Moduli Spaces of K3 Surfaces with Large Picard Number

by

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Abstract

Morrison [21] has constructed a geometric relationship between K3 surfaces with large Picard number and abelian surfaces. In particular, this establishes that the period spaces of certain families of lattice polarized K3 surfaces and lattice polarized abelian surfaces are identical. Therefore, we may study the moduli spaces of such K3 surfaces via the period spaces of abelian surfaces.

In this thesis, we answer the following question: from the moduli space of abelian surfaces with endomorphism structure (either a Shimura curve or a Hilbert modular surface), there is a natural map into the moduli space of abelian surfaces, and hence into the period space of abelian surfaces. What sort of relationship exists between the moduli spaces of abelian surfaces with endomorphism structure and the moduli space of lattice polarized K3 surfaces? We will show that in many cases, the endomorphism ring of an abelian surface is just a subring of the Clifford algebra associated to the Néron-Severi lattice of the abelian surface. Furthermore, we establish a precise relationship between the moduli spaces of rank 18 polarized K3 surfaces and Hilbert modular surfaces, and between the moduli spaces of rank 19 polarized K3 surfaces and Shimura curves.

Finally, we will calculate the moduli space of $E_8 \oplus \langle 4 \rangle$-polarized K3 surfaces as a family of elliptic K3 surfaces in Weierstrass form and use this new family to find families of rank 18 and 19 polarized K3 surfaces which are related to abelian surfaces with real multiplication or quaternionic multiplication via the Shioda-Inose construction.
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Chapter 1

Introduction

1.1 Motivation

Much of the draw towards the study of K3 surfaces and their moduli spaces seems to come from the tension between simplicity and complexity. They are the second simplest examples of Calabi-Yau manifolds, and they sit in between elliptic curves and Calabi-Yau threefolds in terms of dimension.

Elliptic curves have been studied since the antiquity of the 19th century. Their coarse moduli may be described in a familiar way; one just takes the quotient of

$$\mathcal{h} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$$

by $SL_2(\mathbb{Z})$ via fractional linear transformations. This quotient and the related modular forms are quite well understood. They are some of the simplest algebro-geometric objects that may be studied in terms of their moduli.

In dimension three, we have Calabi-Yau threefolds, which may be described as smooth complex algebraic threefolds with trivial canonical bundle and with $h^{1,0} = h^{2,0} = 0$. For various reasons (few of which are known to the author) these threefolds and their period...
spaces are of great interest to string theorists. Due to various Hodge-theoretic facts, the deformations of Calabi-Yau threefolds can be quite complicated; for instance, the dimensions of the deformation spaces of families of Calabi-Yau threefolds may not even be bounded!

In between these two extremes, we find K3 surfaces. By the various Torelli theorems for K3 surfaces, we may reduce the study of K3 surfaces and their moduli to the theory of a single lattice isomorphic to $E_8^2 \oplus U^3$, which we will call the K3 lattice and denote by $\Lambda_{K3}$. Let $X$ be a K3 surface, then the K3 lattice is just the lattice given by the cup product form on $H^2(X, \mathbb{Z})$. This is a rank 22 lattice, and hence it is by no means simple, however, it is this complexity that makes K3 surfaces such a fertile place to study.

In general, the moduli spaces of lattice polarized K3 surfaces are not well understood geometrically. If one wishes to study the moduli space of general K3 surfaces, one obtains a 20 dimensional orbifold; a quotient of an open subset of a quadric in $\mathbb{P}^{21}$ by $O(\Lambda_{K3}, \mathbb{Z})$. To make the problem easier to understand, we may attach more structure to our K3 surfaces. In particular, we will study $N$-polarized K3 surfaces (see Definition 1.2.2). These spaces classify K3 surfaces whose Néron-Severi groups contain a lattice $N$. If $N$ has rank $n$, then the space of $N$-polarized K3 surfaces has dimension $20 - n$. Therefore, if we assume that $n$ is large, our moduli spaces become quite tractable.

We may simplify the problem even further. If a K3 surface $X$ is $N$-polarized for certain lattices $N$ of rank 17,18,19 or 20, then [21] proves that there is an abelian surface $A$ such that the Hodge structure on the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$ is identical to the Hodge structure on the orthogonal complement of $NS(A)$ in $H^2(A, \mathbb{Z})$, and that this Hodge theoretic correspondence may be realized in geometric terms (see Definition 1.2.3). If some K3 surface $X$ satisfies this correspondence, we say that $X$ has Shioda-Inose structure. Therefore, for K3 surfaces with Shioda-Inose structure, the study of moduli spaces of lattice polarized K3 surfaces may be related to the study of the moduli spaces of abelian surfaces.

The elementary question which motivated this thesis was the following:
Question. What is the precise relationship between the moduli space of lattice polarized K3 surfaces with Shioda-Inose structure and the moduli spaces of abelian surfaces with endomorphism structure?

We will provide an answer to this question in the case where $N$ belongs to specific classes of rank 17,18 or 19 lattices. The moduli spaces of rank 20 polarized K3 surfaces are 0-dimensional, hence we do not concern ourselves with their study.

1.2 A review of some existing literature

The question mentioned above (to the author’s knowledge), has never been taken up in full generality, though much of what we will prove should not be unexpected to one who is familiar with the literature on K3 surfaces and their moduli spaces. There are a number of instances in which the moduli spaces of K3 surfaces with large Picard number have been related to the moduli spaces of abelian surfaces.

For instance, [10] show that the moduli space of K3 surfaces polarized by the lattices

$$E_8^2 \oplus \langle 2k \rangle, k \in \mathbb{Z}_{>0}$$

is isomorphic to a quotient of the Siegel modular threefold $\mathfrak{h}_2/Sp_4(k,\mathbb{Z})$ by an involution when $k \neq 1$ and that it is just $\mathfrak{h}_2/Sp_4(1,\mathbb{Z})$ when $k = 1$. This involution will be denoted $\sigma_{GH}$. Therefore, one would expect that once we determine the relationship between the Néron-Severi lattice of an abelian surface and its endomorphism algebra, it should follow that the moduli space of $N$-polarized abelian surfaces (and thus the moduli space of K3 surfaces polarized by $E_8^2 \oplus N$) will be related to the moduli space of abelian surfaces with the given endomorphism structure. This correspondence may be interpreted in terms of the Shioda-Inose structure on K3 surfaces polarized by $N \oplus E_8^2$.

In [6], the authors work out the example of $k = 1$. They show that the moduli space of
K3 surfaces polarized by $E_8^2 \oplus \langle 2 \rangle$ is isomorphic to the Siegel modular threefold, $\mathfrak{h}_2 / \text{Sp}(1, \mathbb{Z})$. Beyond this, the authors exhibit a family of such K3 surfaces as a family of singular quartic hypersurfaces in $\mathbb{P}^3$ which may be written down concretely in terms of four parameters. It is then shown that the parameters of this family may be construed as Siegel modular forms of genus two.

Further in the past, Peters [24] and Dolgachev [7] have independently taken up this question for surfaces $X$ which are lattice polarized by the lattices

$$M_n := \begin{bmatrix} -2n & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus E_8^2 = U \oplus \langle -2n \rangle \oplus E_8^2.$$  

Dolgachev shows that the moduli space of such lattice polarized K3 surfaces is just the classical modular curve $\Gamma_0(n)^+$, and Peters implies that the Shioda-Inose partner of such a K3 surface is a product of a pair of $n$-isogenous elliptic curves.

In a recent article by Hashimoto [12], the author implies such a correspondence for a specific family of K3 surfaces lattice polarized by

$$L := \begin{bmatrix} -4 & -1 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & 20 \end{bmatrix} \oplus E_8^2.$$  

Hashimoto then shows that the moduli space of $L \oplus E_8^2$-polarized K3 surfaces is related to a Shimura curve which comes from the quaternion order isomorphic to $\mathbb{C}^+((1/2)L)$, an order in the even Clifford algebra of $L \otimes \mathbb{Q}$ (see Section 2.1.3 for definition). His approach is very much based on the Kuga-Satake construction, which we should perhaps discuss further.

To any K3 surface, we may associate an isogeny class of abelian varieties $A$. If we take $T(X) := \text{NS}(X)^\perp$, then we have $V(X) := \mathbb{C}^+(T(X) \otimes \mathbb{Q})$, the even Clifford algebra of the
normed space $T(X) \otimes \mathbb{Q}$, which is an even-dimensional $\mathbb{Q}$-vector space. Furthermore, we may put a polarized weight one Hodge structure on $V(X)$ in a canonical way (see [33]). There is a bijection between weight one polarized $\mathbb{Q}$-Hodge structures and isogeny classes of abelian varieties, and hence we obtain an associated isogeny class of abelian varieties. By abuse of language, we call this class of abelian varieties the Kuga-Satake variety of $X$, denote it $KS(X)$ and we will treat it as a single abelian variety.

If $X$ is a K3 surface with Shioda-Inose partner $A_X$, then we have by [20] that $KS(X)$ is isogenous to a product of a number of copies of $A_X$. In this sense, we have that the Kuga-Satake construction is an extension of the Shioda-Inose correspondence. In particular, if $	ext{rank}(\text{NS}(X)^\perp) = 3$, then $KS(X)$ is isogenous to the abelian surface to which $X$ is related by Shioda-Inose structure. Therefore, construction of the moduli spaces of rank 19 polarized K3 surfaces as Shimura curves is tantamount to making the Kuga-Satake correspondence concrete in terms of moduli, and it is this fact that Hashimoto exploits to construct this moduli space of $L \oplus E_8^2$-polarized K3 surfaces.

1.3 Main results.

The most important results in this thesis come from putting the answer to the question stated above to work in a particular example. We will prove;

Theorem 1.1. Let

$$N = E_8 \oplus D_7 \oplus U.$$ 

For $(a, b, c, d) \in \mathbb{C}^4$, denote by $\mathcal{F}(a, b, c, d)$ the family of singular elliptic surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ with $[x : y : z]$ coordinates on $\mathbb{P}^2$ and $[s : t]$ coordinates on $\mathbb{P}^1$ given by the equations

$$\mathcal{F}(a, b, c, d) : y^2z = x^3 + A(s, t)xz^2 + B(s, t)z^3$$
where $A$ and $B$ are polynomials depending on the parameters $a, b, c$ and $d$, given as

\begin{align*}
A(s, t) &= (-3t^2 + ast + (a^2 - 12b)s^2/12)t^4s^2 \\
B(s, t) &= (2t^4 - ast^3 + bs^2t^2 + cs^3t + ds^4)t^5s^3.
\end{align*}

Then

1. If $d \neq 0$, the minimal resolution of $\mathcal{F}(a, b, c, d)$ is a K3 surface with a canonical $N$-polarization coming from the resolution of the singular points in $\mathcal{F}(a, b, c, d)$.

2. Given any $N$-polarized K3 surface $X$, there is a unique point $[a : b : c : d] \in \mathbb{P}^3(1, 2, 3, 4)$ such that $X$ is isomorphic to the minimal resolution of $\mathcal{F}(a, b, c, d)$. In other words, if we put

$$
\mathcal{M}^N_{K3} := \{[a : b : c : d] \in \mathbb{P}^3(1, 2, 3, 4), d \neq 0\}.
$$

Then $\mathcal{M}^N_{K3}$ is a moduli space of K3 surfaces with $N$-polarization. Furthermore we have the isomorphism

$$
\mathcal{M}^N_{K3} \cong \mathfrak{h}_2/(Sp_4(k, \mathbb{Z}) \cup \sigma_{GH}Sp_4(k, \mathbb{Z})).
$$

Furthermore, we will use this example to put to work the results in Chapters 1 to 5. We determine a number of codimension one and two subvarieties of $\mathbb{P}^3(1, 2, 3, 4) \setminus \{d = 0\}$ on which the corresponding K3 surfaces have rank 18 and 19 polarizations respectively. We describe in detail the relationship between the subvarieties we obtain above and classical moduli spaces of abelian surfaces with real multiplication or quaternionic multiplication.
1.4 Organization

Chapter 2 consists of simple exposition. We recall many facts about abelian surfaces and K3 surfaces. We will be more precise on the exact construction of the Shioda-Inose correspondence mentioned already. We will mention the main theorems that allow us to treat the moduli spaces of K3 surfaces in terms of lattice theory. We will discuss the construction of the moduli spaces of K3 surfaces and abelian surfaces, and relate the moduli spaces of abelian surfaces to the study of certain algebras. We will discuss some basic lattice theory and algebra that will allow us to prove later theorems.

In Chapters 3 and 4, we will begin to prove some results. Since we have not found any of them in the literature we have been forced to prove them here. These results mainly pertain to abelian surfaces. In Chapter 3, we prove that with the exception of two cases, if \( A \) is an abelian surface, and \( \text{NS}(A) \) is the Néron-Severi group of \( A \), then we have

\[
\text{End}(A) \otimes \mathbb{Q} \cong C^+(1/2) \text{NS}(A) \otimes \mathbb{Q}.
\]

Furthermore, we show that this relation is true over the integers if \( \text{rank}(\text{NS}(A)) = 3, 1 \), or if \( \text{rank}(\text{NS}(A)) = 2 \) and \( A \) does not have complex multiplication. In Chapter 4, we expose the precise relation between the period spaces of lattice polarized abelian surfaces and the moduli spaces of abelian surfaces with endomorphism structure. In particular, we will see that the moduli space of rank 19 lattice polarized K3 surfaces is either a Shimura curve or a quotient of a Shimura curve by some involution, the moduli space of rank 18 polarized K3 surfaces with Shioda-Inose structure is a quotient of a Hilbert modular surface by some involution.

In Chapters 5 and 6 we make various computations. In Chapter 5, we calculate the Hilbert modular group associated to \( L \), a rank two lattice of signature \((1+, 1-)\). We show how one may embed the Hilbert modular surface associated to \( L \) into the period space of
abelian surfaces, and we calculate the group $O_0(L^\perp, \mathbb{Z})$. In Chapter 6, we then calculate explicitly the quaternion order associated to a rank 3 even lattice. Next we restate some facts about Eichler orders in terms of rank three even lattices of signature $(1+, 2-)$. In particular, given an order $\mathcal{O}(D, n)$, we show how to associate a lattice to this order. Conversely, we give easily checkable necessary and sufficient conditions under which a lattice is associated to an Eichler order. We then prove lemmas which allow one to compute the group $O_0(N^\perp, \mathbb{Z})$ for $N$ a rank three even lattice of signature $(1+, 2-)$. 

In Chapters 7 and 8, we present an example to exhibit our computations. In Chapter 7, we use the theory of elliptic K3 surfaces to write down a family of K3 surfaces which bear a lattice polarization by $\langle 4 \rangle \oplus E_8^2$, determined by Weierstrass equations with four parameters $a, b, c, d$ which are coordinates in $\mathbb{WP}^3(1, 2, 3, 4) \setminus \{d = 0\}$. We then relate this family of elliptic surfaces to a family discussed in [34], which allows us to exhibit the Kummer surfaces associated to our family of K3 surfaces with Shioda-Inose structure. In Chapter 8 we find subvarieties of $\mathbb{WP}^3(1, 2, 3, 4)$ on which the K3 surfaces on these subvarieties have lattice polarization of rank 18. We describe the Néron-Severi groups of the K3 surfaces on these subvarieties and using Chapters 5 and 6, we describe the arithmetic data associated to these K3 surfaces. Finally, we determine components of intersections of hypersurfaces on which we have Néron-Severi rank 19, and compute arithmetic data associated to the K3 surfaces supported on these subvarieties.

1.5 List of notation and conventions.

For reference throughout this thesis, we will fix the following notation.

- $A$ will denote an abelian surface.
- $X$ will denote an algebraic K3 surface.
- Unless otherwise stated, all tensor products will be tensor products of $\mathbb{Z}$-modules. In
the author’s notation, $\otimes$ should usually be read as $\otimes_{\mathbb{Z}}$.

- $N$ will denote a lattice. Specific properties of $N$ will be given in context.

- If $M$ is a matrix, we denote the transpose of $M$ by $M^t$, as opposed to $^tM$.

- The lattices $A_n, n \in \mathbb{Z}_{\geq 0}$, $D_n, n \geq 4$, $E_6, E_7, E_8$ will denote the negative definite lattices associated to the Coxeter-Dynkin diagrams normally associated to the symbols (these are listed in Appendix A.5). Usually, the above symbols will denote a positive definite lattice, however, we find it notationally convenient assume they are negative definite to avoid constant negative signs. The symbols $\tilde{A}_n$, $\tilde{D}_n$ and $\tilde{E}_n$ will denote the extended Coxeter-Dynkin diagram associated to the root systems.

- $U^3$ will be the rank six integral lattice with bilinear form consisting of three orthogonal copies of

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with bases $e_i, f_i$ for each copy of $U$;

- $u_k$ will denote the element $e_3 + kf_3$ in $U^3$.

- $E$ will be a polarization on an abelian surface $A$ (see Section 1.1 for definition).

- $\mathfrak{h}_g$ will denote the $g$-dimensional Siegel upper half plane,

$$\mathfrak{h}_g := \{ \tau \in M_g(\mathbb{C}) : \Im(\tau) \text{ positive definite} \}.$$

We will use $\mathfrak{h}$ as a shorthand for $\mathfrak{h}_1$.

- $\mathcal{O}_\Delta$ will be the quadratic order of discriminant $\Delta$, which is isomorphic to

$$\mathbb{Z} \left[ \frac{\Delta + \sqrt{\Delta}}{2} \right].$$
• \( \mathcal{O} \) will denote a quaternion order. If we write \( \mathcal{O}_N \), we mean the even Clifford ring of \( (1/2)N \) which is a quaternion order. The notation \( \mathcal{O}(D,n) \) denotes the Eichler order of discriminant \( D \) and level \( n \).

• \( \text{Sp}_4(k,\mathbb{Z}) \) will be the subgroup of \( SL_4(\mathbb{Z}) \) made up of matrices \( \gamma \) such that

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -k \\
1 & 0 & 0 & 0 \\
0 & k & 0 & 0
\end{pmatrix}^t = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -k \\
1 & 0 & 0 & 0 \\
0 & k & 0 & 0
\end{pmatrix}.
\]

• A quaternion algebra over \( \mathbb{Q} \) will be an algebra spanned over \( \mathbb{Q} \) by generators \( i, j \) and \( ij \) which satisfy the relations

\[ i^2 = a, \quad j^2 = -b, \quad \text{and} \quad ij = -ji, \]

for \( a, b \in \mathbb{Z}_{>0} \). This algebra will be denoted

\[ (\frac{a}{b}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij. \]

• Let \( \mathcal{Q} \) be a quaternion algebra. Then the canonical involution on \( \mathcal{Q} \) will be denoted \( \overset{\prime}{\cdot} \) and be defined as the involution of \( \mathcal{Q} \) which sends

\[ i \mapsto -i, \quad j \mapsto -j, \quad \text{and} \quad ij \mapsto -ij, \]

and fixes \( \mathcal{Q} \).

• If \( K \) is a quadratic extension of \( K \), then \( \overset{\prime}{\cdot} \) will denote the involution on \( K \) given by the generator of \( \text{Gal}(K/\mathbb{Q}) \).
• For $\alpha$ in either a quaternion algebra or a quadratic extension of $\mathbb{Q}$, we will say $N(\alpha) = \alpha \cdot \alpha'$ is the (reduced) norm of $\alpha$ and $\text{Tr}(\alpha) = \alpha + \alpha'$ is the (reduced) trace of $\alpha$, where $\cdot$ is the Galois involution on a quadratic extension of $\mathbb{Q}$, or the standard involution on a quaternion order.

• $\text{End}(A)$ will be the endomorphism ring of an abelian surface $A$ and $\text{End}^0(A)$ will be the endomorphism algebra of $A$, that is $\text{End}(A) \otimes \mathbb{Q} = \text{End}^0(A)$.

• If $Z$ is a Kähler surface, then $\text{NS}(Z)$ will denote the Néron-Severi group of $Z$. We will write $\text{NS}^0(Z) := \text{NS}(Z) \otimes \mathbb{Q}$.

• If $N$ is a lattice, then $C^+((1/2)N)$ will denote the even Clifford ring associated to the lattice $N$. For a precise definition, see Section 2.1.3.

• Often the objects that we are dealing with will have more than one type of structure attached to them. In this case, it may be ambiguous to say “$A \cong B$”. To get around this, if there is any doubt, we will indicate the type of isomorphism with text. For instance, if two lattices $N$ and $M$ are isometric, then we will sometimes write $N \cong_{\text{isom}} M$. If we have two abelian varieties which are isogenous, we will write $A \cong_{\text{isog}} B$.

• All lattices will be free $\mathbb{Z}$-modules of finite rank equipped with a \textit{nondegenerate, symmetric} bilinear form $\langle \cdot, \cdot \rangle$.

• All lattice embeddings will be primitive. In other words, if $\phi : N \hookrightarrow M$ is an embedding of lattices, then $\phi$ is injective and the cokernel is a free $\mathbb{Z}$-module of finite rank.
1.6 A word to the reader on the mis-use of the phrase “Moduli Space”.

Throughout this thesis (and already throughout the introduction), we have used the phrase “moduli space of K3 surfaces” in a somewhat imprecise manner. The precise term for what we will mean by moduli space of K3 surfaces is “period space”, though it is not completely improper to say moduli space in the case of K3 surfaces. We want to explain why in this brief paragraph.

1.6.1 Period spaces are not moduli spaces

Let us take $X$ to be a K3 surface. There is an integral Hodge structure on $H^2(X, \mathbb{Z})$ provided by Hodge theory. Furthermore, for every K3 surface, this Hodge structure is of exactly the same type, in that $h^{2,0}(X) = 1 = h^{0,2}(X)$ and $h^{1,1}(X) = 20$. This Hodge structure is also polarized by the cup product on $H^2(X, \mathbb{Z})$. As we will explain in Section 1.2, one may classify such polarized Hodge structures up to isomorphism by assigning to each one a point in the quotient of an open subset of a quadratic subvariety of $\mathbb{P}^{21}$. The question then is; to what extent does this point classify $X$ up to isomorphism?

Thanks to [25], we know that the answer is generically yes. First of all, for (almost) every point in the vaguely defined subvariety above, there is a unique K3 surface with the corresponding period point. For some orbifold points of the quotient, one must take care because there may be more than one corresponding K3 surface. This sort of ambiguity necessitates the use of objects called stacks. So the answer that we arrive at here is that the period space of K3 surfaces is nearly the moduli space.

1.6.2 Period spaces are sort of moduli spaces.

However, we will not just be analyzing the moduli spaces of K3 surfaces, but K3 surfaces with extra structure. If we wish to look at the period spaces of K3 surfaces with both a
lattice polarization and on which a specific class in $\text{Pic}(X)$ is ample, then [7] shows that the coarse moduli spaces of such objects are the period spaces of lattice polarized K3 surfaces (which are the objects we will really be describing in this thesis) with a finite number of codimension $n \geq 1$ subvarieties excluded.

This, along with an abhorrence of excessive jargon, motivates our use of the phrase “moduli space” to describe the period spaces of lattice polarized K3 surfaces. We hope that the reader will not object too strongly to this.
Part I

Construction
Chapter 2

Abelian surfaces and K3 surfaces

In this chapter, we will simply present background on abelian surfaces, K3 surfaces and the moduli spaces of the two objects. We start by giving the basic definitions and properties of abelian varieties, their endomorphism algebras and their Néron-Severi groups. Then we discuss concrete ways in which these things may be realized in terms of linear algebra. We will then proceed to describing K3 surfaces and defining Shioda-Inose structure. We will state many of the most important theorems concerning the moduli spaces and period spaces of K3 surfaces. Then we discuss the relationship between the period space of abelian surfaces and the period spaces of K3 surfaces. We will finish the chapter by discussing some lattice theory, and then give the classical description of the moduli spaces of abelian surfaces with endomorphism structure.

2.1 Background on abelian surfaces.

Here we give some basic discussion of abelian varieties and in particular abelian surfaces. Most of the statements made about abelian surfaces are generally statements about abelian varieties.

**Definition 2.1.1.** *By an abelian surface, we will mean a compact complex algebraic group*
of dimension two carrying a fixed embedding into \( \mathbb{P}^n \) for some positive integer \( n \).

Every abelian surface may be expressed as \( \mathbb{C}^2 / \Lambda \) where \( \Lambda \) is a rank four free \( \mathbb{Z} \)-module embedded into \( \mathbb{C}^2 \), equipped with a polarization. By a \textit{polarization}, we mean one of a number of things, and we will sketch out how these are related. The first choice is a real alternating bilinear form \( E \) on \( \mathbb{C}^2 \) which satisfies the (Hodge-Riemann) relations,

- \( E(ix, iy) = E(x, y) \) for any \( x, y \in \mathbb{C}^2 \),
- \( E(x, iy) \) is a positive definite form,
- \( E(\alpha, \beta) \in \mathbb{Z} \) for all \( \alpha \) and \( \beta \) in \( \Lambda \).

Once we have these three conditions satisfied, the form

\[
H(x, y) := E(x, y) - iE(x, iy),
\]

defines a Hermitian form on the Lie algebra of \( A \), which is just \( \mathbb{C}^2 \). Since the tangent space of \( A \) may be identified with \( \mathbb{C}^2 \) at every point via translation, we get a Hermitian form on the tangent bundle of \( A \). Therefore this provides a Kähler form on \( A \). Every Kähler form on a complex manifold \( X \) can be identified with a non-zero class in \( H^2(X, \mathbb{C}) \), and in particular \( H^{1,1}(X, \mathbb{C}) \), the de Rham cohomology classes represented by forms of type \( dz_i \wedge d\bar{z}_j \) for some \( i, j \). Since we have assumed that \( E(\alpha, \beta) \) is integral for all \( \alpha \) and \( \beta \) in \( \Lambda \), this is identified with a non-zero cohomology class in the image of the embedding \( H^2(X, \mathbb{Z}) \) into \( H^2_{DR}(X, \mathbb{C}) \) provided by de Rham’s theorem (see [35] Chapter II, Theorem 3.15).

Thus \( E \) gives us an integral Kähler class, and by the Kodaira embedding theorem (see [35] Chapter IV, Theorem 4.1), this is sufficient to give us an embedding into projective space. Using Chow’s theorem (see [36], pp. 165), we see that \( A \) is therefore projective algebraic.

Going backwards, if \( A \) is embedded into \( \mathbb{P}^n \) for some integer \( n > 0 \) then the restriction of \( \mathcal{O}_{\mathbb{P}^n}(1) \) to \( A \) gives us an element in \( H^{1,1}(A, \mathbb{Z}) \cap H^2(A, \mathbb{Z}) \) and this cohomology class provides
a polarization on $A$. Alternatively, there is a canonical Kähler metric on $\mathbb{P}^n$ whose associated cohomology class is that of $O_{\mathbb{P}^n}(1)$ (called the Fubini-Study metric). The pullback under embedding of this Kähler form to $A$ gives a concrete Kähler form on $A$. These two points of view give identical results.

The point that we must make is that to every cohomology class representing a Kähler form, there is a non-isomorphic embedding of $A$ into projective space, and hence our definition of an abelian surface will simply be: an abelian surface is a complex torus of dimension two, equipped with a polarization. So we will make the identification $A \cong \mathbb{C}^2/\Lambda$ where $\Lambda$ is a maximal rank sub-lattice of $\mathbb{C}^2$.

### 2.1.1 More on polarizations.

What we now want to show is how an abelian surface may be viewed as a linear algebraic object. We will view a polarization as a real alternating form $E$ on $\mathbb{C}^2$ satisfying the Hodge-Riemann bilinear relations and restricting to an integral form on our lattice $\Lambda$. In particular, since we may identify $\Lambda$ with $H_1(A, \mathbb{Z})$, we have that $E$ is an element in $\bigwedge^2(H_1(X, \mathbb{Z}))^*$. By Poincaré duality and the non-degeneracy of the cup-product pairing, we may associate the dual lattice of $H_1(A, \mathbb{Z})$ with $H^1(A, \mathbb{Z})$.

Secondly, one deduces that the cohomology ring $H^*(A, \mathbb{Z})$ is the ring generated by $H^1(A, \mathbb{Z})$ and where multiplication is just the wedge product. Therefore, $\bigwedge^2 H^1(A, \mathbb{Z}) \cong H^2(A, \mathbb{Z})$.

In general then, we can take an element of $H^2(A, \mathbb{Z})$ and get an alternating bilinear form on $H_1(A, \mathbb{Z})$. This proceeds in the following way: we may take $\omega$ a real two form on $A$ such that $\omega$ is a representative of the class of $E$. Then if $\alpha$ and $\beta$ are two $C^\infty$ one-cycles on $A$, then if we have $\alpha \cup \beta$ the topological two-cycle represented by the cup product of $\alpha$ and $\beta$ and $\varphi : \alpha \cup \beta \hookrightarrow A$ is the canonical $C^\infty$ injection, then

$$E_\omega(\alpha, \beta) := \int_{\alpha \cup \beta} \varphi^* \omega.$$
If $\alpha^*$ and $\beta^*$ are elements of $H^1(A, \mathbb{Z})$ which are Poincaré dual to $\alpha$ and $\beta$ respectively, then we may write $E_\omega$ alternatively as

$$E_\omega(\alpha, \beta) = \int_A \omega \wedge \alpha^* \wedge \beta^*.$$  

Since $A$ is a compact manifold of real dimension four, we may identify $H^4(A, \mathbb{Z})$ with $\mathbb{Z}$ by choosing a volume form on $A$. Once we have chosen this volume form, we will omit the integral sign and just write the wedge product.

Next, if we let $\omega$ be a polarization on $A$ then it is an integral alternating form on $\mathbb{Z}^4$. Thus, upon choosing a basis for $H^1(A, \mathbb{Z})$, we may write $E_\omega$ in terms of an integral matrix $J_\omega$ such that $J_\omega^T = -J_\omega$. One may choose a basis of $\mathbb{Z}^4$ such that there is a $k \in \mathbb{N} \setminus \{0\}$ so that

$$J_\omega = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -k \\ 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \end{bmatrix},$$

and $k$ is unique given $E$. We call such a basis a symplectic basis of $\mathbb{Z}^4$. Let $e_1, e_2, e_3, e_4$ be such a basis of $H_1(A, \mathbb{Z})$, and let $H^1(A, \mathbb{Z}) \ni g_i := e_i^*$. Then letting $g_{ij} = g_i \wedge g_j$ and associating $H^1(A, \mathbb{Z})$ with $\bigwedge^i H^1(A, \mathbb{Z})$. If we let $dV := g_1 \wedge g_2 \wedge g_3 \wedge g_4$ represent the volume form on $A$, then we have that $\omega = \sum a_{ij} g_{ij}$ for some integers $a_{ij}$ and thus

$$E_\omega(e_l, e_m) = \omega \wedge g_l \wedge g_m$$  \hspace{1cm} (2.1)

$$= \left( \sum a_{ij} g_{ij} \right) \wedge g_l \wedge g_m$$  \hspace{1cm} (2.2)
And we want to satisfy

\[
\left( \sum a_{ij} g_{ij} \right) \land g_l \land g_m = dV \text{ if } (l, m) = (1, 3) \quad (2.3)
\]
\[
\left( \sum a_{ij} g_{ij} \right) \land g_l \land g_m = kdV \text{ if } (l, m) = (2, 4) \quad (2.4)
\]
\[
\left( \sum a_{ij} g_{ij} \right) \land g_l \land g_m = 0 \text{ otherwise.} \quad (2.5)
\]

Thus \(a_{24} = 1, a_{34} = k\) and \(a_{ij} = 0\) otherwise. Therefore, given a polarization \(E_\omega\) on \(H_1(A, \mathbb{Z})\) and a symplectic basis \(g_i\), the polarization on \(A\) is represented by \(g_{24} + kg_{13} \in H^2(A, \mathbb{Z})\).

**Definition 2.1.2.** An abelian surface which has a polarization \(E\) carrying a symplectic basis under which the form \(E\) is represented by \(g_{24} + kg_{13}\) is called a \((1, k)\)-polarized abelian surface (or a \(k\)-polarized abelian surface for the sake of brevity). The polarization \(E\) will be called a \((1, k)\)-polarization (or a \(k\)-polarization.) If \(k = 1\) this is called a principal polarization.

If \(A\) is \(k\)-polarized, then [5] shows that there is a lattice \(\Lambda \in \mathbb{C}^2\) such that \(A \cong \mathbb{C}^2/\Lambda\), we have that \(\Lambda\) is spanned over \(\mathbb{Z}\) by the column vectors of

\[
\begin{bmatrix}
1 & 0 & \tau_1 & \tau_2 \\
0 & k & \tau_2 & \tau_3
\end{bmatrix},
\]

that

\[
\begin{bmatrix}
\tau_1 & \tau_2 \\
\tau_2 & \tau_3
\end{bmatrix} \in \mathfrak{h}_2 := \{ \tau \in M_2(\mathbb{C}) : \text{Im}(\tau) \text{ is positive definite } \},
\]

and that the polarization \(E\) is represented by the standard symplectic form on this basis, or in other words if this basis is denoted by \(\{\xi_i\}_{i=1}^4\), we have

\[
[E(\xi_i, \xi_j)] = J_k.
\]

We will call \(\mathfrak{h}_2\) the Siegel upper half space of genus two.
2.1.2 Néron-Severi lattices.

Definition 2.1.3. We define a lattice to be a free \( \mathbb{Z} \)-module \( L \) equipped with a symmetric non-degenerate integral bilinear form

\[
L \times L \longrightarrow \mathbb{Z}
\]

often denoted \( \langle \cdot, \cdot \rangle \). A lattice will be called even if for every element \( x \) of \( L \), we have \( \langle x, x \rangle \in 2\mathbb{Z} \). To any even lattice, there is a natural integral quadratic form \( Q \) such that \( \langle x, x \rangle = 2Q(x) \) for any element \( x \in L \), and thus we may write \( \langle \cdot, \cdot \rangle \) in terms of \( Q \).

Example 2.1.1. Let us take \( \mathbb{Z}^2 \) equipped with the bilinear form such that if \( x_1 \) and \( x_2 \) be a minimal generating set, then

\[
\langle x_i, x_j \rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

We call this lattice the hyperbolic lattice of rank two and denote it \( U \). Note that \( U \) is an even lattice, and its associated bilinear form is just \( Q(a_1x_1 + a_2x_2) = a_1a_2 \) for any \( a_1, a_2 \in \mathbb{Z} \).

Now if we let \( A \) be an abelian surface, there is a natural symmetric integral bilinear form on \( H^2(A, \mathbb{Z}) \) which is given by

\[
\langle \eta, \delta \rangle := \eta \wedge \delta
\]

for any \( \eta, \delta \in H^2(A, \mathbb{Z}) \), and where \( \eta \wedge \delta \in H^4(A, \mathbb{Z}) \) is identified via orientation with an element of \( \mathbb{Z} \). This lattice is isometric to three copies of the hyperbolic lattice. This isometry may be seen by taking the bases \( e_i, f_i \) of each copy of \( U \) and making the identifications

\[
U_1 = \{g_{12} = e_1, g_{34} = f_1\}, U_2 = \{g_{14} = e_2, g_{23} = f_2\} \text{ and } U_3 = \{g_{13} = f_3, g_{24} = e_3\}
\]

as bases for each copy of \( U \). Thus, under this natural identification between \( H^2(A, \mathbb{Z}) \),
equipped with the cup product pairing $\langle \cdot, \cdot \rangle$ and the lattice $U^3$ coming from any choice of symplectic basis, our polarization is given by

$$u_k = e_3 + k f_3.$$  

**Definition 2.1.4.** A Hodge structure of weight $n$ on a finite rank free $\mathbb{Z}$-module $H_\mathbb{Z}$ is a linear decomposition of $H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C}$ into sub vector spaces $H^{i,j}$, as

$$H_\mathbb{C} = \bigoplus_{i+j=n} H^{i,j},$$

where our decomposition satisfies $\overline{H^{i,j}(A)} = H^{j,i}(A)$ where $\overline{\cdot}$ denotes complex conjugation.

A polarized Hodge structure of weight $n$ is a Hodge structure of weight $n$ where there is some integral bilinear form,

$$Q : H_\mathbb{Z} \times H_\mathbb{Z} \to \mathbb{Z}$$

whose bilinear extension to $H_\mathbb{C}$ satisfies the (generalized) Hodge-Riemann bilinear relations,

1. $Q(x, y) = (-1)^n Q(y, x)$ for all $x, y \in H_\mathbb{C}$.
2. $Q(x, y) = 0$ for any $x \in H^{p,q}$ and $y \in H^{p',q'}$ satisfying $q \neq p'$ and $p \neq q'$.
3. For any $x \neq 0 \in H^{p,q}$, we have $(\sqrt{-1})^{p-q} Q(x, \overline{x}) > 0$.

**Definition 2.1.5.** We say that two polarized Hodge structures $(H_\mathbb{Z}, Q)$ and $(H'_\mathbb{Z}, Q')$ of weight $n$ are Hodge isometric if there is a map $\phi : H_\mathbb{Z} \to H_\mathbb{Z}$ such that the $\mathbb{C}$-linear extension $\phi_\mathbb{C}$ of $\phi$ sends $H^{i,j}$ to $(H')^{i,j}$, and for every $\alpha, \beta \in H_\mathbb{Z}$, we have $Q(\alpha, \beta) = Q'(\phi(\alpha), \phi(\beta))$.

Upon choosing a complex structure on the torus $(\mathbb{R}/\mathbb{Z})^4$, one obtains a Hodge structure of weight $m$ on $H^m(A, \mathbb{Z})$. The numbers $h^{i,j} = \dim H^{i,j}(A)$ may be arranged in a diamond. This configuration is called the Hodge diamond of $A$, which we draw.
In particular, we have a polarized weight one Hodge structure on $H^1(A, \mathbb{Z})$ where the polarization comes from our polarization on the abelian surface $A$,

$$H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A) \cong \mathbb{C}^2 \oplus \mathbb{C}^2$$

and if $\omega$ is a Kähler form on $A$ and $\mathbb{Z}\omega$ denotes the $\mathbb{Z}$-span of $\omega$ in $H^2(A, \mathbb{Z})$, then we have a weight two polarized Hodge structure on the primitive part of $H^2(A, \mathbb{Z})$, which is denoted

$$PH^2(A, \mathbb{Z}) := \omega^\perp,$$

where the polarization just comes from the restriction of the cup product (or alternatively, wedge product). This Hodge structure splits as

$$PH^2(A, \mathbb{C}) = H^{2,0}(A) \oplus (H^{1,1}(A)/\mathbb{Z}\omega) \oplus H^{0,2}(A) \cong \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}.$$

Since $H^2(A, \mathbb{R})$ embeds into $H^2(A, \mathbb{C})$, it makes sense to define the Néron-Severi group of $A$ to be $H^{1,1}(A) \cap H^2(A, \mathbb{Z})$. We will denote this object by $\text{NS}(A)$. It is a free abelian group of rank $\leq 4$ since $H^2(A, \mathbb{Z})$ is a free abelian group and $H^{1,1}(A)$ has dimension 4. The form $\langle \cdot, \cdot \rangle$ naturally restricts to a bilinear form on $\text{NS}(A)$.

**Definition 2.1.6.** Let $L$ and $M$ be lattices with bilinear forms $\langle \cdot, \cdot \rangle_L$ and $\langle \cdot, \cdot \rangle_M$ respectively. Then if $\phi : L \rightarrow M$ is a map of $\mathbb{Z}$ modules, we say that it is an morphism of lattices if we have that for every $x, y \in L$, $\langle x, y \rangle_L = \langle \phi(x), \phi(y) \rangle_M$. If $\phi$ is injective, we say that it is a
lattice embedding. If the cokernel of an embedding $\phi$ of lattices is a free abelian group, then we say that $\phi$ is a primitive embedding of lattices.

By construction, we have that $\text{NS}(A)$ is a primitive sublattice of $H^2(A, \mathbb{Z})$. Note that since $A$ is algebraic, we have at least one non-trivial element of $\text{NS}(A)$ coming from the polarization. We will study abelian surfaces with fixed Néron-Severi groups in the future. We have another natural sublattice of $H^2(A, \mathbb{Z})$,

**Definition 2.1.7.** Let $A$ be an abelian surface, then we call the orthogonal complement of $\text{NS}(A)$ the transcendental lattice of $A$. We denote it $T(A)$.

The lattice $T(A)$ is contained in $\text{PH}^2(A, \mathbb{Z})$, and carries a polarized Hodge structure, both the Hodge structure and the polarization being the restrictions of polarization and Hodge structure from the primitive cohomology of $H^2(A, \mathbb{Z})$.

2.2 Background on K3 surfaces related to abelian surfaces.

In this section we discuss how we may relate K3 surfaces to abelian surfaces, especially in the case where the Néron-Severi rank of the K3 surface is large. We have

**Definition 2.2.1.** A K3 surface is a smooth compact complex Kähler surface with trivial fundamental group and trivial canonical bundle.

For us, all K3 surfaces $X$ that we consider will be algebraic, or equivalently by the Kodaira embedding theorem, its Kähler form $\omega \in H^{1,1}(X)$ is actually in $H^1(X, \mathbb{Z})$. The Néron-Severi group and transcendental lattices are defined identically to those of an abelian surface, and therefore, we will be considering K3 surfaces with Néron-Severi rank at least 1. A K3 surface has a Hodge diamond,
which may be deduced from standard theorems in complex algebraic geometry (see [1] VIII 3.2.3.2). Notice the important fact that for both an abelian surface and a K3 surface, $h^{2,0} = 1$. Furthermore, we have $\text{Pic}^0(X) = 0$ since $H^2(X, \mathbb{Z}) = 0$, and therefore the equality $\text{NS}(X) = \text{Pic}(X)$ follows from the exact sequence,

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{NS}(X) \longrightarrow 0.$$ 

The cup product pairing on $H^2(X, \mathbb{Z})$ is non-degenerate by Poincaré duality and is an even lattice isomorphic to $\Lambda_{K3} := E_8^2 \oplus U^3$ (see [1] 3.2). There is a polarized Hodge structure on the primitive part of $H^2(X, \mathbb{Z})$. This Hodge structure is what differentiates two K3 surfaces.

**Example 2.2.1** ([3], VII.10). Starting with an abelian surface $A$, we may construct a K3 surface in a very canonical manner. Let

$$\iota : A \longrightarrow A$$

$$x \mapsto -x.$$ 

It is easy to see that $\iota$ has exactly sixteen fixed points (the two-torsion points of $A$). Let $\tilde{A}$ be the surface obtained by performing a monoidal transformation at each of these points, and let $\tilde{\iota}$ be the induced involution on $\tilde{A}$. Then the quotient $Y = \tilde{A}/\tilde{\iota}$ is a smooth minimal surface, and furthermore, it is a K3 surface with Néron-Severi group of rank at least 17. Furthermore, there is a Hodge isometry, $T(Y) \cong 2T(A)$. We call $Y$ the Kummer surface associated to $A$ and denote it $\text{Kum}(A)$,
The most important fact about K3 surfaces that we will use comes in the form of the Strong Torelli theorem for algebraic K3 surfaces, which says;

**Theorem 2.1** ([17] pp. 2). Let $X$ and $X'$ be two algebraic K3 surfaces. Let

$$\phi^*: H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$$

be a map which induces a Hodge isometry between the weight two polarized Hodge structure on the primitive cohomologies of $X$ and $X'$. Then there is a unique isomorphism of algebraic varieties

$$\phi: X' \rightarrow X$$

such that $\phi^*$ is given by the pullback on cohomology.

In other words, if the polarized Hodge structures on two algebraic K3 surfaces are Hodge-isometric, then the underlying K3 surfaces are isomorphic. Secondly, we have a result known as the surjectivity of the period mapping. First we will define the period mapping for a K3 surface. Let us take $X$ a K3 surface, and let $\omega$ be a generator of $H^{2,0}(X)$. Since $H^{2,0}(X)$ is just one-dimensional, $\omega$ is unique up to scaling. We have the following proposition

**Proposition 2.2.1.** Let $\theta$ be the class of some ample divisor in the Picard group of a K3 surface, and let $H_{\mathbb{Z}}$ be the orthogonal complement of $\theta$ with respect to the cup-product pairing, and let $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$. There is a bijection between possible Hodge structures on $H_{\mathbb{Z}}$ and points $[u] \in \mathbb{P}(H_{\mathbb{C}})$ such that $\langle u, u \rangle = 0$ and $\langle u, \overline{u} \rangle > 0$. We will refer to this subvariety of $\mathbb{P}(H_{\mathbb{Z}C})$ as $D(H_{\mathbb{Z}C})$.

**Proof.** First of all, we can show that this is well defined. Let us take $\lambda \in \mathbb{C}$ and $v = \lambda u$ in $[u] \in \mathbb{P}(H_{\mathbb{C}})$. If $u$ satisfies the conditions in the proposition, then $\langle \lambda u, \lambda u \rangle = \lambda^2 \langle u, u \rangle = 0$ as well. Similarly, $\langle \lambda u, \overline{\lambda u} \rangle = |\lambda|^2 \langle u, \overline{u} \rangle > 0$.

Next, assuming that we have some $[u] \in \mathbb{P}(H_{\mathbb{C}})$ satisfying the conditions in the proposition, then we define the decomposition $\mathbb{C}u = H^{2,0}, \mathbb{C}\overline{u} = H^{0,2}$ and let $H^{1,1} = (H^{2,0} \oplus H^{0,2})^\perp$. 
This is a polarized Hodge structure on the pair \((H_Z, \langle \cdot, \cdot \rangle)\).

Finally, if we have a Hodge structure on \(H_Z\), and \(u \in H_C\) spans \(H^{2,0}\), then the point \([u] \in \mathbb{P}(H_C)\) lies inside the variety defined in the proposition. 

Now for a choice of K3 surface \(X\) and a basis of \(H^2(X, \mathbb{Z})\), we may assign a point \(\omega\) in \(\mathcal{D}(H_Z)\) to \(X\) just by taking the point corresponding to the Hodge structure on \(PH^2(X, \mathbb{Z})\) expressed in terms of the given basis. We have the following theorem:

**Theorem 2.2** ([30]). *For each point \(z\) in \(\mathcal{D}(H_C)\) there is a K3 surface whose period is \(z\).*

Not only is this true, but we may actually posit the existence of a universal family of algebraic K3 surfaces over a quotient of the the period domain \(\mathcal{D}(H_C)\) (see [24], Theorem 3.4.1 for details).

A similar game may be played for abelian surfaces. Our construction carries right over, since for an abelian surface, we have \(\dim_C H^{2,0}(A) = 1\) as well. We may then define the period space \(\mathcal{D}_A(H_C)\) in precisely the same manner. In this situation, the analogue of Theorem 3 is not so hard, we have

**Theorem 2.3** ([29] Shioda). *Given any point \(z\) in \(\mathcal{D}_A(H_C)\), there exists an abelian surface \(A\) and a choice of basis of \(H^2(A, \mathbb{Z})\) such that the associated period point of \(A\) and this basis is \(z\).*

In the case of abelian surfaces, however, we cannot quite say that \(A\) is classified by its polarized Hodge structure.

**Definition 2.2.2.** Let \(Y\) denote either an algebraic K3 surface or an abelian surface. \(Y\) is lattice polarized by a lattice \(L\) if there is a primitive embedding of lattices \(\phi : L \rightarrow \text{NS}(Y)\) such that the image of \(L\) contains some ample divisor. The pair \((Y, \phi)\) is called a lattice polarized surface. Two surfaces polarized by the same lattice \(L\), denoted by \((Y', \phi')\) and \((Y, \phi)\) are isomorphic if there is an isomorphism of algebraic surfaces \(\varphi : Y \cong Y'\) such that the pullback on cohomology satisfies \(\varphi^* \cdot \phi' = \phi\).
In [7], Dolgachev establishes the following fact

\textbf{Theorem 2.4 ([7] 3.2, 3.3).} The moduli space $\mathcal{M}_L$ of K3 surfaces carrying a polarization by $L$ is described as follows; let $\eta : L \rightarrow E_8^2 \oplus U^3$ be a primitive lattice embedding. Then let

$$D(L^\perp) := \{\omega \in \mathbb{P}(L^\perp \otimes \mathbb{C}) | \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0\},$$

and define the group

$$O(L^\perp, \mathbb{Z})^* := \{\gamma \in O(E_8^2 \oplus U^3, \mathbb{Z}) : \gamma|_{\eta(L)} = Id\}.$$

Then

$$\mathcal{M}_L = D(L^\perp)/O(L^\perp, \mathbb{Z})^*,$$

where the action of $O(L^\perp, \mathbb{Z})^*$ is the action coming from the restriction of $\gamma$ to $L^\perp \otimes \mathbb{C}$ pulled back along $\eta$.

\textbf{Remark 2.2.1.} If we take abelian surfaces instead of K3 surfaces, then if $L$ is a primitive sublattice of $U^3$ the analogous space $D_A(L^\perp)/O(L^\perp, \mathbb{Z})^*$ is not the moduli space of $L$-polarized abelian surfaces in general. In the generic case, we have that this is a quotient of the moduli space of $L$-polarized abelian surfaces by some involution. Stated more precisely, two abelian surfaces $A$ and $A'$ have period points corresponding to the same point in $D_A(L^\perp)/O(L^\perp, \mathbb{Z})^*$ if and only if $A = A'$ or $A$ is the dual of $A'$ (see Shioda [29]).

Now if we let $L$ be some lattice containing $E_8^2$ and $X$ a K3 surface, then $T(X) \subseteq L^\perp \subseteq U^3$, and hence there exists an abelian surface $A_X$ such that we have a Hodge isometry $T(X) \cong T(A_X)$. In fact this correspondence is not just Hodge theoretic, but geometric. Following Morrison ([21] Def. 6), we define

\textbf{Definition 2.2.3.} A K3 surface is said to have Shioda-Inose structure if there is an involution $i$ on $X$ which fixes precisely eight points, and the rational quotient map $\pi : X \rightarrow Y$
such that $Y$ is a Kummer surface and $\pi_*$ induces a Hodge isometry $2T(X) \cong T(Y)$. Here we denote by $Y$ the minimal resolution of $X/\iota$.

Then Morrison goes on to prove the theorem,

**Theorem 2.5** ([21] 6.3). *Let $X$ be an algebraic K3 surface. The following are equivalent.*

1. $X$ admits Shioda-Inose structure,

2. There is an abelian surface $A$ and a Hodge isometry $T(X) \cong T(A)$,

3. There is a primitive lattice embedding $T(X) \hookrightarrow U^3$,

4. There is a primitive lattice embedding $E_8^2 \hookrightarrow \text{NS}(X)$.

Another way to say this is that if $X$ is a K3 surface such that $\text{NS}(X)$ contains $E_8^2$, then there is an abelian surface $A$ and an involution $j$ on $X$ such that

$$X \dashrightarrow \widetilde{X}/j = \text{Kum}(A) = \widetilde{A}/\iota \dashrightarrow A$$

where $\iota$ is as in Example 1.2.1, and if $Z$ is a singular surface then $\widetilde{Z}$ denotes the minimal resolution of $Z$. The diagram above induces Hodge isometry between $T(A)$ and $T(X)$.

Therefore, if we have a lattice $L$ such that $E_8^2$ embeds primitively into $L$, we have $L^\perp U^3 \oplus E_8^2$ equipped with a primitive embedding into $U^3$, thus there is some $M$ such that

$$\mathcal{D}_A(M^\perp) = \mathcal{D}(L^\perp).$$

Thus for every $L$-polarized K3 surface $X$ there is an $M$-polarized abelian surface $A$ such that there is a Hodge isometry $T(X) \cong T(A)$. Upon knowing that such a Hodge isometry exists, we obtain the Shioda-Inose construction by Morrison’s theorem.

Therefore, if we want to describe the moduli space of $L$-polarized abelian surfaces, it will suffice to describe the period domains of abelian surfaces with $M$-polarization.
2.2.1 Some facts about lattices and their embeddings.

For reference, we will record here some facts and definitions. The following may be found in Nikulin’s wonderful paper, [23].

**Definition 2.2.4.** Let $L$ be a lattice. The Gram matrix of $L$ with respect to a basis $\{\xi_i\}_{i=1}^n$ of $L$ is the matrix whose entries are

$$\langle \xi_i, \xi_j \rangle_L.$$ 

The determinant of this matrix will be called the discriminant of $L$ and denoted $\text{disc}(L)$. We say that $L$ is unimodular if $\text{disc}(L) = 1$.

It is clear that the Gram matrix of a basis for $L$ completely determines the lattice $L$. We will find it quite convenient in the future to consider the Gram matrix of a lattice with respect to a given basis.

**Definition 2.2.5.** Let $L$ be a lattice, then the signature of $L$ records the number of positive and negative eigenvalues of the Gram matrix of $L$ with respect to a given basis. The number of positive eigenvalues will be given by $s_+$ and the number of negative eigenvalues will be given by $s_-$. Often we will write the signature of $L$ as the pair $(s_+, s_-)$.

For example, $U$ has signature $(1+, 1-)$, and therefore $H^2(A, \mathbb{Z})$ has signature $(3+, 3-)$.

The Néron-Severi lattice of $A$ sits inside of $H^{1,1}(A)$. Therefore by Condition 3 in Definition 2.1.4, we have that $\text{NS}(A)$ has signature $(1+, n-)$ for some $n \leq 3$.

We quote a theorem of Nikulin with respect to the number of primitive embeddings of a given lattice into another,

**Theorem 2.6** (Nikulin [23], Theorem 1.4.1). Let $M$ be an even unimodular lattice with signature $(a_1+, a_2-)$ and let $L$ be an even lattice with signature $(b_1+ , b_2-)$, then if we have $a_1 < b_1$, $a_2 < b_2$ and $2 \text{rank}(M) \leq \text{rank}(L) - 2$, then there exists a unique (up to auto-isometry of $L$) primitive embedding of $M$ into $L$. 

Therefore, if we have a lattice $M$ of rank two and signature $(1+, 1-)$, then there is only one possible primitive embedding of $M$ into $U^3$ up to an auto-isometry of $U^3$. Thus there is, up to isomorphism, only one possible way for $M$ to be the Néron-Severi group of an abelian surface $A$.

We will make a standard choice for a rank two lattice embedded in $U^3$. If we have a rank two lattice $M$ embedded primitively in $U^3$, and $U^3$ has a basis $e_1, f_1, e_2, f_2$ and $e_3, f_3$ and $M$ has Gram matrix

$$
\begin{bmatrix}
2a & h \\
h & 2k
\end{bmatrix}
$$

with respect to a basis $u, v$, then we will assume that $M$ is embedded via the basis

$$
v \mapsto e_2 + af_2 + hf_3 \quad (2.6)
$$

$$
u_k \mapsto e_3 + kf_3. \quad (2.7)
$$

We prove the following lemmas which will be useful later.

**Lemma 2.2.1.** Let $N$ be a rank three even lattice with signature $(1+, 2-, 0-)$, and let $u, v, w$ be a basis of $N$ with $\langle u, u \rangle > 0$. Then the lattices $M_1 := \text{span}_\mathbb{Z}\{u, v\}$ and $M_2 := \text{span}_\mathbb{Z}\{u, w\}$ are primitive sublattices of signature $(1+, 1-)$. 

**Proof.** That these are primitive sublattices is by definition. If we had either of $M_i$ of signature $(2+, 0-)$, this would contradict the fact that the signature of $L$ is $(1+, 2-)$. It is clear that $M_i$ cannot have signature $(0+, 2-)$ either, for if this were true, then for any $\gamma \neq 0 \in M_i$, we would have $\langle \gamma, \gamma \rangle < 0$, but we have $u \in M_i$ for $i = 1, 2$, and $\langle u, u \rangle > 0$. \hfill \Box

**Lemma 2.2.2.** Let $N$ be a rank three even lattice of signature $(1+, 2-, 0-)$, then there is a primitive embedding of $N$ into $U^3$. This embedding is not necessarily unique up to isomorphism, however, for any rank three even lattice primitively embedded into $U^3$ and containing
There is a basis of $N$ which looks like

\begin{align*}
  w &= c_1 e_1 + c_2 f_1 + b f_2 + h_2 f_3 \\
  v &= e_2 + a_1 f_2 + h_1 f_3 \\
  u_k &= e_3 + k f_3.
\end{align*}

Proof. It is clear that we may take a basis of $N$ containing $u_k$ as a generator. Then if $v$ is another generator, we must have some $v$ such that $u_k, v$ a primitive lattice of $N$ of rank two and signature $(1+, 1-)$ by the previous lemma. Then $\text{span}_\mathbb{Z}\{u_k, v\}$ is a primitive sublattice of $U^3$ of rank two and signature $(1+, 1-)$, thus we may choose a basis of $U^3$ so that we have

\begin{align*}
  v &= e_2 + a f_2 + h f_3 \\
  u_k &= e_3 + k f_3.
\end{align*}

In this process, we have no control of where the third element of our basis, which we may call $w$, goes. We have

\begin{align*}
  w &= c_1 e_1 + c_2 f_1 + c_3 e_2 + c_4 f_2 + c_5 e_3 + c_6 f_3,
\end{align*}

for some integers $c_i$. Now we may make a basis change,

\begin{align*}
  v' &= v, u'_k = u_k, \text{ and } w' = w - c_5 u_k - c_3 v = c_1 e_1 + c_2 f_1 + b f_2 + h_2 f_3
\end{align*}

for some integers $b, h_2$ and $c_1, c_2$ as before. For this basis of $N$, we have Gram matrix

\[
\begin{pmatrix}
2c_1 c_2 & b & h_2 \\
b & 2a_1 & h_1 \\
h_2 & h_1 & 2k
\end{pmatrix}.
\]
This proves the lemma.

We leave the reader with the following facts, proved in Proposition A.2.1 in the Appendix: by work of Nikulin, we have that if \( M \) is a rank two lattice of signature \((1+, 1-)\) embedded primitively inside of \( U^3 \) and \( N \) is a rank three lattice of signature \((1+, 2-)\), primitively embedded inside of \( U^3 \), then

\[
M_\perp \cong U \oplus -M \quad \text{and} \\
N_\perp \cong -N.
\] (2.11)

\[
N_\perp \cong -N.
\] (2.12)

### 2.3 Endomorphism algebras of abelian surfaces.

We describe the endomorphism algebra of an abelian surface \( A \).

**Definition 2.3.1.** An abelian surface is called simple if it is not isogenous to a product of elliptic curves.

**Definition 2.3.2.** If \( A \) is an abelian surface then an endomorphism of \( A \) is an endomorphism of algebraic groups \( A \rightarrow A \). We denote the ring of endomorphisms of \( A \) by \( \text{End}(A) \). Often it will be easier for us to work with the endomorphism algebra of \( A \), by which we mean \( \text{End}(A) \otimes \mathbb{Q} \), and which we will write as \( \text{End}^0(A) \).

By basic calculations (see \cite{5} 1.2.1), one may show that any holomorphic map from \( A \) to \( A \) may be represented as a homomorphism composed with translation, so this definition of an endomorphism is not so restrictive. Multiplication is provided by composition of homomorphisms and addition is provided pointwise in \( A \).

More concretely, we can think of \( A \) in terms of its representation as a complex torus, \( \mathbb{C}^2/\Lambda \) for a lattice \( \Lambda \). Then what we mean by endomorphism can be said in terms of matrix representations. We want to choose a matrix \( T_{\text{rat}} \in M_4(\mathbb{Z}) \) which will be called the rational representation of the endomorphism of \( A \), which will act on the lattice points within \( \Lambda \) by...
multiplication on the left. Furthermore, since we have $\Lambda \otimes \mathbb{R} \cong \mathbb{C}^2$, this map provides an $\mathbb{R}$-linear map from $\mathbb{C}^2$ to $\mathbb{C}^2$. The second condition is that we have that this be a $\mathbb{C}$-linear map, or in other words, that this commutes with complex structure on $A$. We call the $\mathbb{C}$-linear map associated to $T_{\text{rat}}$ the analytic representation of $T$, denoted $T_{\text{an}}$.

Thus one may imagine that $\text{End}(A)$ is the set of endomorphisms of $H_{\mathbb{Z}}$ inside of $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes \mathbb{R}$ which are equivariant with respect to the complex structure on $H_{\mathbb{R}}$ which defines the Hodge structure on $A$.

If $A$ is a $k$-polarized abelian surface, then there is a natural involution on $\text{End}(A)$ depending on the $k$-polarization.

**Definition 2.3.3.** Let $A$ be an abelian surface and $u_k$ a $k$-polarization on $\text{End}(A)$. The Rosati involution on $A$ with respect to $u_k$ may be defined as

$$
\circ : \text{End}(A) \longrightarrow \text{End}(A)
$$

$$
T \mapsto T^{\circ}, \text{ where } E(T(x), y) = E(x, T^{\circ}(y)), \text{ for all } x, y \in \mathbb{C}^2.
$$

The existence of such an involution is still in question, but we may settle this very concretely. Let us take $\{e_1, \ldots, e_4\}$ be a symplectic basis with respect to the polarization $E$. Then taking two vectors $x$ and $y$ in $\text{span}_{\mathbb{R}}\{e_1, \ldots, e_4\}$, we have

$$
E(T(x), y) = (T_{\text{rat}}x)^t J_k y = x^t (T_{\text{rat}}^t J_k) y = x^t (J_k^{-1} T_{\text{rat}}^t J_k) y = E(x, T^{\circ}(y)).
$$

Thus we have $(T^{\circ})_{\text{rat}} = J_k^{-1} T_{\text{rat}}^t J_k$. We may see that this is actually still an element of $\text{End}^0(A)$ since $J_k$, $J_k^{-1}$ and $T_{\text{rat}}$ preserve complex structure, hence their composition does also. One may show [22] that $\circ$ is actually a positive involution; i.e. if $\text{Tr}$ is the standard matrix trace, then $\text{Tr}(TT^{\circ}) \geq 0$ for all $T \in \text{End}^0(A)$. Thus $\text{End}^0(A)$ is a semisimple algebra equipped with a positive involution. Such algebras are classified by Albert (see [22], pp. 201 or Appendix A.3). We have the following possibilities when $A$ is a simple abelian surface:
1. $\text{End}^0(A)$ is a totally real field which is fixed by the Rosati involution, and isomorphic either to $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{d})$ for $d$ a positive integer.

2. $\text{End}^0(A)$ is a CM field $K$ of degree four (degree two totally imaginary extension of a real quadratic number field $K_0$) and the Rosati involution coincides with complex conjugation on $K$ or equivalently, the action of $\text{Gal}(K/K_0)$.

3. $\text{End}^0(A)$ is an indefinite quaternion algebra over $\mathbb{Q}$ with center isomorphic to $\mathbb{Q}$. In this case, there is a natural anti-involution on $\text{End}^0(A)$, which we denote by $'$ coming from the structure of $\text{End}^0(A)$ as a quaternion algebra. There is an element $\lambda \in \text{End}^0(A)$ such that $\lambda' = -\lambda$ and $\lambda^2 < 0$ such that for any $\alpha \in \text{End}^0(A)$, we have $\alpha^o = \lambda \alpha' \lambda^{-1}$.

**Remark 2.3.1.** Note that these have $\mathbb{Q}$-dimension equal to some power of 2, and bounded by 8. The possible upper bound of $\dim(\text{End}^0(A)) = \dim_{\mathbb{Q}} M_4(\mathbb{Q})$ is not attained.

If we look at the situation where $A$ is not a simple abelian surface, we get slightly different results. In this situation, we have $A \cong_{\text{isog}} E_1 \times E_2$ where $E_1$ and $E_2$ are complex elliptic curves. It is well known that the possible endomorphism algebras of elliptic curves are either $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-d})$ for $d$ a positive squarefree integer. When the second case obtains, we say that the elliptic curve has complex multiplication (often abbreviated “CM”).

Since $E_1$ and $E_2$ are of dimension one, there is a non-constant morphism between them if and only if they are isogenous. Thus we split our analysis into two cases; where $E_1$ and $E_2$ are isogenous elliptic curves and when they are not. First, we will make some definitions.

Let $E_1$ and $E_2$ be isogenous and let $\phi$ represent an element of $\text{End}^0(E_1 \times E_2)$. We want to break $\phi$ up into a number of manageable components. Now if we look at $E_1 \times E_2$, we will restrict an endomorphism $\phi$ to the elliptic curves $E_1 \times \{0\}$ and $\{0\} \times E_2$, we get homomorphisms of algebraic groups,

$$\phi_{E_1} : E_1 \times \{0\} \longrightarrow E_1 \times E_2, \text{ and hence } \phi = \phi_{E_1} + \phi_{E_2}.$$
Therefore, to understand $\phi$, we must just understand $\phi_{E_i}$. We may further decompose this endomorphism by projecting onto the components $E_1$ and $E_2$. The projections $\pi_1$ and $\pi_2$ give us maps

$$\pi_i(\phi_{E_j}) : E_j \rightarrow E_i$$

and hence $\pi_1(\phi_{E_j}) \times \pi_2(\phi_{E_j}) = \phi_{E_j}$.

[Case 1.] Now if we have that $E_1$ and $E_2$ are not isogenous elliptic curves, then $\pi_i(\phi_{E_j}) = 0$ if $i \neq j$. Therefore, we have that $\phi = \pi_1(\phi_{E_1}) \times \pi_2(\phi_{E_2})$, and hence $\text{End}^0(E_1 \times E_2) = \text{End}^0(E_1) \times \text{End}^0(E_2)$. Let $E_1$ and $E_2$ be isogenous via some homomorphism $\eta$ with $\eta^{-1}$ its inverse in $\text{Hom}_Q(E_1, E_2)$. Then we will describe $\pi_i(\phi_{E_j})$ as follows. We write $\pi_1(\phi_{E_2}) = \theta_{12} \cdot \eta^{-1}$ and $\pi_2(\phi_{E_1}) = \eta \cdot \theta_{21}$ for $\theta_{12}$ and $\theta_{21} \in \text{End}^0(E_1)$. We may write $\pi_1(\phi_{E_1}) = \theta_{11} \in \text{End}^0(E_1)$ and we may write $\pi_2(\phi_{E_2}) = \eta \cdot \theta_{22} \cdot \eta^{-1}$ for $\theta_{22}$ in $\text{End}^0(E_1)$. Thus we see that

$$\phi(E_1 \times E_2) = (\pi_1(\phi_{E_1}) + \pi_1(\phi_{E_2})) \times (\pi_2(\phi_{E_1}) + \pi_2(\phi_{E_2}))$$

$$= (\theta_{11} + \theta_{12} \cdot \eta^{-1}) \times (\eta \cdot \theta_{21} + \eta \cdot \theta_{22} \cdot \eta^{-1}).$$

Hence we identify $M_2(\text{End}^0(E_1))$ with $\text{End}^0(E_1 \times E_2)$ by sending the matrix

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \in M_2(\text{End}^0(E_1))$$

to

$$\phi(E_1 \times E_2) = (\theta_{11} + \theta_{12} \cdot \eta^{-1}) \times (\eta \cdot \theta_{21} + \eta \cdot \theta_{22} \cdot \eta^{-1}).$$
2.4 The moduli space of abelian surfaces with endomorphism structure.

In this section we will describe classical construction of the moduli spaces of abelian surfaces with specific endomorphism structure. These moduli spaces are constructed for arbitrary abelian varieties by Shimura [27], and a description may also be found in [5].

2.4.1 Polarized abelian surfaces.

We wish to describe the moduli space of abelian surfaces without any structure beyond a polarization first of all. In particular, let us take $A$ to be an abelian surface with a fixed $k$-polarization. We can think of these objects formally as pairs $(A, E)$ where $A$ is an abelian surface and $E$ is a $k$-polarization on $A$.

Definition 2.4.1. We say that two $k$-polarized abelian surfaces $(A, E)$ and $(A', E')$ are isomorphic if there is an isomorphism

$$\varphi : A \rightarrow A'$$

of complex tori, and such that for any $x, y \in A$, we have $E'((\varphi(x), \varphi(y)) = E(x, y)$.

In other words, our complex isomorphism must preserve the polarizations. Then we have that the space of isomorphism classes of $k$-polarized abelian surfaces is just

$$A_2^k := \mathfrak{h}_2/Sp_4(k, \mathbb{Z}).$$
2.4.2 Abelian surfaces with real multiplication.

Recall that on any Galois extension $K$ of $\mathbb{Q}$, we have both a trace and norm, defined as follows. Let $\sigma$ denote an element of $\text{Gal}(K/\mathbb{Q})$, then

$$\text{Tr}_K(\alpha) = \sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sigma(\alpha)$$

and

$$\text{N}_K(\alpha) = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sigma(\alpha).$$

Let $K$ be a quadratic extension of $\mathbb{Q}$. All quadratic extensions of $\mathbb{Q}$ are Galois. If $\sigma$ is the non-trivial element of $\text{Gal}(K/\mathbb{Q})$, then for $\alpha \in K$, we will denote $\sigma(\alpha)$ by $\alpha'$.

**Definition 2.4.2.** Let $K$ be a quadratic extension of $\mathbb{Q}$. We say that $K$ is a real quadratic extension of $\mathbb{Q}$ if $K \cong \mathbb{Q}(\sqrt{d})$ for $d \in \mathbb{Z}_{>0}$. An order $\mathcal{O}$ in $K$ is a subring such that $\mathcal{O} \otimes \mathbb{Q} = K$, the trace and norm maps restricted to $\mathcal{O}$ have images in $\mathbb{Z}$ and such that $\mathcal{O}$ is a rank two $\mathbb{Z}$-module.

Every quadratic order may be written uniquely as

$$\mathbb{Z} \left[ \frac{\Delta + \sqrt{\Delta}}{2} \right]$$

for some positive integer $\Delta$ which is not a perfect square. If $\mathcal{O}$ may be written this way, then we will often refer to it as $\mathcal{O}_\Delta$.

For the rest of this section, we will follow [32] §1.

**Definition 2.4.3.** Let $A$ be an abelian surface. We say that $A$ has real multiplication by a quadratic order $\mathcal{O}_\Delta$ (or $\text{RM by } \mathcal{O}_\Delta$) if there is an embedding

$$\iota : \mathcal{O}_\Delta \otimes \mathbb{Q} \rightarrow \text{End}^0(A)$$
such that $\text{End}(A) \cap \iota(\mathcal{O}_\Delta \otimes \mathbb{Q}) = \iota(\mathcal{O}_\Delta)$.

We note that if $A$ has real multiplication by $\mathcal{O}_\Delta$, then there is a well-defined action of $\mathcal{O}_\Delta$ on $H^1(A, \mathbb{Z})$ which makes $H^1(A, \mathbb{Z})$ an $\mathcal{O}_\Delta$-module. It is shown in [32] 1.2 that as an $\mathcal{O}_\Delta$-module, $H^1(A, \mathbb{Z})$ is isomorphic to

$$\mathcal{O}_\Delta \oplus \mathcal{I}$$

where $\mathcal{I}$ is some fractional ideal of $\mathcal{O}_\Delta$. Furthermore, we have that the polarization on $A$ may be represented as follows. Let us take points $\lambda_1, \lambda_2 \in H^1(A, \mathbb{Z})$ identified with pairs $\lambda_i = (\alpha_i, \beta_i) \in \mathcal{O}_\Delta \oplus \mathcal{I}$. Then we have

$$E(\lambda_1, \lambda_2) = \text{Tr}(\alpha_1\beta_2 - \alpha_2\beta_1).$$

Now let us take triples $(A, \iota, E)$ of abelian surfaces with $\iota$ denoting real multiplication and $E$ a polarization induced by the trace form we have just indicated.

**Definition 2.4.4.** Two abelian surfaces with real multiplication by some quadratic order $\mathcal{O}_\Delta$, given by $(A, \iota, E)$ and $(A', \iota', E')$ are isomorphic as abelian surfaces with real multiplication by $\mathcal{O}_\Delta$ if there is some isomorphism of $k$-polarized abelian surfaces,

$$\varphi : A \longrightarrow A'$$

and we have that for all $\delta \in \mathcal{O}_\Delta$, we have $\varphi(\iota(\delta)(A)) = \iota'(\delta)(A')$.

Stated differently, let us take $\varphi^* : H^1(A', \mathbb{Z}) \longrightarrow H^1(A, \mathbb{Z})$ the pullback on cohomology. Then we if $\iota^*$ and $(\iota')^*$ represent the representations of $\mathcal{O}_\Delta$ on $H^1(A, \mathbb{Z})$ and $H^1(A', \mathbb{Z})$ discussed above, we have that $(A, \iota, E)$ and $(A', \iota', E')$ are isomorphic as abelian surfaces with real multiplication by $\mathcal{O}_\Delta$ if the map $\varphi^*$ is an isomorphism of representations between $(\iota')^*$ and $(\iota)^*$. 
Again, following [32], we have that the isomorphism classes of abelian surfaces with real multiplication by $\mathcal{O}_\Delta$ such that

$$H^1(A, \mathbb{Z}) \cong \mathcal{O}_\Delta \oplus \mathcal{I}$$

are parameterized by points in

$$\mathfrak{h}^2/PSL_2(\mathcal{O}_\Delta, \mathcal{I})$$

and we have

$$PSL_2(\mathcal{O}_\Delta, \mathcal{I}) := \left\{ \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} : \alpha, \gamma \in \mathcal{O}_\Delta, \delta \in \mathcal{I}^{-1}, \text{ and } \beta \in \mathcal{I} \right\} / \pm \text{Id.}$$

This group acts on $\mathfrak{h}^2$ as

$$\nu \cdot (z_1, z_2) \mapsto (\nu \cdot z_1, \nu' \cdot z_2).$$

Here $\nu \cdot z_i$ indicates a fractional linear transformation and $'$ indicates the Galois involution acting entry-wise on the matrix $\nu \in M_2(\mathbb{Q}(\sqrt{\Delta}))$.

There is a canonical involution on this moduli space provided by fixing abelian surface, but applying the Galois involution to $\iota$, which we will call $\iota'$, so that we have a new embedding

$$\iota' : \alpha \mapsto \iota(\alpha').$$

The action of $\iota$ on points in $\mathfrak{h}^2$ is given by the map

$$\sigma_{\text{Gal}} : (z_1, z_1) \mapsto \left( \frac{-N(\mathcal{I})}{z_2}, -\frac{N(\mathcal{I})}{z_1} \right)$$

where $N(\mathcal{I})$ is the norm of the ideal $\mathcal{I}$. The group generated by $\sigma_{\text{Gal}}$ and $PSL_2(\mathcal{O}_\Delta, \mathcal{I})$ is known as a symmetric Hilbert modular group, and we will denote it $PSL_2(\mathcal{O}_\Delta, \mathcal{I})^\text{sym}$, which we will unimaginatively call a symmetric Hilbert modular surface.
2.4.3 Abelian surfaces with quaternionic multiplication.

Recall the definition of a quaternion algebra.

**Definition 2.4.5.** An indefinite quaternion algebra over \( \mathbb{Q} \) is a dimension four algebra spanned by generators \( i, j \) and \( ij \), where

\[
i^2 = a, j^2 = b \quad \text{and} \quad ij = -ji.
\]

for \( a, b \in \mathbb{Z}_{>0} \). We will denote this algebra by

\[
\left( \frac{a, b}{\mathbb{Q}} \right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij.
\]

There is a canonical anti-involution on an indefinite quaternion algebra. It is denoted by ‘\( \dagger \)’ and defined on generators as

\[
i \mapsto -i, j \mapsto -j, \text{ and } ij \mapsto -ij.
\]

and it fixes \( \mathbb{Q} \). Note that for any \( \alpha \) in an indefinite quaternion algebra, \( \text{Tr}(\alpha) = \alpha + \alpha' \) and \( \text{N}(\alpha) = \alpha \alpha' \) are in \( \mathbb{Q} \).

**Definition 2.4.6.** A quaternion order \( \mathcal{O} \) of an indefinite quaternion algebra \( \mathbb{Q} \) is a subring of \( \mathbb{Q} \) which is also \( \mathbb{Z} \) submodule of \( \mathbb{Q} \). Furthermore, we must have that for every \( \alpha \in \mathcal{O} \), \( \text{Tr}(\alpha) \) and \( \text{N}(\alpha) \) are in \( \mathbb{Z} \), and that \( \mathcal{O} \otimes \mathbb{Q} = \mathbb{Q} \).

**Definition 2.4.7.** Let \( \mathcal{O} \) be an order in an indefinite quaternion algebra \( \mathbb{Q} \). A left ideal \( \mathcal{I} \) of \( \mathcal{O} \) is a \( \mathbb{Z} \)-submodule of \( \mathbb{Q} \) of rank four, which satisfies \( \mathcal{I} \otimes \mathbb{Q} = \mathbb{Q} \), and

\[
\alpha \cdot \beta \in \mathcal{I} \text{ for all } \alpha \in \mathcal{O} \text{ and } \beta \in \mathcal{I}.
\]

One defines both right ideals and two-sided ideals of \( \mathcal{O} \) in a similar manner.
Let $O$ now represent a quaternion order inside of the quaternion algebra $Q$.

**Definition 2.4.8.** Let $A$ be an abelian surface. We say that $A$ has quaternionic multiplication by the quaternion order $O$ (or $QM$ by $O$) if there is an embedding

$$\iota : O \otimes \mathbb{Q} \longrightarrow \text{End}^0(A)$$

such that $O \cong \text{End}(A) \cap \iota(O \otimes \mathbb{Q})$.

If $A$ has QM by $O$, then we have that $H^1(A, \mathbb{Z})$ is an $O$-module of rank one, and that we may represent this $O$-module as an ideal of $O$ in $Q$ which we will call $I$.

If $A$ has QM by $O$, then [27] shows that we may represent the polarization on $A$ as follows. Let $\text{Tr}$ be the standard trace form on $Q$, and $'$ the standard involution. Then there is an element $a \in Q$ such that $a' = -a$, and $a^2 < 0$ such that if we represent $\lambda_i \in H^1(A, \mathbb{Z})$ by elements $\delta_i \in I \subseteq Q$, then we have

$$E(\lambda_1, \lambda_2) = \text{Tr}(\delta_1 a \delta_2 a^{-1}) \in \mathbb{Z}.$$ 

The conditions under which two abelian surfaces are isomorphic are exactly the same as the conditions under which two abelian surfaces with RM are isomorphic (just replace the quadratic order $O_\Delta$ with $O$ a quaternion order).

**Definition 2.4.9.** Two abelian surfaces with quaternionic multiplication by some quaternion order $O$, given by $(A, \iota, E)$ and $(A', \iota', E')$ are isomorphic as abelian surfaces with quaternionic multiplication by $O$ if there is some isomorphism of $k$-polarized abelian surfaces,

$$\varphi : A \longrightarrow A'$$

and we have that for all $\delta \in O$, we have $\varphi(\iota(\delta)(A)) = \iota'(\delta)(A')$.

Recall Finally, we have by [27] that if we take $O$ a quaternion order and $I$ an $O$ ideal,
and $a$ an element of $\mathbb{Q}$ such that $a' = -a$ and $a^2 < 0$, we have that the isomorphism classes of abelian surfaces with $\mathcal{Q}$ by $\mathcal{O}$ such that $H^1(A, \mathbb{Z})$ is isomorphic to $\mathcal{I}$ as an $\mathcal{O}$-module, and such that the polarization $E$ on $A$ is given by the trace form constructed from $a$ as above are parameterized by the points in

$$\mathfrak{h}/\Gamma(\mathcal{O}),$$

where

$$\Gamma(\mathcal{O}) := \{\gamma \in \mathcal{O} : N(\gamma) = 1\}.$$

There is a standard faithful representation of $\mathcal{O}$ inside of $SL_2(\mathbb{R})$, and the group $\Gamma(\mathcal{O})$ acts on $\mathfrak{h}$ by fractional linear transformations via this representation. This representation is constructed in Example 5.1.1
Chapter 3

Algebras and Lattices

Chapter 3 and Chapter 4 present the technical core of our results. As mentioned in the introduction, many things proved here have been floating in the air for quite some time, but to the author’s knowledge, this is the first time most of them have been nailed down. The main theorem of this chapter is as follows

**Theorem 1.** If $A$ is an abelian surface with Néron-Severi rank $\leq 3$, and is neither

\[ A. \]

1. a simple abelian surface with complex multiplication nor

2. isogenous to a product of non-isogenous elliptic curves $E_1 \times E_2$ where at least one of $E_1$ or $E_2$ is a CM elliptic curve,

then we have

$$\text{End}(A) \cong C^+((1/2) \text{NS}(A)).$$

The reader may refer to Section 3.1.3 for the definition of $C^+((1/2) \text{NS}(A))$. Theorem 1 will imply that for an abelian surface satisfying the hypothesis of Theorem 1, the data of the Néron-Severi lattice and the endomorphism ring are equivalent.
3.1 Endomorphisms and the Néron-Severi lattice.

In this section, we want to study the relationship between the Néron-Severi group of an abelian surface and its endomorphism ring. First we will deduce from the classification of endomorphism algebras that except some minor cases, we have \( \text{End}^0(A) \) generated by Rosati-invariant elements. Second, we show that the subring of \( \text{End}^0(A) \) generated by Rosati invariant endomorphisms is isomorphic to the Clifford ring of the Néron-Severi group of \( A \).

3.1.1 Building endomorphisms from algebraic cycles.

**Definition 3.1.1.** Let \( A \) be an abelian surface, and let us fix an isomorphism of complex manifolds \( A \cong \mathbb{C}^2/\Lambda \), and let \( E \) be the polarization of \( A \) represented on \( \mathbb{C}^2 \). Then we define the dual of \( A \) to be \( (\text{Hom}_\mathbb{R}(\mathbb{C}^2, \mathbb{R}))^*/\Lambda^* \) where

\[
\Lambda^* := \{ \lambda \in (\text{Hom}_\mathbb{R}(\mathbb{C}^2, \mathbb{R}))^* : \lambda(\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Lambda \}.
\]

We will write the dual of \( A \) as \( \hat{A} \).

The following theorem is useful. Define \( \text{NS}^0(A) := \text{NS}(A) \otimes \mathbb{Q} \).

**Theorem 3.1.** Let us take \( \text{End}^0(A)^* \) to be the \( \mathbb{Q} \) sub-space of \( \text{End}^0(A) \) which is fixed pointwise by the Rosati involution. Then we have an isomorphism of vector spaces,

\[ \text{NS}^0(A) \cong \text{End}^0(A)^*. \]

Proof of this theorem may be found in [5]. This theorem is quite constructive. In the course of the next few pages, we will describe how one constructs endomorphisms of abelian surfaces from elements of \( \text{NS}^0(A) \).

If we are given \( \gamma \in \text{NS}^0(A) \), we have that, in particular, \( \gamma \in H^2(A, \mathbb{Z}) \cong (\Lambda^2 H_1(A, \mathbb{Z}))^* \).

Thus \( \gamma \) may be associated with some integral alternating two-form on \( H_2(A, \mathbb{Z}) \). Since
\( \gamma \in H^{1,1}(A) \), we may represent it in local coordinates as

\[
\gamma = \sum f_{ij}(z_1, z_2) dz_i \wedge d\bar{z}_j
\]

for \( C^\infty \) functions \( f_{ij}(z_1, z_2) \). Therefore one may check that if \( V \) and \( W \) are local vector fields on \( A \), we have

\[
\gamma(V, iW) = \gamma(W, iV)
\]

and hence \( E_\gamma(x, iy) \) is a symmetric bilinear form on \( \mathbb{C}^2 \) (the Lie algebra of \( A \)). Hence we have that

\[
H_\gamma(x, y) = E(x, iy) + iE(x, y)
\]

is a Hermitian form on \( \mathbb{C}^2 \). Therefore, to any element of \( \text{NS}^0(A) \), we get a map of complex tori

\[
\phi_\gamma : A \longrightarrow \tilde{A}
\]

where \( \mathbb{C}^2 \) is identified with its dual torus, via the map \( z \in \mathbb{C}^2 \mapsto E_\gamma(-, z) \in \text{Hom}_R(\mathbb{C}^2, \mathbb{R}) \). The space \( \text{Hom}_R(\mathbb{C}, \mathbb{R}) \) may be identified with \( \text{Hom}_R(\mathbb{C}^2, \mathbb{C}) \) by sending

\[
\text{Hom}_R(\mathbb{C}^2, \mathbb{R}) \ni \gamma \mapsto \gamma(i-) + i\gamma(-) \in \text{Hom}_R(\mathbb{C}^2, \mathbb{C}).
\]

Since \( E_\gamma(ix, y) \) is an alternating form, we have that \( H_\gamma(x, y) \) is a Hermitian form, therefore,

\[
H_\gamma(-, z)
\]

is actually in \( \text{Hom}_C(\mathbb{C}^2, \mathbb{C}) \). We have a map from \( \Lambda \) to \( \Lambda^* \) given by

\[
\alpha \mapsto E_\gamma(-, \alpha),
\]

which is in \( \Lambda^* \) by the integrality of \( E_\gamma \). If we associate this dual map with its complex
form coming from $H_{\gamma}$, we get a map of complex vector spaces and hence a holomorphic map of complex tori. Now what we would like to do is to construct the dual endomorphism associated to this morphism. In other words, once we have constructed an endomorphism $\phi : A \rightarrow B$ of abelian varieties there is a dual morphism $\hat{\phi} : \hat{B} \rightarrow \hat{A}$, which is induced by dualizing the morphisms of vector spaces and lattices. Therefore, given two elements $\delta, \gamma$ in $\text{NS}^0(A)$, we have an endomorphism,

$$\hat{\phi}_\delta \cdot \phi_{\gamma} : A \rightarrow A.$$  

Assume now that we have fixed the element $u_k = e_3 + kf_3$ in $H^2(A, \mathbb{Z})$ which induces the polarization $E$ on $H^1(A, \mathbb{Q})$, then we can define the map

$$\text{NS}^0(A) \rightarrow \text{End}^0(A)$$

via

$$\gamma \mapsto \hat{\phi}_{u_k} \cdot \phi_{\gamma}.$$  

Theorem 3.1 says that this is an isomorphism onto $\text{End}^0(A)^s$.

### 3.1.2 Concrete representations.

Finally, we will calculate $\phi_\delta$ and $\hat{\phi}_\delta$ as four by four matrices on a symplectic basis of $\Lambda$.

**Definition 3.1.2.** Let $A$ be an abelian surface with $k$-polarization $E$. Let us take the pair $(\mathbb{Z}^4, Q_k)$ where $Q_k$ represents an alternating bilinear form on $\mathbb{Z}$ such that on our standard basis $\{e_i\}$ of $\mathbb{Z}^4$, we have

$$[Q_k(e_i, e_j)] = J_k.$$  

We say that an isomorphism $\iota : \mathbb{Z}^4 \rightarrow H^1(A, \mathbb{Z})$ is an $H^1$-marking if we have $Q_k(e_i, e_j) = E(\iota(e_i), \iota(e_j))$. We will call the pair $(A, \iota)$ an $H^1$-marked abelian surface.
It is clear that any abelian surface admits an $H^1$-marking. By fixing an $H^1$-marking of $A$, we may take representations of $\tilde{\phi}_{u_k} \cdot \phi_\delta$, as four by four matrices acting on the space $\mathbb{Z}^4$. We will refer to the matrix associated to an element $\delta$ of $\text{NS}^0(A)$ as $T_\delta$. Similarly, $T_\delta$ may be represented as an element of $M_2(\mathbb{C})$. We will call this the analytic representation of $T_\delta$ or $(T_\delta)_{\text{an}}$.

We let $\delta = \sum_{i,j,i>j} a_{i,j} g_{i,j}$ where $a_{i,j} \in \mathbb{Q}$, then we fix a symplectic basis $\{e_1, \ldots, e_4\}$, then $H^1(A)$ is naturally associated with $H^1(A)^*$ via the natural bilinear form $\langle e_i, e_j \rangle = \delta_{i,j}$ where $\delta_{i,j}$ here denotes the Kronecker delta symbol. Then both $u_k$ and $\delta$ can be represented by maps on $\mathbb{Z}^4 \otimes \mathbb{R} \cong \mathbb{C}^2$ identified with its dual space. Since we have to each element of $\text{NS}(A)$ some alternating bilinear form, $E_\delta$, we find that $T_\delta$ is just the matrix satisfying on for a given symplectic basis of $H^1$,

$$E_\delta(x, y) = x^T T_\delta y.$$ 

If we fix the orientation in which $g_1 \wedge g_2 \wedge g_3 \wedge g_4$ is positive, and let $g_i \wedge g_j = g_{ij}$ as usual. Then if

$$\delta = a_{12} g_{12} + a_{13} g_{13} + a_{14} g_{14} + a_{23} g_{23} + a_{24} g_{24} + a_{34} g_{34},$$

then $T_\delta$ may be written as

$${T_\delta} := \begin{bmatrix} 0 & a_{34} & -a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & -a_{13} \\ a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & a_{13} & -a_{12} & 0 \end{bmatrix}.$$ 

It may be checked that $\det(T_\delta) = (\langle \delta, \delta \rangle / 2)^2$. As an example, if we have $u_k = g_{24} + kg_{13}$, ...
then

\[
T_{uk} = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -k \\
1 & 0 & 0 & 0 \\
0 & k & 0 & 0
\end{bmatrix} = J_k.
\]

Now we have the fact;

**Proposition 3.1.1** ([5] Prop. 1.2.6). Let \( A \) and \( B \) be abelian varieties and let \( \phi : A \to B \) be an isogeny. Then there exists a unique isogeny \( \tilde{\phi} : B \to A \) and some positive integer \( D \) such that

\[
\tilde{\phi} \cdot \phi = [D], \quad \text{and} \quad \phi \cdot \tilde{\phi} = [D]
\]

where \([D]\) is the multiplication by \( D \) map on \( A \) and \( B \) respectively.

For an abelian surface, and \( \phi_\delta \) as we have described above, \( D \) is actually just the degree of \( \phi_\delta \). We also have

**Proposition 3.1.2** ([5] 2.4.9). Let \( \delta \in \text{NS}(A) \) and \( \phi_\delta \) as above, and let \( H_\delta \) be the analytic matrix associated to \( \phi_\delta \). Then \( \text{deg}(\phi_\delta) = \text{det}(H_\delta) \).

Note that \( \text{det}(H_\delta) \) will just be \( \pm \sqrt{\text{det}(T_\delta)} \). Therefore, we have that for \( \gamma \) such that \( \langle \gamma, \gamma \rangle \neq 0 \), we have \( \tilde{\phi}_\delta \cdot \phi_\delta = [\langle \delta, \delta \rangle] \).

We may extend this construction to \( \gamma \) such that \( \langle \gamma, \gamma \rangle = 0 \) too, in this case, there is a non-unique way to construct \( \tilde{\phi}_\gamma \) such that \( \tilde{\phi}_\gamma \cdot \phi_\gamma = [0] \). To construct \( \tilde{\phi}_\gamma \), we will take

\[
\tilde{\phi} : \begin{bmatrix}
a_{34} & -a_{24} & a_{23} \\
-a_{34} & 0 & a_{14} \\
a_{24} & -a_{14} & 0 \\
-a_{23} & a_{13} & -a_{12} \\
\end{bmatrix} \to \begin{bmatrix}
0 & -a_{12} & -a_{13} & -a_{14} \\
a_{12} & 0 & -a_{23} & -a_{24} \\
a_{13} & a_{23} & 0 & -a_{34} \\
a_{14} & a_{24} & a_{34} & 0
\end{bmatrix}
\]

A computation shows that \( \tilde{T} \cdot T = (a_{24}a_{13} - a_{34}a_{12} + a_{14}a_{23})\text{Id}_4 \) for any integral matrix.
CHAPTER 3. ALGEBRAS AND LATTICES

$T \in A_4(\mathbb{Z})$ where

$$A_4(\mathbb{Z}) := \{ T \in M_4(\mathbb{Z}) : T^t = -T \}.$$ 

Therefore for any $\delta \in U^3$, we have $\tilde{T}_\delta \cdot T_\delta = (\langle \delta, \delta \rangle / 2)\text{Id}_4$. It is not hard to see that $\tilde{\cdot}$ is a linear map from $A_4(\mathbb{Z})$ to $A_4(\mathbb{Z})$ and therefore, $Q(T) := \tilde{T}_T$ a quadratic form on $A_4(\mathbb{Z})$. It canonically extends to a symmetric bilinear form given by

$$Q(\gamma, \delta) = \tilde{\phi}_\gamma \cdot \phi_\delta + \tilde{\phi}_\delta \cdot \phi_\gamma = \langle \gamma, \delta \rangle.$$ 

3.1.3 Clifford algebras.

The reference for this section is Fulton and Harris [9] Lecture 20. We recall that the Clifford algebra of a finite dimensional vector space $V$ over $\mathbb{Q}$ equipped with a $\mathbb{Q}$-bilinear form $B$ is the quotient of the tensor algebra of $V$ by ideal generated by the relations

$$v_1 \otimes \mathbb{Q} v_2 + v_2 \otimes \mathbb{Q} v_1 = 2B(v_1, v_2).$$ 

We denote this $\mathcal{C}(V, B)$ or by abuse of notation just $\mathcal{C}(V)$. If $e_1, \ldots, e_k$ span $V$ then $\mathcal{C}(V)$ is spanned by elements $e_1^{a_1} \cdot e_2^{a_2} \cdot \cdots \cdot e_k^{a_k}$ for $a_i \in \{0, 1\}$, where we are again abusing notation by writing multiplication as $v \cdot u$. We let $\mathcal{C}^+(V)$ be the subalgebra of $\mathcal{C}(V)$ spanned by the basis vectors $e_1^{a_1} \cdot e_2^{a_2} \cdot \cdots \cdot e_k^{a_k}$ where $\sum a_i \equiv 0 \mod 2$.

$\mathcal{C}(V)$ is equipped with a canonical involution $^*$ which satisfies

$$(e_1^{a_1} \cdot e_2^{a_2} \cdot \cdots \cdot e_k^{a_k})^* = e_k^{a_k} \cdot e_{k-1}^{a_{k-1}} \cdot \cdots \cdot e_1^{a_1}.$$ 

We define a second object that will be of use to us in the following chapter. Let $V$ be a $\mathbb{Q}$-vector space and $L$ be a lattice contained in $V$ such that $B$ restricts to $\frac{1}{2}\mathbb{Z}$ on $L$, and let $f_1, \ldots, f_k$ be a basis of $V$ which also spans $L$ as a $\mathbb{Z}$-module. Then we define $\mathcal{C}^+(L)$ to be the subring of $\mathcal{C}^+(V, B)$ spanned over $\mathbb{Z}$ by $f_1^{b_1} \cdot \cdots \cdot f_k^{b_k}$ with $\sum b_i \equiv 0 \mod 2$. We will call
this the even Clifford ring of $L$.

**Example 3.1.1.** Let us take the vector space $V$ to be a two-dimensional vector space over $\mathbb{Q}$ and a basis for $V$ given by $u_1, u_2$, such that

$$[B(u_1, u_2)] = \frac{1}{2} \begin{bmatrix} 2a & h \\ h & 2k \end{bmatrix}$$

for integers $a, h$ and $k$, and so that the bilinear form $B$ has signature $(1+, 1-)$ and that $\text{disc}(B) = -\Delta$ for some positive integer $\Delta$. Then we may calculate the algebra $\mathcal{C}^+(V, B)$. We have $\mathcal{C}^+(V, B)$ generated by $1$ and $u_1 \cdot u_2$ over $\mathbb{Q}$. Therefore, determining the structure of this algebra is just the same as determining the minimal polynomial of $u_1 \cdot u_2$. This relation must be quadratic since $(u_1 \cdot u_2)^2 \in \text{span}\{1, u_1 \cdot u_2\}$. We calculate;

$$(u_1 \cdot u_2)^2 = u_1 \cdot u_2(h - u_1 \cdot u_2)$$

$$= hu_1 \cdot u_2 - ak.$$  

Thus the minimal polynomial of $u_1 \cdot u_2$ is

$$x^2 - hx + ak = 0.$$  

Therefore, we have the isomorphism of rings,

$$\mathcal{C}^+(V, B) \cong \mathbb{Q} \left( \frac{h + \sqrt{\Delta}}{2} \right) = \mathbb{Q}(\sqrt{\Delta}).$$  

If we were to take the Clifford ring associated to the sublattice $L$ of $V$ generated by $u_1$ and $u_2$, then we get

$$\mathcal{C}^+(L) \cong \mathbb{Z} \left[ \frac{h + \sqrt{\Delta}}{2} \right].$$  

We call this ring the quadratic order of discriminant $\Delta$. The Galois group $\text{Gal}(\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q})$
acts on $\mathcal{C}^+(L)$ via this isomorphism. One sees that the action of the induced Galois action on $\mathcal{C}^+(L)$ agrees with the action of the main involution.

We may now prove the theorem;

**Theorem 3.2.** If $\mathcal{C}^+((1/2)\text{NS}^0(A))$ is the Clifford algebra generated by the vector space $\text{NS}^0(A)$ equipped with the inner product $(1/2)\langle \cdot, \cdot \rangle$, then if we let $E^0(A)$ be the sub-algebra of $\text{End}^0(A)$ generated $\text{End}^0(A)^*$, we have the isomorphism

$$\mathcal{C}^+((1/2)\text{NS}^0(A)) \cong E^0(A).$$

**Proof.** For any vector space $V$, we have that $\mathcal{C}^+(V)$ is generated by elements in degree two. Therefore, it suffices to define what our isomorphism does in degree two. We have the degree two elements of $\mathcal{C}^+((1/2)\text{NS}^0(A))$ given by $u \cdot v$ for $u$ and $v$ elements of $\text{NS}^0(A)$. We define the map that sends

$$\phi : u \cdot v \mapsto \hat{\phi}_u \cdot \phi_v,$$

and extends by linearity to a map on $\mathcal{C}^+((1/2)\text{NS}^0(A))$. We have further that any product of pairs of Néron-Severi elements is mapped as one would expect

$$\phi : \prod_i (u_i \cdot v_i) \mapsto \prod_i (\hat{\phi}_{u_i} \phi_{v_i}) \in \text{End}^0(A), \quad (3.1)$$

or in other words, composition of the endomorphisms associated to each pair. If this is well defined, then we have a homomorphism of algebras.

As we noted earlier, we have

$$\hat{\phi}_u \cdot \phi_v + \hat{\phi}_v \cdot \phi_u = \langle [u, v] \rangle.$$

Therefore, it follows that

$$\phi(u \cdot v + v \cdot u) = \langle [u, v] \rangle.$$
Thus the map $\phi$ is actually well-defined since this is the only relation between elements in $C^+((1/2) \text{NS}^0(A))$.

Finally we show that $\phi$ is injective. Recall that $\text{NS}(A)$ has signature $(1+, (\text{rank}(\text{NS}(A)) - 1)-)$. Standard calculations along the lines of Example 3.1.1 (see also Section 5.1 for the rank three case) show that the following cases arise for $A$ an abelian surface:

**Proposition 3.1.3.** Let $A$ be an abelian surface, then the following cases give the correspondence between $\text{NS}^0(A)$ and $C^+((1/2) \text{NS}^0(A))$.

- Case 1. $\text{NS}^0(A)$ is rank one and $C^+((1/2) \text{NS}^0(A))$ is just $\mathbb{Q}$, $\text{NS}^0(A)$ is rank two and $C^+((1/2) \text{NS}^0(A))$ is isomorphic to $\mathbb{Q}[x]/(x^2 + a)$ for $a < 0$ the discriminant of $\text{NS}(A)$ determined up to square. $\text{NS}^0(A)$ is rank three and $C^+((1/2) \text{NS}^0(A))$ is isomorphic to an indefinite quaternion algebra. $\text{NS}^0(A)$ is rank four and $C^+((1/2) \text{NS}^0(A))$ is isomorphic to $M_2(\mathbb{Q}(\sqrt{d}))$ (which is a quaternion algebra over $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}_{<0}$ is the discriminant of $\text{NS}(A)$).

Except in 2 when $-a$ is a square, these are simple algebras over $\mathbb{Q}$ and thus any non-trivial representation must be faithful. In the remaining case, the Rosati invariant elements actually span $E^0(A)$ and therefore, since $\phi$ injects $\text{NS}^0(A)$ into $\text{End}^0(A)$, nothing more needs to be said. □

In [31] pp.214, a similar statement is proved: the author shows that if $A$ is an abelian surface and $\text{NS}^0(A)$ has rank three, then we have $C^+((1/2) \text{NS}^0(A)) \cong \text{End}^0(A)$. Thus the preceding theorem may be thought of an extension of this.

### 3.1.4 Rosati-invariant endomorphisms.

Let $E^0(A) \subseteq \text{End}^0(A)$ be the sub-algebra of $\text{End}^0(A)$ generated by Rosati invariant endomorphisms. We will study when $E^0(A)$ is actually $\text{End}^0(A)$. 


Proposition 3.1.4. For a simple abelian surface without complex multiplication by a CM field of degree four over $\mathbb{Q}$, we have

$$E^0(A) = \text{End}^0(A).$$

4. Proof. This is a consequence of the Albert classification (see Appendix A.3) of involutive algebras. There are three different cases which we must treat (since the case where $\dim(\text{NS}^0(A)) = 1$ is trivial);

[Case 1.] $\text{End}^0(A)$ is a real number field of degree two : In this case it is known [27] that the Rosati-Involution acts as the identity, so the claim is trivial. $\text{End}^0(A)$ is an indefinite quaternion algebra over $\mathbb{Q}$. By Albert’s classification, we have that the Rosati involution on an indefinite quaternion algebra is given as follows. There is an element $a \in \text{End}^0(A)$ such that $a^2 < 0$ and $a' = -a$ (i.e. $a' = -a$) where $'$ is the standard involution on an indefinite quaternion algebra. Then for any $\beta \in \text{End}^0(A)$, we have the Rosati involution given by

$$\beta^o = a^{-1}a'. $$

Therefore $\beta^o = \beta$ is equivalent to $\text{Tr}(a\beta) = a\beta' + \beta a = a\beta' - \beta a = 0$. Since $\text{Tr}$ defines a non-degenerate bilinear form on $\text{End}^0(A)$, we may restate this as the fact,

$$\beta^o = \beta$$ is equivalent to $\beta \perp a$ with respect to $\text{Tr}$.

Since $\text{Tr}$ is non-degenerate, $a^1$ has dimension 3. Therefore, we must have $\dim(\text{NS}^0(A)) = 3$ as well by Theorem 3.1. By Theorem 3.2, we have $E^0(A) \cong C^+((1/2)\text{NS}^0(A))$ which has dimension four over $\mathbb{Q}$. Thus for reasons of dimension, $\text{End}^0(A) = E^0(A)$. $\text{End}^0(A)$ is isomorphic to $M_2(\mathbb{Q}(\sqrt{d}))$ where $d \in \mathbb{Z} < 0$: We see that the Rosati
involution $o$ acts on $\text{End}^0(A)$ via

$$
T = \begin{bmatrix}
\alpha & \beta \\
\delta & \gamma
\end{bmatrix} \mapsto 
\overline{T}' = \begin{bmatrix}
\overline{\alpha} & \overline{\delta} \\
\overline{\beta} & \overline{\gamma}
\end{bmatrix}.
$$

Thus the Rosati invariant elements are just matrices of the form

$$
\begin{bmatrix}
a & \overline{\beta} \\
\beta & b
\end{bmatrix}
$$

where $a, b \in \mathbb{Q}, \beta \in \mathbb{Q}(\sqrt{d})$.

This forms a four dimensional $\mathbb{Q}$-subspace of $\text{End}^0(A)$. By Theorem 3.1, this shows that $\text{NS}^0(A)$ must have rank four, and the algebra $E^0(A)$ is the even Clifford algebra of $\text{NS}^0(A)$, which has dimension eight over $\mathbb{Q}$. Thus for reasons of dimension, we have $E^0(A) \otimes \mathbb{Q} = \text{End}^0(A)$.

It is clear that the result does not hold if $\text{End}^0(A)$ is a quadratic CM extension $K$ of a real quadratic number field $K_0$. In this case, the Albert classification implies that the Rosati involution fixes just $K_0$, which is a subfield of $K$.

**Proposition 3.1.5.** If $A$ is isogenous to the product of two elliptic curves, then $E^0(A) = \text{End}^0(A)$ unless $A$ is isogenous to the product of two non-isogenous curves, at least one of which has complex multiplication.

3. **Proof.** Again, we go about exhausting the classification provided in Chapter 2. In the case where $A \cong_{\text{isog}} E_1 \times E_2$ with $E_1 \cong_{\text{isog}} E_2$, then if $E_1$ is a CM curve, this is taken care of in Case 3 of Proposition 3.1.4. If $A \cong_{\text{isog}} E_1 \times E_2$ and $E_1 \cong_{\text{isog}} E_2$ is not CM then $\text{End}^0(A) \cong M_2(\mathbb{Q})$ and the Rosati involution acts as transposition. Therefore the Rosati fixed part is just the three-dimensional space generated by self-adjoint matrices, which generates the entire space $M_2(\mathbb{Q})$. 
Table 3.1: The relationship between Néron-Severi groups and endomorphism algebras for abelian surfaces.

<table>
<thead>
<tr>
<th>rank(NS(A))</th>
<th>dim(\mathbb{Q}(\mathbb{C}^+((1/2) \text{NS}^0(A))))</th>
<th>End^0(A)</th>
<th>dim(\mathbb{Q}(\text{End}^0(A)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(\mathbb{Q})</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(\mathbb{Q}(\sqrt{\Delta}))</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\mathbb{Q} \times \mathbb{Q})</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\mathbb{Q}(\sqrt{\Delta})(\sqrt{\alpha}))</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\mathbb{Q} \times \mathbb{Q}(\sqrt{-d}))</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\mathbb{Q}(\sqrt{-d_1}) \times \mathbb{Q}(\sqrt{-d_2}))</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>(\begin{pmatrix} a &amp; b \ b &amp; -a \end{pmatrix})</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>(M_2(\mathbb{Q}(\sqrt{-d})))</td>
<td>8</td>
</tr>
</tbody>
</table>

If we have \(A \cong_{\text{isog}} E_1 \times E_2\) with neither \(E_1\) nor \(E_2\) CM and \(E_1\) and \(E_2\) non-isogenous, then the Rosati involution is trivial and the result is therefore trivially true. Finally, when \(A \cong_{\text{isog}} E_1 \times E_2\) with \(E_i\) non-isogenous and at least one of \(E_1\) or \(E_2\) a CM curve, the Rosati involution will act on each component of \(\mathbb{Q}(\sqrt{d_1}) \times \mathbb{Q}(\sqrt{d_1})\) or \(\mathbb{Q} \times \mathbb{Q}(\sqrt{d})\) separately. Since the Rosati involution acts as complex conjugation, it is clear that in both of the above cases, we do not have \(\text{End}^0(A)\) generated by its Rosati-invariant part.

Combining Propositions 3.1.3 and 3.1.4 with Theorem 3.2, we may organize our results into the theorem:

**Theorem 2.** Let \(A\) be an abelian surface such that neither of the following hold:

- \([A/\mathbb{Q}]\) is simple and \(\text{End}^0(A)\) is a CM field of rank four over \(\mathbb{Q}\), \(A\) is isogenous product of non-isogenous elliptic curves, at least one of which has complex multiplication.

Then \(\text{End}^0(A)\) is isomorphic to the even Clifford algebra of \(\text{NS}^0(A)\) equipped with half of the standard intersection form on \(H^2(A, \mathbb{Q})\).

We have summarized this data in Table 3.1 for convenient use.
3.2 Quadratic orders.

We now will look at the case of \( n = 2 \) and thus \( M \) is an even lattice of signature \((1+, 1-), \)
and let \( A \) be an abelian surface without CM and \( \text{NS}(A) \cong M \). From the previous section, we have \( \mathcal{C}^+((1/2) \text{NS}^0(A)) \cong \text{End}^0(A) \). The natural question that now arises is:

**Question.** Let \( A \) be an abelian surface without CM and with \( \text{rank}(\text{NS}(A)) = 2 \). Let \( \text{disc}(\text{NS}(A)) = -\Delta \). Do we have \( \text{End}(A) \cong \mathcal{O}_\Delta \cong \mathcal{C}^+((1/2) \text{NS}(A)) \)?

We will provide an affirmative answer to this question.

3.2.1 Singular relations.

Before proving the claim above, we will make a slight detour into singular relations and Humbert surfaces. By Proposition 2.2.1 we have the period domain associated to \( k \)-polarized abelian surfaces given by

\[
D(\mathcal{U}_k) = \{ [z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 : z_0z_1+z_2z_3-kz_4^2 = 0,
\]

\[
z_0\bar{z}_1 + z_1\bar{z}_0 + z_2\bar{z}_3 + z_3\bar{z}_2 - 2k|z_4| > 0 \}
\]

Another way to describe this is as the set of period points

\[
D(\mathcal{U}^3) := \{ \omega = z_0e_1 + z_1f_1 + z_2e_2 + z_3f_2 + z_4e_3 + z_5f_3 \in \mathbb{P}(\mathcal{U}^3 \otimes \mathbb{C}) : \}
\]

\[
\langle \omega, \omega \rangle = z_0z_1 + z_2z_3 + z_4z_5 = 0 \text{ and } \langle \omega, \overline{\omega} \rangle > 0 \}
\]

which also satisfy \( \langle \omega, e_3 + kf_3 \rangle = 0 \). We see that this is equivalent to \( z_5 = -kz_4 \), thus one deduces the first expression for \( D(\mathcal{U}_k) \). Furthermore, if we have some arbitrary \( \lambda \in \mathcal{U}^3 \) such that

\[
\lambda = te_1 + bf_1 + ce_2 + df_2 + hf_3
\]
then the subvariety of $\mathcal{D}(u_k^\perp)$ satisfying $\langle \omega, \lambda \rangle = 0$ is just the subvariety defined by the intersection of $\mathcal{D}(u_k^\perp)$ and the hyperplane

$$bz_1 + tz_0 + cz_3 + dz_2 + hz_4 = 0 \in \mathbb{P}^4.$$  \hfill (3.2)

The domain $\mathcal{D}(u_k^\perp)$ has precisely two connected components. One maps $\mathfrak{h}_2$ into $\mathcal{D}(u_k^\perp)$ via the equations

$$\wedge^2 : \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \mapsto \left[ k : \tau_2^2 - \tau_1 \tau_3 : \tau_3 : k \tau_1 : \tau_2 \right].$$  \hfill (3.3)

This map is obtained by taking the second wedge power of the columns of

$$\begin{bmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & k & \tau_2 & \tau_3 \end{bmatrix},$$

or in other words, this is the Grassman embedding of the period point into $\mathbb{P}^4$. By [24] Lemma 4.1.1, this provides a biholomorphism between $\mathfrak{h}_2$ and one of the two connected components of $\mathcal{D}(u_k^\perp)$.

This map agrees with the period map of abelian surfaces associating an abelian surface to its period point. The period matrix of an abelian surface is a point in the Grassmannian $G(2, 4)$, the classifying space of Hodge structures on $H^1(A, \mathbb{Z})$. Since the Grassman embedding is essentially the second wedge power of period of $A$ in this case, its image corresponds to the periods of the Hodge structure on $\bigwedge^2 H^1(A, \mathbb{Z})$. Since $A$ is an abelian surface, this is exactly $H^2(A, \mathbb{Z})$.

An abelian surface is said to satisfy a singular relation $\Delta > 0$ if there are integers $t, b, c, d, h$ such that $\gcd(t, b, c, d, h) = 1$ and $tk + b(\tau_1 \tau_3 - \tau_2^2) - c\tau_3 + dk\tau_1 + h\tau_2 = 0$ where $\Delta = h^2 - 4k(bt - cd)$. We note that this is the same as saying that the period point
associated to $A$ is inside the rational hyperplane

$$tkx_0 + bx_1 + cx_2 + dkx_3 + hx_4 = 0.$$  

**Proposition 3.2.1.** Let $A$ be a $k$-polarized abelian surface. Then the period point satisfies a singular relation with discriminant $\Delta > 0$ if and only if there is a lattice $M$ embedded primitively inside of $\text{NS}(A)$ with $\text{disc}(M) = -\Delta$ and the image of $N$ in $\text{NS}(A)$ contains $u_k$.

*Proof.* If $A$ has period point satisfying the singular relation

$$tk + b(\tau_1 \tau_3 - \tau_2^2) - c\tau_3 + dk\tau_1 + h\tau_2 = 0$$

then by Equation 3.2.1 we have that the point $\omega \in \mathbb{P}^4$ representing $H^{2,0}(X)$ satisfies

$$\langle \omega, de_2 + be_1 - cf_2 + hf_3 + tf_1 \rangle = 0,$$

and thus we have $v := de_2 + be_1 - cf_2 + hf_3 + tf_1 \in \text{NS}(A)$ for any $A$ satisfying the singular relation $tk + b(\tau_1 \tau_3 - \tau_2^2) - c\tau_3 + dk\tau_1 + h\tau_2 = 0$. We see that the lattice $M = \text{span}_\mathbb{Z}\{u_k,v\}$ is primitively embedded in $\text{NS}(A)$ since the coefficients of $v$ are coprime. Then one may check that the Gram matrix of $M$ with respect to this basis is

$$\begin{bmatrix} 2(bt - cd) & h \\ h & 2k \end{bmatrix}$$

and hence that $\text{disc}(M) = -\Delta$. Conversely, if we have a primitive embedding of some lattice $M$ into $U^3$ containing $u_k$, we may assume that $M$ is embedded via our standard embedding (see Equations 2.2.1 and 2.2.2), and therefore we have $v = e_2 + af_2 + hf_3$, and therefore

$$\langle \omega, e_2 + af_2 + hf_3 \rangle = 0,$$
and thus $A$ satisfies the singular relation $k\tau_1 + h\tau_2 + a\tau_2 = 0$, which has discriminant $-\text{disc}(M)$.

**Remark 3.2.1.** If we take all points satisfying singular relations with discriminant $\Delta$, then the image in $\mathfrak{h}_2/\text{Sp}_4(k, \mathbb{Z})$ is known as the Humbert surface of discriminant $\Delta$, and is denoted $H_\Delta$. See [31] Chapter IX or Section 3.1.4 for more details.

### 3.2.2 Relation to the endomorphism algebra.

Summing up, for an abelian surface $A$, we have the correspondences:

\[
\{\text{Singular relations}\} \leftrightarrow \{\text{Elements of NS}(A)\} \leftrightarrow \{\text{Rosati-invariant endomorphisms}\}
\]

The first of these two correspondences is just Proposition 3.2.1. The second comes from Proposition 3.2.2 below.

We finally show that not only do we have a Rosati invariant endomorphism of $A$ coming from singular relations, but we actually have RM by $\mathcal{O}_\Delta$ coming from singular relations with discriminant $\Delta$. A calculation shows that if we have $A$ with polarization by $u_k$, and we have some

\[
v = \sum a_{ij}g_{ij}
\]

primitively embedded inside of NS$(A)$ so that $\text{span}_\mathbb{Z}\{u_k, v\}$ is a rank two lattice of signature $(1-, 1+)$, then we have

\[
T_v := (\hat{\phi}_{u_k} \cdot \phi_v)_{\text{rat}} = \begin{bmatrix}
ka_{24} & -ka_{14} & 0 & ka_{12} \\
-a_{23} & a_{13} & -a_{12} & 0 \\
0 & -ka_{34} & ka_{24} & -ka_{23} \\
a_{34} & 0 & -a_{14} & a_{13}
\end{bmatrix}.
\]  

If we take $v$ to be in the normal form provided in Section 2.2.1, then we get $v = e_2 + af_2 + kf_3$, 

and

\[
T_v = \begin{bmatrix}
0 & -ka & 0 & 0 \\
-1 & -h & 0 & 0 \\
0 & 0 & 0 & -k \\
0 & 0 & -a & -h
\end{bmatrix}.
\]

A convenient way for us to look at the endomorphism ring of an abelian surface \( A \) is just to take the rational representations of \( \text{End}^0(A) \) inside \( M_4(\mathbb{Q}) \), and restrict our attention to those which are actually contained in \( M_4(\mathbb{Z}) \). Let \( A_\tau \) be some abelian surface whose period point \( \tau \) only satisfies one singular relation. Then it is clear that if we take the algebra generated by Rosati invariant endomorphisms, which in this case is just \( \mathbb{Q}(T_v) \cong E_0(A_\tau) \), then we have that \( M_4(\mathbb{Z}) \cap E_0(A) = \mathbb{Z}[T_v] \). We have proved;

**Proposition 3.2.2.** Let \( A \) be an abelian surface. If \( A \) satisfies a singular relation with discriminant \( \Delta \), then there is a primitive embedding of \( \mathcal{O}_\Delta \) into \( \text{End}(A) \). Conversely, if we have that \( A_\tau \) is an abelian surface with RM by \( \mathcal{O}_\Delta \) with period point \( \tau \), then \( \tau \) satisfies some singular relation.

### 3.3 Quaternion orders.

If we now take \( A_\tau \) to be an abelian surface carrying an embedding of the lattice \( N \) of rank three and signature \((1+, 2-)\) into its Néron-Severi group and whose image includes the polarization element \( u_k \). First of all, we will extend \( u_k \) to a basis of \( N \), which we will denote \( u_k, v, w \), and we will assume that we have a Gram matrix

\[
\begin{bmatrix}
2a_1 & b & h_1 \\
b & 2a_2 & h_2 \\
h_1 & h_2 & 2k
\end{bmatrix}.
\]
where the first row is associated to \( v \), the second is associated to \( w \) and the third is associated with \( u_k \).

By Lemma 2.2.1, we have both \( M_1 := \text{span}_\mathbb{Z}\{u_k, v\} \) and \( M_2 := \text{span}_\mathbb{Z}\{u_k, w\} \) giving us primitive lattice embeddings of signature \((1+, 1-)\) lattices into \( \text{NS}(A) \). Therefore if \( A_\tau \) carries such a polarization, then \( \tau \) must satisfy two different singular relations, and admit primitive ring embeddings

\[
\mathbb{Z}[T_u] \hookrightarrow \text{End}(A) \quad \text{and} \quad \mathbb{Z}[T_v] \hookrightarrow \text{End}(A).
\]

whose images will span the space of Rosati invariant elements of \( \text{End}^0(A) \).

We have the following;

**Proposition 3.3.1.** Let \( A \) be an abelian surface with Néron-Severi rank three and \( \text{NS}(A) \cong N \). Then we have that \( M_4(\mathbb{Z}) \cap \text{End}^0(A) = \mathbb{Z}[T_u, T_v, \frac{T_u T_v}{k}] \), or in other words, \( C^+((1/2)N) \cong \text{End}(A) \).

**Proof.** In fact, we check the second claim, though the first one is equivalent. We use the basis for \( N \) given in the form of Lemma 2.2.2. We have that \( C^+ (N) \) is generated by \((\hat{\phi}_{u_k} \cdot \phi_v)_{\text{rat}}, (\hat{\phi}_{u_k} \cdot \phi_w)_{\text{rat}}, (\hat{\phi}_w \cdot \phi_v)_{\text{rat}} \) and the identity. One calculates, that these are written as

\[
\alpha = \begin{bmatrix} -h_2 & -kb & 0 & kc_1 \\ 0 & 0 & -c_1 & 0 \\ 0 & -kc_2 & -h_2 & 0 \\ c_2 & 0 & -b & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & -ka_1 & 0 & 0 \\ -1 & -h_1 & -0 & 0 \\ 0 & 0 & 0 & -k \\ 0 & 0 & -a_1 & -h_1 \end{bmatrix}
\]

and

\[
\gamma = \begin{bmatrix} 0 & 0 & a_1 c_2 & -h_1 c_2 \\ h_1 & -a_1 b - h_1 h_2 & 0 & -c_2 \\ -c_1 & h_1 c_1 & -a_1 b & h_1 b \\ 0 & a_1 c_1 & a_1 h_2 & h_1 h_2 \end{bmatrix}.
\]
Now our goal will be to show that the sublattice of $M_4(\mathbb{Z})$ spanned by $\alpha, \beta$ and $\gamma$ is a primitive sublattice, or in other words, that there are no elements $\mu$ in this sublattice and $t \in \mathbb{Z}_{\neq \pm 1}$ such that $\mu = t\theta$ for $\theta \in M_4(\mathbb{Z})$ and $\theta \notin \text{span}_{\mathbb{Z}}\{\alpha, \beta, \gamma\}$. First, if we use the facts that $\gcd(a_1, h_1) = 1$ and $\gcd(c_1, c_2, h_2, b) = 1$, then we may show without much difficulty that none of $\alpha, \beta$ or $\gamma$ are integral multiples of integral matrices. Furthermore, it may be shown using the same facts that the lattices spanned by any pair of elements in $\alpha, \beta, \gamma$ is primitive in $M_4(\mathbb{Z})$.

These two steps are not obvious, however they are elementary, and for that reason, we omit their proof. Finally, assume that we have coprime integers $r_1, r_2, r_3$ such that

$$\mu = r_1\alpha + r_2\beta + r_3\gamma, \text{ and } \mu = t\theta.$$
and it follows that $t = 1$. This completes the proof.

We may now gather our results into the

**Theorem 3.** Let $A$ be an abelian surface with Néron-Severi rank equal to 1, 2 or 3, and such that the abelian surface $A$ is not CM. Then

$$\text{End}(A) = \mathcal{C}^+((1/2)\text{NS}(A)).$$
Chapter 4

Markings and Polarizations.

This chapter will be devoted to proving two results. Let $SO_0(N^\perp, \mathbb{Z})$ be the subgroup of $SO(U^3, \mathbb{Z})$ made up of $\gamma$ such that $\gamma|_N = \text{Id}_N$ for some primitive sublattice $N$ of $U^3$, and such that $\gamma$ fixes the two components of $\mathcal{D}(N^\perp)$. Then we have the following

**Theorem 4.** If $N$ is a rank three lattice with signature $(1+, 2-)$, then we have

$$SO_0(N^\perp, \mathbb{Z}) \cong \Gamma(\mathcal{C}^+((1/2)N)),$$

where

$$\Gamma(\mathcal{C}^+((1/2)N)) := \{ \gamma \in \mathcal{C}^+((1/2)N) : N(\gamma) = 1 \}/ \pm \text{Id}.$$

We also have

**Theorem 5.** If $N$ is a rank two lattice with signature $(1+, 1-)$, then if $\text{disc}(N) = -\Delta$ and is not a square, we have

$$SO_0(N^\perp, \mathbb{Z}) \cong PSL_2(\mathcal{C}^+((1/2)N), \mathcal{I}).$$

for $\mathcal{I}$ some fractional ideal of $\mathcal{C}^+((1/2)N)$ considered as a quadratic order.
CHAPTER 4.  MARKINGS AND POLARIZATIONS.

4.1 Some definitions.

We want to develop the relationship between the moduli of marked lattice polarized abelian surfaces and marked endomorphism polarized abelian surfaces. First, we will say a little bit about what these things are.

A marking on the cohomology of a manifold is just a fixed isomorphism from a free abelian group to a cohomology group of the manifold. For instance,

**Definition 4.1.1.** An $H^1$-marking on an abelian surface is a map $\iota: \mathbb{Z}^4 \to H^1(A, \mathbb{Z})$. If $A$ is $k$-polarized as well, we may take $\mathbb{Z}^4$ equipped with an inner product, then we may say that an $H^1$-marking is an isometry $\iota: \mathbb{Z}^4 \to H^1(A, \mathbb{Z})$ where $H^1(A, \mathbb{Z})$ is equipped with its usual polarization.

Now we will worry about pairs $(A, \iota)$ for $A$ an abelian surface with a fixed $k$-polarization and $\iota$ an $H^1$-marking of $A$.

**Definition 4.1.2.** Two $H^1$-marked abelian surfaces are isomorphic if there is an isomorphism of $k$-polarized abelian surfaces $\varphi: A \to A'$ such that $\varphi^\ast \cdot \iota = \iota'$ where $\varphi^\ast$ is the pullback on $H^1$.

Similarly, we may define an $H^2$-marked abelian surface.

**Definition 4.1.3.** An $H^2$-marking on an abelian surface will be an isometry $j: U^3 \to H^2(A, \mathbb{Z})$ where $H^2(A, \mathbb{Z})$ is equipped with the usual cup product pairing. Two pairs $(A, j)$ and $(A', j')$ are isomorphic as $H^2$-marked polarized abelian surfaces if there is an isomorphism of polarized abelian surfaces $\varphi: A \to A'$ such that $\varphi^\ast \cdot j = j'$ where $\varphi^\ast$ is the pullback on $H^2$.

4.1.1 Moduli of marked polarized surfaces.

Classically (see [5] 8.1) the space of isomorphism classes of $H^1$-marked $k$-polarized abelian surfaces is just the Siegel upper half-space $\mathfrak{H}_2$. If we are given an $H^1$-marking, then if we
take $L$, then following the exposition at the beginning of Chapter 2, $\bigwedge^2 H^1(A) = H^2(A)$, and thus $\iota : L \tilde{\longrightarrow} H^1(A, \mathbb{Z})$ induces an isometry $\bigwedge^2 L \tilde{\longrightarrow} H^2(A, \mathbb{Z})$. Thus to each $H^1$-marked abelian surface, we may associate an $H^2$-marked abelian surface in a canonical way.

If $A$ carries a $k$-polarization on $H^1(A)$, then the discussion in Section 2.1.1 shows that $\text{NS}(A)$ contains $u_k = e_3 + kf_3$ as usual. Thus we have the moduli of $k$-polarized abelian surfaces carrying an $H^2$-marking is just $D(u_k^\perp)$. Associated to the $H^2$ polarization induced by an $H^1$-polarization on a $k$-polarized abelian surface, there is a holomorphic map from $\mathfrak{h}_2$ to $D(u_k)$, which is just the wedge product map that we discussed in Section 3.2.1.

**Remark 4.1.1.** We point out that the period spaces $D(N^\perp)$ all have two connected components (corresponding to different choices of orientation) which we will write as $D(N^\perp)^+$ and $D(N^\perp)^−$. In the future, we will just be interested in $D(N^\perp)^+$.

### 4.1.2 Lattice polarizations.

**Definition 4.1.4.** A $k$-polarized abelian surface is said to be lattice polarized by a lattice $N$ if there is a primitive embedding $\kappa : N \hookrightarrow \text{NS}(A)$ and such that the image of $\kappa$ contains $u_k$, the ample class on $A$ which provides our polarization.

**Remark 4.1.2.** This is identical to the definition of a lattice polarized algebraic K3 surface.

If we have $H^2$-marked lattice polarized abelian surfaces $(A, \kappa)$ and $(A', \kappa')$ then we say that these form isomorphic pairs if there is an isomorphism $\varphi : A \tilde{\longrightarrow} A'$ of marked abelian surfaces such that $\varphi^* \cdot \kappa = \kappa'$. In other words, the embedding of $\kappa$ into $U^3$ determined by the marking is fixed under the isomorphism on cohomology induced by the isomorphism of abelian surfaces.
4.1.3 Endomorphism polarizations.

**Definition 4.1.5.** Let $A$ be an abelian surface with $H^1$-marking and $k$-polarization, and let us take a ring $\mathcal{R}$. We say that $A$ is $\mathcal{R}$-polarized if there is an embedding

$$\theta : \mathcal{R} \hookrightarrow \text{End}(A)$$

such that if we take the induced embedding

$$\theta^0 : \mathcal{R} \otimes \mathbb{Q} \hookrightarrow \text{End}^0(A)$$

then $\theta^0(\mathcal{R} \otimes \mathbb{Q}) \cap \text{End}(A) = \mathcal{R}$ (this is analogous to a primitive lattice embedding).

We may be clear about what it will mean for an abelian surface to be both $H^1$-marked and $\mathcal{R}$-polarized. If $A$ is $H^1$-polarized, then we have chosen an isomorphism $\mathbb{Z}^4 \cong H_1(A, \mathbb{Z})$. Hence if we have $A$ also $\mathcal{R}$-polarized, we get a representation of $\mathcal{R}$ in $M_4(\mathbb{Z})$ by allowing $\mathcal{R}$ to act on $\mathbb{Z}^4$ identified with $H^1(A, \mathbb{Z})$.

4.1.4 Relations between definitions.

We investigate how $N$ polarizations relate to endomorphism polarizations by $C^+((1/2)N)$. If we have $A$ an $N$-polarized abelian surface for $N$ of rank one, two or three, then there is a natural $C^+((1/2)N)$-polarization as described in Chapter 2: if $u, v \in N$ and $\iota$ is our lattice polarization, then $u \cdot v$ is sent to the endomorphism $\widetilde{\phi}_{\iota(u)} \cdot \phi_{\iota(v)}$. Now if we have an isomorphism of abelian surfaces

$$\varphi : A \cong B$$

then any $N$-polarization $\iota$ on $B$ induces an $N$-polarization on $A$ via the pullback isomorphism

$$\varphi^* : \text{NS}(B) \cong \text{NS}(A).$$
Similarly, if we have an endomorphism polarization on $B$ coming from $\iota$, we get an an endomorphism polarization on $A$,

$$
\iota_\varphi : C^+((1/2)N) \ni u \cdot v \mapsto \varphi^{-1} \cdot (\hat{\phi}_u \cdot \phi_v) \cdot \varphi.
$$

If $\hat{\varphi}$ is the dual isomorphism of $\varphi$,

$$
\hat{\varphi} : B \xrightarrow{\sim} A
$$

then we may write

$$
\iota_\varphi(u \cdot v) = (\varphi^{-1} \cdot \hat{\phi}_i(u) \cdot \hat{\varphi}^{-1}) \cdot (\hat{\varphi} \cdot \phi_i(v) \cdot \varphi).
$$

Recalling that we have defined

$$
\phi_i(v) : x \mapsto H_i(v)(x, -)
$$

where $H_i(v)$ is the Hermitian form on $C^2$. Then for $x$ in $C^2$, we have

$$
\hat{\varphi} \cdot \phi_i(v)(x) \cdot \varphi = \hat{\varphi} \cdot H_i(v)(\varphi(x), -) = H_i(v)(\varphi(x), \varphi(-)) = H_{\varphi^*(i(v))}(x, -) = \phi_{\varphi^*(i(v))}(x).
$$

Since we must have $\varphi^{-1} = \hat{\varphi}$, we may write

$$
\varphi^{-1} \cdot \hat{\phi}_i(u) \cdot \hat{\varphi}^{-1} = \hat{\varphi} \cdot \phi_i(v) \cdot \varphi = \phi_{\varphi^*(i(v))}.
$$

We have proved the following;

**Proposition 4.1.1.** Let $A$ and $B$ be isomorphic abelian surfaces, and let $\varphi$ be an isomorphism. If $B$ is $N$-polarized via some primitive embedding $\iota$, then $\varphi^*(\iota)$ gives an $N$-polarization on $A$, and there is an induced $C^+((1/2)N)$-polarization on $A$ given by

$$
C^+((1/2)N) \ni u \cdot v \mapsto \hat{\phi}_{\varphi^*(\iota(u))} \cdot \phi_{\varphi^*(\iota(v))} \in \text{End}(A)
$$
which agrees with the composition of the natural endomorphism polarization of $B$ and the isomorphism of endomorphism rings coming from $\varphi$.

Now let us take $(A, \iota_A)$ and $(B, \iota_B)$ to be $N$-polarized abelian surfaces. Then they are both $C^+((1/2)N)$-polarized in the manner described above. Let $\varphi : A \rightarrow B$ be an isomorphism of $k$-polarized abelian surfaces. When will $\varphi$ induce an isomorphism of $C^+((1/2)N)$-polarized abelian surfaces? According to the definition and Proposition 4.1.1, this is the same thing as saying that for every $u, v \in N$,

$$\widehat{\varphi}_{\iota_A}(u) \cdot \widehat{\varphi}_{\iota_A}(v) = \widehat{\varphi}_{\varphi^*(\iota_B(u))} \cdot \varphi^*(\iota_B(v)).$$

(4.1)

If we have $\iota_A(N) \subseteq \varphi^*(\iota_B(N))$, then there is an induced automorphism $\widetilde{\varphi}$ of $N$ coming from

$$\widetilde{\varphi} = \iota_A^{-1} \cdot \varphi^* \cdot \iota_B.$$

An equivalent version of Equation 4.1.1 is that for every $u$ and $v$ in $N$, we have

$$\widetilde{\varphi}(u) \cdot \widetilde{\varphi}(v) = u \cdot v.$$

**Proposition 4.1.2.** Let $\varphi : A \rightarrow B$ be an isomorphism of abelian surfaces. If $(A, \iota_A)$ and $(B, \iota_B)$ are $N$-polarized abelian surfaces, and $\iota_A(N) \subseteq \varphi^*(\iota_B(N))$, then $A$ and $B$ are isomorphic as $C^+((1/2)N)$-polarized abelian surfaces if and only if

$$\widetilde{\varphi}(u) \cdot \widetilde{\varphi}(v) = u \cdot v.$$

Where $\widetilde{\varphi}$ and the endomorphism polarizations on $A$ and $B$ are as above.
4.2 Some group isomorphisms.

We now wish to interpret the groups $PSL_2(\mathcal{O}, \mathcal{I})$ and $\Gamma(\mathcal{O})$ as subgroups of orthogonal groups.

**Definition 4.2.1.** Let $N$ be some lattice with signature $(1+, n-)$ for $n \leq 2$ and equipped with a primitive embedding into $U^3$. We will define the group

$$SO_0(N^\perp, \mathbb{Z}) := \{ \gamma \in SO(U^3, \mathbb{Z}) : \gamma|_N = Id \}$$

Let $N$ be as above, then we define the group

$$SO_0(N, u_k^\perp, \mathbb{Z}) := \{ \gamma \in SO_0(u_k^\perp, \mathbb{Z}) : \gamma(N) \subseteq N\}.$$

In other words, $SO_0(N^\perp, \mathbb{Z})$ the subgroup of $SO_0(N, u_k^\perp, \mathbb{Z})$ which restricts to the identity on $N$.

Note that $SO_0(N, u_k^\perp, \mathbb{Z})$ is the subgroup of $SO_0(u_k^\perp, \mathbb{Z})$ which stabilizes the subvariety $D^+(N^\perp) \subseteq D^+(u_k^\perp)$.

In particular, we have the following isomorphism, thanks to [24].

**Lemma 4.2.1.** We have a group isomorphism,

$$\Lambda^2 : Sp_4(k, \mathbb{Z})/ \pm Id_4 \longrightarrow SO_0(u_k^\perp, \mathbb{Z}).$$

Furthermore, this isomorphism is equivariant with respect to the actions of $Sp_4(k, \mathbb{Z})$ on $\mathfrak{h}_2$ and of $SO_0(u_k^\perp, \mathbb{Z})$ on $D^+(u_k^\perp)$. In other words, if we have the isomorphism

$$\Lambda^2 : \mathfrak{h}_2 \longrightarrow D^+(u_k^\perp)$$
as given in Equation 3.2.2. then for \( \tau \in \mathfrak{h}_2 \) and \( \gamma \in \text{Sp}_4(k, \mathbb{Z}) \), we have
\[
\wedge^2 (\gamma \cdot \tau) = \wedge^2 (\gamma) \cdot \wedge^2 (\tau).
\]
and an isomorphism of complex analytic spaces,
\[
\mathfrak{h}_2 / \text{Sp}_4(k, \mathbb{Z}) \cong D^+(u_k^+) / \text{SO}_0(u_k^+, \mathbb{Z}).
\]

We will use this isomorphism along with Proposition 4.1.2 to prove the main theorems of this chapter.

### 4.2.1 Hilbert modular groups.

In this subsection, we will let \( N \) be a rank two lattice of signature \((1+, 1-)\) and with an embedding into \( U^3 \) such that the image of \( N \) contains \( u_k = e_3 + kf_3 \). Then the space of marked abelian surfaces with \( N \)-polarization corresponding to the fixed embedding of \( N \) into \( U^3 \cong H^2(A, \mathbb{Z}) \) is the space \( D^+(N^+) \), or, as we have discussed at length already, this may be considered as the space of abelian surfaces \( A_\tau \) where \( \tau \) is a point in \( \mathfrak{h}_2 \) satisfying the singular relation
\[
L_N : k\tau_1 + h\tau_2 + a\tau_3 = 0
\]
for some integers \( h \) and \( a \) such that
\[
N = \begin{bmatrix}
2a & h \\
h & 2k
\end{bmatrix}.
\]
Let us take \( \Gamma(N) \) to be the group such that the moduli space of \( C^+((1/2)N) \)-polarized abelian surfaces is
\[
\mathfrak{h}_2 / \Gamma(N), \text{ where } \mathfrak{h}_2 = \mathfrak{h} \times \mathfrak{h}.
\]
If $N$ has $\text{disc}(N) = -\Delta$ where $\Delta$ is not a perfect square, then this is a Hilbert modular group for $\mathcal{I}$ some fractional ideal of the order $\mathcal{O}_\Delta$. We will determine in Chapter 5 the exact nature of the ideal associated to a given lattice $N$, but we will now content ourselves with knowing that if $A$ is $k$-polarized and $N$-polarized, then there is some ideal depending only on $N$ and $k$ such that as an $\mathcal{O}_\Delta$-module, we have

$$H^1(A, \mathbb{Z}) \cong \mathcal{O}_\Delta \oplus \mathcal{I}_N.$$ 

A similar statement may be made in the case where $\Delta$ is a perfect square (see Section 5.2 for a precise description).

**Remark 4.2.1.** Perhaps it is not without merit to mention that the notation “$\Gamma(N)$” may be found in the literature denoting level $N$ subgroups of the modular group $\text{PSL}_2(\mathbb{Z})$. We hope that the reader is not upset or confused by this choice of notation.

**Proposition 4.2.1.** Let $\gamma \in \text{SO}_0(N, u_k^\perp, \mathbb{Z})$. If $\gamma(\omega) = \omega'$, then $\gamma$ induces an isomorphism between $N$-polarized abelian surfaces $A_\omega$ and $A_{\omega'}$. The isomorphism $\gamma$ acts the identity automorphism on $\text{C}^+((1/2)N)$ under endomorphism polarization if and only if $\gamma \in \text{SO}_0(N^\perp, \mathbb{Z})$.

**Proof.** There is a natural map $\phi : \text{SO}_0(N, u_k^\perp, \mathbb{Z}) \longrightarrow O(N, \mathbb{Z})$ obtained by taking the restriction of $\gamma \in \text{SO}_0(N, u_k^\perp, \mathbb{Z})$ to $N$. The kernel of this map is exactly $\text{SO}_0(N^\perp, \mathbb{Z})$. Furthermore, the image of $\phi$ is contained inside of the subgroup of $O(N, \mathbb{Z})$ made up of elements which fix $u_k$. A calculation shows that if we have a matrix

$$\eta^t \begin{bmatrix} 2a & h \\ h & 2k \end{bmatrix} \eta = \begin{bmatrix} 2a & h \\ h & 2k \end{bmatrix}$$
and \( \eta(u_k) = u_k \), then \( \gamma \) must be of the form

\[
\eta := \begin{bmatrix}
-1 & 0 \\
h/k & 1
\end{bmatrix}.
\]

Thus the group of \( \eta \in O(N, \mathbb{Z}) \) which fixes \( u_k \) has order two if \( k|h \), and is trivial otherwise.

Now if we have that the automorphism of \( N \) induced by \( \gamma \) is equivalent to \( \eta \), then by Proposition 4.2.1, we may calculate how this acts on \( C^+((1/2)N) \). If \( N \) is spanned by vectors \( u_k = e_3 + kf_3 \) and \( v \in U^3 \cong H^2(A, \mathbb{Z}) \) we obtain

\[
 u_k \cdot v \mapsto u_k \cdot (-v + hu_k/k) = -u_k \cdot v + h = v \cdot u_k.
\]

This agrees with the main involution on \( C^+((1/2)N) \), which is non-trivial. Therefore we have that if \( \gamma \) fixes \( C^+((1/2)N) \), then \( \gamma \) must actually be the identity on \( N \).

This allows us to prove the

**Theorem 4.1.** Let \( N \) be a rank two lattice of signature \((1+, 1-)\) with a primitive embedding into \( U^3 \) whose image contains \( u_k = e_3 + kf_3 \). Then we have the isomorphism,

\[
\Gamma(N) \cong SO_0(N^\perp, \mathbb{Z})
\]

induced by \( \wedge^2 \). Furthermore, this isomorphism induces an isomorphism of orbifolds,

\[
h^2/\Gamma(N) \cong D^+(N^\perp)/SO_0(N^\perp, \mathbb{Z}).
\]

**Proof.** There is a forgetful morphism which takes the moduli space of \( k \)-polarized abelian surfaces with RM by \( C^+((1/2)N) \) to the moduli space of \( k \)-polarized abelian surfaces, and
whose image is a finite number of components, each of which consists of a map

\[ \mathfrak{h}^2 \hookrightarrow \mathfrak{h}_2 \]

whose image is a component of some Humbert surface \( H_\Delta \), and a group embedding

\[ \Gamma(N) \hookrightarrow \text{Sp}_4(k,\mathbb{Z})/\text{Id}_4 \]

whose image is contained inside of the stabilizer \( H_\Delta \). Furthermore, these embeddings are equivariant with respect to the actions of \( \Gamma(N) \) on \( \mathfrak{h}^2 \) and the action of \( \text{Sp}_4(k,\mathbb{Z}) \) on \( \mathfrak{h}_2 \) (such embeddings will be constructed in Chapter 5). By definition, the group \( \Gamma(N) \) is the group which preserves \( C^+((1/2)N) \)-markings, or in other words, induces the identity on \( C^+((1/2)N) \) representations. By Proposition 4.2.1, this is isomorphic to the group of \( \text{SO}_0(N^\perp,\mathbb{Z}) \). This proves the theorem.

\[ \square \]

### 4.2.2 Shimura curves.

We will now pull ourselves up by our bootstraps to prove a similar result in the case where \( N \) is a rank three lattice with signature \((1+,2-)\). In this situation, if \( N \) is provided with an embedding \( \iota : N \hookrightarrow U^3 \) such that \( \iota(N) \) contains \( u_k \), then using Lemma 2.2.1, we see that there is a basis of \( \iota(N) \) given by \( u_k, v, w \) such that both \( M_1 := \text{span}\mathbb{Z}\{u_k, v\} \) and \( M_2 := \text{span}\mathbb{Z}\{u_k, w\} \) are rank two lattices of signature \((1+,2-)\) primitively embedded inside of \( U^3 \). Therefore, we have that \( A \) is a rank three polarized abelian surface, then we have that the period point of \( A \) satisfies two different singular relations represented by the intersection of \( D^+(u_k^+) \) and hyperplanes \( L_1 \) and \( L_2 \) in \( \mathbb{P}^4 \).

**Proposition 4.2.2.** Let us take \( \gamma \in \text{SO}_0(u_k^+,\mathbb{Z}) \). We see that \( \gamma|_N = \text{Id} \) if and only if \( \gamma|_{M_1} = \text{Id} \) and \( \gamma|_{M_2} = \text{Id} \). In other words, \( \text{SO}_0(N^\perp,\mathbb{Z}) = \text{SO}_0(M_1^+,\mathbb{Z}) \cap \text{SO}_0(M_2,\mathbb{Z}) \).

**Proof.** It is clear that if \( \gamma \) is to fix \( N \), then it must also restrict to the identity on \( M_1 \) and
CHAPTER 4. MARKINGS AND POLARIZATIONS.

Let \( A_\omega \) be the \( H^2 \)-marked abelian surface represented by the point \( \omega \in D^+(u_k^+) \). If we have the period point \( \omega \) perpendicular to \( M_1 \) and \( \omega \) perpendicular to \( M_2 \), then \( A_\omega \) admits polarization by \( N \), and furthermore, \( A_\omega \) admits endomorphism polarization by \( C^+((1/2)N) \).

In particular, we have injections,

\[
C^+((1/2)M_i) \hookrightarrow C^+((1/2)N) \hookrightarrow \text{End}(A_\omega)
\]

for \( i = 1, 2 \). Therefore, \( A_\omega \) admits \( C^+((1/2)M_1) \) and \( C^+((1/2)M_2) \)-polarization as well. We have that

**Proposition 4.2.3.** Let \( \phi \) be a ring automorphism of \( C^+((1/2)N) \). Then \( \phi \) induces the identity on \( C^+((1/2)M_1) \) and \( C^+((1/2)M_2) \) if and only if \( \phi \) is the identity.

**Proof.** It is certainly necessary that if \( \phi \) is the identity, it acts as the identity on \( C^+((1/2)M_i) \) for \( i = 1, 2 \). Sufficiency follows from the fact that we have that

\[
C^+((1/2)N) = \text{span}_\mathbb{Z}\{1, u_k \cdot v, w \cdot u_k, w \cdot v = \frac{w \cdot u_k \cdot v}{k}\}.
\]

If \( \phi \) fixes \( C^+((1/2)M_1) \), then \( \phi \) restricts to the identity on \( u_k \cdot v \), and if \( \phi \) fixes \( C^+((1/2)M_2) \), it must fix \( w \cdot u_k \). If \( \phi \) fixes both, then it must fix the product

\[
w \cdot v = \frac{w \cdot u_k \cdot v}{k}
\]

as well. \( \Box \)

Combining these two propositions with Proposition 3.5.3, we see that the subgroup of \( SO_0(u_k^+, \mathbb{Z}) \) made up of lattice automorphisms which fix \( N \) are exactly the lattice automorphisms which induce the identity map on \( C^+((1/2)N) \). Therefore, \( \gamma \) preserves lattice
polarization if and only if it preserves endomorphism polarization. Finally, we have

**Proposition 4.2.4.** There is an embedding of $\Gamma(\mathcal{C}^+((1/2)N))$ into the group of elements of $Sp_4(k, \mathbb{Z})$ which preserve endomorphism polarization by $\mathcal{C}^+((1/2)N)$ whose image under $\bigwedge^2$ is $SO_0(N^\perp, \mathbb{Z})$.

**Proof.** There is a forgetful map of moduli spaces

$$\mathfrak{h}/\Gamma(\mathcal{C}^+((1/2)N)) \hookrightarrow \mathfrak{h}_2/Sp_4(k, \mathbb{Z})$$

which sends an endomorphism polarized, $k$-polarized abelian surface to its underlying $k$-polarized abelian surface. This is composed of both a group homomorphism

$$\phi : \Gamma(\mathcal{C}^+((1/2)N)) \hookrightarrow Sp_4(k, \mathbb{Z})$$

and an embedding

$$\tilde{\phi} : \mathfrak{h} \hookrightarrow \mathfrak{h}_2$$

which is compatible with the actions of the groups $\Gamma(\mathcal{C}^+((1/2)N))$ and $Sp_4(k, \mathbb{Z})$ on either side. Since both $\mathcal{D}^+(N^\perp)$ and $\mathfrak{h}$ are connected, we must have the image of $\tilde{\phi}$ the preimage of $\mathcal{D}^+(N^\perp)$, and the image of $\phi$ must therefore lie in the stabilizer of $\mathcal{D}^+(N^\perp)$. The embedding $\tilde{\phi}$ is constructed in Chapter 6, and more details on the embedding $\phi$ may be found in [31] pp.89.

Therefore, we have that $\Gamma(\mathcal{C}^+((1/2)N))$, the largest group whose action on $\mathfrak{h}$ induces isomorphisms on the associated $\mathcal{C}^+((1/2)N)$-marked abelian surfaces is embedded inside of $SO_0(N^\perp, \mathbb{Z})$, and furthermore, that any $\gamma \in SO_0(N^\perp, \mathbb{Z})$ induces an isomorphism of $\mathcal{C}^+((1/2)N)$-polarized abelian surfaces. Thus we have $\Gamma(\mathcal{C}^+((1/2)N)) \cong SO_0(N^\perp, \mathbb{Z})$.

We sum up with the following theorem

**Theorem 4.2.** We have two isomorphisms of groups.
1. If $N$ is a rank two lattice of signature $(1+,1-)$ embedded primitively in $U^3$, then

$$\Gamma(N) \cong SO_0(N^\perp,\mathbb{Z})$$

2. If $N$ is a rank three lattice of signature $(1+,2-)$ embedded primitively in $U^3$, then

$$\Gamma(C^+((1/2)N)) \cong SO_0(N^\perp,\mathbb{Z}).$$

Furthermore, in these two cases we have the groups $O_0(N^\perp,\mathbb{Z})$ either isomorphic to $SO_0(N^\perp,\mathbb{Z})$ or an extension of degree two.

### 4.2.3 The results of Gritsenko and Hulek.

It seems quite necessary at this point that we review briefly the results obtained by [10] on the moduli spaces of $E_8^2 \oplus (2k)$-polarized K3 surfaces, or alternatively, $(2k)$-polarized abelian surfaces. They prove the following theorem

**Theorem 4.3** ([10] Prop. 1.4). Let $N_k := E_8^2 \oplus (2k)$. Then the moduli space of $N$-polarized abelian surfaces is isomorphic to

$$D^+(u_k^\perp)/O_0(u_k^\perp,\mathbb{Z}) \cong \mathfrak{h}_2/(Sp_4(k,\mathbb{Z}) \cup \sigma_{GH}Sp_4(k,\mathbb{Z})).$$

Here $\sigma_{GH} \in M_4(\mathbb{Q})$ has order two and is given by a matrix such that $(-\wedge^2\sigma_{GH})$ is in $O_0(u_k^\perp,\mathbb{Z})$ and has determinant $-1$. Furthermore, if $A_\tau$ is the abelian surface given by $\tau \in \mathfrak{h}_2$, then $A_{\sigma_{GH}(\tau)} = \widehat{A}_\tau$ is the dual abelian surface of $A_\tau$. See [10] for a precise description of $\sigma_{GH}$.

Therefore, if $M$ is some even lattice of signature $(1+,n-)$ for $n \leq 2$, we have by Theorems 3.2 and 3.3 that the moduli space of $E_8^2 \oplus M$-polarized K3 surfaces is isomorphic to the quotient by an involution of the moduli space of abelian surfaces with $C^+((1/2)N)$
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<table>
<thead>
<tr>
<th>rank($N$)</th>
<th>rank($N \oplus E_8^\vee$)</th>
<th>Moduli of $N \oplus E_8^\vee$-polarized K3 surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>$h_2/(Sp_4(k, \mathbb{Z}) \cup \sigma_{GH}Sp_4(k, \mathbb{Z}))$</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>$h^2/(PSL_2(C^+((1/2)N), \mathcal{I}<em>N) \cup \sigma</em>{GH}</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>$h/(\Gamma(C^+((1/2)N)) \cup \sigma_{GH}</td>
</tr>
</tbody>
</table>

Table 4.1: Moduli spaces of lattice polarized K3 surfaces with Shioda-Inose structure.

endomorphism structure. By Section 2.4, these are just quotients of Shimura curves or Hilbert modular surfaces or Siegel modular threefolds. We summarize this in the table. Let $A$ be an abelian surface and $X_A$ be the corresponding K3 surface such that $T(X_A)$ is Hodge isometric to $T(A)$ where we have denoted by $\sigma_{GH}|_N$ the restriction of $\sigma_{GH}$ to $\mathcal{D}^+(N^\perp)$. 
Part II

Computation
Chapter 5

Rank Two Lattices and Quadratic Orders

This chapter is dedicated to putting the theorems proved in the first two chapters to work. We will explore the consequences of what we have proved in detail in the cases where $N$ has rank two. These cases correspond to endomorphism embeddings by quadratic orders which have a rich arithmetic theory. We give an arithmetic characterization of the groups $O_0(N^\perp, \mathbb{Z})$ for $N$ a rank two lattice of signature $(1^+, 1^-)$, and we determine embeddings of the moduli spaces of rank 18 polarized K3 surfaces with Shioda-Inose structure into the moduli spaces of rank 17 polarized K3 surfaces with Shioda-Inose structure.

5.1 Quadratic orders and binary lattices.

In this section we will explore a bit the specific case where $N$ is of rank two and the discriminant of $N$ is non-square. Specifically, we define $N$ to be a lattice such that its
Gram matrix with respect to a basis $u_k, v$ looks like

$$
\begin{bmatrix}
2a & h \\
h & 2k
\end{bmatrix}.
$$

In this case, our moduli spaces are Hilbert modular surfaces. We will give a precise description of the Hilbert modular group associated to $N$-polarized abelian surfaces, and relate this to some classical arithmetic results. We will also discuss how one goes about embedding the moduli space of $N$-polarized abelian surfaces carrying $k$-polarization inside the moduli space of abelian surfaces with $k$-polarization, which, as we mentioned in Section 2.4, is isomorphic to

$$
\mathfrak{h}_2/Sp_4(k, \mathbb{Z}) = A_k.
$$

5.1.1 $H^1$-lattices.

Henceforward, we will refer to the quadratic order associated to the lattice $N$ as $\mathcal{O}$. Since $\mathcal{O}$ is a Dedekind domain, one has a nice description of its projective modules as the fractional ideals of $\mathcal{O}$ in $\mathbb{Q}(-\sqrt{\text{disc}(N)})$. Therefore following [32] §1, we see that any rank two free $\mathcal{O}$-module is isomorphic to

$$
\mathcal{O} \oplus \mathcal{I}
$$

as a module for $\mathcal{I}$ some fractional ideal of $\mathcal{O}$.

If $A$ carries an $N$-polarization, we have a free action of $\mathcal{O}$ on $H^1(A, \mathbb{Z})$, and thus we may imagine that $H^1(A, \mathbb{Z})$ is isomorphic as an $\mathcal{O}$-module to $\mathcal{O} \oplus \mathcal{I}$ for some fractional ideal $\mathcal{I}$. To avoid possible confusion, we will often refer to this module as $\mathcal{I}_N$, if there is ever any confusion as to which lattice $\mathcal{I}$ belongs to.

A natural question to ask now would be how one may assign to a given lattice an order and ideal. The first question we have already answered; by Proposition 3.2.2, this is just the Clifford ring associated to the lattice $N$. Example 3.1.1 shows that this is the quadratic
order

\[ \mathcal{O}_\Delta = \mathbb{Z} \left[ \frac{h + \sqrt{\Delta}}{2} \right], \]

where \( \Delta = - \text{disc}(N) \). Again, from Equation 3.2.3, we see that we have a representation of \( \mathbb{Z} \left[ \frac{h + \sqrt{\Delta}}{2} \right] \) on a symplectic basis of \( A \) given by the assignment,

\[
\frac{h + \sqrt{\Delta}}{2} \mapsto \begin{bmatrix}
0 & -ka & 0 & 0 \\
1 & h & 0 & 0 \\
0 & 0 & 0 & k \\
0 & 0 & -a & h
\end{bmatrix}.
\]

This gives us our module structure. Concretely, we may calculate that if we take the module spanned by \( \mathcal{O} \oplus \mathcal{I}_N \) where \( \mathcal{I}_N \) is the \( \mathbb{Z} \)-module spanned by

\[
\alpha := \frac{k}{\sqrt{\Delta}}, \text{ and } \beta := \frac{1}{2} + \frac{h}{2\sqrt{\Delta}}
\]

then we have

\[
\left( \frac{h + \sqrt{\Delta}}{2} \right) \left( \frac{k}{\sqrt{\Delta}} \right) = k \left( \frac{1}{2} + \frac{h}{2\sqrt{\Delta}} \right)
\]

and

\[
\left( \frac{h + \sqrt{\Delta}}{2} \right) \left( \frac{1}{2} + \frac{h}{2\sqrt{\Delta}} \right) = h \left( \frac{1}{2} + \frac{h}{\sqrt{\Delta}} \right) - a \left( \frac{k}{\sqrt{\Delta}} \right).
\]

Thus, as a \( \mathbb{Z} \left[ \frac{h - \sqrt{\Delta}}{2} \right] \)-representation, \( \mathcal{I} \), is acted on via the matrix

\[
\begin{bmatrix}
0 & ka \\
-a & h
\end{bmatrix}.
\]

In a similar way, we see that if we take basis \(- \left( \frac{h - \sqrt{\Delta}}{2} \right)\) and 1 of \( \mathbb{Z} \left[ \frac{h + \sqrt{\Delta}}{2} \right] \), then we
have
\[
\left( \frac{h + \sqrt{\Delta}}{2} \right) = h - \left( \frac{h - \sqrt{\Delta}}{2} \right)
\]
and we have that
\[
\left( \frac{h + \sqrt{\Delta}}{2} \right) \left( -h + \sqrt{\Delta} \right) = -ak.
\]
and as a \( O \)-representation, \( O \) looks like
\[
\frac{h + \sqrt{\Delta}}{2} \mapsto \begin{bmatrix} 0 & -ak \\ 1 & h \end{bmatrix}.
\]
Thus, as a \( O \)-representation, \( O \oplus I \) is isomorphic to \( H^1(A, \mathbb{Z}) \). Furthermore, on this basis, we may recover the form \( E \) arithmetically, by assigning
\[
E((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \text{Tr}(\alpha_1\beta_1 - \alpha_2\beta_2)
\]
which makes sense since we began with a symplectic basis for \( H^1(A, \mathbb{Z}) \).

### 5.1.2 Ideals and quadratic forms.

There is a classical correspondence between the space of binary quadratic forms of discriminant \( -\Delta \) up to \( SL_2(\mathbb{Z}) \) equivalence and elements in the ideal class group of the quadratic order \( \mathcal{O}_\Delta \). Following [37] Teil II, §10, we may construct this correspondence very concretely. Namely, if we are given an ideal in \( \text{Pic}(\mathcal{O}_\Delta) \), then one may generate a corresponding quadratic form. This is the norm form on the ideal (considered as a rank two \( \mathbb{Z} \)-module) divided by the norm of the ideal, or written in equations, we have for any \( \eta \in \mathcal{I}_N \),
\[
Q(\eta) = \frac{N(\eta)}{N(\mathcal{I})}
\]
Now let us calculate the norm form associated to the ideal $\mathcal{I}_N$ which we have calculated above.

First of all, we calculate that

$$
\begin{bmatrix}
\frac{1}{h + \sqrt{\Delta}} \\
\frac{1}{2}
\end{bmatrix}
= T \cdot \begin{bmatrix}
\frac{k}{\sqrt{\Delta}} \\
\frac{h}{2} + \frac{k}{\sqrt{\Delta}}
\end{bmatrix},
$$

where

$$
T = \begin{bmatrix}
2 & -\frac{h}{k} \\
\frac{h}{\Delta} & \frac{h^2}{2k}
\end{bmatrix}.
$$

Thus $N(\mathcal{I}_N) = \det(T) = \frac{k}{\Delta}$. Secondly, we may calculate the norms of products of elements. Upon doing this, we find

$$
N(\alpha, \alpha) = \frac{k^2}{\Delta},
$$

$$
N(\beta, \beta) = \frac{k\alpha}{\Delta},
$$

$$
N(\beta, \alpha) = \alpha\beta' + \beta\alpha' = \frac{k\hbar}{\Delta}.
$$

Hence the quadratic form we come up with is just

$$
\begin{bmatrix}
2a & h \\
h & 2k
\end{bmatrix}.
$$

Thus the lattice canonically associated to $\mathcal{I}_N$ is just $N$ itself.

**Proposition 5.1.1.** Let $A$ be an abelian surface of Néron-Severi rank two, and let $\Delta$ be the discriminant of the Néron-Severi lattice, and assume $\Delta$ is not a perfect square. Then as an $\mathcal{O}_\Delta$-module, $H^1(A, \mathbb{Z})$ is isomorphic to $\mathcal{O}_\Delta \oplus \mathcal{I}_N$, with $\mathcal{I}_N$ spanned by $\alpha$ and $\beta$ as above. For two lattices $N_1$ and $N_2$ of discriminant $\Delta$, the ideals $\mathcal{I}_{N_1}$ and $\mathcal{I}_{N_2}$ are in the same ideal
class if and only if $N_1$ and $N_2$ are isomorphic as binary quadratic forms.

Proof. The first part of the claim has already been proved. The second part follows from the classical fact cited earlier, that two ideals are in the same ideal class modulo principal ideals if and only if the associated quadratic forms are conjugate modulo $SL_2(\mathbb{Z})$.

Note that this construction is invertible; if we have a fractional ideal $\mathcal{I}$ of $\mathcal{O}_\Delta$ then $\mathcal{O}_\Delta \oplus \mathcal{I}$ carries a canonical polarization coming from the trace form above, and one recovers the lattice polarization just by taking the norm form on $\mathcal{I}$. Thus if $A$ is an abelian surface with RM by $\mathcal{O}_\Delta$, then if we have that $H^1(A, \mathbb{Z})$ is an $\mathcal{O}_\Delta$ module isomorphic to

$$\mathcal{O}_\Delta \oplus \mathcal{I}$$

for some fractional ideal $\mathcal{I}$ of $\mathcal{O}_\Delta$, then one may determine the structure of the Néron-Severi lattice by calculating the form $Q$ on $\mathcal{I}$.

5.1.3 Modular groups.

Now we turn to a geometric application of these ideas. It is well known (see Van der Geer [31] Chapter IX) that the moduli space of $\mathcal{O}$-polarized abelian surfaces carrying $k$-polarization and such that $H^1(A, \mathbb{Z}) \cong \mathcal{O} \oplus \mathcal{I}$ is isomorphic to the (uncompactified) Hilbert modular surface

$$Y(N) := \mathbb{H}^2/PSL_2(\mathcal{O}_\Delta, \mathcal{I}_N)$$

where

$$PSL_2(\mathcal{O}_\Delta, \mathcal{I}_N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \in SL_2(\mathbb{Q}(\sqrt{\Delta}) : \alpha, \gamma \in \mathcal{O}_\Delta, \beta \in \mathcal{I}^{-1}_N, \delta \in \mathcal{I}_N \right\} / \pm \text{Id}$$
acts on \( h^2 \) acts via fractional linear transformation; if \( \cdot \)' represents the natural Galois involution on \( \mathcal{O} \), then \( \alpha \in PSL_2(\mathcal{O}_\Delta, \mathcal{I}_N) \) acts as,

\[
\alpha \cdot (z_1, z_2) \mapsto (\alpha \cdot z_1, \alpha' \cdot z_2).
\]

By Theorem 4.1, \( PSL_2(\mathcal{O}, \mathcal{I}) \cong SO_0(N^\perp, \mathbb{Z}) \). Since if two lattices \( N \) of rank two and signature \( (1+, 1-) \) embedded in \( U^3 \) are isomorphic if and only if there is an element of \( O(U^3, \mathbb{Z}) \) under which they are conjugate (see Theorem 2.6), if \( N \cong N' \), then we must have \( SO_0(N, \mathbb{Z}) \cong SO_0(N', \mathbb{Z}) \) and hence we prove the well-known fact ([31], pp. 12) \( PSL_2(\mathcal{O}, \mathcal{I}_{N_1}) \cong PSL_2(\mathcal{O}, \mathcal{I}_{N_2}) \) if \( \mathcal{I}_{N_1} \) and \( \mathcal{I}_{N_2} \) represent the same ideal class of \( \mathcal{O} \).

### 5.2 Embedding into Siegel threefolds.

There is a natural forgetful map from the moduli space of abelian surfaces with real multiplication by \( \mathcal{O} \) into the moduli space of \( k \)-polarized abelian surfaces. Such a functor induces a map

\[
\psi : h^2 / PSL_2(\mathcal{O}, \mathcal{I}_N) \rightarrow \mathbb{A}^k_2.
\]

This map is not necessarily injective.

**Proposition 5.2.1.** The map \( \psi \) is a \( 2 : 1 \) map onto its image if and only if \( k \) divides \( h \), and otherwise it is injective.

**Proof.** We have an isomorphism between groups \( SO_0(N^\perp, \mathbb{Z}) \) and \( PSL_2(\mathcal{O}_\Delta, \mathcal{I}_N) \) and isomorphisms between \( Sp_4(k, \mathbb{Z}) \) and \( SO_0(u_k^+, \mathbb{Z}) \), and the embedding \( \psi \) is the same as the embedding

\[
\tilde{\psi} : \mathcal{D}^+(N^\perp) / SO_0(N^\perp, \mathbb{Z}) \rightarrow \mathcal{D}^+(u_k^+) / SO_0(u_k^+, \mathbb{Z}).
\]

The group \( SO_0(N^\perp, \mathbb{Z}) \) embeds naturally into the stabilizer of \( \mathcal{D}^+(N^\perp) \subseteq \mathcal{D}^+(u_k^+) \), Thus the question that we must answer is when the stabilizer of \( \mathcal{D}^+(N^\perp) \) in \( SO_0(u_k^+, \mathbb{Z}) \) differs
from the image of \( SO_0(N \perp, \mathbb{Z}) \). In other words, when is there \( \gamma \in O(N, \mathbb{Z}) \) which fixes \( u_k \)?

A basic calculation shows that the only possible matrix is

\[
\gamma = \begin{bmatrix}
-1 & 0 \\
h/k & 1
\end{bmatrix},
\]

which is integral if and only if \( k|h \), and is an involution. Furthermore, another calculation shows that \( \gamma \) induces conjugation on \( \mathcal{C}^+((1/2)N) \). We may then use a lemma of Nikulin ([23] 1.5.2, or see Appendix A.2) to extend \( \gamma \) to an element in \( SO_0(u_k^\perp, \mathbb{Z}) \) (also an involution) which fixes the sublattice \( N \). The extension of \( \gamma \) to \( SO_0(u_k^\perp, \mathbb{Z}) \) is the Galois involution.

Thus the image of a Hilbert modular surface is either birational to a Hilbert modular surface or it is birational to a symmetric Hilbert modular surface (see Section 2.4.2 for definition). Furthermore, the images of these surfaces are the Humbert surfaces discussed in Chapter 2 since the abelian surfaces that they represent have RM. We want to find how to explicitly parameterize subvarieties of \( \mathcal{D}^+(u_k^\perp) \) satisfying single singular relations, and to build an explicit map from the Hilbert modular group into \( SO_0(N \perp, \mathbb{Z}) \). This will be accomplished by first using canonical embeddings of Hilbert modular surfaces into Humbert surfaces and then using the wedge product map from \( h_2 \) to \( \mathcal{D}^+(N \perp) \).

First we recall that if we have a primitive rank two lattice \( N \subseteq U^3 \) then written in our normal form (see Equations 2.2.1, 2.2.2), then we have \( u_k = e_3 + kf_3 \) and \( v = e_2 + af_2 + hf_3 \) providing a basis of \( N \). Therefore, we find that the corresponding singular relation in \( \mathcal{D}^+(u_k^\perp) \) is just \( k\tau_1 + h\tau_2 + a\tau_3 = 0 \). Then by judicious choice of matrix \( S \), we may find an embedding

\[
\mathfrak{h}^2 \longrightarrow \mathfrak{h}_2,
\]

\[
(z_1, z_2) \mapsto S^t \begin{bmatrix} z_1 & 0 \\
0 & z_2 \end{bmatrix} S.
\]
We take the matrix
\[
S = \begin{bmatrix}
-\frac{h - \sqrt{\Delta}}{2} & -1 \\
\frac{h + \sqrt{\Delta}}{2} & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 \\
0 & k
\end{bmatrix}.
\]

Then we have the associated embedding,
\[
\tau_1 = \left( \frac{h^2 - 2h\sqrt{\Delta} + \Delta}{4} \right) z_1 + \left( \frac{h^2 + 2h\sqrt{\Delta} + \Delta}{4} \right) z_2,
\]
\[
\tau_2 = -k \left( \frac{h - \sqrt{\Delta}}{2} \right) z_1 + \left( \frac{h + \sqrt{\Delta}}{2} \right) z_2,
\]
\[
\tau_3 = k^2(z_1 + z_2),
\]
and \( \tau_1 \tau_3 - \tau_2^2 = k^2\Delta z_1 z_2 \). A quick calculation shows that the linear relation \( k\tau_1 - h\tau_2 + a\tau_3 = 0 \) is satisfied by these parameters. We have the associated period points, coming from \( (z_1, z_2) \)
\[
[k : k^2\Delta z_1 z_2 : k^2(z_1 + z_2) : k \left( \frac{h^2 - 2h\sqrt{\Delta} + \Delta}{4} \right) z_1 + k \left( \frac{h^2 + 2h\sqrt{\Delta} + \Delta}{4} \right) z_2 :
- k \left( \frac{h - \sqrt{\Delta}}{2} \right) z_1 + \left( \frac{h + \sqrt{\Delta}}{2} \right) z_2].
\]

Especially, we may allow an action on our period point by the matrix
\[
\tau = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
acts on our period point. Once we normalize so that the equations are again in the correct
form we obtain

\[
[k : \frac{1}{\Delta z_2} : - \frac{k}{\Delta z_1} \mapsto \left( \frac{h^2 - 2h\sqrt{\Delta} + \Delta}{4} \right) \frac{1}{\Delta z_2} + \left( \frac{h^2 + 2h\sqrt{\Delta} - \Delta}{4} \right) \frac{1}{\Delta z_1} \mapsto \left( \frac{h - \sqrt{\Delta}}{2} \right) \frac{1}{\Delta z_2} + \left( \frac{h + \sqrt{\Delta}}{2} \right) \frac{1}{\Delta z_1}.]
\]

We note that this induces the involution on \( h^2 \) given by

\[
\tau : (z_1, z_2) \mapsto \left( \frac{-1}{k\Delta z_2}, \frac{-1}{k\Delta z_1} \right).
\]

We see that this actually fixes the sublattice \( N \subseteq U^3 \) since \( N \) is contained in the two copies of \( U \) which are fixed by \( \tau \), and furthermore, since \( \tau \) induces an automorphism of \( h^2 \), it must fix components of \( D(N) \). Since \( \det(\gamma) = -1 \), it follows that \( \gamma \) generates the cokernel of the injection \( SO_0(N^\perp, \mathbb{Z}) \hookrightarrow O_0(N^\perp, \mathbb{Z}) \). This is summed up by the

**Proposition 5.2.2.** Let \( N \) be a rank two even lattice of signature \((1+, 1-)\). Then the moduli space of \( E_8^\perp \oplus N \)-polarized \( K3 \) surfaces is isomorphic to

\[
h^2/(PSL_2(\mathcal{O}_N, \mathcal{I}_N) \cup \tau PSL_2(\mathcal{O}_N, \mathcal{I}_N))
\]

where \( \tau \) is as above, and \( \mathcal{O}_N \) and \( \mathcal{I}_N \) are the quadratic order and fractional ideal associated to the lattice \( N \) in 4.1.1.

The involution \( \tau \) is not necessarily Galois involution on the moduli space of \( k \)-polarized abelian surfaces with real multiplication by \( O_\Delta \). The two involutions coincide when \( k = 1 \). By [31], pp. 14, the Galois involution is described by

\[
\sigma_{Gal}(z_1, z_2) = \left( \frac{-k}{\Delta z_2}, \frac{-k}{\Delta z_1} \right).
\]

The purpose of \( \tau \), according to [10] is that if \((z_1, z_2)\) represents the period point of some
abelian surface, then \( \tau(z_1, z_2) \) is a point corresponding to the abelian surface dual to the original surface. Therefore, that \( \sigma_{\text{Gal}} \) and \( \tau \) coincide when \( k = 1 \) reflects the fact that principally polarized abelian surfaces are self-dual.

### 5.2.1 Group embeddings.

As a final farewell to this section, we will construct a homomorphism from the group \( PSL_2(\mathcal{O}, \mathcal{T}) \) into \( Sp_4(k, \mathbb{Z}) \). This material follows Van der Geer [31] Chapter IX very closely. Since his results are stated differently and calculations are omitted, we have elected to include them here.

We begin by noting that we may map \( \mathbb{Q}(\sqrt{\Delta}) \) into \( \mathbb{R}^2 \) as a dimension 2 sub-vector space of full rank via

\[
\sigma : \alpha \longrightarrow (\alpha_1, \alpha_2)
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the two \( \mathbb{R} \)-embeddings of \( \alpha \) into \( \mathbb{R} \). Thus we get natural embeddings of \( \mathcal{O} \oplus \mathcal{T} \) into \( \mathbb{R}^4 \), and the symplectic form on \( \mathcal{O} \oplus \mathcal{T} \) induces a symplectic form onto \( \mathbb{R}^4 \). By applying some linear transformation, this may be made to agree with the natural symplectic form on \( \mathbb{R}^4 \). In particular, we want to choose a map that will send symplectic bases to symplectic bases; one such map is

\[
\begin{bmatrix}
R & 0 \\
0 & ((RD_k)^{-1})^t
\end{bmatrix} \sigma(\mathcal{O} \oplus \mathcal{T})^t,
\]

where \( R \) is the matrix

\[
R = \begin{bmatrix}
\frac{h - \sqrt{\Delta}}{2} & \frac{-h - \sqrt{\Delta}}{2} \\
-1 & 1
\end{bmatrix},
\]

and

\[
D_k = \begin{bmatrix}
1 & 0 \\
0 & k
\end{bmatrix}.
\]
One checks that under this map, our symplectic basis of $\mathcal{O} \oplus \mathcal{I}$ goes to the standard basis on $\mathbb{R}^4$, and thus this is a lattice isometry between the lattice $\mathbb{Z}^4 \subseteq \mathbb{R}^4$ equipped with the symplectic form with Gram matrix $J_k$ and the lattice $\mathcal{O} \oplus \mathcal{I}$ equipped with the trace form discussed at the beginning of this chapter.

Next, following [32] §1, one associates to a lattice $\mathcal{O}_\Delta \oplus \mathcal{I}$ and a point $(z_1, z_2) \in \mathfrak{h}^2$ an abelian surface by constructing the lattice in $\mathbb{C}^2$ spanned by $(\alpha_1 z_1 + \alpha_2 z_2, \beta_1 z_1 + \beta_2 z_2)$ for $\alpha \in \mathcal{O}$ and $\beta \in \mathcal{I}$. To express this then as a point in the Siegel upper half-space of genus two, we take the map

$$(z_1, z_2) \mapsto D_k R \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} (D_k R)^t.$$ 

Note now, this was the map that we used in the previous section when embedding $\mathfrak{h}^2$ into $\mathfrak{h}_2$ such that the embedding satisfies specific singular relations. Finally, if we define the map from $\mathbb{Q}(\sqrt{\Delta})$ into $M_2(\mathbb{R})$ that sends

$$\psi : \alpha \mapsto \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix},$$

then we have the relation,

$$\sigma \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} \tau \\ \nu \end{pmatrix} = \begin{bmatrix} \psi(\alpha) & \psi(\beta) \\ \psi(\delta) & \psi(\gamma) \end{bmatrix} \sigma \begin{pmatrix} \tau \\ \nu \end{pmatrix}.$$ 

Thus if we want to embed $PSL_2(\mathcal{O}, \mathcal{I})$ into $Sp_4(k, \mathbb{Z})$, the proper map is

$$\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \mapsto \begin{bmatrix} R & 0 \\ 0 & ((D_k R)^{-1})^t \end{bmatrix} \begin{bmatrix} \psi(\alpha) & \psi(\beta) \\ \psi(\delta) & \psi(\gamma) \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & ((D_k R)^{-1})^t \end{bmatrix}^{-1}. $$
From a previous calculation, we may express $PSL_2(\mathcal{O}, \mathcal{I})$, as matrices

$$
\gamma := \begin{bmatrix}
    a_0 + a_1 \left( \frac{h + \sqrt{\Delta}}{2} \right) & b_0 \sqrt{\Delta} + b_1 \left( \frac{\Delta - h \sqrt{\Delta}}{2k} \right) \\
    c_0 \left( \frac{k}{\sqrt{\Delta}} \right) + c_1 \left( \frac{\sqrt{\Delta} + h}{2\sqrt{\Delta}} \right) & d_0 + d_1 \left( \frac{h + \sqrt{\Delta}}{2} \right)
\end{bmatrix}
$$

where $a_i, b_i, c_i, d_i \in \mathbb{Z}$ and $\det(\gamma) = 1$ (modulo $\pm$Id obviously). Then our map is given by

$$
\gamma \mapsto \begin{bmatrix}
    a_0 & a_1 ak & b_0 h - b_1 ak & -b_0 k \\
    -a_1 & a_0 + a_1 h & b_0 & -b_1 k \\
    c_1 & c_0 k & d_0 & d_1 k \\
    -c_0 & c_0 h + c_1 a & d_1 a & d_0 + d_1 h
\end{bmatrix}.
$$

### 5.3 The case where $\Delta = \Box$.

Surprisingly, (or perhaps unsurprisingly, given our methods), the situation is quite similar if $\Delta$ is a square. In this situation, we have $\mathbb{Q}[x]/(x^2 - hx + ak) \cong \mathbb{Q}^2$. In this case, we represent $SO_0(N^\perp, \mathbb{Z})$ as a discrete subgroup of $SL_2(\mathbb{Q})$, $\mathcal{O}$ may be represented as an order in $\mathbb{Q}^2$, and $\mathcal{I}$ is again just another $\mathbb{Z}$-submodule of $\mathbb{Q}^2$.

To be precise, if we have $\Delta = n^2$, then we may associate elements of $\mathbb{Z}[x]/(x^2 - hx + ak)$ with elements of $\mathbb{Q}^2$ via evaluation at the two roots of $x^2 - hx + ak$ as usual. Therefore, if we have $R_\Box := \mathbb{Z}[x]/(x^2 - hx + ak)$ generated by the polynomials 1 and $x + \frac{h + n}{2}$, then we have

$$
1 \mapsto (1, 1) \in \mathbb{Q}^2 \text{ and } x + \frac{h + n}{2} \mapsto (0, n) \in \mathbb{Q}^2,
$$

and thus the order $\mathcal{O} \in \mathbb{Q}$ is just the subring of $(a, b) \in \mathbb{Z}^2$ satisfying $a \equiv b \mod n$. If one looks closely, the calculations in 4.1.1 do not depend on the fact that $\mathcal{O}$ is a quadratic order and not isomorphic to an order in $\mathbb{Q}^2$. In fact if we replace $\sqrt{\Delta}$ with a proper element in the ring $\mathbb{Q}[x]/(x^2 - hx + ak)$, then all of the calculations follow through effortlessly. Therefore
we will may restate the results in 4.1.1 in the case where $\Delta = \Box$.

We have that $H^1(A, \mathbb{Z})$ is isomorphic as a $\mathcal{R}_\Box$-module to $\mathcal{R}_\Box \oplus \mathcal{I}$, where $\mathcal{I}$ is a $\mathbb{Z}$-submodule of $\mathcal{R}_\Box \otimes \mathbb{Q}$, and may be represented as the $\mathbb{Z}$-module spanned by

$$\frac{k(x - n)}{n^2} \text{ and } \frac{1}{2} + \frac{h}{2n^2}(x - n).$$

Evaluation at $\frac{h + n}{2}$ and $\frac{h - n}{2}$ respectively gives us

$$x - n \mapsto \left(\frac{h - n}{2}, \frac{h - 3n}{2}\right).$$

For completeness, we write out a symplectic basis in $\mathbb{Q}^4$ for $H^1(A, \mathbb{Z})$ as

$$(1, 1, 0, 0), (0, n, 0, 0), (0, 0, k\left(\frac{h - n}{2n^2}\right), k\left(\frac{h - 3n}{2n^2}\right)), (0, 0, \frac{1}{2} + \frac{h}{2n^2}\left(\frac{h - n}{2}\right), \frac{1}{2} + \frac{h}{2n^2}\left(\frac{h - 3n}{2}\right)).$$

In particular, the case where $h = n$ (or in other words, $a = 0$), we have the simplification,

$$(1, 1, 0, 0), (0, h, 0, 0), (0, 0, 0, -k/h), (0, 0, 1/2, 0).$$

Then, as in the Hilbert modular case, we have that the modular group associated to this lattice is just

$$PSL_2(\mathcal{R}_\Box, \mathcal{I}) = \left\{ \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} \in SL_2(\mathbb{Q}^2), \alpha, \gamma \in \mathcal{O}, \beta \in \mathcal{I}^{-1}, \delta \in \mathcal{I} \right\},$$

where $\mathcal{I}^{-1}$ is the module spanned by

$$\left(\frac{h - 3n}{2}, \frac{h - n}{2}\right) \text{ and } \left(\frac{n^2}{2k} + \frac{h}{2k}\left(\frac{h - 3n}{2}\right), \frac{n^2}{2k} + \frac{h}{2k}\left(\frac{h - n}{2}\right)\right).$$
or we will get the subset of the subgroup of \( SL_2(\mathbb{Q})^2 \) given by pairs

\[
G := \left\{ \begin{bmatrix}
    a_0 & b_0 \\
    c_0 \left( \frac{1}{2} + \frac{h}{2n^2} \left( \frac{h-n}{2} \right) \right) + c_1 k \left( \frac{h-n}{2} \right) & d_0 \\
    b_0 \left( \frac{n^2}{2k} + \frac{h}{2k} \left( \frac{h-3n}{2} \right) \right) + b_1 \left( \frac{h-3n}{2} \right) & d_0 + nd_1
\end{bmatrix} \in SL_2(\mathbb{Q})^2 : \right. 
\]

where we have \( a_i, b_i, c_i, d_i \in \mathbb{Z} \)

where \( a_i, b_i, c_i, d_i \in \mathbb{Z} \). The moduli space of abelian surfaces with endomorphism polarization by \( \mathcal{R} \) and where \( H^2(A, \mathbb{Z}) \) is an \( \mathcal{R} \)-module isomorphic to \( \mathcal{R} \oplus \mathcal{I} \) where \( \mathcal{I} \) is as above, is given by

\[
\mathfrak{h}^2 / G
\]

where \( (\gamma_1, \gamma_2) \in G \) acts on \( \mathfrak{h}^2 \) via fractional linear transformation on each component. Therefore, we have from Theorem 4.2,

\[
SO_0(N^\perp, \mathbb{Z}) \cong G,
\]

and as before, we see that we have

\[
\tau = \begin{bmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \in O_0(N^\perp, \mathbb{Z})
\]

generating the quotient \( O_0(N^\perp, \mathbb{Z}) / SO_0(N^\perp, \mathbb{Z}) \) and acting as on \( \mathfrak{h}^2 \) as

\[
\tau : (z_1, z_2) \mapsto \begin{pmatrix}
-1 \\
kn^2 z_2 \\
-1 \\
kn^2 z_1
\end{pmatrix}.
\]
Therefore we have $O_0(N^\perp, \mathbb{Z}) \cong (\mathcal{G} \cup \tau \mathcal{G})$. 
Chapter 6

Rank Three Lattices and Quaternion Orders

In this chapter, we will analyze rank three lattices and their associated quaternion orders in a manner similar to the way rank two lattices and quadratic orders are analyzed in Chapter 4. We first present an explicit representation of the quaternion order associated to a rank three lattice of signature $(1+, 2-)$. We then specialize to the case where $\mathcal{O}_N$ is an Eichler order and relate lattice theory to their arithmetic theory. Finally, we discuss the embeddings of Shimura curves into Siegel modular threefolds.

Henceforward, we let $N$ be a rank three lattice with signature $(1+, 2-)$. Then $\mathcal{C}^+((1/2)N)$ is an order in some indefinite quaternion algebra.
6.1 Quaternion orders and ternary lattices.

We calculate in detail the order associated to such a lattice. Let us take a basis so that $N$ has Gram matrix

$$
\begin{bmatrix}
2a_2 & b & h_2 \\
b & 2a_1 & h_1 \\
h_2 & h_1 & 2k
\end{bmatrix}
$$

and we have that both

$$
\begin{bmatrix}
2a_2 & h_2 \\
h_2 & 2k
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
a_1 & h_1 \\
h_1 & 2k
\end{bmatrix}
$$

are rank two matrices with signature $(1+,1-)$ by Proposition 2.2.1 such that $\Delta_i = h_i^2 - 4a_i k$ and $\delta = h_1 h_2 - 2bk$.

Let us take a $\mathbb{Z}$-basis for $N$ given by $u_k, v_1, v_2$, then we have that $C^+((1/2)N)$ is the lattice in $C^+((1/2)N \otimes \mathbb{Q})$ spanned over $\mathbb{Z}$ by $\alpha = v_1 u_k$, $\beta = u_k v_2$, $\alpha \beta / k = v_1 v_2$. We may find a new $\mathbb{Q}$-basis of $C^+((1/2)N \otimes \mathbb{Q})$ which forms a standard basis $i, j, k$ for the underlying quaternion algebra $\mathbb{Q}$. Expressing $u_k, v_1, v_2$ in terms of this basis will give us an expression for $\alpha, \beta, \delta$ in terms of $i, j, k$. Let $D = \Delta_1 \Delta_2 - \delta^2$, then we may calculate that

$$
\begin{bmatrix}
2k & 0 & -h_2 \\
0 & 2k & -h_1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2a_2 & b & h_2 \\
b & 2a_1 & h_1 \\
h_2 & h_1 & 2k
\end{bmatrix}
= 2k
\begin{bmatrix}
-\Delta_2 & \delta & 0 \\
\delta & -\Delta_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

and that

$$
\begin{bmatrix}
1 & 0 & 0 \\
\delta & \Delta_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2k & 0 & 0 \\
0 & \Delta_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 2k
\begin{bmatrix}
-\Delta_2 & 0 & 0 \\
0 & -D \Delta_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

Thus for the $\mathbb{Q}$-basis of $N$ given by elements labelled $u_k, w_1, w_2$ with the above associated
Gram matrix, we have that $C^+((1/2)N \otimes \mathbb{Q}) \cong \left( \frac{\Delta_2, D}{\mathbb{Q}} \right)$. We have that $i = u_k w_1/k$ and $j = w_1 w_2/(k \Delta_2)$ and hence $ij = w_2 u_k/k$.

Finally, we chase the map back through our linear transformations. We have that our linear map between $\mathbb{Q}$-bases takes the form

$$\begin{bmatrix} 2k & 2k \delta & 0 \\ 0 & 2k \Delta_2 & 0 \\ -h_2 & -h_2 \delta - h_1 \Delta_2 & 1 \end{bmatrix},$$

and has inverse

$$\begin{bmatrix} 1/2k & -\delta/(2k \Delta_2) & 0 \\ 0 & 1/2k \Delta_2 & 0 \\ h_2/2k & h_1/2k & 1 \end{bmatrix}.$$

So we get that

$$v_1 = -\delta w_1/(2k \Delta_2) + w_2/(2k \Delta_2) + h_1 u_k/2k$$

$$v_2 = w_1/2k + h_2 u_k/2k.$$

Thus we have

$$\alpha = v_1 u_k = -\delta w_1 u_k/(2k \Delta_2) + w_2 u_k/(2k \Delta_2) + h_1/2 = \delta i/(2 \Delta_2) + ij/(2 \Delta_2) + h_1/2$$

$$\beta = u_k v_2 = u_k w_1/2k + h_2/2 = i/2 + h_2/2,$$

and $\mathcal{O}_N = \mathbb{Z} \oplus \alpha \mathbb{Z} \oplus \beta \mathbb{Z} \oplus \alpha \beta/k \mathbb{Z} \subset \left( \frac{\Delta_1, D}{\mathbb{Q}} \right).$

Note that the roles played by the above calculation are completely symmetric, and thus one may interchange $\Delta_1$ and $\Delta_2$ with any other pair of non-identical discriminants of Humbert surfaces passing through $\mathcal{D}^+(N^\perp)$.
Example 6.1.1. Let us look at the example of the lattice

\[
M_n := \begin{bmatrix}
-2n & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix}.
\]

We have that \(\Delta_1 = 1\) and that \(D = 4n\), and hence

\[
\mathcal{O}_{M_n} \otimes \mathbb{Q} \cong \left(\frac{1,4n}{\mathbb{Q}}\right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij
\]

where \(i^2 = 1\) and \(j^2 = -4n\). We may take \(\tilde{j} = j/2\) so that \(\tilde{j}^2 = -n\), and then we have

\[
\mathcal{O}_{M_n} \otimes \mathbb{Q} \cong \left(\frac{1,n}{\mathbb{Q}}\right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}\tilde{j} \oplus \mathbb{Q}\tilde{i}\tilde{j}
\]

Thus we have \(\mathcal{O}_{M_n}\) an order embedded into the non-division quaternion algebra \(\left(\frac{1,n}{\mathbb{Q}}\right)\).

We have that \(\delta = 0\) and thus by the calculations above,

\[
\alpha = \tilde{i}\tilde{j}
\]

\[
\beta = (i + 1)/2.
\]

Since \(k = 1\), we have that

\[
\mathcal{O}_{M_n} \cong \mathbb{Z} \oplus \mathbb{Z}\tilde{i}\tilde{j} \oplus \mathbb{Z}\left(\frac{i + 1}{2}\right) \oplus \mathbb{Z}\tilde{i}\tilde{j}\left(\frac{i + 1}{2}\right).
\]

There is a standard matrix representation of the quaternion algebra \(\mathbb{Q} = \left(\frac{a,b}{\mathbb{Q}}\right)\) inside \(M_2(\mathbb{Q}(\sqrt{a}))\), where

\[
j \mapsto \begin{bmatrix} 0 & -1 \\ b & 0 \end{bmatrix} \quad \text{and} \quad i \mapsto \sqrt{a} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]
Under this representation, we have

\[ \alpha \mapsto \begin{bmatrix} 0 & 1 \\ n & 0 \end{bmatrix} \text{ and } \beta \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]

Thus we see that

\[ \alpha \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad 1 - \beta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \beta \alpha - \alpha = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}. \]

and therefore we have that

\[ \mathcal{O}_{M_n} \cong \left\{ \begin{bmatrix} s & t \\ rn & q \end{bmatrix} ; s, t, r, q \in \mathbb{Z} \right\}. \]

**Remark 6.1.1.** Note that \( \Gamma(\mathcal{O}_{M_n}) \cong \Gamma_0(n) \), the congruence subgroup of \( SL_2(\mathbb{Z}) \). The moduli space of abelian surfaces with \( QM \) by \( \mathcal{O}_{M_n} \) is \( \mathfrak{h}/\Gamma(\mathcal{O}_{M_n}) \), however, according to calculations of both Peters [24] and Dolgachev [7], it follows that the period space of abelian surfaces polarized by \( M_n \) is \( \mathfrak{h}/\Gamma_0(n)^+ \) where \( \Gamma_0(n)^+ \) is the group generated by \( \Gamma_0(n) \) and the Atkin-Lehner involution

\[ \iota_n = \begin{bmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{bmatrix}. \]

### 6.2 Eichler orders.

For this section, the reader who is not familiar with quaternion algebras and quaternion orders is advised to recall the definitions at the beginning of Section 2.4.3. Let ‘ denote the standard involution on an indefinite quaternion algebra \( \mathbb{Q} \).

There are many points of comparison between the study of orders in quadratic extensions of \( \mathbb{Q} \) and study of orders of quaternion algebras, however, there are certain differences which
make the theory of a general quaternion order untenable. Therefore if one wishes to do
number theory with quaternion algebras, a convenient class of orders to look at are so
called Eichler orders. We have the following definitions. Let \( \mathbb{Q} \) be an indefinite quaternion
algebra over \( \mathbb{Q} \) and let \( \mathcal{O} \) be an order in \( \mathbb{Q} \).

**Definition 6.2.1.** An order \( \mathcal{O} \) in a quaternion algebra \( \mathbb{Q} \) is called maximal if there are no
other orders in \( \mathbb{Q} \) which properly contain \( \mathcal{O} \).

**Definition 6.2.2.** Let \( \mathcal{O} \) be an order in a quaternion algebra \( \mathbb{Q} \), then we say that \( \mathcal{O} \) is an
Eichler order of \( \mathbb{Q} \) if we have that \( \mathcal{O} \) is the intersection of two maximal orders in \( \mathbb{Q} \).

Any Eichler order in a quaternion algebra over \( \mathbb{Q} \) is classified uniquely by two numbers
called the discriminant and the level, denoted by numbers \( D \) and \( n \). For precise definition
of the level and discriminant of \( \mathcal{O} \), see Hashimoto [13] Definition 2.1. An Eichler order is
defined up to isomorphism by its level and discriminant. We denote the Eichler order of
level \( n \) and discriminant \( D \) by \( \mathcal{O}(D, n) \).

**Example 6.2.1.** Let \( \mathcal{O}_{M_n} \) be as in the previous example. This is an Eichler order of
discriminant 1 and level \( n \).

Our goal for this section will be to classify the lattices \( N \) for which \( \mathbb{C}^+(1/2)N \) is an
Eichler order. First we note that there is a natural rational symmetric bilinear form on a
quaternion algebra \( \mathbb{Q} \) over \( \mathbb{Q} \), which we define,

\[
q(x, y) \mapsto \text{Tr}(xy')
\]

where ’ is the standard involution on \( \mathbb{Q} \). Then we will show

**Proposition 6.2.1.** Let \( N \) be a lattice of rank three and signature \((1+, 2-)\), and choose
some element \( u \) of \( N \) such that \( \langle u, u \rangle \neq 0 \). Then let us define a rational symmetric bilinear
form on \( \mathbb{C}^+(1/2)N \), to be

\[
\bar{q}(x, y) \mapsto \frac{2 \text{Tr}(xy')}{\langle u, u \rangle}.
\]
Then there is an embedding of $N$ into $\mathbb{C}^+((1/2)N)$ which induces an isometry of $N$ onto $\mathbb{C}^+((1/2)N)$.

Proof. Define the embedding of $N$ into $\mathbb{C}^+((1/2)N)$ to be that which acts on $v \in N$ as

$$v \mapsto u \cdot v.$$ 

The quaternionic involution on $\mathbb{C}^+((1/2)N)$ is associated to the main involution, which we recall just sends $uv$ to $vu$. Therefore, we have

$$\bar{q}(uv, uw) = \frac{2 \text{Tr}(vw)u \cdot u}{\langle u, u \rangle} = \langle v, w \rangle,$$

since by the relations on the Clifford algebra, we have $u \cdot u = \frac{(u, u)}{2}$. This establishes that

$$\text{this is an isometry.} \quad \Box$$

Therefore, we have that

**Corollary 6.2.1.** Let $A$ be an abelian surface with $\text{rank}(\text{NS}(A)) = 3$. Then the map

$$\text{NS}(A) \rightarrow \mathbb{C}^+((1/2) \text{NS}(A)) = \text{End}(A)$$

defined in Theorem 3.2 induces an isometry from the Néron-Severi lattice of $A$ onto the Rosati-invariant endomorphisms of $A$ when the Rosati invariant endomorphisms of $A$ are equipped with the form $q'$ as defined in Proposition 6.2.1.

Now if we are given an abelian surface $A$ with $\text{End}(A)$ a quaternion order, then we know that the Rosati involution may be represented as

$$\beta^o \mapsto a\beta' a^{-1}$$

where $'$ represents the standard involution on the quaternion algebra $\text{End}(A)$, and $a \in$
End\(^0\)(A) satisfies \(a' = -a\) and \(a^2 < 0\). Therefore, if we have that \(A\) is \(k\)-polarized and we know which \(a \in \text{End}\(^0\)(A)\) induces the Rosati involution, we may identify the Néron-Severi group inside of \(\text{End}(A)\) by calculating the Rosati-invariant sublattice of \(\text{End}(A)\), and finally, we may calculate the lattice just by calculating the form

\[
\bar{q}(x, y) = \frac{\text{Tr}(xy')}{k}
\]
on \text{NS}(A).

In [13], Hashimoto gives the general form of an Eichler order of discriminant \(D\) and level \(n\). We give his construction more or less verbatim and refer the reader to [13] for proof. We have \(D_0 = \prod_i p_i\) for an even number of distinct primes and \(D = D_0n\), and we will let \(\left( \frac{\cdot}{p} \right)\) denote the Legendre symbol.

Let us take \(p\) to be a prime number satisfying the following conditions;

1. \(p \equiv 1 \mod 4\). If \(2 | D_0\) then \(p \equiv 5 \mod 8\) and if \(2 | n\) then \(p \equiv 1 \mod 8\) (note that both cannot occur at the same time: both \(D_0\) and \(D = D_0n\) are by definition products of distinct primes).

2. \(\left( \frac{p}{p_i} \right) = -1\) for any \(p_i \neq 2\).

3. \(\left( \frac{p}{q} \right) = 1\) for any odd prime \(q | n\).

Furthermore, we ask that \(\left( \frac{D}{p} \right) = 1\). Thus we have the existence of some integer \(a\) such that

\[
a^2D + 1 \equiv 0 \mod p.
\]

Then we have

**Theorem 6.1** ([13] Thm. 2.2). *Let us take the quaternion algebra*

\[
\left( \frac{D, p}{\mathbb{Q}} \right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij
\]
where $i^2 = -D, j^2 = p$ and $ij = -ji$. Then if we let

$$e_1 = 1, e_2 = (1 + j)/2, e_3 = (i + ij)/2 \text{ and } e_4 = (aDj + ij)/p,$$

we have the isomorphism.

$$\mathcal{O}(D, n) \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.$$

Furthermore, [13] Prop. 3.1 shows that we may assume that the Rosati involution is given by $\beta^o = i\beta' i^{-1}$. Now we are in a position to carry out our earlier threat and calculate the lattice associated to an Eichler order. The first step will be to compute the Rosati invariant part of $\mathcal{O}(D, n)$. This is simply done; we see that $e_1, e_2$ and $e_4$ are all fixed by the Rosati involution. That $e_1$ is fixed is trivial, the other two calculations follow.

$$e_2^o = i \left( \frac{i + j}{2} \right)^{i^{-1}} = \left( 1 - iji^{-1} \right) \left( \frac{1 + j}{2} \right),$$

and

$$e_4^o = i \left( \frac{aDj + ij}{p} \right)^{i^{-1}} = \left( \frac{-aDji^{-1} - i^2ji_{-1}}{p} \right) = \left( \frac{aDj + ij}{p} \right).$$

The polarization provided by $i$ is a principal polarization, therefore the intersection form on $\text{NS}(A)$ is just represented by the trace form on the Rosati invariant part of $\text{End}(A)$. By [5] Prop. 5.2.1.(b), the Rosati invariant endomorphisms form a primitive sublattice of $\text{End}(A)$, and thus $\text{End}(A)^s \cong \text{NS}(A)$, and by previous comments, we may recover the lattice structure on $\text{NS}(A)$ via the form given in Proposition 6.2.1.
We perform the following calculations,

\[
\begin{align*}
\text{Tr}(e_1 \cdot e'_1) & = 2 \\
\text{Tr}(e_2 \cdot e'_2) & = \frac{1-p}{2} \\
\text{Tr}(e_4 \cdot e'_4) & = -\frac{a^2D^2p - Dp}{p^2} = -2D \left( \frac{a^2D + 1}{p} \right) \\
\text{Tr}(e_1 \cdot e'_2) & = 1 \\
\text{Tr}(e_1 \cdot e'_4) & = 0 \\
\text{Tr}(e_2 \cdot e_4) & = aD.
\end{align*}
\]

Therefore we have the Gram matrix of this lattice given by

\[
L(D, n) := \begin{bmatrix}
-2D \left( \frac{a^2D + 1}{p} \right) & aD & 0 \\
aD & \frac{1-p}{2} & 1 \\
0 & 1 & 2
\end{bmatrix}.
\]

This is an even integral lattice because \( p \equiv 1 \mod 4 \) and \( a^2D + 1 \equiv 0 \mod p \), and has discriminant \( 2D \). We may now reliably identify the lattices associated to Eichler orders.

We have proved,

**Proposition 6.2.2.** Let \( \mathcal{O}(D, n) \) be an Eichler order of discriminant \( D \) and level \( n \). Then if \( p \) and \( a \) are as in the construction of \( \mathcal{O}(D, n) \), then we have

\[
\mathcal{C}^+((1/2)L(D, n)) \cong \mathcal{O}(D, n).
\]

Finally, we shall make another small detour into lattice theory to make it even more simple to identify the lattices \( L(D, n) \).

**Definition 6.2.3.** If \( L \) is a lattice equipped with inner product \( Q \), then we define the dual...
of $L$ to be the sublattice of $L \otimes \mathbb{Q}$,

$$\{ \gamma \in L \otimes \mathbb{Q}; Q(\gamma, \alpha) \in \mathbb{Z} \text{ for every } \alpha \in L \}$$

and we will denote this $L^*$. We will call the quotient group $L^*/L$ the discriminant group of $L$, which we will denote $A(L)$.

If $L$ is an integral lattice, then we have that $\#A(L) = \text{disc}(L)$. If $L$ is an even integral lattice, then there is a form

$$q_L : A(L) \times A(L) \longrightarrow \mathbb{Q}/2\mathbb{Z},$$

$$(\gamma, \delta) \mapsto Q(\gamma, \delta) \mod 2\mathbb{Z}.$$ 

We calculate the discriminant form of $A(L(D, n))$. A basis of $L^*$ which is dual to the given basis of $L$ is written in terms of the basis $u, v, w$ as

$$u^* = \left( -\frac{p}{2D}, -a, \frac{a}{2} \right), v^* = \left( -a, -\frac{2a^2D - 2}{p}, \frac{a^2D + 1}{p} \right), \text{ and}$$

$$w^* = \left( \frac{a}{2}, \frac{a^2D + 1}{p}, -\frac{a^2D + 1 - p}{2p} \right).$$

One then calculates that we have the discriminant form given by

$$[q_{L(D,n)}] = \begin{bmatrix} -p/(2D) & a \mod 2 & 1 \\ a \mod 2 & 0 & 1/(2a) \\ 1 & 1/(2a) & 1 \end{bmatrix},$$

with respect to the basis $u, v, w$. One notices that if $u^*$ is the generator of $A(L)$ with self-intersection $-p/2D$ then since $2$ does not divide $p$, and $p$ is coprime to $n$ and $D_0$ hence coprime to $D$, the order of $u^*$ in $A(L)$ is $2D$. Therefore, since $\#A(L) = \text{disc}(L) = 2D$, we have that $u^*$ actually generates $A(L)$ as a cyclic group of order $2D$. We have
Proposition 6.2.3 ([23] Cor. 1.13.3). Let $L$ be an even lattice, and let $L$ have invariants $(s_+, s_-, q_L)$, and let $l(A(L))$ be the minimal number of generators for $A(L)$. Then

1. if $s_+ > 0$ and $s_- > 0$, and

2. if we have $l(A(L)) \leq \text{rank}(L) - 2$,

then $L$ is the unique lattice with these invariants up to isometry.

Therefore, if we have $L$ a rank three lattice with signature $(1+, 2-)$ such that $A(L)$ is cyclic, then $L$ is defined up to isometry by the form $q_L$. Thus we may conclude that

Proposition 6.2.4. Let $L$ be an even lattice of rank three and signature $(1+, 2-)$ and with discriminant $2D$ where $D$ is a squarefree natural number. Then we have that $C^+((1/2)L)$ is an Eichler order of discriminant $D$ if and only if we have $A(L)$ a cyclic group and $u$ a generator of $L$ such that $q_L(u^*, u^*) = -p/(2D)$ for some prime $p$. Then the level of $C^+((1/2)L)$ is just the product of all primes $q \neq 2$ dividing $D$ such that

$$\left(\frac{p}{q}\right) = 1.$$ 

Remark 6.2.1. Thm. 1.14.1 in [23] actually shows that there is a unique embedding of $L(D, n)$ into $U^3$ up to automorphism of $U^3$.

Example 6.2.2. Let us take the lattice $L$ whose Gram matrix is

$$\begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & 4
\end{bmatrix}.$$

Then a quick calculation shows that $\text{disc}(L) = 12$ and hence we have $12 = 2 \cdot 6$. Clearly, 6 is squarefree and one may check that there is a primitive element in $L^*$ whose self intersection is $-5/12$. By Proposition 6.2.4 this means that $C^+((1/2)L)$ is an Eichler order of
discriminant 6. Again using Proposition 6.2.4, we may calculate the level of \( C^+((1/2)L) \).

We see that

\[
\left( \frac{5}{3} \right) = 1
\]

and hence \( N = 1 \). Therefore \( C^+((1/2)L) \cong \mathcal{O}(6,1) \) is a maximal order in a quaternion algebra.

### 6.3 Embeddings into Siegel threefolds.

Now we describe \( D^+(L^+) \) in more detail, as a subvariety of \( D^+(u_k^+) \).

Our first goal in this section will be to understand the period space of rank three-polarized abelian surfaces as the intersection of period spaces of rank two polarized abelian surfaces. If we have \( D^+(u_k^+) \) the period space of (marked) \( k \)-polarized abelian surfaces, then if we have a rank two lattice \( M \) which represents \( k \) primitively, then there is a primitive embedding of \( u_k \) into \( M \) and therefore \( M \) may be embedded in \( U^3 \) in such a way that the image of \( M \) contains \( u_k \). Therefore, we can think of \( D^+(M^+) \) as the intersection of \( D^+(u_k^+) \) with a hyperplane in \( \mathbb{P}^4 \).

Now if we take \( L \) a rank three lattice of signature \((1+,2-)\) which represents \( k \), then we may construct a natural embedding of \( D^+(L^+) \) into \( D^+(u_k^+) \) in the same way as before, coming from any primitive embedding of \( L \) into \( U^3 \). Now we are able to take a standard basis of \( L \) as a sublattice of \( U^3 \), which we will write as \( u_k, v, w \). Then by Lemma 2.2.1, we have that the lattices \( M_1 := \text{span}_\mathbb{Z}\{u_k, v\} \) and \( M_2 := \text{span}_\mathbb{Z}\{u_k, w\} \) are rank two primitive sublattices of \( U^3 \) with signature \((1+,1-)\), and thus if \( L \subseteq \text{NS}(A) \) for an abelian surface \( A \), then we have both \( M_1 \) and \( M_2 \) primitive sublattices of \( \text{NS}(A) \) as well. Thus the period point of \( A \) lies in both \( D(M_1^+) \) and \( D(M_2^+) \), and conversely if \( \text{NS}(A) \) contains both \( M_1 \) and \( M_2 \) then \( \text{NS}(A) \) must contain \( L \). Thus \( D^+(M_1^+) \cap D^+(M_2^+) = D^+(L^+) \).
If we recall the complex isomorphism $h_2 \cong D^+(u_k^\perp)$ given by
\[
\wedge^2 : \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \mapsto [k : \tau_2^2 - \tau_1 \tau_3 : \tau_3 : k \tau_1 : \tau_2],
\]
then if $A$ is a $k$-polarized abelian surface bearing a polarization by the lattice $L = \text{span}_\mathbb{Z}\{u_k, v, w\}$, then there is a period point $\tau$ representing $A$ such that $\tau$ satisfies the two singular relations coming from $v$ and $w$. From Lemma 2.2.2, we may choose $v$ and $w$ such that
\[
v = e_2 + a_1 f_2 + h_1 f_3,
\]
\[
w = c_1 e_1 + c_2 f_1 + b f_2 + h_2 f_3,
\]
and the Gram matrix of $L$ is given by
\[
\begin{bmatrix}
2c_1 c_2 & b & h_2 \\
b & 2a_1 & h_1 \\
h_2 & h_1 & 2k
\end{bmatrix}.
\]
Thus we may write the singular relations corresponding to $v$ and $w$ as
\[
k\tau_1 + h_1 \tau_2 + a_1 \tau_3 = 0
\]
\[
br_3 + k c_2 + c_1 (\tau_2^2 - \tau_1 \tau_3) + h_2 \tau_2 = 0.
\]
Let $M_1 = \text{span}_\mathbb{Z}\{u_k, v\}$ be a sublattice of $L$ of discriminant $\Delta_1$ and let $M_2 = \text{span}_\mathbb{Z}\{u_k, w\}$ be a sublattice of $L$ of discriminant $\Delta_2$, with neither $\Delta_1$ or $\Delta_2$ a square. We will let $\alpha = \frac{h_1 - \sqrt{\Delta_1}}{2}$ and $\alpha' = \frac{h_1 + \sqrt{\Delta_1}}{2}$. Assuming that we have the parameterization of singular relations in Section 5.2, then substituting this into the second equation, we get
\[
bk(z_1 + z_2) + c_2 - h_2(\alpha z_1 + \alpha' z_2) + c_1(z_1 z_2 k \Delta_1) = 0.
\]
One rearranges to find the equation

\[
z_2 = \frac{-(bk - h_2\alpha)z_1 - c_2}{c_1k\Delta_1 z_1 + (bk - h_2\alpha')}.
\]

Hence if we let

\[
\gamma = \begin{bmatrix}
-(bk - h_2\alpha) & -c_2 \\
c_1k\Delta_1 & (bk - h_2\alpha')
\end{bmatrix},
\]

then the forgetful map from the Shimura curve with lattice \( N \) maps into \( \h^2/\text{PSL}_2(\mathcal{O}_{\Delta_1}, \mathcal{T}) \) is via \( \phi : z \mapsto (z, \gamma \cdot z) \).

**Remark 6.3.1.** Combining this and the embedding of \( \h^2/\text{PSL}_2(\mathcal{O}_{M_1}, \mathcal{T}_{M_1}) \) into \( \h_2/\text{Sp}_4(k, \mathbb{Z}) \) (Section 5.2), we obtain formulae for the periods associated to \( L \)-polarized abelian surfaces.

Once we have derived this form of the moduli embedding, [31] Chapter IV tells us how we may embed the group \( \Gamma(\mathcal{O}_L) \) into \( \text{PSL}_2(\mathcal{O}_{M_1}, \mathcal{T}_{M_1}) \), and thus using Section 5.2, we may embed \( \Gamma(\mathcal{O}_L) \) into the stabilizer of the image of \( \h \) under the embedding given in this section.

### 6.4 Two propositions on extensions of \( \Gamma(\mathcal{O}_N) \).

We have two groups defined for any lattice \( M \) such that \( M \) embeds primitively into \( U^3 \) and the embedding contains \( u_k \).

\[
SO_0(M, u_k^\perp, \mathbb{Z}) := \{ \gamma \in SO(U^3, \mathbb{Z}) | \gamma(u_k) = u_k \text{ and } \gamma(M) \subseteq M \}.
\]

Notice that \( SO_0(M^\perp, \mathbb{Z}) \) is contained in \( SO_0(M, u_k^\perp, \mathbb{Z}) \). We also define the group

\[
\text{res}_{M^\perp}(SO_0(M, u_k^\perp, \mathbb{Z})) : \text{Image}(SO_0(M, u_k^\perp, \mathbb{Z}) \longrightarrow O(M^\perp)).
\]

Here the map

\[
\text{res}_{M^\perp} : SO_0(M, u_k^\perp, \mathbb{Z}) \longrightarrow O(M^\perp)
\]
is just restriction to \(M^\perp\). We see that \(SO_0(M^\perp, \mathbb{Z})\) may be identified with its image in \(\text{res}_{M^\perp}(SO_0(M, u_k^\perp, \mathbb{Z}))\). If \(\gamma\) is in either \(\text{res}_{M^\perp}(SO_0(M, u_k^\perp, \mathbb{Z}))\) or \(SO_0(M, u_k^\perp, \mathbb{Z})\), we do not require that \(\gamma|_M = \text{Id}\), but just that it maps \(M\) to itself and fixes \(u_k\). This subtle difference sets \(\text{res}_{M^\perp}(SO_0(M, u_k^\perp, \mathbb{Z}))\) and \(SO_0(M^\perp, \mathbb{Z})\) apart. We have containment, \(SO_0(M^\perp, \mathbb{Z}) \subset \text{res}_{M^\perp}(SO_0(M, u_k^\perp, \mathbb{Z}))\), however the reverse containment is more rare. For instance, we proved in Proposition 5.2.1 that if \(M\) is a rank two signature \((1+, 1-)\) even lattice, then we have equality if \(k\) does not divide \(h\), and we have an extra involution otherwise. In the case where \(L\) is a rank three even lattice of signature \((1+, 2-)\), the game is even more complicated, however knowing the difference is important if we wish to describe the image of the forgetful map which sends a Shimura curve into \(b_2/Sp_4(k, \mathbb{Z})\).

**Lemma 6.4.1.** Let \(M\) be a rank three even lattice with signature \((1+, 2-)\), equipped with a primitive embedding into \(U^3\). Then

1. \(M^\perp \cong -M\) and if \(\gamma \in O(M, \mathbb{Z})\) then we have \(\gamma \oplus -\gamma \in O(M \oplus M^\perp, \mathbb{Z})\) extends uniquely to an element of \(O(U^3, \mathbb{Z})\) which restricts to \(\gamma\) on \(M\) and to \(-\gamma\) on \(M^\perp\).

2. If \(\gamma \in O(M, \mathbb{Z})\) and the image of \(\gamma \in O(A(M))\) is the identity, then \(\gamma \oplus \text{Id} \in O(M^\perp, \mathbb{Z})\) extends uniquely to an element \(\tilde{\gamma} \in O(U^3, \mathbb{Z})\).

**Proof.** These are both simple consequences of [23] Thm. 1.4.1. See Appendix A.2. \(\square\)

**Proposition 6.4.1.** Let \(M\) be a rank three even lattice of signature \((1+, 2-)\) embedded primitively in \(U^3\) such that the image contains \(u_k\). Let us take the group \(O_Z(M)_{u_k}\) to be the subgroup of \(O_Z(M)\) which fixes \(u_k\), and let \(\Gamma\) be the image of \(O_Z(M)_{u_k}\) in \(O(A(M))\). Then we have

\[ \text{res}_{M^\perp}(O_0(M, u_k^\perp, \mathbb{Z}))/O_0(M^\perp, \mathbb{Z}) \cong \Gamma. \]

**Proof.** First of all, we note that the kernel of \(\text{res}_{M^\perp}\) is just matrices

\[ \text{Id} \oplus \xi \in O_Z(M^\perp) \oplus O_Z(M). \]
By part 2 of Lemma 6.4.1, this means that $\xi|_{A(M)} = \text{Id}_{A(M)}$. By the same fact, if $\xi \in O_\mathbb{Z}(M)_{u_k}$ then $\text{Id} \oplus \xi$ extends to an element of $O_0(u_k^\perp, \mathbb{Z})$. Thus if we have

$$G := \ker(O_\mathbb{Z}(M)_{u_k} \rightarrow O(A(M)))$$

then $G$ embeds as a normal subgroup of $O_0(M, u_k^\perp, \mathbb{Z})$ via $\xi \mapsto \text{Id} \oplus \xi$, and

$$\text{res}_{M^\perp}(O_0(M, u_k^\perp, \mathbb{Z})) \cong O_0(M, u_k^\perp, \mathbb{Z})/G.$$  

Thought about this way, we have a map from $\text{res}_{M^\perp}(O_0(M, u_k^\perp, \mathbb{Z}))$ coming from taking the restriction of $O_0(M, u_k^\perp, \mathbb{Z})$ to $O(A(M))$ modulo $G$. The kernel of this map is $O_0(M^\perp, \mathbb{Z})$ and the image (by Lemma 6.4.1) is $\Gamma$. This proves the proposition.}

We prove our second proposition, which tells us how to relate the group $O_0(M^\perp, \mathbb{Z})$ to $\text{SO}_0(M^\perp, \mathbb{Z})$.

**Proposition 6.4.2.** Let $M$ be as in the previous proposition. Let

$$\Gamma = O_0(M^\perp, \mathbb{Z})/\text{SO}_0(M^\perp, \mathbb{Z}).$$

Then we have $\Gamma$ nontrivial if and only if there is some $\gamma \in \text{SO}(M^\perp, \mathbb{Z})$ such that the automorphism of $A(M)$ induced by $\gamma$ is $-\text{Id}_{A(M)}$, and $\gamma \neq -\text{Id}_M$. In this case, $\Gamma$ is generated by $\gamma$.

**Proof.** If we have some element $\gamma \in O_0(M^\perp, \mathbb{Z}) \setminus \text{SO}_0(M^\perp, \mathbb{Z})$ then we have that $-\gamma$ restricts to $\text{Id}$ on $A(M)$, and hence $-\gamma$ restricts to $-\text{Id}$ on $A(M)$. Conversely, if we have $\gamma \in \text{SO}(M, \mathbb{Z})$ such that $\gamma$ is $-\text{Id}$ on $A(M)$, then by Lemma 6.4.1 part 2, we may extend $-\gamma \oplus \text{Id} \in O(M^\perp \oplus M, \mathbb{Z})$ to an element of $O(U^3, \mathbb{Z})$ which is the identity on $M$. 

Essentially, these propositions give us a way to find both the modular group $O_0(M^\perp, \mathbb{Z})$
and $SO_0(M^\perp, u_k)$ in specific situations. We will give an easy consequence when our lattice is one of the lattices $L(D, n)$ from Section 6.2.

**Corollary 6.4.1.** Let $M$ be a lattice as in the previous propositions, and suppose that $C^+((1/2)M) \cong \mathcal{O}(D, n)$ for some Eichler order of discriminant $D$ and level $n$. Then the quotient,

$$\text{res}_{M^\perp}(O_0(M, u_k, \mathbb{Z}))/O_0(M^\perp, \mathbb{Z})$$

is contained in the group of square roots of unity modulo $4D$.

**Proof.** By Proposition 6.2.4, we have that $A(L)$ is cyclic and has generator $u^*$ of order $2D$. Clearly, any automorphism of $A(L)$ must be defined by sending $u^*$ to $mu^*$ for some integer $m$, and that this map must satisfy

$$q_L(mu^*, mu^*) = m^2 q_L(u^*, u^*) = -m^2 p/(2D) = -p/(2D) = q_L(u^*, u^*).$$

Thus we must have $-m^2 p/(2D) \equiv -p/(2D) \mod 2\mathbb{Z}$, or equivalently, we have $m^2 \equiv 1 \mod 4D$. Thus $m$ is a square root of unity mod $4D$. \hfill \Box

**Remark 6.4.1.** We note that the group of square roots of unity modulo $2D$ is a finite group whose nontrivial elements all have order two. One notices the relation between the structure of this group and the Atkin-Lehner group of $\mathcal{O}(D, n)$. It should not be hard to discover the relationship between the Atkin-Lehner group and the group $\Gamma$. 

Chapter 7

Elliptic Fibrations and Lattice Polarizations

In this chapter, we will describe in detail a new example of a family of K3 surfaces exhibiting the properties we have discussed in the previous chapters. Specifically, this will be a 4-parameter family of K3 surfaces generically bearing a polarization by the lattice \((4) \oplus E_8^2\) and will be related via Shioda-Inose structure to \((4)\)-polarized abelian surfaces. Furthermore, this will be a family of Jacobian elliptic surfaces given by the equations

\[
\mathcal{F}(a, b, c, d) : y^2 = x^3 + (-3t^2 + at - (a^2 + b)/12)t^4x + (2t^4 + at^3 + bt^2 + ct + d),
\]

where \(a, b, c, d\) are parameters in the weighted projective space \(\mathbb{WP}^3(1, 2, 3, 4)\), and the Néron-Severi group is generically isomorphic to the lattice \(U \oplus D_7 \oplus E_8\). We will commence to find subvarieties of \(\mathbb{WP}^3(1, 2, 3, 4)\) on which the Néron-Severi group gets bigger. This will happen in two ways. First, this occurs when we obtain more reducible singular fibers in our elliptic fibration; say when the fiber representing \(D_7\) degenerates to a \(D_8\)-fiber, or when the implicit \(A_0\)-fibers become \(A_1\)-fibers. The second case is when we obtain non-zero section in the Mordell-Weil group of \(\mathcal{F}(a, b, c, d)\) (we know by work of Shimada [26] that
such elliptic surfaces do not have any torsion sections). Then using basic lattice theory, we may deduce how these sections affect the transcendental lattice of $F(a, b, c, d)$. Once we know the transcendental lattices associated to these degenerate subvarieties, then from work done in Chapter 4, we may associate these to either symmetric Hilbert modular surfaces or quotients of Shimura curves by involutions. From this, we obtain weighted-projective models of certain quotients of symmetric Hilbert modular surfaces and quotients of Shimura curves.

### 7.1 Background on elliptic K3 surfaces.

The absolute best reference for elliptic surfaces is [19], lecture notes by Rick Miranda. Nearly all of the facts in this section on elliptic K3 surfaces may also be found in [19]. The study of elliptic surfaces draws upon algebraic geometry, complex analysis and number theory. The approach of Miranda is focussed on the algebro-geometric. We will follow his approach as it has the strongest bearing on our purposes.

**Definition 7.1.1.** A complex elliptic surface is a smooth minimal algebraic surface $X$ along with a surjective flat morphism $\pi : X \longrightarrow C$ from $X$ to some smooth compact complex curve $C$ such that the generic fiber of $\pi$ is an elliptic curve. An elliptic K3 surface is a K3 surface $X$ along with some map $\pi$ giving it the structure of an elliptic surface. We will denote an elliptic surface by the triple $(X, \pi, C)$. An elliptic surface is called a Jacobian elliptic surface if there is some section $O : C \longrightarrow X$ of the fibration $\pi$.

We will consider exclusively Jacobian elliptic surfaces in the following discussion, and we will often just say “elliptic surface” when what we mean is “Jacobian elliptic surface”.

The reason for this terminology is that if we have that $X$ is a Jacobian elliptic fibration, then we may view $X$ as a curve of genus one defined over the function field of $C$ (see [19]). This is not an elliptic curve over the function field of $C$ unless we actually have a $\mathbb{C}(C)$-rational point on $X$. An elliptic curve may be associated with its own Jacobian, and hence
an elliptic surface with section is actually the Jacobian of a curve over \( \mathbb{C}(C) \) (itself). We may then locally associate to \( X \) a Weierstrass equation

\[
y^2z = x^3 + A(u)xz^2 + B(u)z^3 \in \mathbb{P}^2 \times C
\]

where \( u \) is a local coordinate on \( C \). We will call this the \textit{Weierstrass model} of \( X \).

**Definition 7.1.2.** The discriminant of \( X \) an elliptic surface is the polynomial given locally by the equation

\[
\Delta(u) = 4A(u)^3 + 27B(u)^2.
\]

When \( \Delta(u) = 0 \), the fiber above \( u \) in the Weierstrass fibration is a singular curve.

The surface cut out by the above equation is not necessarily the surface we want, since it may be singular (for example, it is often singular when \( \Delta(u) = 0 \). The singularities we obtain are of ADE type ([19] I.6) and hence may be easily resolved, and the minimal resolution is the surface \( X \).

The local Weierstrass coefficients \( A \) and \( B \) actually patch together globally into sections of some line bundles \( L_A, L_B \) respectively on \( C \) (see [19] II.3). In fact, Miranda shows that there is some line bundle \( L \) such that \( A \) is a section of \( L^\otimes 4 \) and \( B \) is a section of \( L^\otimes 6 \). We will henceforward be interested in the situation where \( C \cong \mathbb{P}^1 \). We then have \( A \) and \( B \) global sections of \( \mathcal{O}_{\mathbb{P}^1}(4n) \) and \( \mathcal{O}_{\mathbb{P}^1}(6n) \), respectively, for some integer \( n \). Miranda then gives the following expression for the Hodge diamond of \( X \),

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & & n-1 & 10n & n-1 \\
0 & & 0 & & \\
1 & & & & \\
\end{array}
\]

From this, we see that if \( n = 2 \) then \( X \) is actually a K3 surface.
Remark 7.1.1. If \( n = 1 \) then \( X \) is a rational surface (\( p_g = 0 \)). Such a surface \( X \) is called an Enriques surface.

Therefore, if \( X \) is to be an elliptic K3 surface, then \( X \) must have a Weierstrass form where \( A \) and \( B \) are represented by polynomials of degree 8 and 12 in projective coordinates respectively.

7.1.1 The Néron-Severi group.

We now investigate the Néron-Severi group of an elliptic K3 surface more deeply. There are two obvious algebraic cycles on \( X \), the first of all being the class of a generic fiber (in other words, the fiber of a point \( z \in \mathbb{P}^1 \) such that \( \pi^{-1}(z) \) is a smooth elliptic curve), which we will call \( F \). We have a well defined map,

\[
\pi^* : \text{Pic}(\mathbb{P}^1) \cong \mathbb{Z} \longrightarrow \text{Pic}(X),
\]

and any two points are linearly equivalent in \( \mathbb{P}^1 \), we must have that any two fibers (regardless of whether they are smooth elliptic curves or not) are linearly equivalent as well. We have that \( F \) is of genus 1, and hence since any two fibers are disjoint, we have that \( \langle F, F \rangle = 0 \).

We also have the class of the section \( O \) on \( X \). We see that \( O \) is a rational curve since \( C \) is. Furthermore, an application of the genus formula for curves on surfaces (see [3] I.15) shows that \( \langle O, O \rangle = -2 \). Finally, we have that a section intersects a fiber in one point by definition, and thus we have that \( \langle F, O \rangle = 1 \). Together, they form a sublattice of \( \text{NS}(X) \),

\[
\begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix} \cong \text{isom} \ U.
\]

Since \( U \) is a unimodular lattice, we have that this is a primitive embedding of \( U \), and that furthermore, we have \( \text{NS}(X) \cong M \oplus U \) for some lattice \( M \) an orthogonal decomposition of \( \text{NS}(X) \). We will see later that the converse actually holds as well.
Next we define a second source of algebraic cycles on an elliptic K3 surface. We have that any two sections $S_1$ and $S_2$ of the fibration $\pi : X \rightarrow C$ give us pairs of points on each fiber. The generic fiber being an elliptic curve, we may add $S_1$ and $S_2$ pointwise on non-singular fibers to obtain a third point on each non-singular fiber. Taking the completion of the the resulting curve gives us a third section of $X$. Under this operation, the sections of $X$ form a group.

**Definition 7.1.3.** The group formed by the sections on an elliptic surface is called the Mordell-Weil group of $X$, which we denote $\text{MW}(X)$. The group $\text{MW}(X)$ is finitely generated and the torsion subgroup of $\text{MW}(X)$ is denoted by $\text{MW}(X)_{\text{tor}}$.

Each section in $\text{MW}(X)$ will correspond to a $\mathbb{C}(C)$ rational point on $X$ as a curve over $\mathbb{C}(C)$. Alternatively, each section gives a rational curve in $X$ and hence gives rise to some class in $\text{NS}(X)$. We will see later how the groups $\text{NS}(X)$ and $\text{MW}(X)$ are related.

There is one more component to the construction of the Néron-Severi group of elliptic K3 surfaces. The extra algebraic cycles we wish to discuss now come from the resolution of singularities in the Weierstrass fibration of $X$. As we mentioned before, the Weierstrass fibration will have fibers which are singular elliptic curves. Often, these will even give rise to singularities in the Weierstrass fibration. If a singular fiber is actually a singularity in the surface, then we must resolve the singularity to find the surface $X$. In this case, since our singularities are of ADE type ([19]), their resolutions may be given as a sequence of monoidal transformations, and thus the divisor on $X$ associated fiber of the resolved Weierstrass fibration will look like an arrangement of rational curves.

Each rational curve will have self-intersection $-2$ (from the genus formula, [3] I.15), and will intersect other components in at most one point. If we draw a diagram where dots represent rational curves, and lines between dots represent simple intersection, then the diagram associated to a singular point in the Weierstrass fibration of $X$ will be the extended Coxeter-Dynkin diagram of an (extended) root system $\tilde{A}_n, n \in \mathbb{N}_{\geq 0}, \tilde{D}_n$ for $n \in \mathbb{N}_{\geq 4}$ or
\( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \) (hence ADE type). We have provided such diagrams for root systems of ADE type in Appendix A.5.

The divisor \( L \) associated to a singular point in the Weierstrass fibration of \( X \) is linearly equivalent to \( F \), and intersects any section of \( \pi \) in exactly one point belonging to a single irreducible component of the blown up singular fiber. We call this component \( L_0 \), and label the remaining irreducible components \( L_1, \ldots, L_n \). If we take the divisors associated to \( L \) except for \( L_0 \), these are inequivalent elements of \( \text{Pic}(X) \).

We have the following theorem.

**Theorem 7.1** (Shioda [28]). Let \( X \) be an elliptic surface. Let \( z_i \) be the points of \( C \) such that \( \pi^{-1}(z_i) \) is not a smooth elliptic curve. Let \( L_j^i, j \geq 1 \) be the components of the resolution of \( \pi^{-1}(z_i) \) not intersecting \( O \). Let \( S_i, i = 1, \ldots, r \) be generators of \( \text{MW}(X)/\text{MW}(X)_{\text{tor}} \) and let \( T_j, j = 1, \ldots, q \) be generators of \( \text{MW}(X)_{\text{tor}} \). Then we have that \( \text{NS}(X) \) is generated over \( \mathbb{Z} \) by \( L_j^i, F \) and \( O \) along with \( S_i \) and \( T_j \), and the only relations among these are satisfied by the torsion section \( T_j \).

What is more, if we take \( M \) to be the lattice generated over \( \mathbb{Z} \) by \( O \) and the vertical classes contained in the fibers of our fibration, then we have an exact sequence of groups

\[
0 \longrightarrow M \longrightarrow \text{NS}(X) \longrightarrow \text{MW}(X) \longrightarrow 0.
\]

and thus

\[
\text{MW}(X) \cong \text{NS}(X)/M.
\]

(See [19] Theorem VII.2).

### 7.1.2 Existence of elliptic fibrations.

We will prove an extension of a theorem of Kondō:
Theorem 7.2 (Kondō [14], §2). Let $X$ be a K3 surface, and let $E$ be an elliptic curve inside of $X$.

1. Then the linear equivalence class of $E$ determines an elliptic fibration of $X$. If we have a rational curve $O$ which intersects $E$ in one point, then $O$ is a section of this fibration.

2. Let $L$ be the sublattice $([E] \oplus [O])^\perp/[E]$ which is spanned by the roots in $([E] \oplus [O])^\perp/[E]$ (by which we mean elements $\gamma$ such that $\langle \gamma, \gamma \rangle = -2$). If $L$ is a sum of negative definite lattices $K_i$ of type $A_1, D_j$ or $E_6, E_7, E_8$, then the Jacobian fibration induced by $O$ and $E$ has singular fibers of corresponding type.

Proof of 1. The first part is proven by Kondō (loc. cit.). We give a sketch of his proof. If we have $E$ the class of some elliptic curve in $X$, then the Riemann-Roch theorem shows that $h^0(O_X(E)) = 2$ and thus the birational map induced by this linear system (see [3] II.6 for construction) is a map

$$\varphi_{[E]} : X \dashrightarrow \mathbb{P}^1.$$ 

This may be completed to a holomorphic map whose fibers are precisely the members of $|E|$. Using Bertini’s theorem ([11] Thm. 8.18) one shows that the generic member of this family is a smooth curve of genus 1. Thus we have obtained an elliptic fibration. If we have $O$ rational such that it intersects the generic member of $E$ in one point, we have that the map $O \dashrightarrow \mathbb{P}^1$ is of degree one, hence has an inverse which induces our section.

We prove the second claim, which is an extension of Kondō’s results (and methods), but first we prove the small lemma.

Lemma 7.1.1. Let $\gamma \in ([E] \oplus [O])^\perp/[E]$ satisfy $\langle \gamma, \gamma \rangle = -2$. Then either $\gamma$ or $-\gamma$ is a sum of irreducible rational components of the reducible fibers of the fibration induced by $E$ and $O$. 
CHAPTER 7. ELLIPTIC FIBRATIONS AND LATTICE POLARIZATIONS

Proof. Since $\gamma$ has self-intersection $-2$, then using the Riemann-Roch theorem for surfaces, we obtain that either $\gamma$ or $-\gamma$ is an effective divisor. Assume $\gamma$ is effective, then

$$\gamma = \sum a_i R_i$$

where $a_i \in \mathbb{Z}_{>0}$ and $R_i$ are classes of curves in $X$. We have that $\langle \gamma, F \rangle = 0$ and hence since $F$ is a NEF divisor and $a_i > 0$ for all $i$, we have $\langle R_i, F \rangle = 0$ for all $i$. Thus $R_i$ are curves which are disjoint from $F$. Now we have that the class of each $R_i$ must be contained in a vertical divisor of the fibration induced by $E$. Since $\gamma$ has no $E$ component, we have that $R_i$ is contained inside of some some reducible fiber. \qed

Now we will finish the proof of Theorem 7.2

Proof of 2. Let us take the sublattice of $((\mathbb{Z} \oplus \mathbb{Z}[O])^\perp)/[E]$ which is generated by roots. By Lemma 7.1.1, we see that this is precisely the lattice generated by components of the singular fibers of the fibration induced by $E$ and $O$. In particular, this lattice may be written uniquely in terms of root lattices $A_n$ for $n \in \mathbb{Z}_{\geq 0}$, $D_n$ for $n \in \mathbb{Z}_{\geq 4}$ or $E_6, E_7$ or $E_8$. On the other hand, the components not intersecting $O$ of each reducible fiber is a lattice of the same type. Thus the decomposition of $L$ is unique and given by the types of reducible fibers on the elliptic fibration induced by $E$ and $O$. \qed

Therefore, we have that the geometric fact of an elliptic fibration of $X$ is equivalent to the lattice theoretic decomposition of the Néron-Severi group of $X$.

7.2 Construction of moduli spaces.

In this section, we will construct various moduli spaces related to K3 surfaces. We begin by constructing the moduli space of elliptic K3 surfaces with reducible fibers of type $E_8$
and $D_7$, then we relate this moduli space to the moduli space of $E_8 \oplus D_7 \oplus U$-polarized K3 surfaces.

7.2.1 Constructing elliptic K3 surfaces with $E_8$ and $D_7$ reducible fibers.

We begin to construct the generic elliptic K3 surface with singular fibers of type $E_8$ at 0 and $D_7$ at $\infty$. The approach will be as follows. We begin by using Kodaira’s classification of irreducible fibers on elliptic surfaces. If we are given the Weierstrass equation associated to an elliptic surface $X$ then fiber above a point $[s_0 : t_0]$ is singular if and only if $\Delta(s_0, t_0) = 0$. Furthermore, Kodaira has provided a classification of the resolutions of singular fibers that depends only on the orders of vanishing $\Delta(s, t), A(s, t)$ and $B(s, t)$ at $[s_0 : t_0]$. We have duplicated this table in Appendix A.4, and another more detailed copy may be found in [19] pp. 41.

In particular, if we have the Weierstrass equation associated to $X$ given by

$$y^2 = x^3 + A(t, s)x + B(t, s)$$

where $A(t, s)$ is a section of $\mathcal{O}_{\mathbb{P}^1}(8)$ and $B(t, s)$ is a section of $\mathcal{O}_{\mathbb{P}^2}(12)$, and $X$ has a reducible fiber of type $E_8$ above 0, then we have that $A(s, t)$ vanishes at 0 to order 5, $B(s, t)$ vanishes at 0 to order 4, and the discriminant

$$\Delta(t, s) = 4A(t, s)^3 + 27B(t, s)^2$$

vanishes at 0 with order 10. If we have the first two conditions, then one may check that we get the third one for free, therefore just to have a singular fiber of type $E_8$ at 0, we must
begin with polynomials of the form

\[ A(t, s) := t^4(a_1 t^4 + a_2 t^3 s + a_3 t^2 s^2 + a_4 t s^3 + a_5 s^4) \]  
(7.1)

\[ B(t, s) := t^5(b_1 t^7 + b_2 t^6 s + b_3 t^5 s^2 + b_4 t^4 s^3 + b_5 t^3 s^4 + b_6 t^2 s^5 + b_7 t s^6 + b_8 s^7) \]  
(7.2)

for complex parameters \( a_i \) and \( b_j \). Now we will locate a singular \( D_7 \) fiber at \( \infty \). We have that a singular \( D_7 \)-fiber is characterized by the fact that \( A(t, s) \) vanishes with order 2 and that \( B(t, s) \) vanishes with order 3, and that \( \Delta(t, s) \) vanishes with order 9. In this case, the first two requirements do not imply the third. We must perform more manipulations. Thus the sections that we must begin with are

\[ A(t, s) := t^4 s^2(a_1 t^4 + a_2 t^3 s + a_3 t^2 s^2) \]

and

\[ B(t, s) := t^5 s^3(b_1 t^7 + b_2 t^6 s + b_3 t^5 s^2 + b_4 t^4 s^3 + b_5 t^3 s^4). \]

With these parameters, we have \( \Delta(s, t) \) vanishing only to order 6 generically. We must specialize the parameters \( a_i \) and \( b_i \) so that \( \Delta(s, t) \) vanishes generically to order 9. In other words, we must find some choice of coordinates \( b_i, a_i \) such that

\[ \Delta(t, s) = t^{10} s^9 \Delta_0(t, s) \]

for some polynomial \( \Delta_0(t, s) \) not vanishing at 0 or \( \infty \) generically. \( \Delta \) is a degree 24 polynomial in \( s \) and \( t \), and if it is to vanish to order 9 at \( \infty \), the coefficients of \( s^i t^j \) must vanish for \( i < 10 \). In general, by the structure of the coefficients \( A(t, s) \) and \( B(t, s) \) above, we have that the coefficients of \( s^i t^j \) vanish for \( i < 7 \), so it remains to find conditions under which
the coefficients of $s^6t^{18}, s^7t^{17}$ and $s^8t^{16}$ vanish. These coefficients are

$$4a_1^3 + 27b_1^2, 12a_1^2a_2 + 54b_1b_2$$

and

$$12a_1^2a_2 + 12a_1a_2^2 + 54b_1b_3 + 27b_2^2.$$ 

We parameterize the intersection of the vanishing locus of these three equations. In order to satisfy the first, we make the coordinate change

$$a_1 = 2q_1^3, a_2 = q_2, a_3 = q_3, b_1 = -3q_1^2, b_2 = p_1, b_3 = p_2, b_4 = p_3, b_5 = p_4. \quad (7.3)$$

and we have that the first equation, expressed in terms of variables $q_i$ and $p_i$ vanishes. Then, under this coordinate change, the second equation becomes

$$108(q_1^4q_2 + q_1^3p_1)$$

To parameterize this solution space, we make (again, injective) the change of variables,

$$q_1 = \lambda_1, q_2 = \lambda_2, q_3 = \lambda_3, p_1 = -\lambda_1\lambda_2, p_2 = \mu_1, p_3 = \mu_2, p_4 = \mu_3.$$ 

Under this change of variables, we are reduced to solving the equation

$$-32\lambda_1^5\lambda_3 + 108\lambda_1^2\mu_1 - 9\lambda_3^2.$$ 

We make the substitution,

$$\lambda_1 = \xi, \lambda_2 = a\xi^2, \lambda_3 = -\xi^2 \left(\frac{a^2 + 12b}{12}\right), \mu_1 = b\xi^3, \mu_2 = c\xi^3, \mu_3 = d\xi^3.$$ 

Therefore, we have the eventual description of our equation as

$$y^2 = x^3 + \xi^2(-3t^2 + ats - (a^2 + 12b)s^2/12)t^4s^2x + \xi^3(2t^4 - at^3s + bt^2s^2 + cts^3 + ds^4)t^5s^3.$$
Note that this is just a quadratic twist by the parameter $\sqrt{\xi}$ of

$$y^2 = x^3 + (-3t^2 + ats - (a^2 + 12b)s^2/12)t^4 s^2 x + (2t^4 - at^3 s + bt^2 s^2 + cts^3 + ds^4)t^5 s^3$$

and so we may remove the parameter $\xi$. We define

$$A(a, b, c, d) := (-3t^2 + ats - (a^2 + 12b)s^2/12)t^4 s^2$$

and

$$B(a, b, c, d) := (2t^4 - at^3 s + bt^2 s^2 + cts^3 + ds^4)t^5 s^3,$$

and arrive at the four-parameter family of elliptic surfaces

$$\mathcal{F}(a, b, c, d) : y^2 = x^3 + A(a, b, c, d)x + B(a, b, c, d).$$

Every elliptic K3 surface with reducible fibers of type $E_8$ at $0$ and $D_7$ at $\infty$ may be written in this form for some choice of $a, b, c, d$.

**The locus $d = 0$.**

We see that if we let the parameter $d$ vanish, then we have an extra power of $t$ coming out of $B(a, b, c, d)$. In other words, $B(a, b, c, 0) = t^6 \bar{B}(a, b, c)$ for the polynomial

$$\bar{B}(a, b, c) = (2t^3 - at^2 s + bt s^2 + cs^3)s^3.$$

Since $A$ does not depend on $d$ at all, nothing changes with respect to $A$. We will write $A$ as $\tilde{A}(a, b, c)t^4$ where

$$\tilde{A}(a, b, c) = (-3t^2 + ats + (a^2 - 12b)s^2/12)s^2.$$
Thus we may write

$$\mathcal{F}(a, b, c, 0) : y^2 = x^3 + \tilde{A}(a, b, c)t^4x + \tilde{B}(a, b, c)t^6.$$  \hspace{1cm} (7.4)

We see that if we make the rational change of variables, we get

$$[s : t] \times [x : y : z] \mapsto [s, t] \times [t^2x : t^3y : z]$$

we get the surface

$$t^6y^2 = t^6(x^3 + \tilde{A}(a, b, c)x + \tilde{B}(a, b, c)).$$  \hspace{1cm} (7.5)

The variety in \( \mathbb{P}^1 \times \mathbb{P}^2 \) represented by this equation is the reducible variety union of (six copies of) the hypersurface \([1 : 0] \times [x : y : z]\) and the rational elliptic surface given by the equation

$$y^2 = x^3 + \tilde{A}(a, b, c)x + \tilde{B}(a, b, c)$$

whose minimal resolution is an Enriques surfaces. Thus we must discard \( d = 0 \) from our base space if we are interested in just the K3 surfaces in this family.

### 7.2.2 Correspondence between elliptic fibrations and lattice polarizations.

The importance is that we now have established a generic form for elliptic K3 surfaces with singular fibers \( E_8 \) and \( D_7 \). Furthermore, Theorem 5.2 allows us to prove the following result.

**Proposition 7.2.1.** Let \( X \) be a K3 surface with lattice polarization by \( E_8 \oplus D_7 \oplus U \cong E_8^2 \oplus \langle 4 \rangle \), then there is an elliptic fibration of \( X \) whose Weierstrass form may be written in terms as \( \mathcal{F}(a, b, c, d) \) for some choice of parameters \( (a, b, c, d) \in \mathbb{C}^4 \).

**Proof.** By Lemma 2.1 of [14], we have that a primitive embedding of \( U \) into \( \text{NS}(X) \) induces an elliptic fibration of \( X \). If we have \( \text{NS}(X) \cong E_8 \oplus D_7 \oplus U \) then by Theorem 7.2 there is
an elliptic fibration of $X$ with reducible fibers of type $E_8$ and $D_7$.

If we have $\text{NS}(X)$ properly containing $E_8 \oplus D_7 \oplus U$, then we again have an elliptic fibration, and we have by Theorem 7.2 that the maximal sub lattice of $U^\perp/[F]$ (where $[F]$ is the class of the fiber) spanned by roots in $\text{NS}(X)$ determines the reducible fibers of the elliptic fibration on $X$. We will call this lattice $L_{\text{root}}$. Since $X$ is $E_8 \oplus D_7 \oplus U$-polarized, we have that $E_8 \oplus D_7$ is a primitive sublattice of $L_{\text{root}}$. We know that $E_8$ embeds as a direct summand of $L_{\text{root}}$ and thus

$$L_{\text{root}} = E_8 \oplus K$$

where $K$ is a root lattice into which $D_7$ embeds primitively. If $L_{\text{root}}$ has rank $\leq 18$, then $K$ has rank $\leq 10$. We determine that the only root lattices of rank $\leq 10$ into which $D_7$ embeds primitively are direct summands

$$D_l \oplus A_i \oplus A_j \oplus A_k$$

where

$$7 \leq l \leq 10, i, j, k \geq 0 \text{ and } 7 \leq l + i + j + k \leq 10.$$

One notes by Kodaira’s classification, that any of these fibrations correspond to $A(s, t)$ vanishing to order 2 at $\infty$, and 4 at 0, $B(s, t)$ vanishing to order 3 at 0 and 4 at $\infty$, and $\Delta(s, t)$ vanishing to order 10 at $\infty$ and 9 at 0 (just as in the case of $E_8$ and $D_7$ reducible fibers), and impose extra conditions on the vanishing of $\Delta(s, t)$ at certain points in $\mathbb{P}^1$. Therefore, they correspond to subfamilies of $\mathcal{F}(a, b, c, d)$. This proves the proposition. \hfill \Box

**Definition 7.2.1.** If $X$ is a $K3$ surface with elliptic fibration whose reducible fibers are $E_8$ and fibers corresponding to the root lattices in Equation 5.2.8, then we say that the elliptic fibration of $X$ has at least $E_8$ and $D_7$ reducible fibers.

We may rephrase Proposition 7.2.1 as the statement that if $X$ is an elliptic K3 surface
with at least $E_8$ and $D_7$ reducible fibers, then there is a point $(a, b, c, d)$ such that $F(a, b, c, d)$ is a Weierstrass equation of $X$.

### 7.2.3 The moduli space of elliptic $K3$ surfaces with at least $E_8$, $D_7$ reducible fibers.

Next it will be shown that after a certain weighted projective scaling, we may take the space $F(a, b, c, d)$ as the moduli space of elliptic $K3$ surfaces with at least $D_7$ and $E_8$ fibers.

**Definition 7.2.2.** Two elliptic $K3$ surfaces $(X_1, \pi_1, \mathbb{P}^1)$ and $(X_2, \pi_2, \mathbb{P}^1)$ are isomorphic if there is an isomorphism of total spaces $f : X_1 \rightarrow X_2$ and an automorphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $\pi_2 \cdot f = \phi \cdot \pi_1$.

In particular, if we want two elliptic surfaces to be isomorphic, they must have the same types of singular fibers, and there must be some map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that if the fiber type of $X_1$ at the point $(s_1, t_1)$ is the same as the fiber type of $X_2$ at $\phi(s_1, s_2)$.

**Proposition 7.2.2.** If we have a complex number $\lambda \neq 0$ and two elliptic surfaces satisfying $X_1 = F(a_0, b_0, c_0, d_0)$ and $X_2 = F(\lambda a_0, \lambda^2 b_0, \lambda^3 c_0, \lambda^4 d_0)$ then $X_1$ and $X_2$ are isomorphic elliptic surfaces.

**Proof.** Define the automorphism of $\mathbb{P}^1$,

$$ [t, s] \mapsto [t/\lambda, s]. $$

Then we may pullback the elliptic surface $X_1$ along this map, to get the surface

$$ y^2 = x^3 + \left( \frac{1}{\lambda} \right)^6 (-3t^2 + \lambda aty - ((\lambda a)^2 + 12(\lambda^2 b))s^2/12)t^4s^2x $$

$$ + \left( \frac{1}{\lambda} \right)^9 (2t^4 - \lambda at^3s + \lambda^2 bt^3s^2 + \lambda^3 ct^3s^3 + \lambda^4 dt^3s^4)t^5 $$

which is a quadratic twist of $X_2$ by $(1/\lambda)^{3/2}$, hence this is isomorphic to $X_2$. 

$\square$
Finally, we prove necessity.

**Proposition 7.2.3.** If two elliptic K3 surfaces $F(a_0, b_0, c_0, d_0)$ and $F(a_1, b_1, c_1, d_1)$ are isomorphic, then there is some $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$(a_1, b_1, c_1, d_1) = (\lambda a_0, \lambda^2 b_0, \lambda^3 c_0, \lambda^4 d_0).$$

In other words, isomorphism classes of elliptic K3 surfaces with singular $D_7$ and $E_8$ fibers are parameterized the Zariski open subset of $\mathbb{W}(1, 2, 3, 4)$ given by $\mathbb{W}(1, 2, 3, 4) \setminus \{d = 0\}$.

**Proof.** Recall that two elliptic surfaces are isomorphic under the condition that there is an isomorphism of their base spaces which induces the isomorphism of total spaces under pullback. Thus there must be some automorphism of $\mathbb{P}^1$ inducing this isomorphism. Secondly, this automorphism must fix 0 and $\infty$ since it must preserve fiber type. The only such automorphisms of $\mathbb{P}^1$ may be represented just by multiplication by a non-zero $\lambda \in \mathbb{C}$. Therefore, we will let $\phi$ be the automorphism of $\mathbb{P}^1$,

$$[t : s] \mapsto [t/\lambda : s].$$

Following [19] II.5 we have that two elliptic fibrations given by $A_i(t, s)$ and $B_i(t, s)$ for $i = 1, 2$ are isomorphic in the sense above if and only if we have that there is some constant $\mu$ such that

$$A_0(t/\lambda, s) = \mu^8 A_1(t/\lambda, s)$$

and

$$B_0(t/\lambda, s) = \mu^{12} B_1(t/\lambda, s).$$

Therefore, we may proceed as in Proposition 7.4.2. We see that we must have $\mu = (\frac{1}{\lambda})^{3/4}$ in order to match up the leading coefficients. Then a calculation following the same lines as Proposition 7.4.2 shows that we must have the coefficients related as in the statement of
the current proposition.

7.2.4 The moduli space of $E_8 \oplus D_7 \oplus U$ polarized K3 surfaces.

In this section, we complete the proof of our main theorem as stated in the introduction. The final piece is as follows

**Proposition 7.2.4.** Let $X$ be an elliptic K3 surface with at least $E_8$ and $D_7$ reducible fibers.

1. The K3 surface $X$ has a canonical $E_8 \oplus D_7 \oplus U$ polarization $\iota$.

2. If $(X, \iota)$ and $(X', \iota')$ are two elliptic K3 surfaces with at least $E_8$ and $D_7$ reducible fibers where $\iota$ and $\iota'$ are the lattice polarizations induced by the elliptic fibrations in 1, then they are isomorphic as elliptic surfaces if and only if they are isomorphic as lattice polarized K3 surfaces.

**Proof of 1.** We construct a lattice polarization by $E_8 \oplus D_7 \oplus U$. From the elliptic fibration, we have distinguished classes $F$ and $(F + O)$ which span a copy of $U$ in $\text{NS}(X)$, and we have distinguished classes $S_1, \ldots, S_8$ which are the classes of the rational curves not intersecting $O$ which make up the $E_8$ reducible fiber. Finally, if $X$ has at least $E_8$ and $D_7$ reducible fibers, then the elliptic fibration of $X$ has reducible fiber of type $D_n$ for $n = 7, 8, 9$ or 10. If $R_1, \ldots, R_n$ are the classes of rational curves which make up $D_n$, then there is a natural subset of $R_i$ which span a copy of $D_7$.

We define $\iota$ to be the lattice polarization that sends

$$F \oplus (F + O) \mapsto U, E_8 \mapsto \text{span}_\mathbb{Z}\{S_1, \ldots, S_8\} \text{ and } D_7 \mapsto \text{span}_\mathbb{Z}\{R_1, \ldots, R_7\}.$$

**Proof of 2.** Assume $X$ and $X'$ are isomorphic as elliptic surfaces, and that $F$ and $F'$ represent the classes of fibers in the fibrations on $X$ and $X'$ respectively, and that $O$ and $O'$
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represent the classes of zero sections on \(X\) and \(X'\) respectively. Then we have Weierstrass fibrations of \(X\) and \(X'\) in which are related via some automorphism of \(\mathbb{P}^1\). Therefore, the isomorphism must take \(F\) to \(F'\), and similarly it must take \(O\) to \(O'\). Furthermore, we must have this isomorphism taking the fibers \(E_8\) and \(D_n\) in \(X\) to the corresponding \(E_8\) and \(D_n\) fibers in \(X'\). These conditions imply that the rational curves spanning \(D_n\) and \(E_8\) in \(X\) are sent to the correct rational curves in \(X'\) spanning \(D_n\) and \(E_8\). Therefore \(X\) and \(X'\) are isomorphic as lattice polarized K3 surfaces.

Conversely, if \((X, \nu)\) and \((X', \nu')\) are isomorphic as lattice polarized K3 surfaces, the isomorphism between \(X\) and \(X'\) which makes them isomorphic as lattice polarized K3 surfaces must send \(F \in \text{NS}(X)\) to \(F' \in \text{NS}(X')\). This implies that the linear system of elliptic curves on \(X\) providing elliptic fibration must be sent to the linear system of elliptic curves on \(X'\) which provide its elliptic fibration. Thus \(X\) and \(X'\) are isomorphic as elliptic surfaces. \(\square\)

Therefore we have proved:

**Theorem 7.3.** Let

\[ N = E_8 \oplus D_7 \oplus U. \]

For \((a, b, c, d) \in \mathbb{C}^4\), denote by \(\mathcal{F}(a, b, c, d)\) the family of singular elliptic surfaces in \(\mathbb{P}^2 \times \mathbb{P}^1\) with \([x : y : z]\) coordinates on \(\mathbb{P}^2\) and \([s : t]\) coordinates on \(\mathbb{P}^1\) given by the equations

\[ \mathcal{F}(a, b, c, d) : y^2 z = x^3 + A(s, t)xz^2 + B(s, t)z^3 \]

where \(A\) and \(B\) are polynomials depending on the parameters \(a, b, c\) and \(d\), given as

\[ A(s, t) = (-3t^2 + ast + (a^2 - 12b)s^2/12)t^4s^2 \] \hspace{1cm} (7.7)

\[ B(s, t) = (2t^4 - ast^3 + bs^2t^2 + cs^3t + ds^4)t^5s^3. \] \hspace{1cm} (7.8)
1. If \( d \neq 0 \), the minimal resolution of \( \mathcal{F}(a,b,c,d) \) is a K3 surface with a canonical \( N \)-polarization coming from the resolution of the singular points in \( \mathcal{F}(a,b,c,d) \).

2. Given any \( N \)-polarized K3 surface \( X \), there is a unique point \([a : b : c : d] \in \mathbb{WP}^3(1,2,3,4)\) such that \( X \) is isomorphic to the minimal resolution of \( \mathcal{F}(a,b,c,d) \). In other words, if we put

\[
\mathcal{M}^N_{K3} := \{[a : b : c : d] \in \mathbb{WP}^3(1,2,3,4), d \neq 0\}.
\]

Then \( \mathcal{M}^N_{K3} \) is a moduli space of K3 surfaces with \( N \)-polarization. Furthermore we have the isomorphism

\[
\mathcal{M}^N_{K3} \cong \mathfrak{h}_2/(Sp_4(k,\mathbb{Z}) \cup \sigma_{GH}Sp_4(k,\mathbb{Z})).
\]

**Remark 7.2.1.** There is a prevailing philosophy that says that once one has a family of lattice polarized K3 surfaces whose moduli space is a classical modular domain, then the parameters of this family should be modular forms on the moduli space of lattice polarized K3 surfaces. For instance, Clingher and Doran [6] have shown that the moduli space of \( E_8^2 \oplus \langle 2 \rangle \) polarized K3 surfaces may be written as the family of singular quartic hypersurfaces in \( \mathbb{P}^3 \)

\[
y^2zw - 4x^3z + 3\alpha xzw^2 + \beta zw^3 + \gamma xz^2w - \frac{1}{2}(\delta z^2w^2 + w^4) = 0
\]

and that the parameters \( \alpha, \beta, \delta \) and \( \gamma \) are classical Siegel modular forms of genus two.

In light of previous comments, we could have just as easily taken the affine chart on \( \mathbb{WP}(1,2,3,4) \) where \( d = 1 \) since \( \mathcal{F} \) degenerates outside of this chart. The reason for using such a construction is that our parameters \( a,b,c,d \) should be represented by modular forms where the weights in weighted projective space correspond to the weights of the respective
modular forms. Using unpublished work of A. Clingher and C. F. Doran, one may indeed express $a, b, c, d$ as restrictions of specific theta functions. These calculations will be featured in future work.

7.3 An alternate fibration.

Recently, work has been done which constructs all of the different elliptic fibrations of certain K3 surfaces. Bertin and Lecacheux [4] have constructed all of the elliptic fibrations of a specific K3 surface with Picard rank 20, and Kumar [15] has presented all of the possible elliptic fibrations of the generic Kummer surface of a principally polarized abelian surface.

We will stop far short of enumerating the possible elliptic fibrations of K3 surfaces with $\text{NS}(X) \cong \langle 4 \rangle \oplus E_8^2$, however, we will construct one example which will be convenient in the future.

First we note the following. By Theorem 7.2, an elliptic fibration on $X$ is equivalent to the linear system associated to an elliptic curve. If we have an elliptic fibration, and the local parameter on $\mathbb{P}^1$ is $t$, then if we take the linear system associated to the function $t$ on $X$, this is the linear system of the divisor associated to the fiber $t = 0$, and hence the linear system associated to the divisor $t = 0$ is the linear system inducing the fibration.

Now the method with which we construct an elliptic fibration of a K3 surface is as follows.

[Step 1.] Identify an effective divisor $D$ in $\text{NS}(X)$ which corresponds to the desired singular fiber. This is done by drawing a diagram of $\text{NS}(X)$ where dots represent rational curves on $X$ coming from reducible fibers or sections, and then finding a configuration of rational curves corresponding to the rational curves in the the desired root lattice (see Figure 7.1 for example). Compute the linear system of $|D|$. If $D$ is the divisor associated to a reducible fiber, then the generic member of $|D|$ is an elliptic curve. Therefore the Nakai-Moishezon ([1] IV 5.4) criterion tells us that $\mathcal{O}_X(D)$ is
an ample line bundle, and therefore, the Riemann-Roch theorem (Beauville [3], I.12) shows that \( h^0(D) = 2 \). One may use [3] Prop. VIII.13 (ii) to see that \( |D| \) is base-point free and hence provides an elliptic fibration. Determine two rational functions \( \xi_1(x) \) and \( \xi_2(x) \) on \( \mathbb{P}^1 \times \mathbb{P}^2 \) which provide global sections of \( \mathcal{O}_X(D) \) when restricted to \( X \). Once this is done, we have that the map to \( \mathbb{P}^1 \) determined by \( |D| \) is given by

\[
x \in X \mapsto [\xi_1(x) : \xi_2(x)].
\]

Therefore, the local coordinate on \( X \) under this new elliptic fibration is given by \( w = \xi_1(x)/\xi_2(x) \). Express \( X \) in Weierstrass form in terms of the variable \( w \). This is perhaps the most difficult part to complete, since it requires that one make nontrivial computations.

### 7.3.1 The Kummer surfaces of (1, 2)-polarized abelian surfaces.

Recall from Chapter 2 (Definition 2.2.3) that when one defines the Shioda-Inose structure on a K3 surface, one begins by finding a symplectic involution \( \iota \) on the K3 surface \( X \) which fixes precisely eight points. Let \( \tilde{X} \) be the surface \( X \) blown up at the fixed points of \( \iota \). If \( Y \) is the quotient of \( \tilde{X} \) by \( \iota \), the induced involution on \( \tilde{X} \), then \( Y \) is a Kummer surface such that \( T(X)(2) \cong T(Y) \) (here \( \cong \) denotes isometry of polarized Hodge structure). The purpose of this section is to exhibit the Kummer surface associated to a K3 surface with \( E_8^2 \oplus (4) \)-polarization.

In [34], Van Geemen and Sarti give a very concrete way to represent this involution when \( X \) is elliptic K3 surface with Shioda-Inose structure and an elliptic fibration in specific cases. Their construction proceeds as follows; we take an elliptic fibration of \( X \) such that the elliptic fibration carries a two-torsion section \( P \). After making a change of variables,
such a surface may be written as

\[ y^2 = x(x^2 + 2f(t,s)x + g(t,s)), \]

where \( f(t,s) \) and \( g(t,s) \) are homogeneous rational functions of degree 4 and 8 respectively, and our two-torsion section is located at \((x,y) = (0,0)\). Then one may take the quotient of such a surface by the two torsion section and the result is again an elliptic surface, now written in the form

\[ y^2 = x(x^2 - 2f(t,s)x + (f(t,s)^2 - 4g(t,s))). \]

In particular, [34] constructs a family of K3 surfaces with elliptic fibration with a reducible fiber of type \( I_{16} \), a two-torsion section, and which has generic Mordell-Weil rank 0. We write down this family in a slightly modified way as

\[ G(\alpha, \beta, \delta, \gamma) : y^2 = x(x^2 - 2(\alpha t^4 + \beta t^3 + \delta t^2 + 1)x + \gamma t^8) \text{ with } \gamma \neq 0. \]

If we had \( \gamma = 0 \), then the surface in question is singular at every fiber. A computation similar to Proposition 7.2.3 shows that if we have \( \lambda \in \mathbb{C}^\times \), then we have \( G(a, b, c, d) \cong G(\lambda^2 a, \lambda^3 b, \lambda^4 c, \lambda^8 d) \). Thus our variables \( \alpha, \beta, \delta, \gamma \) may be taken as parameters \([\delta : \beta : \alpha : \gamma] \in \mathbb{P}^3(2, 3, 4, 8)\).

Furthermore, [34] Proposition 4.7 shows that this family of surfaces has generic Néron-Severi lattice isomorphic to \( (4) \oplus E_8^2 \). Therefore, generically, each K3 surface in this family will admit a fibration with singular fibers \( E_8 \) and \( D_7 \). In other words, to each point in \( \mathbb{P}^3(2, 3, 4, 8) \) representing a elliptic K3 surface with singular \( I_{16} \) fiber and two-torsion section, there is a point in \( \mathbb{P}^3(1, 2, 3, 4) \) which corresponds to an isomorphic K3 surface bearing a fibration with reducible \( E_8 \) and \( D_7 \) fibers.
We will construct a two to one ramified covering,

$$\varphi : \mathbb{P}^3(2,3,4,8) \longrightarrow \mathbb{P}^3(1,2,3,4)$$

which tells us how to write a K3 surface with $I_{16}$ reducible fiber as a K3 surface with $E_8$ and $D_7$ reducible fiber. Furthermore since we have the family of elliptic Kummer surfaces related by to $\mathcal{G}$ by quotient by two-torsion section written as

$$K\mathcal{G} : y^2 = x(x^2 - 2A(t)x + (A(t)^2 - 4\gamma t^8)) \text{ with } A(t) = \alpha t^4 + \beta t^3 + \delta t^2 + 1,$$

then this map will allow us to write down the Weierstrass form of the Kummer surfaces associated any K3 surface in the moduli space of K3 surfaces with $U \oplus E_8 \oplus D_7$ polarization. We will show that $\varphi$ may be written as the two to one map

$$\varphi : [\delta : \beta : \alpha : \gamma] \mapsto [72\delta : 120\delta^2 + 72\alpha : 72\alpha\delta - 27\beta^2 - 2\delta^3 : 216\gamma].$$

Note that this is a well defined map of weighted projective spaces. Furthermore the image of $\delta = 0$ in $\mathbb{P}^3(2,3,4,8)$ is precisely the locus $d = 0$. Finally, notice that this is generically two to one. The fiber of a point $[a : b : c : d]$ is a pair of points if $\beta \neq 0$, and the map is injective if $\beta = 0$. Thus $\varphi$ is a double covering which ramifies on $\beta = 0$. In fact, we may write down the preimage of a point $[a : b : c : d]$,

$$\varphi^{-1}([a : b : c : d]) = \left[ \frac{a}{72} : \pm \sqrt{\frac{ab}{5184} - \frac{61a^3}{13436928} - \frac{c}{72} : \frac{b}{72} - \frac{5a^2}{1552} : \frac{d}{216}} \right]$$

and thus the ramification locus of $\varphi$ is the preimage of the hypersurface defined by the weight 3 homogeneous equation

$$ab - \frac{61a^3}{2592} - 72c = 0.$$
Figure 7.1: The dots in this figure represent roots in our lattice (elements with self-intersection \(-2\)), and lines represent intersection of roots. The large dotted box encloses the roots associated to \(I_{16}\), and the solid roots represent its irreducible components. The white dot represents the identity section of our fibration, the dotted box encloses the roots that will be associated to the generic fiber of our secondary fibration, and the root denoted \(O'\) will provide the identity section on our secondary fibration.

Later, we will demonstrate that this hypersurface is worth considering further. If \(X\) is an elliptic K3 surface with \((a, b, c, d)\) satisfying the equation above, then the fibration of \(X\) has a section of infinite order.

**Theorem 7.4.** The map \(\varphi\) given above gives a correspondence between the space of elliptic fibrations of K3 surfaces with singular fiber \(I_{16}\) at 0 and two torsion section and the space of elliptic fibrations of K3 surfaces with singular fibers \(E_8\) and \(D_7\) at 0 and \(\infty\) respectively.

3. **Proof.** As we explained in the introduction to this section, the problem comes down to finding two sections on the line bundle \(\mathcal{O}(D)\) for some divisor \(D\) in \(\mathcal{G}(\alpha, \beta, \delta, \gamma)\) made up of irreducible components corresponding to the fiber type \(E_8\). One may draw a diagram corresponding to some obvious roots in the Néron-Severi group of an elliptic K3 surface with a singular fiber of type \(I_{16}\). We have the zero section of our fibration of \(X\), which we
will denote by $O$, and we have the components of the fiber. We know how the components of the fiber intersect, and we have that $O$ intersecting the singular fiber in precisely one point. We will call this point the identity component of the singular fiber. Upon drawing this diagram, it is easy to find a subset of these roots which correspond to a root system of type $E_8$, as is exhibited in Figure 6.1. We also see that there is a root that will act as the zero section in our new fibration, which is written as $O'$ and indicated in Figure 6.1. If we let our divisor be

$$D = 3O + 6R_0 + 5R_1 + 4R_2 + 3R_3 + 2R_4 + O' + 2R_{14} + 4R_{15},$$

a section of $\mathcal{O}_\mathbb{P}(D)$ has poles at most of order 3 at the zero section, and a pole of order at most 6 on the identity component of the $I_{16}$ fiber. Since $D$ is effective, it is clear that at least the constant section will be a section of this bundle, thus it remains to find one more. Such a section may be written

$$w(x, y, z) = \frac{y + x(t^2 + 1)}{t^6}.$$

Note that $y$ has a pole of order 3 at $\infty$, and hence has a pole of order 3 on $O$, and that $t^6$ has a pole of order 6 on the identity component of the $I_{16}$ fiber. Rearranging this equation, we find $y = wt^6 - x(t^2 + 1)$. Putting this into the equation for $G(\alpha, \beta, \delta, \gamma)$, and doing a non-trivial amount of rearranging, one finds the equation,

$$t^2 = x^3 + A(w, z)x + B(w, z),$$
where we have the following expressions for $A(w, z)$ and $B(w, z)$,

$$
A(w, z) = \left( -\frac{8\delta \gamma w^3 z}{3} - \frac{\delta w^4}{3} - 4\alpha w^4 - \frac{4\gamma w^2 z^2}{3} \right),
$$

$$
B(w, z) = \left( -\beta^2 w^6 z^6 + 8 w^7 z^5 + \frac{16\delta \gamma^2 w^4 z^8}{9} + \frac{8 \alpha \gamma w^5 z^7}{3} + \frac{16 \delta^2 \gamma^2 w^5 z^7}{27} - \frac{2 \delta^3 w^6 z^6}{27} + \frac{16 \gamma^3 w^3 z^9}{27} \right).
$$

A calculation shows that this family of elliptic surfaces indeed has fibers of type $E_8$ and $D_7$. Furthermore, this is in the standard form for elliptic K3 surfaces with $E_8$ and $D_7$ fibers as determined earlier in this section. The parameters are

$$
a = 9(\delta / \gamma) \quad (7.12)
$$

$$
b = 15(\delta / \gamma)^2 - 9(\alpha / \gamma^2) \quad (7.13)
$$

$$
c = 9(\alpha \delta / \gamma^3) - (27/8)(\beta^2 / \gamma^3) - \varepsilon^3 / (4\gamma^3) \quad (7.14)
$$

$$
d = 27 / \gamma^3. \quad (7.15)
$$

Putting this into weighted projective coordinates and simplifying, we may express our map as

$$
\varphi : [\delta : \beta : \alpha : \gamma] \mapsto \left[ 72\delta : 120\delta^2 + 72\alpha : 72\alpha \delta - 27\beta^2 - 2\delta^3 : 216\gamma \right].
$$

**Remark 7.3.1.** In the course of the preceding variable transformations, one derives a model for the family $\mathcal{G}$ as a family of quartics in $\mathbb{P}^4$. This quartic model is written as

$$
Q\mathcal{G}(\alpha, \beta, \delta, \gamma) : x^3 z + (\alpha - 1)x^2 z^2 - 2x w z^2 - w^2 z^2 + \beta x^2 t z + \gamma x z^3 - \delta x w t^2 = 0
$$

These quartics are not generically smooth: for any $\alpha, \beta, \delta$ and $\gamma$, there are singular points at $x = z = t = 0$ and $x = z = w = 0$. 

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Now we recall that the Kummer surfaces related to the family $\mathcal{G}$ is given by the equations,

$$\mathcal{KG}(\alpha, \beta, \delta, \gamma): y^2 = x(x^2 - 2(\alpha t^4 + \beta t^3 + \delta t^2 + 1) + (\alpha t^4 + \beta t^3 + \delta t^2 + 1)^2 + 4\gamma t^8).$$

To be explicit, given a K3 surface $X$ with elliptic fibration with fiber types $E_8$ and $D_7$, we may write the Kummer surface of $X$ by using our description of $\varphi^{-1}$ This gives a description of the moduli space of Kummer surfaces of abelian surfaces with $(1, 2)$-polarization. Such abelian surfaces have been discussed at length by Barth [2], which will be the topic of discussion in the next subsection.

7.3.2 An extended comment on elliptic fibrations of our Kummer surfaces.

It is clear that the Kummer surface we get from the family $\mathcal{KG}$ is a family of Kummer surfaces related to abelian surfaces with $(1, 2)$-polarization. There is a canonical elliptic fibration on the Kummer surfaces associated to such abelian surfaces, as was exposed by Barth in [2]. To wit: let $D$ be an ample divisor on $A$ which provides the $(1, 2)$-polarization. By the Nakai-Moishezon criterion, we have that $D$ is effective, and a simple application of Riemann-Roch tells us that $\langle D, D \rangle = 4$. Barth proves that if we assume that $A$ is simple, then there are exactly three possible types of divisor in the linear system of $D$, they are ([2] 1.2),

1. A smooth connected curve of genus 3,

2. An irreducible curve of geometric genus 2 which has a double point, and

3. $E + F$ where $E$ and $F$ are elliptic curves and $\langle E, F \rangle = 2$ (note that in this case $A$ is not a simple abelian surface).

Furthermore Barth shows that this linear system is actually a pencil of genus 3 curves, that the basepoint locus is four points (which are located at four of the sixteen two-torsion
points of $A)$, and hence by Bertini’s theorem the generic member in this linear system is a smooth curve of genus 3.

Subsequently, we see ([2] Proposition 1.6) that the involution on $A$ which sends $x \mapsto -x$ fixes each member of the linear system $|D|$ and ([2] 1.7) that if a curve $C$ in $|D|$ is generic, $C/\iota$ is a smooth elliptic curve. Finally, ([2] 1.7) also shows that if a curve $C \in |D|$ passes through one of the sixteen two-torsion points of $A$ which are not in the base locus of $|D|$, then $C$ is a singular curve.

In summation, we obtain a fibration

$$A \longrightarrow \mathbb{P}^1$$

with generic fiber a smooth curve of genus 3. Since $\iota$ preserves fibers of this fibration, and the quotient $C/\iota$ is generically an elliptic curve, this fibration filters as

$$A \longrightarrow \text{Kum}(A) \longrightarrow \mathbb{P}^1,$$

and induces an elliptic fibration on $\text{Kum}(A)$. Furthermore, this is a smooth minimal fibration of a K3 surface, and there are twelve reducible fibers with two components each. Therefore, by the Kodaira classification of reducible fibers of elliptic fibrations, we have that these are singular fibers of type $A_1$, and since each has two irreducible components we have that our reducible fibers have a total of 24 rational components. Therefore by a [19] II.5.7, this fibration does not contain more reducible fibers.

Therefore, a the Kummer surface of any $(1,2)$-polarized abelian surface has an elliptic fibration with 12 singular fibers of type $A_1$. Hence given a member of $K\mathcal{G}$, there exists alternate fibration with twelve fibers of type $A_1$ and Mordell-Weil rank of three (compare to [18] Proposition 5.1).
The Kummer surfaces of abelian surfaces with $\text{(1, 2)}$-polarization given by equations $\mathcal{KG}$ have one singular fiber of type $I_8$ and sixteen fibers of type $I_1$ (see [34] Proposition 4.7). The above comments establish that a more natural fibration on $X$ from the point of view of the geometry of the associated abelian surfaces has 12 fibers of type $A_1$. This is an avenue that will be explored by the author in future work.
Chapter 8

Humbert Surfaces and Their Intersections

In this chapter, we will exhibit a number of families of K3 surfaces contained in $\mathcal{F}(a, b, c, d)$ on which the abelian surfaces to which these K3 surfaces are associated via Shioda-Inose structure exhibit endomorphism structure as well. By the work done in Chapters 2 to 6, this is equivalent to the fact that the generic Néron-Severi rank of these sub-families is either 18 or 19.

8.1 Equations of Humbert surfaces.

In this section, we find the some subvarieties of $\mathbb{W}P^3(1, 2, 3, 4)$ on which $\mathcal{F}(a, b, c, d)$ has Picard rank generically 18. We will put the machinery developed in Chapter 5 to work; we will describe the moduli space of these lattice enhancements in detail, and use this to find weighted projective models for quotients of some symmetric Hilbert modular surfaces.

Before we begin, we must count the number of Humbert surfaces in $\mathfrak{h}_2/Sp_4(k, \mathbb{Z})$, the moduli space of $k$-polarized abelian surfaces. In [31] Thm. 2.4, the following fact is proved,

**Proposition 8.1.1.** Let $H_\Delta$ be the union of all Humbert surfaces in $\mathfrak{h}_2/Sp_4(k, \mathbb{Z})$ with
discriminant $\Delta$. Then the number of irreducible components of $H_\Delta$ is

$$\#\{h \mod 2k : h^2 \equiv \Delta \mod 4k\}.$$ 

In the case that we are interested in, $k = 2$, so this gives us the following proposition,

**Proposition 8.1.2.** Let $\Delta \equiv 0, 1 \mod 4$. The number of components of $H_\Delta$ in $Sp_4(2, \mathbb{Z})$ is given as follows:

[Case 1:] If $\Delta \equiv 0 \mod 8$, then there is just one component of $H_\Delta$. Furthermore if $\Delta = 4 \cdot 2a$ then the lattice representing $H_\Delta$ may be written as

$$\begin{pmatrix} -2a & 0 \\ 0 & 4 \end{pmatrix}.$$

If $\Delta \equiv 4 \mod 8$, then there is again just one component of $H_\Delta$. Furthermore if $\Delta = 4 \cdot 2a + 4$ then we may write the lattice representing $H_\Delta$ as

$$\begin{pmatrix} -2a & 2 \\ 2 & 4 \end{pmatrix}.$$

If $\Delta \equiv 5 \mod 8$, then $H_\Delta$ is empty. If $\Delta \equiv 1 \mod 8$, then $H_\Delta$ has precisely two components. The lattices representing the distinct components of $H_\Delta$ look like

$$\begin{pmatrix} -2a & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2b & 3 \\ 3 & 4 \end{pmatrix}$$

for some integers $b$ and $a$. 
8.1.1 $\Delta = 24$: The ramification locus.

Recall that in 6.3.1 we mentioned that the map $\varphi : \mathbb{P}^3(2, 3, 4) \rightarrow \mathbb{P}^3(1, 2, 3, 4)$ ramifies over the locus $\beta = 0$ where $\beta$ is the variable on $\mathbb{P}^3(2, 3, 4)$. The image of this subvariety is cut out by the homogeneous equation

$$ab - \frac{61a^3}{2592} - 72c = 0.$$ 

We denote this hypersurface $V_{24}$. Now we will show that for a generic element $X = \mathcal{F}(a, b, c, d)$ with $[a : b : c : d] \in V_{24}$, $\text{NS}(X)$ has discriminant 24. If we begin with the family $\mathcal{F}$ and we naively make the variable transformation which sets

$$c = \frac{1}{72} \left(ab - \frac{61a^3}{2592}\right)$$

the resulting equation is a bit daunting. A smarter approach is to approach this via the family $\mathcal{G}(\alpha, \beta, \delta, \gamma)$. We take the sub-family of $\mathcal{G}$ where $\beta = 0$. We have the family

$$\mathcal{G}(\alpha, 0, \delta, \gamma) : y^2 = x(x^2 + (\alpha t^4 + \delta t^2 s^2 + s^4)x + \gamma t^8).$$

We find the section $S$ given by the equations

$$y = \lambda \left(\frac{\delta t^2}{2} + s^2\right) \text{ and } x = \lambda t^4,$$

where $\lambda \in \mathbb{C}$ is a root of

$$x^2 + \left(\alpha + \frac{\delta^2}{4}\right)x + \gamma.$$ 

We see that for $t = 0$, we have the section $S$ intersecting the zero section with intersection index 1, and this is the only value of $t$ for which $x = 0$ and hence the only point of intersection between $S$ and $O$. A calculation following [16] pp. 205 shows that $S$ is generically a section.
of height 2. We now investigate the lattice generated by the singular fibers and $S$.

We calculate at least the discriminant of $NS(X)$. Let us take $F_i$ the lattice corresponding to the non-identity components of a reducible fiber on an elliptic K3 surface $X$, and we will take the lattice $L$ to be the sublattice of $NS(X)$ generated by non-torsion sections and the zero section. Let $MW(X)_{tor}$ denote the torsion part of the Mordell-Weil group of $X$, and let $K_i$ be the lattice generated by the components of the the $i$th reducible fiber, and which do not intersect $O$. Then [28] Cor. 1.7 says that

**Proposition 8.1.3** (Shioda [28] Cor. 1.7). The following equality holds.

$$|\text{disc}(NS(A))| = \frac{|\text{disc}(L)| \cdot (\prod_{i=1}^n |\text{disc}(K_i)|)}{|MW(X)_{tor}|}.$$  

Since we know that in the case under discussion, $|MW(X)_{tor}|$ is generically two and that $\text{disc}(L) = -12$, we deduce that

$$\text{disc}(NS(X)) = -24,$$

since the discriminant of a lattice with signature $(1+, 17-)$ must be negative. Therefore, by Proposition 8.1.2 we see that we have

$$NS(X) \cong \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix} \oplus E_8^2.$$  

We perform calculations following the results of Chapter 5 to obtain,

**Theorem 8.1.** Let $X$ be a K3 surface whose period point lies inside of the ramification locus of $\varphi$. Then $X$ has Shioda-Inose partner $A$ such that

1. $A$ is 2-polarized,

2. $A$ has real multiplication by $\mathbb{Q}(\sqrt{6})$.  

3. There is a primitive embedding of the ring \( \mathbb{Z}[\sqrt{6}] \) into \( \text{End}(A) \),

4. As a \( \mathbb{Z}[\sqrt{6}] \)-module, \( H^1(A, \mathbb{Z}) \) is isomorphic to the direct product,

\[
\mathbb{Z}[\sqrt{6}] \oplus \mathcal{I} \text{ where } \mathcal{I} = \mathbb{Z}_{\sqrt{6}}^2 \oplus \mathbb{Z}_2.
\]

Furthermore, the subvariety \( V_{24} \) of \( \mathbb{W}P^3(1,2,3,4) \) is a compactification of the space

\[
\mathfrak{h}^2/(PSL_2(\mathbb{Z}[\sqrt{6}], \mathcal{I})^{\text{sym}} \cup \tau PSL_2(\mathbb{Z}[\sqrt{6}], \mathcal{I})^{\text{sym}})
\]

where \( \tau \) acts on \( \mathfrak{h}^2 \) via

\[
\tau(z_1, z_2) \mapsto \left( \frac{-1}{96z_2}, \frac{-1}{96z_1} \right),
\]

and \( PSL_2(\mathbb{Z}[\sqrt{6}], \mathcal{I})^{\text{sym}} \) denotes the symmetric Hilbert modular group as defined in Section 2.4.2. This is the group generated by \( PSL_2(\mathbb{Z}[\sqrt{6}], \mathcal{I}) \) and the Galois involution.

### 8.1.2 \( \Delta = 4 \): An easy degeneration.

From Kodaira’s classification of singular fibers of Jacobian elliptic surfaces, we see that the condition under which the order of vanishing of \( \Delta(s, t) \) increases from 9 to 10 at \( \infty \). Writing out the discriminant \( \Delta(s, t) \), we see that the coefficient of \( s^9t^{15} \) is the homogeneous degree 3 polynomial,

\[-2a^3 + 18ab + 108c.\]

Let us denote by \( V_4 \) the subvariety of \( \mathbb{W}P(1,2,3,4) \) given by the vanishing of this polynomial. Therefore, we have that \( V_4 \) gives the subvariety of the moduli space of K3 surfaces with singular fibers at least \( E_8, D_7 \), whose singular fibers are of type \( E_8, D_8 \). Using Proposition 8.1.3 and the facts: \( \text{disc}(D_8) = -4, \text{disc}(E_8) = -1 \) and by the table in [26], a K3 surface bearing \( E_8 \) and \( D_8 \) singular fibers has no torsion sections, we see that for \( X \) a K3 surface
coming from a point in \(V_4\), we have via Proposition 8.1.3 that

\[
\text{disc}(\text{NS}(X)) = -4.
\]

Since the Humbert surface of discriminant 4, denoted \(H_4\), has a single component, we have by Proposition 8.1.2 that

\[
\text{NS}(X) \cong \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \oplus E^2_8
\]

for all \(\mathcal{F}(a, b, c, d)\) with \([a : b : c : d] \in V_4\). We may subsequently list of the data associated to \(V_4\) in Chapter 5. We summarize our results in the following

**Theorem 8.2.** Let \(X\) be a K3 surface whose period point lies inside of the variety \(V_4\). Then \(X\) has Shioda-Inose partner \(A\) such that

1. \(A\) is 2-polarized,
2. \(A\) is 2-isogenous to a product of elliptic curves.
3. \(A\) has real multiplication by \(\mathbb{Q}^2\),
4. There is a primitive embedding of the ring

\[
R_4 := \{(a, b) \in \mathbb{Z}^2; a \equiv b \mod 2\}
\]

into \(\text{End}(A)\),
5. As an \(R_4\)-module, \(H^1(A, \mathbb{Z})\) is isomorphic to the direct product,

\[
R_4 \oplus \mathcal{I} \text{ where } \mathcal{I} = \{(a, b/2) \in \mathbb{Q}^2 : a, b \in \mathbb{Z}\}.
\]
Furthermore, we see that the variety $V_4$ is a compactification of the domain

$$\frac{(h^2/G)}{(\sigma_{\text{Gal}}, \tau)}$$

where we have

$$G = \left\{ \gamma = \begin{bmatrix} a_0 & 2b_1 \\ c_0/2 & d_0 \end{bmatrix} \times \begin{bmatrix} a_0 + 2a_1 & b_0/2 \\ c_1 & d_0 + 2d_1 \end{bmatrix} \in M_2(\mathbb{Q})^2 : a_i, b_i, c, d_i \in \mathbb{Z}, \det(\gamma) = 1 \right\}.$$

and $\sigma_{\text{gal}}$ is the Galois involution

$$\sigma_{\text{Gal}}(z_1, z_2) \mapsto \left( \frac{-2}{z_2}, \frac{-2}{z_1} \right)$$

and where $\tau$ is the involution,

$$\tau(z_1, z_2) \mapsto \left( \frac{-1}{8z_2}, \frac{-1}{8z_1} \right).$$

8.1.3 $\Delta = 8$: The discriminant locus.

We want to determine the subvariety of $\mathbb{W}^3(1, 2, 3, 4)$ on which we have singular fibers of type $E_8$, $D_7$ and $A_1$. By Kodaira’s classification, we have $A_1$ singular fiber if and only if we don’t have the order of vanishing of $A(s, t)$ and $B(s, t)$ increasing, but $\Delta(s, t)$ has vanishes to order two at some point $z \in \mathbb{C}$. We have $\Delta(s, t)$ generically given by $t^{10}s^9\Delta_0(s, t)$ for $\Delta_0(s, t)$ a polynomial of degree 5. Therefore, we have an extra $A_1$ fiber if and only if $\Delta_0(s, t)$ has a double root at some point $z \neq 0$. Generically, this is given by the vanishing of $\text{disc}(\Delta_0(s, t))$. One may calculate the discriminant of $\Delta_0(s, t)$ to be a degree 47 homogeneous polynomial in $a, b, c$ and $d$. We call this polynomial $F_8$. This polynomial will not be written down, given its length and complexity. We will denote the hypersurface defined by this equation by $V_8$.

Thankfully, from [8] we may obtain a very nice parameterization of $V_8$. Given affine
complex parameters, \( \lambda, \mu \in \mathbb{C} \), we parameterize a Zariski open subset of \( V_8 \) in the following way,

\[
\begin{align*}
a &= -6(\lambda + 3\mu), \\
b &= 3(2\lambda^2 + 6\lambda\mu + 9\mu^2 - 6\mu), \\
c &= 2(\lambda^3 - 9\lambda\mu - 27\mu^2), \\
d &= 27\mu^2.
\end{align*}
\]

Then for an elliptic surface in \( \mathcal{F} \) with such parameters, \( \Delta_0(s, t) \) will vanish to order 2 at \( 1 \in \mathbb{P}^1 \) (i.e \( s = t \)). Therefore for these parameters the associated elliptic surfaces will bear at least \( D_7, E_8 \) and \( A_1 \) singular fibers. An application of Proposition 5.4.3 shows that for such K3 surfaces,

\[
\text{disc}(\text{NS}(X)) = -8,
\]

and therefore by Proposition 8.1.2, we have

\[
\text{NS}(X) \cong \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \oplus E_8^2,
\]

We now may make the following calculations coming from Chapter 5.

**Theorem 8.3.** If \( X = \mathcal{F}(a, b, c, d) \) for \( [a : b : c : d] \in V_8 \), we have that \( X \) has Shioda-Inose structure and Shioda-Inose partner \( A \) such that

1. \( A \) is 2-polarized, and admits a principal polarization,

2. \( A \) has real multiplication by \( \mathbb{Q}(\sqrt{2}) \),

3. There is a primitive embedding of the ring \( \mathbb{Z}[\sqrt{2}] \) into \( \text{End}(A) \),
4. As a $\mathbb{Z}[\sqrt{2}]$-module, $H^1(A, \mathbb{Z})$ is isomorphic to the direct product,

$$\mathbb{Z}[\sqrt{2}] \oplus I$$

where $I = \mathbb{Z}[\sqrt{2}] \oplus \mathbb{Z}^1/2$.

Furthermore, we have that $V_8$ is actually a compactification of the symmetric Hilbert modular surface,

$$\mathfrak{h}^2/\text{PSL}_2(\mathbb{Z}[\sqrt{2}], I)^{\text{sym}}$$

since if

$$\tau(z_1, z_2) \mapsto \left( \frac{-1}{16z_2}, \frac{-1}{16z_1} \right)$$

and

$$\sigma_{\text{Gal}}(z_1, z_2) \mapsto \left( \frac{-1}{4z_2}, \frac{-1}{4z_1} \right)$$

then we have

$$\gamma = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix} \in \text{PSL}_2(\mathbb{Z}[\sqrt{2}], I)$$

and $\gamma \cdot \sigma_{\text{Gal}} = \tau$. It follows that $\tau \in \text{PSL}_2(\mathbb{Z}[\sqrt{2}], I)^{\text{sym}}$.

**Proposition 8.1.4.** The twofold covering of moduli spaces of K3 surfaces

$$\varphi : \mathbb{W}^3(2, 3, 4, 8) \longrightarrow \mathbb{W}^3(1, 2, 3, 4)$$

does not correspond to the twofold covering,

$$\mathfrak{h}_2/\text{Sp}_4(k, \mathbb{Z}) \longrightarrow \mathfrak{h}_2/(\text{Sp}_4(k, \mathbb{Z}) \cup \sigma_{\text{GH}}\text{Sp}_4(k, \mathbb{Z})).$$

**Proof.** From [10] Theorem 3.4, one may deduce that $H_8$ is in the ramification locus of the twofold covering,

$$\mathfrak{h}_2/\text{Sp}_4(2, \mathbb{Z}) \mapsto \mathfrak{h}_2/(\text{Sp}_4(2, \mathbb{Z}) \cup \sigma_{\text{GH}}\text{Sp}_4(2, \mathbb{Z})).$$
Therefore the twofold covering \( \phi : \mathbb{P}^3(2, 3, 4, 8) \rightarrow \mathbb{P}^3(1, 2, 3, 4) \) is not the twofold covering corresponding to the relation between the moduli space of abelian surfaces with \( (1, 2) \)-polarization and \( E^2_8 \oplus \langle 4 \rangle \)-polarized K3 surfaces.

\[ \square \]

**Remark 8.1.1** (The geometry of \( A^2_2 \)). We have that \( \mathbb{P}^3(1, 2, 3, 4) \) is a compactification of \( h_2/(Sp_4(2, \mathbb{Z}) \cup \sigma_{GH}Sp_4(2, \mathbb{Z})) \), and we may also describe this space as

\[
\text{Proj}(\mathbb{C}[X_1, X_2, X_3, X_4])
\]

where \( \mathbb{C}[X_1, X_2, X_3, X_4] \) is a graded polynomial ring and where \( X_i \) has weight \( i \). Let \( Y \) be a double cover of \( \mathbb{P}^3(1, 2, 3, 4) \) which ramifies along a divisor \( D \) given by

\[
F_D(X_1, X_2, X_3, X_4) = 0,
\]

with \( F_D \) a weighted homogenous polynomial in \( X_1, X_2, X_3 \) and \( X_4 \) of degree \( d \). Then \( Y \) is just

\[
\text{Proj}(\mathbb{C}[X_1, X_2, X_3, X_4, Z]/(Z^2 - F_D))
\]

where \( Z \) is a variable of degree \( d/2 \). Thus by [10] Theorem 3.4, we have that \( h_2/Sp_4(2, \mathbb{Z}) \) is a double cover of \( \mathbb{P}^3(1, 2, 3, 4) \setminus \{d = 0\} \) ramified along \( V_8 \). Therefore, \( A^2_2 = h_2/Sp_4(2, \mathbb{Z}) \) is birational to

\[
\text{Proj}(\mathbb{C}[X_1, X_2, X_3, X_4, Z]/(Z^2 - V_8))
\]

where \( Z \) is a weighted projective variable of degree \( 47/2 \).

### 8.2 Equations of Shimura curves

In this section we write out the data related to quotients of Shimura curves in our family \( \mathcal{F} \) of K3 surfaces. Following Chapters 2, 3 and 4, these correspond to the moduli spaces of K3 surfaces with generic Picard rank 19. These will be expressed as components of complete
intersections of hypersurfaces. Often these hypersurfaces will correspond to the Humbert surfaces above and sometimes even the self-intersection of Humbert surfaces. We will be able to tell whether the Shimura curves at hand are coming from Eichler orders or not by using Proposition 6.2.4.

In the following, we will often treat the simple case where our rank three lattice is in the form, $\widehat{M} \oplus \langle 4 \rangle$ for some negative definite even lattice $\widehat{M}$ of rank two and prime discriminant, and $\langle 4 \rangle$ the lattice generated by one element with self intersection 4, which are of the form

$$M = \left(\widehat{M} := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right) \oplus [4] \cong \begin{bmatrix} 2a & b & 0 \\ b & 2c & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and we will want to determine the group $O_0(M^\perp, u_k) / SO_0(M^\perp, \mathbb{Z})$, as discussed in Section 6.4. We prove a simple lemma regarding such lattices.

**Lemma 8.2.1.** Let $\gamma \in O_0(M, u_k^\perp, \mathbb{Z})$ for $M \cong \widehat{M} \oplus \langle u_k \rangle$ where $\widehat{M}$ is a rank two even negative definite lattice and $\text{disc}(\widehat{M}) = p$ some prime. Then the image of $O_0(M, u_k^\perp, \mathbb{Z})$ in $O(A(M))$ is either trivial or a group of order two.

**Proof.** We see that if $\gamma \in O_0(M, u_k^\perp, \mathbb{Z})$ then we must have $\gamma = \overline{\gamma} \oplus \text{Id}$ where $\overline{\gamma} \in O(\widehat{M}, \mathbb{Z})$. This is because if $\gamma(u_k) = u_k$, then $\gamma(u_k^\perp) \subseteq u_k^\perp$ and thus $\gamma$ must send $\widehat{M} = u_k^\perp$ to $\widehat{M}$ and preserve inner product. Then we have that the image of $\gamma$ in the quotient $O_0(M, u_k^\perp, \mathbb{Z}) / SO_0(M^\perp, \mathbb{Z})$ is determined by the image of $\gamma$ in $O(A(M))$ by Proposition 6.4.2. We see that $A(\widehat{M} \oplus u_k) = A(\widehat{M}) \oplus A(u_k)$. Since $\gamma$ fixes $u_k$, we have that the class of $\gamma$ in the quotient is determined entirely by its image in $A(\widehat{M})$.

Since $\text{disc}(\widehat{M})$ is prime, we have that $A(\widehat{M})$ is a cyclic group. Furthermore then, we must have that any element of $O(A(\widehat{M}))$ acts as multiplication by some integer mod $p$, and that by an argument very similar to Corollary 5.4.1, it must be a square root of unity modulo $p$, of which there are exactly two. Therefore, the image of $O_0(M, u_k^\perp, \mathbb{Z})$ in $O(A(M))$ is either
trivial or a cyclic group of order two, and hence by Proposition 6.4.2, we have that the extension of \( SO_0(M^+, \mathbb{Z}) \) to \( O_0(M, u_k^+, \mathbb{Z}) \) is either trivial or a degree two extension.

### 8.2.1 The Eichler order \( O(6, 1) \).

We describe the components of the self-intersection of the hypersurface \( V_8 \). Any component of the self-intersection of \( V_8 \) has Néron-Severi group \( NS(X) \cong L_b \oplus E_8^2 \) where \( L_b \) has Gram matrix

\[
L_b := \begin{bmatrix}
-2 & b & 0 \\
b & -2 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]

such that the discriminant of \( L_b \) is positive.

We see that \( \text{disc}(L_b) = 4(4 - b^2) \) and hence \( b = \pm 1 \) or 0. If we let \( b = 0 \), then we have \( \text{disc}(A(L_0)) = 16 \), and hence by Proposition 6.2.4, this cannot correspond to an Eichler order. If we take \( b = \pm 1 \), we have \( A(L_{\pm 1}) \) principally generated, and both have discriminant 12 and a generator of length \(-5/12\). Therefore, we have that both correspond to the same lattice (and hence the same component of the self-intersection).

**The non-Eichler component, \( b = 0 \).**

Geometrically, this corresponds to when two pairs of reducible \( I_1 \) fibers merge to give us two reducible fibers of type \( I_2 \). In the coordinates \( \lambda, \mu \) supplied by [8] and written down in 7.1.3, we have \( \Delta(s, t) = 729\Delta_0(s, t)s^9t^{10}(t - 1)^2 \) with \( \Delta_0(s, t) \) some cubic polynomial. We obtain the degeneration to two reducible \( I_2 \) fibers whenever \( \text{disc}(\Delta_0(s, t)) = 0 \) and \( \Delta_0(s, t) \) does not vanish at 0, 1 or \( \infty \). We calculate,

\[
\Delta_0(s, t) = (4\lambda \mu^2 + 4\mu^3)t^3 + (8\lambda^2 \mu^2 + 36\lambda \mu^3 + 27mu^4 - 4\mu^3)t^2s \\
+ (4\lambda^3 \mu^2 - 36\lambda \mu^3 - 54\mu^4)ts^2 + (27\mu^4)s^3,
\]
which has discriminant

\[
F_0(\lambda, \mu) := 8192\lambda^9\mu^9 + 27648\lambda^8\mu^{10} + 8192\lambda^8\mu^9 + 31104\lambda^7\mu^{11} \\
- 129024\lambda^7\mu^{10} + 11664\lambda^6\mu^{12} - 708480\lambda^6\mu^{11} - 140288\lambda^6\mu^{10} \\
- 1306368\lambda^5\mu^{12} - 293760\lambda^5\mu^{11} - 1049760\lambda^4\mu^{13} + 116640\lambda^4\mu^2 \\
+ 150912\lambda^4\mu^{11} - 314928\lambda^3\mu^{14} + 583200\lambda^3\mu^{13} + 273024\lambda^3\mu^{12} \\
+ 314928\lambda^2\mu^{14} + 23328\lambda^2\mu^{13} - 55728\lambda^2\mu^{12} - 104976\lambda\mu^{14} \\
- 54432\lambda\mu^{13} + 11664\mu^{14} + 6912\mu^{13}.
\]

Thus if in the affine coordinates \(\lambda, \mu\), we have the coordinates of \(X\) satisfying \(F_0(\lambda, \mu)\), then \(X\) has

\[
\text{NS}(X) \cong L_0 \oplus E_8^2.
\]

Furthermore, following calculations in Section 6.1, we have

**Proposition 8.2.1.** Let \(X_0\) be an elliptic K3 surface given by a point \(\lambda_0, \mu_0 \in V_8\) such that \(F_0(\lambda_0, \mu_0) = 0\). Then \(X\) has Shioda-Inose structure, and its Shioda-Inose partner is an abelian surface with quaternionic multiplication by an order

\[
O_{L_0} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij
\]

where \(i^2 = 2\) and \(j^2 = -2\) and \(ij = -ji\).

Finally, we calculate how \(\text{res}_{L_0} (O_0(L_0, u_k^1, Z))\) extends \(SO_0((L_0)^{\perp}, \mathbb{Z})\). We see that if \(L_0\) is spanned by \(v, w, u_k\), then \(L_0^*\) is spanned by \(v/2, u/2\) and \(u_k/4\), and \(A(L_0) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}\). If \(\gamma \in O(A(L_0))\) and \(\gamma\) fixes \(u_k\), then we must have \(\gamma\) permuting \(v/2\) and \(u/2\), hence
\( \gamma \) is induced by the automorphism,

\[
\tau = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Thus we have that \( \text{res}_{L_0}(O_0(L_0, u_k^+, \mathbb{Z})) \) is an extension of \( SO_0(L_0^+, \mathbb{Z}) \) of degree two. Thus, the subvariety of \( V_8 \) cut out by the vanishing locus of \( F_0 \) is birational to

\[
\mathfrak{h}/(\Gamma(\mathcal{O}_{L_0}) \cup \tau \Gamma(\mathcal{O}_{L_0})),
\]

and the natural map from the Shimura curve \( \mathfrak{h}/\Gamma(\mathcal{O}_{L_0}) \) into \( \mathbb{P}^3(1, 2, 3, 4) \) is \( 2 : 1 \).

**The Eichler component.**

We describe this geometrically. We have that \( V_8 \) is the hypersurface of \( \mathbb{P}^3(1, 2, 3, 4) \) along which we have one singular \( I_1 \)-fiber turning into a reducible \( I_2 \)-fiber (giving \( A_1 \) lattice). This component of the self-intersection of \( V_8 \) corresponds to the case when the \( I_2 \)-fiber merges with another \( I_1 \) fiber to form an \( I_3 \) reducible fiber. This gives us lattice type \( A_2 \). Using Proposition 8.1.3 and the fact that \( \text{disc}(A_2) = 3 \) tells us that we have \( \text{disc}(\text{NS}(X)) = 12 \) on the self-intersection of \( V_8 \).

One may exhibit coordinates of this component of the self-intersection by setting \( \lambda = 1 \) in the parameterization of \( V_8 \) given in Section 8.1.3. This degeneration may also be found in [8] Section 3. We notice that in this case, we have that the lattice \( L_1 \) is the lattice in Example 6.2.2, and thus we know that the Clifford ring \( \mathcal{C}^+((1/2)L_{\pm 1}) \) is isomorphic to the maximal order \( \mathcal{O}(6, 1) \).

We may calculate the extension \( \text{res}_{L_1}(O_0(L_1, u_k^+, \mathbb{Z})) \) of \( SO_0(L_1^+, \mathbb{Z}) \). Since we have
$L_1 \cong \bar{L} \oplus u_k$, where

$$\bar{L} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

has discriminant 3, we may bring Lemma 8.2.1 to bear. In particular, we see that this extension is of degree at most two. Furthermore, we may check that the map

$$\tau = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism of $L_1$, and that it acts non-trivially on $A(\bar{L})$. We have that $A(L_1) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$. The automorphism $\tau$ acts as $-\text{Id}$ on the $\mathbb{Z}/3\mathbb{Z}$ component, and clearly as the identity on the $\mathbb{Z}/4\mathbb{Z}$ component. This corresponds to multiplication by 5 on $\mathbb{Z}/12\mathbb{Z}$.

To sum up:

**Proposition 8.2.2.** Let $X_1$ be a K3 surface given by a point in $V_8$ such that $\lambda = 1$, then we have that $X_1$ has Shioda-Inose structure and Shioda-Inose partner $A$ with quaternionic multiplication by $\mathcal{O}(6,1)$. Furthermore, we have that the subvariety of $V_8$ of such points is birational to a compactification of the Shimura curve

$$\mathfrak{h}/(\Gamma(\mathcal{O}(6,1) \cup \tau\mathcal{O}(6,1))).$$
8.2.2 The Eichler order $O(22, 2)$.

Let us take moduli spaces of rank 19 polarized K3 surfaces which are contained in both $V_8$ and $V_{24}$. In such a situation, we have that $\text{NS}(X)$ must look like $M_b \oplus E_8^2$ where

$$M_b := \begin{bmatrix} -6 & b & 0 \\ b & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \tilde{M}_b := \begin{bmatrix} -6 & b \\ b & -2 \end{bmatrix}$$

and $\text{disc}(M_b) = 4(12 - b^2) > 0$. Therefore, we have cases $b = 0, \pm 1, \pm 2, \pm 3$. If we let $b = 0$, then $\text{disc}(M_b) = 48 = 2 \cdot 24$ and 24 is not squarefree. If $b = \pm 2$ then $\text{disc}(M_2) = 32 = 2 \cdot 16$, which is again not squarefree. If we look at $b = \pm 1$, we get $\text{disc}(M_b) = 2 \cdot 22$, and 22 is squarefree. In both cases, $A(M_{\pm 1}) \cong \mathbb{Z}/11\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/44\mathbb{Z}$, and we have $-13/44$ represented primitively by $A(M_{\pm 1})$. Therefore, both of these lattices are isomorphic and we have by Proposition 6.2.4 that $C^+((1/2)M_{\pm 1}) \cong O(22, 2)$. One determines the level by noting that 13 is non-square modulo 11.

Since the lattices $M_1$ and $M_{-1}$ are isomorphic, we proceed just in the case of $M_1$. We calculate how $\text{res}_{M_1}(O_0(M_1, u_k^\perp, \mathbb{Z}))$ extends $SO_0(M_1^\perp, \mathbb{Z})$ by using Lemma 8.2.1 again, since we have that

$$M_1 \cong \left( \widetilde{M} := \begin{bmatrix} -6 & 1 \\ 1 & -2 \end{bmatrix} \right) \oplus \langle 4 \rangle$$

where $\widetilde{M}$ has prime discriminant 11. The group $O(\widetilde{M}, \mathbb{Z})$ contains the automorphism given by the matrix

$$\tilde{\tau} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

and $\tilde{\tau}$ acts as $-\text{Id}$ on $A(\widetilde{M})$, and since $A(M_1) \cong A(\widetilde{M}) \oplus A(u_k) \cong \mathbb{Z}/11\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, we have that the extension,

$$\tau := \tilde{\tau} \oplus \text{Id} \in O(M, \mathbb{Z})$$
extends to $\tau' := -\tau \oplus \tau \in O(U^3, \mathbb{Z})|_{M_1}$ which acts as multiplication by 21 on $A(M_1)$.

**Proposition 8.2.3.** Let $X$ be a $K3$ surface given by some point in the intersection of $V_8$ and $V_{24}$ such that $b = 1$. Then we have that $X$ has Shioda-Inose structure, and the Shioda-Inose partner $A$ of $X$ is a simple abelian surface with quaternionic multiplication by the Eichler order $O(22, 2)$, and that this component of the intersection of $V_8$ and $V_{24}$ is birational to the Shimura curve

$$\mathfrak{h}/\Gamma(O(22, 2) \cup \tau' O(22, 2)).$$

**8.2.3 The Eichler order $O(2, 2)$.**

We now will find the locus of points in $\mathcal{F}(a, b, c, d)$ where the $D_7$ fiber degenerates to a $D_9$. From Kodaira’s classification, this occurs when $X$ is expressed as

$$y^2 = x^3 + A(s, t)x + B(s, t),$$

and where $A(s, t)$ vanishes to order 2 at $\infty$, $B(s, t)$ vanishes to order 3 at $\infty$ and we have $\Delta(s, t)$ vanishes to order 11 at $\infty$. In the course of finding the hypersurface $V_8$, we found the locus on which we have $\Delta(s, t)$ vanishing to order 10 at $\infty$, so it remains to find the locus of $V_8$ where $\Delta(s, t)$ vanishes even further. Calculating the general discriminant of $\mathcal{F}(a, b, c, d)$, we see that the top two coefficients of $\Delta(s, t)$ are

$$-2a^3 + 18ab + 108c,$$

and

$$\frac{3a^4}{4} - 6a^2b - 54ac - 9b^2 + 108d.$$

We have already seen the first equation in 8.1.2, which increases the order of vanishing of $\Delta(s, t)$ at $\infty$ to 10, and the concurrent vanishing of the second equation tells us that $\Delta(s, t)$ vanishes to order 11 at $\infty$. Let $V'_8$ be the degree four hypersurface in $\mathbb{P}^3(1, 2, 3, 4)$ defined
Figure 8.1: The dots in this figure represent roots in our lattice (elements with self-intersection $-2$), and lines represent intersection of roots. Large dotted boxes identify the roots representing components of the singular fibers $I^*_5$ and $II^*$. Roots in the solid boxes represent the two copies of $E_8$ in our lattice. The grey shaded roots are trivially orthogonal to our two copies of $E_8$.

by the vanishing of the second equation above. Then we have that if $X$ is a K3 surface $\mathcal{F}(a, b, c, d)$ which is given by $[a : b : c : d] \in V_8 \cap V'_8$, then $X$ has reducible singular fibers of type $D_9$ and $E_8$. We may now calculate the Néron-Severi of $X$ in such a way that we may apply the results of Chapter 6. We must write $\text{NS}(X) \cong M \oplus E_8^2$.

**Proposition 8.2.4.** Let $X$ be a K3 surface such that $X$ is represented by a point in $V_8 \cap V'_8$. Then there is a primitively embedded lattice,

\[
\begin{bmatrix}
-2 & 0 & 1 \\
0 & -2 & 1 \\
1 & 1 & 0
\end{bmatrix}
\oplus E_8^2 \subseteq \text{NS}(X).
\]

**Proof.** The purpose here is to find a primitive embedding of $E_8^2$ into $U \oplus D_9 \oplus E_8$ and then to calculate the orthogonal complement. We have one obvious copy of $E_8$, the second may be found as a sublattice of the rational components of $\bar{D}_9$ and $O$, as is demonstrated in Figure 7.1. It is easy to see that that two of the roots in the lightly dotted box are
orthogonal to our two copies of $E_8$. We find a third element in our lattice orthogonal to our two copies of $E_8$, we take a linear combination of roots in the top copy of $E_8$ and the root in the lightly dotted box which is not orthogonal to both copies of $E_8$. We determine that $u := D_0 + 2D_1 + 3D_2 + 4D_3 + 5D_4 + 6D_5 + 3D_6 + 4D_7 + 2O$ is also orthogonal to both copies of $E_8$. The lattice spanned by the obvious orthogonal roots and $u$ gives us a lattice $L$ with Gram matrix

$$
\begin{bmatrix}
-2 & 0 & 1 \\
0 & -2 & 1 \\
1 & 1 & 0
\end{bmatrix}.
$$

Since the discriminant of this lattice is 4, and we have $L \oplus E_8^2 \subseteq D_7 \oplus E_7 \oplus U$ and $\text{disc}(D_7 \oplus E_7 \oplus U) = 4$, we must have equality, $L \oplus E_8^2 = D_7 \oplus E_7 \oplus U$, which proves the Proposition.

Now we may check to see that $A(L) \cong \mathbb{Z}/4\mathbb{Z}$ and that it represents $-17/4$ primitively. Therefore, using Proposition 6.2.4, we see that we must have $C^+((1/2)L) \cong O(2,2)$. We know that the lattice

$$M_2 := \begin{bmatrix}
-4 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \cong \begin{bmatrix}
-4 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 4
\end{bmatrix}$$

is also related to this quaternion order, and that indeed we have the two lattices isomorphic, since we have

$$
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 2 \\
-2 & 1 & 3
\end{bmatrix}^t \begin{bmatrix}
-2 & 0 & 1 \\
0 & -2 & 1 \\
1 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 2 \\
-2 & 1 & 3
\end{bmatrix} = \begin{bmatrix}
-4 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 4
\end{bmatrix}.
$$

We have that $M_2 \cong \langle -4 \rangle \oplus U$, and hence $A(M_2) = A(\langle -4 \rangle) \oplus A(U) = A(\langle -4 \rangle)$ since $U$ is unimodular. Thus we have that the extension $\text{res}_{M_2}(O_0(M_2, u^\perp_2, \mathbb{Z}))$ of $SO_0(M_2^\perp, \mathbb{Z})$ is

\[ \text{...} \]
determined entirely by how it acts upon \((-4)\). We have

\[
\tau = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in O(M_2, \mathbb{Z}),
\]

which acts as \(-1\) on \(A(M_2)\). Therefore, by Proposition 6.4.3, we have that we have \(O_0(M_2^+, \mathbb{Z})\) a degree two extension of \(SO_0(M_2^+, \mathbb{Z})\).

Finally, we sum up these results. From [24] we see that \(\tau\) may be represented as a fractional linear transformation by the matrix

\[
\tau := \begin{bmatrix}
0 & -1/\sqrt{2} \\
\sqrt{2} & 0
\end{bmatrix},
\]

and the group generated by \(\Gamma(O(2, 2))\) and \(\tau\) is called \(\Gamma_0(2)^+\).

**Proposition 8.2.5.** Let \(X\) be a K3 surface \(\mathcal{F}(a, b, c, d)\) where \([a : b : c : d]\) is a point in the intersection \(V_8 \cap V_8'\), then \(X\) has Shioda-Inose structure and its Shioda-Inose partner is \(A = E \times E'\) where \(E\) and \(E'\) are 2-isogenous elliptic curves. The intersection \(V_8 \cap V_8'\) is birational to

\(\mathfrak{h}/\Gamma_0(2)^+\).
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Appendix A

Basic results

This appendix is devoted to stating and/or proving some ancillary results which were used throughout our thesis. These are results whose exposition was deemed too far off topic for the main body of our work, but nevertheless, they play important roles.

A.1 A tedious proof.

The following result is an old result, going back even to Humbert in the early 20th century. It is sketched out in [31] IX Prop. 2.3, but we rephrase it, and prove it in detail. This result is used in Section 3.2.

**Proposition A.1.1.** Let $A_\tau$ be a $k$-polarized marked abelian surface, represented by some $\tau \in \mathfrak{h}_2$. Then $\text{NS}(A_\tau)$ contains the element

\[ u = a_{12}g_{12} + a_{34}g_{34} + a_{14}g_{14} + a_{23}g_{23} + a_{13}g_{23} \]
if and only if we have

\[
T_u := \begin{bmatrix}
0 & -ka_{14} & 0 & ka_{12} \\
-a_{23} & a_{13} & -a_{12} & 0 \\
0 & -ka_{34} & 0 & -ka_{23} \\
a_{34} & 0 & -a_{14} & a_{13}
\end{bmatrix}
\]

an endomorphism of \( A_\tau \).

**Proof.** One direction is clear; if we have \( u \in \text{NS}(A_\tau) \), then by using equation (3.2.3), we get that \( T_u \) provides an endomorphism of \( A_\tau \). We proceed to show the contrapositive. If we have \( T_u \in \text{End}(A_\tau) \), then we have that there is some \((T_u)_{an} \in M_2(\mathbb{C})\) such that

\[
(T_u)_{an} \begin{bmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & k & \tau_2 & \tau_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & k & \tau_2 & \tau_3 \end{bmatrix} T_u
\]

or in other words, the \( \mathbb{R} \)-linear endomorphism of \( \mathbb{C}^2 \) provided by \( T_u \) is actually a \( \mathbb{C} \)-linear endomorphism of \( \mathbb{C}^2 \). For the moment, we will represent

\[
T_u = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

for \( A, B, C, D \in M_2(\mathbb{Z}) \). Then we have that if we let

\[
D_k := \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix}
\]

then we have

\[
\begin{bmatrix} D_k & \tau \end{bmatrix} T_u = \begin{bmatrix} D_kA + \tau C & DkB + \tau D \end{bmatrix}.
\]
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We observe what this implies about \((T_u)_{an}\). It follows that

\[
(T_u)_{an} = (D_kA + \tau C)D_k^{-1}.
\]

Furthermore, it must then be true that \((T_u)_{an}\tau = D_kB + \tau D\). Or in other words, we have the equation,

\[
D_kAD_k^{-1}\tau + \tau CD_k^{-1}\tau - D_kB - \tau D = 0.
\]

If we work this out, using our explicit representation of \(A, B, C\) and \(D\), we get

\[
\begin{bmatrix}
0 & ((\tau_2^2 - \tau_1\tau_3)a_{34} + \tau_1a_{23}k + a_{34} - \tau_2a_{13} - \tau_3a_{14} - a_{12}k) \\
-((\tau_2^2 - \tau_1\tau_3)a_{34} + \tau_1a_{23}k + a_{34} - \tau_2a_{13} - \tau_3a_{14} - a_{12}k) & 0
\end{bmatrix}
\]

and hence the condition that \(T_u\) is an endomorphism of \(A_\tau\) is equivalent to the singular relation

\[
((\tau_2^2 - \tau_1\tau_3)a_{34} + \tau_1a_{23}k + a_{34} - \tau_2a_{13} - \tau_3a_{14} - a_{12}k) = 0.
\]

This means that under the period map for \(k\)-polarized abelian surfaces, we have that the period point of \(A_\tau\) lies in the plane

\[-a_{12}x_0 + a_{34}x_1 + a_{41}x_2 + a_{23}x_3 - a_{13}x_4 = 0.\]

Equivalently, we have that if \(\omega_\tau\) is the period point of \(A_\tau\), then

\[
\langle \omega_\tau, a_{34}g_{34} + a_{12}g_{12} + a_{14}g_{14} - a_{23}g_{23} + a_{13}g_{13} \rangle = 0,
\]

which means that

\[
a_{34}g_{34} + a_{12}g_{12} + a_{14}g_{14} - a_{23}g_{23} + a_{13}g_{13} \in \text{NS}(A_\tau).
\]

This completes the proof of our proposition. \(\square\)
A.2 A little (more) lattice theory.

Throughout this thesis we have used a Corollary of Nikulin (see Nikulin [23] Cor. 1.5.2) on numerous occasions to make claims about extensions of lattice automorphisms on sublattices of $U^3$ to automorphisms of $U^3$. In this section, we will explain how this works in detail. First, we will state the claim which appears as Equations 2.2.6 and 2.2.7.

**Proposition A.2.1.** Let $M$ be a lattice primitively embedded into $U^3$. Then we have the following expressions for $M^\perp$.

1. If $M$ has rank two and signature $(1+,1-)$, then $M^\perp \cong (-M) \oplus U$.

2. If $M$ has rank three and signature $(1+,2-)$, then $M^\perp \cong (-M)$.

**Proof.** If we let $q_M$ be the dual form on $A(L)$ (for notation see Definition 4.3.3), then we have by [23] Prop. 1.6.1 that $q_{M^\perp} = -q_M$. In the case 1, we have that $M$ may be assumed to be embedded in $U^2$ by Equations 2.2.1 and 2.2.2, we see that $(M^\perp)_{U^3} \cong N \oplus U$ for some lattice $N$. Furthermore, we see that $N$ must have signature $(1+,1-)$, and that since $U$ is unimodular, $A(N \oplus U) = A(N)$ and hence $q_{N \oplus U} = q_N = -q_M$. We see easily that $q_{M^\perp} = -q_M$. Therefore $N$ has the same signature and discriminant form as $-M$. Therefore by Proposition 6.2.3 (or [23] Cor. 1.13.3), we have that there is an isometry $-M \cong N$. This proves the claim in case 1.

Case 2 follows similarly. We have that $M^\perp$ has signature $(2+,1-)$, and discriminant form $-q_M$. Therefore, an appeal to Proposition 6.2.3 tells us that there is an isometry $M^\perp \cong -M$. \hfill \qed

Now, we have often made the claim that certain automorphisms of a lattice $M$ embedded in $U^3$ may be extended to automorphisms of $U^3$ which restrict to the original automorphism on $M$. We will explain how this works now.

**Definition A.2.1.** Let $L$ be an even lattice and $M$ be an even lattice satisfying $L \subseteq M \subseteq L^*$. We will call $M$ an overlattice of $L$. We will denote the quotient $M/L$ by $H_M$. 
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We have $H_M \subseteq A(L)$ by general principals, and furthermore, if we restrict $q_L$ to $H_M$, we see that $H_M$ corresponds to an isotropic subgroup of $A(L)$ (or in other words, $q_L|_{H_M} = 0$). Conversely, if we have any isotropic subspace $L'$ of $A(L)$, the corresponding subgroup $L'$ in $L^*$ containing $L$ must be integral, that is, if $Q$ is the quadratic form on $L$, then $Q(L') \in 2\mathbb{Z}$. Thus there is a bijection between isotropic subspaces of $A(L)$ and even overlattices of $L$ ([23] Prop. 1.4.1).

Now if we take $M$ a lattice of rank $n$ and signature in $(1+, (n-1)-)$ primitively embedded into $U^n$ for $n = 2$ or $3$, then by Proposition A.2.1 above, we have $M^\perp = -M$. Then we have

$$H_M = U^n/(M \oplus -M) \subseteq A_M \oplus -A_M$$

Furthermore, we may determine what $H_M$ looks like. We have a natural isotropic subspace of $A_M \oplus -A_M$ given by

$$\{ \alpha + \alpha' : \alpha \in A_M, \alpha' \in -A_M \text{ and } \phi(\alpha) = \alpha' \}$$

where $\phi$ is the natural isomorphism of the underlying groups such that $-q_M(\alpha) = q_{-M}(\phi(\alpha))$ for all $\alpha \in A(M)$. This subgroup has order $\text{disc}(M)$. We may also calculate the discriminant of the associated overlattice. Since $H_M$ has order $\text{disc}(M)$ we have $M \oplus (-M)$ a lattice of index $\text{disc}(M)$ in the overlattice $L$, and $\text{disc}(M \oplus -M) = \text{disc}(M)^2$. There is a classical formula [23] which tells us that

$$\text{disc}(L) = \frac{\text{disc}(M \oplus -M)}{|L : M \oplus -M|^2}$$

and hence $\text{disc}(L) = 1$. Since $L$ has signature $(n+, n-)$, we must have $L$ isometric to $U^n$ by Milnor’s classification of even indefinite unimodular lattices. Thus this is the subgroup of $A(M \oplus -M)$ associated to the primitive embedding into $U^n$ of $M$ given at the outset.

Finally, there is a natural isomorphism $A(M) \cong H_M \cong A(-M)$ of groups, which Nikulin
calls $\gamma_{M,-M}$. This may be given by

$$\alpha \in A(M) \mapsto \alpha \oplus \phi(\alpha) \mapsto \phi(\alpha)$$

and hence the composition agrees with $\phi$ on $A(M)$. We are now in a position to state [23] Corollary 1.5.2 in our particular situation. We have

**Corollary A.2.1.** Let $M$ be a rank $n$ sublattice of signature $(1+, (n-1)-)$ embedded primitively into $U^n$ for $n = 1, 2$. Then if we have $\varphi$ an automorphism of $M$ and $\tilde{\varphi}$ its restriction to $A(M)$, then there is some $\varphi'$ an automorphism of $O(U^n)$ which restricts to $\varphi$ on $M$ if and only if there is some automorphism $\psi$ of $-M$ such that $\tilde{\psi}(\phi(\alpha)) = \phi(\tilde{\varphi}(\alpha))$ for all $\alpha \in A(M)$.

More simply put, if we have $\varphi$ acting via some matrix $T$, on a basis $\{v_i\}_{i=1}^n$ for $M$, then $\psi$ must act via the same matrix on the basis $\{\phi(v_i)\}_{i=1}^n$ of $-M$.

### A.3 The Albert classification of endomorphism algebras.

Albert has provided a complete classification of algebras with positive involution.

**Definition A.3.1.** A $k$-algebra $\mathcal{R}$ is an algebra with positive involution over a field $k$ if there is a well defined $k$ linear form $\text{Tr}$ defined on $\mathcal{R}$, and that furthermore, there is an anti-involution denoted $^o$ which turns $\text{Tr}$ into a positive definite bilinear form defined as

$$Q(\alpha, \beta) := \text{Tr}(\alpha \beta^o).$$

**Example A.3.1 (Quadratic extensions of $\mathbb{Q}$).** Let us take a real quadratic extension of $\mathbb{Q}$, which may be written as $\mathbb{Q}(\sqrt{\Delta})$ for $\Delta$ a positive squarefree integer. Then there is a natural trace form on $\mathbb{Q}(\sqrt{\Delta})$, given by $\text{Tr}(\alpha) = \alpha + \alpha'$, where $'$ indicates the Galois involution on $\mathbb{Q}(\sqrt{\Delta})$. It may be tempting to try to interpret the Galois involution $'$ on $\mathbb{Q}(\sqrt{\Delta})$ as
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a positive involution, but this does not work. Instead, what we must do is take the trivial involution as our positive involution.

Example A3.2 (Indefinite Quaternion algebras over \( \mathbb{Q} \)). Let us take a \( \mathbb{Q} \)-algebra defined in the following way. We have

\[
\mathcal{R} := \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij
\]

where \( i^2 = -a \) and \( j^2 = b \) for \( a, b \in \mathbb{Z}_{>0} \), and such that \( ij = -ji \). We also denote \( \mathcal{R} \) by

\[
\left( \frac{-a, b}{\mathbb{Q}} \right).
\]

There is a natural anti-involution, which we will call \( ' \) on \( \mathcal{R} \) which satisfies

\[
\begin{align*}
    j &\mapsto -j, \\
i &\mapsto -i, \\
ij &\mapsto -ij, \\
1 &\mapsto 1.
\end{align*}
\]

Then we have the trace defined as \( \text{Tr}(\alpha) = \alpha + \alpha' \). Again, the natural involution is not a positive involution. To obtain a positive involution, we do the following. Choose some \( a \in \mathcal{R} \) such that \( a' = -a \) and \( a^2 < 0 \). Then the involution given by

\[
\beta \mapsto a\beta' a^{-1}
\]

is a positive anti-involution on \( \mathcal{R} \).

We may now present in detail the classification of division \( k \)-algebras with positive involution. In particular, if \( A \) is a simple abelian variety, then its endomorphism algebra is a division algebra, so we have the following, ([22] pp. 201)

**Theorem A.1.** Let \( \mathcal{R} \) be a division algebra of finite rank over \( k \) with involution \( ^o \) such that the induced trace form is positive definite. Let \( K \) be the center of \( \mathcal{R} \) and let \( K_0 \) be the subfield of \( K \) which is fixed by \( ^o \). Then \( \mathcal{R} \) belongs to one of the four classes of algebras;
[Type I.] \( \mathcal{R} = K = K_0 \) is a totally real algebraic number field, and the involution is the identity. \( K = K_0 \) is a totally real algebraic number field and \( \mathcal{R} \) is an indefinite quaternion algebra over \( K \). The involution \( ^o \) is as in Example A.3.2 above. Denote by \( ' \) the natural involution on \( \mathcal{R} \). There is some \( a \in \mathcal{R} \) such that \( a^2 \in K_0 < 0 \) and \( a' = -a \) such that
\[
\beta^o = a \beta' a^{-1}.
\]

\( K = K_0 \) is a totally real algebraic number field and \( \mathcal{R} \) is a quaternion division algebra over \( K \) such that the tensor product over \( \mathbb{Q} \) with \( \mathbb{R} \) is isomorphic to a finite product of the real Hamilton quaternions, and upon choosing an embedding \( \sigma \) of \( K \) into \( \mathbb{R} \), we have \( \mathbb{R} \otimes_K \mathcal{R} \) is isomorphic to the Hamilton quaternions. The involution \( ^o \) agrees with the involution on the Hamilton quaternions. \( K_0 \) is a totally real algebraic number field and \( K \) is a totally imaginary quadratic extension of \( K_0 \). The action of \( ^o \) agrees with complex conjugation on \( K \). Then \( \mathcal{R} \) is some division algebra over \( K \) satisfying arithmetic conditions.

Finally, we have that the third case does not occur when \( A \) is an abelian surface, and that the third case occurs only when \( K = \mathcal{R} \), or in other words, \( \mathcal{R} \) is a CM extension of \( \mathbb{Q} \) of degree four.

### A.4 The singular fibers of elliptic surfaces and their invariants.

In this section, we will record the description of some singular fibers of elliptic surfaces. The type of singular fiber that we obtain is determined by the order of vanishing of parameters \( A(s,t) \), \( B(s,t) \) and \( \Delta(s,t) \) where the Weierstrass equation of our elliptic surface is written as
\[
y^2 = x^3 + A(s,t)x + B(s,t)
\]
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<table>
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<th>Symbol</th>
<th>Curve</th>
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</tr>
<tr>
<td>$II$</td>
<td>Cuspidal rational curve</td>
<td>none</td>
<td>2</td>
<td>$a \geq 1$</td>
<td>1</td>
</tr>
<tr>
<td>$II^*$</td>
<td>Two tangent rational curves</td>
<td>$A_1$</td>
<td>3</td>
<td>1</td>
<td>$b \geq 2$</td>
</tr>
<tr>
<td>$IV$</td>
<td>Three rational curves</td>
<td>$A_2$</td>
<td>4</td>
<td>$a \geq 2$</td>
<td>2</td>
</tr>
<tr>
<td>$IV^*$</td>
<td>Rational curves</td>
<td>$E_6$</td>
<td>8</td>
<td>$a \geq 3$</td>
<td>4</td>
</tr>
<tr>
<td>$III^*$</td>
<td>Rational curves</td>
<td>$E_7$</td>
<td>9</td>
<td>3</td>
<td>$b \geq 5$</td>
</tr>
<tr>
<td>$II^*$</td>
<td>Rational curves</td>
<td>$E_8$</td>
<td>10</td>
<td>$a \geq 4$</td>
<td>5</td>
</tr>
</tbody>
</table>

Table A.1: List of the numerical characteristics of specific singular fibers in Weierstrass fibrations of elliptic surfaces.

and

$$\Delta(s, t) = 4A(s, t)^3 + 27B(s, t)^2.$$  

We will give Kodaira’s name for the singularity type, a short description of the resolved curve, the associated root system in NS($X$), and the numerical constants given as the vanishing order of $A$, $B$ and $\Delta$ at the point $p$, which is denoted by $v_p(A)$, $v_p(B)$ and $v_p(\Delta)$ respectively. These are written down in Table A.1. Numerical invariants and descriptions of the resolution of singular fibers of a Weierstrass fibration.

A.5 Root diagrams.

In Figures A.1 A.2, A.3, A.4 and A.5, we will reproduce drawings of the various root diagrams used throughout this thesis. We draw roots of the standard root systems (denoted $A_n, D_n, E_6, E_7, E_8$) represented by black dots, and the extra roots corresponding to the
extended Dynkin diagrams (denoted $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$) correspond to the white diamonds. The lines between roots indicate that the intersection product of the associated rational curves is a single point, and a dotted line indicates intersections in the extended root system, but not in the original system. The dotted lines indicate how the extra root in the extended diagram intersects the roots in the standard root system.

![Diagram](image.png)

Figure A.1: $A_n$ and $\tilde{A}_n$

![Diagram](image.png)

Figure A.2: $D_n$ and $\tilde{D}_n$
Figure A.3: \( E_6 \) and \( \tilde{E}_6 \)

Figure A.4: \( E_7 \) and \( \tilde{E}_7 \)

Figure A.5: \( E_8 \) and \( \tilde{E}_8 \)