A MATHEMATICAL DISCUSSION

OF

COROTATIONAL FINITE ELEMENT MODELING

based on

The Element Independent Corotational (EICR) Finite Element Method
(Rankin, Brogan, Nour-Omid)

and

A Unified Formulation of Small-Strain Corotational Finite Elements
(Felippa, Haugen)

by

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in conformity with the requirements for
the degree of Master of Science

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Abstract

This thesis discusses the mathematics of the Element Independent Corotational (EICR) Method and the more general Unified Small-Strain Corotational Formulation. The former was developed by Rankin, Brogan and Nour-Omid [106]. The latter, created by Felippa and Haugen [49], provides a theoretical frame work for the EICR and similar methods and its own enhanced methods.

The EICR and similar corotational methods analyse non-linear deformation of a body by its discretization into finite elements, each with an orthogonal frame rotating (and translating) with the element. Such methods are well suited to deformations where non-linearity arises from rigid body deformation but local strains are small (1-4%) and so suited to linear analysis. This thesis focuses on such small-strain, non-linear deformations.

The key concept in small-strain corotational methods is the separation of deformation into its rigid body and elastic components. The elastic component then can be analyzed linearly. Assuming rigid translation is removed first, this separation can be viewed as a polar decomposition \( F = vR \) of the deformation gradient \( F \) into a rigid rotation \( R \) followed by a small, approximately linear, stretch \( v \). This stretch usually causes shear as well as pure stretch.

Using linear algebra, Chapter 3 explains the EICR Method and Unified Small-Strain Corotational Formulation initially without, and then with, the projector operator, reflecting their historical development. Projectors are orthogonal projections which simplify the isolation of elastic deformation and improve element strain invariance to rigid body deformation.

Turning to Lie theory, Chapter 4 summarizes and applies relevant Lie theory to explore rigid and elastic deformation, finite element methods in general, and the EICR Method in particular. Rigid body deformation from a Lie perspective is well represented in the literature which is summarized. A less developed but emerging area in differential geometry (notably,
Marsden/Hughes [82]), elastic deformation is discussed thoroughly followed by various Lie aspects of finite element analysis. Finally, the EICR Method is explored using Lie theory. Given the available research, complexity of the area, and level of this thesis, this exploration is less developed than the earlier linear algebraic discussion, but offers a useful alternative perspective on corotational methods.
This thesis gratefully is dedicated to:

family and friends for their support, encouragement and help, especially my nephew Kerry and my fellow grad students;

those who so generously and extensively have discussed their work for this thesis, especially Dr. Carlos Felippa (Colorado - Boulder), Dr. Peyman Khosravi (Concordia), Dr. Andrew Lewis (Queen’s), Dr. Charles Rankin (Rhombus Consultants), and Valdemar Tsanov (Queen’s);

and

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Sincere thanks also are expressed to Dr. Andrew Lewis who generously has helped with this thesis and other aspects of my studies. In particular, our frequent discussions have been invaluable in my developing a working understanding of Lie Theory for this thesis. Dr. Lewis’ insightful advice and, no less so, his wry humour have been appreciated very much.

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List of Notation

The following conventions and notation are used generally in this thesis unless defined otherwise locally in the text. Other notation used less frequently is defined only in the body of the thesis.

Conventions

lowercase letters, regular face
- Scalars (e.g. \( x \))

lowercase letters, italics
- Group elements (e.g. \( g \))

lowercase letters, bold face
- Column vectors – spatial if relevant (e.g. \( \mathbf{x}^{(a)} \)), spatial tensors e.g. \( \mathbf{b} \) (see in notation)

uppercase letters, regular face
- Sets, manifolds (e.g. \( M \)), reference frames (e.g. \( E_k \))

uppercase letters, italics
- Groups (e.g. \( G \))

uppercase letters, bold face
- Material column vectors (e.g. \( \mathbf{X}^{(a)}_G \)), matrices, material or mixed tensors, vector or tensor fields

uppercase letters, regular french script
- Configurations, e.g. \( \mathcal{C}^0 \)

uppercase letters, bold french script
- Body/system states, e.g. \( \mathcal{C}^0 \)

material / spatial position vectors
- Table 3-1

\( \cdot (i) \)
- For iteration \( i \)

\( \cdot (\text{def}), \cdot (\text{rig}), \cdot (\text{tot}) \)
- Elastic, rigid, both components of \( \cdot \)

\( \cdot, \cdot, \cdot \)
- First, Second Time Derivative of \( \cdot \)

\( |\cdot| \)
- Norm (Length) of Vector \( \cdot \)

Notation

\( \gamma \)
- Curve, Trajectory or Integral Curve

\( \gamma_{xy} \)
- Shear

\( \Gamma \)
- Variations of Base Vectors of \( E_k \) with respect to Nodal Displacements: local (Rankin/Nour-Omid), like \( \mathbf{G} \)

\( \delta \cdot \)
- Infinitesimal change in \( \cdot \)

\( \delta_{ab} \)
- Kronecker delta

\( \Delta \cdot \)
- Finite change in \( \cdot \)

\( \Psi \)
- Cross-Product Operator (skew-symmetric matrix) corresponding to current relative Nodal Position Vector (Rankin/Nour-Omid), like \( \mathbf{S}^D \)
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Q_free: Free Configuration Manifold
R: Rotation Tensor, Rotator
R_A: Axis Rotation, e.g. R_{A:Ek→G}
R_V = R_A^T: Vector Rotation, e.g. R_V: = R_{A:Ek→G}
R^n: Euclidean n-space
R_g: Right Action, Translation or Multiplication Map
S: Configuration Space, see also Q
S_0, S_k: Nodal Surface Reference Frames, iteration k

\[ \overline{S}^D = \begin{bmatrix} S_1^D \\ \vdots \\ S_N^D \end{bmatrix} \] (Felippa/Haugen), like \( \Gamma \)

\[ \overline{S}_a^D \] Spin-lever or Moment-arm Matrix for c^D with respect to c^R: local (Felippa/Haugen)

se(n, R): Lie Algebra of SE(n, R)
SE(n, R): Special Euclidean Group
so(n, R): Lie Algebra of SO(n, R)
SO(n, R): Special Orthogonal Group
TF_p: Tangent Map, point p
TM: Tangent Bundle of manifold M
T_aM: Tangent Space to manifold M at point a
TM^*, T^*_aM: Cotangent Bundle of M, Cotangent Space to M at a

TR: Diag \[ \begin{bmatrix} R_{A:Ek→G}^T & \cdots & R_{A:Ek→G}^T \end{bmatrix} \] for N nodes of element

u, \( \overline{u} \): Translational Displacement Vector: global, local
v, \( \overline{v} \): Total Displacement Vector: global, local \( v = (u, \omega) \), \( \overline{v} = (\overline{u, \omega}) \)
V*: Dual Space or Dual of Vector Space V

X_{G}^{(a,R)} , X_{G}^{(a,D)}: Nodal Position Vector, see Table 3-1
X_{0}^{(a)} , X_{R,k}^{(a,R)} , X_{D,k}^{(a,D)}: Nodal Position Vector, see Table 3-1
X_{0}^{(a)} , X_{R,k}^{(a,R)} , X_{D,k}^{(a,D)}: Nodal Position Vector, see Table 3-1
X_{G}^{(a)}: Nodal Position Vector, see Table 3-1
X(f/g)(m): Linear Map for (D_f/g)(m)
Chapter 1

Introduction

This thesis stems from an interest in hanger cables on the Great Belt East Bridge which vibrate significantly in certain weather conditions. This interest began as a practical one focused on understanding the phenomenon, evolved into a desire to model it with a particular finite element method (Element-Independent Corotational (EICR) Method), and later deepened into a fascination with the mathematics of this and similar corotational methods. Those mathematics are the focus of this thesis.

The literature contains a variety of work on corotational methods. This thesis brings together two major groups of corotational work. The first group includes three papers by Rankin with Brogan [104] and with Nour-Omid [91,105] which collectively explain the EICR Method initially without, and later enhanced with, a projector operator. This group is referred to as the Rankin/Brogan/Nour-Omid research in this thesis. The other group, herein referred to as Felippa/Haugen research, consists of broader work by Felippa [44,45], Haugen [57] and both [47]. This work provides a unified theoretical framework (Unified Small-strain Corotational Formulation) for the EICR and similar corotational methods; explains and compares these corotational methods relative to this unified formulation; and offers three enhanced methods using a more sensitive projection operator. Both groups draw significantly upon work on corotational methods by Horrigrmoe/Bergan [60], Belytschko et al. [14], and Argyris et al. [8,9]; on rotations by Argyris [7] and Hughes/Winget [62]; and on projectors by Belytschko et al.[15]. Other notable corotational work includes Battini [12], Crisfield [34], and Khosravi [67-69]. All of this research uses linear algebra without any reference to Lie theory.

This thesis seeks to modestly contribute to the existing literature in two ways. It first draws together the two groups of research mentioned above to thoroughly explain the EICR and
similar corotational methods mathematically, using linear algebra. It then uses Lie theory and other differential geometry to discuss such methods as well as rigid and elastic deformation. These contributions hopefully will benefit applied researchers wanting a deeper understanding of corotational methods as well as those interested in applied geometric approaches.

As background, this introductory chapter briefly discusses the motivating problem (Great Belt East Bridge hanger cable vibrations), its suspected cause (wind-ice induced cable galloping), mathematical approaches to its analysis, and the EICR and similar corotational methods. Chapter 2 provides further background for later chapters by exploring iterative solution techniques, spatial rotation theory, and nodal rotations. Chapter 3 explains the EICR and similar corotational methods using linear algebra. The mathematical focus shifts in Chapter 4 which uses Lie theory and other differential geometry to discuss corotational methods as well as rigid and elastic deformation. Chapter 5 summarizes the main thesis goals and results, draws some general conclusions, and suggests future work. Reviews of the relevant literature are included in individual chapters as appropriate.

1.1 Motivating Problem: Great Belt East Bridge Hanger Cable Vibrations

The Great Belt East Bridge is a 2.7 km suspension bridge with a 1.6 km main span. It is part of the Great Belt Link carrying vehicles between eastern and western Denmark. The steel box girder deck is supported by hanger cables as shown in Figure 1-1. The cables are 80-114 mm in diameter, as long as 170m, and arranged in pairs separated by spacers every 40 metres [6].

These hanger cables (like any long cable) are particularly susceptible to wind excitation due to their flexibility, relatively small mass and low inherent damping [123]. During and since the bridge’s construction, there have been several occurrences of large amplitude vibrations with cables of 100 metres or longer. In these cases, large amplitude vibrations are defined as any amplitude exceeding 0.2% of cable length (or 0.2 metres for a 100 metre cable). This threshold is the lesser of the amplitudes at which cable vibration can be seen by the motoring public (0.2% of
cable length) and at which cable wear exceeds design specifications (0.33% of cable length). However, much larger amplitudes actually have been observed (ranging from 0.5 to 2.0 metres). Such vibrations are an immediate concern for public perception, a significant maintenance issue given increased cable fatigue and so reduced life, and, if not eventually addressed, a threat to the expected life of the bridge.

![Great Belt East Bridge showing Hanger Cable](image)

Figure 1-1: Great Belt East Bridge showing Hanger Cable [6]

These large amplitude vibrations usually occur in wet, near-freezing conditions with moderate laminar winds when cable icing occurs. Given these conditions and the character of the vibrations, their cause is thought to be wind-ice induced galloping. Research has shown that, for the first 40 vibration modes, the cables’ inherent damping is less than 0.2% of critical damping. It also has established that 0.6% of critical damping is needed to dampen the vibrations sufficiently to avoid large amplitude vibrations [6,51]. Critical damping $c_{cr}$ is defined as $c_{cr} = 2k\omega$ where $k =$ stiffness and $\omega =$ natural frequency of a given system [99]. It represents the damping level which results in the quickest re-equilibration of the system after its being perturbed [54].

1.2 Suspected Cause: Wind-Ice Induced Cable Galloping

As bridges and their components have become longer, lighter and more flexible, their susceptibility to excitation also has increased whether from loading, weather or seismic activity. This is particularly true of long, slender cables which have very low inherent damping (i.e. below
0.5% and often, as above, much lower) and which, with wind and precipitation, often experience changes in their cross sectional shape and/or texture from rain rivulets or icing. This altered cross section can change the cable’s aerodynamic properties and the resulting aerodynamic forces on the cable from the wind. This can excite the cable resulting in wind-ice (or wind-rain) induced cable galloping. Cable galloping is defined as a low-frequency, high amplitude oscillation of the cable occurring in a steady-state wind [51,123,135,136].

The aerodynamic forces and resulting galloping vary with wind intensity and direction. As the cable cross section changes, its centre of mass also shifts producing a moment and twisting which further affect the dynamics. Research has shown that this twisting significantly increases the vibration amplitude due to coupling between the equations of motion [40,41,53]. This coupling results from the aerodynamic forces twisting the cable which, in turn, alters the angle of attack and the aerodynamic forces acting on the cable. The resulting vibration can be quite large especially when the exciting frequency is close to the natural frequency of the cable.

The following aspects of wind-induced galloping are important to the mathematics discussed later in this thesis:

- The vibration amplitudes are much larger than the cable diameter necessitating non-linear analysis. It is helpful in analyzing such vibrations that the nonlinearity arises from rigid body motion with elastic deformation being small (up to 1-4%). Hence, this vibration is a case of geometric non-linearity where the non-linearity arises from large displacements not a non-linear stress-strain relationship [89].

- The vibration consists of significant rotational as well as translational motion. Both the magnitude and the non-commutativity of the rotations complicate the analysis.

- The cable twisting results in the equations of motion for the three degrees of freedom of interest (translation and rotation in the plane perpendicular to the cable’s longitudinal axis) being coupled and, hence, difficult to solve.
1.3 Mathematical Approaches to Analyzing Wind-Ice Induced Galloping

Wind-ice induced cable galloping is challenging to analyze mathematically for the reasons discussed above (nonlinearity, rotation as well as translation, and coupling). Gjelstrup et al.[52] have developed a model of the Great Belt East Bridge hanger cable as shown in Figure 1-2. The model has three degrees of freedom consisting of translation and cable rotation in the plane perpendicular to the cable’s longitudinal axis.

![3 DOFs Model of Hanger Cable, Great Belt East Bridge](image)

Using the energy approach and the Euler-Lagrange equation, Gjelstrup et al.[52] have produced equations of motion as well as equations for aerodynamic forces as shown below.

**Equations of Motion** [52]

\[
m_{\text{tot}} \ddot{x} + C_{\text{sx}} \dot{x} + k_x x + m_{\text{tot}} L_e \left( \cos(\varphi) \dot{\theta}^2 + \sin(\varphi) \dot{\theta} \right) = F_x
\]

\[
m_{\text{tot}} \ddot{y} + C_{\text{sy}} \dot{y} + k_y y + m_{\text{tot}} L_e \left( -\sin(\varphi) \dot{\theta}^2 + \cos(\varphi) \dot{\theta} \right) = F_y
\]

\[
\left( L_e^2 m_{\text{tot}} + J \right) \ddot{\theta} + C_{\text{so}} \dot{\theta} + k_0 \theta + m_{\text{tot}} L_e \left( \sin(\varphi) \ddot{x} + \cos(\varphi) \ddot{y} \right) = F_\theta
\]

where:

- \( m_{\text{tot}} \) = total mass of system (cable plus ice)
- \( C_{\text{sx}}, C_{\text{sy}}, C_{\text{so}} \) = structural damping in x, y & \( \theta \) directions
- \( k_x, k_y, k_0 \) = structural stiffness in x, y & \( \theta \) directions
- \( L_e \) = mass centre offset
- \( \varphi \) = change in angle offset of mass centre
  \( (\varphi = y_0 - \theta) \) where \( y_0 \) = initial offset, \( \theta \) = cable twist
- \( J \) = rotational inertia about mass centre
- \( x, \dot{x}, \ddot{x} \) = displacement, velocity, acceleration in x direction
- \( y, \dot{y}, \ddot{y} \) = displacement, velocity, acceleration in y direction
- \( \theta, \dot{\theta}, \ddot{\theta} \) = displacement, velocity, acceleration in \( \theta \) direction
External Aerodynamic Forces [52]

\[ F_x = \frac{1}{2} \rho U_R^2 D \left( C_D(\alpha_R, Re_R, \phi_R) \cos(\alpha_R) + C_L(\alpha_R, Re_R, \phi_R) \sin(\alpha_R) \right) \]

\[ F_y = \frac{1}{2} \rho U_R^2 D \left( C_L(\alpha_R, Re_R, \phi_R) \cos(\alpha_R) - C_D(\alpha_R, Re_R, \phi_R) \sin(\alpha_R) \right) \]

\[ F_\theta = \frac{1}{2} \rho U_R^2 D^2 C_M(\alpha_R, Re_R, \phi_R) \]

where: \( F_x, F_y, F_\theta \) = aerodynamic forces in \( x, y, \theta \) directions
\( \rho \) = air density
\( U_R \) = relative wind velocity
\( D \) = diameter of cable
\( C_D \) = drag coefficient
\( C_L \) = lift coefficient
\( C_M \) = moment coefficient
\( \alpha_R \) = wind angle of attack, cable surface
\( Re_R \) = Reynolds Number, longitudinal axis of cable
\( \phi_R \) = wind angle of attack, longitudinal axis of cable

The equations of motion shown above reflect the challenges discussed earlier, namely nonlinearity (\( \theta^2 \) term), both translation (\( x, y \) terms) and rotation (\( \theta \) terms), and coupling (4th term on left side of each equation).

Although an active research area [100], finding closed-form solutions to such non-linear, coupled equations can be very challenging and often is impossible. Hence, most efforts have focused on estimating solutions using various numerical techniques. Gjelstrup et al.[52] have used two approaches: i) using an ordinary differential equation solver to estimate deflections with and without twist and ii) linearizing the equations to create and analytically solve a simplified eigenvalue problem predicting instability ranges (based on the wind’s angle of attack and velocity) but not actual deflections.

Other researchers have estimated deflections for wind-ice induced cable galloping situations through finite element methods [39-41,80,84,85,127]. In particular, Di Pilato et al. [40, 41,84,85] have been using the element-independent corotational method. This work first brought the EICR method to the author’s attention for modelling purposes. Exploring that method led to
other corotational formulations [8,12,34] and the more general corotational work by Felippa [44, 45], Haugen [57] and both [47]. This, in turn, stimulated the mathematical interest in the corotational approach which motivates this thesis.

1.4 Non-Linear Finite Element Analysis

This discussion of non-linear finite element analysis is summarized from Khosravi [68]. Non-linear finite element analysis is more challenging than the linear case since the structure’s configuration (in geometric non-linearity), stiffness (in material non-linearity), loads, boundary conditions, and/or other properties are affected as the structure deflects. As already discussed, geometric non-linearity arises from large rigid body displacements. With material non-linearity, the non-linearity results from stiffness being a function of deformation requiring specialized non-linear elements having stiffnesses that can be varied with deformation. Whatever its source, non-linearity means that equilibrium must be established relative to the most recent configuration (which is unknown and called the Eulerian frame of reference) rather than, as with linear analysis, the initial configuration (which is known and called the Lagrangian frame of reference), requiring iterative solution methods like the Newton-Raphson Method discussed in Chapter 2. Eulerian and Lagrangian frames of reference are discussed in Chapter 4 under kinematics (Section 4.3).

1.5 Element-Independent Corotational (EICR) Method & Similar Corotational Finite Element Methods

A variety of finite element methods have been developed for nonlinear analysis. Many of these methods use specialized nonlinear elements with variable stiffness that become increasingly complex as non-linearities and rotational degrees of freedom become larger. This complexity can make the elements computationally expensive, specialized and limited in number, and often incompatible with standard elements.
For certain non-linear cases, the EICR and similar corotational methods offer a simpler approach without such specialized elements and their disadvantages. Corotational methods are well suited to cases where global displacements (translations and/or rotations) are nonlinear due to large rigid body deformation but where, for any given element, local strains (stretches and/or shears) are small. Corotational methods remove the rigid deformation before analyzing the small local strain and then reintroduce the rigid component. Since local strain is small, the deformed element is comparable to the initial (undeformed) element and, hence, strain can be analyzed linearly using standard elements.

What constitutes small strain is somewhat situational depending on the accuracy desired. However, common practice seems to consider strains of up to 1 - 4% (of the dimension being deformed) as small strain. In the case of shear rotation, this would mean local rotations of 4 - 14 degrees or 0.02π - 0.08π radians (4% of 360 degrees or 2π radians). According to Rankin/Brogan [104], the EICR Method yields satisfactory results for even larger local rotations of up to 30 degrees (0.167π or π/6) which their paper calls moderate strain as discussed in Chapter 3. However, in general, strain in excess of 4% is considered large strain.

It should be noted that small strain is sometimes inappropriately called infinitesimal strain even though the two strains are not the same. Small strain is a finite elastic deformation while the infinitesimal strain is a linear approximation of how that deformation changes if perturbed slightly. The common intermingling of the two terms occurs because, for small deformations, strain often is approximated linearly using the Infinitesimal Strain Tensor rather than the non-linear Green Lagrange Strain Tensor. The difference between finite (small) and infinitesimal strain is clear from the definitions of these tensors given below and discussed in Section 4.3.3.

**Green Lagrange Strain Tensor** \( \mathbf{E} \) for Finite Strain [19]

\[
\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) = \frac{1}{2} (\text{Grad}^T \mathbf{u} + \text{Grad} \mathbf{u} + \text{Grad}^T \mathbf{u} \text{Grad} \mathbf{u}) \quad \text{since} \quad \mathbf{F} = \text{Grad} \mathbf{u} + \mathbf{I}
\]

**Linear Strain Tensor** \( \mathbf{\varepsilon} \) for Infinitesimal Strain [19]

\[
\mathbf{\varepsilon} = \lim_{\text{Grad} \mathbf{u} \to 0} \mathbf{E} = \frac{1}{2} (\text{Grad}^T \mathbf{u} + \text{Grad} \mathbf{u}) \quad \text{since} \quad \text{Grad}^T \mathbf{u} \text{Grad} \mathbf{u} \to 0 \quad \text{faster than} \quad \text{Grad} \mathbf{u} \to 0
\]
where \( E = \) Green Lagrange Strain Tensor
\( \varepsilon = \) Linear Strain Tensor
\( F = \) Deformation Gradient
\( I = \) Identity
\( \text{Grad}\mathbf{u} = \) Displacement Gradient

Similarly, Argyis [7] explains that the difference between infinitesimal and finite strain is whether second order effects (i.e. \( \text{Grad}^T\mathbf{u}\text{Grad}\mathbf{u} \)) need to be considered.

Corotational methods are well suited to wind-ice induced cable galloping which results in non-linear motion consisting mainly of rigid motion with only small local strains. Since these methods analyze local strain at the element level, small local strains can be assured by choosing as fine a finite element mesh (i.e. discretization of the body) as needed to achieve them. If local strains are too large, one simply uses a finer mesh with more and smaller elements [45].

Corotational methods also have broader applications such as the modelling of post-buckling behaviour and collapse. However, buckling cannot be predicted solely from geometry so such models are more complex than using corotational methods alone as discussed in Haugen [57].

Corotational (CR) approaches to analyzing the deformation or motion of a body are characterized by a coordinate frame which is attached to and so corotates (and also co-translates) with the body or, for finite element analysis, with each element of the body. Rankin/Nour-Omid [105] and Felippa/Haugen [47] attribute the origins of corotational finite element methods to Wempner [128], Belytschko/Hsieh [14], and Horrigmoe/Bergen [60]. All of this work occurred in the decade from 1969 to 1978.

The Element Independent Corotational (EICR) Method was published in 1986 by Rankin and Brogan [104]. Over the next five years, Rankin and Nour-Omid [91,105] greatly enhanced the method with the introduction of a projector which simplifies isolating local strains and ensures that such strains are invariant to rigid rotation. Similar work by Crisfield [34], but limited to spatial beams, occurred about the same time. Crisfield called this formulation A Consistent Co-rotational Formulation for Non-Linear, Three-Dimensional, Beam-Elements with consistent
referring to a regularly updated tangent stiffness matrix. This updating was missing from the original EICR Method but was addressed when Rankin/Nour-Omid enhanced the method with the projector [91,105].

The reference to *element independence* in the naming of the EICR Method refers to the fact that the removal (and later reintroduction) of rigid deformation occurs before (and after) the actual assembly and analysis steps of the finite element analysis as shown in Figure 1-3. Further, this treatment of rigid deformation is purely geometric, depending only on the number of element nodes and degrees of freedom not other aspects of the element. Hence, all elements sharing the same number of nodes and degrees of freedom are processed the same way which is called element independence [45].

![Diagram](image.png)

Figure 1-3: EICR Method pictured as CR “Filters” between the local element analysis and the global assembly/solving steps ([45], modified by * additions)

Between 1994 and 2005, Felippa [44], Haugen [57] and both [47] significantly generalized and enriched the discussion of corotational finite element methods including the EICR Method. Their contributions included formulating a theoretical framework for such methods known as the *unified formulation of small-strain corotational elements*; explaining, reconciling and classifying existing methods within that unified formulation; enhancing and explaining the projector; proposing more sensitive variants of the formulation using this enhanced projector; and summarizing the rotational mathematics needed for CR methods. These
contributions stemmed from doctoral research by Haugen [57] while studying with Felippa, their joint paper [47], and work by Felippa [44].

The most sensitive CR variant developed by Haugen [57] based on this work was the Consistent Symmetrizable Self-Equilibrated Co-rotated Formulation (CSSE). Its title can be understood through the five criteria established by Haugen [57] to evaluate the performance of corotational finite element formulations used for non-linear analysis. These criteria are listed below in order of increasing importance for modeling accuracy and are discussed more thoroughly in Section 3.3.3.

1) **Self-Equilibrium** which refers to formulations in which each element’s internal force vector $\mathbf{F}$ is in equilibrium with respect to the deformed configuration. $\mathbf{F}$ represents the local internal forces (and moments) induced by the elastic deformation of the element.

2) **Consistency** which refers to formulations having tangent stiffness matrices which are regularly updated to remain equal to the gradient of the internal forces with respect to the degrees of freedom. Such updating is needed only with material non-linearity.

3) **Invariance** which refers to formulations where element stresses are independent of element orientations or, in other words, invariant under rigid motion.

4) **Symmetrizability** which refers to formulations that converge as quickly when their non-symmetric consistent tangent stiffness matrices are made symmetric as with the unaltered non-symmetric stiffness matrices.

5) **Element Independence** refers to formulations processing all elements with the same number of nodes and degrees of freedom in the same way, as discussed earlier.

Table 1-1 modified from Haugen [57] summarizes and contrasts the various corotational methods using these criteria. From this table, it can be seen that the formulations by Nour-Omid/Rankin (EICR Method with rotational projector) and by Haugen/Felippa (CSSE Method with rotational/translational projector) basically are the same except for the enhanced projector. The Haugen/Felippa projector addresses sensitivity to both rigid translation and rigid rotation while the Nour-Omid/Rankin projector is limited to only the latter.
This projector difference only matters with linear elements sensitive to rigid translation [57]. Such sensitivity is quite rare especially amongst the lower order linear elements normally used in corotational methods. Hence, it seems appropriate to merge the explanations in the Rankin/Brogan/Nour-Omid research [106] with those of the Felippa/Haugen research [49] as done in this thesis.

Table 1-1: Summary of Corotated Formulations ([57], with Rankin/Brogan added)

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Self-Equilibrium</th>
<th>Consistency</th>
<th>Invariance</th>
<th>Symme-trizable</th>
<th>Element Independence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bergan et al. [60]</td>
<td></td>
<td></td>
<td>√ (all rigid)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rankin/Brogan [104]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>(EICR, no projector)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nour-Omid/Rankin [91,105]</td>
<td></td>
<td>√</td>
<td>√ rotation only</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>(EICR w. projector)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Haugen [57]</td>
<td></td>
<td>√</td>
<td>√ (all rigid)</td>
<td></td>
<td>√</td>
</tr>
<tr>
<td>(CSSE Formulation)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Assuming invariance, rigidly translating and/or rotating an element as in Figure 1-4 does not affect its local stress state. Hence, rigid deformation can be removed prior to analyzing the stress state of an element without altering that state. If any nonlinearity rests in the rigid motion, this leaves only small elastic deformation which can be analyzed linearly. This is the key characteristic of the corotational approach used in the EICR, CSSE and similar methods.

Figure 1-4: Rigid Body Translation & Rotation [47, modified]

As explained by Felippa ([45]) and shown in Figure 1-5, corotational methods use two orthogonal coordinate frames: a global frame and an associated corotated body frame. Elements having nodal rotational degrees of freedom also require a surface frame at each node as discussed in Chapter 3. These frames give rise to three configurations: an initial configuration ($\mathbf{c}^0$), a
corotated configuration ($C^R$) which is a rigid transformation of $C^0$, and a deformed configuration ($C^D$) which introduces local elastic deformation. $C^0$ and $C^D$ are actual physical configurations while $C^R$ is estimated based on “best fit” criteria discussed in Chapter 3.

Using these frames and configurations, corotational methods separate (and remove) any rigid motion from the local pure deformation before performing the local finite element analysis. This leaves only pure elastic deformation which can be analyzed linearly in local coordinates. The results then are transformed back into the global coordinates by reintroducing the rigid body deformation (nonlinearity).

![Figure 1-5: Initial, Corotated, Deformed Configurations](image)

A key aspect of the EICR and CSSE methods is their use of the same material stiffness matrix ($\overline{K}_M$) throughout the analysis. A constant $\overline{K}_M$ avoids the computational expense of reforming it for each new deformed configuration but also poses challenges to maintaining element equilibrium and invariance under rigid motion. $\overline{K}_M$ is calculated based on the local dimensions in the initial undeformed configuration ($C^0$ in Figure 1-5). These local dimensions and, hence, $\overline{K}_M$ do not change in the corotated configuration ($C^R$) because $C^R$ results from only rigid deformation with no incremental strain. However, the local dimensions in the deformed configuration ($C^D$) do change given the elastic deformation with its stretching and bending of the element. This normally would require calculating a new stiffness matrix to maintain element internal force equilibrium.
Rather than doing so, the internal force vector $\bar{f}$ is adjusted. First, $\bar{K}_M$ is used to calculate a “non-equilibrating” internal force vector $\bar{f} = \bar{K}_M \bar{u}$ for elastic deformation $\bar{u}$. $\bar{f}$ then is converted to a “self-equilibrating” internal force vector $\bar{f}_{\text{equil}}$. This adjustment involves scaling those components of the force vector affected by the dimensional changes as illustrated below by Khosravi [67] for a 2-noded beam element.

$$\bar{f} = (P_1, V_1, M_1, P_2, V_2, M_2)$$

$$\bar{f}_{\text{equil}} = \left( P_1, V_1 \left( \frac{L_o^2}{L^2} \right), M_1 \left( \frac{L_o}{L} \right), P_2, V_2 \left( \frac{L_o^2}{L^2} \right), M_2 \left( \frac{L_o}{L} \right) \right)$$

where

- $P_i =$ internal axial force at node $i$
- $V_i =$ internal normal force at node $i$
- $M_i =$ internal moment at node $i$
- $L_o =$ initial (undeformed) beam element length
- $L =$ current (deformed) beam element length (calculated as difference between node (1) and node (2) positions)

The projector $P$ developed by Rankin and Nour-Omid [91,105] and enhanced by Felippa and Haugen [47,58] greatly facilitate this equilibrating of the internal force vector $\bar{f}$. It also benefits several other aspects of the EICR, CSSE and similar corotational methods through ensuring that the element material stiffness matrix $\bar{K}_M$ is insensitive to rigid motion and simplifying the separation of local elastic translational deformations $\bar{u}$ from global displacements $\bar{u}$. Being an orthogonal projection, $P^2 = P$ and so likewise $P^n = P$ for any positive integer $n$. Hence, applying $P$ repeatedly or to an already satisfactory $\bar{u}, \bar{f}$ or $\bar{K}_{\text{tan}}$ results in no change or complications. The projector $P$ is discussed more thoroughly in Chapter 3.

As an aside, it should be noted that, even though $\bar{K}_M$ remains constant, stress stiffening $\bar{K}_S$ may be added in some cases. Stress stiffening represents the effect on transverse stiffness of longitudinal loading with tension increasing and compression decreasing stiffness. If stress stiffening is considered, the tangential stiffness matrix $\bar{K}_{\text{tan}}$ includes both $\bar{K}_M$ and $\bar{K}_S$. 
Rotations pose another challenge in the EICR and similar corotational methods which allow large global rotations subject only to the local rotation thresholds already discussed. Rotations in three dimensions are not commutative. This means that the order of composed rotations affects the final configuration and that rotations cannot be composed as vectors since vector addition is commutative. Corotational methods manage this challenge by exploiting the fact that infinitesimal rotations “approximately” commute provided higher order powers are ignored [9]. Spatial rotations are discussed more thoroughly in later chapters.

Khosravi developed code [69] and supporting materials [68] for applying the EICR method to a spatial beam using 2-noded beam elements. This method incorporates the Newton Raphson iterative process, a choice of force or arc length load control, and optional stress stiffening. The Khosravi code also nicely illustrates the EICR, CSSE and similar corotational methods and, hence, is summarized in Figure 1-6. This figure is presented here as an overview of the method with detailed discussion of the figure occurring in later chapters.

The EICR and similar corotational methods use lower order elements such as the two-noded beam element, the three-noded triangular shell element, and the four-noded quadrilateral shell element with the nodes having both translational and rotational degrees of freedom. Rotational degrees of freedom or nodal rotations are explained in Chapter 2. Such elements can model constant strain or, given their nodal rotations, linear strain but not non-linear strain. For example, such elements are not suitable for modeling incompressible materials like rubber. Rankin/Brogan/Nour-Omid [106] and Haugen [57] provide several planar and spatial examples of applying corotational methods numerically.

Although all these EICR elements are discussed initially, this thesis focuses on the three noded triangular element with drilling rotations to explain the EICR and similar corotational methods. A drilling rotation is a rotation about an axis normal to the element plane which passes through a node [103] as discussed in Chapter 2 (Section 2.3). This element has been chosen for reasons of clarity, illustration and available literature. Regarding clarity, triangular elements avoid
the special challenges associated with beam elements (collinear nodes requiring a constructed reference node) and quadrilateral elements (too few nodes to represent full order shape functions). On the other hand, triangular elements illustrate nodal rotations much better than beam elements. Finally, the literature, both general and in the corotational context, is far richer for triangular than other elements.

Hence, the three-noded triangular element with drilling rotations seems the best choice for explaining corotational methods in this thesis. However, if the goal instead was to model the actual hanger cable, then a spatial beam element would be the preferred choice as found in the beam examples by Rankin/Brogan/Nour-Omid [106] and the models of cable galloping by Di Pilato et al.[40,41,84,85]. While cables differ from beams in lacking transverse stiffness and resistance to compression, hanger cables are in tension which creates stress stiffness and means compression is not a factor. The “locked coil” construction of the hanger cables also makes them denser and stiffer than ordinary cable. For example, a typical 112mm diameter locked coil cable has a density (88% of its cross section being steel), Young’s modulus (163 versus 200 GPa) and Poisson’s ratio (0.30) close to solid steel [81]. All of these factors favour modelling a hanger cable using beam elements.

The chapter which follows presents some further background needed in later chapters to explain the EICR and similar corotational methods.
<table>
<thead>
<tr>
<th>Process Steps</th>
<th>Coordinate Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate material stiffness matrix ($\mathbf{K}_M$) for all elements</td>
<td>local</td>
</tr>
<tr>
<td>(saved to be used throughout the analysis)</td>
<td></td>
</tr>
<tr>
<td><strong>LOAD STEP LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>Initiate / increment Load Step “$t + \Delta t$” with load $\mathbf{L}^{(t+\Delta t)}$</td>
<td></td>
</tr>
<tr>
<td>Check termination criteria for load steps</td>
<td></td>
</tr>
<tr>
<td><strong>ITERATION LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>Initiate / increment Iteration “$i$”</td>
<td>global</td>
</tr>
<tr>
<td><strong>ELEMENT LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>For each/all element(s):</td>
<td></td>
</tr>
<tr>
<td>Update nodal coordinates by adding total nodal deflection/rotation ($\mathbf{v}$)</td>
<td>global</td>
</tr>
<tr>
<td>from previous iteration, see below. ($\mathbf{v}=(\mathbf{u}^{\text{tot}}, \omega^{\text{rot}})$), $\mathbf{v}$ initially 0</td>
<td></td>
</tr>
<tr>
<td>Form “best fit” local element frame from global nodal coordinates</td>
<td>global/local</td>
</tr>
<tr>
<td>Use “best fit frame” to:</td>
<td></td>
</tr>
<tr>
<td>Estimate rigid rotation $\mathbf{R}$ of element, $\mathbf{c}^R$ to $\mathbf{c}^0$</td>
<td>global/local</td>
</tr>
<tr>
<td>Use $\mathbf{R}$ to change global $\mathbf{v}^{\text{tot}}$ to local $\mathbf{v}^{\text{tot}} = \mathbf{R}^T \mathbf{v}^{\text{tot}}$</td>
<td>global/local</td>
</tr>
<tr>
<td>Calculate projector $\mathbf{P}$ using first variations of “best fit frame”</td>
<td>global/local</td>
</tr>
<tr>
<td>Use projector $\mathbf{P}$ to:</td>
<td></td>
</tr>
<tr>
<td>Isolate local deform.($\mathbf{v}^{\text{def}}$) from total ($\mathbf{v}^{\text{tot}}$), $\mathbf{v}^{\text{def}} = \mathbf{P} \mathbf{v}^{\text{tot}}$</td>
<td>local</td>
</tr>
<tr>
<td>Calculate non-equilibrated internal force vector, $\mathbf{f} = \mathbf{K}_M \mathbf{v}$, $\mathbf{v} = (\mathbf{u}, \omega)$</td>
<td>local</td>
</tr>
<tr>
<td>Ensure self-equilibrating internal force vector, $\mathbf{f}_{\text{equil}} = \mathbf{P}^T \mathbf{f}$</td>
<td>local</td>
</tr>
<tr>
<td>Ensure $\mathbf{K}<em>M$ invariant to rigid rotation, $\mathbf{K}</em>{M,\text{clean}} = \mathbf{P}^T \mathbf{K}_M \mathbf{P}$</td>
<td>local</td>
</tr>
<tr>
<td>Calculate stress stiffness ($\mathbf{K}_S$) for element</td>
<td>local</td>
</tr>
<tr>
<td>Calculate tangential stiffness $\mathbf{K}<em>{\text{tan}} = \mathbf{K}</em>{M,\text{clean}} + \mathbf{K}_S$</td>
<td>local</td>
</tr>
<tr>
<td>Transform back to global coordinates using $\mathbf{R}$, as follows:</td>
<td>global</td>
</tr>
<tr>
<td>$\mathbf{f} = \mathbf{R}^T \mathbf{f}<em>{\text{equil}}$, $\mathbf{K}</em>{\text{tan}} = \mathbf{R}^T \mathbf{K}_{\text{tan}} \mathbf{R}$</td>
<td></td>
</tr>
<tr>
<td>Update global internal force matrix $\mathbf{F}$ with $\mathbf{f}$ for given element</td>
<td>global</td>
</tr>
<tr>
<td>Update global tangent stiffness matrix ($\mathbf{K}$) with $\mathbf{K}_{\text{tan}}$ for given element</td>
<td>global</td>
</tr>
<tr>
<td><strong>ELEMENT LOOP End</strong></td>
<td></td>
</tr>
<tr>
<td>(Looping through all elements assembles global internal force &amp; stiffness matrices)</td>
<td></td>
</tr>
<tr>
<td>Calculate unbalanced force $\Delta \mathbf{F}<em>{(i)} = \mathbf{L} - \mathbf{F}</em>{(i)}$</td>
<td>global</td>
</tr>
<tr>
<td>Calculate incremental nodal deflection $\Delta \mathbf{v}<em>{(i)} = (\mathbf{K}</em>{(i)})^{-1} \Delta \mathbf{F}_{(i)}$</td>
<td>global</td>
</tr>
<tr>
<td>Update total nodal deflections $\mathbf{v}<em>{(i+1)} = \mathbf{v}</em>{(i)} + \Delta \mathbf{v}_{(i)}$</td>
<td>global</td>
</tr>
<tr>
<td>Update total nodal rotations (matrix multiplication or Argyris formula)</td>
<td>global</td>
</tr>
<tr>
<td>Check criteria for convergence of $\Delta \mathbf{v}$ (or practical termination if not converging)</td>
<td>global</td>
</tr>
<tr>
<td><strong>ITERATION LOOP End</strong></td>
<td></td>
</tr>
<tr>
<td>Update global coordinates</td>
<td>global</td>
</tr>
<tr>
<td>Plot deformed shape</td>
<td>global</td>
</tr>
<tr>
<td><strong>LOAD STEP LOOP End</strong></td>
<td></td>
</tr>
</tbody>
</table>

Figure 1-6: Summary of EICR Method based on 3-D Beam Formulation by Khosravi [31]
Chapter 2

Background on Selected Topics

This chapter provides background on three topics which are required when explaining the EICR and similar corotational methods in later chapters. These topics include i) iterative solution methods, ii) basic rotation theory, and iii) nodal rotations. Each topic is discussed below including its role in corotational methods, the relevant literature, and a brief explanation.

2.1 Iterative Solution Techniques

This discussion of iterative solution techniques is based on Khosravi [68]. Iterative solution techniques are needed for corotational methods because, although linear analysis with constant material stiffness is used at the element level, the global displacement is non-linear. This means that, globally, the current deformed configuration differs too significantly from the initial (material) configuration for that initial configuration to serve as the reference configuration for the analysis.

To overcome this challenge, an iterative procedure must be used to estimate the global equilibrium solution path. Such procedures include a stepped control technique (to control progress along the path), an iterative process (to estimate a solution for each step), and some termination criteria for the iterations (at convergence or, if not obtained, after a finite number of iterations) and for the steps (once completed). The step control can use displacement, load or arc length.

For a given step, the iterative process repeatedly estimates a new result (e.g. incremental displacement) and compares the new and previous results until they converge with a desired accuracy. One such technique, commonly used for the EICR and similar corotational methods, is the Newton-Raphson iterative method shown in Figure 2-1.
Figure 2-1: Newton-Raphson iterative method, force load control [68, modified]

For iteration \( i = 1, 2, \ldots \) of load step \( (t + \Delta t) \) using global frame:

- \( \mathbf{L}^{(t)} \), \( \mathbf{L}^{(t+\Delta t)} \), \ldots: Load Vector for Step (forces &/or moments)
- \( \mathbf{K}^{(t+\Delta t)} \) = Tangential Stiffness Matrix (actually a cubic array as includes all elements)
- \( \Delta \mathbf{v}^{(i)} \) = Incremental Displacement Vector (translations \( \Delta \mathbf{u} \) &/or rotations \( \Delta \theta \))
- \( \mathbf{v}^{(t+\Delta t)} \) = Total Displacement Vector (translations \( \mathbf{u} \) &/or rotations \( \theta \))
- \( \mathbf{F}^{(t+\Delta t)} \) = Internal Force Vector (forces &/or moments)
- \( \Delta \mathbf{F}^{(t+\Delta t)} \) = Unbalanced Force Vector (forces &/or moments)

\( (\Delta \mathbf{F}^{(t+\Delta t)} = \mathbf{L}^{(t+\Delta t)} - \mathbf{F}^{(t+\Delta t)} ) \)

As discussed in Section 2.2.2, rotations \( \theta \) and \( \Delta \theta \) may be represented by rotation pseudo-vectors \( \omega \) and \( \Delta \omega \) through the normalization process. The \( \Delta \)-notation used above refers to a finite change in the given variable whereas the \( \delta \)-notation introduced in Chapter 3 refers to an infinitesimal change. Despite this difference, the \( \Delta \)- and \( \delta \)-notations sometimes are used interchangeably as discussed in Chapter 3.

The Newton-Raphson method (NR) can be explained in the context of the EICR Method by jointly discussing Figure 1-6 (EICR Method Summary) and Figure 2-1 (NR Method). For a given iteration \( i \) in Figure 1-6, the EICR element loop calculates the internal force vector and tangential stiffness matrix for each element, transforms them to global coordinates, and then assembles them.
into the global internal force vector $\mathbf{F}^{(i+\Delta t)}$ and tangential stiffness matrix $\mathbf{K}^{(i+\Delta t)}$.

After processing all elements for the iteration, the EICR Method moves from the element loop into the lower part of the iteration loop where $\mathbf{F}^{(i+\Delta t)}$ and $\mathbf{K}^{(i+\Delta t)}$ are used as shown in Figure 2-1 (NR Method). First, the NR method subtracts $\mathbf{F}^{(i+\Delta t)}$ from the load vector for the step $\mathbf{L}^{(i+\Delta t)}$ to obtain the unbalanced force vector $\Delta \mathbf{F}^{(i+\Delta t)} = \mathbf{L}^{(i+\Delta t)} - \mathbf{F}^{(i+\Delta t)}$. Then, the inverted tangent stiffness matrix is used to find the incremental displacement resulting from this unbalanced force as in

$$\Delta \mathbf{v}^{(i)} = \left(\mathbf{K}^{(i+\Delta t)}\right)^{-1} \Delta \mathbf{F}^{(i+\Delta t)}.$$

Next, the total displacement vector $\mathbf{v}^{(i-1)}$ is updated to $\mathbf{v}^{(i)}$ using the incremental displacement vector $\Delta \mathbf{v}^{(i)}$. Simple vector addition is used for translations as in $\mathbf{u}^{(i)} = \mathbf{u}^{(i-1)} + \Delta \mathbf{u}^{(i)}$. However, updating rotations is more complicated. Although specifics vary, corotational methods convert rotation pseudo-vectors ($\theta$) to full rotation matrices ($\mathbf{R}$) before updating and then revert to pseudo-vectors. Rotations, their representations and the updating approaches used by the EICR and similar corotational methods are discussed in Section 2.2 and Chapter 3.

If $\Delta \mathbf{v}^{(i)}$ exceeds the convergence criteria, the solution has not yet converged for that load step so iterations continue with $\mathbf{v}^{(i)}$ as the displacement vector for the next iteration. If $\Delta \mathbf{v}^{(i)}$ falls below the convergence criteria, the solution has converged so $\mathbf{v}^{(i)}$ is stored as the displacement for that load step, the load step is increased, and iteration begins again with $\mathbf{v}^{(i)}$ and $\Delta \mathbf{v}^{(i)}$ reset to zero.

### 2.2 Some Basic Spatial Rotation Theory

Corotational methods including the EICR Method use various representations for global rotations (rotators) and local rotations (spinors or twists). For this reason, some basic spatial rotation theory is needed to understand corotational methods. The discussion which follows is based on work by Felippa [45, Appendix R] and Argyris [7]. It also should be noted that the terminology used (e.g. rotator $\mathbf{R}$) reflects common usage in corotational papers rather than the Lie terminology (e.g. $\mathbf{R} \in \text{SO}(3)$) found in Chapter 4.
2.2.1 Rotation Representations

By Euler’s Displacement Theorem, the general displacement of a rigid body with one point fixed can be represented as a rotation about some axis passing through that point. Figure 2-2 shows such a rotation \( \theta \) about a rotation axis \( \omega \). With appropriate scaling (e.g. \(|\omega| = \theta \) or some function of \( \theta \)) and sign convention (e.g. the right hand rule), \( \omega \) can represent the rotation much like a vector. However, spatial (3-D) rotations cannot be composed as vector quantities since, unlike vector addition, successive rotations do not commute (i.e. their order matters as shown in Figure 2-3). Very small rotations sometimes are treated as commuting but this is only an approximation. For this reason, \( \omega \) is usually called a rotation pseudo-vector rather than a vector.

![Figure 2-2: Rotation of angle \( \theta \) about axis of rotation \( \omega \) ([45])](image)

![Figure 2-3: Order of Spatial Rotations matters (Argyris [7])](image)

Other rotation representations exist which do reflect the non-commutativity of spatial rotations using matrices, Euler angles or quaternions. There are two such matrix representations. A rotator or rotation tensor is a 3 x 3 orthogonal rotation matrix (\( \mathbf{R} \in SO(3) \)) which uses matrix multiplication to spatially rotate any vector \( \mathbf{v} \) (e.g. \( \mathbf{Rv} \)). A rotation also can be expressed as the matrix exponential of a spinor or spin tensor \( \mathbf{\Omega} \in so(3) \) which is a 3 x 3 skew-symmetric matrix.
Being skew symmetric, three parameters uniquely define a spinor. Hence, a spinor can be represented by a pseudo-vector $\mathbf{\omega} \in \mathbb{R}^3$. If $\mathbf{\omega}$ is not normalized (i.e. neither expressed as a unit vector or function of $\theta$), a rotation $\mathbf{R}$ of $\theta$ radians about an axis $\mathbf{\omega} = (\omega_1, \omega_2, \omega_3)^T$ can be represented as follows.

$$
\mathbf{R} (\Omega) = e^{\Omega \theta} = I + \frac{\sin \theta}{|\mathbf{\omega}|} \mathbf{\Omega} + \frac{2 \sin^2 (\theta / 2)}{|\mathbf{\omega}|^2} \mathbf{\Omega}^2 = I + \frac{\sin \theta}{|\mathbf{\omega}|} \mathbf{\Omega} + \frac{1 - \cos(\theta)}{|\mathbf{\omega}|^2} \mathbf{\Omega}^2
$$

where the spinor is $\mathbf{\Omega} = \text{Spin}(\mathbf{\omega}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

and the pseudo-vector is $\mathbf{\omega} = (\omega_1, \omega_2, \omega_3)^T$

Figure 2-4 from Felippa [45] summarizes the various rotation representations (pseudo-vector, spinor, rotator) and the transformations between them. The various transformations also are discussed below.

![Figure 2-4: Representations of Finite Spatial Rotations ([45])]()

The spin transformation converts pseudo-vectors to spinors with its inverse being the axial operator. Equivalent notation from Bullo/Lewis [21] is $\Lambda$ for Spin and $\Lambda^\gamma$ for axial. Spin and axial are defined below.

$$
\text{Spin}(\mathbf{\omega}) = \mathbf{\hat{\omega}} = \mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \text{axial}(\mathbf{\Omega}) = \mathbf{\Omega}^\gamma = (\omega_1, \omega_2, \omega_3)^T = \mathbf{\omega}
$$

where $\mathbf{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is a pseudo-vector.

The following properties of spinors and pseudo-vectors are clear from the above definitions, in particular:

- Being skew symmetric, the transpose of a spinor is simply its negative (e.g. $\mathbf{\Omega}^T = - \mathbf{\Omega}$).
For any vector \( \mathbf{v} \in \mathbb{R}^3 \), the product \( \mathbf{\Omega} \mathbf{v} = \mathbf{\omega} \times \mathbf{v} \) or, in other words, the cross product of \( \mathbf{\omega} \) and \( \mathbf{v} \) where \( \mathbf{\omega} \) is the pseudo vector associated with \( \mathbf{\Omega} \).

The \textbf{Rot} and \textbf{Skew} transformations link rotators and spinors as in \( \mathbf{R} = \text{Rot}(\mathbf{\Omega}) = e^{\mathbf{\Omega}} \) and \( \mathbf{\Omega} = \text{Skew}(\mathbf{R}) = \log(\mathbf{R}) \). The formulae for \textbf{Rot} (and the associated \textbf{Skew}) depend on how the pseudo-vectors and, hence, spinors are normalized. These normalizations and the resulting \textbf{Rot} and \textbf{Skew} formulae are discussed below.

2.2.2 Normalizations

The case of pseudo-vector \( \mathbf{\omega} \) not being normalized was discussed one page previous. However, pseudo-vectors usually are normalized to exploit certain advantages of the chosen normalization. Such advantages can include simplifying the associated \textbf{Rot} and \textbf{Skew} formulae, avoiding singularities for various rotation angles, and/or making \( |\mathbf{\omega}| \) a function of \( \theta \). Table 2-1 adapted from Felippa [45] summarizes the various normalizations and their associated spinor and rotator formulae. The left-most column also shows that these formulae are equivalent after adjusting for normalizations.

Normalizing \( \mathbf{\omega} \) as a unit vector results in a familiar formula known as \textit{Rodrigues Rotation Formula} and shown below. With this normalization, the pseudo-vector becomes \( \mathbf{n} = \mathbf{\omega}/|\mathbf{\omega}| \) with \( |\mathbf{n}| = 1 \) and accordingly the spinor becomes \( \mathbf{N} = \mathbf{\Omega}/|\mathbf{\omega}| \).

\[
\mathbf{R}(\mathbf{N},\theta) = e^{\mathbf{\Omega} \theta} = I + \frac{\sin \theta}{|\mathbf{\omega}|} \mathbf{\Omega} + \frac{2 \sin^2 (\theta/2)}{|\mathbf{\omega}|^2} \mathbf{\Omega}^2 = I + \sin(\theta) \mathbf{N} + (1-\cos(\theta)) \mathbf{N}^2
\]

where \( \mathbf{N} = \text{Spin}(\mathbf{n}) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}, \mathbf{n} = (n_1, n_2, n_3)^T = \frac{\mathbf{\omega}}{|\mathbf{\omega}|}, |\mathbf{n}| = 1 \)

The Rodrigues-Cayley normalization approximates \( \theta/2 \) by \( |\mathbf{\omega}| = \tan(\theta/2) \) uniquely representing any \( \theta \) (because \(-\infty < \tan(\theta/2) < +\infty \)) except for \( \theta = (2n+1)\pi \) for any integer where \( |\mathbf{\omega}| \) is singular. The de Veubeke approach removes this singularity problem by approximating \( \theta/2 \) by \( |\mathbf{\omega}| = \sin(\theta/2) \) but limits the angles which can be represented uniquely by \( |\mathbf{\omega}| \) to \(-\pi \leq \theta \leq \pi \) since
The Exponential Map normalization uses $|\omega| = 0$ and has no singularities so works for any $\theta$ and, hence, is favoured for larger rotations.

### Table 2-1: Rotator Formulae for various normalizations [45, adapted]

<table>
<thead>
<tr>
<th>Normalization</th>
<th>Scale Factor $(\gamma)$</th>
<th>Spinor $\omega$</th>
<th>Rotator $\mathbf{R} = \text{Rot}(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>$\frac{1}{</td>
<td>\omega</td>
<td>} = \frac{1}{\omega}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(= e^{\Omega\sin(\theta)})^0 = e^{N^0} = e^{\Theta}$</td>
</tr>
<tr>
<td>Unit Axial-Vector</td>
<td>$\frac{1}{</td>
<td>\omega</td>
<td>} = \frac{1}{\omega}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(= e^{N\sin(\theta/2)})^0 = e^{N^0} = e^{\Theta}$</td>
</tr>
<tr>
<td>Rodrigues - Cayley</td>
<td>$\tan(\theta/2)/</td>
<td>\omega</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(= (\Sigma + \Sigma^{-1})^{-1} = (\Sigma - 1)(\Sigma + 1)^{-1})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rankin-Brogan use: $\mathbf{I} + \Sigma + 0.5 \Sigma^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>as $</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rankin-Brogan use: $\mathbf{I} + \Sigma + 0.5 \Sigma^2$</td>
</tr>
<tr>
<td>de Veubeke</td>
<td>$\frac{\sin(\theta/2)}{</td>
<td>\omega</td>
<td>}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(= e^{\Omega_p\sin(\theta/2)})^0 = e^{N^0} = e^{\Theta}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rankin-Brogan use: $\mathbf{I} + \mathbf{\Omega}_p \sqrt{1 + 0.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>as $</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rankin-Brogan use: $\mathbf{I} + \mathbf{\Omega}_p \sqrt{1 - 0.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>as $</td>
</tr>
<tr>
<td>Exponential Map</td>
<td>$\frac{\theta}{</td>
<td>\omega</td>
<td>}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(= e^{\Theta_{\sin(\theta/2)})^0 = e^{N^0} = e^{\Theta}}$</td>
</tr>
</tbody>
</table>

$\mathbf{I}$ is the identity matrix.
The **Skew** operator is used to obtain a spinor \( \Omega \) from a rotator \( R \) as in \( \Omega = \text{Skew}(R) \). Like the **Rot** operator, the **Skew** formulation depends upon the normalization used. Adapted from Felippa [45], Table 2-2 shows **Skew** operators for the various normalizations.

Table 2-2: Spinor Formulae for various normalizations ([45], adapted)

<table>
<thead>
<tr>
<th>Normalization</th>
<th>Spinor</th>
<th>Spinor = Skew(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>( \Omega )</td>
<td>( \Omega = \left</td>
</tr>
<tr>
<td>Unit Axial-Vector</td>
<td>( N = \frac{1}{</td>
<td>\Omega</td>
</tr>
<tr>
<td></td>
<td></td>
<td>where ( R^a = \frac{1}{2} (R - R^T) ) = antisymmetric of ( R )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A more robust quaternion algorithm also exists for ( \theta = n\pi, n = 0, 1, \ldots ) where ( \sin \theta = 0 ). [45]</td>
</tr>
<tr>
<td>Rodrigues-Cayley</td>
<td>( \Sigma = \frac{\tan (\theta / 2)}{</td>
<td>\Omega</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Sigma_{RB} = 2\tan \left( \frac{\theta}{2} \right) \Omega )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Sigma_{RB} = 2(R - I)(R + I)^{-1} = \frac{4R^a}{1 + \text{trace}(R)} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Omega_{P} = \frac{\sin (\theta / 2)}{</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Omega_{P} = (R - I)(R + I)^{-1} = \frac{R^a}{\text{trace}(R) - 1} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Omega_{RB} = \frac{2 \sin (\theta / 2)}{</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Omega_{RB} = 2(R - I)(R + I)^{-1} = \frac{2R^a}{\text{trace}(R) - 1} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Theta = \log_e (R) = \frac{\arcsin \tau}{2\tau} (R - R^T) = \frac{\arcsin \tau}{\tau} R^a )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>where ( \tau = \frac{1}{2} \left</td>
</tr>
</tbody>
</table>

All of the above formulations require the rotation angle \( \theta \). This is found from the rotator using the property that, for any rotation matrix \( R \), \( \text{trace}(R) = 1 + 2 \cos \theta \). Reorganizing, this gives \( \cos \theta = 0.5 \left( \text{trace}(R) - 1 \right) \) with \( 0 \leq \theta \leq 2\pi \) by convention for uniqueness. [45]

Different researchers prefer different rotation pseudo-vector normalizations, each with its own relationship to the actual rotation \( \theta \). Table 2-3 shows various normalizations, the resulting relationship of their pseudo-vectors with rotation \( \theta \), and what rotations can be represented. For the
EICR Method, Rankin/Brogan [104] combines their own variants of the Rodrigues-Cayley normalization for cumulative rotations \( \overline{\omega} = 2 \tan(\theta/2) \approx \theta \) and of the de Veubeke normalization for incremental rotations \( \overline{\omega} = 2 \sin(\theta/2) \approx \theta \) duly modifying the updating process. Alternatively, Felippa/Haugen [47] uses the Exponential Map normalization \( \overline{\omega} = \theta \). Both approaches are discussed in detail in Chapter 3.

| Normalization                                           | Scale Factor (\( \gamma \)) \( \overline{\omega} = \gamma \overline{\omega} \) | Relationship of \( |\overline{\omega}| \) to \( \theta \) | Rotations \( \theta \) which can be represented                                                                 |
|----------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------|---------------------------------------------------------------------------------------------------------------|
| None                                                     | 1                                                                              | \( |\overline{\omega}| = \sin(\theta) \approx \theta \) | \( |\overline{\omega}| \) can represent \( \theta = \pm n\pi \) but limited to \(-1 \leq |\overline{\omega}| = \theta \leq 1 \) or \(-0.32\pi \leq \theta \leq 0.32\pi \) |
| Unit Axial-Vector                                        | \( \frac{1}{|\overline{\omega}|} \)                                           | \( |\overline{\omega}| = \sin(\theta/2) \approx \theta/2 \) |                                                                                                              |
| Rodrigues-Cayley                                          | \( \frac{\tan(\theta/2)}{|\overline{\omega}|} \)                             | \( |\overline{\omega}| = \tan(\theta/2) \approx \theta/2 \) | \( |\overline{\omega}| \) can represent any \( \theta \) except \( \theta = \pm n\pi \) which can occur for large nodal rotations |
| Rankin – Brogan Variant of Rodrigues-Cayley              | \( \frac{2 \tan(\theta/2)}{|\overline{\omega}|} \)                           | \( |\overline{\omega}| = 2\tan(\theta/2) \approx 0 \) |                                                                                                              |
| de Veubeke                                               | \( \frac{\sin(\theta/2)}{|\overline{\omega}|} \)                             | \( |\overline{\omega}| = \sin(\theta/2) \approx \theta/2 \) | \( |\overline{\omega}| \) can represent \( \theta = \pm n\pi \) but limited to \(-2 \leq |\overline{\omega}| = \theta \leq 2 \) or \(-0.64\pi \leq \theta \leq 0.64\pi \) |
| Rankin – Brogan Variant of de Veubeke                    | \( \frac{2 \sin(\theta/2)}{|\overline{\omega}|} \)                           | \( |\overline{\omega}| = 2\sin(\theta/2) \approx 0 \) |                                                                                                              |
| Exponential Map                                          | \( \frac{\theta}{|\overline{\omega}|} \)                                     | \( |\overline{\omega}| = 0 \)                      | \( |\overline{\omega}| \) can represent any \( \theta \) without singularities                                    |

Except for the exponential map \( |\overline{\omega}| = \theta \), all other normalizations only approximate \( \theta \) by \( |\overline{\omega}| \). However, these approximations are quite good for small (less than \( \pi/12 \)) or even moderate (less than \( \pi/6 \)) rotations as shown in Figures 2-5(a) and 2-5(b). For example the errors with the tangent normalizations for \( \theta = \pi/6 \) are 0.006 radians (1.1%) for Rodrigues-Cayley and 0.012 radians (2.2%) for the Rankin/Brogan variant of Rodrigues-Cayley. The sine normalization errors for \( \theta = \pi/6 \) are even smaller, ranging from 0.003 radians (0.57%) for de Veubeke to 0.006 radians (1.1%) for the Rankin/Brogan variant of de Veubeke.
2.2.3 Nature of the Spinor

Spinor $\mathbf{N}$ represents the derivative with respect to $\theta$ evaluated at $\theta = 0$ of rotator $\mathbf{R}$ or, in other words, an infinitesimal rotation. Hence, $\mathbf{N}\theta$ approximates the small rotation for $\theta$ near $\theta = 0$. This is shown below using Rodrigues Formula and help from Aguilar [2] and Lewis [79]. It also is shown more elegantly in Chapter 4 when rotations are discussed using Lie Theory.

Rodrigues Rotation Formula:

$$\mathbf{R}(\mathbf{N},0) = \mathbf{I} + \sin(\theta) \mathbf{N} + (1-\cos(\theta)) \mathbf{N}^2$$

where $\mathbf{N} = \text{Spin}(\mathbf{n}) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$, $\mathbf{n} = (n_1, n_2, n_3)^T = \frac{\omega}{|\omega|}$, $|\mathbf{n}| = 1$

$$\mathbf{R}(\mathbf{N},0) = \mathbf{I} \text{ where } \mathbf{I} = 3 \times 3 \text{ identity matrix for } \theta = 0$$

By replacing $\mathbf{N}$ with its full matrix and expanding:

$$\mathbf{R}(\mathbf{n},0) = \begin{bmatrix} 1 + (1-\cos \theta)(n_1^2 - 1) & (1-\cos \theta)n_1n_2 - n_3 \sin \theta & (1-\cos \theta)n_1n_3 + n_2 \sin \theta \\ (1-\cos \theta)n_2n_1 + n_3 \sin \theta & 1 + (1-\cos \theta)(n_2^2 - 1) & (1-\cos \theta)n_2n_3 - n_1 \sin \theta \\ (1-\cos \theta)n_3n_1 - n_2 \sin \theta & (1-\cos \theta)n_3n_2 + n_1 \sin \theta & 1 + (1-\cos \theta)(n_3^2 - 1) \end{bmatrix}$$

By taking the first derivative of $\mathbf{R}(\mathbf{n},0)$ with respect to $\theta$, namely $\mathbf{R}^{(1)}(\mathbf{n},\theta)$:
\[
\mathbf{R}^{(1)}(n, \theta) = \begin{bmatrix}
\sin \theta (n_1^2 - 1) & \sin \theta n_2 - n_1 \cos \theta & \sin \theta n_3 - n_2 \cos \theta \\
\sin \theta n_2 n_1 + n_3 \cos \theta & \sin \theta (n_2^2 - 1) & \sin \theta n_3 + n_1 \cos \theta \\
\sin \theta n_3 n_1 - n_2 \cos \theta & \sin \theta n_3 + n_2 \cos \theta & \sin \theta (n_3^2 - 1)
\end{bmatrix}
\]

By evaluating at \( \theta = 0 \):

\[
\mathbf{R}^{(1)}(n, 0) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = \mathbf{N}
\]

By further direct calculation, the powers of \( \mathbf{N} \) and derivatives of \( \mathbf{R}(\mathbf{N}, \theta) \) can be shown to cycle as follows for integers \( k \geq 0 \) reflecting the cyclical nature of derivatives of sine and cosine.

For \( \mathbf{N}^a \), \( a \)th power of \( \mathbf{N} \):

\[
\begin{align*}
\mathbf{N}^1, \mathbf{N}^5, \ldots, \mathbf{N}^{(4k+1)} &= \mathbf{N} \\
\mathbf{N}^2, \mathbf{N}^6, \ldots, \mathbf{N}^{(4k+2)} &= \mathbf{N}^2 \\
\mathbf{N}^3, \mathbf{N}^7, \ldots, \mathbf{N}^{(4k+3)} &= -\mathbf{N} \\
\mathbf{N}^4, \mathbf{N}^8, \ldots, \mathbf{N}^{(4k+4)} &= -\mathbf{N}^2
\end{align*}
\]

For \( \mathbf{R}^{(a)}(n, 0) \), \( a \)th derivative of \( \mathbf{R}(\mathbf{n}, \theta) \):

\[
\begin{align*}
\mathbf{R}^{(1)}(n, 0), \mathbf{R}^{(5)}(n, 0), \ldots, \mathbf{R}^{(4k+1)}(n, 0) &= \mathbf{R}^{(1)}(n, 0) = \mathbf{N} \\
\mathbf{R}^{(2)}(n, 0), \mathbf{R}^{(6)}(n, 0), \ldots, \mathbf{R}^{(4k+2)}(n, 0) &= \mathbf{R}^{(2)}(n, 0) = \mathbf{N}^2 = \mathbf{N}^{(4k+2)} \\
\mathbf{R}^{(3)}(n, 0), \mathbf{R}^{(7)}(n, 0), \ldots, \mathbf{R}^{(4k+3)}(n, 0) &= \mathbf{R}^{(3)}(n, 0) = -\mathbf{N} = \mathbf{N}^{(4k+3)} \\
\mathbf{R}^{(4)}(n, 0), \mathbf{R}^{(8)}(n, 0), \ldots, \mathbf{R}^{(4k+4)}(n, 0) &= \mathbf{R}^{(4)}(n, 0) = -\mathbf{N}^2 = \mathbf{N}^{(4k+4)}
\end{align*}
\]

Using these relationships and a Taylor Series centred at \( \theta = 0 \), the rotator \( \mathbf{R}(n, \theta) \) can be approximated (by a finite series) or computed exactly (by an infinite series). The infinite series with normalization \( \omega/|\omega| \) and spinor \( \mathbf{N} = \mathbf{\Omega}/|\omega| \) gives rise to Rodrigues Formula. This is not surprising as the starting point for \( \mathbf{N} \) on the previous page was that formula. With the alternate normalization \( \omega/(|\omega|/\theta) \) and \( \Theta = \mathbf{\Omega}/(|\omega|/\theta) \), the infinite series also yields the exponential map.

Both of these outcomes are demonstrated below starting with the following infinite Taylor Series about \( \theta = 0 \).

\[
\mathbf{R}(n, \theta) = \mathbf{I} + \mathbf{R}^{(1)}(0) \frac{\theta}{1!} + \mathbf{R}^{(2)}(0) \frac{\theta^2}{2!} + \mathbf{R}^{(3)}(0) \frac{\theta^3}{3!} + \mathbf{R}^{(4)}(0) \frac{\theta^4}{4!} + \mathbf{R}^{(5)}(0) \frac{\theta^5}{5!} + \ldots
\]

Rodrigues Formula results from replacing \( \mathbf{R}^{(0)}(0) \) by \( \pm \mathbf{N} \) or \( \pm \mathbf{N}^2 \) as appropriate, regrouping and substituting \( \sin \theta \) and \( 1 - \cos \theta \) as their series as shown below.

\[
\mathbf{R}(\mathbf{N}, \theta) = \mathbf{I} + \mathbf{N} \frac{\theta}{1!} + \mathbf{N}^2 \frac{\theta^2}{2!} + \mathbf{N}^3 \frac{\theta^3}{3!} + \mathbf{N}^4 \frac{\theta^4}{4!} + \mathbf{N}^5 \frac{\theta^5}{5!} + \mathbf{N}^6 \frac{\theta^6}{6!} + \mathbf{N}^7 \frac{\theta^7}{7!} + \mathbf{N}^8 \frac{\theta^8}{8!} + \ldots
\]
The exponential map results from replacing \( R^{(4k+a)}(0) \) by \( N^{(4k+a)} \), regrouping, and substituting \( \Theta = N \theta \) and \( e^{\Theta} \) for its series.

\[
R(N, \theta) = I + N \theta + \frac{N^2 \theta^2}{2!} + \frac{N^3 \theta^3}{3!} + \frac{N^4 \theta^4}{4!} + \frac{N^5 \theta^5}{5!} + \frac{N^6 \theta^6}{6!} + \frac{N^7 \theta^7}{7!} + \frac{N^8 \theta^8}{8!} + \ldots
\]

\[
R(N, \theta) = I + \sin \theta N + (1 - \cos \theta) N^2
\]

The exponential map also can be expressed in the closed form shown below since the powers of \( \Theta = N \theta \) cycle just like \( N \).

\[
R(\Theta, \theta) = e^{\Theta} = I + \frac{\Theta}{1!} + \frac{(\Theta)^2}{2!} + \frac{(\Theta)^3}{3!} + \frac{(\Theta)^4}{4!} + \frac{(\Theta)^5}{5!} + \frac{(\Theta)^6}{6!} + \frac{(\Theta)^7}{7!} + \frac{(\Theta)^8}{8!} + \ldots
\]

\[
R(\Theta, \theta) = e^{\Theta} = I + \sin \frac{\Theta}{\theta} \Theta + \frac{(1 - \cos \theta)}{\theta^2} \Theta^2
\]

The fact that both Rodrigues Formula and the exponential map can be developed from the same Taylor series only reinforces the earlier assertion that all rotator formulae are equivalent once adjusted for their differing normalizations. Further, as explored in Chapter 4, the exponential relationship between a spinor \( \Omega \) and a rotator \( R \) (e.g. \( R(\Omega) = e^{\Omega \theta} \)) shows that rotators and spinors respectively are elements of the Lie group SO(3) and its Lie algebra so(3).

2.2.4 Vector & Axis Rotations

Affine transformations consisting of rigid translations and rotations are used to move between the global, base and corotated coordinate frames. Two types of transformations are possible.

- **Axis Transformations** rigidly translate and rotate one coordinate frame to another, not moving vectors but rather re-expressing them in the new coordinate frame.
- **Vector Transformations** rigidly translate and rotate a vector (without changing its length) from one configuration to another thereby aligning it with the new configuration but still expressing it in the original coordinate frame.

In both cases, if the rigid translation is removed first as is the case in the EICR and similar corotational methods, the affine transformation becomes a linear transformation. These linear transformations are rigid rotations including both axis and vector rotations as discussed below.

The EICR and similar corotational methods use orthogonal coordinate frames formed by three orthogonal vectors. These vectors form a basis of \( \mathbb{R}^3 \) as independent (since orthogonal) and spanning (since some linear combination represents any point in \( \mathbb{R}^3 \)). Hence, if their origins coincide (i.e. rigid translation removed first), change of basis transformations can be used to transform coordinates and vectors between the various coordinate frames. Given two bases \( E_k \) and \( G \) of \( \mathbb{R}^3 \), the basis transformation matrix from \( E_k \) to \( G \) (\( R_{A: E_k \rightarrow G} \)) has columns consisting of the \( E_k \) basis vectors expressed in the \( G \) basis. \( R_{A: E_k \rightarrow G} \) is an axis rotation matrix rotating \( E_k \) to \( G \) and so expressing any vector in \( G \). For example, Figure 2-6 shows point \( A \) (and its position vector \( v \)) with coordinate frames \( G \) and, rotated by \( \theta \), \( E_k \).

![Figure 2-6: Coordinate Frames G & E_k, rotated by \( \theta \), E_k](image)

![Figure 2-7: Basis Vectors of E_k expressed in basis of G](image)

The coordinates and position vector of \( A \) can be expressed in \( G \) or \( E_k \) and transformed between the two frames using \( R_{A: E_k \rightarrow G} \). \( R_{A: E_k \rightarrow G} \) rotates \( E_k \) to \( G \) (\( \theta \) radians counter-clockwise or...
+θ by the right hand rule) transforming the E_k coordinates and position vector of point A to
point A to frame G whereas R^{T}_{A:E_k→G} does the opposite. Figure 2-6 shows the basis vectors of
E_k expressed in the standard basis of G as in e^k_1 = cos θ e^g_1 - sin θ e^g_2 + 0 e^g_3 = (cosθ, -sinθ, 0)^T.
Assembling e^k_1, e^k_2, and e^k_3 as columns of a matrix yields the R_{A:E_k→G} axis rotation matrix used to
transform coordinates or vectors from the E_k frame to the G frame. Hence, v_G = R_{A:E_k→G} v_{Ek} where:

\[
R_{A:E_k→G} = \begin{bmatrix}
\cos θ & \sin θ & 0 \\
-\sin θ & \cos θ & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Similarly, vector rotations can be used to rotate a vector (versus its coordinate frame) but
with one significant difference. In transforming coordinates (e.g. R_{A:E_k→G}), the vector remains fixed
while the coordinate frame rotates. With vector rotation (R_{V,θ}), the vector rotates by θ relative to a
fixed coordinate frame as seen in Figure 2-8. The box on the next page illustrates finding R_{V,θ}.

Successive axis/vector rotations can be expressed as a single transformation using matrix
multiplication. Assuming R_{A:B→C} followed by R_{A:C→D} and then R_{A:D→E}, the combined rotation is
R_{A:B→E} = R_{A:D→E} R_{A:C→D} R_{A:B→C} transforming coordinates/vectors from frame B to E. Any
number of transformations can be composed in this way. However, for spatial rotations about
differing axes, rotations do not commute so the composition order matters. Small rotations some-
times are treated as commuting in R^3 but only by approximating cosθ_i ≈ 1, sinθ_j ≈ 0, and higher order
terms as zero (θ^n_i ≈ 0, θ^n_j…θ^n_k ≈ 0).
\( R_{V:0} \) can be found as follows.

Let \( |v| = |v_1| = |v_2| \) since vector length is constant with rigid rotation.

Using \( v_1 \) in Figure 2-8, find \( \cos \psi \) and \( \sin \psi \).

\[
\begin{align*}
  x_1 &= |v| \cos \psi \quad \text{so} \quad \cos \psi = x_1 / |v| \\
  y_1 &= |v| \sin \psi \quad \text{so} \quad \sin \psi = y_1 / |v|
\end{align*}
\]

Substituting \( x_1, y_1 \) from above, find \( v_2 = (x_2, y_2, z_2)^T \) in terms of \( v_1 \).

\[
\begin{align*}
  x_2 &= |v| (\cos \theta \cos \psi - \sin \theta \sin \psi) \\
  y_2 &= |v| (\sin \theta \cos \psi + \cos \theta \sin \psi) \\
  z_2 &= z_1 \\
  &= |v| (\cos \theta (x_1 / |v|) - \sin \theta (y_1 / |v|)) \\
  &= |v| (\sin \theta (x_1 / |v|) + \cos \theta (y_1 / |v|)) \\
  &= x_1 \cos \theta - y_1 \sin \theta \\
  &= x_1 \sin \theta + y_1 \cos \theta
\end{align*}
\]

Collect \( v_2 = (x_2, y_2, z_2)^T \).

\[
R_{V:0} R_{zV: \theta} v_1 = v_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}
\]

Given the above:

\[
R_{V:0} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{A: Ek \rightarrow G}^T
\]

and, using this matrix,

\[
v_2 = R_{zV: \theta} v_1 = R_{zA: Ek \rightarrow G}^T v_1.
\]

Hence, a vector rotation is equivalent to an equal but opposite axis rotation as Figure 2-9 shows.

2.3 Elastic Deformational Nodal Rotations

The EICR and similar corotational methods use elements which have rotational degrees of freedom (dofs) at each node. These rotational dofs represent rotational (elastic) deformation in the neighbourhood of a node, known as nodal rotations. Hence, understanding such rotations is important to explaining corotational methods.

A linear strain element is one for which strain can vary linearly across the element. The earliest such elements used mid-side nodes to achieve this linearity. Nodal rotations are an alternative formulation for linear strain elements that avoids mid-side nodes by allowing rotation
about the nodes. This formulation has more degrees of freedom than a constant strain element (with neither mid-side nodes nor nodal rotations) but fewer than a linear element with mid-side nodes. It also can cause some challenges with boundary continuity.

The literature defines nodal rotations in a variety of ways as summarized in the contribution by Zi-Fei/Song to the book *Advanced Finite Element Analysis* [137]. One approach is to define nodal rotations using displacement gradients as developed by Bergan/Felippa [16] who called it the *continuum mechanics definition* of nodal rotations. Later work by Allman [3-5] and Cook [30,31] also used this approach and forms the basis of its discussion here. An alternate approach defines nodal rotation as “extra” rigid nodal rotation, namely the difference between the rigid rotations of the element (i.e. from $c^0$ to $c^R$) and the global position vector of the node in $c^N$. This approach is well explained by Zi-Fei/Song [137] and that discussion is summarized here.

Nodal rotations also can be subdivided into drilling and bending rotations. A *drilling rotation* is a rotation about an axis normal to the element plane which passes through the node [103]. The element plane is formed by the element nodes for triangular elements with three non-collinear nodes; the diagonals connecting pairs of element nodes for quadrilateral elements; or, for beam elements, by the collinear nodes and the reference node used to orient the beam cross section. *Bending rotations* are rotations about the local element axes lying in the plane of the element. The EICR and similar corotational methods accept both drilling and bending rotations. For shell elements, only drilling rotations usually are considered (i.e. bending rotations assumed to be 0) as such elements have little or no stiffness normal to the element plane. All three nodal rotations must be considered for spatial solid elements.

Both definitions of a nodal rotation are explained below using a planar 3-noded triangular element with drilling rotations as an example. These explanations then are extended to the 3-noded noded shell element with nodal rotations used in the EICR Method in Section 2.3.3.
2.3.1 Nodal Rotations defined using Displacement Gradients

Two common planar triangular elements are the 3-noded (or constant strain - CST) triangle with only corner nodes and the 6-noded (or linear strain - LST) triangle also having mid-side nodes. The resulting degrees of freedom (dofs) are six for the CST (2 translations at 3 nodes) or twelve for the LST (2 translations at 6 nodes). The “drilling” triangle has three nodes, each with three dofs (2 translational, 1 rotational), for a total of nine dofs. It can be viewed as either an enhanced CST or a modified LST as discussed below.

Allman [4] describes a “drilling” triangle as a CST enhanced through modifying its shape functions. The new functions are constructed to include rotations which represent the nodal tangential rotations resulting from a mid-side deflection. For example, in Figure 2-11(b) below, the nodal rotations \( \omega_i \) (\( i = \) node) for adjoining nodes 1 and 2 are \( \omega_1 = -\omega_2 \). Drilling rotations are defined based on shear strain. Shear strain \( \gamma \) in continuum mechanics is defined as the change in angle between two lines originally orthogonal in the material configuration. It can be defined mathematically as follows [4].

\[
\gamma_{sy} = \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \theta_x - \theta_y \text{ where }
\]

\( \gamma_{sy} \) or \( \gamma_{yx} \) = shear strain in x-y plane  
\( \frac{\partial v}{\partial x} \) = shearing of \( v \) with respect to \( x \)  
\( \frac{\partial u}{\partial y} \) = shearing of \( u \) with respect to \( y \)  
\( \theta_x = \frac{\partial v}{\partial x} \) = rotation of x-axis (\( \theta_x \))  
\( \theta_y = -\frac{\partial u}{\partial y} \) = rotation of y-axis  
\( u \) (or \( v \)) = displacement along the x (or y) axis

Similarly, the drilling rotation at node \( i \) is defined as follows [4].

\[
\omega_i = \Omega(x_i, y_i) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_i} - \frac{\partial u_i}{\partial y_i} \right) = \frac{1}{2} (\theta_x + \theta_y) = \frac{1}{2} \left( \frac{\delta v_i}{\delta x_i} - \frac{\delta u_i}{\delta y_i} \right)
\]

where \( \omega_i \) = rotation for node \( i \)
\[ x_i, y_i = \text{coordinates of node } i \]
\[ u_i, v_i = \text{displacements of node } i \]
\[ \delta \bullet = \text{increment of } \bullet \text{ for a given iteration} \]

Hence, defining nodal rotations using displacement gradients makes nodal rotations dependent upon displacements.

The modified shape functions reduce to the standard CST functions at the nodes but elsewhere permit linear strain through their extra quadratic and cubic terms. Resulting displacements are compatible along element boundaries for all but the cubic terms. These cubic terms can be managed satisfactorily but explain why the “drilling” triangle formulated in this way performs less well than the LST although much better than the CST.

Cook [31] describes the “drilling” triangle as the result of a coordinate transformation from a twelve dof LST triangle to a nine dof “drilling” triangle. This transformation is shown in Figure 2-10 below. Using the “drilling” triangle, LST configurations \((u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, u_5, v_5, u_6, v_6)\) in \(R^{12}\) can be expressed as \((u_1, v_1, \omega_1, u_2, v_2, \omega_2, u_3, v_3, \omega_3)\) in \(R^9\). Hence, the set of all possible deformations of the nodes is a nine-dimensional manifold. This manifold and manifolds in general are discussed in Chapter 4.

\[
\begin{bmatrix}
u_1 \\
v_1 \\
u_2 \\
v_2 \\
u_3 \\
v_3 \\
u_4 \\
v_4 \\
u_5 \\
v_5 \\
u_6 \\
v_6
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1/2 & 0 & b_3 & 1/2 & 0 & -b_3 & 0 & 0 & 0 \\
0 & 1/2 & a_3 & 0 & 1/2 & -a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & b_1 & 1/2 & 0 & -b_1 \\
0 & 0 & 0 & 0 & 1/2 & a_1 & 0 & 1/2 & -a_1 \\
1/2 & 0 & -b_2 & 0 & 0 & 0 & 1/2 & 0 & b_2 \\
0 & 1/2 & -a_2 & 0 & 0 & 0 & 0 & 1/2 & a_2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_1 \\
u_2 \\
v_2 \\
u_3 \\
v_3 \\
u_4 \\
u_4 \\
u_5 \\
u_5 \\
u_6 \\
u_6
\end{bmatrix}
\]

where
\[ a_1 = (x_3 - x_2)/8, a_2 = (x_1 - x_3)/8, a_3 = (x_2 - x_1)/8 \]
\[ b_1 = (y_2 - y_3)/8, b_2 = (y_3 - y_1)/8, b_3 = (y_1 - y_2)/8 \]

Figure 2-10: Cook’s Coordinate Transformation [30]
Figure 2-11 illustrates the various triangular elements, nodal rotations and side deflections discussed above.

![Figure 2-11: Triangular Elements with Drilling Rotations or Mid-Side Nodes](image)

(a) 3-Noded Triangle with Drilling Rotations
(b) Deflection of Side when \( \omega_1 = -\omega_2 \)
(c) 6-Noded LST Triangle with Mid-Side Nodes

2.3.2 Nodal Rotations defined as “Extra” Rigid Rotation

Alternatively, nodal rotations can be defined as the difference between the rigid rotation of the nodal global position vector in \( \mathbf{c}^D \) and the rigid rotation of the element from \( \mathbf{c}^0 \) to \( \mathbf{c}^R \) (estimated through the best fit procedure described in Chapter 3). In this case, local nodal rotations are independent of local nodal displacements. Hence, the local element displacement field has two independent translational components: one determined by the nodal translational displacements and the other by the incremental rigid rotation. Figure 2-12 illustrates these “decoupled” components. The element’s deformation can be specified by \((u_1, v_1, \theta_1, u_2, v_2, \theta_2, u_3, v_3, \theta_3)\) so the set of all possible nodal deformations is a nine-dimensional manifold as expected.

2.3.3 Nodal Rotations in EICR and Similar Corotational Methods

Both the EICR Method by Rankin/Brogan/Nour-Omid [106] and the unified corotational formulation by Felippa/Haugen [49] use nodal rotations defined as extra rigid rotation but using shell rather than planar elements. With shell elements, each node has six degrees of freedom (3
translational, 3 rotational). In particular, the three-noded triangular shell element with nodal rotations discussed in this thesis has eighteen degrees of freedom. With such elements, only drilling rotations often are considered as bending rotations are negligible and assumed to be zero.

Figure 2-12: Nodal Displacements from (a) Translation and (b) Extra Rigid Rotation [137]

Drawing on the background provided in this and the introductory chapter, Chapter 3 explains the EICR and similar corotational methods using linear algebra.
Chapter 3

Explanation of EICR & Similar Corotational Methods using Linear Algebra

This chapter explains the EICR and similar corotational methods mathematically using linear algebra. The explanation follows the same progression as the development of these methods. It first discusses the original EICR Method (without a projector) developed by Rankin/Brogan [104]. It then moves to the enhanced EICR Method using a projector developed by Rankin and Nour-Omid [91,105] which ensures that element stress is invariant to rigid rotation. The subsequent projector discussion includes a more sensitive projector by Haugen [57] and Felippa/Hogan [47] which distinguishes the CSSE Method. This enhanced projector ensures translational as well as rotational invariance to rigid deformation. It reduces to the EICR projector if the translational component is ignored.

The original EICR Method proved adequate for most cases of large global deformations with small local strains (1-4%). Rankin/Brogan [104] actually showed its suitability for moderate local nodal rotations up to 30 degrees (π/6 radians) or 8% shear strain. However, this method did not enforce element invariance to rigid motion. Hence, for some shell elements sensitive to even infinitesimal rigid rotation, the original method often overestimated stiffness and so did not perform as well as with other elements. A common example of such stiffening is warping sensitivity which is over stiffening encountered when modeling curved surfaces with flat shell elements [91].

Such challenges led to the addition of the projector operator to the EICR Method as initially developed in Rankin/Nour-Omid [105], clarified and applied in Nour-Omid/Rankin [91], and later refined by Haugen [57] and Felippa/Haugen [47]. Not only does the EICR projector enforce stress invariance to rigid rotation but it also ensures that the deformed element is in force equilibrium and greatly simplifies the isolation of local elastic deformations. Hence the projector
has resulted in a much more robust EICR Method. As noted earlier, the EICR projector addresses element sensitivity to rigid rotation which is the most common issue. The projector and CSSE Method developed by Haugen [57] and Felippa/Haugen [47] also addresses element sensitivity to rigid translation which is far less common but, in such cases, the CSSE Method is preferable.

This chapter begins by reviewing the literature regarding corotational finite element methods, large rotation techniques, and projector operators (in the finite element context). It then discusses these topics individually and jointly to explain the EICR, CSSE, and similar corotational methods. It is important to note the following when reading this chapter:

- The EICR and similar corotational methods use the initial element material stiffness \( \bar{K}_M \) throughout the analysis. Hence, a material stiffness array (cube) for all elements is created as soon as the elements are formed in the base configuration. The appropriate elemental stiffness matrix from the array is used when processing a given element. This chapter assumes that this array already has been formed.

- The separation of pure deformations (and their later remerging) by transforming between global and local coordinate frames occurs at the element level in the inner most element loop as highlighted in Figure 3-1. Hence, an element is formed in global coordinates; transformed into local coordinates to find its local pure deformations, internal forces and tangential stiffness; and then returned to global coordinates to update the global internal force vector and tangent stiffness array for that element. All elements are processed in this way for a given iteration before solving globally and checking for convergence. Section 3.2.1 through 3.2.9 focuses on this process for one element in a given iteration.

- Section 3.2.10 moves to the global level to discuss the global solution process once all elements have been processed creating a fully updated global internal force vector and tangent stiffness array. This represents the iteration loop and, ultimately, the load step loop in Figure 3-1.

- The projector discussion in Section 3.3 is at the element level. The blocks used to assemble the projector correspond to element nodes not all body nodes. The application of the projector also occurs in the element loop prior to returning to global coordinates.

- As explained in Chapter 1, the explanations in this chapter are presented mainly using the three-noded triangular element with nodal rotations.
Table 3-1: Summary of EICR Method based on 3-D Beam Formulation by Khosravi [69]

<table>
<thead>
<tr>
<th>Process Steps</th>
<th>Coordinate Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate material stiffness matrix (( \overline{K}_M )) for all elements</td>
<td>local</td>
</tr>
<tr>
<td>(saved to be used throughout the analysis)</td>
<td></td>
</tr>
<tr>
<td><strong>LOAD STEP LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>Initiate / increment Load Step “( t + \Delta t )” with load ( \mathbf{L}^{(t+\Delta t)} )</td>
<td></td>
</tr>
<tr>
<td>Check termination criteria for load steps</td>
<td></td>
</tr>
<tr>
<td><strong>ITERATION LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>Initiate / increment Iteration “( i )”</td>
<td></td>
</tr>
<tr>
<td><strong>ELEMENT LOOP Start</strong></td>
<td></td>
</tr>
<tr>
<td>For each/all element(s):</td>
<td></td>
</tr>
<tr>
<td>Update nodal coordinates by adding total nodal deflection/rotation (( \mathbf{v} )) from previous iteration, see below. (v=(( \mathbf{u}^{(\text{def})} ), ( \omega^{(\text{def})} )), ( \mathbf{v} ) initially 0)</td>
<td>global</td>
</tr>
<tr>
<td>Form “best fit” local element frame from global nodal coordinates</td>
<td>global/local</td>
</tr>
<tr>
<td>Use “best fit frame” to:</td>
<td></td>
</tr>
<tr>
<td>Estimate rigid rotation ( \mathbf{R} ) of element, ( \mathbf{C}^0 ) to ( \mathbf{C}^0 )</td>
<td>global/local</td>
</tr>
<tr>
<td>Use ( \mathbf{R} ) to change global ( \mathbf{v}^{(\text{tot})} ) to local ( \mathbf{v}^{(\text{tot})} = \mathbf{R}^T \mathbf{v}^{(\text{tot})} )</td>
<td>global/local</td>
</tr>
<tr>
<td>Calculate projector ( \mathbf{P} ) using first variations of “best fit frame”</td>
<td>global/local</td>
</tr>
<tr>
<td>Use projector ( \mathbf{P} ) to:</td>
<td></td>
</tr>
<tr>
<td>Isolate local deform. (( \mathbf{v}^{(\text{def})} ) ) from total (( \mathbf{v}^{(\text{tot})} )), ( \mathbf{v}^{(\text{def})} = \mathbf{P} \mathbf{v}^{(\text{tot})} )</td>
<td>local</td>
</tr>
<tr>
<td>Calculate non-equilibrated internal force vector, ( \mathbf{\bar{f}} = \mathbf{\overline{K}}_M \mathbf{v} ), ( \mathbf{v} = (\mathbf{u} , \mathbf{\omega}) )</td>
<td>local</td>
</tr>
<tr>
<td>Ensure self-equilibrating internal force vector, ( \mathbf{\bar{f}}_{\text{equil}} = \mathbf{P}^T \mathbf{\bar{f}} )</td>
<td>local</td>
</tr>
<tr>
<td>Ensure ( \mathbf{K}_M ) invariant to rigid rotation, ( \mathbf{K}<em>M</em>{\text{clean}} = \mathbf{P}^T \mathbf{K}_M \mathbf{P} )</td>
<td>local</td>
</tr>
<tr>
<td>Calculate stress stiffness (( \overline{K}_S )) for element</td>
<td>local</td>
</tr>
<tr>
<td>Calculate tangential stiffness ( \mathbf{K}_\text{tan} = \mathbf{K}<em>M</em>{\text{clean}} + \mathbf{K}_S )</td>
<td>local</td>
</tr>
<tr>
<td>Transform back to global coordinates using ( \mathbf{R} ), as follows:</td>
<td>global</td>
</tr>
<tr>
<td>( \mathbf{f} = \mathbf{R}^T \mathbf{\bar{f}}<em>{\text{equil}} ), ( \mathbf{K}</em>\text{tan} = \mathbf{R}^T \mathbf{K}_\text{tan} \mathbf{R} )</td>
<td></td>
</tr>
<tr>
<td>Update global internal force matrix ( \mathbf{F} ) with ( \mathbf{f} ) for given element</td>
<td>global</td>
</tr>
<tr>
<td>Update global tangent stiffness matrix (( \mathbf{K} )) with ( \mathbf{K}_\text{tan} ) for given element</td>
<td>global</td>
</tr>
<tr>
<td><strong>ELEMENT LOOP End</strong></td>
<td></td>
</tr>
<tr>
<td>(Looping through all elements assembles global internal force &amp; stiffness matrices)</td>
<td></td>
</tr>
<tr>
<td>Calculate unbalanced force ( \Delta \mathbf{F}^{(i)} = \mathbf{L} - \mathbf{F}^{(i)} )</td>
<td>global</td>
</tr>
<tr>
<td>Calculate incremental nodal deflection ( \Delta \mathbf{v}^{(i)} = (\mathbf{K}^{(i)})^{-1} \Delta \mathbf{F}^{(i)} )</td>
<td>global</td>
</tr>
<tr>
<td>Update total nodal deflections ( \mathbf{v}^{(i+1)} = \mathbf{v}^{(i)} + \Delta \mathbf{v}^{(i)} )</td>
<td>global</td>
</tr>
<tr>
<td>Update total nodal rotations (matrix multiplication or Argyris formula)</td>
<td>global</td>
</tr>
<tr>
<td>Check criteria for convergence of ( \Delta \mathbf{v} ) (or practical termination if not converging)</td>
<td>global</td>
</tr>
<tr>
<td><strong>ITERATION LOOP End</strong></td>
<td></td>
</tr>
<tr>
<td>Update global coordinates</td>
<td>global</td>
</tr>
<tr>
<td>Plot deformed shape</td>
<td>global</td>
</tr>
<tr>
<td><strong>LOAD STEP LOOP End</strong></td>
<td></td>
</tr>
</tbody>
</table>
3.1 Literature Review

A wide variety of approaches are found in the literature for analyzing the deformation of a body (or a finite element) using frames rotating (and translating) with it. These approaches vary mainly in the number and type of coordinate frames used. Some approaches use a single frame for the body while others, in conjunction with finite element analysis, have a local frame for each element. As for frames types, corotational-coordinate methods use orthogonal frames which remain so even as the body or element deforms elastically. Convected methods, on the other hand, use frames which deform elastically with the body. This results in a curvilinear coordinate system reflecting the deformed metric so that coordinate values remain constant [47].

During the 1960s and 1970s, corotating frames appeared in the work of Wempner [129], Argyris et al.[8,9], and Horrigmoe/Bergen [60] for orthogonal element frames; Beltytschko/Hsieh [14] for convected element frames; and De Veubeke [38] for a single orthogonal body frame. This work expanded over the next two decades. In particular, Rankin with Brogan [104] and later Nour-Omid [91,105], Crisfield [34], Hsaio et al. [61], Felippa [19], Haugen [57] and Felippa/Haugen [47] focused on corotational approaches using orthogonal element frames which form the basis of this thesis. This focus sought to exploit orthogonality ($A^T = A^{-1}$ for matrices, $a \cdot b = 0$ for vectors) for computational simplicity and many elements each with its own local frame to obtain elastic deformations small enough to permit linear analysis locally despite nonlinearity existing globally.

Most of the corotational literature mentioned above starts with infinitesimal rotations but quickly extends to finite rotations. In so doing, the corotational literature draws heavily upon the very comprehensive work on rotations by Argyris [7] as well as supporting work by Hughes/Winget [62]. In particular, Argyris [7] proposed a pseudo-vectorial representation for rotations and an additive formula for updating such pseudo-vectors for finite rotations without first converting to full rotation matrices. Hughes/Winget [62] showed that any sized rotation has a
pseudo-vectorial representation and provided a formula for obtaining it. All of this rotation literature makes considerable use of classical work by Euler (e.g. Euler’s Rotation Theorem), Rodrigues (e.g. Rodrigues’ Rotation Formula), and Hamilton (e.g. quaternions).

A projector operator in the corotational finite element context appeared in Belytschko et al. [15] in 1984. Rankin/Nour-Omid [105] and Nour-Omid/Rankin [91] introduced a projector operator for the EICR method in 1988. About the same time, Crisfield [34] independently derived a projector operator for a spatial beam equivalent to that of Rankin and Nour-Omid. Haugen [57] and Felippa/Haugen [47] later developed a more sensitive projector operator (ensuring invariance to rigid body translation as well as rotation) which simplifies to Rankin and Nour-Omid’s (rotational) projector.

The derivation of these projectors required a key condition called biorthonormality or bi-orthogonality. Nour-Omid and Rankin [105] proved this condition only for specific local element choices (frame positioning, nodal numbering, etc.). In 1999, Rankin [102] proved it generally, based on the best fit criteria for local frames by De Veubeke [38], for all elements except those having only collinear nodes. Felippa/Haugen [47] sketched a general argument for all elements including those with collinear nodes. Biorthonormality is defined and discussed in Section 3.3.2.

The original EICR method was published by Rankin and Brogan [104] in 1986 for beam, triangular and quadratic elements, without the projector operator. After introducing the projector [105], Nour-Omid and Rankin [91] reformulated the EICR method in 1990 to fully incorporate the projector operator. In the same year, Crisfield [34] published a similar approach with a projector for spatial beams. Haugen [57] and Felippa/Haugen [47] expanded the area significantly providing a unified theoretical formulation and more sensitive corotational methods. Subsequent research, application and/or explanation by Battini [12], Felippa [44,45], Khosravi [67], Lapeira [73], Martinez [86], and Yaw [134] have made significant contributions to refining, understanding and/or promoting the method.
This thesis focuses on the Rankin/Brogan/Nour-Omid research [106] as well as the Felippa/Haugen research [49] so their publications are the main resources used herein. Extensive use also has been made of supporting corotational work by Felippa [44,45] and Khosravi [67-69] as well as work by Argyris [7] on large rotations. The published literature has been enriched significantly by generous personal help from Drs. Felippa, Khosravi and Rankin on various aspects of corotational methods and rotations.

It should be noted that, in 2006, Rankin [101] extended the EICR method to large strain cases. Rankin [103] currently is pursuing a modified corotational formulation using an irrotational local displacement field at the element centroid. An irrotational vector field is one with a curl of zero (i.e. no infinitesimal rotation at any point in the vector field) or, in other words, with no rotational elastic deformation. Unfortunately, these large strain and irrotational formulations are beyond the scope of this thesis.

3.2 Explanation of the Original EICR Method (before Projector)

The EICR and similar corotational methods separate the motion of an element into rigid and elastic deformation so that the small elastic deformation can be analyzed linearly. Both the rigid and elastic deformation may have translational and rotational components resulting in six global degrees of freedom for spatial cases, three translations along and three rotations about the principle axes. Various configurations, orthogonal coordinate frames, and associated transformations between these frames are used to achieve the separation of rigid and elastic deformation. These are discussed below. In this discussion, the centroid in statics can be taken as the centre of mass for the dynamic case unless otherwise indicated.

3.2.1 Configurations

As explained by Felippa [44], the EICR Method uses three main configurations to describe an individual finite element undergoing rigid body motion and local deformation with
respect to some global inertial frame. These are discussed below and shown in Figure 3-2.

i) The base configuration ($C^0$) is the initial configuration of the element before any rigid body or elastic deformations. It also is the reference configuration for comparing and analyzing all of the element’s subsequent elastic deformation, with significant computational savings over other non-linear methods using updated configurations.

ii) The corotated (or shadow) configuration ($C^R$) is the configuration resulting from rigid body deformation (translation and/or rotation) of the base configuration. As explained later, this configuration is estimated from deformed configuration so is mainly conceptual.

iii) The deformed (or current) configuration ($C^D$) is the configuration resulting from elastic as well as rigid deformation of the base configuration. Conceptually, it can be viewed as the elastic deformation of the corotated configuration.

![Figure 3-2: Configurations [44]](image)

For an element simultaneously undergoing rigid body and elastic deformation, the $C^R$ configuration often does not physically occur. Hence, $C^R$ (or, more particularly, the equivalent rigid body transformation of the base configuration) is estimated from the deformed configuration in order to isolate the rigid body motion. This estimation is accomplished by creating local corotational element frames ($E_o$ and $E_k$) as discussed in the next section.

Felippa/Haugen [47] describes two other configurations which should be mentioned for later use: the perturbed and iterated configurations. A perturbed configuration ($C^0 + \delta C$) represents,
as in variational calculus, a first variation \( \delta C \) of a configuration \( C \). As such, the perturbation \( \delta C \) is infinitesimal. The perturbed configuration is used in the definition and derivation of the projector operator in Section 3.3. An *iterated configuration* \( (C + \Delta C) \) represents, for iterative techniques like Newton-Raphson, a small but finite change \( \Delta C \) in a configuration \( C \). If a high convergence threshold is chosen, the number of iterations becomes very large (i.e. \( \Delta C \) becomes very small) and the iterated configuration approaches the perturbed configuration of \( C \). For this reason, \( \delta \)-notation often is used for both iterated and perturbed configurations as is done later in this thesis.

### 3.2.2 Coordinate Frames

The EICR Method and similar corotational methods use various *orthogonal* coordinate frames to track an element’s rigid body and local elastic deformations.

The three frames used by the EICR and similar corotational methods are shown in Figure 3-3 and described below with associated notation. This notation is drawn from the Rankin/Brogan/Nour-Omid research [106] for frame designations and the Felippa/Haugen research [49] for vector notation.

1. A single *global inertial frame* (“G”) for the *entire system*. This frame remains fixed throughout the analysis linking all parts of the system. Global basis vectors are designated as \( e^G_i \) with \( i = 1, 2, 3 \).

2. A *local corotational element frame* (“Ek”) for *each element* of the discretized system, which follows the element’s rigid body motion. For example, \( E_0 \) is the local frame for the base configuration \( c^0 \) and, hence, the reference configuration for the element. Likewise, for iteration \( k \), \( E_k \) approximates the orientation of the body’s best fit \( c^R \) configuration and tracks its rigid body motion. Local basis vectors are designated as \( e^0_i \) for \( E_0 \) and \( e^k_i \) for \( E_k \) with \( i = 1, 2, 3 \). Often for convenience, \( E_0 \) is aligned with the global frame in which case the base configuration \( (c^0) \) sometimes is then called the *global configuration* \( (c^G) \) [47].

3. A *global nodal surface frame* (“Sk”) for *each node* of the discretized system with “k” being as in (ii). The surface frame is tied rigidly to the node in the sense of rotating with
the node. Nodal rotations are defined and discussed in Chapter 2. $S_k$ represents the current rotational state compared to that of the base configuration ($S_0$). $S_0$ is arbitrary and usually taken as $I_3$. Comparing frame $S_k$ to $S_0$ in Figure 3-3 shows how surface frames move with nodes and change with rotational deformation. Surface frame basis vectors are designated as $s^0_i$ for $S_0$ and $s^k_i$ with $i = 1, 2, 3$.

![Figure 3-3: Global (G), Element (E₀, Eₖ) & Surface (S₀, Sₖ) Coordinate Frames with Triangular Shell Element ([67], notation modified)](image)

In their initial paper [104], Rankin/Brogan used a fourth frame type, local rotational frames ($Q_s^k, Q_e^k$), as an aid in developing their equations. However, these local frames were linked rigidly to $S_k$ (for $Q_s^k$) and $E_k$ (for $Q_e^k$); $Q_e^k$ ultimately became $E_k$ (for flat elements) or $S_k$ (for solid elements); and neither $Q_s^k$ nor $Q_e^k$ appeared in the final result of the initial paper [104] or at all in later EICR papers [91,105]. Hence, these local rotational frames are not discussed here.

### 3.2.3 Vector Notation for Corotational Frames

With corotational methods, two aspects need to be specified to fully describe a vector, namely:

i) the frame in which the vector is expressed in terms of its components, and

ii) the configuration to which the position/displacement vector relates physically or, as expressed by Felippa/Haugen [47], with which the vector is aligned.
The first consideration is a coordinate choice whereas the second is a physical one. For example, a nodal position vector physically in \( \mathbb{C}^D \) can be expressed in either local or global coordinates.

Hence, the vector notation used must reflect these choices as well as the particular node. The vector notation in this thesis is based on Felippa/Haugen [47] but with the EICR frame designations (i.e. \( G, E_0, E_k \)). The resulting notational conventions for this thesis are outlined below.

- Lower-case variables (e.g. \( x \)) are used for spatial vectors meaning those relating to \( \mathbb{C}^R \) or \( \mathbb{C}^D \). Upper-case variables (e.g. \( X \)) designate material vectors meaning those relating to \( \mathbb{C}^0 \). Material and spatial vectors are discussed more thoroughly in Chapter 4.

- A particular node “\( a \)” is designated by superscripting the vector variable as follows: \( x^{(a)} \) in \( \mathbb{C}^0 \), \( x^{(a,R)} \) in \( \mathbb{C}^R \), and \( x^{(a,D)} \) in \( \mathbb{C}^D \).

- The configuration to which the position/displacement (represented by the vector) relates physically is designated by the subcripting the vector variable as \( x_0 \) for \( \mathbb{C}^0 \), \( x_{R,k} \) for \( \mathbb{C}^D \) or \( x_{D,k} \) for \( \mathbb{C}^D \) where \( k \) is the iteration. The subscript \( G \) refers to the global frame \( G \).

- The frame used to express the vector (i.e. the coordinate choice) is designated by capping the vector variable as follows: no cap if \( G \) (e.g. \( x \)), a tilde if \( E_0 \) (e.g. \( \bar{x} \)), and a bar if \( E_k \) (e.g. \( \bar{x} \)).

Table 3-1 summarizes this notation for the various frame and configuration combinations.

Table 3-1: Vector notation reflecting frames and configuration choices

<table>
<thead>
<tr>
<th>Configuration being used</th>
<th>Physically Representing vector representing position of node “( a )” relative to origin of</th>
<th>G (( “G” ) subscript)</th>
<th>E_0 (( “E_0” ) subscript)</th>
<th>E_k (( “E_k” ) subscript)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mathbb{C}^0 )</td>
<td>( \mathbb{C}^R )</td>
<td>( \mathbb{C}^D )</td>
<td>( \mathbb{C}^0 )</td>
</tr>
<tr>
<td>G (no cap, ( x ))</td>
<td>( X^{(a)}_G )</td>
<td>( X^{(a,R)}_G )</td>
<td>( X^{(a,D)}_G )</td>
<td>( X^{(a)}_0 )</td>
</tr>
<tr>
<td>E_0 (tilde cap, ( \bar{x} ))</td>
<td>not applicable (n.a.)</td>
<td>( \bar{x}^{(a)}_0 )</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td>E_k (bar cap, ( \bar{x} ))</td>
<td>n.a.</td>
<td>n.a.</td>
<td>( \bar{x}^{(a,R)}_{R,k} )</td>
<td>( \bar{x}^{(a,D)}_{D,k} )</td>
</tr>
</tbody>
</table>
3.2.4 $E_o$ and $E_k$ Local Corotational Element Frames

As discussed earlier, $E_o$ is the local corotational element frame for the base configuration as well as the reference frame for the element’s subsequent motion. It is formed at node (1) from the global nodal coordinates using standard FEM conventions for the particular element type. The origin of $E_o$ is left either at node (1) in the Rankin/Brogan/Nour-Omid research [106] or moved to the element centroid in the Felippa/Haugen research [49]. $E_k$ is formed using the same approach as $E_o$. The same conventions apply to simpler bar, triangular membrane and quadrilateral membrane elements. Figure 3-4 summarizes the forming of $E_o$ and $E_k$ for linear beam (2-noded), triangular (3-noded) shell and quadrilateral (4-noded) shell elements.

This approach reasonably approximates the element’s rigid body motion relatively quickly using simple geometry and nodal positions. More accurate, but computationally expensive, estimates can be obtained using best fit criteria to position $C^R$. De Veubeke [38] developed two best fit criteria for positioning $C^R$ relative to $C^D$ based on minimizing either the kinetic energy of their relative motion or the Euclidean norm of their relative deformations. De Veubeke [38] also showed that the centroids of $C^R$ and $C^D$ coinciding is equivalent to a best fit using either criterion. Unfortunately, according to Felippa [44], De Veubeke’s best fit approaches sometimes result in an eigenvalue problem with many but no unique solution(s). Hence, the simpler geometric approach is preferable.

3.2.5 Transformations between Global, Base and Corotated Frames

The EICR method uses the following rotation matrices and their transposes for axis and vector rotations between its various coordinate frames: $R_{A:Eo \rightarrow G}$, $R_{A:Ek \rightarrow G}$ and $R_{A:Ek \rightarrow Eo}$ for the $G$, $E_o$ and $E_k$ coordinate frames as well as $R_{A:So \rightarrow G}$, $R_{A:Sk \rightarrow G}$ and $R_{A:Sk \rightarrow So}$ for the $G$, $S_o$ and $S_k$ frames. Figure 3-5 for axis rotations and Figure 3-6 for vector rotations show these rotations and their inter-relationships such that any particular rotation can be expressed as the product of two others. For example, $R_{A:Ek \rightarrow Eo} = R_{A:Ek \rightarrow G}^T R_{A:Eo \rightarrow G}$ is an axis rotation from $E_k$ to $E_o$.  

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<table>
<thead>
<tr>
<th></th>
<th>Beam Element</th>
<th>Triangular Element</th>
<th>Quadrilateral Element</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FEM Convention</strong></td>
<td>Origin: Node 1</td>
<td>Basis Vectors: $e^k_1$ in node 1 to 2 direction</td>
<td>Origin: Node 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q^*$ is position vector of x-section reference node $q$ &amp; may not be normal to $e^k_1$; hence above construction</td>
<td></td>
</tr>
<tr>
<td><strong>EICR Convention</strong></td>
<td>Origin: Centroid or Node (1)</td>
<td>Basis Vectors: as above</td>
<td>Origin: Centroid or Node (1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Forming $E_o$</strong></td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Forming $E_k$</strong></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td><img src="image7" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Figure 3-4: Forming of $E_o$ & $E_k$ (based on [44,104])

Figure 3-5: Axis rotations between various EICR coordinate frames ([47], modified)
These rotations can be used to express a vector in a new frame through re-expressing its components (i.e. axis rotation) or to rotate a vector to align it with a new frame without re-expressing the vector (i.e. vector rotation). Examples are given below based on Figure 3-7.

**Axis Transformation Example**

For any given configuration, local position vectors of point P in global coordinates can be transformed into local coordinates as shown below. Likewise, the coordinate transformation can be inverted to transform from local to global coordinates.

For $c^0$: $\mathbf{x}^{(P)} = R^T_{A:E_0} \mathbf{x}^{(P)}$  

For $c^R$: $\mathbf{x}^{(P,R)} = R^T_{A:E_0} \mathbf{x}^{(P,R)}$

where $\mathbf{x}^{(P)} = P$ position vector in $c^0$ relative the local origin expressed in global frame  
$\mathbf{x}^{(P)} = P$ position vector in $c^0$ relative the local origin expressed in local frame  
$\mathbf{x}^{(P,R)} = P$ position vector in $c^R$ relative the local origin expressed in global frame  
$\mathbf{x}^{(P,R)} = P$ position vector in $c^R$ relative the local origin expressed in local frame  
$R^T_{A:E_0} = $ axis rotation from G to $E_0$  
$R^T_{A:E_0} = $ axis rotation from G to $E_k$

Figure 3-8 provides a numerical example of global and local coordinates in $c^0$, $c^R$ and, adding elastic deformation, $c^D$. 

---

**Figure 3-6:** Vector Rotations for equal but opposite angles to axis rotations in Figure 3-5 ([47], modified)
The following properties of axis rotations are illustrated in Figure 3-7 and Figure 3-8 and should be noted for future use.
- The nodal position vectors in local coordinates for \( C^0 \) and \( C^R \) are equal as the configurations differ only by rigid (axis) rotation. This does not extend to \( C^D \) where elastic deformation is involved.

\[ \mathbf{x}^{(P)}_0 = \mathbf{x}^{(P,R)}_{R,k} \neq \mathbf{x}^{(P,D)}_{D,k} \]

- The lengths of local and global nodal position vectors for a given configuration are equal since the same vector with only components (i.e. bases) differing. Likewise, the lengths of nodal position vectors in \( C^0 \) and \( C^R \) are equal since only rigidly rotated but again this does not extend to \( C^D \).

\[ |\mathbf{x}^{(P)}_o| = |\mathbf{x}^{(P)}_0| = |\mathbf{x}^{(P,R)}_{R,k}| = |\mathbf{x}^{(P,R)}_{R,k}| \neq |\mathbf{x}^{(P,D)}_{D,k}| = |\mathbf{x}^{(P,D)}_{D,k}| \]

where \( |\bullet| \) = length of \( \bullet \)

**Vector Transformation Example**

In global coordinates, the affine transformation of the P position vector \( \mathbf{x}^{(P)}_o \) in \( C^0 \) to \( \mathbf{x}^{(P,R)}_{R,k} \) in \( C^R \) is as follows:

\[ \mathbf{x}^{(P,R)}_{R,k} = \mathbf{R}_{A:E,R,k \rightarrow G} \mathbf{R}^T_{A:E,o \rightarrow G} \mathbf{x}^{(P)}_o + \mathbf{u}_{Eo \rightarrow Ek} \] where, in expressed in global frame,

- \( \mathbf{x}^{(P)}_o = \) P position vector in \( C^0 \) relative the local origin (centroid)
- \( \mathbf{x}^{(P,R)}_{R,k} = \) P position vector in \( C^R \) relative the local origin (centroid)
- \( \mathbf{R}_{A:E,R,k \rightarrow G} \mathbf{R}^T_{A:E,o \rightarrow G} = \) vector rotation from \( C^0 \) to \( C^R \)
- \( \mathbf{u}_{Eo \rightarrow Ek} = \) rigid body translation from \( C^0 \) to \( C^R \)

If the rigid translation is removed first as done in the EICR and similar corotational methods, the above transformation simplifies to a rigid rotation (i.e. linear transformation) as shown below.

\[ \mathbf{x}^{(P,R)}_{R,k} = \mathbf{R}_{A:E,k \rightarrow G} \mathbf{R}^T_{A:E,o \rightarrow G} \mathbf{x}^{(P)}_o \]

It should be noted that, understandably, differences exist between the frame and transformation conventions used by the Rankin/Brogan/Nour-Omid research \[106\] and those used by the Felippa/Haugen research \[49\]. Since this can be confusing, the major differences and their treatments in this thesis are summarized in Table 3-2 and also highlighted where appropriate.
Table 3-2: Notational & Other Conventions used by various authors

<table>
<thead>
<tr>
<th>Formulation</th>
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<td>columns (axis rotation)</td>
<td>Eₖ</td>
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<td>rows (vector rotation)</td>
<td>Tₐ⁻¹</td>
<td>Rₐ⁻¹ Tₐ⁻¹</td>
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<tr>
<td>columns (axis rotation)</td>
<td>Rₐ,Eₖ→G</td>
<td>Rₐ,E₀→G</td>
</tr>
</tbody>
</table>

3.2.6 Iterative Solution Notation

The EICR and similar corotational methods use iterative solution methods (such as Newton-Raphson) which require slightly expanded notation. In particular, notation is required to represent both cumulative and incremental displacements. This notation is introduced here based on iterated configurations which were discussed earlier. As mentioned then, δ-notation commonly is used for both iterated and perturbed configurations as is done in the discussion which follows. The v and Δv used in the Newton-Raphson explanation in Chapter 2 also are respectively v and δv herein. The expanded notation which follows is based on that used by Haugen [57].

As Khosravi [68] highlights, iterative solution methods such as the Newton-Raphson Method apply a control increment (step) and then iterate within that step until the iterations converge as defined by some pre-determined convergence criterion. The stepped control usually is external load or arc length. The convergence criteria can be based on incremental displacement, force, or energy being below a certain threshold. The discussion which follows assumes load control with an incremental displacement convergence criterion.

Each element has a base configuration or state (E₀) that is defined by a state vector of nodal positions and rotations in global coordinates as shown below for the spatial case.

Element: \[ \mathbf{E}^0 = (\mathbf{e}_1^0, \mathbf{e}_2^0, \ldots, \mathbf{e}_n^0)^T [57] \]

where
The initial state vector \( \mathbf{e}^o \) is updated to the deformed state vector \( \mathbf{e}^d \) once the iterations for a given load step have converged. It should be noted that, since spatial rotations \( \mathbf{R}_a \) do not transform as vectors, \( \mathbf{e}^o \) and \( \mathbf{e}^d \) really are arrays rather than vectors [47].

Any displacement vector must have the same components as the state vector. For an iterative solution, two displacements vectors actually are needed: the cumulative displacement for the load step and the incremental displacement for any given iteration. The cumulative displacement is updated after each iteration and, once convergence is achieved, reset to zero for the start of the next load step. To simplify the updating and since iterated rotations are quite small, the incremental rotations are expressed as pseudo-vectors \( \mathbf{\omega} = (\omega_1, \omega_2, \omega_3)^T \) based on their spin representations and the normalization used. These pseudo vectors reduce storage (as \( R^3 \)) compared to 3x3 rotation or spinor matrices. Typical cumulative and incremental displacement vectors are shown below.

Cumulative Displacement Vector of Element from iteration \( k \)

\[
\mathbf{v}^k = \left( v_1^k, v_2^k, \ldots, v_a^k \right)^T \tag{57}
\]

where

\( a = \) element node number

\[
v_a^k = \left( u_a^k, R_a^k R_o^a \right)^T \text{ expressed as } \left( u_a^k, \omega_a^k \right)^T \text{ before converting to rotation}
\]

\[
u_a^k = \left( u_{a1}^k, u_{a2}^k, u_{a3}^k \right)^T = \text{nodal translation vector}
\]

\[
R_a^k R_o^a = \text{nodal rotation matrix, simplifying to } R_a^k \text{ if } R_o^a \text{ chosen as } I_{3x3}
\]

\[
\left( R_a^k \right)^T = \text{from Figure 3-5}
\]

\[
\omega_a^k = \left( \omega_{a1}^k, \omega_{a2}^k, \omega_{a3}^k \right)^T = \text{nodal rotational pseudo-vector}
\]

For a particular iteration (i.e. \( k = i \)) of a given time step (e.g. \( (t+\Delta t) \)), \( \mathbf{v}^k \) is the same as \( \mathbf{v}_i^{(t+\Delta t)} \) used in the Newton-Raphson explanation in Chapter 2.
Incremental Displacement Vector of Element from iteration k

\[ \delta v^k = \left( \delta v_1^k, \delta v_2^k, \ldots, \delta v_a^k \right) \] \hspace{1cm} \text{[57]}

where

- \( a \) = element node number

\[ \delta v_a^k = \left( \delta u_a^k, \delta (R_a^k R_0^k) \right)^T \]

expressed as \( \left( u_a^k, \omega_a^k \right)^T \) before converting to rotator

\[ \delta u_a^k = (\delta u_{a1}, \delta u_{a2}, \delta u_{a3})^T \]

= nodal incremental translation vector

\[ \delta (R_a^k, R_0^k) = \text{nodal incremental rotation matrix} \]

\[ \delta \omega_a^k = (\delta \omega_{a1}, \delta \omega_{a2}, \delta \omega_{a3})^T \]

= nodal incremental rotational pseudo-vector

For a particular iteration (e.g. \( k = i \)) of a given time step (e.g. \( t+\Delta t \)), \( \delta v^k \) is the same as \( \Delta v_i^{(t+\Delta t)} \) used in the Newton-Raphson explanation in Chapter 2.

The above vectors represent rigid and elastic deformational displacement (both translations and rotations) in the global/inertial frame. These vectors can be transformed to the local frame by an axis rotation in which case they represent only elastic deformation and are designated using the usual bar notation as shown below.

\[ \bar{v}^k = \left( \bar{v}_1^k, \bar{v}_2^k, \ldots, \bar{v}_a^k \right)^T \]

where \( \bar{v}_a^k = \left( \bar{u}_a^k, \bar{R}_a^k R_0^k \right)^T \) or \( \left( \bar{u}_a^k, \bar{\omega}_a^k \right)^T \) \hspace{1cm} \text{[57]}

\[ \delta \bar{v}_a^k = \left( \delta \bar{v}_1^k, \delta \bar{v}_2^k, \ldots, \delta \bar{v}_a^k \right)^T \]

where \( \delta \bar{v}_a^k = \left( \delta \bar{u}_a^k, \delta \bar{\omega}_a^k \right)^T \) \hspace{1cm} \text{[57]}

3.2.7 Isolating Translational Elastic Deformation from Total Displacement

To quote Rankin/Brogan [104], “... any displacement field in a deforming body can be decomposed into a rigid body translation, a rigid body rotation, and a strain-producing deformation”. If rigid body translation is removed first, this statement is equivalent to saying that any deformation gradient (\( F \)) can be decomposed into a rigid body rotation (\( R \)) and elastic stretch (\( U \) or \( v \)) as in the polar decomposition \( F = RU = vR \). The elastic stretch results in both translational (i.e. pure stretch) and rotational (i.e. shear) deformation. The deformation gradient and its polar decomposition are discussed fully in Chapter 4. This section focuses on isolating elastic translational deformation following the EICR and similar corotational methods (without projectors). Similarly, the next section discusses isolating rotational elastic deformation.
For any total global displacement vector for a node “a” \( (u^{(a, tot)}) \) in the field, the statement by Rankin/Brogan can be expressed as the following equation [104].

\[ u^{(a, \text{tot})} = u^{(a, \text{def})} + u^{(a, \text{rig})} \]  \( (1a) \)
or, alternatively, \( u^{(a, \text{def})} = u^{(a, \text{tot})} - u^{(a, \text{rig})} \)  \( (1b) \)

where:

- \( u^{(a, \text{tot})} \) = any given total displacement vector (rigid body & deformational)
- \( u^{(a, \text{def})} \) = the associated deformational displacement vector
- \( u^{(a, \text{rig})} \) = the associated rigid body displacement vector

For the purposes of this section, \( u^{(a, \text{tot})} \) is the translational component of the total global displacement vector (\( v^k \) or \( v_i^{(t+\Delta t)} \)) from the previous iteration as described in the Newton-Raphson method in Chapter 2. The goal is to isolate the elastic component of this translation initially in global coordinates and ultimately in local coordinates. This elastic deformation then can be analyzed linearly. The simpler notation found above is used in the discussion which follows but refers to the following displacements consistent with Table 3-1 and Table 3-2.

Global Coordinates:

\[ u^{(a, \text{tot})}_{G, E_0 \rightarrow E_k}, u^{(a, \text{def})}_{G, E_0 \rightarrow E_k}, u^{(a, \text{rig})}_{G, E_0 \rightarrow E_k} \]

Local Coordinates:

\[ u^{(a, \text{def})}_{D, E_0 \rightarrow E_k} = R^T_{A; E_k \rightarrow G} u^{(a, \text{def})}_{G, E_0 \rightarrow E_k} \]

Modified from Felippa/Haugen [47], Figure 3-9 shows a triangular element with drilling rotations which has been rigidly translated and rotated as well as deformed from its base configuration \( C^0 \) to a current (deformed) configuration \( C^D \). The best fit corotated configuration \( C^R \) also is shown for illustration. This best fit \( C^R \) assumes that the centroids of \( C^R \) and \( C^D \) coincide at \( C \) with the local frame origin also being located there.

In **global** coordinates, Figure 3-9 shows node (1) position vectors relative to the element centroid in \( C^0 \) (\( x^{(1)}_o \)), \( C^R \) (\( x^{(1,R)}_{R,k} \)) and \( C^D \) (\( x^{(1,R)}_{R,k} + u^{(1,\text{def})}_e \)), node (1) total (\( u^{(1,\text{tot})} \)), rigid (\( u^{(1,\text{rig})} \)) and deformational (\( u^{(1,\text{def})} \)) displacements from \( C^B \) to \( C^D \); and, for local element frame \( E_k \) at \( C \), its rigid body translation (\( u^{(C, \text{rig})} \)) and axis rotation (\( R^T_{A; E_k \rightarrow E_0} \)) from \( C^0 \) to \( C^R \).
Using Figure 3-9, an expression for node (1)’s deformational displacement \( (u^{(1, \text{def})}) \) in global coordinates can be found using equation (1b) \( u^{(1, \text{def})} = u^{(1, \text{tot})} - u^{(1, \text{rig})} \). First, \( u^{(1, \text{rig})} \) is isolated as follows:

\[
x^{(1)} + u^{(1, \text{rig})} = x^{(C, \text{rig})} + x^{(1, R)}
\]

from Figure 3-9.[47]

Since the axis rotation from \( E_o \) (in \( C^0 \)) to \( E_k \) (in \( C^R \)) is \( \theta \), the vector \( x^{(1, R)} \) (in \( C^R \)) can be expressed as a vector rotation of \( x^{(1)} \) (in \( C^B \)) by \( \theta \) and, from Figure 3-6(a), \( R_{V,0} = R_{A,E_k \rightarrow G} R_{A,E_0 \rightarrow G}^{T} \) giving:

\[
u^{(1, \text{rig})} = u^{(C, \text{rig})} + R_{A,E_k \rightarrow G} R_{A,E_0 \rightarrow G}^{T} x^{(1)} - x^{(1)} \]

which simplifies to

\[
u^{(1, \text{rig})} = u^{(C, \text{rig})} + \left( R_{A,E_k \rightarrow G} R_{A,E_0 \rightarrow G}^{T} - I \right) x^{(1)} \]  \hspace{1cm} (2) [47]

Equation (2) shows that the rigid body translation of a node consists of the rigid body translation of the element (i.e. centroid) plus the extra nodal rigid translation resulting from the rigid rotation of the element as shown below.

\[
\begin{align*}
\text{Total Nodal Rigid Translation} & = u^{(c, \text{rig})} \\
\text{Rigid Translation from Element Rigid Rotation} & = \left( R_{A,E_k \rightarrow G} R_{A,E_0 \rightarrow G}^{T} - I \right) x^{(1)}
\end{align*}
\]
This extra translation is the difference between the local position vectors in \( C^0 (x^{(1)}_0) \) and \( C^R (x^{(1)}_{R,1}) \), by vector rotation, as shown in Figure 3-10. This is consistent with the expected motion of a point (not at the centroid) on a rigid body undergoing rigid motion.

Figure 3-10: Extra Translation from Rotation

The deformational displacement of node (1) \((u^{(1), \text{def}})\) now can be found in global coordinates as follows:

\[
    u^{(1, \text{def})} = u^{(1, \text{tot})} - u^{(1, \text{rig})} \quad \text{from (1b)}
\]

replacing \(u^{(1, \text{rig})}\) by (2)

\[
    u^{(1, \text{def})} = u^{(1, \text{tot})} - u^{(C, \text{rig})} - \left( R_{A:Ek \to G} R_{A:Eo \to G}^T - I \right) x^{(1)}_0 \quad (3)
\]

Equation (3) shows that the elastic translational deformation in global coordinates simply is the node’s total translational displacement less its rigid translation, as shown below.

\[
    \begin{align*}
        u^{(1, \text{def})} & = u^{(1, \text{tot})} - u^{(C, \text{rig})} - \left( R_{A:Ek \to G} R_{A:Eo \to G}^T - I \right) x^{(1)}_0 \\
        & = \underbrace{\text{Elastic Nodal Deformation}}_{\text{Total Nodal Displacement}} - \underbrace{\text{Rigid Element Translation}}_{\text{Extra Rigid Translation}}
    \end{align*}
\]

This global deformation can be transformed to local coordinates in \( E_k \) by an axis rotation from \( G \) to \( E_k \) using \( R_{A:Ek \to G}^T \) from Figure 3-5.

\[
    \overline{u}^{(1, \text{def})} = R_{A:Ek \to G}^T u^{(1, \text{def})} \quad \text{using axis rotation}
\]

which, substituting (3), becomes

\[
    \overline{u}^{(1, \text{def})} = R_{A:Ek \to G}^T \left[ u^{(1, \text{tot})} - u^{(C, \text{rig})} - \left( R_{A:Ek \to G} R_{A:Eo \to G}^T - I \right) x^{(1)}_0 \right]
\]
which, by expanding using $R^T_{A;Ek \rightarrow G}$ and then simplifying, becomes

$$\bar{\mathbf{u}}^{(1, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(1, \text{tot})} - \mathbf{u}^{(C, \text{rig})} \right) - R^T_{A;Ek \rightarrow G} \left( R^T_{A;Ek \rightarrow G} R^T_{A;Eo \rightarrow G} - \mathbf{I} \right) \mathbf{x}_o^{(l)}$$

$$\bar{\mathbf{u}}^{(1, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(1, \text{tot})} - \mathbf{u}^{(C, \text{rig})} \right) - R^T_{A;Eo \rightarrow G} \mathbf{x}_o^{(l)} + R^T_{A;Ek \rightarrow G} \mathbf{x}_o^{(l)}$$

which, by grouping $R^T_{A;Ek \rightarrow G}$ terms, simplifies to

$$\bar{\mathbf{u}}^{(1, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(1, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(l)} \right) - R^T_{A;Eo \rightarrow G} \mathbf{x}_o^{(l)}$$

which, by $R^T_{A;Eo \rightarrow G} \mathbf{x}_o^{(l)} = \tilde{\mathbf{x}}_o^{(l)}$ (i.e. axis rotation), simplifies to

$$\bar{\mathbf{u}}^{(1, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(1, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(l)} \right) - \tilde{\mathbf{x}}_o^{(l)}$$

This equation isolates the translational elastic deformation in local coordinates as desired. It can be generalized for any node $(a)$ as shown below.

$$\bar{\mathbf{u}}^{(a, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(a)} \right) - \tilde{\mathbf{x}}_o^{(a)} \quad (4) \quad [91]$$

where $\mathbf{x}_o^{(a)} = \text{node “a” position vector expressed in G}$

$$\tilde{\mathbf{x}}_o^{(a)} = R^T_{A;Eo \rightarrow G} \mathbf{x}_o^{(a)} \quad (4a)$$

which is node a’s local position vector in $C^0$ found by axis rotation

Equation (4) can be understood physically by simplifying it still further into equation (4d) shown below. To do so, it must be recalled from Section 3.2.5 that the local nodal position vectors in $C^0$ and $C^R$ are equal as only rigid (axis) rotation involved, specifically:

$$R^T_{A;Eo \rightarrow G} \mathbf{x}_o^{(a)} = \mathbf{x}_o^{(a)} = \overline{\mathbf{x}}^{(a,R)} = R^T_{A;Ek \rightarrow G} \mathbf{x}^{(a,R)} \quad (4b)$$

Further, as can be seen from Figure 3-9 (with $a=1$),

$$(\mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(a)} \mathbf{x}^{(a,D)}_{(D,k)} \quad (4c)$$

Substituting (4b) and (4c) into (4) gives the following:

$$\bar{\mathbf{u}}^{(a, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(a)} \right) - \tilde{\mathbf{x}}_o^{(a)} \quad (4)$$

$$\bar{\mathbf{u}}^{(a, \text{def})} = R^T_{A;Ek \rightarrow G} \left( \mathbf{x}^{(a,D)}_{(D,k)} \right) - \tilde{\mathbf{x}}^{(a,R)}_{(R,k)} \quad \text{from (4b) and (4c)}$$

$$\bar{\mathbf{u}}^{(a, \text{def})} = \mathbf{x}^{(a,D)}_{(D,k)} - \tilde{\mathbf{x}}^{(a,R)}_{(R,k)} \quad \text{from (4b)} \quad (4d)$$

Equation (4d) shows that the local elastic translational deformation is the difference between the local nodal position vectors for $C^D$ and $C^R$ in $E_k$ coordinates as shown in Figure 3-9.

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For computations, equation (4) is preferred as it expresses $\mathbf{u}^{(a,\text{def})}$ in terms of two rotations $\mathbf{R}_{A: \text{Eo} \to G}^T$, $\mathbf{R}_{A: \text{Ek} \to G}^T$ and three displacements $(\mathbf{u}^{(a,\text{tot})}, \mathbf{u}^{(C,\text{rig})}, \mathbf{x}^{(a)}_0)$ that can be found from the global nodal coordinates for $\mathbf{c}^0$ and $\mathbf{c}^D$. Further, since $\mathbf{c}^0$ is the reference configuration, its rotation $\mathbf{R}_{A: \text{Eo} \to G}$ and nodal position vectors $\mathbf{x}^{(a)}_0$ can be found initially and used for the entire analysis [104]. These computations can be illustrated by adding the inertial frame (G) to Figure 3-9 which becomes Figure 3-11. The $\mathbf{c}^D$ global nodal coordinates simply are the $\mathbf{c}^0$ coordinates updated for the previous iterations.

Using Figure 3-11, the rotations and displacements required for equation (4) can be found as follows based on the summary found in Felippa/Haugen [47, Table 3].

Step 1: Compute the following in $\mathbf{c}^0$ using global nodal coordinates

a) Position Vector of Centroid ($\mathbf{X}^{(C_0)}_G$)

The centroid position is computed by averaging the position vectors of the nodes which, according to Felippa [44], is accurate for 2-noded beams and 3-noded triangular elements and a workable approximation for quadrilateral elements.

So, generally: $\mathbf{X}^{(C_0)}_G = (1/N) \sum_{a=1}^{n} \mathbf{X}^{(a)}_G$ where $N =$ number of element nodes [47]

Or, specifically for Figure 3-11: $\mathbf{X}^{(C_0)}_G = 1/3 (\mathbf{X}^{(1)}_G + \mathbf{X}^{(2)}_G + \mathbf{X}^{(3)}_G)$

For dynamics with evenly distributed mass (i.e. equal mass lumped at each node), the centre of mass will coincide with the centroid so can be computed in the same way. With unevenly distributed mass, the computation must be weighted for nodal masses as follows: $\mathbf{X}^{(C_0)}_G = (1/N)(1/M)\sum_{a=1}^{n} \mathbf{X}^{(a)}_G m^{(a)}$ where $n =$ number of element nodes, $M =$ total mass, $m^{(a)} =$ mass at node “a”.

b) Local Elemental Frame $\mathbf{E}_0$ & $\mathbf{R}_{A: \text{Eo} \to G}$

$\mathbf{E}_0$ is formed based on the convention for the type of element as in Figure 3-4. Using the triangular element in Figure 3-11 and following Rankin/Nour-Omid [105], the basis vectors of $\mathbf{E}_0$ expressed in G are:

$\mathbf{e}^0_2 = \frac{\mathbf{X}^{(3)}_G - \mathbf{X}^{(1)}_G}{| \mathbf{X}^{(3)}_G - \mathbf{X}^{(1)}_G |}$ to obtain unit basis vector in nodes (1) - (3) direction

$\mathbf{e}^0_3 = \frac{\mathbf{X}^{(2)}_G \times \mathbf{X}^{(3)}_G}{| \mathbf{X}^{(2)}_G \times \mathbf{X}^{(3)}_G |}$ to obtain unit basis vector normal to nodal plane
\[ e_1^0 = e_2^0 \times e_3^0 \] to obtain unit basis vector orthogonal to \( e_2^0 \) and \( e_3^0 \). 

\( \mathbf{R}_{A: E_0 \rightarrow G} \) is found by simply assembling these basis vectors into the basis transformation (rotation) matrix 
\[ \mathbf{R}_{A: E_0 \rightarrow G} = \begin{bmatrix} e_1^0 & e_2^0 & e_3^0 \end{bmatrix}. \]

c) Nodal Position Vectors in \( G \) (\( x_o^{(a)} \)) and \( E_0 \) (\( \tilde{x}_o^{(a)} \))

Using the global position vectors for the nodes (\( x_G^{(a)} \)) and the centroid (\( x_G^{(C)} \)), the local nodal position vectors can be computed in \( G \) using \( x_o^{(a)} = x_G^{(a)} - x_G^{(C)} \) and then expressed in \( E_0 \) as \( \tilde{x}_o^{(a)} = \mathbf{R}_{A:E_0 \rightarrow G}^T x_o^{(a)} \) using axis rotation \( \mathbf{R}_{A: E_0 \rightarrow G} \) from above and equation (4a). Using node (1) in Figure 3-11 as an example:

\[ x_{o}^{(1)} = x_{G}^{(1)} - x_{G}^{(C)} \text{ and } \tilde{x}_{o}^{(1)} = \mathbf{R}_{A:E_0 \rightarrow G}^T x_o^{(1)} \]

\( \tilde{x}_o^{(a)} \) is required later for computing local deformations using equation (4).

Step 2: Compute the following in \( E \) using global nodal coordinates

a) Updated Nodal Position Vectors in \( G \) (\( x_G^{(a,D)} \))

For the next load step or iteration (\( k \)), the global nodal position vectors from \( E_0 \) are updated using the total displacement vector (\( u^{(a,tot)} \)) from the previous iteration (\( k-1 \)) as in \( x_G^{(a,D)} = x_G^{(a)} + u^{(a,tot)} \) [47]. Again using node (1), \( x_{G}^{(1,D)} = x_{G}^{(1)} + u_{G}^{(1,tot)} \).

b) Updated Centroid Position Vector in \( G \) (\( x_G^{(C)} \))

The updated \( x_G^{(C)} \) is found as in Step 1(a) but using the updated nodal position vectors (\( x_G^{(a,D)} \)) from Step 2(a).

c) Rigid Body Displacement of the Centroid in \( G \) (\( u^{(C,rig)} \))

Having the centroid’s position vectors in \( E_0 \) (\( x_G^{(C)} \)), Step 1(a)) and \( E \) (\( x_G^{(C)} \)), Step (b)), its rigid body displacement (\( u^{(C,rig)} \)) is computed as 
\[ u^{(C,rig)} = x_{G}^{(C)} - x_{G}^{(C^0)}. \]

d) Updated Local Elemental Frame \( E_k \) & \( \mathbf{R}_{A: E_k \rightarrow G} \)

The updated frame \( E_k \) and \( \mathbf{R}_{A: E_k \rightarrow G} \) are found as in Step 1(a) but using the updated nodal position vectors (\( x_G^{(a,D)} \)). In particular for Figure 11, the basis vectors are:

\[ e_2^k = \frac{x_{G}^{(3,D)} - x_{G}^{(1,D)}}{\|x_{G}^{(3,D)} - x_{G}^{(1,D)}\|} \text{ to obtain unit basis vector in nodes (1) - (3) direction} \]
\[ e_3^k = \frac{x_{G}^{(2,D)} \times x_{G}^{(3,D)}}{\|x_{G}^{(2,D)} \times x_{G}^{(3,D)}\|} \text{ to obtain unit basis vector normal to nodal plane} \]
\[ e_1^k = e_2^k \times e_3^k \text{ to obtain unit basis vector orthogonal to } e_2^k \text{ and } e_3^k \]

So, 
\[ \mathbf{R}_{A: E_k \rightarrow G} = \begin{bmatrix} e_1^k & e_2^k & e_3^k \end{bmatrix}. \]
Figure 3-11: Triangular Element Translated, Rotated & Deformed showing Global (Inertial) Frame ([47], modified)

C is element centroid of both $\mathbb{R}$ & $\mathbb{D}$

(1), (2), (3) & (1D), (2D), (3D) are element nodes in $\mathbb{R}$ & $\mathbb{D}$ respectively.
Step 3: Compute the Deformation in Local Coordinates ($E_k$)

Equation (4) is used to find the deformation in local coordinates ($E_k$), namely

$$
\vec{u}^{(a,\text{def})} = R^T_{A:E_k \rightarrow G} \left( u^{(a,\text{tot})} - u^{(C,\text{rig})} + x_o^{(a)} \right) - \tilde{x}_o^{(a)} \tag{4}
$$

with $R^T_{A:E_k \rightarrow G}$ from Step 2(d)

$u^{(a,\text{tot})}$ from the previous iteration

$u^{(C,\text{rig})}$ from Step 2(c)

$x_o^{(a)}$, $\tilde{x}_o^{(a)}$ from Step 1(c)

Hence, using the above algorithm, the translational deformation can be isolated from the total displacement in preparation for the linear analysis. Rotational deformation also may be present and its isolation is discussed in the next section.

### 3.2.8 Isolating Rotational Elastic Deformations from Total Rotations

In addition to translations, global and local rotations can occur. Global rotations can be separated into rigid and elastic components. The elastic components are called nodal rotations. As discussed in Section 2.3, the EICR and similar corotational methods define a nodal rotation as the “extra” nodal rigid rotation above and beyond the rigid rotation of the element. It is only these nodal rotations that involve strain and, hence, are needed for the linear finite element analysis at the element level done in the EICR and similar corotational methods.

Corotational methods use rotators for global rotations and, for local rotations, pseudo-vectors based on spinors. Employing pseudo-vectors reduces computation and storage requirements since a pseudo-vector has three parameters per node versus nine parameters for a rotator. Corotational methods first isolate local rotation as a rotator and then convert it to a pseudo-vector. Since local rotations are small but finite (i.e. not infinitesimal), this conversion uses the work of Hughes/Winget [62] to formulate first a spinor and then a pseudo-vector from the rotator.

The pseudo-vector is used for the local finite element analysis and for calculating global incremental rotations from unbalanced moments. However, the global cumulative rotations cannot be updated with vector addition as spatial rotations do not commute. The Felippa/Haugen
research [49] resolves this challenge by converting back to a rotator before updating global rotations. In the EICR Method, Rankin/Brogan [104] uses a pseudo-vector updating formula from Argyris [7] which is equivalent to using rotators but produces a new pseudo-vector. This updated pseudo-vector is converted to a rotator once the iterations have converged.

The sections which follow discuss isolating local rotations as rotators, their conversion to and use as pseudo-vectors, and the updating process.

### 3.2.8.1 Isolating Rotational Deformation as a Rotator

As with translations, the total global rotation of a node (e.g. \( R^{(a,\text{tot})} \) for node (a)) can be decomposed into a rigid rotation (\( R^{(a,\text{rig})} \)) followed by a rotational deformation (\( R^{(a,\text{def})} \)) as shown in Figure 3-12 adapted from Felippa/Haugen [47]. This order of composition (rigid followed by deformational) is a convention shared by the Felippa/Haugen [49], the Rankin/Brogan/Nour-Omid [49], and other [60] research. Such a convention is needed given that spatial rotations do not commute so their order of composition matters.

\( R^{(a,\text{tot})} \) is the rotational component of the total global displacement vector (\( v^k \) or \( v_{i}^{(t+\Delta t)} \)) from the previous iteration as described in the Newton-Raphson method in Chapter 2. The goal is to isolate the elastic component of this rotation initially in global coordinates (\( R^{(a,\text{def})} \)) and then in local coordinates (\( R^{(a,\text{def})}_{L} \)). Once \( R^{(a,\text{def})} \) is isolated, calculating \( R^{(a,\text{def})}_{L} \) is a standard tensor change of basis from global basis \( G \) to local basis \( E_k \), specifically:

\[
\bar{R}^{(a,\text{def})}_{L} = R_{A:E_k\rightarrow G} \left( R^{(a,\text{def})} \right) R_{A:E_k\rightarrow G}^T
\]

The general notation described above relates to the EICR coordinate frames (Section 3.2.2) and rotators (Section 3.2.5) as follows:

- \( R^{(a,\text{tot})} \) is the rotation of the node’s surface frame from \( S_0 \) in \( C^0 \) to \( S_k \) in \( C^D \), specifically:
  \[
  R^{(a,\text{tot})} = R_{A:S_k\rightarrow S_0}^T = R_{A:S_k\rightarrow G}^T R_{A:S_0\rightarrow G}
  \]
gives the current rotational state of the system (nodes) in global coordinates compared to an arbitrary initial state (\( S_0 \)). Commonly, \( S_0 \) is chosen to be \( I_3 \) so that \( R_{A:S_k\rightarrow S_0}^T \) simplifies to \( R_{A:S_k\rightarrow G}^T \) or, if \( G \) also is chosen as \( I_3 \), simply \( S_k \).
- \( R^{(a,\text{rig})} \) is the rigid rotation of the element frame from \( C^0 \) to \( C^R \), specifically:
  \[
  R^{(1,\text{rig})} = R_{A: Ek \rightarrow Eo}^T R_{A: Eo \rightarrow G} R_{A: Ek \rightarrow G}
  \]
  from Figure 3-5
  - \( R^{(a,\text{def})} \) is the deformational rotation between the element frame \( E_k \) and the surface frame \( S_k \) in \( C^D \), specifically:
  \[
  R^{(1,\text{def})} = R_{A: Sk \rightarrow Ek}^T
  \]

Using Figure 3-12 and following Felippa/Haugen [47], the rotational elastic deformation can be found as follows:

**In global coordinates:**

\[
R^{(1,\text{tot})} = R^{(1,\text{def})} R^{(1,\text{rig})}
\]

\[
R^{(1,\text{def})} = R^{(1,\text{tot})} (R^{(1,\text{rig})})^T
\]

\[
R^{(1,\text{def})} = R^{(1,\text{tot})} (R_{A: Ek \rightarrow G} R_{A: Eo \rightarrow G})^T \quad \text{because} \quad R^{(1,\text{rig})} = R_{A: Ek \rightarrow G} R_{A: Eo \rightarrow G}
\]

\[
R^{(1,\text{def})} = R^{(1,\text{tot})} R_{A: Eo \rightarrow G} R_{A: Ek \rightarrow G} \quad [47,58] \quad (5)
\]

**In local coordinates:**

\[
\bar{R}^{(1,\text{def})} = R_{A: Ek \rightarrow G} (R^{(1,\text{def})}) R_{A: Ek \rightarrow G} \quad \text{to change tensor basis}
\]

\[
\bar{R}^{(1,\text{def})} = R_{A: Ek \rightarrow G} (R^{(1,\text{tot})} R_{A: Eo \rightarrow G} R_{A: Ek \rightarrow G}) R_{A: Ek \rightarrow G} \quad \text{by substituting (5)}
\]

\[
\bar{R}^{(1,\text{def})} = R_{A: Ek \rightarrow G} R^{(1,\text{tot})} R_{A: Eo \rightarrow G} \quad \text{by simplifying}
\]

\[
\bar{R}^{(1,\text{def})} = R_{A: Ek \rightarrow G} R_{A: Sk \rightarrow G} R_{A: So \rightarrow G} R_{A: Eo \rightarrow G} \quad \text{because} \quad R^{(1,\text{tot})} = R_{A: Sk \rightarrow G} R_{A: So \rightarrow G}
\]

which can be generalized for any node “\( a \)” as

\[
\bar{R}^{(a,\text{def})} = R_{A: Ek \rightarrow G} R_{A: Sk \rightarrow G} R_{A: So \rightarrow G} R_{A: Eo \rightarrow G} \quad [47,58] \quad (6)
\]

Equation (6) expresses local rotational deformation in terms of the initial (\( k=0 \)) and latest (\( k \)) rotations of the elemental (\( E_k \)) and surface (\( S_k \)) frames. Computing \( R_{A: Eo \rightarrow G} \) and \( R_{A: Ek \rightarrow G} \) from the global nodal coordinates already has been discussed with respect to translational deformations. \( R_{A: So \rightarrow G} \) can be chosen arbitrarily with \( I_3 \) usually being used. This leaves only \( R_{A: Sk \rightarrow G} \) which represents to the total global rotation for node (\( a \)) from the previous iteration. As shown for the Newton-Raphson Method in general (Section 2.2) and the EICR Method in particular (Figure 3-1), each iteration ends with global displacements and rotations being updated based on the unbalanced forces including moments (load step minus internal forces). This gives \( R_{A: Sk \rightarrow G} \).
Figure 3-12: Rotational Transformations between Frames ([47], modified)
3.2.8.2 Conversion to and use as Pseudo-Vectors

The local rotational deformation found in equation (6) is expressed as a rotator $\bar{R}^{(a,\text{def})}$. To reduce computations and storage, corotational methods use pseudo-vectors rather than rotators for analyzing local rotational deformation. Hence, the rotator $\bar{R}^{(a,\text{def})}$ is converted first to a spinor using the appropriate Skew formula from Table 2-2 and subsequently to a pseudo-vector using the axial operator. With this conversion, rotator equation (6) becomes a pseudo-vector equation as shown below.

$$\bar{R}^{(a,\text{def})} = R_{A:Ek\rightarrow G} R_{A:Sk\rightarrow G} R_{A:So\rightarrow G} R_{A:Eo\rightarrow G}$$  \hspace{1cm} (6)

$$\omega^{(a,\text{def})} = \omega^{(a,\text{tot})} - \omega^{(\text{rig})}$$  \hspace{1cm} (7)

where expressed in local frame

- $\omega^{(a,\text{def})}$ = instantaneous deformational rotation at node (a)
- $\omega^{(a,\text{tot})}$ = instantaneous total rotation at node (a)
- $\omega^{(\text{rig})}$ = instantaneous rigid body rotation of element (i.e. of frame $E_i$)

Pseudo-vector $\omega^{(a,\text{def})}$ is used in the calculation of the local internal moments. In the global frame, the incremental rotation resulting from the unbalanced moments also is expressed as a pseudo-vector ($\delta\omega$). However, the cumulative rotation cannot be updated by simply adding pseudo-vector $\delta\omega$ as spatial rotations do not commute.

The formation and updating of pseudo-vectors varies between corotational methods depending upon the normalization and updating approaches used for rotations. The normalization chosen affects the Skew formula and how well the resulting pseudo-vector ($\omega$) approximates the actual nodal rotation ($\theta$) as shown in Table 2-3 and Figure 2-5. Cumulative rotations may be updated by either i) first converting pseudo-vectors to rotators and then multiplying matrices or ii) using an equivalent pseudo-vector updating formula developed by Argyris [7]. In this latter case, the resulting pseudo-vector is converted to a rotator once the solution has converged for a given iteration.
The Felippa/Haugen research [49] uses the $|\omega| = \theta$ normalization and, hence, the logarithmic map to convert rotators to spinors. Likewise, before updating rotations, these papers convert $\delta \omega$ to first a spinor and then, using the exponential map, to a rotator. Cumulative rotation (stored as a rotator) is updated by matrix multiplication. The $|\omega| = \theta$ normalization gives an exact as opposed to an approximate representation of $\theta$ and has no singularities.

For the EICR Method, Rankin/Brogan [104] use pseudo-vectors for both cumulative and incremental rotations using the Argyris pseudo-vector update formula [7]. To achieve this, the cumulative and incremental pseudo-vectors are normed differently to ensure that sufficiently large angles can be represented uniquely without singularities. The Argyris update formula also is modified to combine the two normalizations. The various normalizations are summarized below and discussed in detail in Chapter 2. The cumulative pseudo-vector is converted to a rotator once the solution has converged for a given iteration.

Rankin/Brogan [104] normalize incremental rotations using a variant of the Rodrigues-Cayley normalization, specifically $|\delta \omega| = 2\tan(\delta \theta/2)$. This normalization can represent any angle except for $\delta \theta = (2n+1)\pi$ for any integer $n$ resulting in a singularity. This limitation poses no problem for incremental rotations (since well less than $\pi$) but is problematic for cumulative rotations. Hence, for cumulative rotations, a variant of the de Veubeke normalization is used which has no singularities, namely $|\omega| = 2\sin(\theta/2)$. Unfortunately, this normalization cannot represent all possible cumulative rotations since limited to $-2 \leq |\omega| \approx \theta \leq 2$ or $-0.64\pi \leq \theta \leq 0.64\pi$. However, the cumulative rotations are updated by the incremental rotations using a modified Argyris updating formula and, hence, are no longer are limited to $-0.64\pi \leq \theta \leq 0.64\pi$. This results in an updated pseudo-vector which can represent sufficiently large cumulative rotation angles without singularities.

The modified Argyris pseudo-vector update formula used in the EICR Method is shown below from Rankin/Brogan [104] with notation adjusted to be consistent with this thesis.
\[ \omega_{k+1} = \pm \left( \delta \omega \sqrt{1 - 0.25|\omega_k|^2} + \omega_k - 0.5 (\omega_k \times \delta \omega) \right) \left( \sqrt{1 + 0.25|\delta \omega|^2} \right) \]  

[104]

with sign determined by sign of 

\[ \sqrt{1 - 0.25|\omega_k|^2} - 0.25 \omega_k \cdot \delta \omega \]

where

- \( \omega_{k+1} \) = updated cumulative pseudo-vector, \( |\omega_{k+1}| = 2\sin(\theta/2) \)
- \( \omega_k \) = previous cumulative pseudo-vector, \( |\omega_k| = 2\sin(\theta/2) \)
- \( \delta \omega \) = incremental pseudo-vector, \( |\delta \omega| = 2\tan(\delta \theta/2) \)

### 3.2.9 Elemental Internal Force Vector & Tangent Stiffness Matrix

The local rotational deformation for each node can be isolated and approximated by the pseudo-vector \( \overline{\omega}^{(a,\text{def})} \) as described above. The local translational deformation (\( \overline{u}^{(a,\text{def})} \)) for each node also can be found as discussed earlier in Section 3.2.7. These collectively form the local deformation vector \( \overline{v}^k \) for the element. As mentioned earlier, the same material stiffness matrix \( \overline{K}_M \) is used for the element throughout the analysis. The internal force vector \( \overline{f} \) for the element is calculated using \( \overline{f} = \overline{K}_M \overline{v}^k \). In the original EICR Method (without the projector), \( \overline{f} \) is left in this form (without ensuring that it’s self-equilibrating) and transformed to global coordinates using the following vector rotation: \( \overline{f} = R^T_{A:Ek \rightarrow G} \overline{f} \).

Likewise, the original EICR Method does not “cleanse” \( \overline{K}_M \) to ensure its invariance to rigid motion. Depending on the type of analysis, a stress stiffening matrix \( \overline{K}_S \) may be added to \( \overline{K}_M \). \( \overline{K}_M \) with or without \( \overline{K}_S \) forms the tangent stiffness matrix \( \overline{K}_{\text{tang}} \) which is transformed to global coordinates using the following change of basis: \( \overline{K}_{\text{tang}} = R^T_{A:Ek \rightarrow G} \left( \overline{K}_{\text{tang}} \right) R_{A:Ek \rightarrow G} \).

The roles of the projector in equilibrating the internal force vector and cleansing the tangent stiffness matrix are discussed in Section 3.3.3.
3.2.10 Solving in Global Coordinates

The discussion to this point has been about processing a single element through the element loop (Figure 3-1) which calculates its internal force vector and tangent stiffness matrix locally \((\mathbf{f}, \mathbf{K}_{\text{tan}})\), transforms both to global coordinates \((\mathbf{f}, \mathbf{K}_{\text{tan}})\), and uses them to update that element in the global internal force vector \(\mathbf{F} (\mathbf{F}_k = \mathbf{F}_{k-1} + \mathbf{f})\) and global tangent stiffness matrix \(\mathbf{K} (\mathbf{K}_k = \mathbf{K}_{k-1} + \mathbf{K}_{\text{tan}})\). Once the element loop has processed all elements for a given iteration \(k\), \(\mathbf{F}_k\) and \(\mathbf{K}_k\) are fully assembled. The analysis then can move to the next task, namely solving the system in global coordinates and testing for convergence.

As explained for the Newton-Raphson Method in Chapter 2, subtracting \(\mathbf{F}_k\) from the external load \((\mathbf{L})\) for the current load step gives the unbalanced force/moment remaining \((\Delta \mathbf{F}_k = \mathbf{L} - \mathbf{F})\). This unbalanced force/moment is used to calculate the incremental displacement \((\Delta \mathbf{v}_k)\) needed to balance it, as follows.

\[
\Delta \mathbf{v}_k = (\mathbf{K}_k)^{-1} \Delta \mathbf{F}_k\]

\[
\text{or, using } \delta\text{-notation, } \delta \mathbf{v}_k = (\mathbf{K}_k)^{-1} \delta \mathbf{F}_k
\]

The cumulative displacement \((\mathbf{v}_{k-1})\) is updated using \(\delta \mathbf{v}_k\). This updating process is discussed in Section 3.2.8.2. Incremental displacement \(\delta \mathbf{v}_k\) also is checked against the convergence criteria to determine whether the solution has converged or not. If not converged, another iteration occurs using the updated cumulative displacement \((\mathbf{v}_k)\). \(\mathbf{v}_k\) contains the translational displacements \((\mathbf{u}^{(a)}_{\text{tot}})\) and rotational displacements \((\mathbf{R}_{\mathbf{A}_{sk} \rightarrow \mathbf{G}})\) needed for the next iteration. If convergence has occurred, displacement \(\mathbf{v}_k\) is stored for that load step as a rotator. The cumulative and incremental displacement vectors then are reset to zero, the load is increased to the next load step, and iteration begins again. Either way, the process described to this point in the chapter simply repeats.

3.2.11 Comparison of Local Elastic Deformation Equations

Section 3.2 has discussed the EICR and similar corotational methods prior to their being enhanced by a projector. A major focus has been the explanation of the equations for isolating
local translational and rotational elastic deformations in Sections 3.2.7 and 3.2.8 respectively.

Except for notational differences, these equations are the same whether found in the Rankin/Brogan/Nour-Omid research [106], the Felippa/Haugen [48] research, or this thesis. For clarity, Table 3-3 summarizes these equations as presented in the various sources.

Table 3-3: Comparison of Local Elastic Deformation Equations

<table>
<thead>
<tr>
<th>Source</th>
<th>Translational Elastic Deformation</th>
<th>Rotational Elastic Deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rankin/Brogan 1986 [104]</td>
<td>$u_{c}^{\text{def}} = E_{k}^{T}(u_{g} + X_{g}) - X_{c}$</td>
<td>$D_{k}^{c} = E_{k}^{T}S_{k}S_{o}^{T}E_{o}$</td>
</tr>
<tr>
<td></td>
<td>Centroid translation $u_{g}^{o}$ removed first</td>
<td></td>
</tr>
<tr>
<td>Rankin/Nour-Omid 1988 [105]</td>
<td>$\bar{u}<em>{c}^{e} = E</em>{k}^{T}(u_{g} - u_{g}^{o} + X_{g}^{o}) - X_{a}$</td>
<td>$\bar{T}<em>{a} = E</em>{k}^{T}S_{a}E_{o}, S_{a} = S_{k}S_{o}^{T}$</td>
</tr>
<tr>
<td>Nour-Omid/Rankin 1991 [91]</td>
<td>$\bar{u}<em>{c}^{e} = E</em>{k}^{T}(u_{g} - u_{g}^{o} + X_{g}^{o}) - X_{a}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>In [91], $X_{g}^{o}$ expressed as $(X_{g}^{o} - X_{g}^{o})$</td>
<td></td>
</tr>
<tr>
<td>Haugen 1995 [57]</td>
<td>$\tilde{u}<em>{d} = T</em>{n}(u - u_{c} - (R_{on} - I)x^{o})$</td>
<td>$\tilde{R}<em>{d} = T</em>{n}R_{T_{o}^{T}}$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{u}<em>{d} = T</em>{n}(u - u_{c} + x^{o}) - \tilde{x}^{o}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>By simplifying with $R_{on} = T_{n}^{T}T_{0}$</td>
<td></td>
</tr>
<tr>
<td>Felippa/Haugen 2005 [47]</td>
<td>$\bar{u}<em>{da} = T</em>{R}(u - c - (R_{o} - I)(x^{o} - a))$</td>
<td>$R_{da}^{c} = T_{R}^{c}R_{o}^{c}T_{R_{o}^{T}}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{u}<em>{da} = T</em>{R}(u - c + x^{o}) - \tilde{x}^{o}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>By simplifying with $R_{o} = T_{R_{o}^{T}}T_{0}$</td>
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Reconciliation of Notation

<table>
<thead>
<tr>
<th>Thesis</th>
<th>$u^{(a,\text{def})}$</th>
<th>$R_{A:Ek\rightarrow G}$</th>
<th>$u_{a}^{(a,\text{tot})} - u_{a}^{(C_{rk},G)} + x_{a}^{(a)} - x_{a}^{(a)}$</th>
<th>$R_{A:Ek\rightarrow G}^{(a,\text{def})}$</th>
<th>$E_{k}^{T}$</th>
<th>$E_{T_{o}^{T}}$</th>
<th>$T_{n}$</th>
<th>$T_{R}$</th>
<th>$R_{A:Ek\rightarrow G}^{(a,\text{def})}$</th>
<th>$R_{A:Ek\rightarrow G}$</th>
<th>$R_{A:Sk\rightarrow G}$</th>
<th>$R_{A:So\rightarrow G}$</th>
<th>$R_{A:Ek\rightarrow G}^{(a,\text{def})}$</th>
<th>$R_{A:Ek\rightarrow G}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>This Thesis</td>
<td>$u_{c}^{(a,\text{def})}$</td>
<td>$R_{A:Ek\rightarrow G}^{-1} \left( u_{c}^{(a,\text{tot})} - u_{c}^{(C_{rk},G)} + x_{c}^{(a)} - x_{c}^{(a)} \right)$</td>
<td>$R_{A:Ek\rightarrow G}^{-1} \left( u_{c}^{(a,\text{def})} \right)$</td>
<td>$D_{k}^{c}$</td>
<td>$\bar{T}_{a}^{c}$</td>
<td>$\tilde{T}^{c}$</td>
<td>$\tilde{R}_{d}^{c}$</td>
<td>$R_{da}^{c}$</td>
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<td>$u_{c}$</td>
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<td>$x_{c}^{(a)}$</td>
<td>$T_{o}^{T}$</td>
</tr>
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<td>$X_{a}$</td>
<td>$X_{e}$</td>
<td>$\tilde{x}^{o}$</td>
<td>$\tilde{x}^{e}$</td>
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<td>$\tilde{R}_{da}^{c}$</td>
<td>$T_{o}^{T}$</td>
<td>$\tilde{T}_{a}^{c}$</td>
<td>$\tilde{T}^{c}$</td>
<td>$\tilde{R}_{d}^{c}$</td>
<td>$R_{da}^{c}$</td>
</tr>
</tbody>
</table>
3.3 Addition of Projector to EICR Method

In 1988, Rankin and Nour-Omid [91,105] significantly improved the EICR Method through the introduction of projectors building on earlier work by Belytschko et al.[15]. Haugen [57] and Felippa/Haugen [47] subsequently significantly generalized and expanded the discussion of projectors and associated corotational methods. As the name suggests, projectors are orthogonal projections which decompose a given vector space into two orthogonal subspaces. These projectors simplify computations and ensure that element deformations (and, hence, internal stresses and strain energy) are invariant to rigid body motion. This invariance broadens the method’s applicability to arbitrarily large rigid body rotations and translations as well as to elements having significant sensitivity to even infinitesimal rigid body motion (e.g. warping-sensitivity) [105].

This section discusses projectors, their derivation and use in detail. Throughout this discussion, δ-notation refers to a perturbation or first variation in the sense of variational calculus unless otherwise indicated.

3.3.1 Projector Formulations

Various projector formulations are used by different authors but most are equivalent to or simplifications of the same projector. Haugen [57] and Felippa/Haugen [47] defined this projector P as follows.

\[
P = I - P_T - P_R \quad \text{by Haugen}
\]

\[
P = P_u - P_\omega \quad \text{by Felippa/Haugen}
\]

where

- \(I\) = Identity Matrix (3 x 3)
- \(P_T\) = Projector which extracts rigid translation from global deformation
- \(P_u = I - P_T\) = Translational Projector (T-Projector) which extracts local (elastic) stretch from global deformation
- \(P_R = P_\omega\) = Rotational Projector (R-Projector) which extracts rigid translation of node due to rigid rotation of element
The projector developed by Rankin/Nour-Omid [91,105] is equivalent to \( P = I - P_R \). It ignores \( P_T \) given that most simple finite elements are strain invariant under rigid body translation.

When doing an iterative solution, incremental rigid body motions arise in each step since the local element frame \( E_k \) is calculated from the nodal positions. Projector \( P \) removes the rigid body translations from the total incremental translations for each node. These rigid body translations have two components: (i) translations arising from the rigid body translation of the element and, since nodes are not coincident with the centroid, (ii) translations arising from the rigid body rotations of the element. \( P_T \) and \( P_R \) remove (i) and (ii) respectively and, hence, \( P = I - P_T - P_R \) extracts the deformational translations.

Projectors can be formulated with respect to \( \mathcal{C}^0 \), \( \mathcal{C}^R \) or \( \mathcal{C}^D \) and expressed in global or local coordinates. Felippa/Haugen [47] offers a detailed discussion of the six possible projector formulations (3 configurations, local and global coordinates). It should be noted that, amongst these formulations, the forms of \( P_R \) and so \( P \) change but \( P_T \) does not. Haugen [57] also defines projectors for perturbations of \( \mathcal{C}^R \) relative to \( \mathcal{C}^0 \) and \( \mathcal{C}^D \) relative to \( \mathcal{C}^R \) in local and global coordinates.

The EICR Method uses the projector to decompose perturbations of \( \mathcal{C}^D \) relative to \( \mathcal{C}^R \) in local coordinates as shown in Figure 3-1. Hence, the discussion of the projector which follows is in this context with the projector designated as \( \overline{P} \). It should be noted that the initial projector paper by Rankin/Nour-Omid [105] mixed local and global coordinates in defining their projector. In reality, this formulation was projector \( \overline{P} \) multiplied on the right by coordinate transformation \( T_R \) to transform from global to local coordinates.

\[
T_R = \text{diag} \left[ R_{T,G}^{T_{A,Ek}} \cdots R_{T,G}^{T_{A,Ek}} \right]_{N \text{ nodes of element}}
\]

In the later paper by Nour-Omid/Rankin [91], the projector was clarified and defined as \( \overline{P} \).

Like all finite element matrices, \( \overline{P}_T \), \( \overline{P}_R \) and so \( \overline{P} \) are assembled node-by-node from nodal level blocks. This means that these matrices have dimension \( 6N \times 6N \) for an element with

73
N nodes (e.g. 18x18 for 3-noded triangle). $\mathbf{P}$, $\mathbf{P}_T$ and $\mathbf{P}_R$ are defined as follows based on Haugen [57].

$$
\mathbf{P} = \mathbf{I} - \mathbf{P}_T - \mathbf{P}_R
$$

where

$$
a,b = 1 \ldots N \text{ for N-noded element}
$$

$$
\mathbf{I} = \text{diag } \begin{bmatrix} 1_{6x6} & 1_{6x6} & \cdots & 1_{6x6} \end{bmatrix}_{N \times 1_{6x6}}
$$

$$
\mathbf{P}_T = \mathbf{P}_T = \begin{bmatrix} P_{11} & \cdots & P_{1N} \\
\vdots & \ddots & \vdots \\
P_{N1} & \cdots & P_{NN} \end{bmatrix}
$$

where

$$
\mathbf{P}_{ab} = \begin{bmatrix} 1 & -I_{3x3} & 0_{3x3} \\
-\frac{1}{n} & 0_{3x3} & 0_{3x3} \\
0_{3x3} & 0_{3x3} & 0_{3x3} \end{bmatrix}
$$

$$
\mathbf{P}_R = \mathbf{S}^D \mathbf{G}
$$

where

$$
\mathbf{S}^D = \begin{bmatrix} \mathbf{S}_1^D \\
\mathbf{I}_{3x3} \\
\vdots \\
\mathbf{S}_N^D \\
\mathbf{I}_{3x3} \end{bmatrix}
$$

which is $6N \times 3$

$$
\mathbf{S}_a^D = \text{Spin}(\mathbf{x}_a^D) = \begin{bmatrix} 0 & -\mathbf{x}_{3a}^D & \mathbf{x}_{2a}^D \\
\mathbf{x}_{3a}^D & 0 & -\mathbf{x}_{1a}^D \\
-\mathbf{x}_{2a}^D & \mathbf{x}_{1a}^D & 0 \end{bmatrix}
$$

called the spin-lever or moment-arm matrix

$$
\mathbf{x}_a^D = (x_{a1}^D, x_{a2}^D, x_{3a}^D) = \text{node (a) position vector in } \mathbb{C}^D, \text{ local coordinates}
$$

$$
\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \cdots & \mathbf{G}_N \end{bmatrix}
$$

called the spin-fitter matrix ($3 \times 6N$) as gives spin variations of $E_k$ resulting from nodal spin variations as shown below.

$$
\delta \mathbf{\omega}_E^k = \mathbf{G} \delta \mathbf{v}_E^k = \sum_{b=1}^N \mathbf{G}_b \delta \mathbf{v}_b^k \text{ in local coordinates in } \mathbb{C}^D, \text{ where}
$$

$$
\delta \mathbf{\omega}_E^k \text{ = variations in rigid body rotation of the element represented by variations of the basis vectors of } E_k
$$

$$
\delta \mathbf{v}_E^k = \text{variations in nodal translations (}\delta \mathbf{u}_E^k\text{), spins (}\delta \mathbf{\omega}_E^k\text{) at all nodes}
$$

$$
\mathbf{G}_b = \text{spin fitter submatrix giving } \delta \mathbf{\omega}_E^k \text{ rigid body rotation variation as function of varying node (b) translations/spins (}\delta \mathbf{v}_b^k\text{)}
$$
As highlighted in Nour-Omid/Rankin [105], $\overline{G}_b$ has the form of a Jacobian matrix, namely

$$
\overline{G}_b = \frac{\partial G_k}{\partial u_b} \frac{\partial G_k}{\partial \omega_b} = \begin{bmatrix}
\frac{\partial G_k}{\partial u_{b1}} & \frac{\partial G_k}{\partial u_{b2}} & \frac{\partial G_k}{\partial u_{b3}} & \frac{\partial G_k}{\partial \omega_{b1}} & \frac{\partial G_k}{\partial \omega_{b2}} & \frac{\partial G_k}{\partial \omega_{b3}}
\end{bmatrix}
$$

where, in local coordinates in $\mathcal{C}$,

- $\omega_E^k = (\omega_{c1}^k, \omega_{c2}^k, \omega_{c3}^k)^T$ = rigid body rotation of the element
- $u_b^k = (u_{b1}^k, u_{b2}^k, u_{b3}^k)^T$ = translational deformation vector for node (b)
- $\omega_b^k = (\omega_{b1}^k, \omega_{b2}^k, \omega_{b3}^k)^T$ = nodal spin (instantaneous rotation) for node (b)

In forming the R-Projector ($\overline{P}_R$), $\overline{S}_D$ is computed from the geometry of the element in $\mathcal{C}$ using the global nodal coordinates to find the centroid position vector and then nodal position vectors in local coordinates $x_{(b)}^{(a,D)}$. However, $\overline{G}$ depends not only on the element geometry but also how $\mathcal{C}$ is fitted to $\mathcal{C}$ since this determines the relationship between the local element frame $E_k$ and the nodes required for $\overline{G}$. Hence, $\overline{G}$ must be computed based on the specific best fit, local frame positioning (at node 1 or C), and nodal numbering choices made, resulting in some loss of invariance to such choices [47,58].

The Rankin and Nour-Omid [91,105] projector differs from the Felippa/Haugen [47,58] projector in one substantial way and two matters of convention. Rankin and Nour-Omid ignore $P_T$ under the assumption that the element deformation is invariant under rigid body translation as required by the finite element patch test. Rankin and Nour-Omid also use node (1) rather than the centroid as the origin for rigid body rotations. This convention affects the computed values of $\overline{S}$ and $\overline{G}$. In their initial paper [105], Rankin/Nour-Omid also used incremental nodal rotations $\delta \theta^k_a$ rather than $\delta \omega^k_a$ but their later paper [91] used $\delta \omega^k_a$. The two projectors are compared below.
HAUGEN AND FELIPPA [47,58]

\[ \mathbf{P} = I - \mathbf{P}_T - \mathbf{P}_R \] or, if \( \mathbf{P}_T \) omitted

\[ \mathbf{P} = I - \mathbf{P}_R = I - \mathbf{S}^D \mathbf{G} \]

where

\[ \mathbf{I} = \text{diag} \left[ \mathbf{I}_{6 \times 6} \mathbf{I}_{6 \times 6} \ldots \mathbf{I}_{6 \times 6} \right] \]

\[ \mathbf{S}^D = \begin{bmatrix} -\text{Spin}(\mathbf{x}_i^D) \\
\mathbf{I}_{3 \times 3} \\
\vdots \\
-\text{Spin}(\mathbf{x}_N^D) \\
\mathbf{I}_{3 \times 3} \end{bmatrix} \] with \( \mathbf{x}_i^D = \) node (a) position vector in \( \mathbf{e}^D \) referred to centroid \( \mathbf{C} \), local coordinates

\[ \mathbf{G} = \left[ \mathbf{G}_1 \ldots \mathbf{G}_N \right] \]

\[ \mathbf{G}_b = \begin{bmatrix} \frac{\partial \omega^k_e}{\partial u^k_b} & \frac{\partial \omega^k_e}{\partial \omega^k_b} \end{bmatrix} \] with \( \mathbf{E}_k \) positioned at centroid \( \mathbf{C} \)

NOUR-OMID AND RANKIN (no \( \mathbf{P}_T \)) [91]

\[ \mathbf{P} = I - \psi \mathbf{F}^T \]

where

\[ \mathbf{I} = \text{diag} \left[ \mathbf{I}_{6 \times 6} \mathbf{I}_{6 \times 6} \ldots \mathbf{I}_{6 \times 6} \right] \]

\[ \psi = \begin{bmatrix} -\text{Spin}(\mathbf{x}_i^D) \\
\mathbf{I}_{3 \times 3} \\
\vdots \\
-\text{Spin}(\mathbf{x}_N^D) \\
\mathbf{I}_{3 \times 3} \end{bmatrix} \] with \( \mathbf{x}_i^D = \) node (a) position vector in \( \mathbf{e}^D \) referred to node (1), local coordinates

\[ \mathbf{F}^T = \left[ \mathbf{F}_1^T \ldots \mathbf{F}_N^T \right] \]

\[ \mathbf{F}_b = \begin{bmatrix} \frac{\partial \omega^k_e}{\partial u^k_b} & \frac{\partial \omega^k_e}{\partial \omega^k_b} \end{bmatrix} \] with \( \mathbf{E}_k \) positioned at node (1)

### 3.3.2 Nature of the Projector

The projector is best understood as an orthogonal projection. A projection is a linear transformation \( \mathbf{P} \) from a finite dimensional vector space \( \mathbf{V} \) to itself (\( \mathbf{P} : \mathbf{V} \rightarrow \mathbf{V} \)) that is indempotent (\( \mathbf{P} = \mathbf{P}^2 \) and so \( \mathbf{P}^n = \mathbf{P} \)). If subspaces \( \mathbf{U} \) and \( \mathbf{W} \) respectively are the range and kernel of \( \mathbf{P} \), then:

i) \( \mathbf{P} \mathbf{u} = \mathbf{u} \) for any \( \mathbf{u} \in \mathbf{U} \) since \( \mathbf{P}^2 \mathbf{u} = \mathbf{P} \mathbf{u} \) giving \( \mathbf{P}(\mathbf{P} - \mathbf{I})\mathbf{u} = 0 \) and thus \( \mathbf{P} = \mathbf{I} \)

ii) \( (\mathbf{I} - \mathbf{P}) \mathbf{w} = \mathbf{w} \) for any \( \mathbf{w} \in \mathbf{W} \) since \( \mathbf{P} \mathbf{w} = \mathbf{0} \). \( (\mathbf{I} - \mathbf{P}) \) is also a projection because

\( (\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - \mathbf{P}) \) but with range \( \mathbf{W} \) and kernel \( \mathbf{U} \).

iii) Any \( \mathbf{v} \in \mathbf{V} \) is decomposable into \( \mathbf{v} = \mathbf{u} \oplus \mathbf{w} \) with \( \mathbf{u} = \mathbf{P} \mathbf{v} \), \( \mathbf{w} = (\mathbf{I} - \mathbf{P}) \mathbf{v} \), \( \mathbf{u} \in \mathbf{U} \) and \( \mathbf{v} \in \mathbf{V} \).
The above decomposition is not unique because there may be several projections with the same range and kernel. However, uniqueness can be achieved by introducing an inner product to the vector space \( V \) making it an *inner product space* where orthogonal subspaces can be defined. An *orthogonal projection* for a finite vector space \( V \) (with inner product \( \langle u, w \rangle \), \( u, w \in V \)) is a projection having a range (\( U \)) and kernel (\( W \)) which are orthogonal with respect to the inner product. By the Orthogonal Decomposition Theorem, the decomposition \( v = u \oplus w \) obtained with an orthogonal projection is unique (since \( P \) is unique) and often expressed as \( v = u \oplus u^\perp \) where \( u^\perp \) replaces \( w \) and \( U^\perp \) replaces \( W \).

The projector \( \left( \bar{P} = I - \bar{P}_T - \bar{P}_R \right) \) is an orthogonal projection. It decomposes the vector space \( V \) of spatial motions (translations and instantaneous rotations or spins) into two orthogonal subspaces: \( U \) of rigid motions (extracted by \( \bar{P}_T \) for translations and \( \bar{P}_R \) for rotations) and \( U^\perp \) of deformational motions (extracted by \( \bar{P} \)). The inner product for statics is the Euclidean inner product (dot product). Using this inner product, any given rigid body or deformational mode shape (whether translational or rotational) is orthogonal to any other such mode shape \([48]\). The mode shapes can be made orthonormal by norming the mode shapes to 1. Figure 3-13 illustrates modal orthogonality for a 2-D beam element. Bisplinghoff et al.\([17]\) and Bower \([19]\) have shown that, in the dynamic case, the rigid body and vibration modes also are orthogonal.

As must be the case with any projection, \( \bar{P} \) satisfies \( \bar{P}^2 = \bar{P} \) as shown below. Showing this involves modal orthogonality plus another condition called *biorthonormality* \([105]\) or *bi-orthogonality* \([47]\). Bi-orthogonality is frame and coordinate independent so can be defined using any of the projector formulations. Using the R-Projector \( \bar{P}_R = \bar{S}_D \bar{G} \), *bi-orthogonality* is the condition that \( \bar{G} \bar{S}_D = I_{3x3} \) for all finite elements of order 2 or higher.

Felippa and Haugen \([47]\) explain \( \bar{G} \bar{S}_D = I_{3x3} \) based on the construction of \( \bar{S}_D \) and \( \bar{G} \) as follows. The columns of the \( \text{Spin}(x_a^D) \) blocks of \( \bar{S}_D \) represent the displacement vectors associated with rigid body rotation \( \delta \omega^k_k = (\delta \omega_{c1}, \delta \omega_{c2}, \delta \omega_{c3})^{T} \) of the corotational frame \( E_k \). The rows
of the blocks of $\mathbf{G}$ represent the elemental frame variations ($\delta \mathbf{\omega}^k_{\text{el}}, \delta \mathbf{\omega}^k_{\text{el}}, \delta \mathbf{\omega}^k_{\text{el}}$). Thus, $\mathbf{G}^D$ has on its diagonal the amplitudes of the three rigid body rotational modes which equal 1 as normalized. The off-diagonal entries are 0 because the three rigid body rotational modes are orthogonal.

This explanation assumes that the blocks in $\mathbf{G}$ representing the variation of the corotational frame with nodal drilling rotations $\partial \mathbf{\omega}^k / \partial u^k$ are $\theta_{3x3}$ which is the case for any element with 2 or more non-colinear nodes such as 3-noded triangular or 4-noded quadrilateral elements. For the same element types and associated drilling rotation condition, Rankin [46] recently proved bi-orthogonality using de Veuke’s best fit criteria.

With modal basis vectors of the form $(u_{(1)}, v_{(1)}, \theta_{(1)}, u_{(2)}, v_{(2)}, \theta_{(2)})^T$:

- Rigid Body Modes: $(1, 0, 0, 1, 0, 0)^T, (0, 1, 0, 0, 1, 0)^T, (0, 1, 0, -1, 0)^T$
- Deformational Modes: $(1, 0, 0, -1, 0, 0)^T, (0, 0, 1, 0, 0, -1)^T, (0, 0, 1, 0, 0, 1)^T$

All of the above vectors are orthogonal with respect to the euclidean inner (dot) product and, hence, linearly independent.
This leaves only elements of order less than 2 (e.g. a bar element) and those of higher order with all nodes collinear (e.g. beam element). In the former case, \( \mathbf{G}^D \) will contain a zero order with all nodes collinear (e.g. beam element). In the latter case, \( \partial^k / \partial \omega^k \neq 0_{3 \times 3} \) but Rankin/Nour-Omid [105] have shown that, for the best fit and nodal numbering conventions currently in use, biorthogonality holds. The lack of a general proof with an associated unique best local frame results from beam elements using an arbitrary third reference node (q) to orient the beam cross-section meaning multiple solutions.

Using bi-orthogonality and modal orthogonality, \( \mathbf{p} = \mathbf{p}^2 \) can be proven as follows by combining and elaborating on the explanations found in Haugen [57] and Felippa/Haugen [47].

\[
\mathbf{p}^2 = (I - \mathbf{p}_T - \mathbf{p}_R)^2
\]

which, expanding, gives

\[
\mathbf{p}^2 = I - 2\mathbf{p}_T - 2\mathbf{p}_R + \mathbf{p}_T \mathbf{p}_R + \mathbf{p}_R \mathbf{p}_T + \mathbf{p}_R^2 + \mathbf{p}_T^2
\]

which gives

\[
\mathbf{p}_R^2 = \mathbf{p}_R - \mathbf{p}_R^2
\]

since, as shown below,

\[
\mathbf{p}_R^2 = \mathbf{p}_R, \mathbf{p}_R = \mathbf{p}_T, \mathbf{p}_T \mathbf{p}_R = \mathbf{p}_R \mathbf{p}_T = 0_{6 \times 6N} [57]
\]

\[
\mathbf{p}_T^2 = \mathbf{p}_T\] because:

\[
\mathbf{p}_T^2 = \mathbf{p}_T^2 = \begin{bmatrix}
\mathbf{P}_{T11} & \cdots & \mathbf{P}_{T1N} \\
\vdots & \ddots & \vdots \\
\mathbf{P}_{TNN} & \cdots & \mathbf{P}_{TN1} \\
\end{bmatrix}^2 \\
= \begin{bmatrix}
\mathbf{P}_{T11} & \cdots & \mathbf{P}_{T1N} + \cdots + \mathbf{P}_{TIN} \mathbf{P}_{TN1} \\
\vdots & \ddots & \vdots \\
\mathbf{P}_{TNN} & \cdots & \mathbf{P}_{TN1} + \cdots + \mathbf{P}_{TN1} \mathbf{P}_{TNN} \\
\end{bmatrix} \\
= \begin{bmatrix}
N \mathbf{P}_{T11}^2 & \cdots & N \mathbf{P}_{T1N}^2 \\
\vdots & \ddots & \vdots \\
N \mathbf{P}_{TN1}^2 & \cdots & N \mathbf{P}_{TNN}^2 \\
\end{bmatrix} \\
= N \mathbf{P}_{Tab}^2 = N \begin{bmatrix}
\frac{1}{N^2} \mathbf{I}_3 & \mathbf{O}_3 \\
\mathbf{O}_3 & \mathbf{O}_3 \\
\end{bmatrix} = \mathbf{P}_{Tab} [57]
\]
Using $P = 2$ because, using bi-orthogonality $(GS^D = I_{3x3})$

$$
\bar{P}_R^2 = \bar{P}_R \text{ because, using bi-orthogonality } (GS^D = I_{3x3})
$$

$$
\bar{P}_R^2 = (S^D G)^2 = S^D (G S^D) G = S^D (I_{3x3}) G = S^D G = \bar{P}_R \quad [57]
$$

For elements of order less than 2 without bi-orthogonality (e.g. bar element where $GS^D = (0,1,1)^T \neq I_{3x3}$), Felippa/Haugen [47] show $\bar{P}_R^2 = \bar{P}_R$ by row/column reducing $\bar{P}_R^2$

$$
\bar{P}_T \bar{P}_R = 0_{6Nx6N} \text{ because: }
$$

$$
\bar{P}_T \bar{P}_R = (\bar{P}_T S^D) G = 0_{6Nx3} G = 0_{6Nx6N} \text{ since } \bar{P}_T S^D = 0_{6Nx3} \text{ as shown below. }
$$

$$
\bar{P}_T S^D = \begin{bmatrix}
  p_{T11} & \cdots & p_{T1N} \\
  \vdots & \ddots & \vdots \\
  p_{TN1} & \cdots & p_{TNN}
\end{bmatrix}
\begin{bmatrix}
  -\text{Spin}(\bar{x}_1^D) \\
  I_{3x3} \\
  \vdots \\
  -\text{Spin}(\bar{x}_N^D) \\
  I_{3x3}
\end{bmatrix}
$$

$$
\begin{bmatrix}
  \frac{1}{N} \begin{bmatrix}
  I_{3x3} \\
  0_{3x3}
\end{bmatrix} \\
  \frac{1}{N} \begin{bmatrix}
  0_{3x3} \\
  I_{3x3}
\end{bmatrix}
\end{bmatrix} \\
+ \cdots + \\
\begin{bmatrix}
  \frac{1}{N} \begin{bmatrix}
  I_{3x3} \\
  0_{3x3}
\end{bmatrix} \\
  \frac{1}{N} \begin{bmatrix}
  0_{3x3} \\
  I_{3x3}
\end{bmatrix}
\end{bmatrix}
$$

which, by expanding and reorganizing the blocks and spins, becomes as shown below.

$$
\bar{P}_T S^D = \begin{bmatrix}
  -\frac{1}{N} \text{Spin}(\bar{x}_1^D) + \cdots + -\frac{1}{N} \text{Spin}(\bar{x}_N^D) \\
  0_{3x3} & 0_{3x3}
\end{bmatrix}
$$

$$
\begin{bmatrix}
  \frac{1}{N} \begin{bmatrix}
  0_{3x3} \\
  I_{3x3}
\end{bmatrix} \\
  \frac{1}{N} \begin{bmatrix}
  I_{3x3} \\
  0_{3x3}
\end{bmatrix}
\end{bmatrix}
+ \cdots + \\
\begin{bmatrix}
  \frac{1}{N} \begin{bmatrix}
  0_{3x3} \\
  I_{3x3}
\end{bmatrix} \\
  \frac{1}{N} \begin{bmatrix}
  I_{3x3} \\
  0_{3x3}
\end{bmatrix}
\end{bmatrix}
$$

Using $\bar{x}_C^D = \frac{1}{N} \sum_{a=1}^{N} x_a^D = \text{local centroid position vector} = (0,0,0)^T$, $\text{Spin}(\bar{x}_C^D) = 0_{3x3}$ [57]

which gives

$$
\bar{P}_T S^D = \begin{bmatrix}
  -\text{Spin}(\bar{x}_1^D) \\
  0_{3x3} \\
  \vdots \\
  -\text{Spin}(\bar{x}_N^D) \\
  0_{3x3}
\end{bmatrix} = 0_{6Nx3} \quad [57]
$$
\[ \mathbf{P_R P_T} = 0_{6\times 6N} \text{ because} \]
\[ \mathbf{P_R P_T} = \mathbf{S}^D (\mathbf{G P_T}) = \mathbf{S}^D (\mathbf{0}_{3\times 6N}) = \mathbf{0}_{6\times 6N} \text{ since } (\mathbf{G P_T}) = \mathbf{O}_{3\times 6N} \text{ as shown below.} \]

\[ \mathbf{G P_T} = [\mathbf{G}_1 \; \cdots \; \mathbf{G}_N] \begin{bmatrix} \mathbf{P}_{T11} & \cdots & \mathbf{P}_{TIN} \\ \vdots & \ddots & \vdots \\ \mathbf{P}_{TN1} & \cdots & \mathbf{P}_{TNN} \end{bmatrix} \]

\[ \mathbf{G P_T} = \begin{bmatrix} \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_1} & \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}_1} & \cdots & \frac{\partial \mathbf{u}_N}{\partial \mathbf{u}_1} \\ \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_2} & \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}_2} & \cdots & \frac{\partial \mathbf{u}_N}{\partial \mathbf{u}_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_N} & \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}_N} & \cdots & \frac{\partial \mathbf{u}_N}{\partial \mathbf{u}_N} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{O}_{3\times 3} \\ \mathbf{O}_{3\times 3} & \mathbf{O}_{3\times 3} \\ \vdots & \vdots \\ \mathbf{O}_{3\times 3} & \mathbf{O}_{3\times 3} \end{bmatrix} \]

\[ \mathbf{G P_T} = \begin{bmatrix} \left( \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_1} \right)_{13\times 3} + \frac{\partial \mathbf{u}_1}{\partial \mathbf{u}_1} \mathbf{0}_{3\times 3} \\ \left( \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}_1} \right)_{13\times 3} + \frac{\partial \mathbf{u}_2}{\partial \mathbf{u}_2} \mathbf{0}_{3\times 3} \\ \vdots \\ \left( \frac{\partial \mathbf{u}_N}{\partial \mathbf{u}_N} \right)_{13\times 3} + \frac{\partial \mathbf{u}_N}{\partial \mathbf{u}_N} \mathbf{0}_{3\times 3} \end{bmatrix} \]

\[ \mathbf{G P_T} = \begin{bmatrix} \frac{1}{N} \sum_{a=1}^{N} \frac{\partial \mathbf{u}_a}{\partial \mathbf{u}_1} \mathbf{I}_{3\times 3} \mathbf{0}_{3\times 3} \\ \vdots \\ \frac{1}{N} \sum_{a=1}^{N} \frac{\partial \mathbf{u}_a}{\partial \mathbf{u}_N} \mathbf{I}_{3\times 3} \mathbf{0}_{3\times 3} \end{bmatrix} \]

and, because the variations sum to zero,
\[
\begin{bmatrix} \frac{1}{N} \sum_{a=1}^{N} \frac{\partial \mathbf{u}_a}{\partial \mathbf{u}_a} \mathbf{I}_{3\times 3} \mathbf{0}_{3\times 3} \end{bmatrix} = \mathbf{0}_{3\times 3} \text{ giving} \]

\[ \mathbf{G P_T} = \mathbf{0}_{3\times 6N} \]

### 3.3.3 Roles of Projector

The projector has the following roles in the EICR and similar corotational methods:
i) simplifying the isolation of incremental translational deformations, ii) fixing “polluted”
elemental stiffness matrices that incorrectly produce internal forces from rigid body motion, and
iii) converting non-equilibrating internal forces/moments to self-equilibrating ones so that the
deformed configuration is in force equilibrium. The last two improvements ensure that the
element’s deformation, internal stresses and strain energy are invariant to rigid body motion as
should be the case. These projector roles and their theoretical basis are discussed below.

(i) **Isolation of Local Elastic Deformations**

The local elastic deformations \( \mathbf{v}^{(\text{def})} \) can be isolated from the total deformation \( \mathbf{v}^{(\text{tot})} \)
using the projector as shown below.
This is proven in the derivation of the projector (Section 3.3.5).

(ii) Converting Internal Forces/Moments from Non-equilibrating to Self-equilibrating

An element is in force equilibrium in its base configuration ($C^0$) and, assuming a clean $\mathbf{K}_M$ (see (iii)), also in its corotated configuration ($C^R$). However, its deformed configuration ($C^D$) is not in force equilibrium unless a new stiffness matrix is computed. To avoid this and to continue using $\mathbf{K}_M$ for $C^D$, the internal force/moment vector ($\mathbf{f}$) components calculated with $\mathbf{K}_M$ are adjusted so that the deformed element is in force equilibrium. The modified internal force/moment vector ($\mathbf{f}_{\text{equil}}$) is said to be self-equilibrating [67].

The local internal force/moment vector ($\mathbf{f}$) in a deformed configuration (calculated using $\mathbf{K}_M$) becomes self-equilibrating when pre-multiplied by $\mathbf{P}^T$ as below [47,58,91].

$$\mathbf{f}_{\text{equil}} = \mathbf{P}^T \mathbf{f}$$

(10)

According to Rankin/Nour-Omid [105], this multiplication by $\mathbf{P}^T$ is equivalent to adjusting for the unbalanced moments in $C^D$ as follows.

$$\mathbf{f}_{\text{equil}} = \mathbf{P}^T \mathbf{f} = \sum_{a=1}^{N} \mathbf{x}_a^e \times \mathbf{f}_a^u + \mathbf{f}_a^\theta$$  \[105\]

where $\mathbf{x}_a^e$ = node (a) position vector in $C^D$, local coordinates

$\mathbf{f}_a^u$ = force component of node (a) internal force vector $\mathbf{f}_a$

$\mathbf{f}_a^\theta$ = moment component of node (a) internal force vector $\mathbf{f}_a$

This presumably assumes that stress stiffening is addressed elsewhere, namely through increasing $\mathbf{K}_M$ by $\mathbf{K}_S$ as discussed in Section 3.2.9. These two adjustments are consistent with the modified internal force vector by Khosravi [67] discussed in Chapter 1.

The equilibrating of the internal force vector by $\mathbf{f}_{\text{equil}} = \mathbf{P}^T \mathbf{f}$ can be proven in two steps as outlined below based on Felippa/Haugen [47].
Step 1: Show any equilibrated \( \bar{f} \) (i.e. \( C^D \) in equilibrium) has the property \( \bar{f} = P^T \bar{f} \)

Assuming a “clean” stiffness matrix, internal force and so virtual work should be invariant with respect to coordinate frames. Hence, the virtual work evaluated in the local and global frames should be equal as assumed below [47].

\[
f^{(a)} \left( \frac{\delta u^{(a),\text{tot}}}{\delta \omega^{(a),\text{tot}}} \right) = \bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right)
\]

assuming virtual work invariant to frame

\[
\left( R^T \bar{f}^{(a)} \right)^{T} \left( \frac{\delta u^{(a),\text{tot}}}{\delta \omega^{(a),\text{tot}}} \right) = \bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) \text{ because } f^{(a)} = R^T \bar{f}^{(a)}
\]

\[
\bar{f}^{(a)} \left( \frac{\delta u^{(a),\text{tot}}}{\delta \omega^{(a),\text{tot}}} \right) = \bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) \text{ by transposing } \left( R^T \bar{f}^{(a)} \right)^{T}
\]

\[
\bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) = R \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) \text{ because } \bar{f}^{(a)} = R \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right)
\]

\[
\sum_{a=1}^{N} \bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) = \sum_{a=1}^{N} \bar{f}^{(a)} \left( \frac{\delta \bar{u}^{(a),\text{def}}}{\delta \overline{\omega}^{(a),\text{def}}} \right) \text{ by summing all element nodes}
\]

\[
\bar{f}^T \left( \frac{\delta \bar{u}^{\text{tot}}}{\delta \overline{\omega}^{\text{tot}}} \right) = \bar{f}^T \left( \frac{\delta \bar{u}^{\text{def}}}{\delta \overline{\omega}^{\text{def}}} \right) \text{ by evaluating the sums}
\]

\[
\bar{f}^T \delta \bar{V}^{\text{tot}} = \bar{f}^T \delta \bar{V}^{\text{def}} \text{ because } \delta \bar{V} = \left[ \frac{\delta \bar{u}}{\delta \overline{\omega}} \right]
\]

\[
\bar{f}^T \delta \bar{V}^{\text{tot}} = \bar{f}^T \bar{P} \delta \bar{V}^{\text{tot}} \text{ because } \delta \bar{V}^{\text{def}} = \bar{P} \delta \bar{V}^{\text{tot}} \text{ as shown in derivation of } \bar{P}
\]

(Section 3.3.5)

\[
\bar{f}^T = \bar{f}^T \bar{P} \bar{f}
\]

\[
\bar{f} = \bar{P}^T \bar{f} \text{ for any internal force vector } \bar{f} \text{ equilibrated in } C^D
\]

Step 2: Show any non-equilibrating \( \tilde{f} \) (i.e. \( C^D \) not in equilibrium) can be made self-equilibrating by \( \tilde{f}_{\text{equil}} = P^T \tilde{f} \)

Let \( \tilde{f} = P^T \tilde{f} \) for \( \tilde{f} \) in any \( C^D \) not in equilibrium

\[
\tilde{f} = P^T \left( P^T \tilde{f} \right) \text{ because } \bar{P}^2 = \bar{P} \text{ since a projector} \quad [91]
\]

\[\hat{f} = P^T(\tilde{f}) \text{ because } \hat{f} = P^T \tilde{f} \text{ assumed initially} \]

\[\hat{f} \text{ is self-equilibrating from Step 1} \]

\[
\tilde{f}_{\text{equil}} = P^T \tilde{f}
\]
(iii) Fixing Elemental Stiffness Matrices “Polluted” with respect to Rigid Body Motion

Since rigid motion results in no elastic deformation, no internal forces/moments result. Hence, the computed internal forces/moments should be zero with \( \bar{f} = \bar{K}_M \bar{R} = 0 \) where \( \bar{K}_M \) = element stiffness and \( \bar{R} \) = matrix of rigid body mode basis vectors. If this is the case, \( \bar{K}_M \) is said to be “clean” with respect to rigid body motion. If not, \( \bar{K}_M \) is said to be “polluted” but can be fixed by pre- and post-multiplying respectively by \( P^T \) and \( P \) as shown below [48,91].

\[
\bar{K}_{M,\text{clean}} = \bar{P}^T \bar{K}_M \bar{P} \tag{11}
\]

This can be proven in two steps as outlined below from Felippa/Haugen [47].

Step 1: Show any “clean” stiffness matrix has the property \( \bar{K}_M = \bar{P}^T \bar{K}_M \bar{P} \).
Assuming a clean local stiffness matrix (\( \bar{K}_M \)) and a self-equilibrating internal force vector (\( \bar{f}_{\text{equil}} \)), the computed internal force/moments from rigid body motion should be zero.

\[
\bar{f}_{\text{equil}} = \bar{K}_M \bar{v}^{(\text{tot})} = \bar{K}_M \bar{v}^{(\text{def})} \tag{11a}
\]

\[
\bar{f}_{\text{equil}} = \bar{P}^T \bar{f}_{\text{equil}} = \bar{P}^T \bar{K}_M \bar{v}^{(\text{tot})} \text{ because } \bar{f}_{\text{e}} \text{ is self-equilibrating}
\]

\[
\bar{f}_{\text{equil}} = \bar{P}^T \bar{K}_M \bar{P} \bar{v}^{(\text{tot})} \text{ because } \bar{v}^{(\text{def})} = \bar{P} \bar{v}^{(\text{tot})} \text{ by (9)}
\]

\[
\bar{f}_{\text{equil}} = \left( \bar{P}^T \bar{K}_M \bar{P} \right) \bar{v}^{(\text{tot})} = \bar{K}_M \bar{v}^{(\text{tot})} \text{ by (11a)}
\]

\[
\bar{P}^T \bar{K}_M \bar{P} = \bar{K}_M
\]

Step 2: Show any “polluted” stiffness matrix can be “cleaned” by \( \bar{K}_{M,\text{clean}} = \bar{P}^T \bar{K}_M \bar{P} \)
Let \( \hat{K}_M = \bar{P}^T \bar{K}_M \bar{P} \) for any polluted stiffness matrix

\[
\hat{K}_M = \bar{P}^T \hat{K}_M \bar{P} = \bar{P}^T \left( \bar{P}^T \bar{K}_M \bar{P} \right) \bar{P} \text{ because } \bar{P}^2 = \bar{P} \text{ since a projector}
\]

\[
\hat{K}_M = \bar{P}^T \left( \bar{P}^T \bar{K}_M \bar{P} \right) \bar{P} = \bar{P}^T \left( \hat{K}_M \right) \bar{P} \text{ because } \hat{K}_M = \bar{P}^T \bar{K}_M \bar{P} \text{ assumed initially}
\]

\( \hat{K}_M \) is clean stiffness matrix by Step 1

\[
\bar{K}_{M,\text{clean}} = \bar{P}^T \bar{K}_M \bar{P}
\]
(iv) Computing Consistent Tangent Stiffness Matrix

The EICR Method uses iterative solution techniques. With such techniques, an updated tangent stiffness matrix is needed for each iteration. A tangential stiffness matrix \( \mathbf{K}_{\text{tan}} \) represents the “instantaneous” stiffness at any particular displacement expressed as below [47].

\[
\mathbf{K}_{\text{tan}} = \frac{\partial \mathbf{f}}{\partial \mathbf{v}} \quad \text{where}
\]

\[
\mathbf{f} = \text{internal force/moment vector}
\]

\[
\mathbf{v} = \text{displacement vector}
\]

\( \mathbf{K}_{\text{tan}} \) is said to be consistent with respect to the internal forces if it relates variations of \( \mathbf{f} \) and \( \mathbf{v} \) as given below [57].

\[
\delta \mathbf{f} = \mathbf{K}_{\text{tan}} \delta \mathbf{v}
\]

\( \mathbf{K}_{\text{tan}} \) includes both material and geometric stiffness. Felippa/Haugen [47] have shown that a consistent \( \mathbf{K}_{\text{tan}} \) can be computed using the projector and varying the internal force/moment vector with respect to the total displacement vector. The material stiffness component, already discussed above, is \( \mathbf{K}_{M,\text{clean}} = \mathbf{P}^T \mathbf{K}_M \mathbf{P} \). The geometric stiffness \( \mathbf{K}_s \) has three components (moment-correction, equilibrium-projection, and rotational geometric stiffness) which also can be computed using the projector. These computations are quite involved but can be found in Felippa/Haugen [47].

(v) Uniform Use of Projector

Since \( \mathbf{P} \) is a projection, its being applied to vectors or matrices already having the desirable properties described above fortunately leaves the vectors and matrices unchanged. For example, an “unpolluted” stiffness matrix \( \mathbf{f} = \mathbf{K}_M \mathbf{R} = \mathbf{0} \) adjusted by equation (11) will remain unchanged \( \mathbf{K}_{M,\text{clean}} = \mathbf{P}^T \mathbf{K}_M \mathbf{P} = \mathbf{K}_M \). Hence, equations (9) - (11) can be applied uniformly to all elements in a given finite element analysis.
3.3.4 Incorporation of Projector into EICR Method

Figure 3-1 summarizes where the above projector roles occur in the EICR Method, all being within the element loop. As a result, the internal force vector and tangential stiffness matrix respectively are self-equilibrating and clean when transformed to global coordinates for the assembly and solving steps. These steps were described earlier in Section 3.2.10 under the original EICR Method (without projector) so are not repeated here.

3.3.5 Derivation of Projector

The projector can be derived through either its role in equilibrating internal forces or its role in isolating elastic deformations. The first approach can be found in Nour-Omid/Rankin [91] and Felippa/Haugen [47]. This discussion focuses on the latter approach based on Haugen [57] and Rankin/Nour-Omid [105].

This chosen approach is based on taking the first variation of the elastic deformation vector with respect to the nodal degrees of freedom. The first variation results in a projection matrix which relates infinitesimal changes in elastic deformation to infinitesimal changes in the configuration of the nodes. This relationship and the projector can be expressed in either local or global coordinates with the local case shown below.

\[
\delta V^{(\text{def})} = \overline{P} \delta V^{(\text{tot})} \quad \text{where}
\]

\[
\delta V^{(*)} = \begin{bmatrix}
\delta V^{(1,*)} \\
\vdots \\
\delta V^{(N,*)}
\end{bmatrix}
\]

\[
\overline{\delta V}^{(a,*)} = \begin{bmatrix}
\delta \overline{u}^{(a,*)} \\
\delta \overline{\omega}^{(a,*)}
\end{bmatrix}
\]

\[
\delta \overline{u}^{(a,*)} = \text{first variation of } \overline{u}^{(a,*)} \text{ with respect to nodal freedoms}
\]

\[
\delta \overline{\omega}^{(a,*)} = \text{first variation of } \overline{\omega}^{(a,*)} \text{ with respect to nodal freedoms}
\]

This projector then can be used to estimate finite changes in elastic deformation for finite changes in nodal configurations during the iterative solution process as shown below.
\[ \vec{v}^{(\text{def})} \approx \overline{P} \vec{v}^{(\text{tot})} \text{ where} \\
\vec{v}^{(*)} = \begin{bmatrix} \vec{v}^{(1,*)} \\ \vdots \\ \vec{v}^{(N,*)} \end{bmatrix} \\
\vec{v}^{(a,*)} = \begin{bmatrix} \vec{u}^{(a,*)} \\ \overline{\omega}^{(a,*)} \end{bmatrix} \]

\[ \vec{u}^{(a,\text{def})} = R^T_{A:E_k \rightarrow G} \left( u^{(a,\text{tot})} - u^{(C,\text{rig})} + x^{(a)}_o - \bar{x}^{(a)}_o \right) \text{ from equation (4)} \]

\[ \overline{\omega}^{(a,\text{def})} = \overline{\omega}^{(a,\text{tot})} - \overline{\omega}^{(\text{rig})} \text{ from equation (7)} \]

This approximation approaches an equality as the number of iterations becomes very large (i.e. approaches infinity).

Projector \( \overline{P} \) is found by taking the first variations of \( \vec{v}^{(a,\text{def})} \). Hence, the main tasks are to find \( \delta \vec{u}^{(a,\text{def})} \) and \( \delta \overline{\omega}^{(a,\text{def})} \), assemble them for all nodes, and use the results to form \( \vec{v}^{(a,\text{def})} \).

Computing these variations requires various spinor properties and centroid relationships presented below.

**Spinor Properties**

**SP1:** Given any \( \text{Spin}(\omega) \) for \( \omega = (v_1, v_2, v_3)^T \), it can be shown by expanding that:

\[ (\text{Spin}(\omega))^T = - \text{Spin}(\omega) \text{ because Spin}(\omega) \text{ is skew symmetric} \]  \[105\]

**SP2:** Given any vector \( \mathbf{v} = (v_1, v_2, v_3)^T \) and any \( \text{Spin}(\omega) \) for \( \omega = (\omega_1, \omega_2, \omega_3)^T \), it can be shown by expanding that:

\[ \omega \times \mathbf{v} = \text{Spin}(\omega) \mathbf{v} = -\text{Spin}(\mathbf{v}) \omega = -\mathbf{v} \times \omega \]  \[105\]

**SP3:** Given any orthogonal matrix \( R \) such that \( R = e^{\text{Spin}(\omega_R)} \), it can be shown by varying \( R \) and expanding that:

\[ \delta R = \delta \text{Spin}(\omega_R) e^{\text{Spin}(\omega_R)} = \text{Spin}(\delta \omega_R) e^{\text{Spin}(\omega_R)} = \text{Spin}(\delta \omega_R) R \]  \[105\]

\[ \delta R^T = R^T (\text{Spin}(\delta \omega_R))^T = -R^T (\text{Spin}(\delta \omega_R)) \text{ by SP1} \]

**Centroid Relationships**

**CR1:** As mentioned earlier, the centroid position vector is calculated from the nodal position vectors as follows:

\[ X^{C_0} = \left( \frac{1}{N} \right) \sum_{a=1}^{N} x^{(a)} \text{ where } N = \text{number of element nodes} \]  \[47\]

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CR2: Similarly, the centroid’s translational displacement vector is calculated from the nodal displacement vectors as follows:

\[ \mathbf{u}^C = \left( \frac{1}{N} \right) \sum_{a=1}^{N} \mathbf{u}^a \text{ where } N = \text{number of element nodes} \]  \[47\]

CR3: Using CR2, the following expression can be developed.

\[ \mathbf{u}^a - \mathbf{u}^C = \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) \mathbf{u}_b \text{ where } \delta_{ab} = \text{Kronecker delta} \]  \[47\]

as follows:

\[ \mathbf{u}^a - \mathbf{u}^C = \mathbf{u}^a - \left( \frac{1}{N} \right) \sum_{a=1}^{N} \mathbf{u}^a \text{ by CR2} \]

\[
\begin{bmatrix}
\mathbf{u}_{a1} \\
\mathbf{u}_{a2} \\
\mathbf{u}_{a2}
\end{bmatrix}
- \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{11} \\
\mathbf{u}_{12} \\
\mathbf{u}_{12}
\end{bmatrix}
+ \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{a1} \\
\mathbf{u}_{a2} \\
\mathbf{u}_{a2}
\end{bmatrix}
+ \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{a1} \\
\mathbf{u}_{a2} \\
\mathbf{u}_{a2}
\end{bmatrix}
+ \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{N1} \\
\mathbf{u}_{N2} \\
\mathbf{u}_{N3}
\end{bmatrix}
\]

\[
= \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{11} \\
\mathbf{u}_{12} \\
\mathbf{u}_{12}
\end{bmatrix}
- \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{a1} \\
\mathbf{u}_{a2} \\
\mathbf{u}_{a2}
\end{bmatrix}
- \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{a1} \\
\mathbf{u}_{a2} \\
\mathbf{u}_{a2}
\end{bmatrix}
- \frac{1}{N}
\begin{bmatrix}
\mathbf{u}_{N1} \\
\mathbf{u}_{N2} \\
\mathbf{u}_{N3}
\end{bmatrix}
\]

\[
= \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) \mathbf{u}_b
\]

Using the above resources, the translational deformation given by equation (4) can be varied as shown below following Nour-Omid Rankin [91] and Haugen [57].

\[
\bar{\mathbf{u}}^{(a, \text{def})} = \mathbf{R}_{A:Ek \rightarrow G}^T \left( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o^{(a)} \right) - \bar{\mathbf{x}}_o^{(a)} \quad (4)
\]

\[
\delta\bar{\mathbf{u}}^{(a, \text{def})} = \mathbf{R}_{A:Ek \rightarrow G}^T \delta \left( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} \right) + \delta \mathbf{R}_{A:Ek \rightarrow G}^T \left( \mathbf{u}^{(a, \text{tot})} + \mathbf{x}_o^{(a)} - \mathbf{u}^{(C, \text{rig})} \right)
\]

because \( \delta \mathbf{x}_o^{(a)} = \delta \bar{\mathbf{x}}_o^{(a)} = 0 \) since constants [91]

\[
\delta\bar{\mathbf{u}}^{(a, \text{def})} = \mathbf{R}_{A:Ek \rightarrow G}^T \delta \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) \mathbf{u}_b \right) - \mathbf{R}_{A:Ek \rightarrow G}^T \mathbf{Spin}(\delta \omega_E) \left( \mathbf{x}_o^{(a)} - \mathbf{u}^{(C, \text{rig})} \right)
\]

using CR3 to replace \( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} \)

using SP3 to replace \( \delta \mathbf{R}^{T}_{A:Ek \rightarrow G} \)
\[ \delta \mathbf{u}^{(a, \text{def})} = R_{A;E}^T \delta \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \mathbf{u}_b \right) - R_{A;E}^T \text{Spin}(\delta \mathbf{w}_E) R_{A;E}^T (\mathbf{x}_D^{(a,D)} - \mathbf{u}^{(C, \text{rig})}) \]

by inserting \( R_{A;E}^T R_{E}^T = I \) [91]

\[ \delta \mathbf{u}^{(a, \text{def})} = \delta \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \mathbf{u}_b \right) - \text{Spin}(\delta \mathbf{w}_E) (\mathbf{x}_D^{(a,D)} - \mathbf{u}^{(C, \text{rig})}) \]

because \( R_{A;E}^T \mathbf{u}_b = \mathbf{u}_b, R_{A;E}^T \mathbf{x}_a = \mathbf{x}_a \) by axis rotation

\[ R_{A;E}^T \text{Spin}(\delta \mathbf{w}_E) R_{A;E}^T = \text{Spin}(\delta \mathbf{w}_E) \]

by tensor basis change

\[ \delta \mathbf{u}^{(a, \text{def})} = \delta \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \mathbf{u}_b \right) - \text{Spin}(\delta \mathbf{w}_E) (\mathbf{x}_D^{(a,D)}) \]

because \( \mathbf{u}^{(C, \text{rig})} = 0 \) since best fit assumes same centroid for \( \mathcal{C}^R \) and \( \mathcal{C}^D \) and \( \mathcal{C}^D \) being perturbed with respect to \( \mathcal{C}^R \) [57]

\[ \delta \mathbf{u}^{(a, \text{def})} = \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \mathbf{u}_b \right) + \text{Spin}(\mathbf{x}_D^{(a,D)}) \delta \mathbf{w}_E \]

by SP2

\[ \delta \mathbf{u}^{(a, \text{def})} = \left( \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \mathbf{u}_b \right) + \text{Spin}(\mathbf{x}_D^{(a,D)}) \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \]

because \( \delta \mathbf{w}_E = G \delta \mathbf{v}_E = \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \) by definition [57]

\[ \delta \mathbf{u}^{(a, \text{def})} = \sum_{b=1}^{N} \left[ \left( \delta_{ab} - \frac{1}{N} \right) I_{3 \times 3} 0_{3 \times 3} \right] \delta \mathbf{v}_b + \text{Spin}(\mathbf{x}_D^{(a,D)}) \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \] because

\[ \delta \mathbf{v}_b = \left[ \delta \mathbf{u}_b \right] \]

giving \( \sum_{b=1}^{N} \left[ \left( \delta_{ab} - \frac{1}{N} \right) I_{3 \times 3} 0_{3 \times 3} \right] \delta \mathbf{v}_b = \sum_{b=1}^{N} \left( \delta_{ab} - \frac{1}{N} \right) I \delta \mathbf{u}_b \) [57]

\[ \delta \mathbf{u}^{(a, \text{def})} = \sum_{b=1}^{N} \left[ \left[ \left( \delta_{ab} - \frac{1}{N} \right) I_{3 \times 3} 0_{3 \times 3} \right] + \text{Spin}(\mathbf{x}_D^{(a,D)}) (G_b) \right] \delta \mathbf{v}_b \] [57] (9a)

Likewise, the variation of the rotational deformation given by equation (7) can be found as follows using Haugen [57].

\[ \mathbf{w}^{(a, \text{def})} = \mathbf{w}^{(a, \text{tot})} - \mathbf{w}^{(\text{rig})} \] (7)

\[ \mathbf{w}^{(a, \text{def})} = \mathbf{w}^{(a, \text{tot})} - \mathbf{w}_E \] as nodal rigid body rotation equal to element rotation

\[ \delta \mathbf{w}^{(a, \text{def})} = \delta \mathbf{w}^{(a, \text{tot})} - \delta \mathbf{w}_E \]

\[ \delta \mathbf{w}^{(a, \text{def})} = \delta \mathbf{w}^{(a, \text{tot})} - \mathbf{G} \delta \mathbf{v} = \delta \mathbf{w}^{(a, \text{tot})} - \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \] as \( \delta \mathbf{w}_E = \mathbf{G} \delta \mathbf{v} = \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \) by definition

\[ \delta \mathbf{w}^{(a, \text{def})} = \delta \mathbf{w}^{(a, \text{tot})} - \mathbf{G} \delta \mathbf{v} = \delta \mathbf{w}^{(a, \text{tot})} - \sum_{b=1}^{N} (G_b) \delta \mathbf{v}_b \]
\[
\delta \omega^{(a, \text{def})} = \delta \vec{v}_a - \sum_{b=1}^{N} G_b \delta v_b \text{ because } \delta \omega_a = G_a \delta v_a \\
\delta \omega^{(a, \text{def})} = G_a \delta v_a - (G_1 \delta v_1 + \ldots + G_a \delta v_a + \cdots + G_N \delta v_N) \\
\delta \omega^{(a, \text{def})} = \left\{ \begin{array}{l}
G_1 \delta v_1 + \cdots + G_{a-1} \delta v_{a-1} + 0_{3 \times 6} + G_{a+1} \delta v_{a+1} + \cdots + G_N \delta v_N \\
\end{array} \right. \\
\delta \omega^{(a, \text{def})} = \sum_{b=1}^{N} (\delta_{ab} [0_{3 \times 3} 1_{3 \times 3}] - G_b) \delta v_b \text{ [57]} \tag{9b}
\]

Equation (9b) is true because, for

\( b = a : (\delta_{ab} [0_{3 \times 3} 1_{3 \times 3}] - G_b) \delta v_b = ([0_{3 \times 3} 1_{3 \times 3}] - G_a) \delta v_a = \delta \omega_a - G_a \delta v_a = \delta \omega_a - \delta \omega_a = 0_{3 \times 1} \)

\( b \neq a : (\delta_{ab} [0_{3 \times 3} 1_{3 \times 3}] - G_b) \delta v_b = -G_b \delta v_b \)

Equations (9a) and (9b) then can be combined into the incremental deformation vector \( \delta \vec{v}^{(a, \text{def})} \) for node (a) as follows:

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
\delta \vec{u}^{(a, \text{def})} \\
\delta \vec{\omega}^{(a, \text{def})}
\end{array} \right] = \left[ \begin{array}{c}
\left( \sum_{b=1}^{N} \left( \begin{array}{c}
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right) \right) + \text{Spin}(\vec{x}_{(D)}^{(a)}) G_b \\
\left( \sum_{b=1}^{N} (\delta_{ab} [0_{3 \times 3} 1_{3 \times 3}] - G_b) \right) \delta \vec{v}_b
\end{array} \right]
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] - \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] - \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] + \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] - \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] + \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

\[
\delta \vec{v}^{(a, \text{def})} = \left[ \begin{array}{c}
0_{3 \times 3} \\
\left( \delta_{ab} \frac{1}{N} \right) I_{3 \times 3} - O_{3 \times 3}
\end{array} \right] - \text{Spin}(\vec{x}_{(D)}^{(a)}) \frac{1}{G_b} \delta \vec{v}_b
\]

Equations (9a) and (9b) then can be combined into the incremental deformation vector \( \delta \vec{v}^{(a, \text{def})} \) for node (a) as follows:
\[ \delta \mathbf{v}^{(a), \text{def}} = \sum_{b=1}^{N} \delta_{ab} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{N} \begin{bmatrix} I_{3x3} & 0_{3x3} \\ 0_{3x3} & I_{3x3} \\ 0_{3x3} & 0_{3x3} \end{bmatrix} \delta \mathbf{G}_b \delta \mathbf{v}_b \]

\[ \delta \mathbf{v}^{(a), \text{def}} = \sum_{b=1}^{N} \delta_{ab} \mathbf{I}_{6x6} - \mathbf{P}_{\text{Tab}} - \delta \mathbf{G}_b \delta \mathbf{v}_b \text{ where } \mathbf{P}_{\text{Tab}} = \frac{1}{N} \begin{bmatrix} I_{3x3} & 0_{3x3} \\ 0_{3x3} & I_{3x3} \end{bmatrix} \]

Finally, these \( \delta \mathbf{v}^{(a), \text{def}} \) blocks can be assembled for all nodes to form the full incremental deformation vector \( \delta \mathbf{v} \) and the projector \( \mathbf{P} \) as shown below.

\[ \delta \mathbf{v}^{\text{def}} = \sum_{a=1}^{N} \sum_{b=1}^{N} \delta_{ab} \mathbf{I}_{6x6} - \mathbf{P}_{\text{Tab}} - \delta \mathbf{G}_b \delta \mathbf{v}_b \]

\[ \delta \mathbf{v}^{\text{def}} = \begin{pmatrix} \mathbf{I}_{6x6} & \cdots & \mathbf{I}_{6x6} \end{pmatrix}_{N \times \text{times}} - \begin{bmatrix} \mathbf{P}_{T11} & \cdots & \mathbf{P}_{T1N} \\ \vdots & \ddots & \vdots \\ \mathbf{P}_{TNN} & \cdots & \mathbf{P}_{TNN} \end{bmatrix} - \begin{bmatrix} \delta \mathbf{G}_1 & \cdots & \delta \mathbf{G}_N \\ \vdots & \ddots & \vdots \\ \delta \mathbf{G}_N & \cdots & \delta \mathbf{G}_N \end{bmatrix} \delta \mathbf{v} \]

\[ \delta \mathbf{v}^{\text{def}} = \left( \mathbf{I} - \mathbf{P}_{\text{Tab}} - \bar{\mathbf{P}}_R \right) \delta \mathbf{v} \text{ because } \mathbf{P}_{\text{Tab}} = \mathbf{P}_{\text{Tab}} \]

\[ \delta \mathbf{v}^{\text{def}} = \bar{\mathbf{P}} \delta \mathbf{v} \quad [57] \]  

where

\[ \bar{\mathbf{P}} = \mathbf{I} - \mathbf{P}_T - \bar{\mathbf{P}}_R \]

\[ \mathbf{I} = \text{diag} \left[ \mathbf{I}_{6x6} \cdots \mathbf{I}_{6x6} \right] \]

\[ \bar{\mathbf{P}}_T = \mathbf{P}_T = \begin{bmatrix} \mathbf{P}_{T11} & \cdots & \mathbf{P}_{T1N} \\ \vdots & \ddots & \vdots \\ \mathbf{P}_{TNN} & \cdots & \mathbf{P}_{TNN} \end{bmatrix}, \quad \mathbf{P}_{\text{Tab}} = \begin{bmatrix} \frac{1}{N} \mathbf{I}_{3x3} & \mathbf{0}_{3x3} \\ \mathbf{0}_{3x3} & \mathbf{0}_{3x3} \end{bmatrix} \]

\[ \bar{\mathbf{P}}_R = \delta \mathbf{G} \]

with

\[ \delta \mathbf{v}_E = \bar{\mathbf{G}} \delta \mathbf{v}^k = \sum_{b=1}^{N} \bar{\mathbf{G}}_b \delta \mathbf{v}_b^k \text{ in local coordinates in } \mathbb{C}^D, \]

\[ \delta \mathbf{v}_E^k = \text{variations in rigid body rotation of the element (i.e. variations of } E_k \text{ basis vectors) } \]

\[ \delta \mathbf{v}^k = \text{variations in nodal translations ( } \delta \mathbf{u}^k \text{), rotations ( } \delta \mathbf{\omega}^k \text{) at all nodes } \]
The above definition is the same as that given initially in Section 3.3.1. It can be seen as equivalent to the definitions in Haugen [57] with $\chi^{(a,D)}_{(D)} = \chi^a$ and in Felippa/Haugen [47] with $\chi^{(a,D)}_{(D)} = \chi^D_a$. A comparison with the Nour-Omid/Rankin projector [91] is given in Section 3.3.1.

Informed by this chapter’s explanations, the next chapter explores rigid and elastic deformations, finite element analysis in general, and the EICR and similar corotational methods in particular from a Lie theory perspective.
Chapter 4

Lie Theory Discussions of Rigid & Elastic Deformation, Finite Element Methods in general, and the EICR Method

The explanation of the EICR Method found in Chapter 3 requires coordinates and various coordinate choices. These choices include the frame type (e.g. orthogonal versus convecting), how local frames are defined and positioned using the element nodes, and even the elemental nodal numbering convention. Different coordinate choices change the vector or matrix representations of the various translations, rotations, coordinate transformations, and, hence, the projector.

However, the same physical rigid and elastic deformational motions occur regardless of how they are represented. In other words, the numerical representations of a motion may change but the actual physical (or geometric) motion does not. In this sense, the physical motions are more fundamental (and arguably simpler) than their numerical representations. For example, the algorithms in Chapter 3 are dependent upon and, hence, change with the coordinate choices made. Algorithms based only on the physical motions are invariant to coordinate changes. Furthermore, simpler algorithms often highlight and help explain more fundamental coordinate invariant properties.

Such advantages explain the growing popularity of differential geometry in kinematics and also the reason for this chapter. Smooth manifolds and Lie groups are key tools in differential geometry and considerable literature exists on their application to rigid body kinematics. Discussion of elastic deformations in terms of Lie theory is far less developed but an emerging research area as demonstrated by works like Marsden/Hughes [82].
This chapter explores rigid and elastic deformations, finite element analysis in general, and the EICR Method in particular from a Lie theory perspective. After reviewing the literature, it provides some basic theory (manifolds, Lie groups/algebras, Lie derivative, etc.). It then uses this theory to discuss the kinematics of rigid and elastic deformation, finite element methods in general, and the EICR Method in particular. It should be noted that this chapter seeks a middle ground between more theoretical and more applied discussion of Lie theory. As examples, manifolds are discussed briefly as independent structures (i.e. without an ambient space) but later applied only in $\mathbb{R}^3$; Lie groups are defined generally but then the focus shifts to simpler matrix Lie groups; and elastic deformations are recognized as an infinite-dimensional vector spaces but simplified to finite-dimensional spaces through discretization and linearization.

4.1 Literature Review

Differential geometry, manifolds and Lie theory are rich areas extensively discussed, separately and jointly, in a wide range of literature, both abstract and applied. More theoretical texts include Boothby [18], Burns/Gidea [22], and O’Neill [93] for differential geometry; Lee for topological [76] and smooth [75] manifolds; and Varadarajan [125] for Lie theory. Hall [56], Murray et al. [90] and Stillwell [115] focus on matrix (versus general) Lie theory which is more easily understood (requiring less mathematical background) and extensively used in areas like robotics. Texts by Bullo/Lewis [21], Abraham et al. [1], and Marsden/Ratiu [83] offer a very useful blending of Lie theory and its application. Tensors and their importance in physical (coordinate invariant) motion are well explained in Bower [19], Mase [87] and Kolecki [71].

The literature discussing rigid body kinematics using Lie theory also is quite extensive. Two excellent sources using largely matrix Lie groups include Bullo/Lewis [21] and, relative to robotics, Murray et al.[90]. Other noteworthy robotics works are Kumar [72] and Park et al. [98]. Two papers by Kecskeméthy [65,66] discuss rigid deformation using general Lie groups blending the applied with the more abstract.
Considerable literature exists on the kinematics of elastic deformation using linear algebra. Gurtin et al. [55] explains elastic deformation theory very clearly as do Bower [19] and Chadwick [27]. Thesis work and course materials by Brannan [20], Pandolfi [96], and Weinberg [128] also are very useful. Two papers by Yanao, Koon et al. (including Marsden) [131,132] discuss decomposing elastic deformations using singular value decompositions albeit in a chemical engineering context. Casey [24-26] models bodies deformed homogeneously as pseudo-rigid bodies with twelve degrees of freedom.

Less literature is available using Lie theory to discuss elastic deformation. Comprehensive but advanced discussions can be found in Marsden/Hughes [82]; Abraham, Marsden, and Ratiu [1]; Marsden/Ratiu [83]; and Simo/Hughes [111]. More basic discussion can be found in Yavari, Marsden et al. [133] and, in the pseudo-rigid body context, in Slawianowski [112-114]. Pandolfi [96,97] and others [11,59] also provide useful background, especially about the Lie derivative.

Personal discussions have supplemented and enriched the above literature greatly. Generous discussions have occurred with Drs. Scott Carnahan, Reuven Segev, and Jan Slawianowski on Lie theory and elastic deformation; Drs. Ivan Dimitrov and Jack Lee on Lie theory, orthogonal projections and Killing forms; Drs. Gerhard Holzapfel, Tom Hughes, Jerry Marsden, and Anna Pandolfi on the Lie derivative; doctoral candidate Valdemar Tsanov on Lie theory; Drs. Mark Green and Ian Moore on finite element analysis; Drs. Carlos Felippa and Charles Rankin on the EICR Method; and Dr. Andrew Lewis on a wide range of issues kindly discussed at length. It should be noted that these topics are complex so a rich understanding of them is challenging, especially at the level of this thesis and its author. Hence, any oversimplification rests with the author, not those who so kindly have helped.
4.2 Basic Theory

This section includes basic theory on groups, manifolds, Lie groups and Lie algebras progressing from less to more structured objects. For example, a Lie group is both a smooth manifold and a group. The appropriate morphisms (maps) between objects of a particular type are discussed as are the roles of tensors in coordinate invariance. The Lie derivative then is explored. This section is guided by the “middle of the road” approach discussed earlier and content which is truly essential for later discussions. The specifics of rigid body kinematics and elastic deformation are left to the later section on kinematics.

4.2.1 Groups

A group is an algebraic structure consisting of a set and an operation that, when combining any two elements of the set, produces an element within the set. More formally, a group and its properties can be defined as follows based on Humphreys/Prest [63].

A group $G$ is a set $G$, together with an operation $*$, which satisfies the following properties:

(G1) for all elements $g$ and $h$ of $G$, $g * h$ is an element of $G$ (closure);
(G2) for all elements $g$, $h$ and $k$ of $G$, $(g * h) * k = g * (h * k)$ (associativity);
(G3) there exists an element of $e$ of $G$, called the identity (or unit) of $G$, such that for all $g$ in $G$ we have $e * g = g * e = g$ (existence of identity);
(G4) for every $g$ in $G$ there exists an element $g^{-1}$ called the inverse of $g$, such that $g * g^{-1} = e$ (existence of inverse)

These properties are discussed later relative to Lie groups. In these discussions, the operation symbol $*$ often is omitted for simplicity (e.g. $g * h$ is written simply as $gh$).

4.2.2 Manifolds

This discussion draws primarily upon books by Lee on topological [76] and smooth [75] manifolds supplemented by Bullo/Lewis [21] and Hall [56]. It follows Lee’s progression beginning with topological manifolds (first described informally, then defined embedded in an
ambient Euclidean space $\mathbb{R}^k$, and lastly defined abstractly absent of any ambient space) and later adding smooth structure to the topological manifold to explain a smooth manifold. Finally, Riemann manifolds which combine smooth manifolds with a Riemann metric are discussed.

4.2.2.1 Topological Manifolds

Topology, as a branch of mathematics, is concerned with spatial properties that are preserved when objects are deformed continuously. Since objects are sets of points, topology combines continuity and set theory [76]. Informally, an $n$-dimensional topological manifold is an object (like a curve, surface or higher dimension object) that looks locally like $\mathbb{R}^n$ [21,76]. This means that points on the manifold can be represented locally in $\mathbb{R}^n$. The manifold’s dimension (n) can be thought of as the number of independent parameters (e.g. numbers, degrees of freedom, etc.) needed to specify a point reasonably close to some reference point. How close is reasonable depends on how extremely the object’s shape (or whatever other characteristic is represented by the manifold) changes moving away from the reference point.

Using an example from Lee [76] shown in Figure 4-1, the position of a rigid body moving through space can be specified by three non-collinear points (P, Q, R). Three points with three degrees of freedom result in nine degrees of freedom (i.e. $\mathbb{R}^9$). However, being a rigid body, P, Q and R are somewhat interdependent. The distance from P to Q ($d_{PQ}$) is fixed so, for any point P (3 coordinates), Q can be uniquely determined by its latitude ($Q_1$) and longitude ($Q_2$) on a sphere of radius $d_{PQ}$ (2 more coordinates). Finally, R can be specified by the rotation $\theta$ (1 more coordinate) of the plane PQR about the line PQ relative to some reference plane. So, the body’s position can be determined by only 6 coordinates ($P_1, P_2, P_3, Q_1, Q_2, \theta$) in $\mathbb{R}^6$ with the set of all possible positions being a 6-dimensional manifold. It is important to note the following: i) this manifold represents the possible positions not the rigid body itself; ii) P is the reference point; and iii) distance from reference point P is not a factor because the object is rigid and thus well behaved.
Topological manifolds can be defined formally in two ways, either embedded in an ambient space (e.g. $\mathbb{R}^k$, $k \geq n$, $n =$ dimension of manifold) or abstractly as their own topological space without an ambient space. For either definition, “looks locally like $\mathbb{R}^n$” needs to be formalized using the concepts of homeomorphic spaces and locally Euclidean. For the simpler embedded case, Lee [76] defines these concepts and an n-dimensional manifold embedded in ambient space $\mathbb{R}^k$ ($k \geq n$) as follows:

Two subsets of Euclidean spaces $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^n$ are topologically equivalent or homeomorphic if there exists a one-to-one correspondence $\varphi: U \to V$ such that both $\varphi$ and its inverse are continuous maps.

A subset $M$ of some Euclidean space $\mathbb{R}^k$ is locally Euclidean of dimension $n$ if every point of $M$ has a neighbourhood in $M$ that is homeomorphic to a ball in $\mathbb{R}^n$.

An n-dimensional manifold (n-manifold for short) is a subset of some Euclidean space $\mathbb{R}^k$ that is locally Euclidean of dimension $n$.

The rigid body example given earlier is a 6-dimensional manifold embedded in $\mathbb{R}^9$.

For the abstract case, the loss of the Euclidean ambient space (and, hence, its norm) requires a shift to topological spaces based on open sets to manage issues like convergence and continuity without norms. A topological space is a pair $(X, \tau)$ consisting of a set $X$ and a topology $\tau$ on $X$ [76]. This topology on the set $X$ is a collection $\tau$ of subsets of $X$, called open sets,
satisfying the following properties [76]:

(i) the set \( X \) and the null set \( \{ \} \) are elements of \( \tau \).

(ii) \( \tau \) is closed under finite intersections: If \( U_1, \ldots, U_n \in \tau \), then their intersection \( U_1 \cap \ldots \cap U_n \) is in \( \tau \).

(iii) \( \tau \) is closed under arbitrary unions: If \( A \) is an arbitrary index set and \( \{ U_a \}_{a \in A} \) is any (finite or infinite) collection of elements of \( \tau \), then their union \( \bigcup_{a \in A} U_a \) is in \( \tau \).

If \((X, \tau)\) is a topological space and \( x \in X \), a neighbourhood of \( x \) is an open set \( U \in \tau \) for which \( x \in U \) [76].

Likewise, definitions appropriate to topological spaces are needed for convergence, continuity, homeomorphic and locally Euclidean in order to consider abstract manifolds.

If \((X, \tau)\) is a topological space and \( \{ q_i \} \) is any sequence of points in \( X \), the sequence converges to \( q \in X \) if, for every neighbourhood \( U \) of \( q \), there exists \( N \) such that \( q_i \in U \) for all \( i \geq N \) [76].

If \((X, \tau)\) and \((Y, \mu)\) are topological spaces, a map \( f: X \rightarrow Y \) is continuous if, for every open set \( U \subset Y \), \( f^{-1}(U) \) is open in \( X \) [76].

If \((X, \tau)\) and \((Y, \mu)\) are topological spaces, \( X \) and \( Y \) are homeomorphic if there exists a continuous bijective map \( \varphi: X \rightarrow Y \) with a continuous inverse. Such a map is called a homeomorphism [76].

A topological space \((M, \tau)\) is said to be locally Euclidean of dimension \( n \) if every point \( q \in M \) has a neighbourhood that is homeomorphic to an open subset of \( \mathbb{R}^n \) [up to and including \( \mathbb{R}^n \) itself] [76].

Hence, a topological space \((M, \tau)\) being locally Euclidean of dimension \( n \) means that every point has a neighbourhood \( U \) (\( U \subset M \) and \( U \in \tau \)) together with a homeomorphism \( \varphi: U \rightarrow \tilde{U} \) where \( \tilde{U} \) is some open subset of \( \mathbb{R}^n \) [76,122].

Two more properties are needed in defining abstract manifolds to ensure that the topology has enough but not too many open sets. Requiring that the topological space be a Hausdorff space provides sufficient sets so that points can be separated similar to a metric space. At the lesser extreme, a second-countable topological space minimizes the number of open sets required to cover the topological space. It also ensures (through a countable basis) that the
manifold can be embedded in some finite-dimensional Euclidean space thereby linking abstract and embedded manifolds [76]. Lee [76] defines these final supporting concepts and an abstract manifold as follows.

A topological space $X$ is said to be a **Hausdorff space** if given any pair of distinct points $q_1, q_2 \in X$, there exists neighbourhoods $U_1$ of $q_1$ and $U_2$ of $q_2$ with $U_1 \cap U_2 = \{\}$. A topological space is **second countable** if it admits a countable basis.

| An $n$-dimensional topological manifold is a second countable Hausdorff space that is locally Euclidean of dimension $n$. |

Having now defined an $n$-dimensional topological manifold, the open sets and homeomorphisms implicit in $M$ being locally Euclidean can be discussed more thoroughly as coordinate charts. From Lee [76], a **coordinate chart** (or just a **chart**) on an $n$-dimensional topological manifold $M$ is a pair $(U, \phi)$, where $U$ is an open subset of $M$ and $\phi: U \rightarrow \tilde{U}$ is a homeomorphism from $U$ to an open subset $\tilde{U} = \phi(U) \subset \mathbb{R}^n$ called a **coordinate map**. This map $\phi$ defines a local coordinate system $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Several open sets $(U_a, U_b, \text{etc.}, \text{often overlapping})$ may be needed to cover all of $M$, each with its own homeomorphism $(\phi_a, \phi_b, \text{etc.})$ and local coordinates as shown in Figure 4-2 from Bullo/Lewis [21].

![Figure 4-2: Coordinate Charts ([21], modified)](image)
These charts collectively form a coordinate atlas for M. If two charts \((U_a, \phi_a)\) and \((U_b, \phi_b)\) overlap \((U_a \cap U_b \neq \emptyset)\), the composite map \(\phi_{ab} = \phi_b \circ \phi_a^{-1}|_{\phi_a(U_a \cap U_b)}\) is called the transition map, overlap map or change of coordinates map from \(\phi_a(U_a \cap U_b)\) to \(\phi_b(U_a \cap U_b)\) [21]. Such transition maps ensure that points covered by two charts have compatible local representations. Being a composition of homeomorphisms, this transition map also is a homeomorphism and, hence, continuous.

Coordinate charts, atlases and transition maps (duly upgraded to smoothness versus continuity) are important in defining smooth manifolds which are discussed next.

4.2.2.2 Smooth Manifolds

Topological manifolds are based on topological spaces and homeomorphisms. While homeomorphisms are continuous, they are not always smooth (i.e. differentiable) so are not suited to calculus. Hence, interpreting the derivatives of real-valued functions and curves on manifolds or of maps between manifolds requires a smooth manifold. Informally, a smooth manifold is a topological manifold with some extra smooth structure which supports the definition of smooth functions on and between manifolds [75].

Hall [56] describes this smooth structure as “... a collection of local coordinate systems [charts] that cover the whole [topological] manifold and such that whenever two coordinate systems are defined in overlapping regions, the expression for one set of coordinates in terms of the other [i.e. the transition map] is always smooth.” Since the coordinate systems are in Euclidean space, the usual definition of smoothness applies as given below.

If \(U\) and \(V\) are open subsets of Euclidean spaces \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively, a function \(F: U \rightarrow V\) is said to be smooth (or \(C^\infty\), or infinitely differentiable) if each of its component functions has continuous partial derivatives of all orders [75].

Creating a smooth structure can be viewed as upgrading homeomorphisms, coordinate charts, atlases, and transition functions to be smooth as well as continuous. Since smoothness is defined above in Euclidean space \(\mathbb{R}^n\), it is most readily applied to transition maps (which are in
R^n) and, through them, to coordinate maps (which are not in R^n). This is done by requiring the transition maps to be diffeomorphic as defined below.

If U and V are open subsets of Euclidean spaces R^n and R^m, respectively, U and V are **diffeomorphic** if there exists a smooth bijective function F: U→V with a smooth inverse. Such a function is called a **diffeomorphism** [75].

Given a topological manifold M, two charts (U_a, \( \phi_a \)) and (U_b, \( \phi_b \)) are **smoothly compatible** if either \( U_a \cap U_b = \{ \} \) or the transition map \( \phi_{ab} = \phi_b \circ \phi_a^{-1} | \phi_a(U_a \cap U_b) \) is a diffeomorphism [75].

A **smooth atlas** \( (A) \) is a collection of charts such that its domain covers M (i.e. \( A \) is an atlas) and any two charts in \( A \) are smoothly compatible with each other. A manifold can have multiple smooth atlases resulting in the same collection of smooth functions on M which poses a problem in defining its smooth structure uniquely. This is overcome by defining either a **maximal atlas** (Lee [75]) or an **equivalence class of smooth atlases** (Bullo/Lewis [21]). Definitions of these concepts, smooth coordinate maps and charts, and smooth structure are given below.

**Smooth Structure defined as a Maximal Atlas by Lee [75]**

A smooth atlas \( A \) on a topological manifold M is **maximal smooth atlas** if it is not contained in any strictly larger smooth atlas.

Any chart \((U, \phi)\) contained in the smooth atlas of M is called a **smooth chart** with its coordinate map \( \phi \) being called a **smooth coordinate map**.

A **smooth structure** on a topological n-manifold M is a maximal smooth atlas.

**Smooth Structure defined as an Equivalence Relation by Bullo/Lewis [21]**

Two \( C^r \)-atlases [smooth atlases] \( A_1 \) and \( A_2 \) are **equivalent** if \( A_1 \cup A_2 \) is also a \( C^r \)-atlas.

A **\( C^r \)-differentiable structure** [smooth structure] on S [a set from a topological manifold] is an equivalence class of atlases under this equivalence relation.

Any chart \((U, \phi)\) contained in this equivalence class is called a **smooth chart** with its coordinate map \( \phi \) being called a **smooth coordinate map**.

Now having the smooth structure, a smooth manifold can be defined as follows.
A smooth manifold is a pair \((M, A)\) where \(M\) is a topological manifold and \(A\) is a smooth structure on \(M\) [75].

While a maximal atlas (or equivalence class) is used to uniquely define a smooth structure and manifold, any individual smooth atlas normally is sufficient when working with a smooth manifold. It also should be noted that, in the literature and this thesis, a smooth manifold \((M, A)\) usually is denoted simply as \(M\) rather than \((M, A)\).

Using its smooth structure, smooth functions (maps) can be defined on and between smooth manifolds. As discussed in Lee [75], a function \(f: M\rightarrow \mathbb{R}^k\) on a smooth \(n\)-manifold \(M\) is smooth if, for every \(p\in M\), there exists a smooth chart \((U, \phi)\) for \(M\) whose domain contains \(p\) and such that the composite function \(f\circ \phi^{-1}\) is smooth on the open subset \(\phi(U)\subset \mathbb{R}^n\) as illustrated in Figure 4-3. This definition covers both real-valued functions \(f: M\rightarrow \mathbb{R}\) (i.e. \(k = 1\)) and vector-valued functions \(f: M\rightarrow \mathbb{R}^k\) \((k > 1)\). The set of all real-valued smooth functions on \(M\) is designated \(C^\infty(M)\) and, since summed or scaled smooth functions remain smooth, is a vector space. The function \(\hat{f}: \phi(U)\rightarrow \mathbb{R}^k\) such that \(\hat{f}(x) = f\circ \phi^{-1}(x)\) is the coordinate representation of \(f\). For a smooth function, smooth coordinate representations exist for every smooth coordinate chart. Or, in other words, a smooth function has a smooth representation in any coordinate system [75].

![Figure 4-3: Definition of Smooth Functions ([75], modified)](image)

Smooth maps between manifolds can be defined similarly using Lee [75]. Assuming two smooth manifolds \(M\) and \(N\) not necessarily of the same dimension, any \(F: M\rightarrow N\) is a smooth map
if, for every $p \in M$, there exists two smooth charts i) $(U, \varphi)$ containing $p$ and ii) $(V, \psi)$ containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ as illustrated in Figure 4-4. The map $\hat{F} : \varphi(U) \psi(V)$, such that $\hat{F}(x) = \psi \circ F \circ \varphi^{-1}(x)$, is the coordinate representation of $F$. So, $F$ is smooth if $\hat{F}$ is smooth or, more simply, if $F$ is smooth in local coordinates. Compositions of smooth maps between smooth manifolds also are smooth.

![Figure 4-4: Definition of Smooth Maps [75]](image)

If a smooth map $F : M \rightarrow N$ between smooth manifolds $M$ and $N$ also is bijective and has a smooth inverse, it is a diffeomorphism between $M$ and $N$ with the manifolds being diffeomorphic. A diffeomorphism can exist between any two $n$-dimensional smooth manifolds including $\mathbb{R}^n$. This means that, for any smooth coordinate chart $(U, \varphi)$ on an $n$-dimensional smooth manifold $M$, the smooth coordinate map $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism between open subsets of $M$ and $\mathbb{R}^n$ [75].

A diffeomorphism for smooth manifolds is analogous to a homeomorphism for topological manifolds in that two diffeomorphic smooth manifolds are essentially the “same”. The physical (versus numeric or coordinate-specific) properties of smooth manifolds are those that are invariant under diffeomorphisms.

Given a smooth manifold’s smooth structure, smooth curves connecting any two points as well as tangents at any point can be defined on the manifold [75]. For example, if a smooth
manifold represents positions of a rigid body, trajectories between configurations (i.e. smooth curves) and infinitesimal changes in a given configuration (i.e. differentials) can be obtained. It also is possible to construct linear approximations (i.e. tangents) of a curve (trajectory) at any given point [75]. These ideas can be formalized through defining a tangent vector, tangent space and tangent bundle in the context of smooth manifolds.

Informally, a tangent vector is an infinitesimal displacement at a specific point P on a smooth manifold. The tangent space at point P is a real vector space of all the tangent vectors at that point or, more simply, of all the possible “directions” one can tangentially pass through P [75]. The tangent bundle is the collection of all the tangent spaces on a manifold and itself forms a manifold of twice the dimension of the original manifold [75].

These concepts can be defined formally both extrinsically (for a smooth manifold in an ambient space R^n) or intrinsically (for any smooth manifold). Intrinsically, there are two equivalent definitions: one based on the approximate directions (linearizations) of curves (trajectories) through a point and the other based on derivations which are an abstraction for general manifolds of directional derivatives in R^n. Bullo/Lewis [21] use the former whereas Lee [75] and Hall [56] use the latter approach.

**Extrinsic Definitions (i.e. R^n ambient space) [56]**

For a smooth manifold S embedded in R^n, the tangent space at a point a ∈ S is the set of vectors v in R^n such that v = dγ/dt|_{t=0}, where γ(t) is a smooth curve lying in S and satisfying γ(0) = a.

A tangent vector at point a is an element of this set.

Figure 4-5 illustrates tangent spaces for two smooth manifolds. In Figure 4-5(a), sphere S^2 is a smooth manifold M embedded in R^3 with a tangent space shown at point a ∈ S^2, T_aM. T_aM is a plane containing the various tangent vectors shown in blue and thus is two dimensional. Figure 4-5(b) shows S^1 (a circle) and its tangent space at point e, T_eM. T_eM is one-dimensional
consisting of tangent vectors of varying lengths and orientations \[122\]. As an aside, \(S^1\) and \(S^2\) both are smooth manifolds but \(S^1\) also is a Lie group which is discussed later.

![Figure 4-5: Tangent Spaces to \(S^2\) and \(S^1\)]

**Intrinsic Definitions using Curves** [21]

Given a manifold \(M\) and \(x \in M\), a curve at \(x\) is a curve \(\gamma: I \to M\), where \(I\) is an interval containing 0 in its interior, and for which \(\gamma(0) = x\).

Two curves \(\gamma_1\) and \(\gamma_2\) are equivalent at \(x\) \((\gamma_1 \sim_x \gamma_2)\) if, in a coordinate chart \((U, \varphi)\) with \(x \in U\), the local representatives of \(\gamma_1\) and \(\gamma_2\) have the same derivative at 0. The equivalence class of \(\gamma\) is denoted \([\gamma]\).

A tangent vector at \(x\) is an equivalence class of curves under the equivalence relation \(\sim_x\).

The tangent space \((T_xM)\) is the collection of all tangent vectors at \(x\).

These definitions are illustrated in Figure 4-6 from Bullo/Lewis [21].

![Figure 4-6: Equivalent curves determine a tangent vector [21]]
Intrinsic Definitions using Derivations [56,75]

This approach generalizes the directional derivative (in R^n) to any smooth manifold (not necessarily embedded in R^n). To do so, it focuses on the directional derivative for a product of smooth functions which uses the usual product rule for derivatives.

The directional derivative of smooth function \( f \) at point \( m \) in direction of vector \( v \) is

\[
D_v(f(m)) = \frac{d}{dt} (f(\gamma(t))) \bigg|_{t=0} = \frac{d}{dt} (f(m + tv)) \bigg|_{t=0}
\]

So, the directional derivative for smooth product \( fg \) for same point and direction is

\[
D_v((fg)(m)) = \frac{d}{dt} (fg(m + tv)) \bigg|_{t=0} = f(m) (D_v(g))(m) + g(m) (D_v(f))(m)
\]

Replacing the directional derivative (\( D_v \)) by a linear map \( X: C^\infty(M) \rightarrow \mathbb{R} \) gives

\[
X((fg)(m)) = f(m) X(g)(m) + g(m) X(f)(m)
\]

and the definitions which follow.

Given a smooth manifold \( M \) and a point \( m \) of \( M \), a derivation at \( m \) is a linear map \( X: C^\infty(M) \rightarrow \mathbb{R} \) which satisfies \( X((fg)) = f(m) X(g) + g(m) X(f) \) for all \( f, g \in C^\infty(M) \).

Given a smooth manifold \( M \) and a point \( m \in M \), the tangent space \( (T_m(M)) \) at \( m \) to \( M \) is the set of all linear maps \( X: C^\infty(M) \rightarrow \mathbb{R} \) satisfying the following conditions.

1. the “product rule”: \( X((fg)) = f(m) X(g) + g(m) X(f) \) for all \( f, g \in C^\infty(M) \);
2. “localization”: If \( f = g \) in a neighbourhood of \( m \), then \( X(f) = X(g) \).

An element of \( T_m(M) \) is called a tangent vector at \( m \).

The tangent space is a vector space. The disjoint union of all tangent spaces of a n-dimensional smooth manifold \( M \) is called the tangent bundle (TM) which itself is a smooth manifold. Its dimension is 2n given that, for each dimension, two parameters are needed: the point \( m \) and its tangent space. For every tangent bundle, there exists a local smooth projection map \( \pi: TM \rightarrow M \) such that \( \pi(x, v_x) = x \) where \( x \) is a point on a smooth manifold \( M \) and \( v_x \) is any tangent vector in \( T_xM \).

If a smooth map \( F: M \rightarrow N \) exists between two smooth manifolds \( M \) and \( N \), then a linear map can be defined between their tangent spaces from a given point \( p \) on \( M \) to \( F(p) \) on \( N \). This map is called the tangent map \( TF_p \) (since it maps a tangent at \( p \) to a tangent at \( F(p) \)), the differential \( dF_p \) (since it is the best linear approximation of \( F \) near \( p \)), or the pushforward \( F_* \) (since it “pushes
forward” tangent vectors on M to tangent vectors on N) of F at point p on M. As with earlier concepts, it can be defined extrinsically using the derivative of F or intrinsically using equivalence classes or the derivation of F.

**Extrinsic Definition (Euclidean Space)** [75]

If F: M→N is a smooth map between smooth manifolds M and N which are R^m and R^n respectively, then the total derivative of F at point p is a linear map given by its Jacobian matrix.

**Intrinsic Definition using Curves** [21]

If F: M→N is a smooth map between smooth manifolds M and N, the tangent map of F at p is a map TF: TM→TN such that TF([γ]_p) = [F © γ]_F(p) where [γ]_p ∈ T_pM and [F © γ]_F(p) ∈ T_{F(p)}N are tangent vectors at p on M and F(p) on N respectively.

**Intrinsic Definition using Derivations** [56,75]

If F: M→N is a smooth map between smooth manifolds M and N, the differential or pushforward of F at p is the linear map F_*: T_pM→T_{F(p)}N such that F_* (X)(f) = X(f © F) where X ∈ T_pM and F_* (X) ∈ T_{F(p)}N respectively are tangent vectors at m on M and F(p) on N and f is a smooth (real-valued) function on N.

Figure 4-7 from Lee [75] illustrates the intrinsic definition using derivations or, more specifically, a pushforward. Replacing X by [γ]_p for X and F_* (X) by [F © γ]_{F(p)}, the same figure also represents the definition using curves.
Tangent spaces and bundles include multiple tangent vectors at a given point on a smooth manifold. A vector field assigns a particular tangent vector at each point. A vector field is a continuous map $X$ that associates to each point $m$ in $M$ a tangent vector $X_m \in T_m(M)$ such that $X: m \mapsto X_m$ [56]. In any local coordinate system $(x_1, \ldots, x_n)$, a vector field can be expressed locally as

$$X_m(f) = \sum_{k=1}^{n} a_k(m)(\partial f / \partial x_k) \quad [56]$$

where $a_k$ are real-valued coefficient functions.

A vector field is smooth if the coefficient functions are smooth in each local coordinate system. Or, without referring to coordinates, a smooth vector field is a map $X: C^\infty(M) \rightarrow C^\infty(M)$ such that $X(fg) = fX(g) + X(f)g$ (i.e. the “product rule”).

The following points about vector fields are useful later in this thesis when discussing Lie algebras.

- Multiplying two vector fields $X$ and $Y$ does not generally yield a vector field but composing them via their commutator $[X,Y] = XY - YX$ does result in a vector field. The commutator is a Lie bracket which is discussed in the sequel.

- A separate vector field can be defined including each tangent vector in a given tangent space leading to multiple vector fields associated with a given smooth manifold. Assuming composition via a Lie bracket giving another vector field, these vector fields form a vector space [122].

- Pushing forward a vector field does not generally yield another vector field unless the smooth map (between the two smooth manifolds) is a diffeomorphism.

From Lewis [78], an integral curve of a vector field $V$ at $x \in M$ is a curve $\gamma$ at $x$ having the property that $\gamma'(t) = V(\gamma(t))$ for all times $t$ for which $\gamma(t)$ is defined, as illustrated in Figure 4-8. In other words, for every point $p$ on the curve ($p = \gamma(t)$ at time $t$), the vector field $V$ at $p$ ($V(p)$) gives the tangent vector to the curve at point $p$ [122]. Hence, an integral curve exists locally for whatever interval the vector field values match the tangent vectors, but not necessarily globally (if the vector field values and tangent vectors differ in other intervals).
If an integral curve at \( x \) is extended as far as possible covering an interval \( I(V, x) \), it is called the \textit{maximal integral curve} of \( V \) through \( x \). The \textit{flow} of \( V \) is the map \( (t, x) \rightarrow \Phi^V_t(x) \) where \( (t, x) \in \text{dom}(V) = \{(t, x) \in \mathbb{R} \times M \mid t \in I(V, x)\} \) and \( \Phi^V_t(x) \) is the point in \( M \) given by the maximal integral curve \( I(V, x) \) for \( V \) through \( x \). A vector field \( V \) is called \textit{complete} if \( \gamma(t) \) can be defined for all \( t \) for all initial points \( x \) (i.e. globally) [21].

![Figure 4-8: An Integral Curve of a Vector Field [75]](image)

Since the tangent space \( (T_mM) \) at point \( m \) on smooth manifold \( M \) is a vector space, it has a dual (vector) space called the \textit{co-tangent space} \( (T^*_mM) \). Given any real vector space \( V \), a \textit{dual space or dual} \( (V^*) \) is the set of all linear maps \( V^*: V \rightarrow \mathbb{R} \) or, in set notation, \( V^* = \{f: V \rightarrow \mathbb{R} \mid f(au + bv) = af(u) + bf(v) \text{ for } a, b \in \mathbb{R} \& u, v \in V\} \) [122]. The definitions of cotangent vectors \( (X^*) \), spaces \( (T^*_mM) \) and bundles \( (T^*M) \) are analogous to these concepts for tangent spaces.

While tangent bundles have vector fields (a tangent vector assigned to each point), cotangent bundles have \textit{differential 1-forms}, a linear map (at each point) from the cotangent space to \( \mathbb{R} \). These may be defined in Euclidean space or, more generally, on smooth manifolds as follows based on [29].

\begin{itemize}
  \item A \textit{smooth 1-form} \( \phi \) on \( \mathbb{R}^n \) is a real valued function \( \phi: \mathbb{R}^n \rightarrow \mathbb{R} \) such that i) \( \phi \) is linear on \( T_x \mathbb{R}^n \) for each \( x \in \mathbb{R}^n \) and ii) \( \phi(v): \mathbb{R}^n \rightarrow \mathbb{R} \) is smooth for any smooth vector field \( v = v(x) \).

  For a given point \( x \in \mathbb{R}^n \) and its tangent vector \( v = (v_1, \ldots, v_n) \), the map \( \phi_x: T_x \mathbb{R}^n \rightarrow \mathbb{R} \) has the form \( \phi_x(v) = (\mathbf{d}x^1)v = (\mathbf{d}x_1, \ldots, \mathbf{d}x_n) v \) such that \( \mathbf{d}x_i v = v_i \) where \( v_i \in \mathbb{R} \).

  A \textit{smooth 1-form} \( \phi \) on smooth manifold \( M \) is a real valued function \( \phi: T_x M \rightarrow \mathbb{R} \) such that i) \( \phi \) is linear on \( T_x M \) for each \( x \in M \) and ii) \( \phi(v): M \rightarrow \mathbb{R} \) is smooth for any smooth vector field \( v = v(x) \) on \( M \). If \( f: M \rightarrow \mathbb{R} \) is a differentiable function, for a given point \( x \in M \) and its tangent vector \( v = (v_1, \ldots, v_n) \), the map \( \phi_x: T_x M \rightarrow \mathbb{R} \) has the form \( \phi_x(v) = (df)_x(v) = v(f) \).
\end{itemize}
Differentials are used to define the exterior derivative \((df)\) of as follows [29].

If \(f: M \to R\) is smooth, then the exterior derivative of \(f\) is the 1-form \(df\) with the property that, for any \(x \in M\) and \(v \in T_xM\), \((df)_x(v) = v(f)\).

More particularly, if \(M = \mathbb{R}^n\), \(df_x(v) = \frac{\partial f}{\partial x_1}dx_1 v_1 + \ldots + \frac{\partial f}{\partial x_n}dx_n v_n\)

Analogous to pushforwards for vector fields but oppositely directed and more flexible, pullbacks exist for differentials and are defined by Lee [75] as follows.

Given smooth manifolds \(M\) and \(N\) with smooth map \(F: M \to N\) and any point \(p \in M\), there exists a linear map called a pullback \(F^*: T^*_pN \to T^*_pM\) which maps the 1-form at \(F(p)\) on \(N\) to a 1-form at \(p\) on \(M\).

Unlike pushforwards which require diffeomorphisms between manifolds to preserve vector fields, only smooth maps are needed for pushbacks to preserve differential forms and exterior derivatives. This allows coordinate invariant information expressed in differential forms to be moved readily between manifolds via the pullback. If the function is diffeomorphic as well as smooth, then pushforwards and pullbacks can be combined as with the Lie derivative discussed later.

4.2.2.3 Riemann Manifolds & Metrics

A Riemann manifold is a pair \((M, g)\) where \(M\) is a smooth manifold and \(g\) is a Riemann metric on \(M\) [75]. A Riemann metric is defined as follows by Marsden/Hughes [82].

A Riemann metric on \([\text{smooth manifold}]\ A\) is a \(C^\infty\) covariant two-tensor tensor \(G\) (i.e., \(G\) is a tensor of type \(\binom{0}{2}\) ) such that for each \(X \in A\):

(i) \(G(X)\) is symmetric; that is, for \(W_1, W_2 \in T_XA\), \(G(X)(W_1, W_2) = G(X)(W_2, W_1)\).

(ii) \(G(X)\) is positive definite: \(G(X)(W, W) > 0\) for \(0 \neq W \in T_XA\); in other words, \(G(X)\) is an inner product on \(T_XA\). If there is no danger of confusion, this often is written \(G(X)(W_1, W_2) = \langle W_1, W_2 \rangle_X\).

Given a Riemann metric \(G\) on \(A\), there is an inner product \(G_p\) on \(T_p(A)\) for each \(p \in A\) and these local inner products are smoothly compatible with each other similar to local charts. If Euclidean
space, the Riemann manifold is $\mathbb{R}^n$ with the Euclidean metric. This discussion is continued later under kinematics.

4.2.3 Lie Theory

This section discusses Lie groups, Lie algebras, associated adjoint operators, Lie group actions (transformations), and Lie derivatives. Discussion of the first four areas is based on Bullo/Lewis [21], Hall [56], Kecseméthy [65], and Lee [75] supplemented by generous personal explanations by Lewis [79] and Tsanov[122]. Lie derivatives are discussed based on Holzapfel [59] and Pandolfi [96,97] reinforced by explanations from Abrahamson et al. [1], Bullo/Lewis [21], Marsden/Hughes [82], Marsden/Ratiu [83], and Simo/Hughes [111].

4.2.3.1 Lie Groups

Hall [56] defines a Lie group as “... a smooth manifold $G$ together with a smooth map from $G \times G \rightarrow G$ that makes $G$ into a group and such that the inverse map $g \rightarrow g^{-1}$ is a smooth map of $G$ to itself.”.\(^{42}\) Requiring the multiplicative and inverse maps to be smooth ensures differentiability. This requirement also could be viewed as $G$ being diffeomorphic to itself. This differentiability (of the group operations) is key to considering infinitesimal changes and linearizing maps as discussed later.

Kecseméthy [65] defines a Lie group less precisely as “... a set of elements embodying simultaneously the properties of a group and a smooth manifold”\(^{43}\). Hence, a Lie group simply is a smooth manifold the points (or elements) of which also form a group under some group operation with that group operation (map) and its inverse being smooth. For example, shown in Figure 4-9, the circle group ($S^1$) is a Lie group for complex numbers of absolute value 1 ($z = a + bi, |z| = 1$) under multiplication or, alternatively, for angles $\theta$ modulo $2\pi$ under addition [122]. SO(3) also is a Lie group for $3 \times 3$ rotation matrices under matrix multiplication with identity element $e = I$. 

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Kecskeméthy [65] nicely explains how the group and manifold properties of a Lie group interact and the value of their doing so. He begins by summarizing the group properties including the identity (or neutral) element $e$ of a Lie group $G$ and, for any element $g$ of $G$, its inverse $g^{-1}$ is also in $G$. He then adds that, as a smooth manifold, any two points in a Lie group can be connected by a smooth trajectory and, at any given point $g$, a tangent vector or differential ($dg$) can be defined. He highlights that differentials at the identity element $e$ are special as they represent infinitesimal perturbations ($\delta g$) from $e$. For example, if points in $G$ represent positions, then tangent space $T_e$ represents perturbations of position $e$ such as angular velocities and twists. At other points ($g \neq e$), tangent spaces $T_g$ represent only first-order approximations of perturbations at $g$ based on the actual perturbations at $e$. These approximations become equalities in the case of the Lie algebra of a Lie group. Lie algebras are discussed shortly.

Lie groups also may be thought of as smoothly varying families of symmetries [65]. A symmetry (or invariance) is “an intrinsic property of a mathematical object which causes it to remain invariant under certain classes of transformations” [4-4]. This symmetry (invariance) comes from the properties of a group and the smoothness is proved by the smooth manifold. In the case of Lie group SO(3), rotations about a fixed axis are symmetries because they preserve distances between points.
4.2.3.2 Lie Group Actions (Transformations)

A group action can be either a left or right action of a group on a set. It is a mapping as defined below based on Lee [75].

If \( G \) is a group and \( X \) is a set, then

- A left action of \( G \) on \( X \) is a map \( L_g : G \times X \rightarrow X \), often written \( L_g(x) : (g, x) \rightarrow g \cdot x \), that satisfies i) \( g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \) for all \( g_1, g_2 \in G \), \( x \in X \) and ii) \( e \cdot x = x \) for \( e \) = identity element of \( G \) and all \( x \in X \).

- A right action of \( G \) on \( X \) is a map \( R_g : X \times G \rightarrow X \), often written \( R_g(x) : (x, g) \rightarrow x \cdot g \), that satisfies i) \( (x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2) \) for all \( g_1, g_2 \in G \), \( x \in X \) and ii) \( x \cdot e = x \) for \( e \) = identity element of \( G \) and all \( p \in X \).

A left (right) action can be expressed as a right (left) action by using the group’s inverse. As an example, \( R_g(x) = L_{g^{-1}}(x) \).

When \( G \) is a Lie group, such group actions are called Lie group actions or transformations. In this thesis, Lie groups act on themselves, other Lie groups, smooth manifolds and vector spaces. An example of a Lie group acting on itself is the composition of two group elements \( L_g : G \times G \rightarrow G \) such that \( L_g(h) = gh \) where \( g, h, gh \in G \). An example of a Lie group action on a smooth manifold is given below. The discussion in Section 4.2.3.6 of the adjoint representations \( Ad_g \) is an example of a Lie group acting on a vector space.

Kecseméthy [65] provides an applied example of a Lie group action. As shown in Figure 4-10, Lie group \( G \) acts on a reference configuration \( \mathcal{C}^0 \) to produce a new configuration \( \mathcal{C}^N \) with both configurations being smooth manifolds embedded in \( X \subseteq R^n \). This occurs by group elements \( g \) on trajectory \( C_g(t) \subseteq G \) acting on \( X \) causing points \( x \in X \) to move smoothly along trajectory \( C' \). The left action of \( G \) on \( X \) is given by \( \sigma(g, x) = L_g(x) = gx = x' \) for all \( g \) on \( C_g(t) \) and all \( x \) in \( \mathcal{C}^0 \). If \( g \) is fixed, this mapping is a space mapping \( \sigma_g : X \rightarrow X \) such that \( x \rightarrow \sigma_g(x) = gx = x' \) maps all points of \( X \) to their new positions in \( \mathcal{C}^N \) for a given \( g \). The associated tangent map is \( T_g X \rightarrow T_{gx} X \).

Fixing \( x \) gives a point mapping \( \sigma_x : G \rightarrow X \) such that \( g \rightarrow \sigma_x(g) = gx = x' \) maps trajectory \( C'_x \) of \( x \in X \) based on trajectory \( C_g(t) \) followed by \( g \in G \). The associated tangent map is \( T_{g} G \rightarrow T_{gx} X \).
4.2.3.3 Lie Algebras

As with manifolds, Lie algebras can be defined extrinsically (as an independent object) or intrinsically (relative to a Lie group or manifold) [122]. Varadarajan [125] defines a Lie algebra extrinsically as follows.

A vector space \( g \) over a [field] \( k \) is called a *Lie algebra* over \( k \) if there is a map \((X,Y) \mapsto [X,Y]\) of \( g \times g \) into \( g \) with the following properties:

(i) \((X,Y) \mapsto [X,Y]\) is bilinear

\[
\text{[ For } a,b \in \mathbb{R}, [aX+bY, Z] = a[X, Z]+b[Y, Z], [Z, aX+bY] = a[Z, X]+b[Z, Y] \]

(ii) \([X,Y] + [Y,X] = 0 \) [i.e. antisymmetric \([X, Y] = -[Y, X]\)]

(iii) \([X, [Y, Z]] + [Y, [Z,X]] + [Z, [X,Y]] = 0 \) [Jacobi identity]

where \([X, Y]\) is called the *Lie bracket* of \( X \) with \( Y \)

\( X, Y, Z, [X,Y], [Y,Z], [Z,X] \in g \)

For this thesis, it is more useful to define a Lie algebra relative to a Lie group (as done here) or, even more specifically, relative to a matrix Lie group (as done in Section 4.2.3.5).

Relative to a Lie group \( G \), a Lie algebra \( g \) can be defined using either left invariant vector fields or the tangent space \((T_eG)\) at the group identity \((e)\).
Lie Algebra Definition using Left Invariant Vector Fields

Using left invariant vector fields, Lee [75] defines a Lie algebra $\mathfrak{g}$ for a Lie Group $G$ as follows.

A Lie algebra is a real vector space $\mathfrak{g}$ of smooth left invariant vector fields $(X, Y, Z, \ldots)$ on a Lie group $G$ endowed with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket and usually denoted by $(X, Y) \rightarrow [X, Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:


(ii) Antisymmetry: $[X, Y] = -[Y, X]$

(iii) Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

A vector field $X$ on $G$ is left-invariant if $X$ satisfies $\left( L_g \right)_h(X_h) = X_{gh}$ for all $g, h \in G$ [75].

Figure 4-11 uses Lie group $S^1$ (for $z = a + bi, |z| = 1$ under multiplication) to illustrate a vector field which is left invariant (Figure 4-11(a)) and another which is not (Figure 4-11(b)). As discussed in Section 4.2.2.2, the Lie bracket ensures that compositions of these (left invariant) vector fields also are vector fields and, hence, form a vector space as required in the extrinsic definition of a Lie algebra.

As discussed earlier under Lie group actions, $\left( L_g \right)_h$ is the tangent map or pushforward associated with the left group action $L_g$ which is a diffeomorphism. Hence, left-invariance means that any tangent vector $X_h$ at $h \in G$, when pushed forward to $gh$, is equal to the tangent vector $X_{gh}$ at $gh$. $gh$ is the composition of $g$ and $h$ using the group operation and, thus, also an element of $G$. In other words, $X_{gh}$ can be replaced by $\left( L_g \right)_h(X_h)$ [122]. It should be highlighted that $X_{gh}$ is being replaced (since an equality) not only approximated.

If identity element $e$ of the group is used for $h$ ($h = e$), the above relationship simplifies as shown below.

$\left( L_g \right)_e(X_e) = X_{ge} \text{ from above}$

$\left( L_g \right)_e(X_e) = X_{ge} \text{ because } h = e$

$\left( L_g \right)_e(X_e) = X_g \text{ because } ge = g \text{ for elements of a group}$
This means that, for any left invariant vector field $\mathbf{V}$, tangent vector $X_g \in \mathbf{V}$ can be expressed by the left translation of $X_e \in \mathbf{V}$ which also an element of $T_eG$. Hence, the vector space of all left-invariant vector fields is isomorphic to $T_eG$, the tangent space at the identity element [56]. This gives rise to the equivalent $T_eG$ definition of a Lie algebra given on the next page.

**Figure 4-11:** Vector Fields on $S^1$, both
(a) Left Invariant and (b) Not Left Invariant [based on 122]
Hall [56] defines a Lie algebra, using the tangent space at the group identity \((T_eG)\), as given below.

The Lie algebra \(\mathfrak{g}\) of a Lie group \(G\) is the tangent space at the identity with the [Lie] bracket operation defined by \([v,w] = [X^v, X^w]_e\), where

\[
v, w \in T_eG
\]

\[
X^v = X^v_g = (L_g)_* (v) \text{ for all } g \in G \text{ which is the set (also a vector space)}
\]

of all left invariant vector fields containing \(v \in T_eG\)

\([\cdot,\cdot]_e = \text{Lie bracket} [\cdot,\cdot] \text{ evaluated at the identity element } e\)

Hence, a Lie algebra \(\mathfrak{g}\) for Lie group \(G\) can be defined simply as the tangent space \(T_eG\) at the identity element \(e\) of \(G\) endowed with the Lie bracket described above. However, use of \(\mathfrak{g}\) is not limited to \(T_eG\) because any tangent vector \(X_g \in T_gG\) at any element \(g\) can be expressed in terms of a tangent vector \(X_e \in T_eG\). Returning to Figure 4-5(b), the Lie algebra of Lie group \(S^1\) can be visualized as the various tangent vectors at the identity element \(e\), namely \(T_eM\).

### 4.2.3.4 Exponential & Logarithmic Maps

The exponential map links a Lie algebra and its Lie group, providing a canonical smooth map from the Lie algebra into the Lie group. Since every left invariant vector field on a Lie group \(G\) is complete, exponential maps can be defined using flows along these fields or, in other words, integral curves of the field starting at the identity \(e\) of \(G\) at \(t = 0\) [56,76]. Hall [56] defines the exponential map as follows and as illustrated in Figure 4-12 from Lee [75].

Let \(G\) be a Lie group and \(\mathfrak{g} = T_e(G)\) be its Lie algebra. For each \(v \in \mathfrak{g}\), let \(X^v\) be the associated left invariant vector field and \(\Phi^v_t\) be the associated flow. Then, the exponential mapping is the map \(\exp: \mathfrak{g} \rightarrow G\) defined by \(\exp(v) = \Phi^v_t(e)\).

Assuming a Lie group \(G\) and its Lie algebra \(\mathfrak{g}\), the exponential map has various properties [75] which prove useful for later discussions.

i) The exponential map is a smooth map from \(\mathfrak{g}\) to \(G\).
ii) \( \exp(tX) \) is a one-parameter subgroup of \( G \) generated by \( X \) for any \( X \in \mathfrak{g} \). The one parameter subgroups of Lie group \( G \) are the integral curves of left-invariant vector fields starting at the identity \( e \) of \( G \).

iii) \( \exp(s+t)X = (\exp sX)(\exp tX) \) for any \( X \in \mathfrak{g} \).

iv) The pushforward \( \exp_* : T_o \mathfrak{g} \rightarrow T_o G \) is the identity map where \( O \) and \( e \) are the identity elements of \( \mathfrak{g} \) and \( G \) respectively.

v) The exponential map is a local diffeomorphism from some neighbourhood of \( O \) in \( \mathfrak{g} \) to a neighbourhood of \( e \) in \( G \). This means that \( \mathfrak{g} \) and \( G \) are “essentially” the same in this neighbourhood.

![Figure 4-12: The Exponential Map ([75], modified)](image)

Since the exponential map is a local diffeomorphism, it has a smooth local inverse which provides a smooth map from the Lie group into the Lie algebra. This inverse map is defined as follows by Murray et al.[90].

Let \( U \) be a small neighbourhood of the identity element \( e \) of Lie group \( G \). The logarithmic map is the map \( \log : U \rightarrow \mathfrak{g} \) such that \( \exp \circ \log(g) = g \) for all \( g \in U \subset G \).

Specific exponential and logarithmic maps are discussed in the next section.

### 4.2.3.5 Matrix Lie Groups with Associated Lie Algebras and Exponential & Logarithmic Maps

Matrix Lie groups are a common and important type of Lie group. A (real) matrix Lie group is defined as a closed subgroup of the real general linear group \( \text{GL}(n, R) \) which, in turn, is an open subset of the vector space of real \( n \times n \) matrices \( M(n, R) \) [56]. Before discussing this definition, it is useful to first explore the character and relationship of \( M(n, R) \) and \( \text{GL}(n, R) \).
Being a finite dimensional vector space under matrix addition and scalar multiplication, \( M(n, \mathbb{R}) \) is a \( n^2 \)-dimensional smooth manifold with its topology generated by any of its norms and its smooth structure determined by any of its (ordered) bases [75]. The \textit{real general linear group} \( \text{GL}(n, \mathbb{R}) \) is the group of all real \( n \times n \) invertible matrices. It is a group under matrix multiplication since, for all \( A, B, C, I \in \text{GL}(n, \mathbb{R}), A(BC) = (AB)C \) (associativity), \( AI = IA = A \) (\( I \) being the identity element) and \( A^{-1}A = AA^{-1} = I \) (inverse elements). \( \text{GL}(n, \mathbb{R}) \) is an open subset of \( M(n, \mathbb{R}) \) since \( \text{GL}(n, \mathbb{R}) \) excludes matrices with zero determinants which are not invertible [75]. Being an open subset of \( M(n, \mathbb{R}) \), \( \text{GL}(n, \mathbb{R}) \) also is a \( n^2 \)-dimensional smooth manifold which derives its topology and smooth structure from \( M(n, \mathbb{R}) \) [75]. Hence, since both a group and a smooth manifold, \( \text{GL}(n, \mathbb{R}) \) is a Lie group.

Returning to the original definition, a (real) \textit{matrix Lie group} is defined as any closed subgroup of \( \text{GL}(n, \mathbb{R}) \). A \textit{subgroup} of any group \( G \) is a subset \( H \) of \( G \) with the following properties: i) The identity of \( G \) is an element of \( H \); ii) If \( h \in H \), then \( h^{-1} \in H \); and iii) If \( h_1, h_2 \in H \), then \( h_1h_2 \in H \). A subgroup \( P \) of \( \text{GL}(n, \mathbb{R}) \) being \textit{closed} is defined as follows.

Let \( A_m \) be a sequence of real matrices in \( M_n(\mathbb{R}) \), the space of all real \( n \times n \) matrices. \( A_m \) \textit{converges} to a matrix \( A \) if each entry of \( A_m \) converges (as \( m \to \infty \)) to the corresponding entry of \( A \).

If \( A_m \) is any sequence of matrices in any \textit{closed} subgroup \( G \) of \( \text{GL}(n, \mathbb{R}) \) and \( A_m \) converges to some matrix \( A \), then either \( A \in G \) or \( A \) is not invertible.

Any real matrix Lie group is a group (since a subgroup of \( \text{GL}(n, \mathbb{R}) \)) and also a \( n^2 \)-dimensional smooth manifold (since an open subset of \( M(n, \mathbb{R}) \)) and, thus, respects the general definition of a Lie group.

As Lie groups, matrix Lie groups have matrix Lie algebras and, to connect them, exponential and logarithmic mappings. These mappings are the matrix exponential and matrix logarithm defined in the usual ways, specifically using Hall [56]:

\[
e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}, \text{ where } X = \text{any } n \times n \text{ real (or complex) matrix}
\]

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\[
\log(A) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (A-I)^m
\]
whenver series converges, where

\[A = \text{any } n \times n \text{ real (or complex) matrix} \]

For all \(A\) with \(\|A - I\| < 1\), \(e^{\log A} = A\)

For all \(X\) with \(\|X - I\| < \log 2\) and \(\log e^X = X\). Using the matrix exponential, a matrix Lie algebra can be defined as follows.

The Lie algebra \(g\) of a matrix Lie group \(G\) is the set of all matrices \(X\) such that \(A = e^{tX}\) is in \(G\) for all real numbers \(t\) [56].

Requiring \(e^X\) versus simply \(e^X\) is important as it gives a smooth curve in \(G\) as \(t\) is varied rather than a single point in \(G\). (i.e. \(e^X = e^{tX}\) for \(t = 1\)) [122].

These definitions can be shown to be equivalent to the more general definitions of Lie algebras (using the Lie bracket) and exponential maps (using flows). If \(e^{tX}\) and \(e^{sX}\) are elements of group \(G\), then the composition \(e^{tX} e^{sX} = e^{(t+s)X}\) also must be in \(G\) or, in other words, \(e^{tX}\) and \(e^{sX}\) are one-parameter subgroups of \(G\) (integral curves of left invariant vector field \(X\) starting at the identity of \(G\)) representing flows as in the more general exponential map. If \(X\) and \(Y\) are elements of Lie algebra \(g\), closure under the Lie bracket \([X, Y]\) (required by the more general Lie algebra definition) ensures that their composition also is in \(g\).

Some common matrix Lie groups of interest for this thesis are defined below with their Lie algebras.

**Real General Linear Group GL\((n, R)\) and its Lie algebra gl\((n, R)\):**

\(GL(n, R) = \{M \in \mathbb{R}^{n \times n} | \det M \neq 0\}\) including identity transformations, rotations, reflections, and volumetric (stretch) as well as deviatoric (shear) deformations.

\(gl(n, R) = \{g \in \mathbb{R}^{n \times n} | [g_1, g_2] = g_1 g_2 - g_2 g_1 \in \mathbb{R}^{n \times n}\}\)

**Proper Real Linear Group GL\(^{\ast}\)(\(n, R\)) (determinant > 0) and its Lie algebra gl\((n, R)\):**

\(GL^\ast(n, R) = \{M \in \mathbb{R}^{n \times n} | \det M > 0\}\) including \(GL(n, R)\) transformations except reflections.

\(gl(n, R) = \{g \in \mathbb{R}^{n \times n} | [g_1, g_2] = g_1 g_2 - g_2 g_1 \in \mathbb{R}^{n \times n}\}\)
Affine (Sub-)Group $\text{Aff}^+(n, \mathbb{R})$ and its Lie algebra $\mathfrak{aff}(n, \mathbb{R})$:

$\text{Aff}^+(n, \mathbb{R}) = \left\{ G \in \mathbb{R}^{(n+1)\times(n+1)} \mid G \in \begin{bmatrix} M & r \\ 0_{1\times n} & 1 \end{bmatrix}, \det G > 0, M \in \text{GL}^+(n, \mathbb{R}), r \in \mathbb{R}^n \right\}$

combining $M \in \text{GL}^+(n, \mathbb{R})$ with a rigid translation $r$.

$\mathfrak{aff}(n, \mathbb{R}) = \left\{ \begin{bmatrix} g & v \\ 0_{1\times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \mid g \in \mathfrak{gl}(n, \mathbb{R}), v \in \mathbb{R}^n \right\}$

Special Orthogonal Group $\text{SO}(n)$ and its Lie algebra $\mathfrak{so}(n)$:

$\text{SO}(n, \mathbb{R}) = \{ R \in \mathbb{R}^{n\times n} \mid RR^T = I_n, \det R = 1 \}$ representing rotations

$\mathfrak{so}(n, \mathbb{R}) = \{ S \in \mathbb{R}^{n\times n} \mid S^T = -S \}$

Special Euclidean Group $\text{SE}(n)$ and its Lie algebra $\mathfrak{se}(n)$:

$\text{SE}(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{(n+1)\times(n+1)} \mid g \in \begin{bmatrix} R & r \\ 0_{1\times n} & 1 \end{bmatrix}, \det g = 1, R \in \text{SO}(n), r \in \mathbb{R}^n \right\}$

combining rotation and translation $R$.

$\mathfrak{se}(n, \mathbb{R}) = \left\{ \begin{bmatrix} S & v \\ 0_{1\times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \mid S \in \mathfrak{so}(n), v \in \mathbb{R}^n \right\}$

Given that a Lie algebra $\mathfrak{g}$ is associated with the tangent space at the identity $e$ of Lie group $G$ or, in other words, infinitesimal transformations of $G$ at $e$, it is natural that derivatives evaluated at $e$ (i.e. $t = 0$) can be used to find matrix Lie algebras from their groups. Using $\text{SO}(n)$ as an example,

$R(t) R^T(t) = I_n$, by definition of $\text{SO}(n)$, where $R \in \text{SO}(n)$

$R(t) R^T(t) + R(t) \dot{R}^T(t) = 0_n$, by taking the derivative

$\dot{R}(t) R^T(t) = -R(t) \dot{R}^T(t)$ by reorganizing

$\dot{R}(0) R^T(0) = -R(0) \dot{R}^T(0)$ by evaluating at $t = 0$

$\dot{R}(0) I_n = -I_n \dot{R}^T(0)$ using $R^T(0) = R(0) = I_n$ at $t = 0$

$\dot{R}(0) = -\dot{R}^T(0)$ simplifying

$S = -S^T$ using $S = \dot{R}(0)$, where $S \in \mathfrak{so}(n)$

This calculation of matrix Lie algebra $\mathfrak{so}(n)$ is a more elegant version of the explanation of the spinor $\mathfrak{N} \in \mathfrak{so}(3)$ using the derivative of rotator $R \in \text{SO}(3)$ found in Section 2.2.3. Similarly, the definition of $\mathfrak{se}(n)$ uses $S \in \mathfrak{so}(n)$ and $v = \dot{r}(0)$.
Given that $\mathbf{S} \in \text{so}(3)$ is skew symmetric, it is uniquely defined by three parameters. As explained by Bullo/Lewis [21], $\text{so}(3)$ is isomorphic to $\mathbb{R}^3$ based on the linear map $\cdot : \mathbb{R}^3 \to \text{so}(3)$ such that $\hat{\omega} \times \hat{y} = \hat{\omega} \times y$ for all $\omega, y \in \mathbb{R}^3$. The inverse isomorphism is $\cdot^\vee : \text{so}(3) \to \mathbb{R}^3$. These maps are explained below from Bullo/Lewis [21] and are equivalent to $\text{Spin}(\omega)$ and its inverse $\text{axial}(\Omega)$ found in Chapter 2.

For $\omega = (\omega_1, \omega_2, \omega_3)$:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}^\vee = (\omega_1, \omega_2, \omega_3)^T = \omega$$

Likewise, $\hat{\xi} \in \text{se}(3)$ can be represented uniquely by six parameters: three each from $\omega \in \text{so}(3)$ and $v \in \mathbb{R}^3$. Again from Bullo/Lewis [21], $\text{se}(3)$ is isomorphic to $\mathbb{R}^3 \oplus \mathbb{R}^3$ based on the linear map $\cdot : \mathbb{R}^3 \oplus \mathbb{R}^3 \to \text{se}(3)$ such that, for $\xi = (\omega, v) \in \mathbb{R}^3 \oplus \mathbb{R}^3$, $\hat{\xi}$ is as shown below.

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0_{1 \times n} & 0 \end{bmatrix}$$

The inverse isomorphism is $\cdot^\vee : \text{se}(3) \to \mathbb{R}^3 \oplus \mathbb{R}^3$ such that

$$\begin{bmatrix} \hat{\omega} & v \\ 0_{1 \times n} & 0 \end{bmatrix}^\vee = \xi = (\omega, v) \in \mathbb{R}^3 \oplus \mathbb{R}^3$$

Using this notation, the matrix exponentials of and matrix logarithms are expressed as follows by Bullo/Lewis [21].

Matrix Exponential of $\hat{\omega} \in \text{so}(3)$ such that $\exp(\hat{\omega}) \in \text{SO}(3)$ (Rodrigues' Formula):

$$\exp \hat{\omega} = \begin{cases} I_3, & \omega = 0 \\ I_3 + \frac{\sin \|\omega\|_{\mathbb{R}^3}}{\|\omega\|_{\mathbb{R}^3}} \hat{\omega} + \frac{1 - \cos \|\omega\|_{\mathbb{R}^3}}{\|\omega\|_{\mathbb{R}^3}^2} \omega^2, & \omega \neq 0 \end{cases}$$

Matrix Logarithm of $R \in \text{SO}(3)$:

$$\log R = \begin{cases} 0_{3 \times 3}, & R = I_{3 \times 3} \\ \frac{\phi(R)}{2 \sin(\phi(R))} (R - R^T), & R \neq I_{3 \times 3} \end{cases}$$

where, for $U_{\text{SO}(3)} = \{ R \in \text{SO}(3) | \text{tr}(R) \neq -1 \}$,

$$\phi : U_{\text{SO}(3)} \to [0, \pi] \text{ is defined as } \phi(R) = \arccos\left(\frac{1}{2}(\text{tr}(R) - 1)\right)$$
Matrix Exponential of $\hat{\xi} \in \mathfrak{se}(3)$ such that $\exp(\hat{\xi}) \in \text{SE}(3)$:

$$
\exp(\hat{\xi}) = \begin{bmatrix}
\exp \hat{\omega} & B(\omega) \nu \\
0_{1 \times 3} & 1
\end{bmatrix},
$$

where

$$
B(\omega) = \begin{cases}
I_3, & \omega = 0 \\
I_3 + \frac{(1 - \cos\|\omega\|_R^3)}{\|\omega\|_R^3} \hat{\omega} + \left(1 - \frac{\sin\|\omega\|_R^3}{\|\omega\|_R^3}\right) \frac{\hat{\omega}^2}{\|\omega\|_R^3}, & \omega \neq 0
\end{cases}
$$

Matrix Logarithm of $(\mathbf{R}, \mathbf{r}) \in \text{SE}(3)$ such that $\log(\mathbf{R}, \mathbf{r}) \in \mathfrak{se}(3)$:

$$
\log(\mathbf{R}, \mathbf{r}) = \begin{bmatrix}
\log(\mathbf{R}) & B^{-1}(\log(\mathbf{R})^\nu) \mathbf{r} \\
0_{1 \times 3} & 0
\end{bmatrix},
$$

where

$$
B^{-1}(\psi) = \begin{cases}
I_3, & \psi = 0 \\
I_3 - \frac{1}{2} \psi + \left(1 - \frac{\|\psi\|_R^3}{2} \cot \left(\frac{\|\psi\|_R^3}{2}\right)\right) \frac{\psi^2}{\|\psi\|_R^3}, & \psi \neq 0
\end{cases}
$$

$\psi \in \mathbb{R}^3$ with $\|\psi\|_R^3 < \pi$

SE(3) is discussed further under rigid body motion.

4.2.3.6 Adjoint Representations

Representations are based on Lie group actions which are discussed in Section 4.2.3.2.

Lee [75] defines a representation, its relationship to Lie group actions, and the related concept of a conjugation map as follows:

If $G$ is a Lie group, a \textit{(finite-dimensional) representation} of $G$ is a Lie group homomorphism $\rho: G \rightarrow \text{GL}(V)$ for some finite-dimensional real or complex vector space $V$.

Any representation $\rho$ yields a smooth left action of $G$ on $V$, defined by $g \cdot \mathbf{v} = \rho(g) \mathbf{v}$ for $g \in G$, $\mathbf{v} \in V$. This action is said to be \textit{linear} if, for each $g \in G$, the map $\mathbf{v} \mapsto g \cdot \mathbf{v}$ is linear for all $\mathbf{v} \in V$.

For any $g \in G$, the \textit{conjugation map} $C_g: G \rightarrow G$ given by $C_g(h) = ghg^{-1}$ for $h \in G$ is a Lie group homomorphism.

As shown below, the conjugate map $C_g(h) = ghg^{-1}$ is equivalent to $C_g(h) = L_g R_g^{-1}(h)$ where $L_g$ and
The conjugation map \( C_g : G \to G \) alters every \( h \in G \) except for the identity element \( e \) since \( C_g(e) = geg^{-1} = e \). This means that \( C_g : G \to G \) maps \( G \) onto a subgroup of \( G \) centred at \( e \) [122].

The tangent map or pushforward \( (C_g)_* \) of \( C_g \) is a map \( (C_g)_* : g \to g \), \( g \) being the Lie algebra of \( G \), such that \( (C_g)_*(X) = (L_g R_g^{-1})_*(X) \) where \( X \in T_e G \) [75]. Hence, \( (C_g)_* \) maps elements from the tangent space \( T_e G \) at any point \( h \) in \( G \) to the tangent space \( T_e G \) at the identity element of \( G \). In other words, \( (C_g)_* \) represents every point \( h \) in \( G \) as an element of its Lie algebra \( g \) (i.e. \( T_e G \)) providing a linearized representation of \( G \). This representation using \( (C_g)_* \) is called the adjoint representation of \( G \) which is defined below based on Bullo/Lewis [21].

If \( G \) is a Lie group with Lie algebra \( g \) and \( g \in G \), the adjoint representation of \( G \) is \( \text{Ad}_g : g \to g \) such that \( \text{Ad}_g(h) = (L_g R_g^{-1})_*(h) \) for all \( h \in G \).

If \( G \) is a matrix Lie group with Lie algebra \( g \), \( \text{Ad}_A(X) = AXA^{-1} \) where \( A \in G, X \in g \) [56]. It should be noted that, \( A \) in \( \text{Ad}_A(X) = AXA^{-1} \), is an element of the matrix Lie group \( G \) not its Lie algebra \( g \). This may seem surprising given that the adjoint representation uses the push-forward \( (C_g)_* \) not \( C_g \). However, for example, a rotation \( R \in \text{SO}(3,\mathbb{R}) \) acting on another Lie group \( G \) causes its tangent vectors and, hence, elements \( X \) of its Lie algebra \( g \) to undergo the same rotation \( R \). Thus, tangent vector \( X \in g \) becomes \( RX \) where \( R \in \text{SO}(3,\mathbb{R}) \), not \( \hat{R}X \) with \( \hat{R} \in \text{so}(3) \) [122].

Similarly, it is possible to represent a Lie algebra \( g \) by a subgroup of \( g \). This representation is designated \( \text{ad} \) and defined as follows based on Bullo/Lewis [21].

If \( g \) is a Lie algebra and \( \xi \in g \), the adjoint representation of \( g \) is the map \( \text{ad}_\xi : g \to g \) such that \( \text{ad}_\xi(\eta) = [\xi, \eta] \) for all \( \eta \in g \).
For matrix Lie group $G$ and its Lie algebra $\mathfrak{g}$, the adjoint representation of $\mathfrak{g}$ can be conceptualized as representing every vector (field) $Y \in \mathfrak{g}$ as another vector (field) $[X, Y]$ using a given $X \in \mathfrak{g}$. In so doing, $Y = X$ gives $[X, Y] = [X, X] = 0$ which corresponds to the identity element of matrix Lie group $G$ ($e^0 = I$). Hence, $\text{ad}_X$ represents $\mathfrak{g}$ as a Lie subalgebra centered at $0 \in T_e G$.

### 4.2.3.6.3 Relationship between $\text{Ad}$ and $\text{ad}$

For any given representation (including adjoint representations), the representations of a Lie group $G$ and of its Lie algebra $\mathfrak{g}$ are related as described below by Lee [75].

Suppose $G$ is a Lie group and $\mathfrak{g}$ is its Lie algebra. If $\rho: G \to \text{GL}(V)$ is any representation of $G$, then the pushforward at the identity $\rho_*: \mathfrak{g} \to \text{gl}(V)$ is . . . a representation of $\mathfrak{g}$.

This general relationship can be applied to the adjoint representations $\text{Ad}$ and $\text{ad}$ in the matrix Lie group context as shown below based on Hall [56].

Let $G$ be a matrix Lie group $G$ and $\mathfrak{g}$ be its Lie algebra. For $A \in G$ and $X, Y \in \mathfrak{g}$, the adjoint representations of $G$, $\text{Ad}_A(X) = AXA^{-1}$, and of $\mathfrak{g}$, $\text{ad}_X(Y) = [X, Y]$, are related as follows:

$$\text{ad}_X = \frac{d}{dt} \text{Ad}_{e^{tX}} \Big|_{t=0} \text{ for } A = e^{tX}, \text{ such that } \text{ad}_X(Y) = \frac{d}{dt} \text{Ad}_{e^{tX}}(Y) \Big|_{t=0}$$

$$e^{\text{ad}_X} = \text{Ad}_{e^X} \text{ for } A = e^X, \text{ such that } e^{\text{ad}_X}(Y) = e^{[X,Y]} = \text{Ad}_{e^X}(Y) = e^XYe^{-X}$$

These interrelationships are not surprising given that $\text{Ad}$ and $\text{ad}$ both are centered at $0 \in T_e G$ which corresponds to the identity element of $G$.

### 4.2.3.7 Lie Derivatives

Lie derivatives can be defined various ways (using directional derivatives, pushforwards and pullbacks, Lie brackets, etc.) for a variety of objects (scalar, vector or tensor fields). Lee [75] informally defines a Lie derivative as a “directional derivative” of a vector (or tensor) field with respect to another vector field. If applied to scalar fields, the Lie derivative reduces to the traditional directional derivative [59].

The challenge in defining the Lie derivative formally for smooth manifolds is that the tangent spaces change as one flows along the reference vector field $V$ as illustrated in Figure 4-
13 for the Lie derivative of vector field $\mathbf{W}$ with respect to vector field $\mathbf{V}$ [75]. This is addressed by pushing back the values of $\mathbf{W}$ to the initial tangent space using the flow of $\mathbf{V}$ and then differentiating. Using this approach, Lee [75] defines the Lie derivative as follows.

$$(\mathcal{L}_V \mathbf{W})_p = \frac{d}{dt} \bigg|_{t=0} (\theta_{-t})_* \mathbf{W}_{\theta_t(0)} = \lim_{t \to 0} \frac{(\theta_{-t})_* \mathbf{W}_{\theta_t(0)} - \mathbf{W}_p}{t},$$

where $(\theta_{-t})_*$ is the pullback of $\mathbf{W}$ from $\theta_t(p)$ to $p$ using the flow of $\mathbf{V}$, as $(\theta_{-t})_* = (\theta_t)^*$.

Figure 4-13: Lie Derivative of Vector Field $\mathbf{W}$ with respect to Vector Field $\mathbf{V}$ [75, notation modified]

Lee [75] further shows that the Lie derivative of a tensor field $\tau$ with respect to a vector field $\mathbf{V}$ is defined similarly as shown below and in Figure 4-14.

$$(\mathcal{L}_V \tau)_p = \frac{d}{dt} \bigg|_{t=0} (\theta_{-t})_* \tau_{\theta_t(0)}$$

where $\tau$ is a smooth covariant tensor field on a smooth manifold $M$.

It is important to note that the above definitions give the Lie derivative of a vector/tensor field at point $p$ (i.e. at $-t$ or $t = 0$). If desired, the resulting vector/tensor can be returned to point $\theta_t(p)$ by pushing it forward using the flow of $\mathbf{V}$. Holzapfel [59] defines the Lie derivative in this way. In this case, the earlier definitions become the following.

$$(\mathcal{L}_V \mathbf{W})_{\theta_t(p)} = (\mathcal{L}_V \mathbf{W})_p = (\mathcal{L}_V \tau)_{\theta_t(p)}$$

where $(\mathcal{L}_V \bullet)_{\theta_t(p)}$ is pushed forward from $p$ to $\theta_t(p)$ using the flow of $\mathbf{V}$. 

$$(\theta_{t})_* (\bullet) = \text{push forward of (\bullet) using flow of } \mathbf{V}$$
4.3 Kinematics & Lie Theory

Kinematics is a branch of classical mechanics that describes motion of objects without consideration of the causes leading to this motion [128]. As such, it deals purely with the geometry of the motion (and the mathematics required to represent it) without regard to forces, mass, momentum, or energy. This geometric approach fits well with manifolds and Lie theory.

This section discusses the kinematics of first rigid body and then elastic deformation. Rigid body kinematics is well developed using not only linear algebra but also manifolds and Lie theory (Bullo/Lewis [21], Murray et al. [90] and others). The kinematics of elastic deformation is well understood as well, mostly through linear algebra (Gurtin et al. [55], Holzapfel [59], Bower [19] and others) but also more recently using Lie theory and differential geometry (most notably, Marsden/Hughes [82]). For the less initiated, useful resources include an elasticity paper by Yavari et al. [133] summarizing geometric mechanics, a doctoral thesis by Segev [108] on a simple geometrical formulation of statics, and a paper by Segev/Pressburger [109] on the same topic as well as Lie theory asides in more traditional works by Holzapfel [59] and Pandolfi [96]. For both rigid body motion and elastic deformation, this section attempts to focus as quickly as possible on a geometric discussion of kinematics. Unless stated otherwise, this discussion uses matrix Lie groups which are defined and discussed in Section 4.2.3.5.
In discussing kinematics using manifolds, it is important to distinguish between the physical spaces where motions occur and the sets of motions occurring within those spaces [79]. Physical spaces, whether subsets (e.g. a body) or all of $\mathbb{R}^3$, are Riemann manifolds with the Euclidean metric. Sets of motions (e.g. rotations and/or translations) are smooth manifolds but not necessarily Riemann manifolds in a natural way. For example, a rigid body lives in $\mathbb{R}^3$ but rigid rotations of that body are elements of $\text{SO}(3)$.

The set of rigid body motions is finite dimensional since such motion has only six degrees of freedom, namely translation and rotation of the body about the three standard orthogonal axes. This means that rigid body motions can be represented by the matrix Lie group $\text{SO}(3)$ if only rotation or $\text{SE}(3)$ if translation as well, both of which are finite-dimensional. The set of elastic deformations generally is infinite dimensional since every particle of an elastic body can deform relative to all other particles. However, elastic deformation is finite dimensional when the deformation is homogeneous (uniform throughout the body) or, even if not homogeneous, when the body is discretized as in finite element analysis since only a finite number of points (nodes) are tracked. If finite dimensional, elastic deformation can be represented by Lie group $\text{GL}^+(3, \mathbb{R})$ for homogeneous deformation or cross product $\text{GL}^+(3, \mathbb{R}) \times \ldots \times \text{GL}^+(3, \mathbb{R})$ combining nodal deformations. This cross product of Lie groups also is a Lie group under matrix multiplication [75].

Some basic concepts needed for both rigid and elastic deformational kinematics are defined below using various sources.

A **point** ($x$ or $X$) is a position in a space, having no mass or volume [59].

A **particle** ($P$) is an object having mass and position but no volume [21].

A **body** ($\mathcal{B}$) is a set whose elements (particles) are in one-to-one correspondence with points of a region $\mathcal{B} \subset \mathbb{S} \subset \mathbb{R}^3$ where $\mathbb{S}$ is the configuration space defined below. $\mathcal{B}$ is a Riemann manifold with a Euclidean metric $\mathbb{G}$ with, since $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B}$ being a submanifold of $\mathcal{A}$. As defined here, a body can be rigid or deformable [128,133].

A **configuration** or **deformation** of body $\mathcal{B}$ is an injective mapping $\phi : \mathcal{A} \rightarrow \mathbb{S}$ where herein $\phi$ is assumed to be a diffeomorphism onto its image. Since $\mathcal{A}$ and $\mathbb{S}$ are Riemann
manifolds, an equivalent definition of a configuration or deformation is a $C^1$ embedding $\phi: B \rightarrow \mathcal{S}$ where $\phi$ is a diffeomorphism [133]. The injective map precludes tearing or interpenetration of the body. Under either definition, deformation includes both rigid and elastic deformation [20].

The initial (undeformed) configuration is called the reference or material configuration $\mathcal{S}$ whereas the deformed configuration is called the current or spatial configuration $\phi(B)$ as shown in Figure 4-15 from Marsden/Hughes [82].

The configuration space $\mathcal{S} \subset \mathbb{R}^3$ is the space of all possible space configurations given the body’s constraints [59]. $\mathcal{S}$ is a Riemann manifold with Euclidean metric $g$.

![Figure 4-15: Configuration or Deformation of a Body [82]](image)

The next concepts involve motion or, in other words, moving from a material configuration $\mathcal{S}$ to a spatial configuration $\phi(B)$ over time. In so doing, there is a choice as to whether $\phi(B)$ is compared to $\mathcal{S}$ (Lagrangian description) or vice-versa (Eulerian description).

A motion of body $\mathcal{S}$ is a curve in configuration space $\mathcal{S}$; that is, a mapping $t \mapsto \phi_t$ of $\mathbb{R}$ to $\mathcal{S}$ (or of some open interval of $\mathbb{R}$ to $\mathcal{S}$) where $\phi_t = \phi(X, t) \in \mathcal{S}$ for $t \in \mathbb{R}$ [82]. Equivalently, Gurtin et al. [55] defines a motion of $\mathcal{S}$ as $\phi_t: B \times \mathbb{R} \rightarrow \mathcal{S}$. $\phi_t$ maps all particles in the body from their material position $X$ to their new spatial position $x$ at a given time $t$.

A trajectory of a particle is a mapping $p: \mathcal{S} \rightarrow \mathcal{S}$ such that $X \mapsto \phi_X$ where $\phi_X = \phi(X, t) \in \mathcal{S}$ for some $X \in \mathcal{S}$ [55]. $\phi_X$ maps a given particle $X$ over time $t$.

The free configuration manifold $Q_{\text{free}}$ of the motions of a body (not its physical space configurations) is the set of all possible motions assuming no constraints. The configuration manifold $Q$ is the set of all possible motions permitted by any constraints and is a submanifold of $Q_{\text{free}}$. Both $Q_{\text{free}}$ and $Q$ are assumed to be smooth manifolds.
herein [21]. As an example, for the rigid motion of a body, \( Q_{\text{free}} \) is the Lie group SE(3) with \( Q \) being some subset of SE(3).

The \textit{Lagrangian (material or reference) description} represents a motion (or any other quantity) with respect to the reference configuration \( \mathcal{B} \). If coordinates are introduced, \textit{material (or reference) coordinates} \((X_1, X_2, X_3)\) represent the position of a particle in the reference configuration [59].

The \textit{Eulerian (current or spatial) description} represents a motion (or any other quantity) with respect to the current configuration \( \phi(\mathcal{B}) \). If coordinates are introduced, \textit{current (or spatial) coordinates} \((x_1, x_2, x_3)\) represent the position of a particle in the current configuration [59].

A motion of a body generally alters its position, orientation and/or shape/size. A rigid body motion changes position and/or orientation but not shape or size. An elastic deformation affects the shape and/or size of the body.

Several vector fields are associated with a deformation. These include the displacement field for a deformation, the velocity field for a motion (the \textit{temporal} derivative of the displacement field), and their gradients (their \textit{spatial} derivatives). A \textit{displacement field} relates positions in the material and spatial configurations at each time \( t \) and specifies the deformation of a solid [20]. The \textit{velocity field}, describes the motion over time of each particle and, hence, the solid. Displacement and velocity fields are illustrated in Figure 4-16 and defined in Table 4-1, based on Marsden/Hughes [82] and Holzapfel [59].

![Figure 4-16: Motion and Velocity Field [82]](image)
Table 4-1: Material & Spatial Displacements & Velocities

<table>
<thead>
<tr>
<th>Material (Lagrangian) Representation</th>
<th>Spatial (Eulerian) Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Material Displacement Field</strong> $\mathbf{U}(\mathbf{X}, t)$</td>
<td><strong>Spatial Displacement Field</strong> $\mathbf{u}(\mathbf{x}, t)$</td>
</tr>
<tr>
<td>$\mathbf{U}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} = \phi(\mathbf{X}, t) - \mathbf{X}$</td>
<td>$\mathbf{u}: \phi(\mathbb{R}^3) \rightarrow \mathbb{R}^3$ such that $\mathbf{u}(\mathbf{x}, t) = \mathbf{X}(\mathbf{x}, t) - \mathbf{x} = \phi^{-1}(\mathbf{x}, t) - \mathbf{x}$</td>
</tr>
<tr>
<td>or, if $t$ is fixed: $\mathbf{U}_t(\mathbf{X}) = \phi_t(\mathbf{X}) - \mathbf{X}$</td>
<td>or, if $t$ is fixed: $\mathbf{u}_t(\mathbf{x}) = \phi^{-1}_t(\mathbf{x}) - \mathbf{x}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Material Velocity Field</strong> $\mathbf{V}(\mathbf{X}, t)$</th>
<th><strong>Spatial Velocity Field</strong> $\mathbf{v}(\mathbf{x}, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{V}(\mathbf{X}, t) = \partial\mathbf{U}(\mathbf{X}, t) \partial t = \partial\phi(\mathbf{X}, t) \partial t$</td>
<td>$\mathbf{v}(\mathbf{x}, t) = \partial\mathbf{u}(\mathbf{x}, t) \partial t = \partial\phi^{-1}(\mathbf{x}, t) \partial t$</td>
</tr>
<tr>
<td>or, if $\mathbf{X}$ is fixed: $\mathbf{V}_t(\mathbf{X}) = \frac{D\phi_X(t)}{Dt}$</td>
<td>or, if $\mathbf{x}$ is fixed: $\mathbf{v}_t(\mathbf{x}) = \frac{d\phi^{-1}_x(t)}{dt}$</td>
</tr>
</tbody>
</table>

Substituting $\mathbf{x} = \phi(\mathbf{X}, t)$, it can be shown that $\mathbf{U}(\mathbf{X}, t) = \mathbf{u}(\mathbf{x}, t)$ and $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$ but not so with their components (e.g. $U_a \neq u_a$). In other words, the material and spatial vectors are equal but differ in their local representations. The velocities in Table 4-1 are taken within the material (or spatial) configuration and, thus, are referred to as the material (or spatial) time derivatives of a smooth material (or spatial) field. It also is possible to take the material time derivative of a spatial field $\mathbf{f}(\mathbf{X}, t)$ using the Lie derivative discussed earlier.

The derivative of a field with respect to position yields the gradient of that field. Gradients of a material field $\mathbf{H}(\mathbf{X}, t)$ or a spatial field $\mathbf{h}(\mathbf{x}, t)$ can be taken in either configuration or between configurations as defined below by Holzapfel [59].

**Material Gradient of a Material Field** maps $T_{\mathbf{X}^A} \rightarrow T_{\mathbf{X}^A}$ such that

$$\text{Grad}\mathbf{H}(\mathbf{X}, t) = \frac{\partial\mathbf{H}(\mathbf{X}, t)}{\partial \mathbf{X}}$$

**Spatial Gradient of a Spatial Field** maps $T_{\phi(\mathbf{X})} \mathbb{S} \rightarrow T_{\phi(\mathbf{X})} \mathbb{S}$ such that

$$\text{grad}\mathbf{h}(\mathbf{x}, t) = \frac{\partial\mathbf{h}(\mathbf{x}, t)}{\partial \mathbf{x}}$$

**Material Gradient of a Spatial Field** maps $T_{\mathbf{X}^A} \rightarrow T_{\phi(\mathbf{X})} \mathbb{S}$ such that

$$\text{Grad}\mathbf{h}(\mathbf{x}, t) = \frac{\partial\mathbf{h}(\mathbf{x}, t)}{\partial \mathbf{X}}$$
Spatial Gradient of a Material Field \( T_{\phi(X)} \mathcal{S} \rightarrow T_{X}(\mathcal{B}) \) such that

\[
\text{grad} H(X, t) = \frac{\partial H(X, t)}{\partial x}
\]

The material gradient of the spatial displacement field is known as the displacement gradient defined below [19].

\[
\text{Grad} u(x, t) = \text{Grad} (X(x, t) - X) = \text{Grad} - I = F(X, t) - I
\]

where \( \text{Grad} x = \frac{\partial x}{\partial X} = F(X, t) \) known as the deformation gradient.

Marsden/Hughes [82] define the deformation gradient more precisely as follows.

Let \( \phi: \mathcal{B} \rightarrow \mathcal{S} \) be a \( C^1 \) configuration of \( \mathcal{B} \) in \( \mathcal{S} \). The tangent [map] of \( \phi \) is denoted \( F \) and is called the deformation gradient of \( \phi \); thus \( F = T\phi \). For \( X \in \mathcal{B} \), let \( F_X \) or \( F(X) \) denote the restriction of \( F \) to \( T_X \mathcal{B} \). Thus \( F(X): T_X \mathcal{B} \rightarrow T_{\phi(X)} \mathcal{S} \) is a linear transformation for each \( X \in \mathcal{B} \).

For finite-dimensional deformations, \( F \in \text{GL}^+(n, \mathbb{R}) \) or, in \( \mathbb{R}^3 \), \( \text{GL}^+(3, \mathbb{R}) \) [112-114, 133]. \( \text{GL}^+(n, \mathbb{R}) \) is a matrix Lie group and chosen rather than \( \text{GL}(n, \mathbb{R}) \) to exclude reflections.

The deformation gradient \( F(X, t) \) plays an important role in kinematics as it relates the material tangent vector \( dX \) to the spatial tangent vector \( dx \) as shown below and in Figure 4-17.

This role makes sense as \( F \) is the tangent map of deformation \( \phi \), \( F = T\phi \).

\[
F(X, t) = \frac{\partial \phi(X, t)}{\partial X} = \frac{\partial x}{\partial X} = \text{Grad} x(X, t), \text{ such that } dx = F(X, t) dX
\]

Figure 4-17: Deformation of Material Curve \( \Gamma \) into Spatial Curve \( \gamma \) ([59], modified)
Using a polar decomposition, the deformation gradient $\mathbf{F}$ can be decomposed into a stretch $\mathbf{U}$ followed by rotation $\mathbf{R}$ or a rotation $\mathbf{R}$ followed by stretch $\mathbf{v}$ as shown in Figure 4-18.

This polar decomposition of $\mathbf{F}$ produces a unique $\mathbf{R}(\mathbf{F})$ but not a unique $\mathbf{U}$ (or $\mathbf{v}$) unless $\mathbf{F}$ is invertible. Fortunately, $\mathbf{F}$ is invertible since deformation $\phi$ is assumed to be a diffeomorphism in its definition earlier in this section. Hence, the polar decomposition of $\mathbf{F}$ can be expressed as below with $\mathbf{R}$ and $\mathbf{U}$ (or $\mathbf{v}$) being unique for a given $\mathbf{F}$. It also should be noted that $\mathbf{R}(\mathbf{F})$ is the same whether used with $\mathbf{U}(\mathbf{F})$ or $\mathbf{v}(\mathbf{F})$.

Stretch then Rotation: $\mathbf{F} = \mathbf{R}(\mathbf{F})\mathbf{U}(\mathbf{F})$
Rotation then Stretch: $\mathbf{F} = \mathbf{v}(\mathbf{F})\mathbf{R}(\mathbf{F})$

where

$\mathbf{R}(\mathbf{F}) \in \text{SO}(3)$ is the rotation.
$\mathbf{U}(\mathbf{F}), \mathbf{v}(\mathbf{F}) \in \text{Symm}^+(3, \mathbb{R})$ are the right and left stretch tensors respectively.
$\text{Symm}^+(3, \mathbb{R})$ is the set of all positive definite symmetric $3 \times 3$ matrices.
$\mathbf{U}(\mathbf{F}) = \sqrt{\mathbf{F}^T \mathbf{F}} = \sqrt{\mathbf{C}}$ where $\mathbf{C}$ = right Cauchy-Green (or Green) strain tensor
$\mathbf{v}(\mathbf{F}) = \sqrt{\mathbf{F}^T \mathbf{F}} = \sqrt{\mathbf{b}}$ where $\mathbf{b}$ = left Cauchy-Green (or Finger) strain tensor

It can be shown that manifolds $\text{Symm}^+(3, \mathbb{R})$ and $\text{GL}^+(3, \mathbb{R})/\text{SO}(3)$ are naturally diffeomorphic. The latter is the set of all right (or left) cosets of $\text{SO}(3)$ in $\text{GL}^+(3, \mathbb{R})$ or, in other words, the set of equivalence classes $[\mathbf{F}] = \{\mathbf{RF} | \mathbf{R} \in \text{SO}(3), \mathbf{F} \in \text{GL}^+(3, \mathbb{R})\}$ when $\text{GL}^+(3, \mathbb{R})$ is partitioned using $\text{SO}(3)$ [79]. $\text{GL}^+(3, \mathbb{R})/\text{SO}(3)$ focuses on stretches and shears for a given

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rotation. \(GL^+(3, R)/SO(3)\) and \(Symm^+(3, R)\) are not the same set because all members of a given
equivalence class \([F]\) in \(GL^+(3, R)/SO(3)\) have the same \(U(F)\) or \(v(F)\) in \(Symm^+(3)\) as per this
equation [79].

\[
[F] = U(RF) = \sqrt{(RF)^T(RF)} = \sqrt{F^T R^T R F} = \sqrt{F^T F} = U(C) = U(F)
\]

where \([F] = U(RF) \in GL^+(3, R)\), \(U(F) \in Symm^+(3, R) \equiv GL^+(3, R)/SO(3, R)\)

\(GL^+(3, R)/SO(3)\) is only a smooth manifold (not a Lie group) because \(SO(3)\) is a subgrou+ of \(GL^+(3, R)\) but not a normal one. \(Symm^+(3, R)\) also is only a smooth manifold because com-
positions of its elements are not necessarily symmetric so it fails as a group under closure. Hence,
\(F = R(F)U(F)\) and \(F = v(F)R(F)\) respectively represent left and right Lie group actions of \(SO(3)\)
on the smooth manifold \(Symm^+(3, R)\) since \(R(F) \in SO(3)\) and \(U(F), v(F) \in Symm^+(3, R)\). Also,
since \(F \in GL^+(3, R)\) which is a matrix Lie group, \(x(X, t) = F(X, t) X\) represents a left Lie group
action of \(GL^+(3, R)\) on \(R^3\).

\(F(X, t)\) also can be used to transform material vectors into spatial vectors with \(F\)
mapping material vectors to spatial vectors and \(F^{-1}\) the inverse. Likewise, \(F^T\) maps material
covectors to spatial covectors and \(F^{-T}\) the inverse. Since \(F(X, t)\) and its polar decomposition are
defined point-wise for a given \(X\) at time \(t\), such vector transformations are exact for hom-
geogeneous deformation but only approximate for inhomogeneous deformation as explained below.

If \(F\) depends only on \(t\), it produces \textit{homogeneous} deformation which is uniform throughout
the body. In this case, both the polar decomposition and the vector transformation are exact as in
\(F = RU = vR\) and, for the vectors, \(\phi(Y) - \phi(X) = F(Y - X) = F(Y) - F(X) = y - x\). The motion
associated with homogeneous deformation, such that \(x(X, t) = F(t)X + e(t)\) where \(e(t) = \text{rigid}
translation\), is an affine transformation and so an element of the \(\text{Aff}^+(3, R)\) matrix Lie group. Rigid
body motion is a special case of this affine transformation where \(F(t) = I\) (if only translation) or
\(F(t) = R \in SO(3)\) (if also rotation) since distances remain constant within the body (i.e. no elastic
deformation).
If $F(X, t)$ depends on $X$ (as well as $t$), deformation varies with position and is said to be **inhomogeneous**. In this case, the deformation of $\phi(Y) - \phi(X) = y - x$ can be approximated by a first order Taylor expansion of $\phi(X)$ about $X$ which yields the following:

$$\phi(Y) - \phi(X) = F(X, t)(Y - X) + o(|Y - X|) \text{ as } |Y - X| \to 0$$  \[55\]

$$\phi(Y) - \phi(X) \approx F(X, t)(Y - X) \text{ within error of } o(|Y - X|)$$

where $o(|Y - X|) \to 0$ faster than $|Y - X| \to 0$

So, within a neighbourhood of $X$, an inhomogeneous deformation can be approximated by a homogeneous deformation within an error of $o(|Y - X|)$.

$F(X, t)$ is a second-order, two-point tensor since it converts vectors to vectors (second-order) and involves points in two distinct configurations (two-point). $F(X, t)$ is a linear transformation and smooth since the motion $\phi: (X,t) \to (x, t)$ is a diffeomorphism. The local instantaneous change of volume of the body is given by $J = \det(F(X, t))$ meaning that $J > 0$ and, thus, $F(X, t)$ is locally invertible with its inverse defined as shown below.

$$F^{-1}(X, t) = \frac{\partial \phi^{-1}(x, t)}{\partial x} = \text{Grad } X(x, t), \text{ such that } dX = F^{-1}(X, t) \, dx$$

The associated **spatial gradient of the material displacement** is $\text{grad } u = \text{grad } x - \text{grad } X(x, t) = I - F^{-1}(x, t)$.

The **spatial gradient** of the **spatial velocity** is known as the **(spatial) velocity gradient** $L(x, t)$ and is defined below [55].

$$L(x, t) = \text{grad } v(x, t) = \frac{\partial v(x, t)}{\partial x} \text{ such that, } dv = L(x, t) \, dx$$

$L(x, t)$ quantifies the relative spatial velocities of particles in the neighbourhood of $x$, as in $v_i(x+dx) = v_i(x) + L(x, t)dx$.

Alternatively, $L(x, t)$ can be expressed in terms of the deformation gradient $F(X, t)$ and its time derivative $\dot{F}$ as $L = \dot{F} F^{-1}$ where it can be shown that $\dot{F} = \text{Grad } V(X, t) = \text{Grad } v(x, t)$ since $V(X, t) = v(x, t)$ [55]. $L$ can be decomposed additively into $L = D + W$ where
\( \mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T) \) is the rate of deformation (strain or stretch) tensor representing the instantaneous rate of stretching of an infinitesimal fibre in the spatial configuration. \( \mathbf{D} \) is symmetric as expected from its formula [55].

\( \mathbf{W} = \frac{1}{2} (\mathbf{L} - \mathbf{L}^T) \) is the spin or rate of rotation tensor with \( \mathbf{\omega} = \mathbf{W}^\vee \) representing the average angular velocity of all infinitesimal fibres passing through a point in the spatial configuration. \( \mathbf{W} \in \mathfrak{so}(3) \) is non-symmetric as expected from its formula and more particularly skew-symmetric [55].

\( \mathbf{D} \) and \( \mathbf{W} \) are discussed further under non-linear elastic deformations (Section 4.3.2) including explaining why \( \mathbf{W} \) is skew-symmetric.

As with \( \mathbf{F} \), \( \mathbf{L}(x, t) \) is defined point-wise for a given \( x \) at time \( t \) but also can transform a spatial vector (e.g. \( y - x \)) into a spatial vector (e.g. \( \mathbf{v}(x - y) \)) and \( \mathbf{L}^{-1} \) the inverse. Likewise, \( \mathbf{L}^T \) and \( \mathbf{L}^{-T} \) transform spatial covectors. These transformations are exact if \( \mathbf{L} \) depends only on \( t \) (i.e. independent of \( x \)) but approximate otherwise. \( \mathbf{L} \) independent of \( x \) means homogeneous deformation since \( \mathbf{L} = \mathbf{FF}^{-1} \). When only rigid body motion occurs, \( \mathbf{L} = \mathbf{W} \) (thus, skew-symmetric) and \( \mathbf{L} \) is independent of position depending only on time [55]. In the case of only elastic deformation, \( \mathbf{L} = \mathbf{D} \) and may depend on just time or both position and time. The resulting spatial velocity field is symmetric if \( \mathbf{L}(t) \) and non-symmetric if \( \mathbf{L}(x, t) \) [55].

### 4.3.1 Rigid Body Motion & Lie Theory

Bullo/Lewis [21] define a rigid body as a collection of particles whose relative position to each other is fixed. This definition precludes elastic deformation so rigid body motion includes only changes of position (translation) or orientation (rotation or reflection) of a body. Using some fixed point on the body (usually its centroid or centre of mass) and an inertial frame, position can be established by a vector. However, specifying orientation requires a body frame attached to the body at the chosen point to establish orientation relative to the inertial frame. If the inertial and body frames are both right (or left) handed, this precludes reflections leaving only rotations and translations as possible rigid body motions.
Bullo/Lewis [21] model rigid body motion using the following frames, position and displacement vectors, and coordinate transformation (rotation) matrix:

- an orthogonal spatial frame $\Sigma_{\text{spatial}} = (O_{\text{spatial}}, \{s_1, s_2, s_3\})$ as the inertial reference frame, with $O_{\text{spatial}}$ being any chosen point in space and $\{s_1, s_2, s_3\}$ being the basis vectors of the frame;
- an orthogonal body frame $\Sigma_{\text{body}} = (O_{\text{body}}, \{b_1, b_2, b_3\})$ for the body, with origin $O_{\text{body}}$ and basis vectors $\{b_1, b_2, b_3\}$;
- a displacement vector $\mathbf{r}$ from $O_{\text{spatial}}$ to $O_{\text{body}}$;
- a coordinate transformation (rotation) matrix $\mathbf{R}$ from $\{b_1, b_2, b_3\}$ to $\{s_1, s_2, s_3\}$; and
- position vectors $\mathbf{X}$ and $\mathbf{x}$ in the spatial and body frames respectively.

Figure 4-19 adapted from Bullo/Lewis [21] illustrates this construction for an arbitrary point P. It should be noted that spatial frame in Bullo/Lewis means an inertial reference frame whereas this thesis generally uses spatial frame to refer to a deformed (current) configuration.

Using Figure 4-19, it can be seen that position vectors $\mathbf{X}$ and $\mathbf{x}$ are related by the affine transformation given below in both regular ($\mathbf{x} \in \mathbb{R}^3$) and homogeneous ($\langle \mathbf{x}, 1 \rangle \in \mathbb{R}^4$) coordinates as per Bullo/Lewis [21]. The left column presents the Eulerian description of the material coordinates expressing them in the current coordinates. Conversely, the right column gives the Lagrangian description of the current coordinates in material coordinates.
Eulerian
\[ X_s = R \cdot x_b + r \]
\[ \tilde{X}_s = \begin{bmatrix} R & r \\ 0_{1 \times 3} & 1 \end{bmatrix} \tilde{x}_b \]

Lagrangian
\[ x_b = R^T (X_s - r) \]
\[ \tilde{x}_b = \begin{bmatrix} R^T & -R^T r \\ 0_{1 \times 3} & 1 \end{bmatrix} \tilde{X}_s \]

Vector \( r \) and matrix \( R \) represent rigid body translation and rotation respectively. Both rigid translation and rotation are symmetries in that they preserve distances between particles of the body. Hence, they can be represented by the appropriate Lie groups, namely SO(3) for rotations and \( \mathbb{R}^3 \) for translations. Using the earlier SO(n) definition, SO(3) is defined as \( S0(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid RR^T = I_n, \det R = 1 \} \) where \( RR^T = I_n \) ensures orthogonality and \( \det R = 1 \) limits \( R \) to only rotations (not reflections).

Hence, the free configuration manifold for rigid body motion is \( SE(3) = \text{SO}(3) \times \mathbb{R}^3 \).

Based on \( SE(n, R) \) from Section 4.2.3.5, \( SE(3, R) \) is defined as follows.

\[ SE(3, R) = \left\{ g \in \mathbb{R}^{4 \times 4} \mid g = \begin{bmatrix} R & r \\ 0_{1 \times n} & 1 \end{bmatrix}, \det g = 1, R \in \text{SO}(3, R), r \in \mathbb{R}^3 \right\} \]

It is important to note that the configuration manifold for rigid body motion is \( SE(3) = \text{SO}(3) \times \mathbb{R}^3 \), the elements of which associate a given rigid rotation (in SO(3)) with a given rigid translation (in \( \mathbb{R}^3 \)) reflecting the fact that the translation partially depends upon the rotation. In physical terms, this interdependence (or coupling) stems from the fact that any rigid rotation also translates any point not on the axis of rotation. This is discussed extensively in Chapter 3. The algebraic explanation is that \( \text{SO}(3) \), but not \( \mathbb{R}^3 \), is a normal subgroup of \( SE(3) \) such that \( SE(3)/\text{SO}(3) = \mathbb{R}^3 \) but \( SE(3)/\mathbb{R}^3 \neq \text{SO}(3) \) since part of the total translation (\( \mathbb{R}^3 \)) comes from the Lie group action of \( \text{SO}(3) \). Hence, in terms of both physical reality and group structure, rigid translation must be considered jointly with rigid rotation not as an independent motion.

The Lie algebras as well as the matrix exponentials and logarithms associated with \( SE(3) \) (and se(3)) and \( \text{SO}(3) \) (and so(3)) are defined in Section 4.2.3.5. \( \mathbb{R}^3 \) (or any \( \mathbb{R}^n \)) is a Lie group under vector addition which commutes making the group abelian. The Lie algebra for Lie group \( \mathbb{R}^3 \) is \( \mathbb{R}^3 \) with the vector cross-product as the Lie bracket and the identity map \( \text{id}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as the expon-
ential map [122]. The identity map \( \text{Id}_X: X \to X \) is defined as \( \text{Id}_X(x) = x \) for all \( x \in X \).

Since \( \mathbb{R}^3 \) is an abelian Lie group, the left and right Lie group actions of \( \mathbb{R}^3 \) are equivalent. Using this fact, the affine transformations representing rigid body motion can be explained as Lie group actions of \( \text{SE}(3) \) acting on \( \mathbb{R}^3 \). \( \mathbf{x}_s = \mathbf{R}_x + \mathbf{r} \) represents \( \text{SE}(3) \) acting on \( \mathbb{R}^3 \) where \( \mathbf{(R, r)} \in \text{SE}(3) \), \( x \in \mathbb{R}^3 \). Alternatively, \( \mathbf{x}_b = \mathbf{R}^T(\mathbf{x}_s - \mathbf{r}) \) is a right action of \( \text{SE}(3) \) on \( \mathbb{R}^3 \) as shown below.

This makes sense since, as observed under Lie group actions (Section 4.2.3.2), \( R \circ g(x) = L_{g^{-1}}(x) \).

Given its free configuration manifold \( \text{SE}(3) = \text{SO}(3) \times \mathbb{R}^3 \), motion of a rigid body is given by the curve \( t \to \mathbf{g}(t) \in \text{SE}(3) \). From Bullo/Lewis [21], the velocity of the rigid body then is \( \dot{\mathbf{g}}(t) = \mathbf{g}(t)\mathbf{A}(t), \mathbf{g}(0) = \mathbf{g}_0 \) where \( \mathbf{g} \in \text{SE}(3) \) if and only if \( \mathbf{A} \in \mathfrak{se}(3) \) with \( \mathbf{g}: \mathbb{R} \to \mathbb{R}^{4x4} \) and \( \mathbf{A}: \mathbb{R} \to \mathbb{R}^{4x4} \). The same is true for \( \dot{\mathbf{g}}(t) = \mathbf{A}(t)\mathbf{g}(t), \mathbf{g}(0) = \mathbf{g}_0 \). Solving for \( \mathbf{A}(t) \) gives the spatial velocity \( \dot{\mathbf{\xi}}_s(t) \) (with respect to the inertial frame) and the body velocity \( \dot{\mathbf{\xi}}_b(t) \) (with respect to the body frame) as described below.

\[
\begin{align*}
\text{Lagrangian Rep. (right action)} & \quad \text{Eulerian Rep. (left action)} \\
\dot{\mathbf{g}}(t) &= \mathbf{g}(t)\mathbf{A}(t) & \dot{\mathbf{g}}(t) &= \mathbf{A}(t)\mathbf{g}(t) \\
\mathbf{A}(t) &= \mathbf{g}^{-1}(t)\dot{\mathbf{g}}(t) & \mathbf{A}(t) &= \dot{\mathbf{g}}(t)\mathbf{g}^{-1}(t) \\
\dot{\mathbf{\xi}}_b(t) &= \mathbf{g}^{-1}(t)\dot{\mathbf{g}}(t) & \dot{\mathbf{\xi}}_s(t) &= \dot{\mathbf{g}}(t)\mathbf{g}^{-1}(t) \\
\text{where } \mathbf{A}(t) &= \dot{\mathbf{\xi}}_b(t) \in \mathfrak{se}(3) & \text{where } \mathbf{A}(t) &= \dot{\mathbf{\xi}}_s(t) \in \mathfrak{se}(3)
\end{align*}
\]

Hence, the Lie algebra \( \mathfrak{se}(3) \) represents infinitesimal rigid body transformations at the identity of \( \text{SE}(3) \) or, in other words, velocities at \( t = 0 \). Given the resulting motion combining infinitesimal rotation and translation, elements of \( \mathfrak{se}(3) \) are called \textit{twists}. Elements of \( \text{SE}(3) \) are called \textit{screws}.

The spatial velocity and body velocity can be more easily understood if their defining equations are expanded as follows using the definition of \( \mathbf{g} \in \text{SE}(3) \).
Since \( g(t) = \begin{bmatrix} R & r \\ 0_{1x3} & 1 \end{bmatrix} \), \( \det g = 1 \), \( R \in \text{SO}(3) \), \( r \in \mathbb{R}^3 \), then
\[
\dot{g} = \begin{bmatrix} \dot{R} & \dot{r} \\ 0_{1x3} & 0 \end{bmatrix}
\] and
\[
g^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}r \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} R^T & -R^T r \\ 0_{1x3} & 1 \end{bmatrix}
\]

Using the above,
\[
\dot{\xi}_b(t) = g^{-1}(t)\dot{g}(t) = \begin{bmatrix} R^T & -R^T r \\ 0_{1x3} & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{r} \\ 0_{1x3} & 0 \end{bmatrix} = \begin{bmatrix} R^T(t)\dot{R}(t) & -R^T(t)\dot{r}(t) \\ 0_{1x3} & 0 \end{bmatrix}
\]
\[
\dot{\xi}_s(t) = \dot{g}(t)g^{-1}(t) = \begin{bmatrix} \dot{R} & \dot{r} \\ 0_{1x3} & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T r \\ 0_{1x3} & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}(t)R^T(t) & -\dot{R}(t)R^T(t)r(t) + \dot{r}(t) \\ 0_{1x3} & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} \dot{\omega}(t) & r(t) \times \omega(t) + \dot{r}(t) \\ 0_{1x3} & 0 \end{bmatrix}
\]
where \( \dot{\omega}(t) = \dot{R}(t)R^T(t) = \) spatial angular velocity
\( \omega(t) = \) rotational velocity
\( r(t) \times \omega(t) + \dot{r}(t) = \) total translational velocity

The “mixed” terms in \( \dot{\xi}_b(t) \) and \( \dot{\xi}_s(t) \) which combine \( R(t) \in \text{SO}(3) \) and \( r(t), \dot{r}(t) \in \mathbb{R}^3 \) reflect the coupling of translations and rotations (and their rates of change) in rigid body motion. This illustrates why \( \text{SE}(3) \) is the correct configuration manifold for rigid body motion rather than considering \( \mathbb{R}^3 \) independently of \( \text{SO}(3) \).

As explained in Bullo/Lewis [21], the adjoint map \( \text{Ad}_g \) transforms coordinates from one frame to another via the Lie group. If a rigid body has a configuration given by \( g \in \text{SE}(3) \), body twists can be transformed into spatial twists using the following adjoint map.
Ad_{g(t)} : se(3) \to se(3) \text{ such that } \dot{\hat{\xi}}_s(t) = Ad_{g(t)} \dot{\hat{\xi}}_b(t) = g(t) \dot{\hat{\xi}}_b(t) g^{-1}(t),  \\
where g(t) = (R, r) \in SE(3), \ Ad_{g(t)} = Ad_{(R, r)} = \begin{bmatrix} R & 0_{3 \times 3} \\ \dot{r}R & R \end{bmatrix} \\
SE(3) \text{ and } se(3) \text{ are related by the matrix exponential and logarithm already discussed. The matrix exponential of a twist in } se(3) \text{ is a rigid body transformation or screw in } SE(3). \text{ Likewise, the matrix logarithm provides local coordinates in } se(3) \text{ for rigid body transformations in } SE(3).  \\

4.3.2 Non-Linear Elasticity & Lie Theory  \\

\textit{Elasticity} is the physical property of a material such that it returns to its original shape after the stress causing the deformation is removed. For small deformations, the constitutive relation between stress and strain can be approximated as linear for many materials but generally this relationship is non-linear. This nonlinearity means that spatial configurations (and the associated deformation) are not readily comparable to the material configuration. Since this comparability worsens over time and iterative solution techniques are used, constitutive relations for non-linear elasticity usually are expressed in rate-form. This section discusses general (non-linear) elasticity.  \\

With elastic deformation, the body actually changes shape and/or size with the deformation gradient \( F = vR = RU \neq R \) and, likewise, the velocity gradient \( L = D + W \neq W \). So, \( F(X, t) \) and \( L(X, t) \) involve a stretch or a stretch and a rotation. If \( F \) is independent of \( X \), then the deformation is homogeneous and vectors can be transformed exactly by \( F \) and \( L \). Otherwise, the deformation is inhomogeneous and vectors can be transformed only approximately by \( F \) and \( L \).  

An important property of homogeneous deformation is that straight lines are preserved but not necessarily lengths of lines (if stretch) and angles between them (if shear). Hence, the deformation is the same at all points in the solid. For this reason, a body which deforms homogeneously sometimes is modeled as a pseudo-rigid body. A \textit{pseudo-rigid body} or \textit{affine-rigid body} assumes that all internal elastic deformations occur uniformly according to a fixed affine transformation. Hence, such a body has twelve degrees of freedom (6 rigid, 6
deformational) resulting in a 12-dimensional configuration manifold. For inhomogeneous deformation of a continuum, all points can deform differently requiring an infinite dimensional configuration manifold [24-26]. However, this configuration manifold becomes finite dimensional when a body is discretized in finite element analysis.

\( F \) (or \( F^{-1} \)) can be used to define strain tensors (representing stretch only) in either the material (with \( F \)) or spatial (with \( F^{-1} \)) configurations. Some examples from Holzapfel [59] and Marsden/Hughes [82] are given below where \((\mathcal{B}, \mathcal{G})\) and \((\mathcal{S}, \mathcal{g})\) are the Riemann manifolds representing the material and spatial configurations respectively and \( \phi : \mathcal{B} \rightarrow \mathcal{S} \) is the deformation.

**Right Cauchy-Green or Green deformation(or strain) tensor** \( \mathbf{C} \):
\[
\mathbf{C} : \mathcal{T} \mathcal{B} \rightarrow \mathcal{T} \mathcal{B} \text{ such that } \mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2 \text{ or, for } \mathbf{X} \in \mathcal{B} :
\]
\[
\mathbf{C}(\mathbf{X}) : \mathcal{T} \mathcal{X} \mathcal{B} \rightarrow \mathcal{T} \mathcal{X} \mathcal{B} \text{ such that } \mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^T \mathbf{F}(\mathbf{X}) = \mathbf{U}(\mathbf{X})^2
\]

**Piola, Cauchy or Finger deformation tensor** \( \mathbf{B} \):
\[
\mathbf{B} : \mathcal{T} \mathcal{B} \rightarrow \mathcal{T} \mathcal{B} \text{ such that } \mathbf{B} = (\mathbf{F}^T \mathbf{F})^{-1} = \mathbf{F}^{-1} \mathbf{F}^T \text{ or, for } \mathbf{X} \in \mathcal{B} :
\]
\[
\mathbf{B}(\mathbf{X}) : \mathcal{T} \mathcal{X} \mathcal{B} \rightarrow \mathcal{T} \mathcal{X} \mathcal{B} \text{ such that } \mathbf{B}(\mathbf{X}) = \mathbf{F}^{-1}(\mathbf{X}) \mathbf{F}^{-T}(\mathbf{X})
\]

**Left Cauchy-Green or Finger deformation(or strain) tensor** \( \mathbf{b} \):
\[
\mathbf{b} : \mathcal{T} \mathcal{S} \rightarrow \mathcal{T} \mathcal{S} \text{ such that } \mathbf{b} = \mathbf{F} \mathbf{F}^T = \mathbf{v}^2 \text{ or, for } \mathbf{x} \in \mathcal{S} :
\]
\[
\mathbf{b}(\mathbf{x}) : \mathcal{T} \mathcal{X} \mathcal{S} \rightarrow \mathcal{T} \mathcal{X} \mathcal{S} \text{ such that } \mathbf{b}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \mathbf{F}(\mathbf{x})^T = \mathbf{v}(\mathbf{x})^2
\]

**Green-Lagrange strain tensor** \( \mathbf{E} \):
\[
\mathbf{E} : \mathcal{T}^* \mathcal{B} \rightarrow \mathcal{T}^* \mathcal{B} \text{ such that } \mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) \text{, for } \mathbf{X} \in \mathcal{B} :
\]
\[
\mathbf{E}(\mathbf{X}) : \mathcal{T}^* \mathcal{X} \mathcal{B} \rightarrow \mathcal{T}^* \mathcal{X} \mathcal{B} \text{ such that } \mathbf{E}(\mathbf{X}) = \frac{1}{2} (\mathbf{F}(\mathbf{X})^T \mathbf{F}(\mathbf{X}) - \mathbf{I}) = \frac{1}{2} (\mathbf{C} - \mathbf{I})
\]

**Euler – Almansi strain tensor** \( \mathbf{e} \):
\[
\mathbf{e} : \mathcal{T}^* \mathcal{S} \rightarrow \mathcal{T}^* \mathcal{S} \text{ such that } \mathbf{e} = \frac{1}{2} (\mathbf{I} - \mathbf{F}^T \mathbf{F}^i) \text{ or, for } \mathbf{x} \in \mathcal{S} :
\]
\[
\mathbf{e}(\mathbf{x}) : \mathcal{T}^* \mathcal{X} \mathcal{S} \rightarrow \mathcal{T}^* \mathcal{X} \mathcal{S} \text{ such that } \mathbf{e}(\mathbf{x}) = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T}(\mathbf{x}) \mathbf{F}^{-i}(\mathbf{x}))
\]

Yavari/Marsden [133] provide the following relationship between the spatial \( (\mathcal{g}) \) and material \( (\mathcal{G}) \) Riemann metric using strain tensors \( \mathbf{C} \) and \( \mathbf{B} \):
\[
g(d\mathbf{x}) = <d\mathbf{x}, d\mathbf{x}> = \mathbf{C}<d\mathbf{X}, d\mathbf{X}> = \mathbf{C}\mathbf{G} \neq \mathbf{G}
\]
\[
\mathbf{G}(d\mathbf{X}) = <d\mathbf{X}, d\mathbf{X}> = \mathbf{B}<d\mathbf{x}, d\mathbf{x}> = \mathbf{B}\mathbf{g} \neq \mathbf{g}
\]
As discussed later, this means that metric tensors \( g \) and \( G \) are not objective (i.e. not frame indifferent) when non-linear elastic deformation occurs.

\( C \) and \( b \) live respectively in the material and spatial configurations and represent stretch \( U \) and \( v \). So, unlike \( F \), they do not capture the rigid rotation (transformation) between the reference and spatial configurations. However, \( U \) (or \( v \)) produces not only stretch but also shear.

As shown by Marsden/Hughes [82], for curves \( \sigma_1 \) and \( \sigma_2 \) in \( \mathcal{A} \) deformed into curves \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) in \( \mathcal{S} \) by deformation \( \phi : \mathcal{A} \rightarrow \mathcal{S} \), the length of \( \bar{\sigma}_1 \) (or \( \bar{\sigma}_2 \)) and the angle between \( \bar{\sigma}_1 \) and \( \bar{\sigma}_2 \) depend only on the initial curve(s) and the stretch tensor \( U \) (or strain tensor \( C \)). This means that both volumetric (stretch) and deviatoric (shear) strain depend only on \( U \), not \( R \). This makes sense because lengths and angles are invariant under rigid rotations.

Given that \( RU = vR \) (i.e. \( F = F \)), it can be seen that \( U = R^T vR \) and \( v = R^T vR^T \) reflecting tensor changes of basis. Hence, \( U \) and \( v \) are similar with the same eigenvalues \( (\lambda_1, \lambda_2, \lambda_3) \). These eigenvalues represent the principal stretches along three orthogonal axes (principal directions) determined by the eigenvectors of \( U \) (or \( v \)) [59]. It is important to note that these principle stretch directions do not necessarily coincide with the basis vectors of the reference (undeformed) configuration. When not coinciding, a stretch \( U \) (or \( v \)) results in both volumetric (pure stretch) and deviatoric (shear) strain. This is illustrated in Figure 4-20.

![Figure 4-20: Effect of “Stretch” \( U \) on Basis Vectors (deforms angles and length) and on Principal Stretches (deforms length but not angles) ([130], modified)](image-url)
The pure stretch component can be isolated using a singular value decomposition [130]. Since $U$ is symmetric and positive definite, its singular value decomposition is given by its eigenvalue decomposition shown below.

$$U = P \Sigma P^T$$

where

$$P = \begin{bmatrix} \lambda_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

where $\lambda_1, \lambda_2, \lambda_3$ = eigenvalues of $U$

This decomposition can be understood by the actions of its component matrices including the matrix product $\Sigma P^T$ as shown below.

\[
\begin{pmatrix}
\text{Stretch} & \text{Rot'n} \\
\text{U} & \Sigma & P & P^T
\end{pmatrix}
\]

Hence, $U$ (or $v$) can be a pure stretch or $I_{3x3}$ (if $U$ or $v$ diagonal), a stretch plus shear (if off-diagonal non-zero entries), or only a shear (if all diagonal entries =1 with off-diagonal non-zero entries).

Following Gurtin et al. [55], $L$ can be better understood through the time derivative of $F$ similar to the approach used in the earlier discussion of $\mathfrak{so}(n, R)$.

$$F(X, t) = \text{Grad} x$$, by definition

$$\dot{F}(X, t) = \frac{\partial}{\partial t} \text{Grad} x$$

$$\dot{F}(X, t) = \text{Grad} \frac{\partial}{\partial t} x = \text{Grad} v(x, t)$$, by $\frac{\partial}{\partial t} \text{Grad} x = \text{Grad} \frac{\partial}{\partial t} x$ from Holzapfel [59]

$$\dot{F}(X, t) = \text{grad} v(x, t) F(x, t)$$ by $\text{Grad} g = (\text{grad} g) F$ for any vector field $g$ from Gurtin et al. [55]

$$\dot{F} = L F$$, by definition of $L$

Hence, $L$ relates $F$ to its time derivative $\dot{F}$.

Likewise, $L$ can defined as $L = \dot{F} F^{-1}$. The structure of $L$ can be explored by expanding this definition as done below [55].
\[
L = \dot{FF}^{-1} \\
L = ( \dot{R}U)(RU)^{-1} \\
L = (\dot{RU} + RU)(U^{-1}R^T) \\
L = \dot{RR}^T + RUU^{-1}R^T
\]

where \( \dot{RR}^T \) shown as skew earlier

\[
L = (\dot{RR}^T + R\text{skew}(UU^{-1})R^T) + R\left( \text{symm}(UU^{-1}) \right) R^T 
\]

grouping into skew and symm

\[
L = W + D = D + W
\]

\[
D = R \left( \text{symm}(UU^{-1}) \right) R^T \\
W = \dot{RR}^T + R\text{skew}(UU^{-1})R^T = W_{\text{rot}} + W_{\text{str}}
\]

where

\[
W_{\text{rot}} = \dot{RR}^T, W_{\text{str}} = R\text{skew}(UU^{-1})R^T
\]

Gurtin et al. [55] shows that \( D \) characterizes the (instantaneous) rates at which deforming fibres stretch and at which the angle between deforming fibres change. On the other hand, \( W \) consists of the “rigid” rotational spin \( W_{\text{rot}} \) from rigid rotation \( R \) plus the “shear” spin \( W_{\text{str}} \) from stretch \( U \).

\( W_{\text{str}} \) is either invariant (if rigid body motion) or zero (if pure stretch) so only spin \( W_{\text{rot}} \) varies and, thus, \( W \) acts like a spin overall [55]. Hence, \( D \in \text{gl}(3, \mathbb{R}), W \in \text{so}(3) \subset \text{gl}(3, \mathbb{R}) \) and, since \( L = D + W, L \in \text{gl}(3, \mathbb{R}) \).

These all are Lie algebras of Lie groups, namely \( \text{gl}(3, \mathbb{R}) \) for \( \text{GL}^+(3, \mathbb{R}) \) and \( \text{so}(3) \) for \( \text{SO}(3, \mathbb{R}) \). This makes sense given the relationship \( L = \dot{FF}^{-1} \) with \( F \in \text{GL}^+(3, \mathbb{R}) \).

### 4.3.2.1 Homogeneous Deformation with Constant Velocity Gradient

As shown above, \( F \) and \( L \) respectively are subgroups of \( \text{GL}^+(3, \mathbb{R}) \) and \( \text{gl}(3, \mathbb{R}) \). A natural question is when \( L \) might be in the Lie (sub)algebra of the Lie (sub)group containing \( F \). \( \text{GL}^+(3, \mathbb{R}) \) and its subgroups are matrix Lie groups. For any matrix Lie group \( G \), its Lie algebra \( g \) is defined as “. . . the set of all matrices \( X \) such that \( e^X \) is in \( G \) for all real numbers \( t \).” \( ^{4,5} \)

So, the essential question is how might \( L \) be defined so that \( F = e^{L \cdot t} \). This, combined with \( \dot{F} = L \cdot F \) (as \( L = \dot{FF}^{-1} \)), requires that \( \dot{F} = L \cdot e^{L \cdot t} \). By the properties of matrix exponentials and their derivatives, this occurs only if \( L \) is constant (independent of both time and position). Hence, the set of all constant matrices \( L \) form the Lie (sub)algebra for the Lie (sub)group of all homogeneous deformation matrices \( F \).
The physical significance of \( \mathbf{L} \) being constant is that strain is constant within the body or element [89]. A constant \( \mathbf{L} \) also means that spatial acceleration is constant which is a common assumption in finite element models and reasonable for small time steps [54].

Gurtin et al. [55] discusses this class of motions in detail albeit without reference to Lie theory. In particular, Gurtin et al. shows that spatial motion with a constant \( \mathbf{L} \) can be expressed as follows.

\[
\varphi(\mathbf{X}, t) = \mathbf{x}_0 + \mathbf{F}(t)(\mathbf{X} - \mathbf{X}_0) \text{ for } -\infty < t < +\infty \text{ if } \varphi(\mathbf{X}_0, t_0) = \mathbf{x}_0
\]

\[
\varphi(\mathbf{X}, t) = \mathbf{x}_0 + e^{(t - t_0)\mathbf{L}} \mathbf{F}_0(\mathbf{X} - \mathbf{X}_0) \text{ for } -\infty < t < +\infty \text{ if } \varphi(\mathbf{X}_0, t_0) = \mathbf{x}_0 \quad [55]
\]

or, if \( t_0 = 0 \),

\[
\varphi(\mathbf{X}, t) = \mathbf{x} = \mathbf{x}_0 + e^{t\mathbf{L}} \mathbf{F}_0(\mathbf{X} - \mathbf{X}_0) \text{ if } \varphi(\mathbf{X}_0, 0) = \mathbf{x}_0
\]

Since \( \varphi(\mathbf{X}_0, 0) = \mathbf{x}_0 \) (i.e. material particle \( \mathbf{X}_0 \) passes through spatial point \( \mathbf{x}_0 \) at \( t = 0 \)), the various terms in \( \varphi(\mathbf{X}, t) \) can be understood as follows.

\( \mathbf{F}_0(\mathbf{X} - \mathbf{X}_0) \) represents the deformation (rigid and/or elastic) of particle \( \mathbf{X} \) as it moves from its undeformed position \( \mathbf{X} \) (in reference configuration) to its deformed position \( \mathbf{x} \) (in the deformed configuration).

\( \mathbf{L}t \) represents the instantaneous spin (rigid) and/or stretch (elastic) as \( \mathbf{X} \) moves from \( \mathbf{x} \) to \( (\mathbf{x} + \mathbf{t}\mathbf{d}\mathbf{x}) \) where \( (\mathbf{x} + \mathbf{t}\mathbf{d}\mathbf{x}) \) is a linearization of the motion.

\( e^{t\mathbf{L}} \) represents the instantaneous deformation (rigid and/or elastic) as \( \mathbf{X} \) moves from \( \mathbf{x} \) to \( (\mathbf{x} + \mathbf{t}\mathbf{d}\mathbf{x}) \) where \( (\mathbf{x} + \mathbf{t}\mathbf{d}\mathbf{x}) \) is a linearization of the motion.

\( \varphi(\mathbf{X}, t) \) can represent both rigid and elastic (homogeneous) deformation as shown in Table 4-2.

4.3.2.2 Objectivity, Constitutive Relations & Lie Theory

It is important to note that \( \mathbf{F} \) relates deformation between the reference and current configurations relative to the reference configuration whereas \( \mathbf{L}t \) gives deformation in (and relative to) the current frame. Solid mechanics usually uses a Langrangian representation whereby deformations (and resulting strains and stresses) are referenced to the reference configuration (or updated reference configuration if large strains). Hence, somehow the spatial deformations (strains) and associated stresses must be transformed to the material (reference) configuration.
Table 4-2: Homogeneous Deformations with Constant Velocity Gradient

<table>
<thead>
<tr>
<th>$F_0$</th>
<th>$L$</th>
<th>$\varphi(X, t) = x_0 + e^{L t} F_0(X - X_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_0 = I$</td>
<td>$L = 0$</td>
<td>$\varphi(X, t) = x_0 + (X - X_0)$ Rigid Translation</td>
</tr>
<tr>
<td>$F_0 = R \in SO(3)$</td>
<td>$L = W_{rot} \in \text{so}(3)$</td>
<td>$\varphi(X, t) = x_0 + e^{W_{rot} t} R (X - X_0)$ Rigid Rotation</td>
</tr>
<tr>
<td>$SO(3) \subset GL^+(3, R)$</td>
<td>$L = D \in \text{gl}(3, R)$</td>
<td>$\varphi(X, t) = x_0 + e^{D t} U (X - X_0)$ Pure Stretch, basis/stretch axes coincide</td>
</tr>
<tr>
<td>$F_0 = U \in GL^+(3, R)$</td>
<td>$L = D + W_{str}$</td>
<td>$\varphi(X, t) = x_0 + e^{(D + W_{str}) t} U(X - X_0)$ Stretch with Shear, basis/stretch axes differ</td>
</tr>
<tr>
<td>$F_0 = RU$</td>
<td>$L = D + W$</td>
<td>$\varphi(X, t) = x_0 + e^{(D + W) t} RU(X - X_0)$ Rigid Rotation &amp; Stretch</td>
</tr>
<tr>
<td>$F_0 \in GL^+(3, R)$</td>
<td>$L = D \in \text{gl}(3, R)$</td>
<td>$\varphi(X, t) = x_0 + e^{D t} U (X - X_0)$ Pure Stretch, basis/stretch axes coincide</td>
</tr>
</tbody>
</table>

The most convenient way would be to represent $L$ (and associated strains and stresses) in the reference configuration through a tensor change of basis. This would require that these tensors be objective or materially frame indifferent which is defined as follows based on Holzapfel [59].

Let $Q(t)$ represent a change of coordinates and a time shift ($\tilde{t} = t + \alpha, \tilde{t}, \alpha \in \mathbb{R}$), or equivalently a rigid body motion, from one configuration (upper case symbols) to another configuration (lower case symbols) with $A(X, t)$ and $a(x, \tilde{t})$ being regular tensor fields (i.e. not two-point), $H(X, t)$ and $h(x, \tilde{t})$ being two-point tensor fields, $Z(X, t)$ and $z(x, \tilde{t})$ being vectors fields, and $\Phi(X, t)$ and $\varphi(x, \tilde{t})$ being scalar fields. The transformation $Q(t)$ could be between two spatial configurations or between the material and spatial configurations. Objectivity requires that these fields transform as follows:

<table>
<thead>
<tr>
<th>Initial to New</th>
<th>New to Initial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(x, \tilde{t}) = Q(t) A(X, t) Q(t)^T$</td>
<td>$A(X, t) = Q(t)^T a(x, \tilde{t}) Q(t)$</td>
</tr>
<tr>
<td>$h(x, \tilde{t}) = Q(t) H(X, t)$</td>
<td>$H(X, t) = Q(t)^T h(x, \tilde{t})$</td>
</tr>
<tr>
<td>$z(x, \tilde{t}) = Q(t) Z(X, t)$</td>
<td>$Z(X, t) = Q(t)^T z(x, \tilde{t})$</td>
</tr>
<tr>
<td>$\varphi(x, \tilde{t}) = \Phi(X, t)$</td>
<td>$\Phi(X, t) = \varphi(x, \tilde{t})$</td>
</tr>
</tbody>
</table>

Marsden/Hughes [82] defines objectivity more generally as follows:

Let $t$ be a tensor field (or tensor density) on a manifold $\tilde{S}$ and $\psi$ a diffeomorphism from $\tilde{S}$ to another manifold $\tilde{S}'$. We say that [the push-forward of] $t' = \psi_* t$ is the objective transformation of $t$, i.e., $t$ transforms in the usual way under map $\psi$. 

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It can be shown that the deformation gradient (\(F\)) and its associated material and spatial strain tensors (\(U, v\), etc.) are objective [55] although not the Riemann metric [133]. Furthermore, the time derivatives of tensor and vector fields (required for rate-form constitutive relations) generally are not objective including the material/spatial velocities, vector fields and the (spatial) velocity gradient \(L\) although its \(D\) component is objective [55]. It similarly can be shown that the material/spatial stress tensors (associated with the strains) are objective but generally not their rate forms. This poses a significant challenge since the elastic response rate, being a property of the material, must be objective [55].

This challenge is addressed using the Lie time derivative defined earlier. This derivative pulls back the spatial strain (or stress) at a given time into the material configuration; takes its time derivative there and finds the associated stress (or strain) rate using the rate form constitutive relationship; and then pushes this stress (or strain) rate forward again into the spatial configuration. Hence, with the constitutive relationship being applied in the material configuration, the elastic response is frame indifferent. This use of the Lie derivative underlies well known objective rates such as the Jaumann-Zaremba or Juamann objective corotational rate (for stress or strain tensors) and the Green-Naghdi, Oldroyd and Truesdell objective stress rates summarized in Figure 4-21 adapted from Tian [120]. Non-linear finite element analysis commonly uses the Juamann rate.

For example, the Lie time derivative of the contravariant Cauchy stress tensor (\(\sigma^\#\) in the material configuration) gives the Oldroyd stress rate (\(\text{Oldr}(\cdot)\)) as shown below.

\[
\text{Oldr}(\sigma^\#) = \mathbf{L}_V(\sigma^\#) = \mathbf{F} \left[ \frac{\mathbf{D}(\mathbf{F}^{-1}\sigma \mathbf{F}^{-\mathbf{T}})}{Dt} \right] \mathbf{F}^\mathbf{T} \\
= \mathbf{F} \left[ \mathbf{F}^{-\mathbf{T}} \mathbf{F}^{-1}\sigma \mathbf{F}^{-\mathbf{T}} + \mathbf{F}^{-\mathbf{T}} \mathbf{F}^{-1}\sigma \mathbf{F}^{-\mathbf{T}} + \mathbf{F}^{-\mathbf{T}} \mathbf{F}^{-1}\sigma \mathbf{F}^{-\mathbf{T}} \right] \mathbf{F}^\mathbf{T} \\
= \dot{\sigma} - \mathbf{L} \sigma - \sigma \mathbf{L}^\mathbf{T}
\]

It similarly can be shown that the Oldroyd stress rate (\(\text{old}(\cdot)\)) based on \(\sigma^\#\) in the current configuration is given by \(\mathbf{Q}\text{Old}(\sigma^\#)\mathbf{Q}^\mathbf{T}\) so that the Oldroyd stress tensor is objective [55].
Hence, for general (non-linear) elastic deformations, stresses (or strains) rates are computed in the current configuration and then, via the Lie derivative, are converted to an objective rate so that the elastic response can be calculated in the material configuration and then returned to the current configuration.

4.3.3 Linear Elasticity & Lie Theory

Linear elasticity is a simplification (and, hence, usually an approximation) of non-linear elasticity. Linear elastic theory can be derived from basic principles (using linearization theory and balance laws) as done by Gurtin et al. [55] or from the linearization of general non-linear elasticity theory as done by Marsden/Hughes [82]. This section focuses on the key results, not the derivation of the theory.
The fundamental simplifying assumption in linear elasticity is that the deformation gradient $F$ is very close to $I$. This means that deformed (spatial) configurations are comparable to the material configuration. So, for all deformed configurations, the material configuration is used as the reference configuration. The material configuration is used given its natural state, namely undeformed so free of deformational stress [82].

This simplification permits the use of the following linear constitutive relationship for linear elasticity.

$$\sigma = C\varepsilon$$ or, component-wise, $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ where

$\sigma$ = Cauchy stress tensor (or linear stress tensor) where $\sigma = 2 \dfrac{\partial \psi(C)}{\partial C}$ for strain-energy function $\psi(C)$, Green strain tensor $C$

$\varepsilon$ = Elasticity tensor where $\varepsilon = 2 \dfrac{\partial \sigma(C)}{\partial C} \bigg|_{C=I}$

$\varepsilon$ = Linear (or infinitesimal) strain tensor where $\varepsilon = \frac{1}{2}(\text{Grad}^T u + \text{Grad} u)$ where Grad $u$ = displacement gradient

As defined above, the Cauchy stress tensor ($\sigma$) depends only on the Green strain tensor ($C$) and free energy $\psi$. Since both are objective, so is $\sigma$. Likewise, the elasticity tensor ($\varepsilon$) depends only on $\sigma$ and $C$ (which are objective) so also is objective as well as symmetric and positive definite. Since the reference configuration is natural, any rigid body translation ($F = C = I$) produces no stress so $\sigma(I) = 0$ [55].

The linear strain tensor ($\varepsilon$) is a linearization of the Green-Lagrange strain tensor ($E$) as shown below by Gurtin et al.[55].

$$E = \frac{1}{2} (F^T F - I)$$
$$E = \frac{1}{2} \{(\text{Grad} u + I)^T (\text{Grad} u + I) - I\}$$ as $F = \text{Grad} u + I$
$$E = \frac{1}{2} (\text{Grad}^T u + \text{Grad} u + \text{Grad}^T u \text{Grad} u)$$
$$\varepsilon = \lim_{\text{Grad} u \to 0} E = \frac{1}{2} (\text{Grad}^T u + \text{Grad} u)$$ since $\text{Grad}^T u \text{Grad} u \to 0$ before $\text{Grad} u \to 0$
$$\varepsilon = \frac{1}{2} (\text{Grad}^T u + \text{Grad} u)$$
Yavari et al. [133] expresses $\varepsilon$ in terms of the material Riemann metric $G$ and Green strain tensor $C$ as in $\varepsilon = \frac{1}{2} (C - G)$. Marsden/Hughes [82] define $\varepsilon$ using the Lie derivative of the spatial Riemann metric ($g$), $\varepsilon = \frac{1}{2} \mathcal{L}_{u} g$, and also show that $\varepsilon$ is objective. So, the linearization of $E$ is equivalent to making $g$ objective using the Lie derivative.

Hence, for small deformations, linear elasticity provides a linear (first order approximation of the) constitutive relationship which is objective (i.e. frame indifferent). This enables spatial and material strains and stresses to be compared and combined directly.

4.4 Finite Element Methods in General, EICR Method & Lie Theory

This section first explores Lie theory that is common to all finite element methods including the EICR and similar corotational methods. It then discusses and explains the EICR Method in a Lie theory context. Since corotational methods focus on the individual element, this section also does so – specifically the 3-noded triangular shell element with drilling rotations. This element also forms the basis of Chapter 3.

This section draws heavily upon earlier sections; Holzapfel [59] and Cook et al. [32] for general finite element analysis; and Felippa/Haugen [47] for variations in the corotational context. Further, the personal discussions described earlier in the literature review (Section 4.1) have been especially helpful.

4.4.1 Finite Element Methods in General & Lie Theory

All finite element methods discretize the body being analyzed, representing it as a mesh of finite elements defined by nodes (particles) and shape functions connecting them. As explained by Cook et al. [32], “… a fully continuous field is represented by a piecewise continuum field defined by nodal quantities and simple interpolation within each element.” Hence, finite element methods convert an infinite-dimensional field into a finite-dimensional one. This is especially helpful with elastic deformation fields which normally are infinite dimensional.
A key concept underlying finite element methods is the virtual displacement field ($\delta u$) used in the calculation of virtual work. Based on Holzapfel [59], a virtual displacement is “...an arbitrary, infinitesimal . . . and virtual change” in the configuration of a system, body or element. It represents a perturbation of the configuration or a first variation of the displacement field $u$. Using Figure 4-22, a first variation $\delta u$ of displacement field $u$ is defined as follows by Holzapfel [59].

$$\delta u = \bar{u} - u = \varepsilon w$$

where

- $u$ = unperturbed displacement field
- $\bar{u}$ = perturbed displacement field
- $w$ = perturbation field
- $\varepsilon$ = scalar parameter (e.g. time)

A first variation $\delta u$ is an infinitesimal (i.e. $\varepsilon \to 0$) change at a fixed instant of time. It linearizes the motion around $\varepsilon = 0$. Being the difference of two displacement fields ($\delta u = \bar{u} - u$), $\delta u$ is the same in the material and spatial frames such that $\delta u(x) = \delta U(X)$ [59].

![Figure 4-22: Virtual configuration in neighbourhood of $u$ [59]](image)

Given its definition, a virtual displacement (as a first variation) lives in the tangent space of the configuration manifold at the unperturbed configuration. If the configuration manifold also is a Lie group, the virtual displacement can be identified with the Lie algebra associated with that group because any point in the tangent space can be represented through a left (or right) translation of the Lie algebra. Hence, the virtual displacements are elements of that Lie algebra [79].
Assuming a homogeneous deformation with constant strain (i.e. \( L \) constant) as discussed in Section 4.3.2.1, matrices \( L \) are a Lie (sub)algebra of the Lie (sub)group of matrices \( F \) as already discussed. The resulting equation from Gurtin can be reorganized as shown below for displacements.

\[
\begin{align*}
    x &= x_0 + e^{L t} F_0 (X - X_0), \text{ if } \varphi (X_0, 0) = x_0 \\
    x - x_0 &= e^{L t} F_0 (U) \\
    u &= e^{L t} F_0 (U)
\end{align*}
\]

This reorganized equation has the matrix Lie group/algebra structure discussed in Section 4.3.2.1 whereby \( F = e^{L t} \). Hence, in this case, virtual displacements are elements of the Lie (sub)algebra.

Although the material configuration is perturbed initially, subsequent iterations perturb the spatial configuration. For such spatial perturbations, the Lie derivative is involved in calculating virtual displacements. Following Holzapfel [59], this is explained below for the first variation of any smooth vector function whether mapping to scalars, vectors or tensors.

**Material Configuration: First Variation of \( \bar{\varphi} = \bar{\varphi} (U) \) in direction \( \delta U \)**

\[
\begin{align*}
    \delta \bar{\varphi} (U, \delta U) &= D_{\delta U} \bar{\varphi} (U) = \frac{d}{d\varepsilon} \bar{\varphi} (U + \varepsilon \delta U)|_{\varepsilon = 0}
\end{align*}
\]

- \( \delta \bar{\varphi} (U, \delta U) \) = first variation of \( \bar{\varphi} \) in direction \( \delta U \)
- \( D_{\delta U} \bar{\varphi} (U) \) = directional derivative of \( \bar{\varphi} \) at any fixed \( U \) in direction of \( \delta U \)
- \( \bar{\varphi} (U) \) = material scalar, vector or tensor function
- \( U \) = material displacement field
- \( \delta U \) = first variation of material displacement field, \( \delta U = \delta u \)

**Spatial Configuration: First Variation of \( f = f (u) \) in direction \( \delta u \)**

\[
\begin{align*}
    \delta f (u, \delta u) &= \varepsilon_{\delta u} f (u) = \phi_* (D_{\delta u} \phi^* (f (u))) \text{ where}
\end{align*}
\]

- \( \delta f (u, \delta u) \) = first variation of \( f \) in direction \( \delta u \)
- \( \varepsilon_{\delta u} f (u) \) = Lie derivative of \( f \) in direction \( \delta u \)
- \( \phi_* (\bullet), \phi^* (\bullet) \) = push forward, pull back of \( (\bullet) \)
- \( D_{\delta U} (\bullet) \) = directional derivative of \( (\bullet) \) in direction of \( \delta U \)
- \( f (u) \) = spatial scalar, vector or tensor function
- \( u \) = material displacement field
- \( \delta u \) = first variation of material displacement field, \( \delta u = \delta U \)
Hence, calculating the spatial variation of $f$ above involves pulling $f(u)$ back into the material configuration ($\mathfrak{F}(U) = \phi^* (f(u))$), then taking the variation of $\mathfrak{F}(U)$ in the direction of $\delta U$, and finally pushing it forward to the current configuration. In other words, $\delta f(u)$ is the Lie derivative of $f(u)$ in the direction of $\delta u$. This is yet another application of the Lie derivative to ensure objectivity as already observed in non-linear elasticity (Section 4.3.2) and linear elasticity (Section 4.3.3).

Epstein/Segev [43] have proposed a more ambitious geometric description of the elastic deformation of a continuum. It incorporates virtual displacements (as generalized strains), forces (as generalized stresses), and the principle of virtual work. Their description combines a global model of the body (as a set of points forming an infinite dimensional configuration manifold), a local model (as a set of neighbourhoods of these points forming an infinite dimensional configuration manifold), and appropriate compatibility conditions (maps) between the two manifolds, their tangent bundles and their cotangent bundles. For each manifold, virtual displacements and forces are considered as elements of the tangent and cotangent bundles respectively. Epstein/Segev then show that the required compatibility between the tangent and cotangent bundles is equivalent to the principle of virtual work, expressed as follows.

$$\sigma(\delta \chi) = \mathcal{f}(\delta \kappa)$$

where

- $\sigma(\cdot)$ = local force (in cotangent bundle, local model)
- $\mathcal{f}(\cdot)$ = global force (in cotangent bundle, global model)
- $\delta \chi$ = local virtual displacement (in tangent bundle, local model)
- $\delta \kappa$ = global virtual displacement (in tangent bundle, global model)

### 4.4.2 EICR Method & Lie Theory

As discussed in Chapter 1, the EICR Method (like other similar corotational methods) essentially is a finite element analysis combined with a “filter” that removes rigid body deformation thereby isolating elastic deformation before the analysis occurs [45]. The “filter” then reintroduces the rigid body deformation after the analysis. Because corotational methods are used to analyze non-linear motion where the non-linearity arises from rigid body motion with local
strains being small (i.e. approximately linear), the small strain remaining (after rigid body motion is removed) can be analyzed linearly. Hence, the EICR and similar corotational methods use linear finite element analysis and elements.

The roles of Lie theory in finite element methods in general are discussed in the previous section. Hence, this section focuses on Lie theory contributions to other aspects of the EICR and similar corotational methods. The section begins by describing the EICR Method (and its "filter") geometrically which leads naturally to the associated global and local configuration manifolds. Nodal rotations, the treatment of rotations in general, and the projector also are discussed geometrically. As with corotational methods themselves, these topics are considered at the element level using the three noded triangular element with nodal drilling rotations. This element was introduced in Chapter 3 and is shown in Figure 4-23. Unless otherwise indicated, these discussions also are based on matrix Lie groups.

### 4.4.2.1 Nodal Rotations in EICR Method

As discussed in Chapter 2, the EICR and similar corotational methods use elements with nodal rotational degrees of freedom (dofs) to model linear strain within the element. This approach uses fewer dofs than introducing mid-side nodes. The three-noded shell element with nodal rotations discussed in this thesis six dofs at each node (3 translational, 3 rotational) with its configurations forming an 18-dimensional manifold. If only drilling rotations are considered, then the (usually very small) nodal bending rotations are set to zero.

As also discussed in Chapter 2, nodal rotations can be defined using either displacement gradients or "extra" rigid rotation. The EICR and similar corotational methods use the latter approach, defining deformational nodal rotations as the additional rigid rotation of the global nodal position vector versus the estimated rigid rotation of the element. Hence, the local element displacement field has two translational components: one determined by the nodal translational displacements and the other by the incremental rigid rotations. This again illustrates the partial
dependence of rigid translation on rigid rotation and, hence, the need to model rigid body motion with SE(3, R) as discussed in Section 4.3.1.

Since nodal rotations allow linear strain within the element (i.e. “bowing” of its sides), the velocity gradient \( \mathbf{L} \) is not necessarily constant across the element. Hence, the deformation does not fit the Lie group/algebra model discussed in Section 4.3.2.1. It also means that the deformation gradient \( \mathbf{F} \) can vary with nodal position so deformation is inhomogeneous. As an aside, it should be noted that the smaller the analysis iteration steps, the more the element performs like a constant strain element undergoing homogeneous deformation.

### 4.4.2.2 Geometric Description of EICR Method

Assuming inhomogeneous deformation, the relationship between the deformed and undeformed position of node (a) of a finite element in the EICR Method is as follows.

\[
\mathbf{x}^{(a)}(aD) = \mathbf{F} \mathbf{x}^{(a)} + \mathbf{u}^{(C, \text{rig})}
\]

where

\[
\mathbf{x}^{(a)}(aD) = \mathbf{x}^{(a)}(\mathbf{x}^{(a)}, t) = \text{position of node (a) in } \mathbb{C}^D \text{ at time } t
\]

with respect to element centroid

\[
\mathbf{F} = \mathbf{F}(\mathbf{x}^{(a)}, t) = \text{deformation gradient for node (a) at time } t
\]

\[
\mathbf{x}^{(a)} = \text{position of node (a) in } \mathbb{C}^0 \text{ with respect to element centroid}
\]

\[
\mathbf{u}^{(C, \text{rig})} = \text{rigid translation of element centroid at time } t
\]

As discussed earlier (Section 4.3), \( \mathbf{F} \in \text{GL}^+(3, \mathbb{R}) \) which is a Lie group. Hence, \( \mathbf{F} \mathbf{x}^{(a)} \) is a left Lie group action of \( \text{GL}^+(3, \mathbb{R}) \) on \( \mathbb{R}^3 \). \( \mathbf{F}(\mathbf{X}, t) \) can be decomposed using a polar decomposition such that \( \mathbf{F}(\mathbf{x}^{(a)}, t) = \mathbf{v}(\mathbf{F}) \mathbf{R}(\mathbf{F}) \) as explained in Section 4.3.2. This decomposition includes a rigid rotation \( \mathbf{R} \) followed by a left stretch \( \mathbf{v} \) with \( \mathbf{v} \) producing shear (nodal rotation) as well as pure stretch. \( \mathbf{v} \) is an element of \( \text{Symm}^+(3, \mathbb{R}) \) which is diffeomorphic to \( \text{GL}^+(3, \mathbb{R})/\text{SO}(3) \). The use of the left stretch \( \mathbf{v} \) rather than right stretch \( \mathbf{U} \) reflects the convention used in the EICR and similar corotational methods, namely rigid rotation followed by elastic deformation. This is discussed in Chapter 3.
Figure 4-23: Deformation of 3-noded Triangle Element with Drilling Rotations ([47] modified)
The EICR and similar corotational methods estimate (by the best fit process described in Chapter 3) the rigid rotation $\mathbf{R}$ in order to approximate the polar decomposition of $\mathbf{F(X, t)}$. Since the same $\mathbf{R}$ is used for all nodes (i.e. $\mathbf{R}$ independent of nodal position), $\mathbf{F} = \mathbf{v(F)} \mathbf{R(F)}$ becomes $\mathbf{F} = \mathbf{v(F)} \mathbf{R(t)}$ and the affine transformation becomes the following:

$$\mathbf{x}^{(aD)} = \mathbf{F} \mathbf{x}^{(a)} + \mathbf{u}^{(C, \text{rig})}$$

$$\mathbf{x}^{(aD)} = \mathbf{v(F)} \mathbf{R(t)} \mathbf{x}^{(a)} + \mathbf{u}^{(C, \text{rig})}$$

where

- $\mathbf{x}^{(aD)} = \mathbf{x}^{(aD)}(\mathbf{x}^{(a)}, t) = \text{position of node (a) in } c^D \text{ at time } t$ with respect to element centroid
- $\mathbf{v(F)} = \mathbf{\text{stretch of position vector for node (a), } x^{(a)}, \text{ at time } t}$
- $\mathbf{R(t)} = \mathbf{\text{rigid rotation of element (i.e. of “best fit” element frame)}}$
- $\mathbf{x}^{(a)} = \mathbf{\text{position of node (a) in } c^0 \text{ with respect to element centroid}}$
- $\mathbf{u}^{(C, \text{rig})} = \mathbf{\text{rigid translation of element centroid at time } t}$

Using global coordinates, the rigid translation $\mathbf{u}^{(C, \text{rig})}$ and the “best fit” rigid rotation $\mathbf{R(t)}$ are removed leaving nodal stretch and nodal rotation (shear) represented by $\mathbf{v}$ [47]. This combined effect of $\mathbf{v}$ can be seen by using a singular value decomposition to decompose $\mathbf{v}$ as below.

$$\mathbf{v} = \mathbf{P} \begin{pmatrix} \mathbf{P} \mathbf{\Sigma} \mathbf{P}^T \end{pmatrix} \begin{pmatrix} \mathbf{\text{Stretch}} \mathbf{\text{& Rot'n}} \end{pmatrix}$$

where

- $\mathbf{P} = 3 \times 3 \text{ matrix with eigenvectors of } \mathbf{v} (\mathbf{\sigma_1}, \mathbf{\sigma_2}, \mathbf{\sigma_3}) \text{ as columns}$
- $\mathbf{\Sigma} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $\lambda_1, \lambda_2, \lambda_3 = \text{eigenvalues of } \mathbf{v}$

Given inhomogeneous deformation, $\mathbf{v}$ varies between nodes. It represents the local elastic nodal deformation found in Chapter 3 and shown below.

$$\mathbf{v}^{(a,\text{def})} = \begin{pmatrix} \mathbf{u}^{(a,\text{def})} \\ \mathbf{\omega}^{(a,\text{def})} \end{pmatrix}$$

where, for node a,

- $\mathbf{u}^{(a,\text{def})} = \mathbf{\text{translational elastic deformation}}$
- $\mathbf{\omega}^{(a,\text{def})} = \mathbf{\text{rotational elastic deformation (shear), as a pseudo-vector}}$

As also described in Chapter 3, the EICR Method uses these local elastic deformations in the linear finite element analysis before returning to global coordinates.
4.4.2.3 Configuration Manifolds for EICR Method

The configuration in $\mathbb{R}^3$ of node (a) of an element with nodal rotations can be specified by its position vector in $\mathbb{R}^3$ and its nodal rotation in $\text{SO}(3, \mathbb{R})$. This is illustrated in Figure 4-24 where the configuration of node (a) with respect to the element centroid is given by $(x^{(aD)}, R^{(a, \text{def})})$ or, using pseudo-vectors, $(x^{(aD)}, \omega^{(a, \text{def})})$. Figure 4-24 shows two deformed configurations of node (a) as examples, both with the same $x^{(aD)}$ but one with and the other without a nodal rotation. As explained in Chapter 3, the nodal position vector is calculated from the updated nodal coordinates whereas the nodal rotation is the difference between the total nodal rotation (from the previous iteration) and the best fit element rotation (for the current iteration).

![Figure 4-24: Deformed Configurations with and without nodal rotation](image)

Hence, the configuration of a three noded triangular element with nodal rotations can be specified in $\mathbb{R}^{18}$ (3 nodes each with 6 dimensions). If only nodal drilling rotations are considered, then the configuration still is in $\mathbb{R}^{18}$ but with the nodal bending rotations equal to 0.

However, the configuration of node (a) (and, hence, the element) also can be expressed in terms of the affine transformation discussed earlier, namely: $x^{(aD)} = v(F) R(t) x^{(e)} + u^{(C, \text{rig})}$. In other words, the configuration of a node can be specified by a rigid translation $u^{(C, \text{rig})}$, a rigid rotation $R(t)$, and a stretch $v(F)$ as long as their representations are unique. The rigid translation and rotation transform $c^0$ to best fit $c^R$ and are common to all nodes. As with any rigid body
deformation, \( \mathbf{R} \) also translates any point not at the centroid and, hence, \( \mathbf{S} = (\mathbf{R}, \mathbf{u}^{(C, \text{rig})}) \) is an element of \( \text{SE}(3, \mathbb{R}) \). The stretch \( \mathbf{v}(\mathbf{F}) \) transforms \( \mathbf{c}_\mathbb{R} \) to \( \mathbf{c}_\mathbb{D} \) and varies from node to node with inhomogeneous deformation.

Uniqueness is not a challenge with rigid rotation \( \mathbf{R}(t) \) since it transforms one orthogonal frame into another with a one-to-one correspondence between such transformations and rotation matrices [79]. The rigid translation \( \mathbf{u}^{(C, \text{rig})} \) also is unique. Unfortunately, stretch \( \mathbf{v} \) is not unique as shown below.

Let \( \mathbf{w} \in \text{GL}^+(3, \mathbb{R}) \) such that \( \mathbf{w}\mathbf{x}^{(a)} = \mathbf{x}^{(a)} \), then \( \mathbf{v}\mathbf{w}\mathbf{x}^{(a)} = \mathbf{v}\mathbf{x}^{(a)} \) because \( \mathbf{w}\mathbf{x}^{(a)} = \mathbf{x}^{(a)} \).

Hence, \( \mathbf{vw} \) and \( \mathbf{v} \) result in the same stretching of \( \mathbf{x}^{(a)} \) and so \( \mathbf{v} \) is not unique [79].

This challenge with the uniqueness of \( \mathbf{v} \) can be resolved by replacing \( \mathbf{v} \) with equivalence class \( [\mathbf{v}] \) defined as follows:

\[
[v] = \{\mathbf{w} \mid \mathbf{w}\mathbf{x}^{(a)} = \mathbf{x}^{(a)}, \mathbf{w} \in \text{GL}^+(3, \mathbb{R}), \mathbf{x}^{(a)} \in \mathbb{R}^3, \mathbf{v} \in \text{Symm}^+(3, \mathbb{R}) \cong \text{GL}^+(3, \mathbb{R})/\text{SO}(3, \mathbb{R})\}
\]

\[
[v] \in \text{GL}^+(3, \mathbb{R})/N \text{ where } N = \{\mathbf{w} \mid \mathbf{w}\mathbf{x}^{(a)} = \mathbf{x}^{(a)}, \mathbf{w} \in \text{GL}^+(3, \mathbb{R}), \mathbf{x}^{(a)} \in \mathbb{R}^3\} [79]
\]

The condition \( \mathbf{w}\mathbf{x} = \mathbf{x} \) fixes \( \mathbf{x} \) creating a one-dimensional subspace of \( \mathbb{R}^3 \) with its complement being the two-dimensional subspace of planes normal to \( \mathbf{x} \). For any \( \mathbf{v} \) in equivalence class \( [\mathbf{v}] = \mathbf{vw} \) where \( \mathbf{w}\mathbf{x} = \mathbf{x} \), \( \mathbf{x} \) is an eigenvector of \( \mathbf{v} \) [79]. Using \( [\mathbf{v}] \), the affine transformation discussed earlier becomes \( \mathbf{x}^{(aD)} = [\mathbf{v}] \mathbf{R}(t) \mathbf{x}^{(a)} + \mathbf{u}^{(C, \text{rig})} \) and is illustrated in Figure 4-25.

\( \text{N} \) partitions \( \text{GL}^+(3, \mathbb{R}) \). \( \text{GL}^+(3, \mathbb{R})/\text{N} \) is a smooth manifold but not a Lie group as \( \text{N} \) is not a normal subgroup of \( \text{GL}^+(3, \mathbb{R}) \) [107]. For a specific vector \( \mathbf{x}^{(a)} \), elements of \( \text{N} \) are in \( \text{GL}^+(3, \mathbb{R}) \) but have only 6 free variables making them six dimensional. This is shown below from Helmer [58].

Let \( \mathbf{x} \in \mathbf{V} \subseteq \mathbb{R}^3 \) such that \( \text{dim}(\mathbf{V}) = 1 \)

Assume \( \mathbf{x} = (1, 0, 0)^T \) without loss of generality since \( \mathbf{x} \) is arbitrary
under an appropriate change of basis

Consider the matrices \( \mathbf{w} \in \mathbf{N} \subseteq \text{GL}+(3, \mathbb{R}) \) such that \( \mathbf{w}\mathbf{x} = \mathbf{x} \)

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & k
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
a \\
d \\
g
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
= \mathbf{x}
\]
\[
\mathbf{w} = \begin{bmatrix} 1 & b & c \\ 0 & e & f \\ 0 & h & k \end{bmatrix} \text{ because } \begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ above}
\]

With only six free variables (b, c, e, f, h, k), \(\mathbf{w} \in N\) is 6-dimensional.

Hence, N is 6-dimensional.

Given that N is 6-dimensional, \(\text{GL}^+(3, \mathbb{R})/N\) is 3 dimensional including pure stretching of vector \(\mathbf{x}\) (1-dimensional) and its reorientation through transforming the space of planes normal to \(\mathbf{x}\) (2-dimensional) [107].

![Figure 4-25: Affine Transformation from \(\mathbb{C}^0\) to \(\mathbb{C}^D\) using \([\mathbf{v}]\) for uniqueness](image)

Based on the above discussion, the configuration of node (a) with respect to the global frame can be expressed as \((\mathbf{S}, [\mathbf{v}])\) where \(\mathbf{S} = (\mathbf{R}, \mathbf{u}^{(\text{C, rig})}) \in \text{SE}(3, \mathbb{R})\) common to all nodes and \([\mathbf{v}] \in \text{GL}^+(3, \mathbb{R})/N\) specific to node (a). This results in the global configuration manifold shown below for a three-noded triangular element with nodal rotations.

\[
\text{SE}(3, \mathbb{R}) \times \frac{\text{GL}^+(3, \mathbb{R})/N \times \text{GL}^+(3, \mathbb{R})/N \times \text{GL}^+(3, \mathbb{R})/N}{\text{nodal global deformation (3 nodes)}} \quad (G1)
\]

where \(N = \{ \mathbf{w} | \mathbf{w}^{(a)} = \mathbf{x}^{(a)}, \mathbf{w} \in \text{GL}^+(3, \mathbb{R}), \mathbf{x}^{(a)} \in \mathbb{R}^3 \}\)

Since \(\text{SE}(3, \mathbb{R})\) and \(\text{GL}^+(3, \mathbb{R})/N\) respectively are 6- and 3-dimensional, (G1) is a 15-dimensional smooth manifold. It is not a Lie group since the product manifold includes \(\text{GL}^+(3, \mathbb{R})/N\) which is only a smooth manifold [75].
The configuration of node(a) with respect to the local frame can be described by \([v]\) where \([v] \in GL^+(3, \mathbb{R})/N\) specific to node (a). This results in the local configuration manifold shown below for a three-noded triangular element with nodal rotations.

\[
\frac{GL^+(3, \mathbb{R})/N \times GL^+(3, \mathbb{R})/N \times GL^+(3, \mathbb{R})/N}{\text{nodal local deformation (3 nodes)}} (L1)
\]

where \(N = \{w \mid wx^{(a)} = x^{(a)}, w \in GL^+(3, \mathbb{R}), x^{(a)} \in \mathbb{R}^3\}\).

Since \(GL^+(3, \mathbb{R})/N\) is 3-dimensional, \(L1\) is a 9-dimensional smooth manifold but not a Lie group for the same reason as above.

Although the literature discusses configuration manifolds for homogeneous elastic deformation [112-114], no such discussion has been found for inhomogeneous deformation. Hence, the above configuration manifolds cannot be considered relative to the literature. In general, manifolds for inhomogeneous deformation merit further study.

4.4.2.4 Treatment of Rotations in EICR Method

As explained in Chapter 3, the EICR Method uses rotators for global rotations and pseudo-vectors (representing spinors) for local rotations. Rotators and spinors respectively are elements of Lie group \(SO(3, \mathbb{R})\) and its Lie algebra \(so(3)\). Since \(so(3)\) is isomorphic to \(\mathbb{R}^3\), a spinor \(\Omega \in so(3)\) can be represented by a pseudo-vector \(\omega \in \mathbb{R}^3\). Further, since small rotations approximately commute, local rotations within a given iteration can be stored and composed as pseudo vectors.

A pseudo vector \(\omega \in \mathbb{R}^3\) representing a rotation \(R \in SO(3)\) is parallel to the axis of rotation of \(R\) and its magnitude \(|\omega|\) is proportionate to the rotation angle. This proportionality depends upon how the pseudo vector is normed as discussed in Chapter 2. Pseudo vectors are converted back to rotations (in \(SO(3)\)) to update nodal rotations before the next iteration.

As discussed under matrix Lie groups (Section 4.2.3.5), the matrix logarithm and matrix exponential respectively map \(so(3)\) to \(SO(3)\) and \(SO(3)\) to \(so(3)\). The formulation of these maps depend on how the pseudo vector is normed but are equivalent (once adjusted for norming) as shown in Chapter 2 (Table 2-1). The EICR Method uses a modified Rodrigues-Cayley norm \(\langle |\omega| \rangle\)
= 2 \tan(\theta/2) \approx \theta or |\omega| \approx \theta for 0 \leq \pi/6) whereas the unified corotational formulation by Felippa/Haugen [47] uses the norm |\omega| = 0. The latter approach results in the usual matrix exponential and logarithm formulation found in Bullo/Lewis and earlier in Section 4.2.3.5.

### 4.4.2.5 Projector in EICR Method

As explained in Chapter 3, the EICR Method and similar corotational methods use a projector to isolate nodal elastic deformation. An orthogonal projection (with respect to the Euclidean inner product), this projector decomposes the first variation of a given configuration (i.e. an incremental displacement in a Newton-Raphson iteration) into rigid and elastic deformations. As discussed earlier, first variations of a configuration live in the tangent space of the configuration manifold (at that configuration point) which is a vector space. Like any orthogonal projection, the projector decomposes this vector space (of virtual displacements) into two orthogonal vector subspaces (i.e. rigid and elastic deformation).

The projector can be defined either implicitly or explicitly as discussed in Section 3.3 based on Haugen [57] and summarized below.

**Implicitly:** \( \frac{\delta \mathbf{v}^{\text{def}}}{\delta \mathbf{v}} = \mathbf{P} \delta \mathbf{v} \) from Section 3.3.5, equation (9d), where

\[
\mathbf{P} = \frac{\delta \mathbf{v}^{\text{def}}}{\delta \mathbf{v}}
\]

\( \delta \cdot \) = first variation of \( \cdot \)

\[
\left( \mathbf{v}^{\text{def}} \right)^T = \begin{bmatrix} \left( \mathbf{u}^{(1,\text{def})} \right)^T, \left( \omega^{(1,\text{def})} \right)^T, \ldots, \left( \mathbf{u}^{(N,\text{def})} \right)^T, \left( \omega^{(N,\text{def})} \right)^T \end{bmatrix}
\]

where with respect to the local frame

- \( \mathbf{u}^{(a,\text{def})} \) = translational elastic deformations at node a
- \( \omega^{(a,\text{def})} \) = nodal rotation at node a, as pseudo-vector

\[
\left( \mathbf{v} \right)^T = \begin{bmatrix} \left( \mathbf{u}^{(1,\text{tot})} \right)^T, \left( \omega^{(1,\text{tot})} \right)^T, \ldots, \left( \mathbf{u}^{(N,\text{tot})} \right)^T, \left( \omega^{(N,\text{tot})} \right)^T \end{bmatrix}
\]

where with respect to the local frame

- \( \mathbf{u}^{(a,\text{tot})} \) = total translational displacement at node a
- \( \omega^{(a,\text{tot})} \) = total nodal rotation at Node a, as pseudo vector
Explicitly: \[ \bar{P} = I - \bar{P}_T - \bar{P}_R \]
from Section 3.3.1, where

\( I \) = identity matrix

\( \bar{P}_T \) = component isolating rigid translations of element centroid
(element centroid determined by averaging nodal positions)

\( \bar{P}_R \) = component isolating nodal translations from element rigid rotation
(element rotation determined by best fit process in Chapter 3)

\( \bar{P}_T, \bar{P}_R \) are defined in Section 3.3.1

The first definition shows that the projector represents the relationship (or change of coordinate map) between the variation of the local nodal elastic deformations \( \delta \mathbf{v}^{\text{def}} \) and the variation of the total nodal displacements \( \delta \mathbf{v} \). In this sense, \( P \) is similar to a Jacobian matrix [103]. The second definition reflects the “extra” rigid rotation approach to defining nodal rotations discussed in Section 4.4.2.3.

The projector is derived by taking the first variation of a given nodal configuration after the element rigid rotation \( R \) and translation \( e(t) \) have been removed. In other words, one takes the first variation of that configuration about \( R = I \) and \( e(t) = 0 \). As presented in Chapter 3, this is accomplished by taking the first variation of the local deformation equations shown below.

\[
\begin{align*}
\mathbf{u}^{(a, \text{def})} &= R^T_{A:Ek \rightarrow G} \left( \mathbf{u}^{(a, \text{tot})} - \mathbf{u}^{(C, \text{rig})} + \mathbf{x}_o \right) - \mathbf{x}_o^{(a)} \quad \text{Section 3.3.5, equation (4)} \\
\mathbf{\omega}^{(a, \text{def})} &= \mathbf{\omega}^{(a, \text{tot})} - \mathbf{\omega}^{(\text{rig})} \quad \text{Section 3.3.5, equation (7)}
\end{align*}
\]

For any rigid rotation \( R \in SO(3) \) in these equations, its variation about \( R = I \) is \( \delta R \in so(3) \) as expected from Section 4.2.3.5 since so(3) is Lie algebra of Lie group SO(3). However, this Lie structure does not necessarily hold for the variation of the full equations because the nodal deformations are interdependent. This interdependence arises from the element rigid rotation and translation being based on the orientation and position of the best fit element frame which is estimated using all nodal positions. This is consistent with the more complex configuration manifolds discussed earlier.
As discussed in Chapter 3, the projector has several interesting properties including biorthonormality. Biorthonormality is proven in corotational papers for specific coordinate choices and/or element types but not generally. While unfortunately beyond this thesis, it is thought that a more thorough geometric understanding of corotation and the projector may lead to a general proof of biorthonormality.

4.5 Summary of Lie Theory Contributions

As this chapter demonstrates, Lie theory is very relevant to rigid and elastic deformations as well as the finite element methods used to analyze them. Given this, it is not surprising that research into and use of Lie theory (and differential geometry in general) in such areas is growing. This growth stems from the general advantages of geometric approaches as well as various area-specific advantages. Discussed in the introduction to this chapter, such general advantages include a focus on physical phenomena rather than their coordinate representations and on relationships that are coordinate invariant. Area-specific advantages are thoroughly discussed within the individual chapter sections, with some major ones highlighted below.

For rigid body deformation, Lie approaches are well-developed and used extensively in fields like robotics and aeronautics. The configuration manifold for rigid body motion is Lie group $\text{SE}(3, \mathbb{R}) = \text{SO}(3, \mathbb{R}) \times \mathbb{R}^3$ with $\text{se}(3, \mathbb{R})$ being its Lie algebra. This simple configuration manifold permits the ready discussion of rigid body motions and their velocities for the body as a whole, much like it were a single particle [21]. This discussion is further simplified by the associated geometric machinery (Lie group actions, adjoint representations, etc.).

While less advanced, the study of elastic deformations from a geometric perspective is increasingly popular whether more theoretical (most notably Marsden/Hughes [82]) or more applied (flexible structures, robots or vehicles; image fitting; etc.). Elastic deformation generally is infinite dimensional. However, it becomes finite-dimensional when i) a continuous body deforms homogeneously in which case the configuration manifold is $\text{GL}^+(3, \mathbb{R})$ or ii) a continuous
body is discretized in finite element analysis in which case the configuration manifold is $\text{GL}^+(3, R) \times \ldots \times \text{GL}^+(3, R)$ (i.e. one copy of $\text{GL}^+(3, R)$ for each node).

$\text{GL}^+(3, R)$ is a larger group than and includes $\text{SO}(3)$ (rigid rotations), making its role less straightforward in elastic deformation than $\text{SE}(3, R)$ in rigid body deformation. The polar decomposition $F = [\mathbf{v}] R$ can be used to isolate the elastic deformation $[\mathbf{v}] \in \text{GL}^+(3, R)/N$ where $N = \{ w | \mathbf{w} \mathbf{x}^{(a)} = \mathbf{x}^{(a)}, \mathbf{w} \in \text{GL}^+(3, R), \mathbf{x}^{(a)} \in \mathbb{R}^3 \}$ [79]. While unfortunately not a Lie group, $\text{GL}^+(3, R)/N$ is a smooth manifold thereby retaining some advantages (velocities as tangent space, etc.). An added challenge is that most elastic deformation is inhomogeneous resulting in more complex configuration manifolds combining copies of $\text{GL}^+(3, R)/N$. These product manifolds are smooth manifolds but not Lie groups. However, such configuration manifolds may be lower in dimension and, hence, simpler than specifying configurations in $\mathbb{R}^n$. For example, configurations of the 3-noded triangular shell element with nodal rotations in the EICR Method can be described with a 15-dimensional manifold using Lie group $\text{SE}(3, R)$ and smooth manifolds $\text{GL}^+(3, R)/N$ rather than in $\mathbb{R}^{18}$.

Attempting to exploit such advantages, several geometric models have been developed for elastic deformation. Examples include Chao et al. [28], Epstein/Segev [43], and Lei/Bume [77]. Lie theory also contributes much to elasticity in areas such as frame objectivity and linearization.

For finite element analysis generally, the Lie derivative again figures prominently in objectivity and linearization. For configuration manifolds that are Lie groups, deformations and virtual displacements respectively are elements of the Lie group and its Lie algebra. A more holistic geometric model incorporating virtual work also has been proposed by Epstein/Segev [43] as discussed earlier in this chapter.

As for the EICR and similar corotational methods, Lie theory provides geometric descriptions of the associated rigid and elastic deformation leading to simpler, lower dimensional configuration manifolds. With further work, such manifolds potentially can provide a deeper
understanding of other aspects of corotational methods such as projector biorthonormality. Such corotational methods also exploit Lie groups/algebras to manage rotations using SO(3) globally and so(3) locally.

General conclusions as well as suggestions for future work are discussed in the final chapter which follows.
Chapter 5

Summary, Conclusions & Future Work

This chapter summaries the work done in this thesis, presents some general conclusions arising from it, and suggests future work.

5.1 Summary of Work

This thesis has explained the EICR and similar corotational finite element methods through both linear algebra and geometric methods including manifolds and Lie groups. This has led naturally to considerable discussion of supporting theory in differential geometry, rotations, rigid and elastic deformations, and finite element analysis (elements, virtual displacements, etc.). The resulting contribution is a thorough mathematical explanation of corotational methods for researchers first encountering these methods as well as those interested in a geometric discussion of such methods and/or elasticity.

Corotational methods were developed initially and have evolved through linear algebra. Using linear algebra, Chapter 3 has drawn together, interpreted and, where beneficial, elaborated upon existing corotational and supporting literature. Research has been brought together on the EICR Method by Rankin, Brogan and Nour-Omid [106]; on the unified small-strain corotational formulation by Felippa and Haugen [49]; on projectors by Rankin and Nour-Omid [91,105] and by Felippa and Haugen [49]; and on rotations by Argyris [7]. Elaborations have included detailed explanations of key corotational equations, of the various rotational updating approaches, of nodal rotations, and of the projector. Hence, this chapter contributes a thorough and broad explanation of the EICR and similar corotational methods using linear algebra.
The geometric explanation of corotational methods in Chapter 4 has required a broader scope of discussion than Chapter 3. Considerable geometric and kinematic theory has been needed to support the later geometric discussion of corotational methods. Manifolds, Lie groups and their application to rigid deformation are well understood so their theoretical discussions mainly have summarized existing literature. However, geometric approaches to elasticity and virtual displacement/work are far less developed. Finding and presenting this theory has been considerably more challenging but also quite rewarding. Some exploration of traditional elasticity theory also has been required. Hence, Chapter 4 has drawn together and presented (at an introductory level) geometric discussions of elasticity by Marsden/Hughes [82], Slawianowski [112-114] and Holzapfel [59]; of virtual displacement/work by Epstein/Segev [43] and Holzapfel [59]; and, to support these discussions, of traditional elasticity theory by Gurtin et al. [55] and of Lie derivatives by Holzapfel [59], Marsden/Hughes [82], and Pandolfi [96,97]. The resulting contribution is an initial, introductory geometric explanation of corotational methods, elasticity, and various aspects of finite element analysis. Such explanations obviously merit more exploration and elaboration than has been feasible at the level of this thesis and so hopefully will be the subject of future research.

The above contributions also should be contextualized. They are intended for applied researchers seeking introductory rather than full explanations of the various topics. Given the complexity of the topics, this seems reasonable for the intended audience and level of this thesis. This having been said, there seems to be considerable interest in the broader mathematics and engineering communities about even simple geometric approaches to such topics, especially elasticity. Seven posts by the author on two moderated, professional web forums, the more applied iMechanica and the more theoretical mathoverflow, resulted in over 4,000 reads. These led to productive exchanges with seventeen researchers including grad students, professors, and professionals sharing and seeking information.
5.2 General Conclusions

The various chapters and sections of this thesis offer observations and conclusions specific to their topics. However, some general conclusions also have emerged. These are discussed here.

5.2.1 EICR & Similar Corotational Methods

EICR and similar corotational methods provide non-linear modelling with standard linear elements, resulting in better compatibility with other elements and less computing resources than with specialized non-linear elements. This certainly is true for the small strain with large rotation case discussed in this thesis but, given more advanced formulations by Rankin [101,103] and others, also for large strain cases. Hence, corotational methods seem a viable, but seemingly an underused and not well understood, alternative to other non-linear finite element methods.

The development of corotational methods has been an interesting combination of application and theory. Like many finite element methods, corotational methods involve practical accommodations that produce results consistent with physical reality but are not necessarily well explained theoretically. Examples include nodal rotations (Chapter 2), locating the local element frame at the element centroid or a node (Chapter 3), and projector biorthonormality (Chapter 4). A better understanding of these accommodations might encourage greater use of corotational methods. It should be noted that Felippa [44,45] and Haugen [57] individually and together [47] have made significant contributions in this area. Further progress might be aided by a more thorough geometric understanding of corotational methods.

5.2.2 Geometric Approaches - Elasticity & Virtual Displacement/Work

Traditional elasticity theory and geometric models for rigid deformation are well understood. However, geometric discussions of elasticity have been emerging only recently. These range from quite theoretical (e.g. Marsden/Hughes [82]) to more applied (e.g. Holzapfel
from the more specialized (e.g. pseudo-rigid bodies, Slawianowski [112-114]) to the more general (e.g. any elastic continuum, Epstein/Segev [43]); and over a variety of applications (e.g. deformable images, Trouvé/Younes [121]) and approaches (e.g. (co)tangent bundles in Epstein/Segev [43], distance from SO(3) group in Chao et al. [28]). The Epstein/Segev model [43] also incorporates virtual displacement and virtual work found in finite element analysis. It seems that such models have potential, especially with expanding research in applied differential geometry.

5.2.3 Geometric Approaches - Generally

Geometric approaches to applied topics are a growing research area. These approaches often can simplify but also sometimes can cloud such topics especially for the novice. Simplification arises when physical phenomena become clearer absent their coordinate representations as, for example, with rigid body motion. However, confusion sometimes can cloud the picture. Initially, this confusion can stem from the considerable investment needed in understanding basic differential geometry. This learning curve is evident from Chapter 4 where almost half of the chapter is geometric theory. Later, properly reformulating and understanding the geometric model also can be challenging. This seems to stem mainly from geometric elements (manifolds, (co)tangent spaces and bundles, maps, etc.) not being intuitive for the novice. It also is not surprising as manifolds generalize concepts which traditionally are presented in a purely Euclidean context. Hence, considerable confusion can arise in formulating and, once done, understanding the geometric formulation. For example, an interesting elasticity model by Epstein/Segev uses (co)tangent bundles of (co)tangent bundles which certainly requires careful thought.

In some cases, it also may be unclear whether the geometric formulation is superior to the traditional one. For example, the degree of simplification may be reduced significantly when the resulting manifold is not also a Lie group. Further, some engineering problems naturally reside in
Euclidean space so their generalization to the “curved” space of the manifold may or may not be useful. A key aspect of this decision may be whether some physical property of the problem (e.g. motion, temperature distribution, etc.) forms a lower-degree manifold than the traditional Euclidean representation.

It is recognized that such uncertainties may be related as much to relevant experience as to the geometric approaches themselves. However, it seems reasonable for the applied researcher using traditional models to consider both the benefits and challenges of geometric models. If geometric modelling appears attractive, the researcher should be prepared for the significant investment (and/or collaboration) needed to fully realize its considerable benefits.

5.3 Future Work

Directions for future work emerge naturally from the above discussion as well as from the balance of this thesis.

5.3.1 EICR & Similar Corotational Methods

This thesis has presented a thorough explanation of the EICR and similar corotational methods using linear algebra. It also has initiated a geometric discussion of these methods which seems new to the literature. However, this thesis has not explored this area as completely as might be possible with greater expertise and/or as geometric models evolve for elasticity and finite element analysis. As examples, further research into configuration manifolds for inhomogeneous elastic deformation, a geometric formulation of the projector, and/or geometric virtual work models would be beneficial.

This thesis has focussed on the small strain, large rotation case with nodal rotations. However, corotational methods have been extended to the large strain case by Rankin [101] and others. Another modification by Rankin [103] uses an irrotational local displacement field at the element centroid involving only pure strain without nodal rotations. This approach may suggest
homogeneous deformation and, if so, a simpler configuration manifold - possibly even a Lie group structure. Hence, further research on the linear algebraic and/or geometric aspects of such extended and modified corotational methods is merited.

As discussed in Section 5.2.2, corotational methods use several accommodations which work physically but have not necessarily been well explained theoretically. Some of these have been addressed by Felippa [44,45] and Haugen [57] both individually and together [47] as well as, more modestly, by this thesis. However, some accommodations remain unexplained or merit more complete discussion. Examples include small artificial stiffnesses used in certain modelling situations, nodal rotations, and projector biorthonormality.

5.3.2 Geometric Approaches to Elasticity

This thesis has explored geometric approaches to elasticity through summarizing recent literature and applying it to corotational methods. However, as seen in this summary, geometric elasticity is a developing area which merits more attention. Such models also are closely related to better understanding configuration manifolds for corotational methods such as those developed in this thesis.

5.3.3 Introductory Resources for Differential Geometry

In Introduction to Topological Manifolds [76], Lee observes that “... it is possible to get through an entire undergraduate mathematics education, at least in the United States, without ever hearing the word ‘manifold’ ”. While not true universally or at Queen’s (given joint graduate / undergraduate courses), this comment does highlight that introductory differential geometry may not be as readily accessible as might be desirable, especially from an applied perspective. This also seems the case with published resources although, once again, there are notable exceptions such as books by Bullo/Lewis [21], Burns/Gidea [22], Epstein [42] and Lee [75,76] as well as works by Segev [108] and Segev/Pressburger [109]. However, to quote Epstein in motivating
Geometrical Language of Continuum Mechanics [42], there generally seems to a need for publications which “... provide some familiarity with the basic ideas of Differential Geometry as they become actualized in the context of Continuum Mechanics so that the reader can feel more at home with the masters [in Differential Geometry and Continuum Mechanics].” Presumably, the same can be said for other applied areas using differential geometry.

Hence, the expansion of introductory, applied resources in differential geometry is a desirable area of work which would benefit both students and, through them, future applied research.
Footnotes

Chapter 3


Chapter 4


Chapter 5


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