)) \geq \frac{1}{2} (\|u\|_{L^2}^2 - |E(t)|).
\]
Here $E(t)$ is a function of $t$ that Hirschorn and Lewis [2002] show to satisfy a bound
\[
|E(t)| \leq t E_{01}(\|u\|_{L^2}^2)^2
\]
for some $E_{01} > 0$. For $t$ sufficiently small this shows that $\phi_q(\gamma(t))$ has constant sign, provided that $u_1, \ldots, u_k$ are not zero a.e. If $u_1, \ldots, u_k$ are zero a.e., then since $Y_{k+1}, \ldots, Y_m$ are decoupling, it follows that $\phi_q(\gamma(t)) = 0$ along the corresponding controlled trajectory. This shows that $\Sigma$ is not STLCC from $q_0$.

Now let us show that our above construction also precludes $\Sigma$ from being STLCC. Choose a coordinate chart $(\Lambda, \chi)$ for $Q$ around $q_0$ with the following properties: (1) $\chi(q_0) = 0$ and (2) $\text{d}q^a = \lambda(q_0)$. Let us define a function $\psi_q$ on the coordinate domain $U$ by $\psi_q(q) = q^a$, such that
\[
\Psi^\parallel_q = \{ q \in Q | \psi_q(q) > 0 \}, \quad \Psi^\perp_q = \{ q \in Q | \psi_q(q) < 0 \}
\]
each intersect any neighbourhood of $q_0 \in Q$. Along any nonstationary trajectory $t \rightarrow \gamma(t)$ we have
\[
\frac{\text{d} \phi_q(\gamma(t))}{\text{d}t} \bigg|_{t=0} = \phi_q(\gamma(t)) \phi_q(0_q) = \phi_q^0(t) < 0,
\]
Since $\phi_q(0_q) = 0$, this means that for sufficiently small $t$, $\phi_q(\gamma(t)) < 0$ and this shows that the points in $\Psi^\parallel_q$ are not reachable in small time, and so $\Sigma$ is not STLCC.

Now we use the preceding result to characterise a class of systems, of which our hovercraft model is one. It is helpful to have some terminology. Let us say that $\Sigma$ is weakly STLCC from $q_0$ if $q_0$ is in the interior of $\mathbb{R}^2_{\mathbb{Z}}(0_q \leq T)$, relative to the orbit topology on the integral manifold of $\Sigma_0$, of the accessibility distribution. Thus weakly controllable means, roughly, that the system is “as controllable as it can be.” In like manner we have the notion of $\Sigma$ being weakly STLCC. An analytic subset of $Q$ is a subset $S \subset Q$ with the property that for each $q \in Q$ there exists a neighbourhood $N$ of $q$ for which $N \cap S$ is the common zero set of a finite number of analytic functions on $N$.

With this language, we have the following result.

4.3 Theorem: Let $\Sigma = (Q, \nabla, \Psi, U)$ be an affine connection control system for which
(i) $Y$ has constant rank 2,
(ii) there exists a rank 1 kinematic reduction $X$ for $\Sigma$, and
(iii) $U$ is compact.
Then for any $q_0 \in Q$, $B_{\lambda q_0}$ is not definite. Furthermore, we have the following dichotomy:
(v) there exists two rank 1 kinematic reductions $X_1 = X$ and $X_2$; or
(vi) there exists a proper analytic subset $S$ of $Q$ with the property that if $q_0 \in Q$ then $\Sigma$ is weakly STLCC from $q_0$, and if $q_0 \in Q \setminus S$ then $Q$ is not STLCC from $q_0$. 

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To check the definiteness or indefiniteness of that for analytic functions

\[ Y_a : Y_b = \beta_{ab}^1 Y_1 + \beta_{ab}^2 Y_2 + B_{ab}^1 X_1 + \cdots + B_{ab}^{n-2} X_{n-2}, \quad a, b \in \{1, 2\}, \]

for analytic functions \( \beta_{ab}^d, B_{ab}^d, a, b, c \in \{1, 2\}, d \in \{1, \ldots, n-2\} \). Since \( Y_2 \) is a decoupling vector field, the functions \( B_{ab}^d \), \( d \in \{1, \ldots, n-2\} \), are identically zero. We then define \( n-2 \) symmetric matrix functions on \( Q \) by

\[ B^d = \begin{bmatrix} B_{11}^d & B_{12}^d \\ B_{12}^d & B_{22}^d \end{bmatrix}, \quad a \in \{1, \ldots, n-2\}. \]

To check the definiteness or indefiniteness of \( B_Y \), we use the following observations, valid since we are considering \( \dim(Y) = 2 \) and \( Y_2 \) is decoupling:

1. \( B_Y \) is indefinite if and only if for every \( (\lambda_1, \ldots, \lambda_{n-2}) \in \mathbb{R}^{n-2} \), the symmetric matrix \( \lambda_1 B^1(q) + \cdots + \lambda_{n-2} B^{n-2}(q) \) is either identically zero or has strictly negative determinant;

2. \( B_Y \) is definite if and only if there exists \( (\lambda_1, \ldots, \lambda_{n-2}) \in \mathbb{R}^{n-2} \) so that the symmetric matrix \( \lambda_1 B^1(q) + \cdots + \lambda_{n-2} B^{n-2}(q) \) has strictly positive determinant.

Note that

\[ \lambda_1 B^1 + \cdots + \lambda_{n-2} B^{n-2} = \begin{bmatrix} \lambda_1 B_{11}^1 + \cdots + \lambda_{n-2} B_{11}^{n-2} & 0 \\ \lambda_1 B_{12}^1 + \cdots + \lambda_{n-2} B_{12}^{n-2} \\ 0 & 0 \end{bmatrix}, \]

so that

\[ \det(\lambda_1 B^1 + \cdots + \lambda_{n-2} B^{n-2}) = -\lambda_1 (\lambda_1 B_{11}^1 + \cdots + \lambda_{n-2} B_{11}^{n-2})^2. \]

This shows that \( B_Y \) is never definite. One can then check for indefiniteness. To do so it is convenient to introduce two linear maps \( L_{11}(q) \) and \( L_{12}(q) \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \) defined by

\[ L_{11}(q)(\lambda_1, \ldots, \lambda_{n-2}) = \lambda_1 B_{11}^1(q) + \cdots + \lambda_{n-2} B_{11}^{n-2}(q) \]

\[ L_{12}(q)(\lambda_1, \ldots, \lambda_{n-2}) = \lambda_1 B_{12}^1(q) + \cdots + \lambda_{n-2} B_{12}^{n-2}(q). \]

This leads to the following cases and subcases.

1. \( \dim(\ker(L_{11}(q))) = n - 2 \):
   
   (a) \( \dim(\ker(L_{11}(q))) = n - 2 \): \( B_Y \) is indefinite;
   
   (b) \( \dim(\ker(L_{11}(q))) = n - 3 \): \( B_Y \) is semidefinite.

2. \( \dim(\ker(L_{12}(q))) = n - 3 \):
   
   (a) \( \ker(L_{12}(q)) \cap \ker(L_{11}(q)) \neq \{(0, 0)\} \): \( B_Y \) is indefinite;
   
   (b) \( \ker(L_{12}(q)) \cap \ker(L_{11}(q)) = \{(0, 0)\} \): \( B_Y \) is semidefinite.

There are therefore two cases when \( B_Y \) is indefinite.

1. \( L_{12}(q) = L_{11}(q) = 0 \): Since \( L_{11} \) and \( L_{12} \) are analytic functions of \( q \) this will happen on the set of coincident zeros of the analytic functions \( B_{11}^d, B_{12}^d, d \in \{1, \ldots, n-2\} \).

2. \( \dim(\ker(L_{12}(q)) \cap \ker(L_{11}(q))) = 1 \): This will happen on the set of zeros of the analytic functions

\[ f_1(q) = L_{11}(q)(\lambda_1(q), \ldots, \lambda_{n-2}(q)), \]

\[ f_2(q) = L_{12}(q)(\lambda_1(q), \ldots, \lambda_{n-2}(q)), \]

where \( (\lambda_1(q), \ldots, \lambda_{n-2}(q)) \) is a basis for \( \ker(L_{12}(q)) \cap \ker(L_{11}(q)) \).

This proves that we have indefiniteness of \( B_Y \) either for all \( q \in Q \) or for \( q \) lying in a proper analytic subset \( S \) of \( Q \). Furthermore, when \( B_Y \) is indefinite for all \( q \in Q \) we have \( B_{11}^d(q) = 0 \) for all \( q \in Q \) and \( d \in \{1, \ldots, n-2\} \). This means that \( Y_1 \) is decoupling, and so gives the existence of another rank 1 kinematic reduction. It remains to show that when \( B_Y \) is indefinite only on a proper analytic subset \( S \), then from \( q_0 \in Q \setminus S \) the system is not STLCC. This, however, follows from Theorem 4.2.

### 4.4 Remarks:

The basic idea of the theorem is that for systems of the type under consideration, either the system is “nice” (it essentially possesses two decoupling vector fields), or it is “not nice” (it is uncontrollable except on a proper analytic subset). Note that the “nice” case is not the same as the system being kinematically controllable, since one would also have to check that the two decoupling vector fields have a maximal involutive closure. □

**4.3. Controllability of the hovercraft model.** To use the controllability conditions of Theorem 4.3 we need to compute \( B_{Y_{\phi}} \). To do so we need a model for the quotient \( T_{\phi}Q/Y_{\phi} \). It turns out to be convenient to use the \( g \)-orthogonal complement for \( Y_{\phi} \), and one can check that the vector fields

\[ X_1 = \sin \phi \frac{\partial}{\partial x} - \cos \phi \frac{\partial}{\partial y}, \quad X_2 = h \cos \phi \sin \phi \frac{\partial}{\partial x} + h \sin^2 \phi \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta} \]

give a basis for this orthogonal complement at \( q_0 = (0, 0, \phi) \) for every \( \phi \). Pushing these forward by left-translation will then give a global basis for the orthogonal complement to \( Y \), although its value at \((0, 0, \phi)\) is all we need. To compute \( B_{Y_{\phi}} \) using the basis of vector fields \( \{Y_1, Y_2, X_1, X_2\} \), we write

\[ (Y_a : Y_b) = \beta_{ab}^1 Y_1 + \beta_{ab}^2 Y_2 + B_{ab}^1 X_1 + B_{ab}^2 X_2, \quad a, b \in \{1, 2\}, \]

and then the matrices

\[ B^1 = \begin{bmatrix} B_{11}^1 & B_{12}^1 \\ B_{12}^1 & B_{22}^1 \end{bmatrix}, \quad B^2 = \begin{bmatrix} B_{11}^2 & B_{12}^2 \\ B_{12}^2 & B_{22}^2 \end{bmatrix} \]

may then be used to characterise \( B_{Y_{\phi}} \). It turns out that when one does this computation, in all terms appears the positive denominator

\[ C_2 = J_{\tan C_4 M} (J_{\tan M} + C_1 + m_{\text{mod}} B^2 \sin^2 \phi). \]
Pulling this out of all matrices will not change their relationship to the definiteness or indefiniteness of $B_{\varphi_{\omega}}$, so we do this to simplify the analysis. Doing the computations gives

$$B^{3} = \begin{bmatrix} J_{\text{fan}}n_{\text{body}}^{2}m_{\text{fan}}h^{3}\cos\phi\sin(2\phi) & C_{3} \\ C_{3} & 0 \end{bmatrix},$$

$$B^{2} = \begin{bmatrix} -J_{\text{fan}}m_{\text{body}}h^{2}\sin(2\phi) & -m_{\text{body}}M(C_{1} + J_{\text{fan}}M)\cos\phi \\ -m_{\text{body}}M(C_{1} + J_{\text{fan}}M)\cos\phi & 0 \end{bmatrix},$$

where

$$C_{3} = \frac{1}{2}M(2h^{2}m_{\text{body}}J_{\text{fan}} + h^{2}m_{\text{body}}m_{\text{fan}} + J_{\text{body}}^{2}M)\sin^{2}\phi + 2J_{\text{body}}J_{\text{fan}}M + J_{\text{body}}(h^{2}m_{\text{body}}(m_{\text{body}} + 3m_{\text{fan}})).$$

We wish now to apply Theorem 4.3. First we make the following observation.

4.5 Lemma: $Y_{2}$ is a decoupling vector field for $\Sigma_{\varphi_{\omega}}$.

4.6 Remark: We can ascertain that $Y_{2}$ is decoupling since we have computed $\nabla_{Y_{1}}Y_{2} = 0$.

However, this observation is also consistent, and may indeed be derived from, Lemma 2.4. What’s more, Lemma 2.4 gives the nice interpretation of the integral curves of $Y_{2}$ as being zero SE(2)-momentum moves for the system.

Thus we are indeed in the case described by Theorem 4.3. It turns out that we are in case (v) of the theorem, and that it is possible to explicitly describe the analytic subset $S$. The following result summarises this.

4.7 Proposition: Let $q_{0} = (0, 0, 0, \phi)$. The following statements hold:

(i) $Y_{\varphi_{\omega}}$ is STLC from $q_{0}$ if $\sin(2\phi) = 0$;

(ii) $Y_{\varphi_{\omega}}$ is not STLCC from $q_{0}$ if $\sin(2\phi) \neq 0$.

Thus the system is STLC only from the configurations shown in Figure 2.

Proof: We refer to the notation used in the proof of Theorem 4.3. Also, throughout the proof we take points in $Q$ of the form $q = (0, 0, 0, \phi)$, this being without loss of generality by SE(2)-invariance. With $\{Y_{1}, Y_{2}, X_{1}, X_{2}\}$ as described above we note that $\dim(\ker(\Lambda_{2}(q))) = 1$ for all $q \in Q$ since $B_{2} = C_{3} \neq 0$. Now let $(\lambda_{1}(q), \lambda_{2}(q)) = (-B_{2}(q), B_{2}(q))$, and note that $(\lambda_{1}(q), \lambda_{2}(q)) \in \ker(\Lambda_{2}(q))$ for all $q \in Q$. Therefore

$$\lambda_{1}(q)B_{1}(q) + \lambda_{2}(q)B_{2}(q) = \begin{bmatrix} \alpha(q) \\ 0 \\ 0 \end{bmatrix},$$

where we compute

$$\alpha(q) = -h^{2}J_{\text{fan}}n_{\text{body}}^{2}(m_{\text{body}} + m_{\text{fan}})(h^{2}m_{\text{body}}m_{\text{fan}} + J_{\text{body}}(m_{\text{body}} + m_{\text{fan}})) \times (J_{\text{body}}m_{\text{body}} + J_{\text{fan}}m_{\text{fan}} + J_{\text{body}}m_{\text{fan}} + J_{\text{fan}}m_{\text{fan}} + h^{2}m_{\text{body}}m_{\text{fan}} + 2h^{2}m_{\text{body}}\sin^{2}\phi)\sin(2\phi).$$

This function is zero only when $\sin(2\phi) = 0$, directly giving the result via the analysis of Theorem 4.3.

4.8 Remarks: 1. The system has the interesting property of possibly being STLC at a point (points where $\sin(2\phi) = 0$), but of not being STLCC at points in a neighbourhood of the point. A similar, but simpler, example was examined by Shen, Sanjay, and McClamroch [2002].

2. It is interesting to contrast our hovercraft model with the simpler ones considered in, for example, [Bullo, Leonard, and Lewis 2000, Bullo and Lynch 2001, Lewis and Murray 1997], and is furthermore kinematically controllable [Bullo and Lynch 2001]. The addition of inertial properties to the fan destroys these controllability properties.

References


