SHORT-TIME ASYMPTOTICS OF HEAT KERNELS OF HYPOEIIPTIC LAPLACIANS ON LIE GROUPS

by

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A thesis submitted to the
Department of Mathematics and Statistics
in conformity with the requirements for
the degree of Master of Science

Queen’s University
Kingston, Ontario, Canada
October 2011

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Abstract

This thesis suggests an approach to compute the short-time behaviour of the hypoelliptic heat kernel corresponding to sub-Riemannian structures on unimodular Lie groups of type I, without previously holding a closed form expression for this heat kernel. Our work relies on the use of classical non-commutative harmonic analysis tools, namely the Generalized Fourier Transform and its inverse, combined with the Trotter product formula from the theory of perturbation of semigroups. We illustrate our main results by computing, to our knowledge, a first expression in short-time for the hypoelliptic heat kernel on the Engel and the Cartan groups, for which there exist no closed form expression.
Acknowledgments

First and foremost I wish to thank my supervisor Professor Mansouri, for his guidance regarding the direction of this research, for his patience through long and frequent discussions, and for thoroughly reviewing all the chapters of this thesis. Mostly, I thank him for sharing with me his effervescent passion and enthusiasm for research and teaching during the two years of my master’s studies. I would like to express particular thanks for his generosity in allowing me to attend the 5th Summer School on Geometry, Mechanics and Control in Spain, in the summer of 2011, which was an incredibly enriching experience in every aspect, and was the opportunity for me to meet inspiring students and mathematicians.

I am thankful to the members of my committee for taking the time to read and comment my work.

I want to thank Professor Richard Montgomery, for sharing with me his thoughts about my research, Professor Ole Nielsen for discussing some elements of representation theory, and Dr. Mauricio Godoy Molina for answering some questions via email at the speed of light and with much kindness.

I owe a lot to my friends from Montreal who believed in me deeply and encouraged me to come to Kingston to pursue these studies, and stayed extremely present and supportive despite the distance.

I was then lucky enough to meet wonderful people in Kingston, who gave so much meaning to my time here, and turned my office and apartment into wonderful learning and living environments. Particularly I thank those of you who have worked very hard in the last months to transform the graduate library into the fantastic work place it is today. The free coffee and large sunlit tables have contributed greatly in the writing of this thesis!

I wish to thank all the staff from the Department, but specifically Jennifer Read for kindly
helping me through all the administrative steps, since the organization of my first visit to Queen’s University in the fall of 2008.

Finally, I cannot express how grateful I am towards my parents, my two sisters and Andrew, for being so loving and encouraging throughout the past two years, and especially the last weeks. I could never have done it without you.

I gratefully acknowledge the financial support that I received from the National Sciences and Engineering Research Council, as well as from the Fonds Québécois de la Recherche sur la Nature et les Technologies, for my master’s studies.
Co-authorship

The work within this thesis is co-authored by my supervisor Professor Mansouri, who suggested the consecutive steps of the research, corrected uncountable mistakes, and made recommendations that strongly improved the final results of Chapter 5.
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Chapter 1

Introduction

1.1 Motivation

Solving the heat equation is probably one of the oldest and most classical problems of applied mathematics. This parabolic partial differential equation modeling the diffusion of heat on a surface, was introduced by Fourier in the beginning of the 19th century in his paper “Théorie de la propagation de la chaleur dans les solides”. The impact of his work was immense: it was the starting point for the understanding of numerous diffusion-type problems existing in various fields, like biology, physics, and economics, only to name a few. The quantity and diversity of these problems motivated a growing interest towards studying the diffusion of heat on more and more complex geometric structures.

The geometric context which we will consider in this thesis is the one of sub-Riemannian geometry. Sub-Riemannian geometry emerged through various fields, such as Riemannian geometry, optimal control theory, quantum physics, and classical mechanics. Roughly speaking, it can be described as the study of a smoothly varying distribution $\mathcal{H}$ endowed with a positive definite quadratic form at each point on a $C^\infty$ manifold $M$. Riemannian manifolds, therefore, arise as the specific case of a sub-Riemannian manifold with distribution $\mathcal{H}$ whose rank is everywhere equal to the dimension of $M$, and where the metric is non-degenerate.

How is this geometrical context a natural one? In classical mechanics, one can think about considering a system with restricted directions of motion, while leaving the configuration space
untouched. Control theory leads to similar systems: for example let $M$ be our state manifold, with control set $U \subset \mathbb{R}^m$, and consider the driftless control affine system $x'(t) = \sum_{a=1}^{m} u^a(t)X_a(x(t))$, where the $X_a$ are smooth vector fields on $M$. One can then define a distribution $\mathcal{H}$ on $M$ by $\mathcal{H}(p) = \text{span}_\mathbb{R}\{X_a(p), a = 1, \ldots, m\}$, $p \in M$. A solution $x(t)$ to the system is then a curve having tangent vector in $\mathcal{H}$ at every point. Moreover, one should recall that the Heisenberg group, by far the most studied sub-Riemannian manifold, has countless applications in quantum physics. All these fields have used specific vocabulary to describe these geometries, resulting in different names for the theory: Carnot-Carathéodory geometry, singular Riemannian geometry, etc.

The first result of differential geometry that later became considered as part of the sub-Riemannian literature is probably the Chow–Rashevskii’s theorem([Cho39], [Ras38]), that solved in the 1930’s a particular case of the “accessibility problem” in control theory. It is, however, only in the 1980’s and mostly through the work of Brockett ([Bro80]) that sub-Riemannian geometry became a field of its own. The work of others such as Strichartz ([Str86], [Str89]) and Mitchell([Mit85]) also strongly contributed in laying the foundations of the theory, which has become today a very active field of research, as the literature keeps expanding (good surveys can be found in Montgomery [Mon02] and Calin and Chang [CC09]), and many problems remain open. Moreover, problems from various domains that can be modelled in the sub-Riemannian geometry setting keep emerging, therefore reinforcing the raison d’être of the field. An interesting and recent application in neuro-biology can be found in [SCP08], where the authors use a sub-Riemannian manifold to represent the structure of the human cortex.

In a natural way, many concepts and problems in Riemannian geometry have been extended to sub-Riemannian manifolds: the definition of length and of distance, the existence and properties of geodesics, the structure of the cut-locus, etc. However, there exist important differences between the two kinds of geometric structures. For example, all the geodesics in Riemannian geometry are solution to the Hamiltonian differential equations, whereas in the sub-Riemannian context, those geodesics are called “normal”, and the remaining ones are called “abnormal”. The existence of the latter remained an open problem for a long time and was actually denied multiple times in the literature, until an example was exhibited by Montgomery in [Mon94]. Subsequently Liu and Sussman ([LS95]) showed, by studying the particular case of rank two distributions, that these extremals were
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not an exceptional phenomenon. On another level, it is well-known that the Gaussian curvature is a local invariant of the metric on a Riemannian manifold, but in sub-Riemannian geometry there exist no invariant of the metric, only of the distribution. Another important distinction is made when studying the tangent bundle: indeed, the tangent space at a (regular) point on a sub-Riemannian manifold won’t be a Euclidean space, but can be shown to be itself a sub-Riemannian manifold, having the algebraic structure of a graded nilpotent Lie algebra with dilations.

1.2 Main problem and review of the literature

It is natural to wonder how diffusion happens on sub-Riemannian manifolds. This first leads to first defining a new intrinsic “Laplacian” operator $\Delta_{sr}$ associated to a sub-Riemannian structure, which will only be hypoelliptic, and will be called hypoelliptic Laplacian or subLaplacian. Associated to this operator is the partial differential equation

$$\frac{\partial}{\partial t} \phi(t, x) = \Delta_{sr} \phi(t, x), \quad t > 0, x \in M$$

$$\phi(0, x) = \phi_0(x),$$

which we call the hypoelliptic heat equation. In the cases which will be studied in this thesis, we will suppose that $M = G$ is a unimodular Lie group, and it will be shown that the subLaplacian is the sum of squares of left-invariant vector fields, considered as differential operators. In that case, the solution to the system 1.1 will admit a right-convolution kernel, i.e. there exists a function $p_t$ such that

$$\phi(t, x) = e^{t\Delta_{sr}} \phi_0(x) = \phi_0 * p_t(x).$$

We will call hypoelliptic heat kernel the fundamental solution $p_t$.

Much work has been done towards finding the hypoelliptic heat kernel on sub-Riemannian manifolds. The first calculations were done on the Heisenberg group, at the end of the 1970’s, and the heat kernel was computed simultaneously by various authors, using different methods: Gaveau ([Gav77]) used probabilistic methods while Hulanicki ([Hul76]) made use of representation theory. Explicit formulas were then obtained for general step-2 nilpotent groups ([Gav77], [Cyg79], [BGG00]) and finally for H-type groups ([Ran96], [YZ08], [CCFI11]). The complexity of this problem has forced
researchers to restrict themselves to those Lie groups that enjoy specific algebraic structures, and the methods they have used all strongly relied on that specific structure.

Varadhan, in [Var67], proved a fundamental result regarding the small time asymptotic of the heat kernel $p_t(x, y)$ on a Riemannian manifold $M$:

$$\lim_{t \to 0} t \log p_t(x, y) = -\frac{1}{4} d(x, y)^2,$$

(1.2)

where $d(\cdot, \cdot)$ denotes the distance induced by the Riemannian metric. He was, therefore, the first to introduce the idea of a strong connection between the short-time behaviour of the heat kernel and the Riemannian distance. Work has been pursued in this direction, and in recent work Neel and Stroock ([NS04], [Nee07]) have studied the short-time asymptotics of the gradient and Hessian of the heat kernel at the cut-locus on Riemannian manifolds.

Regarding sub-Riemannian geometry as a natural generalization of Riemannian geometry, the next logical question that one should ask is whether such relations exist in the sub-Riemannian context. This was first answered by Léandre, who established in [Lea92] that the limit (1.2) holds for the sub-Riemannian heat kernel, therefore explicitly showing how the heat kernel yields information on the local geometry of the sub-Riemannian manifold.

This result motivated researchers to study the behaviour in short-time of the hypoelliptic heat kernel, usually by finding upper and lower bounds for it. For example, Ben Arous studied the short-time behaviour of a hypoelliptic Laplacian in $\mathbb{R}^n$, first outside of the cut locus ([Aro88]), and later along the diagonal ([Aro89]), while Gaveau ([Gav77]) found estimates in the case of the Heisenberg group. Very recently, Eldredge, in ([Eld09]), obtained bounds for the hypoelliptic heat kernel on H-type groups, while Brockett and Mansouri ([BM09]) studied short-time asymptotics for a particular class of step-two nilpotent Lie groups on the cut locus. One should note that all this study of the short-time behaviour takes as starting point the closed form expression of the heat kernel, therefore strongly restricting the sub-Riemannian manifolds that can be considered.

Agrachev et al. were the first to suggest, in [ABGR09], the use of classical non-commutative harmonic analysis tools to compute the hypoelliptic heat kernel, in the particular case of unimodular Lie groups of type I. Let $G$ denote such a group. The method, which has long been used in the study of partial differential equations (see for example [Tay86]), consists in first applying a Generalized
(non-commutative) Fourier Transform to the sub-Riemannian heat equation, and obtaining a family of operator equations on the representation space of the unitary irreducible representations of $G$, indexed by the equivalence classes (denoted $\lambda$) of these representations:

$$\frac{\partial}{\partial t} \hat{u}^\lambda(t, \xi) = \hat{\Delta}_{sr}^\lambda \hat{u}^\lambda(t, \xi).$$

(1.3)

The powerful properties of the Fourier transform on Euclidean spaces, such as diagonalizing differential operators, and transforming a differential equation into an algebraic one, exist in an analogous way for the study of certain kinds of operators on arbitrary unimodular Lie groups of type I. For example, left-invariant operators will be simplified as a consequence of the spectral decomposition of the left-regular representation by the non-commutative Fourier transform.

Assuming the family of operator equations (1.3) can be solved, the original heat kernel can then be recovered from the fundamental solutions $e^{t\hat{\Delta}_{sr}^\lambda}$ through a Generalized Inverse Fourier transform. The reality is that for the vast majority of groups $G$, equations (1.3) are completely intractable, and these steps, therefore, only offer a formal method of computation of the hypoelliptic heat kernel.

1.3 Contribution of this thesis

In this thesis we suggest an approach similar in spirit to the one of Agrachev et al., but leading instead to a computation of the short-time behaviour of the hypoelliptic heat kernel.

Our main idea is to use the theory of perturbation of semigroups, and in particular the Trotter product formula, to extract the short-time behaviour of the fundamental solutions $e^{t\hat{\Delta}_{sr}^\lambda}$ of these new equations 1.3, in such a way that after applying an Inverse non-commutative Fourier transform, we recover the short-time asymptotic of the original hypoelliptic heat kernel. We think about our method as a short-cut from the harmonic analysis of the Lie group that we are studying to the short-time asymptotic behaviour of the heat kernel. From a technical point of view, the difficulty relies in keeping track of the error terms along the various steps, as well as ensuring that the final estimate of the heat kernel that we present, which is given as the Inverse Fourier Transform of an approximation in short-time of the semigroups $e^{t\hat{\Delta}_{sr}^\lambda}$, is a $C^\infty$ function, and not only a distribution.

One should note that our method only makes use of the harmonic analysis of $G$, and in particular does not require an already existing closed form expression of the heat kernel. This in fact
constitutes the main advantage of our approach: it can be used to gain insight into the short-time behaviour of a much larger class of left-invariant sub-Riemannian manifolds than the ones for which we have a closed form expression for the heat kernel.

As an application of our main theorem, we will show how to compute the short-time behaviour of the hypoelliptic heat kernel corresponding to two left-invariant sub-Riemannian structures on step-3 nilpotent Lie groups, namely the Engel and the Cartan groups. There exist, to our knowledge, neither a closed form expression nor an expression in short time of these heat kernels. Very recently, Boscain et al., in [BGRar], applied the theory developed in [ABGR09] to these two groups, but the intractability of the transformed equations forced the authors to leave the final expression in terms of unknown functions. Our work therefore offers a first explicit expression for the hypoelliptic heat kernels on the Engel and Cartan groups.

The relevance of using nilpotent Lie groups as our examples does not only originate from the fact that their particularly nice algebraic structure simplifies the calculations, but more so because it has been shown (for example in [RS76]) that sub-Riemannian geometries can be locally approximated at regular points by a left-invariant sub-Riemannian structure on a nilpotent Lie group (more precisely a Carnot group). Therefore, the analysis of the heat kernel on such groups can lead to a local understanding of the heat diffusion on a general sub-Riemannian manifold.

1.4 Outline of the thesis

This thesis is organized as follows: we start by outlining the theory of sub-Riemannian geometry in Chapter 2, and by recalling the classic definitions and theorems. In particular, we recall how to build a natural left-invariant sub-Riemannian structure on Lie groups. We then detail the intrinsic construction of a new Laplacian on a left-invariant sub-Riemannian manifold, which will be a hypoelliptic operator and will coincide with the usual sum of squares on unimodular Lie groups. We conclude by stating the formula given by Agrachev et al. in [ABGR09] for the computation of the hypoelliptic heat kernel. In Chapter 3, we review the basic concepts of representation theory, before studying elements of harmonic analysis on abelian, compact, and nilpotent groups. Our main result is described in Chapter 4: after reviewing some elements of functional analysis, we show how the
Trotter product formula can be combined with the harmonic analysis within Agrachev’s work, to obtain an expression of the short-time behaviour of the hypoelliptic heat kernel. In Chapter 5, to illustrate our work as well as highlight its strengths and weaknesses, we apply our main theorem to the classical Heisenberg group. We then do the same for the Engel and Cartan groups, therefore suggesting a continuation of the work done in [BGRar] on those groups. Finally, Chapter 6 summarizes the main steps of this thesis and suggests future work.
Chapter 2

Sub-Riemannian Geometry

2.1 Basic definitions

We start with a few definitions:

**Definition 1.** Let $M$ be a smooth manifold of dimension $n$. A smooth distribution $\mathcal{H}$ of constant rank $m \leq n$ on $M$ is a smooth map that to each point $q$ of $M$ associates an $m$-dimensional subspace $\mathcal{H}(q)$ of $T_qM$. Explicitly, this means that $\forall \ p \in M$, there exists an open neighborhood $U$ of $p$ and smooth vector fields $X_1, X_2, \ldots, X_m$ on $U$ such that $(X_1(q), \ldots, X_m(q))$ is a basis of $\mathcal{H}(q)$, $\forall q \in U$.

**Definition 2.** Denote by $\text{Vec}(M)$ the space of smooth vector fields on $M$, i.e. the smooth maps $X : M \rightarrow TM$ such that $X(p) \in T_pM$, $\forall \ p \in M$. The distribution $\mathcal{H}$ is called bracket generating (also called non-holonomic, or completely non-integrable, or verifying Hörmander’s condition) if at each point, the vector fields of $\mathcal{H}$, together with a finite number of iterated Lie brackets span the whole tangent space:

$$\text{span}\{[X_1, [X_2, \ldots [X_{k-1}, X_k]]](q)| X \in \mathcal{H}\} = T_qM, \ \forall q \in M,$$

where $\mathcal{H} = \{X \in \text{Vec}(M)| X(q) \in \mathcal{H}(q), \forall q \in M\}$. We call $\mathcal{H}$ the horizontal distribution, and $\mathcal{H}$ the set of horizontal vector fields.

**Definition 3.** The curves in $M$ having tangent vector at each point in that distribution, i.e absolutely continuous curves $\gamma(t)$ such that for almost all $t$, $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$, are called horizontal curves.
We can now define a sub-Riemannian manifold:

**Definition 4.** A sub-Riemannian manifold is a triple \((M, \mathcal{H}, g)\), where \(M\) is a smooth manifold, \(\mathcal{H}\) a smooth bracket-generating distribution, endowed with a positive-definite, non-degenerate metric \(g\). This metric is called the sub-Riemannian metric or also the metric of Carnot-Carathéodory.

This metric enables us to define the length of a horizontal curve \(\gamma : [0, T] \to M\):

\[
    l(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \tag{2.1}
\]

The following theorem of Chow and Rashevskii was the first in sub-Riemannian geometry, and is of fundamental importance:

**Theorem 1.** *(Chow 1939, Rashevskii 1938)* Any two points in a connected sub-Riemannian manifold \(M\) can be joined by a horizontal curve.

Chow’s theorem, together with the sub-Riemannian metric on a connected manifold \(M\), enables us to talk about a sub-Riemannian distance on \(M\), which we define in the following way:

\[
    d(x, y) := \inf \left( \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \right), \tag{2.2}
\]

where the infimum is taken over all horizontal curves \(\gamma\) such that \(\gamma(0) = x\) and \(\gamma(T) = y\).

We now want to show how to construct a natural sub-Riemannian structure on Lie groups:

**Definition 5.** Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\) and let \(\mathfrak{P}\) be a subspace of \(\mathfrak{g}\) that verifies the bracket-generating condition:

\[
    \text{Lie } \mathfrak{P} := \text{span} \{ [p_1, [p_2, \ldots, [p_{n-1}, p_n]]] | p_i \in \mathfrak{P} \} = \mathfrak{g}
\]

Endow \(\mathfrak{P}\) with a positive-definite quadratic form \(\langle \cdot, \cdot \rangle\). We can therefore define a sub-Riemannian structure on \(G\) in the following way:

1. We define our distribution as: \(\mathcal{H}(g) := g\mathfrak{P}\)

2. We define a quadratic form \(g\) on the distribution: \(g_g(v_1, v_2) := \langle g^{-1}v_1, g^{-1}v_2 \rangle\), for \(v_1, v_2 \in \mathcal{H}(g)\).

With this construction, we will say that \((G, \mathcal{H}, g)\) is a left-invariant sub-Riemannian manifold.
We also recall the definition of a regular sub-Riemannian manifold \((M, \mathcal{H}, g)\), and introduce some notation that will be useful later on.

For a distribution \(\mathcal{H}\), we write \(X \tilde{\in} \mathcal{H}\) if \(X\) is a smooth vector field on \(M\) such that \(X(q) \in \mathcal{H}(q)\), \(\forall q \in M\). We can then define a sequence of distributions \(\mathcal{H}_n\) by
\[
\mathcal{H}_1 := \mathcal{H}, \quad \mathcal{H}_{n+1} = \mathcal{H}_n + [\mathcal{H}, \mathcal{H}] \quad \forall n \geq 1,
\]
which means that, at every point \(q \in M\)
\[
\mathcal{H}_{n+1}(q) := \mathcal{H}_n(q) + [\mathcal{H}_n, \mathcal{H}](q) = \{X(q) + [Y, Z](q) \mid X, Y \in \mathcal{H}_n, \ Z \in \mathcal{H}\}.
\]
Note that we clearly have the inclusion \(\mathcal{H}_n(q) \subset \mathcal{H}_{n+1}(q)\), for all \(n \in \mathbb{N}\). In the case where \(\mathcal{H}\) is bracket generating, for each point \(q\) there exists an positive integer \(r\) such that \(\mathcal{H}_r(q) = T_qM\), and therefore at each point we have a sequence of subsets:
\[
\mathcal{H}(q) \subset \mathcal{H}_2(q) \subset \cdots \subset \mathcal{H}_r(q) = T_qM.
\]
(2.3)

Also denote by \(n_i(q) := \dim(\mathcal{H}_i(q))\), for \(i \geq 1\).

**Definition 6.** Let \((M, \mathcal{H}, g)\) be a sub-Riemannian manifold, and consider the sequence (2.3) of subspaces of \(T_qM\) defined above. We call small flag of \(\mathcal{H}\) at \(q\) this sequence, step of the distribution \(\mathcal{H}\) at \(q\) the smallest \(r\) such that \(\mathcal{H}_r(q) = T_qM\), and growth vector of \(\mathcal{H}\) at \(q\) the list of integers: \((n_1(q), n_2(q), \ldots, n_r(q))\). We say that a sub-Riemannian manifold is regular if the growth vector is the same at every point of the manifold \(M\). Finally, we say that \(q\) is a regular point of \(M\) if the growth vector is constant in a neighborhood of \(q\).

It is important to notice that, from their construction, left-invariant sub-Riemannian manifolds \((G, \mathcal{H}, g)\) are always regular, and they are also trivializable.

We end this section with the definition of a particular type of sub-Riemannian manifold:

**Definition 7.** A contact distribution on a manifold \(M\) is a distribution \(\mathcal{H}\) which is the kernel of a one-form \(\theta\), such that the restriction of the two-form \(d\theta\) to the distribution planes \(\mathcal{H}(p)\) is symplectic, i.e. non-degenerate. If we have a metric \(g\) on the contact planes \(\mathcal{H}\) we say that \((M, \mathcal{H}, g)\) is a sub-Riemannian manifold of contact type.

Note that the conditions in the definition imply that \(\mathcal{H}\) is a distribution of even rank, and has codimension 1.
2.2 The hypoelliptic Laplacian

Consider a left-invariant sub-Riemannian manifold \((G, \mathcal{H}, g)\) of rank \(m\), and let \((p_i)_{i=1}^m\) be an orthonormal frame for the subspace \(\mathcal{H}(e)\) of \(T_e G = g\). We construct a family \((X_i)_{i=1}^m\) of vector fields on \(G\) by defining \(X_i(g) = g \cdot p_i\), and from the construction of a left-invariant sub-Riemannian manifold we have that the vectors \((X_i(g))_{i=1}^m\) form an orthonormal frame for the subspace \(\mathcal{H}(g)\) of \(T_g G\), for all \(g \in G\).

Now consider the second order differential operator on \(G\):

\[
\Delta_{sr} = \sum_{i=1}^m L_{X_i}^2,
\]

As the vector fields form an orthonormal frame for the distribution, this operator will be elliptic when \(m = n\), but will not be otherwise. Indeed, let us consider a local system of coordinates \((x_1, x_2, \ldots, x_n)\), and write the vector fields

\[
X_i = \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, m,
\]

where the \(a_{ij}\) are \(C^\infty\) functions on \(G\). We, therefore, know that in the local frame, the vectors \(a_i(x) = (a_{ij}(x))_{j=1}^n, \quad i = 1, \ldots, m\) are linearly independent for all \(x = (x_1, \ldots, x_n)\). The second order differential operator \(L_{X_i}^2\) can be written in coordinates as

\[
L_{X_i}^2(x) = \sum_{j,k=1}^n a_{ij}(x) \left[ \frac{\partial a_{ik}(x)}{\partial x_j} \frac{\partial}{\partial x_k} + a_{ik}(x) \frac{\partial^2}{\partial x_j \partial x_k} \right],
\]

and, by taking the sum over \(i\), we obtain the expression in coordinates of the operator \(\Delta_{sr}\). Taking only its higher order terms, we obtain the operator

\[
\sum_{j,k=1}^n \left( \sum_{i=1}^m a_{ij}(x) a_{ik}(x) \right) \frac{\partial^2}{\partial x_j \partial x_k}
\]

\[
= \sum_{j,k=1}^n B_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k},
\]

where \(B_{jk}(x) = \sum_{i=1}^m a_{ij}(x) a_{ik}(x)\). Notice that, in fact, \(B(x) = \sum_{i=1}^m a_i(x) a_i(x)^T\), and therefore taking \(y\) a vector in \((\mathbb{R}^n)^*\), we compute

\[
y^T B(x) y = \sum_{i=1}^m y^T a_i(x) a_i(x)^T y = \sum_{i=1}^m (y^T a_i(x))^2,
\]

which is strictly positive for all \(y \in (\mathbb{R}^n)^*\) if \(m = n\) by the linear independence of the \((a_i(x))\), and the matrix \(B\) is definite, therefore proving the ellipticity of \(\Delta_{sr}\). If however \(m < n\), the equality...
\( y^T B(x) y = 0 \) does not imply that the vector \( y \) is 0, but only that it belongs to the orthogonal complement of the subspace of \( \mathbb{R}^n \) spanned by the \( (a_i(x)) \). The matrix \( B(x) \) is therefore not definite, which means that \( \Delta_{sr} \) is only subelliptic.

We now define another kind of operator.

**Definition 8.** Let \( U \subset \mathbb{R}^n \) be open, and \( \mathcal{F}(U) \) be the set of all \( \mathbb{R} \)-valued functions on \( U \). Let \( X_1, \ldots, X_m \) be vector fields on \( U \), and \( \mathcal{L} = \sum_{i=1}^{m} L_{X_i}^2 \). The differential operator \( \mathcal{L} \) is said to be hypoelliptic on \( U \) if \( \forall f \in \mathcal{F}(U), \forall u \in D'(U), \) and \( \forall V \subset U \) open, having that \( \mathcal{L} u = f \) and \( f \in C^\infty(V) \) implies \( u \in C^\infty(V) \).

That an elliptic operator is also hypoelliptic is a consequence of the Elliptic Regularity Theorem. The operator \( \Delta_{sr} \) defined above will however be hypoelliptic even in the subelliptic case \( m < n \), as a result of a classical theorem of Hörmander ([Hör67]):

**Theorem 2.** Consider a differential operator on \( \mathbb{R}^n \): \( P = \sum_{j=1}^{r} X_j^2 + X_0 + c \) where \( X_0, \ldots, X_r \) denote smooth vector fields in an open set \( \Omega \subset \mathbb{R}^n \), and \( c \in C^\infty(\Omega) \). Assume that, among the vector fields \( X_{j_1}, [X_{j_1}, X_{j_2}], \ldots, [X_{j_1}, [X_{j_2}, \ldots, [X_{j_{k-1}}, X_{j_k}]]]) \) where \( j_i = 0, 1, \ldots, r \), there exist \( n \) which are linearly independent at any given point in \( \Omega \). Then, it follows that \( P \) is hypoelliptic.

We are, therefore, ready to give the definition:

**Definition 9.** Let \( (G, \mathcal{H}, g) \) be a left-invariant sub-Riemannian manifold of rank \( m \) and let \( X_1, \ldots, X_m \) be left-invariant \( C^\infty \) sections of the distribution \( \mathcal{H} \) such that, at each point \( g \in G \), the vectors \( (X_i(g))_{i=1}^{m} \) form an orthonormal frame for \( \mathcal{H}(g) \) with respect to the sub-Riemannian metric at \( g \). The left-invariant second order differential operator \( \Delta_{sr} \) defined by

\[
\Delta_{sr} = \sum_{i=1}^{m} L_{X_i}^2
\]

is called the hypoelliptic Laplacian (or simply sub-Laplacian) associated to the left-invariant sub-Riemannian structure \( (G, \mathcal{H}, g) \).

It is worth pointing out that this operator is independent of the choice of orthonormal frame \( (X_i)_{i=1}^{m} \), and that, for left-invariant sub-Riemannian structures defined on unimodular Lie groups, the hypoelliptic Laplacian can be constructed intrinsically, i.e. without resorting to any vector field.
We will give a rough outline of how this can be done, and for this we follow Sections 2.2 and 2.3 in [ABGR09] (this was first done by Brockett [Bro80] and can also be found in [Mon02]).

We construct the sub-Laplacian as the natural generalisation of the Laplace–Beltrami operator in the Riemannian setting. Remember that this last operator is defined as the divergence of the gradient:

$$\Delta \phi = \text{div}(\nabla \phi).$$

The gradient $\nabla$ is the unique operator from $C^\infty(M)$ to $\text{Vec}(M)$ that verifies:

$$\langle \nabla \phi, X \rangle(q) = d\phi(X)(q), \quad \forall q \in M, \forall X \in \text{Vec}(M),$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric, and the divergence of a vector field is the unique function satisfying

$$L_X(\mu) = (\text{div } X)\mu,$$

where $\mu$ is the Riemannian volume form. We now want to define a sub-Riemannian gradient $\nabla_{sr}$, and a sub-Riemannnian divergence $\text{div}_{sr}$, and express the sub-Laplacian as

$$\Delta_{sr} \phi = \text{div}_{sr}(\nabla_{sr} \phi), \quad (2.7)$$

We want this operator to be defined intrinsically, i.e. to be independent of a choice of local orthonormal frame. Let us first define $\nabla_{sr}$, which we call the horizontal gradient.

**Definition 10.** Let $(M, \mathcal{H}, g)$ be a sub-Riemannian manifold. The horizontal gradient is the unique operator $\nabla_{sr}$ from $C^\infty(M)$ to the set of horizontal vector fields $\mathcal{H}$, verifying

$$g(\nabla_{sr} \phi, X)(q) = d\phi(X)(q), \quad \forall q \in M, \forall X \in \mathcal{H}$$

It is easy to check that if $\{X_1, X_2, \ldots, X_n\}$ is a local orthonormal frame for $(M, \mathcal{H}, g)$, we get the formula

$$\nabla_{sr} \phi = \sum_{i=1}^{n} X_i(\phi)X_i.$$

Before defining the sub-Riemannian divergence, we will need to define an intrinsic sub-Riemannian volume form. Denote it $\mu_{sr}$. It was first defined for general regular and orientable sub-Riemannian manifolds by Montgomery in [Mon02], under the name of Popp’s measure. Recall that if $(M, \mathcal{H}, g)$
is regular, then we have the filtration:

$$0 \subset H_1(q) \subset \ldots H_k(q) = T_q M, \forall q \in M.$$  

The main idea is to use the natural isomorphism:

$$\bigwedge^n (T_q M)^\ast = \bigwedge^n \left( \bigoplus_{i=1}^k \left( H_i / H_{i-1} \right) \right)^\ast,$$

and to construct a volume form $\nu_i$ on each $H_i / H_{i-1}$.

This construction simplifies in the case of an orientable 3 dimensional contact sub-Riemannian manifold, and the resulting intrinsic volume form is given in [ABGR09]:

**Proposition 1.** Let $(M, \mathcal{H}, g)$ be an orientable contact sub-Riemannian manifold of dimension 3, and let $(X_1, X_2)$ be a local orthonormal frame for $\mathcal{H}$. Let $X_3 := [X_1, X_2]$ and denote as usual $dX_1$, $dX_2$ and $dX_3$ the dual basis. Then: $\mu_{sr} = dX_1 \wedge dX_2 \wedge dX_3$ is an intrinsic volume form, i.e is invariant under an orientation preserving change of orthonormal frame.

Suppose now that our manifold is in fact a Lie group $G$ together with a bracket generating subset $\mathfrak{P}$ of $\mathfrak{g}$, with the resulting left-invariant sub-Riemannian manifold $(G, \mathcal{H}, g)$ as defined in the previous section. Assume that $\{p_1, \ldots, p_m\} \subset \mathfrak{g}$ is an orthonormal basis for $\mathfrak{P}$. Then, the steps above can be translated in global coordinates and lead to the following intrinsic global expression for the sub-Laplacian:

$$\Delta_{sr} \phi = \sum_{i=1}^m \left( L_{X_i}^2 \phi + L_{X_i} \phi \text{Tr}(\text{ad} p_i) \right),$$

(2.8)

where $X_i$ is the vector field defined by $X_i(g) = gp_i, \forall g \in G$ (we refer to [ABGR09] for this computation).

In the case of unimodular Lie groups, the sub-Laplacian reduces to the usual sum of squares, and surprisingly, this particular form of the sub-Laplacian characterizes the unimodular Lie groups:

**Proposition 2.** Let $(G, \mathcal{H}, g)$ be a left-invariant sub-Riemannian manifold generated by the orthonormal basis $\{p_1, \ldots, p_m\} \subset \mathfrak{g}$. Define the smooth vector fields $X_i(g) = gp_i, i = 1, \ldots m$. Then, $G$ is unimodular if and only if $\Delta_{sr} \phi = \sum_{i=1}^m L_{X_i}^2 \phi$.

We again refer to [ABGR09] for a proof of this proposition.
CHAPTER 2. SUB-RIEMANNIAN GEOMETRY

2.3 A formula for the hypoelliptic heat kernel

The goal of this section is to state the main result of Agrachev et al. ([ABGR09]), which motivated our work in Chapter 4.

From now on we will suppose that $G$ is a unimodular Lie group of type I, endowed with its natural sub-Riemannian structure generated by an orthonormal set $\{p_1, \ldots, p_m\}$ of the Lie algebra $g$. We construct the left-invariant vector fields: $X_i(g) = gp_i, \forall g \in G, i = 1, \ldots, m$. By the previous section, we know that the sub-Laplacian operator is the sum of squares of these vector fields, i.e: $\Delta_{sr} \phi = \sum_{i=1}^{m} L_{X_i}^2 \phi$, and is hypoelliptic by Hörmander’s theorem. The reasons for restricting ourselves to groups of type I will be discussed in the next chapter. We consider the new heat equation associated to this operator:

$$\frac{\partial}{\partial t} \phi(t, g) = \Delta_{sr} \phi(t, g),$$

$$\phi(0, g) = \phi_0(g). \quad (2.9)$$

Since each $L_{X_i}$ is skew-adjoint, the operator $-\Delta_{sr}$ is symmetric and positive, and therefore it has a canonical self-adjoint extension, called the Friedrich extension. We can, therefore, suppose that $-\Delta_{sr}$ is positive and self-adjoint, which implies that $\Delta_{sr}$ is the generator of a contraction semigroup which we denote $(e^{t\Delta_{sr}})_{t \geq 0}$. (All these notions will be developed in more detail in Chapter 4). When $\phi(0, g) = \phi_0(g) \in L^1(G)$, then by construction $e^{t\Delta_{sr}} \phi_0(g)$ is solution to the partial differential equation (2.9).

Notice that the operators $L_{X_i}$ are left-invariant, and consequently so is $\Delta_{sr}$, which implies that for every $t$, the operator $e^{t\Delta_{sr}}$ is left-invariant. As it is also bounded since we have a contraction semigroup, by [Hör60] it can therefore be written as a right-convolution by a distribution. Let us denote by $p_t$ the distribution which is the right-convolution kernel of the operator $e^{t\Delta_{sr}}$:

$$\phi(t, g) = e^{t\Delta_{sr}} \phi_0(g) = \phi_0 * p_t(g) = \int_G \phi_0(h)p_t(h^{-1}g)dh.$$ 

We will call $p_t$ the hypoelliptic heat kernel. Note that the hypoellipticity of $\Delta_{sr}$ implies that $p_t(g)$ is a $C^\infty$ function on $G$ for all $t > 0$. Also, notice that by the left-invariance of the sub-Laplacian, it is sufficient to consider the case where the initial condition is the Dirac distribution at the identity: $\phi(g) = \delta_e(g)$, which implies: $\phi(t, g) = p_t(g)$.

We now briefly introduce a few notions and theorems of harmonic analysis that will be necessary
for the statement of the main result of Agrachev et al. They will be developed in detail in Chapter 3.

**Definition 11.** The dual of a Lie group $G$ is the set of equivalence classes of irreducible unitary representations of $G$, and is denoted by $\hat{G}$. Letting the variable $\lambda$ index the elements of $\hat{G}$, a representative of the class $\lambda$ is a unitary representation $\mathcal{X}_\lambda$ of $G$, i.e. a homomorphism from $G$ to unitary operators on a Hilbert space $H^\lambda$.

**Definition 12.** Let $G$ be a unimodular Lie group of type I, and $f \in L^1(G, \mathbb{C})$. The Generalized Fourier Transform (GFT) of $f$ is the map $\hat{f}$ (or $\mathcal{F}(f)$) that associates to each element $\lambda$ of the dual the linear operator on $H^\lambda$:

$$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_G f(g) \mathcal{X}_\lambda(g^{-1})dg$$  \hspace{1cm} (2.10)

It can be shown that for all $\lambda \in \hat{G}$, $\hat{f}(\lambda)$ is a Hilbert–Schmidt operator on $H^\lambda$ and in particular is a bounded operator (see for example Section 4.2 of [CG90]). We will denote by $\mathbf{HS}^\lambda$ the set of Hilbert-Schmidt operators on $H^\lambda$.

The following proposition gives us the well-known Plancherel formula and the inverse Generalised Fourier Transform (inverse GFT):

**Proposition 3.** Let $G$ be a unimodular Lie group of type I. Then there exists on $\hat{G}$ a positive measure $dP(\lambda)$ called the Plancherel measure, such that for every $f \in L^1(G, \mathbb{C}) \cap L^2(G, \mathbb{C})$, we have

$$\int_G |f(g)|^2d\mu(g) = \int_{\hat{G}} \text{Tr}(\hat{f}(\lambda) \circ \hat{f}(\lambda)^*)dP(\lambda)$$  \hspace{1cm} (2.11)

$$f(g) = \int_{\hat{G}} \text{Tr}(\hat{f}(\lambda) \circ \mathcal{X}_\lambda(g))dP(\lambda)$$  \hspace{1cm} (2.12)

The definition of the Generalised Fourier Transform of a left-invariant vector field follows in a natural way from that of a function:

**Definition 13.** Let $G$ be a unimodular Lie group of type I, and $X$ a left-invariant vector field on $G$. The GFT of $X$ is defined by

$$\hat{X} = \mathcal{F}L_X\mathcal{F}^{-1}$$  \hspace{1cm} (2.13)

Clearly, $\hat{X}$ acts on the Hilbert-Schmidt operators $\hat{f}(\lambda)$, for $\lambda \in \hat{G}$. From [ABGR09] we have the following useful proposition:
Proposition 4. Let $G$ be a unimodular Lie group of type I with Lie algebra $\mathfrak{g}$, $X$ a left-invariant vector field on $G$ and $\hat{X}$ its GFT. Define by $dX^\lambda$ the differential of the representation $X^\lambda$:

$$dX^\lambda(X) := \left. \frac{d}{dt} \right|_{t=0} X^\lambda(e^{t p}),$$

where $X = gp$ ($p \in \mathfrak{g}, g \in G$). We have that $\hat{X}$ splits into the Hilbert sum of operators $\hat{X}^\lambda$, each of which acts on $\mathbf{HS}^\lambda$. We usually write this as $\hat{X} = \int_{\hat{G}} \hat{X}^\lambda$. Moreover,

$$\hat{X}^\lambda \Xi = dX^\lambda(X) \circ \Xi, \forall \Xi \in \mathbf{HS}^\lambda,$$

(2.14)

which means that $\hat{X}^\lambda$ acts as a left translation over $\mathbf{HS}^\lambda$.

Proof. As $G$ is unimodular, we consider $\mu$ a right and left-invariant measure on the group. We call right-regular representation $R$ the homomorphism from $G$ to the the set of unitary representations on $L^2(G, \mu)$, where $G$ acts by right translation: $[R_g f](h) = f(hg)$. Let us compute the Fourier transform of this representation. We let $p \in \mathfrak{g}$, $t \in \mathbb{R}$, $g_0 \in G$ and $\lambda \in \hat{G}$, and compute

$$\mathcal{F}(R_{e^{tp}} f)(\lambda) = \int_G [R_{e^{tp}} f](g_0)X^\lambda(g_0^{-1})d\mu(g_0)$$

$$= \int_G f(g_0 e^{tp})X^\lambda(g_0^{-1})d\mu(g_0)$$

$$= \int_G f(g)X^\lambda(e^{tp})X^\lambda(g^{-1})d\mu(g) \text{ by the right-invariance of } \mu,$$

$$= X^\lambda(e^{tp})\hat{f}(\lambda).$$

We have therefore shown that the General Fourier transform disintegrates the right-regular representation: $\hat{R}_{e^{tp}} = \int_{\hat{G}} \hat{X}^\lambda(e^{tp})$.

We now compute the GFT of a left-invariant vector field $X$, such that $X(g) = gp$ for an element $p \in \mathfrak{g}$, for all $g \in G$. The curve $\gamma(t) = ge^{tp}$ is, therefore, a representative of the equivalence class of
curves corresponding to \( X(g) \):

\[
[X^\lambda f](\lambda) = [\mathcal{F}X^\lambda f](\lambda) = \mathcal{F}[X(f)](\lambda) = \int_G X(f)(g) \mathcal{X}^\lambda (g^{-1}) d\mu(g)
\]

\[
\mathcal{X}^\lambda e^{tH} \mathcal{X}^\lambda (g^{-1}) d\mu(g) = \frac{d}{dt} \bigg|_{t=0} \int_G f(g^t) \mathcal{X}^\lambda (g^{-1}) d\mu(g)
\]

\[
\mathcal{X}^\lambda (e^{tH} f)(\lambda) = d\mathcal{X}^\lambda (X) \mathcal{X}^\lambda f(\lambda).
\]

The passage of the limit outside the integral is justified by the Dominated Convergence Theorem, assuming that \( f \) has compact support. We have, therefore, shown the decomposition of the operator \( \hat{X}^\lambda = \int_{\hat{G}} \mathcal{X}^\lambda \) on the spaces \( \mathbf{H}^\lambda \), and also that on each of these spaces \( \hat{X}^\lambda \Xi = d\mathcal{X}^\lambda (X) \circ \Xi, \forall \Xi \in \mathbf{H}^\lambda \).

Note that as for a left-invariant vector field \( X \), the operator \( \hat{X}^\lambda \) acts as a left translation over \( \mathbf{H}^\lambda \), we can in fact consider \( \hat{X}^\lambda \) as an operator over \( H^\lambda \), more precisely as the operator \( d\mathcal{X}^\lambda (X) \).

We are finally ready to state the main result of [ABGR09].

**Theorem 3.** Let \( G \) be a unimodular Lie group of type I, and \((G, \mathcal{H}, g)\) a left-invariant sub-Riemannian manifold generated by the orthonormal basis \( \{p_1, \ldots, p_m\} \). The intrinsic hypoelliptic Laplacian is \( \Delta_{sr} = L^2_{X_1} + \ldots + L^2_{X_m} \) where \( X_i = gp_i \). Let \( \{\mathcal{X}^\lambda\}_{\lambda \in \hat{G}} \) be the set of equivalence classes of irreducible unitary representations of the group \( G \), each acting on the Hilbert space \( H^\lambda \), and \( dP(\lambda) \) the Plancherel measure on \( \hat{G} \). We have the following:

(i) The GFT of \( \Delta_{sr} \), \( \hat{\Delta}_{sr} = \mathcal{F} \Delta_{sr} \mathcal{F}^{-1} \), splits into the Hilbert sum of operators \( \hat{\Delta}_{sr}^\lambda \), each one of which leaves \( \mathcal{H}^\lambda \) invariant:

\[
\hat{\Delta}_{sr} = \int_{\hat{G}} \hat{\Delta}_{sr}^\lambda, \quad \text{where} \quad \hat{\Delta}_{sr}^\lambda = \sum_{i=1}^m \left( \hat{X}_i^\lambda \right)^2.
\]

(ii) The operator \( \hat{\Delta}_{sr}^\lambda \) is self-adjoint and is the infinitesimal generator of a contraction semigroup \( e^{t\hat{\Delta}_{sr}^\lambda} \) over \( \mathbf{H}^\lambda \), i.e \( e^{t\hat{\Delta}_{sr}^\lambda} \Xi^\lambda_0 \) is the solution for \( t > 0 \) to the operator equation \( \partial_t \Xi^\lambda(t) = \hat{\Delta}_{sr}^\lambda \Xi^\lambda(t) \) in \( \mathbf{H}^\lambda \), with initial condition \( \Xi^\lambda(0) = \Xi^\lambda_0 \).
(iii) The hypoelliptic heat kernel is given by:

\[ p_t(g) = \int_{\hat{G}} \text{Tr} \left( e^{t\Delta_{sr}^\lambda} \mathcal{X}^\lambda(g) \right) dP(\lambda), \quad t > 0. \]  

(2.15)

In the case where for each \( t \in \mathbb{R}^+ \) and each \( \lambda \in \hat{G} \), \( e^{t\Delta_{sr}^\lambda} \) is an integral operator with integral kernel \( Q^\lambda_t(\cdot, \cdot) \), we have the following corollary.

**Corollary 1.** Under the hypotheses of Theorem 3, if, for all \( \lambda \in \hat{G} \), we have \( H^\lambda = L^2(X^\lambda, d\theta^\lambda) \) for some measure space \((X^\lambda, d\theta^\lambda)\), and

\[ \left[ e^{t\Delta_{sr}^\lambda} f^\lambda \right](\theta) = \int_{X^\lambda} f^\lambda(\bar{\theta}) Q^\lambda_t(\theta, \bar{\theta}) d\bar{\theta} \]

then

\[ p_t(g) = \int_{\hat{G}} \int_{X^\lambda} \mathcal{X}^\lambda(g) Q^\lambda_t(\theta, \bar{\theta}) |_{\theta = \bar{\theta}} d\bar{\theta} dP(\lambda) \]  

(2.16)

Let us now give a proof of Theorem 3 (following the one given in [ABGR09]):

**Proof.** (i) By Proposition 4, the General Fourier Transform of a left-invariant vector field \( X \) splits into the direct integral \( \hat{X} = \int_{\hat{G}}^G \hat{X}^\lambda \), and the same is then true for the square of such a vector field: \( (\hat{X})^2 = \int_{\hat{G}}^G (\hat{X}^\lambda)^2 \). As on a unimodular Lie group \( G \), \( \Delta_{sr} \) is the sum of squares of left-invariant vector fields, the same decomposition follows for the hypoelliptic Laplacian.

(ii) To prove that \( \Delta_{sr} \) is a self-adjoint operator and is the generator of a contraction semigroup over \( \mathcal{H}^\lambda \), we shall use the following theorem of functional analysis:

**Theorem 4.** The generators of self-adjoint contraction semigroups \((T(t))_{t \geq 0}\) are precisely the operators \((-A)\), where \(A\) are positive self-adjoint operators.

(For the definition of these notions, we refer to the first section of Chapter 4). Following [VSCC92], we start by recalling that for a left-invariant vector field \( X \) on a Lie group \( G \) verifying the assumptions of the theorem, with \( \gamma(t) \) an integral curve of \( X \) such that \( \gamma(0) = g \in G \), and for \( f \in C_0^\infty(G) \), we have

\[
\int_G (Xf)(g) dg = \int_G \lim_{t \to 0} f(\gamma(t)) \frac{d}{dt} dg = \int_G \lim_{t \to 0} \frac{f(g \cdot e^{tX}) - f(g)}{t} dg = \lim_{t \to 0} \frac{1}{t} \int_G [f(g \cdot e^{tX}) - f(g)] dg = 0,
\]
where we used the unimodularity of $G$. Also, the continuity of $f$ at $g$ combined with the Dominated Convergence theorem justify passing the limit outside of the integral. This shows that $X = L_X$ is a skew-adjoint differential operator: $\int_G (Xf)(g)h(g)dg = - \int_G f(g)(Xh)(g)dg$, and hence that $-X^2$ is both symmetric and positive. The negative of the hypoelliptic Laplacian $-\Delta_{sr}$ is therefore also symmetric and positive, and identifying it with its Friedrich’s extension, we can conclude from Theorem 4 that $\Delta_{sr}$ is the generator of a self-adjoint contraction semigroup. Let us now use these results to study the transformed sub-Laplacian $\hat{\Delta}_{sr}^\lambda$.

Recall that we proved in (i) that $\hat{X}$ splits into the direct integral $\int_G^{oplus} \hat{X}^\lambda$. Also, the Plancherel formula (given at the beginning of Chapter 3) is equivalent to saying that the Generalized Fourier Transform $\mathcal{F}$ is an isometry from the set of square integrable functions from $G$ to $\mathbb{C}$ with respect to the Haar measure on $G$, $L^2(G)$, to the set $\int_G^{oplus} \mathcal{HS}^\lambda$ of Hilbert-Schmidt operators on the spaces $H^\lambda$, with respect to the Plancherel measure. Remember that on $\mathcal{HS}^\lambda$ we can define the inner product $\langle A, B \rangle = \text{Tr}(A \circ B^*)$. Therefore, denoting $\langle \cdot, \cdot \rangle$ the inner product of both spaces $L^2(G)$ and $\mathcal{HS}^\lambda$, and using the properties of the operator $X^2$, we obtain

$$\langle (\hat{X}^\lambda)^2 \hat{f}(\lambda), \hat{h}(\lambda) \rangle = \langle \mathcal{F}(X^2f)(\lambda), \mathcal{F}(h)(\lambda) \rangle$$

$$= \langle X^2f, h \rangle = \langle f, X^2h \rangle$$

$$= \langle \mathcal{F}(f)(\lambda), \mathcal{F}(X^2h)(\lambda) \rangle = \langle \hat{f}(\lambda), (\hat{X}^\lambda)^2 \hat{h}(\lambda) \rangle.$$ 

We conclude that $-(\hat{X}^\lambda)^2$ is symmetric and positive, and, therefore, so is $\hat{\Delta}_{sr}^\lambda$. Identifying the transformed sub-Laplacian with its Friedrich’s extension, we have shown that $\hat{\Delta}_{sr}^\lambda$ is self-adjoint and is the infinitesimal generator of a contraction semigroup.

(iii) The inverse Generalized Fourier Transform 2.12 applied to the heat kernel $p_t$ yields

$$p_t = \int_G \text{Tr}(\hat{p}_t(\lambda) \circ \hat{X}^\lambda(\lambda)) dP(\lambda).$$

However, by definition, $u(t, g) = \phi_0 \ast p_t(g)$ is the solution to the heat equation $\frac{\partial}{\partial t} u(t, g) = \Delta_{sr} u(t, g)$ with initial condition $u(0, g) = \phi_0(g)$. Therefore, by taking the GFT and using the formula for the GFT of a convolution, we obtain that $\hat{p}_t(\lambda) \hat{\phi}_0(\lambda)$ is a solution to the transformed heat equation $\frac{\partial}{\partial t} \hat{u}(t, \lambda) = \hat{\Delta}_{sr} \hat{u}(t, \lambda)$ with initial condition $u(0, \lambda) = \hat{\phi}_0(\lambda)$, which we can write as $\hat{p}_t(\lambda) \hat{\phi}_0(\lambda) = e^{t\hat{\Delta}_{sr}} \hat{\phi}_0(\lambda)$. Letting $\hat{\phi}_0(g) = \delta_g(g)$ results in $\hat{p}_t(\lambda) = e^{t\hat{\Delta}_{sr}}$. 

(since $\hat{\delta}_e(\lambda) = Id_{H^\lambda}$), which proves (iii).

\[\square\]

Agrachev et al., therefore, give in [ABGR09] a formal method of computation for the hypoelliptic heat kernel on a unimodular Lie group of type I, based on classical tools of non-commutative harmonic analysis. We will focus on this theory in the next chapter.
Chapter 3

Harmonic Analysis

3.1 Motivation within the thesis

The subject of harmonic analysis being extremely broad and the amount of literature in this area being gigantic, it would be impossible and quite naive to try to cover it in a thesis chapter. Our goal throughout this chapter is to review the notions of harmonic analysis, and more specifically non-commutative harmonic analysis, that will be necessary to a clear understanding of the results within this thesis. More specifically, we will want to take a closer look at the following notions for $G$ a connected, unimodular, type I locally compact group:

1. The (unitary) dual $\hat{G}$ of $G$:

   **Definition 14.** The dual of a Lie group $G$ is the set of equivalence classes of irreducible unitary representations of $G$, and is denoted $\hat{G}$.

2. The Generalized Fourier Transform (GFT):

   **Definition 15.** Let $G$ be a unimodular Lie group of type I, with $dg$ a fixed Haar measure. The Generalized Fourier Transform (GFT) of $f$ is the map $\hat{f}$ (or $\mathcal{F}(f)$) that associates to each element $\lambda$ of the dual the linear operator on $H^\lambda$:

   $$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_G f(g)X^\lambda(g^{-1})dg,$$

   where $X^\lambda$ a representative of the class $\lambda$. (3.1)
3. The Abstract Plancherel Theorem:

**Theorem 5.** There exists a unique measure $\mu$ on $\hat{G}$, called the Plancherel measure, such that:

$$f(g) = \int_{\hat{G}} \text{Tr}(\hat{f}(\lambda)X^\lambda(g))d\mu(\lambda), \quad \text{(Inverse Generalized Fourier Transform)}$$  

(3.2)

$$\|f\|_2^2 = \int_{\hat{G}} \|X^\lambda(f)\|_{HS}^2 d\mu(\lambda), \forall f \in L^1(G) \cap L^2(G).$$  

(3.3)

$F$ is therefore an isometry, and extends to an onto isometry from $L^2(G)$ to $\mathcal{H} = \int_{\hat{G}} HS^\lambda$.

The proof of this theorem in all its generality exceeds the breadth of this thesis and the knowledge of the author, and we refer to Dixmier’s book [Dix64], section 18, for its complete exposition. We will start by studying these three notions on abelian and compact groups, and show how on these groups harmonic analysis reduces to classical Fourier analysis. We then move on to nilpotent Lie groups, on which we only treat the first item, and explain how the dual can be computed from the dual of lower dimensional groups using an induction process.

We will only consider finite dimensional Lie groups and Lie algebras. All the groups of interest to us will be locally compact topological group, which means that for every element $g$ of $G$, we can find a neighbourhood of $g$ which has compact closure in the topology of $G$. Let us start by recalling a few properties of these groups (for a complete treatment of locally compact groups, see for example [Die69]). There exists on $G$ a unique (up to a multiplicative constant) non-zero right-invariant positive measure $\mu$, called the right-Haar measure. This measure is a positive function on the $\sigma$-algebra generated by the compact sets in $G$, verifying the following properties:

1. right-invariance: $\mu(Ag) = \mu(A), \forall A \subset G, \forall g \in G$;

2. $\mu(K)$ is finite for all compact $K$ in $G$;

3. $\mu$ is outer-regular and inner-regular.

This natural measure enables us to integrate over the group $G$, and the right-invariance of $\mu$ results in the equality $\int_{G} R(h)f(g)d\mu(g) = \int_{G} f(g)d\mu(g)$, where $[R(h)f](g) = f(gh)$. In the same way, there exists on $G$ a left-Haar measure, which is left-invariant and verifies the two last properties of the right-Haar measure listed above. It is also unique up to a multiplicative constant.

Let $\mu$ be a right Haar measure on the locally compact group $G$. Define a new measure $\tilde{\mu}$ by $\tilde{\mu}(A) = \mu(g^{-1}A)$ for compacts $A \subset G$. Clearly, $\tilde{\mu}$ is also right-invariant, and, therefore, there exists a
positive constant $\alpha$ depending on $g$ such that $\tilde{\mu}(A) = \alpha\mu(A)$. We can therefore introduce the modular function $\Delta$ from $G$ to the multiplicative group of real numbers, which verifies: $\tilde{\mu}(A) = \Delta(g)\mu(A)$ and can be proved to be a continuous homomorphism. Note that in the case where $G$ is a Lie group, it is known that in fact $\Delta(g) = \det(\text{Ad}(g))$, where $\text{Ad}$ is the adjoint map from $G$ to $\text{Aut}(T_eG)$ (defined further in the chapter).

We will say that $G$ is unimodular if the modular function $\Delta$ is identically equal to 1. For example, if $G$ is a compact topological group, then $\Delta(G)$ is a compact subgroup of $(\mathbb{R}^*, \cdot)$ containing $\Delta(e) = 1$, and therefore $\Delta$ has to be identically 1, and $G$ is unimodular. It can also be shown that abelian, discrete, semisimple and connected nilpotent groups are also unimodular. All the groups studied in this thesis will share this property.

### 3.2 Basic notions of representation theory

#### 3.2.1 Definitions

**Definition 16.** Let $G$ be a group and $V$ a finite-dimensional vector space over $\mathbb{R}$. Denote by $GL(V)$ the space of linear invertible operators on $V$. We say that $\rho$ is a representation of $G$ on the space $\mathcal{K}$ if $\rho$ is a homomorphism from $G$ to $GL(V)$, i.e. for all $v_1, v_2 \in V$, we have

$$\rho(v_1)\rho(v_2) = \rho(v_1v_2).$$

$V$ is called the representation space and $\text{dim}(V)$ the dimension of the representation. We will often use the notation $(\rho, V)$ for the representation.

Instead of a finite-dimensional vector space, it is in possible to consider a representation space $V$ which is a Banach or Hilbert space, therefore giving rise to infinite-dimensional representations. Independently of the choice of space, we shall always assume that the representation $(\rho, V)$ is continuous, meaning that the map $\rho : (g, v) \rightarrow \rho(g)v$ is continuous in both $g$ and $v$. However, for our purposes we will be mostly interested in the following kind of infinite-dimensional representations:

**Definition 17.** Let $G$ be a group and $H$ a Hilbert space. We say that $\pi$ is a unitary representation of $G$ in $H$, and we denote it $(\pi, H)$ if $\pi$ is a strongly continuous homomorphism from $G$ to the group $U(H)$ of unitary operators on $G$, i.e $\pi : G \rightarrow U(H)$ such that
1. \( \pi(g)\pi(h) = \pi(gh) \), \( \forall g, h \in G \).

2. The map \( g \mapsto \pi(g)x \) is a continuous mapping from \( G \) to \( H \), \( \forall x \in H \).

An example of such a representation which will be useful later on is the left-regular representation. For a locally compact group \( G \) endowed with a Haar measure \( \mu \), we consider the space of square-integrable functions on \( G \):

\[
L^2(G, \mu) = \{ f : G \to \mathbb{C} | \int_G \| f(g) \|^2 d\mu(g) < \infty \}.
\]

We call left-regular representation of \( G \) the homomorphism \( L : G \to U(L^2(G, \mu)) \) where the group acts by left-translations on the space of functions: \( [L(g)f](h) = f(gh) \). Similarly we can define the right-regular representation \( R \) where \( G \) acts by right-translations: \( [R(g)f](h) = f(hg) \).

Let us now consider an arbitrary representation \((\pi, V)\) of a group \( G \). If \( W \) is a subspace of \( V \) that is invariant under \( \pi \), i.e. \( \pi(G)W \subset W \), then \((\pi, W)\) is called a subrepresentation of \((\pi, V)\). Clearly, \((0)\) and \( V \) are invariant subspaces of \( V \), and are called the trivial subrepresentations. If a representation \((\pi, V)\) admits no non-trivial subrepresentation, we say that it is irreducible.

**Definition 18.** Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) be two representations of a group \( G \). We say that \( T \) is an intertwining operator between \( \pi_1 \) and \( \pi_2 \) if \( T \) is an invertible linear operator from \( V_1 \) to \( V_2 \) such that \( T \circ \pi_1(g) = \pi_2(g) \circ T \) for all \( g \in G \).

We will denote by \( \text{Hom}_G(V_1, V_2) \) the set of intertwining operators between the representations \( \pi_1 \) and \( \pi_2 \). Two representations \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are called equivalent if there exists a (non-trivial) intertwining operator between \( \pi_1 \) and \( \pi_2 \). In the same way, two unitary representations \((\pi_1, H_1)\) and \((\pi_2, H_2)\) are called unitarily equivalent if there exists a unitary (non-trivial) intertwining operator between \( \pi_1 \) and \( \pi_2 \).

We can now state a very useful criterion for the irreducibility of a unitary representation.

**Lemma 1.** (Schur’s lemma) A unitary representation \((\pi, H)\) of a group \( G \) is irreducible if and only the set of intertwining operators from \( H \) to \( H \) is one-dimensional: \( \text{Hom}_G(H, H) = \mathbb{C}Id \)

We also recall and prove the particular property of finite-dimensional representation of finite groups:

**Theorem 6.** (Maschke’s Theorem) A finite dimensional representation of a finite group \( G \) is equivalent to a direct sum of irreducible representations.
Proof. Let us consider a finite dimensional representation \((\rho, V)\). We first need to show that \(\rho\) is equivalent to a unitary representation. Suppose that \((\cdot, \cdot)_0\) is an inner product on \(H\). Define a new one \((\cdot, \cdot)\) in the following way: \((v, w) = \sum_G (\rho(g)v, \rho(g)w)_0\). Then, for all \(g' \in G\), we have: 
\[(\rho(g')v, \rho(g')w) = \sum_G (\rho(g'g)v, \rho(g'g)w)_0 = (v, w),\]
and the representation \(\rho\) acting on \(V\) with this new inner product is unitary. We will, therefore, assume that \((\rho, V)\) is a unitary representation of \(G\), and we are ready to prove the theorem.

We proceed by induction on the dimension of \(V\). If \(V\) is of dimension 1, i.e \(\rho\) is a one-dimensional representation, then it is irreducible. Suppose now that \(\dim(V) = k > 1\), and that any representation of degree less than \(k\) is equivalent to a direct sum of irreducible representations. If \(\rho\) is irreducible, then we are done. If not, consider \(W_2\) the orthogonal complement of \(W\) in \(V\). Let \(w\) and \(w_2\) be elements of \(W\) and \(W_2\) respectively. Then we have: 
\[0 = (w, w_2) = (\rho(g)w, \rho(g)w_2) = (w', \rho(g)w_2)\]
where \(w' \in W\) by invariance of \(W\) under \(G\). Therefore \(\rho(g)w_2 \in W_2\) and therefore \(W_2\) is also invariant under \(G\), and \(V = W \oplus W_2\). As both \(W\) and \(W_2\) have dimension \(< k\), by our induction hypothesis \((\rho, W)\) and \((\rho, W_2)\) are both equivalent to a direct sum of irreducible representation, hence so is \((\rho, V)\).

An immediate corollary to this theorem is that a finite-dimensional unitary representation of an arbitrary group \(G\) is equivalent to a direct sum of irreducible unitary representation. Also, as finite groups only have finite dimensional representations, Maschke’s theorem tells us that in order to study the representation theory of a finite group, we only need to compute its irreducible unitary representations. We will show how this theorem can be extended to infinite-dimensional groups, and infinite-dimensional representations. In all cases, we will see that the irreducible unitary representations are always at the center of the representation theory of a type I Lie group. It is this idea that leads to studying the dual of a group \(G\).

We end this section by explaining how representations of closed subgroups of a group \(G\) can be naturally extended to representations of \(G\), and how representations of a Lie group \(G\) can lead to a representation of the corresponding Lie algebra.

**Definition 19.** Let \(H\) be a closed subgroup of a group \(G\), such that \(H \setminus G\) is endowed with a right-invariant measure \(d\mu\), and denote by \((\pi, \mathcal{H}_\pi)\) a representation of \(H\). Let us define a new Hilbert
space $H_{\sigma}$, consisting of functions $f : G \rightarrow H_{\pi}$ such that
\[ f(hg) = \pi(h)f(g), \]
and
\[ \int_{H \backslash G} \|f([g])\|_{L^2(H_{\pi})}^2 \, d\mu([g]) < \infty, \]
where $[g]$ denotes the equivalence class of $g \in G$ in $H \backslash G$. We then define the representation $\sigma$ on $H_{\sigma}$ in the following way:
\[ \sigma(g)f(x) = f(xg). \]
We call $(\sigma, H_{\sigma})$ the representation of $G$ induced by the representation $\pi$ of $H$, and we denote it by $\sigma = \text{Ind}(H \uparrow G, \pi)$ or $\sigma = \text{Ind}_G^H \pi$.

Note that $H \backslash G$ is endowed with a right-invariant measure whenever the modular functions on $H$ and on $G$ agree, which is always the case when the modular function on both is equal to one, i.e. when $G$ is unimodular. As we said earlier, all of the groups we will study in this thesis share this property.

Now, suppose that we have a representation $(\pi, H_{\pi})$ of a Lie group $G$. We create a representation of its Lie algebra $\mathfrak{g}$, which we also denote $\pi$, in the following way:
\[ \pi(l) v = \frac{d}{dt} \bigg|_{t=0} \pi(e^{tl})v, \text{ for all } l \in \mathfrak{g}, \]
where as usual $e^{tl}$ denotes the flow at $t$ of the left-invariant vector field generated by $l$. The space of this representation is therefore the subspace of elements $v \in H_{\pi}$ for which the limit $\lim_{t \to 0} \frac{1}{t} (\pi(e^{tl})v - v)$ exists. We denote this subspace $C^\infty(\pi)$ and call its elements $C^\infty$ vectors of the representation $\pi$. If in addition $\pi$ is a unitary representation, i.e. $\pi(g)$ is a unitary operator for all $g \in G$, what properties does the resulting representation on $\mathfrak{g}$ possess? Note first that for an element $l$ of $\mathfrak{g}$, $(\pi(e^{tl}))_{t \geq 0}$ is a strongly continuous semigroup of unitary operators. Indeed, $\pi(e^{tl})\pi(e^{sl}) = \pi(e^{t+s}l)$, and the map $t \mapsto \pi(e^{tl})x$ is continuous for every $x$ in $H_{\pi}$ by composition of the continuous maps $t \mapsto e^{tl}$ and $e^{tl} \mapsto \pi(e^{tl})x$. Hence by definition, $\pi(l)$ is the infinitesimal generator of the semigroup $(\pi(e^{tl}))_{t \geq 0}$. By Stone’s theorem, it is therefore a skew-adjoint operator. (The definitions of semigroups of operators and infinitesimal generators, as well as the statement and proof of Stone’s Theorem can be found in Section 4.2 of the following Chapter).
3.3 Abelian groups

In this section we will consider a locally compact abelian group \( G \).

### 3.3.1 Construction of the dual

Let \((\rho, H)\) denote a unitary representation of \( G \). We notice that by the commutativity of the elements of \( G \):
\[
\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g), \quad \forall g, h \in G.
\]
Therefore, every operator \( \rho(g) \) is an intertwining operator, and by Schur’s lemma \((\rho, H)\) is irreducible if and only if \( \rho(g) \) is a scalar operator, for all \( g \in G \). Also, it must have norm 1 since it is unitary. To conclude, every irreducible unitary representation of \( G \) is a character, i.e., a homomorphism \( \chi: G \to \mathbb{T} \), where \( \mathbb{T} \) is the multiplicative group of all complex numbers of norm 1. The dual \( \hat{G} \) of \( G \) is therefore the set of characters on \( G \).

We can construct a group structure on \( \hat{G} \) in the following way. Let \( \chi_1 \) and \( \chi_2 \) denote two characters on \( G \). We define the group operation as the pointwise multiplication
\[
(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g), \quad \forall g \in G,
\]
and the inverse of an element as:
\[
\chi^{-1}(g) = \frac{1}{\chi(g)} = \overline{\chi(g)}, \quad \text{since} \quad \chi(g) \text{ is a complex number of norm 1.}
\]
We can therefore refer to \( \hat{G} \) as the dual group of \( G \).

### 3.3.2 Plancherel theorem

As \( \hat{G} \) is a locally compact abelian group, it can therefore be endowed with a right Haar measure. We will call this measure the dual measure, and denote it \( \hat{\mu} \).

For a function \( f \in L^1(G) \), the Fourier transform on a locally compact abelian group will be the following function on the group \( \hat{G} \):
\[
\hat{f}(\chi) = \mathcal{F}(f)(\chi) = \int_G f(g)\chi(g^{-1})d\mu(g) = \int_G f(g)\overline{\chi(g)}d\mu(g), \quad \forall \chi \in \hat{G},
\]
and the inverse Fourier transform of a function \( h \in L^1(\hat{G}) \) is:
\[
\mathcal{F}^{-1}(h)(g) = \int_{\hat{G}} h(\chi)\chi(g)d\hat{\mu}(\chi)
\]
3.3.3 Pontryagin Duality

Let us now consider $\hat{\hat{G}}$, the dual of the dual group $\hat{G}$. By the same arguments as in Section 3.3.1, $\hat{\hat{G}}$ is also a locally compact abelian group, and consists of characters on $\hat{G}$. There is a canonical homomorphism between $G$ and $\hat{\hat{G}}$, which we denote by $\delta: g \mapsto \hat{\delta}_g$, where $\hat{\delta}_g(\chi) := \chi(g), \forall g \in G$. Notice that $\hat{\delta}_g(\chi_1 \chi_2) = (\chi_1 \chi_2)(g) = \chi_1(g)\chi_2(g) = \hat{\delta}_g(\chi_1)\hat{\delta}_g(\chi_2)$, and that $\hat{\delta}_g(\chi) = \chi(g)$ is a complex number of norm 1. This proves that $\hat{\delta}_g$ is an element of $\hat{\hat{G}}$. The map $\delta$ is called the Pontryagin map.

We have the theorem:

**Theorem 7.** The map $\delta : G \rightarrow \hat{\hat{G}}$ is an isomorphism of locally compact abelian groups.

*Proof. See [DE09].*

3.3.4 Example: $G = \mathbb{R}$

Let us consider the additive group of the real numbers $(\mathbb{R}, +)$. By Section 3.3.1, we know that the dual of $\mathbb{R}$ is the group of characters of $\mathbb{R}$, i.e. the homomorphisms $\chi : \mathbb{R} \rightarrow \mathbb{T}$. Choosing one of these characters, let $y \in \mathbb{R}$ be such that $\chi(1) = e^{iy}$. We then have that for $n \in \mathbb{N}$, $\chi(n) = \chi(1)^n = e^{iny}$, and similarly that $\chi(1/n) = \chi(1)^{1/n} = e^{iyn/n}$. From this we deduce how $\chi$ acts on fractions, and by the density of the rational numbers in $\mathbb{R}$ and the continuity of $\chi$, we obtain the behaviour on any real number: $\forall x \in \mathbb{R}: \chi(x) = e^{ixy}$. The number $y$ completely characterizes the representation $\chi$, which we will therefore denote $\chi_y$, and in fact any representation $\chi_y$ is a character of $\mathbb{R}$. Moreover, we can show that $\chi_y \approx \chi_z$ if and only if $y = z$. This proves the isomorphism of groups:

$\hat{\mathbb{R}} \approx \mathbb{R}$

$\chi_y \sim y$

Similarly, on the additive groups $(\mathbb{R}^n, +)$, we have the isomorphisms: $\hat{\mathbb{R}}^n \approx \mathbb{R}^n$.

Let us now compute the usual Fourier Transform of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$:

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-ixy}dx = \int_{\mathbb{R}} f(x)\chi_y(x)^{-1}dx, \quad (3.4)$$

and therefore we witness that the Generalized Fourier Transform agrees with the usual Fourier Transform on Euclidean spaces. Let us now see how the Fourier transform behaves together with
the left regular representation $L$:

$$[L(z)f](y) = \int_{\mathbb{R}} [L(z)f(x)]e^{-ixy}dx = \int_{\mathbb{R}} f(z + x)e^{-ixy}dx$$

$$= \int_{\mathbb{R}} f(\hat{x})e^{-iy(\hat{x} - z)}d\hat{x}$$

$$= e^{iyz}\hat{f}(y).$$

This means that the Fourier transform of the left regular representation splits into the direct integral of the multiplication operators, or in other words that the left regular representation on $\mathbb{R}$ is unitarily equivalent to the direct integral of all the irreducible unitary representations of $\mathbb{R}$: $L(z) \cong \int_{\mathbb{R}} e^{iyz}$.

Taking the Generalized (non-commutative) Fourier transform of $\hat{f}$, one gets

$$f(x) = \int_{\mathbb{R}} \text{Tr}(\hat{f}(y)\chi_y(x))dP(y) = \int_{\mathbb{R}} \hat{f}(y)\chi_y(x) = \int_{\mathbb{R}} \hat{f}(y)e^{ixy}dP(y)$$

as both $\hat{f}(y)$ and $\chi_y(x)$ are scalars. From the classical inverse Fourier formula,

$$f(x) = \int_{\mathbb{R}} \hat{f}(y)e^{ixy}dy,$$

the equality of the two formulas and the uniqueness of the Plancherel measure, we conclude that the Plancherel measure on $\mathbb{R}$ is just the Lebesgue measure. (Again, the same is true on any additive group $(\mathbb{R}^n, +)$).

### 3.4 Compact groups

Let us consider a compact topological group $G$. As it is locally compact, it can be endowed with a left or right Haar measure. Moreover, we had seen that such a group is unimodular, meaning that the right and left Haar measures coincide and are unique up to multiplication by a positive constant. In particular, let us choose the Haar measure $\mu$ on $G$ such that $\int_{G} d\mu(g) = 1$.

The theory of non-commutative harmonic analysis on compact groups was developed during the 1920’s, with main contributions by the mathematicians H. Weyl and F. Peter, and follows naturally from the study of finite groups. For example, we can generalize the idea of averaging over the group by integrating over the group, which is a tool continuously used in proofs. In this section we only give a brief summary of this theory.

We start by recalling a strong property that compact groups share with finite groups, which is
that of complete reducibility of unitary representations:

**Theorem 8.** Any unitary representation \( \pi \) of a compact group \( G \) is equivalent to a direct sum of finite-dimensional irreducible unitary representations.

**Proof.** See [Sug90], theorem 3.1, p.16, or [Fol95], theorem 5.2 p.126.

A direct consequence of this theorem is that every irreducible unitary representation \( U \) of \( G \) is finite-dimensional. Let \( V \) denote the vector space on which \( G \) acts through \( U \). By choosing a basis for \( V \) and writing \( U(g) \) with respect to that basis, we get a matrix realization of the representation \( U \), which we call a matricial representation of \( G \). We will therefore always assume that the irreducible unitary representations are matricial, and denote by \( \{U_{i,j}(g)\} \) the \((i,j)\) element of the matrix \( U(g) \), for \( g \in G \). Those matrix elements are therefore continuous functions on \( G \), verifying the following orthogonality property:

**Theorem 9.** Let \( U \) and \( V \) be two irreducible unitary representations of the compact group \( G \). We have

\[
(U_{i,j}, V_{k,l}) = \begin{cases} 
0, & \text{if } U \not\cong V, \\
\frac{1}{\dim(U)} \delta_{ik} \delta_{jl}, & \text{if } U \cong V.
\end{cases}
\]

Here \((\cdot, \cdot)\) denotes the standard inner product in \( L^2(G) \).

For a proof of this result we refer to [Sug90], theorem 3.2 p.18. Another way to state this result is in terms of characters. Recall that the character of a finite dimensional representation \((U, W_U)\) is defined as: \( \chi_U(g) = \text{Tr}(U(g)) \). If we consider that \( U \) is a matricial representation of dimension \( n \) with matrix elements \( U_{i,j} \), then we can write \( \chi_U(g) = \sum_{i=1}^n U_{i,i}(g) \). Hence, considering two irreducible unitary representation \( U \) and \( V \), which we now know are finite-dimensional, we can use the previous theorem to obtain the orthogonality of characters, a well known result in the representation theory of finite groups:

\[
(\chi_U, \chi_V) = \begin{cases} 
0, & \text{if } U \not\cong V, \\
1, & \text{if } U \cong V.
\end{cases}
\]

The matrix elements of all unitary irreducible representations in fact verify a very special property.
Theorem 10. (Peter–Weyl) Let $\lambda$ denote an equivalence class in $\hat{G}$. Choose a matricial realization $U^\lambda$, and denote by $d^\lambda$ its dimension.

The set of functions: \( \{ \dim(U^\lambda)U^\lambda_{i,j} | 1 \leq i, j \leq \dim(U^\lambda), \lambda \in \hat{G} \} \) is a complete orthonormal family in $L^2(G)$.

Moreover, for $i \in \{1, \ldots, d^\lambda\}$, the space of the $i$-th column matrix elements $C^\lambda_i = \text{span}\{U^\lambda_{1,i}, \ldots, U^\lambda_{d^\lambda,i}\}$ is invariant under the left-regular representation, and on this space the left-regular representation is equivalent to $U^\lambda$.

Explicitly, we have the following direct sum decomposition of the Hilbert space $L^2(G)$ and of the left-regular representation:

\[
L^2(G) = \bigoplus_{\lambda \in \hat{G}} \bigoplus_{i=1}^{d^\lambda} C^\lambda_i \quad (3.5)
\]

\[
L \cong \bigoplus_{\lambda \in \hat{G}} d^\lambda U^\lambda \quad (3.6)
\]

Proof. See for example [Sug90], theorem 3.3 p.20.

We want to deduce from this theorem the Plancherel formula and Plancherel measure for a compact group. We know that the Fourier transform of a function in $L^2(G)$ is defined by

\[
\hat{f}(\lambda) = \int_G f(g)U^\lambda(g^{-1})dg = \int_G f(g)U^\lambda(g)^*dg.
\]

which is, therefore, a matrix with coefficients $(\hat{f}(\lambda))_{i,j} = \int_G f(g)U^\lambda_{j,i}(g)dg$. Let us compute this last integral, but before recall that by Peter-Weyl’s theorem, there exist constants $c^\lambda_{i,j}$ such that

\[
f = \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d^\lambda} c^\lambda_{i,j}U^\lambda_{i,j}. \quad \text{We therefore have}
\]

\[
\int_G f(g)U^\lambda_{k,l}(g)dg = (f, U^\lambda_{k,l}) = \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d^\lambda} c^\lambda_{i,j}(U^\lambda_{i,j}, U^\lambda_{k,l}) = c^\lambda_{k,l}/d^\lambda,
\]

and hence we have the matrix coefficients:

\[
(\hat{f}(\lambda))_{i,j} = c^\lambda_{j,i}/d^\lambda.
\]
We can use this to compute the trace of the product of matrices:

\[
\text{Tr}(\hat{f}(\lambda)U^\lambda(g)) = \sum_{i,j}(\hat{f}(\lambda)_{i,j}U^\lambda_{j,i}) = \sum_{i,j}c_{j,i}^\lambda / d^\lambda U^\lambda_{j,i}(g)
\]

\[
\Rightarrow d^\lambda \text{Tr}(\hat{f}(\lambda)U^\lambda(g)) = \sum_{i,j}c_{j,i}^\lambda U^\lambda_{j,i}(g).
\]

Combining all these calculations we recover the inverse Generalized Fourier Transform

\[
f(g) = \sum_{\lambda \in \hat{G}} d^\lambda \text{Tr}(\hat{f}(\lambda)U^\lambda(g)), \tag{3.7}
\]

where the convergence of the series is understood in the \(L^2\) sense. In the same way, we compute the norm of \(f\):

\[
\|f\|_2^2 = \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d^\lambda} \frac{1}{d^\lambda} |c_{i,j}^\lambda|^2, \quad \text{which we notice can be expressed in the following way:}
\]

\[
\|f\|_2^2 = \sum_{\lambda \in \hat{G}} d^\lambda \text{Tr}(\hat{f}(\lambda)^* \hat{f}(\lambda)) = \sum_{\lambda \in \hat{G}} d^\lambda \|\hat{f}(\lambda)\|_{HS}^2.
\]

By uniqueness of the Plancherel measure, we can see that the Plancherel measure of a compact group \(G\) verifies for each \(\lambda \in \hat{G}\):

\[
\mu(\lambda) = d^\lambda, \quad \text{which means that the Plancherel measure is a counting measure on} \ \hat{G} \ \text{with weights given by the numbers} \ d^\lambda.
\]

### 3.4.1 Example: \(G = \mathbb{T}\)

Let us consider the group \(\mathbb{T}\) of complex numbers of norm 1, with multiplication. It is clearly abelian and compact, and therefore the dual of \(\mathbb{T}\) will be the group of characters: \(\chi : \mathbb{T} \rightarrow \mathbb{T}\). We choose a character \(\chi\), and by identifying \(\mathbb{T} \approx \mathbb{R}/\mathbb{Z}\), we can consider \(\chi\) as a continuous homomorphism from \(\mathbb{R}\) to \(\mathbb{T}\), of period 1. By our study in Section 3.3.4, we now that there exist a real number \(y\) such that \(\chi(1) = e^{iy}\), and \(\chi(x) = e^{ixy}\) for all \(x \in \mathbb{R}\). However, as in addition \(\chi\) is periodic of period 1, we must have that \(\chi(1) = 1\), and therefore \(y = k \in \mathbb{Z}\). We will then denote \(\chi = \chi_k\), and we proved that every character of \(\mathbb{T}\) is equivalent to a \(\chi_k\) for a \(k \in \mathbb{Z}\). Conversely, every \(\chi_k\) is a character of \(\mathbb{T}\), and we have \(\chi_{k_1} \approx \chi_{k_2}\) if and only if \(k_1 = k_2\). We have obtained the isomorphism:

\[
\mathbb{T} \cong \mathbb{Z}
\]

\[
\chi_k \sim k
\]

Note that since \(\mathbb{T} \approx \mathbb{R}/\mathbb{Z}\), we can identify \(\mathbb{T}\) with the interval \([0, 1]\), where we identify the end points.

Let us now compute the Fourier transform of a function \(f \in L^1(\mathbb{T}) \cap L^2(\mathbb{T})\), which we will consider
in a natural way as a function on \( \mathbb{Z} \):

\[
\hat{f}(k) = \int_0^1 f(x) \chi_k(x)^{-1} \, dx = \int_0^1 f(x) e^{-ikx} \, dx
\]

Applying the Peter-Weyl theorem to the group \( G = \mathbb{T} \), we obtain that the set of functions \( \{ \chi_k, k \in \mathbb{Z} \} \) is a complete orthonormal family in \( L^2(\mathbb{T}) \), and that for each \( k \in \mathbb{Z} \), the space span\( \{ \chi_k \} \) is invariant under the left-regular representation, the left-regular representation being equivalent to \( \chi_k \) on each space span\( \{ \chi_k \} \):

\[
[L(y)\chi_k](n) = \int_0^1 [L(y)\chi](x)e^{-inx} \, dx = \int_0^1 e^{ik(y+x)}e^{-inx} \, dx = e^{iky}\chi_k(n).
\]

Moreover, the Inverse Fourier Transform for compact groups (3.7) yields:

\[
f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \chi_k(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx},
\]

which is the usual expression of a function as its Fourier series.

### 3.5 Nilpotent groups

We would like to be able to extend the ideas used in the previous sections for abelian and compact groups in a natural way. To do this we will restrict ourselves to type I groups, which are the groups having the special property that a multiplicity free representation \( \pi \) on a separable Hilbert space can be decomposed uniquely in a direct integral \( \pi = \int \oplus \pi_x \mu(dx) \) of irreducible representations \( \pi_x \).

However, it is nearly impossible to say anything about the dual or Plancherel measure on arbitrary unimodular Lie group of type I.

We therefore choose to restrict our study to the case of nilpotent Lie groups, whose particular algebraic structure simplifies the harmonic analysis, and we will only detail how to find their irreducible unitary representations. For the computation of the Plancherel measure on these groups we refer to [Tay86] or [Puk67].

Our goal is to describe how one can reduce the problem of finding the irreducible unitary representations of a nilpotent Lie group \( G \) to the same problem for a nilpotent group of smaller dimension. This means that by knowing the dual of the Heisenberg group (the only non-abelian nilpotent Lie group of dimension 3), one can compute the dual of any nilpotent Lie group. This induction process is detailed for example in Chapter 6 of [Tay86], and we will follow this exposition. We start by
recalling a few basic definitions about nilpotent Lie groups and Lie algebras, before detailing the
dual of the Heisenberg group, and finally showing how induction can be used to compute the dual
of arbitrary nilpotent Lie groups of dimension greater than 3. We finish the section by using this
induction process to compute the dual of the Engel group.

3.5.1 Properties of nilpotent Lie groups

Consider a Lie algebra \( g \). We define the descending central series of \( g \) to be the series of subalgebras
\( g^{(j)}, j \in \mathbb{N} \), where \( g^{(1)} = g \), and \( g^{(j+1)} = [g, g^{(j)}] \) and therefore verifying: \( g = g^{(1)} \supseteq g^{(2)} \supseteq \ldots \). We say that \( g \) is a nilpotent Lie algebra if there exists an \( n \in \mathbb{N} \) such that \( g^{(n+1)} = 0 \). If in addition we have that \( g^{(n)} \neq 0 \), we say that \( g \) is step-\( n \) nilpotent. A nilpotent Lie group \( G \) is
the unique simply connected Lie group with nilpotent Lie algebra \( g \). Note that subalgebras and
quotient algebras of \( g \) remain nilpotent Lie algebras. Finally, in the nilpotent case the exponential
map \( \exp : g \rightarrow G \) becomes an analytic diffeomorphism, which enables us to identify \( G \) with \( \mathbb{R}^n \) where \( n = \dim(g) \). Indeed, we define coordinates on \( G \) by \( \exp(t_1X_1 + \ldots + t_nX_n) \sim (t_1, \ldots, t_n) \), and by the Baker–Campbell–Hausdorff formula \( \exp(X) \exp(Y) = \exp(Z) \), where:
\[
Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots, \tag{3.8}
\]
(see for example Section 1.2 of [CG90] for the exact formula), which is a finite expression when \( G \) is
a nilpotent group, we obtain a polynomial operation on the vectors \( (t_1, \ldots, t_n) \).

It is useful for the study of nilpotent Lie groups to state results in terms of the adjoint represen-
tation, of which we recall the definition. Conjugation by an element \( g \in G \) leads to an automorphism
of \( G \): \( \delta_g : h \mapsto ghg^{-1} \), which fixes the identity element \( e \). We can then take the differential of \( \delta_g \)
at \( e \), and obtain an automorphism of the Lie algebra \( g \). We denote this new map by \( Ad(g) \), which
results in a map \( Ad : G \rightarrow Aut(g) \) from \( G \) to the group of automorphisms of \( g \). This latter group
can also be thought of as \( GL(g) \), since \( g \cong T_eG \) is a vector space. Moreover, \( Ad \) is a homomor-
phism, and hence is a representation of \( G \) on its Lie algebra: it is called the adjoint representation.
Again, we can take the representation of this representation at the identity, and we obtain a homo-
morphism \( ad : g \rightarrow End(g) \) which is a representation of the Lie algebra \( g \). Notice that using the
Baker–Campbell–Hausdorff formula, we compute

\[
\begin{align*}
ad(X)Y &= \left. \frac{d}{dt} \right|_{t=0} \left. \left( \frac{d}{dh} \right|_{h=0} \exp(tX) \exp(hY) \exp(-tX) \right) \\
&= \left. \frac{d}{dt} \right|_{t=0} \left. \left( \frac{d}{dh} \right|_{h=0} \exp(tX + hY + th/2[X,Y] + \ldots) \exp(-tX) \right) \\
&= \left. \frac{d}{dt} \right|_{t=0} \left. \left( \frac{d}{dh} \right|_{h=0} \exp(hY + th[X,Y] + \ldots) \right) \text{ (the remaining terms have a factor of } t^2 \text{ or } h^2) \\
&= \left. \frac{d}{dt} \right|_{t=0} Y + t[X,Y] + t^2P(X,Y) \\
&= [X,Y].
\end{align*}
\]

for all \(X, Y\) in \(\mathfrak{g}\), and where \(P(X,Y)\) is a sum of Lie brackets of the elements \(X\) and \(Y\).

It is therefore immediate that \(G\) is nilpotent if and only if there exists a positive integer \(k\) such that \(ad(X)^k = 0\) for all \(X \in \mathfrak{g}\).

It is important to recall the following theorem:

**Theorem 11. (Birkhoff embedding theorem)** Let \(\mathfrak{g}\) be a nilpotent Lie algebra over \(\mathbb{R}\). There is a finite dimensional vector space \(V\), together with a monomorphism of Lie algebras \(\alpha : \mathfrak{g} \to gl(V)\), such that, for all \(X \in \mathfrak{g}\), \(\alpha(X)\) is nilpotent.

An important addition to this theorem is the following:

**Theorem 12. (Engel’s theorem)** Let \(\mathfrak{g}\) be a Lie algebra with a homomorphism \(\alpha : \mathfrak{g} \to gl(V)\) for a vector space \(V\), such that \(\alpha(X)\) is nilpotent for all \(X \in \mathfrak{g}\). Then there exists a flag of subspaces:

\[(0) = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V, \quad \text{with } \dim V_j = j,
\]

such that \(\alpha(X)V_j \subseteq V_{j-1}\) for all \(j \geq 1\) and \(X \in G\). In particular, \(\alpha(\mathfrak{g})\) is a nilpotent matrix Lie algebra.

We omit the proofs of these theorems, which can be found for example in Chapter 1 of [CG90]. Recall that a nilpotent matrix \(A\) is such that \(A^k = 0\) for a positive integer \(k\), and is therefore equivalent to a strictly upper or lower triangular matrix. Identifying an element of \(\mathfrak{g}\) with its image by the map \(\alpha\), we can therefore identify the Lie algebra \(\mathfrak{g}\) with a family of strictly upper triangular matrices. It is therefore always possible to look at a Lie algebra as a nilpotent matrix algebra, which induces a matrix realization of the group \(G\). The exponential map becomes the usual matrix
exponential, and the adjoint map reduces to the conjugation: \( Ad(g)X = gXg^{-1} \), for \( g \in G \) and \( X \in \mathfrak{g} \).

### 3.5.2 The Heisenberg group

The Heisenberg group \( H_1 \) will be studied in detail in Chapter 5, but let us briefly summarize its main properties. The Heisenberg group is the unique simply connected Lie group whose Lie algebra is spanned by elements \( t_1, t_2 \) and \( t_3 \), where \( [t_1, t_2] = t_3 \) is the only non-zero bracket relation between these generators. Also, \( H_1 \) can be identified with \( \mathbb{R}^3 \) endowed with the operation \( (a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + \frac{1}{2}(ay - bx)) \).

Let us as usual denote by \( \hat{H}_1 \) the dual of \( H_1 \), i.e. the set of equivalence classes of unitary irreducible representations on the Heisenberg group. We shall consider first how an equivalence class in \( \hat{H}_1 \) behaves on the center \( Z(H_1) \). Choose \( \pi \) a representative of such an equivalence class, and identify \( Z(H_1) \) with \( \mathbb{R} \) in the natural way: \( (0, 0, z) \sim z \). Restricting \( \pi \) to \( Z(H_1) \), we obtain a unitary representation of the center, which is an abelian group, and as a consequence of Schur’s lemma it will be unidimensional. From this we conclude that \( \chi_\pi := \pi|_{Z(H_1)} \) is a character, and we call it the central character of the representation \( \pi \).

The first case to consider is when the central character of the representation \( \pi \) is the identity on all of the center: \( \chi_\pi(z) = 1 \), \( \forall z \in Z(H_1) \). Such a representation factors through a representation of the quotient group \( H_1/Z(H_1) \), which is isomorphic to \( \mathbb{R}^2 \). As we have shown that the dual of \( \mathbb{R}^2 \) can be identified with \( \mathbb{R}^2 \) itself, and hence we conclude that \( \hat{H}_1^0 \), the subset of \( \hat{H}_1 \) consisting of representations that are trivial on the center, can be identified with \( \mathbb{R}^2 \). This is done in the following way:

\[
\mathbb{R}^2 \longrightarrow \hat{H}_1^0 \quad \quad (a, b) \longmapsto \chi_{a,b}
\]

where:

\[
\chi_{a,b} : H_1 \longrightarrow \mathbb{T} \quad \quad (x, y, z) \longmapsto e^{2\pi i(ax + by)}
\]

Now, let us consider a representative \( \pi \) of an equivalence class in \( H_1 \) whose central character
\( \chi_\pi \) is not trivial. The element \( t \neq 0 \) verifying \( e^{it} = \chi_\pi(1) \) completely characterizes \( \chi_\pi \), as we have \( \chi_\pi(z) = e^{it}z \), \( z \in Z(H_1) \). It is therefore natural to identify these characters with \( \mathbb{R}^* \):

\[
t \neq 0 \sim \chi_\pi \text{ such that } \chi_\pi(1) = e^{2\pi it}.
\]

For a \( t \) in \( \mathbb{R}^* \), we now construct representations \( \pi_t \) that have central character \( \chi_t \) and representation space \( L^2(\mathbb{R}) \): for \( \phi \) in \( L^2(\mathbb{R}) \), we define the operator

\[
\pi_t(a,b,c)\phi(x) = e^{2\pi i(c - bx + \frac{\Delta}{2})t} \phi(x - a).
\] (3.9)

The well-known theorem of Stone–von Neumann then establishes that these unitary representations \( \pi_t \) are irreducible, and that they are the only ones with central characters \( \chi_t \). We will not prove it here, but it will be useful later on to have the explicit statement. There exists many versions, but the one we use is taken from [CG90]:

**Theorem 13. (Stone–von Neumann)** Let \( \rho_1 \) and \( \rho_2 \) be two unitary representations of \( \mathbb{R} \) on the same Hilbert space \( \mathcal{H} \) satisfying the commutation relation:

\[
\rho_1(x)\rho_2(y)\rho_1(x)^{-1}\rho_2(y)^{-1} = e^{2\pi i\lambda xy}, \ \forall x, y \in \mathbb{R}, \ \lambda \neq 0.
\] (3.10)

Then \( \mathcal{H} \) is a direct sum \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \) of subspaces that are invariant and irreducible under the joint action of \( \rho_1 \) and \( \rho_2 \). For each \( \mathcal{H}_k \) there exists an isometry \( J_k : \mathcal{H}_k \rightarrow L^2(\mathbb{R}) \) which transforms \( \rho_1 \) and \( \rho_2 \) into the actions on \( L^2(\mathbb{R}) \):

\[
[\hat{\rho}_1(x)f](t) = f(t + x), \ [\hat{\rho}_2(y)f](t) = e^{2\pi i\lambda yt} f(t).
\]

For each \( \lambda \neq 0 \), the pair \( \hat{\rho}_1, \hat{\rho}_2 \) acts irreducibly on \( L^2(\mathbb{R}) \), so \( \rho_1, \rho_2 \) act irreducibly on each \( \mathcal{H}_k \).

In the case of the Heisenberg group, define two representations of \( \mathbb{R} \) on \( \mathcal{H}_\pi \): \( \rho_1(x) = \pi(x,0,0) \), and \( \rho_2(y) = \pi(0,y,0) \), and we easily find that they verify the commutation relation 3.10. Moreover, notice that these two elements \((x,0,0)\) and \((0,y,0)\) generate all of \( H_1 \), and hence as \( \pi \) is irreducible, the joint action of \( \rho_1 \) and \( \rho_2 \) is irreducible. Applying the Stone–von Neumann theorem, we obtain that there exists an isometry between \( \mathcal{H} \) and \( L^2(\mathbb{R}) \), such that we can consider

\[
[\rho_1(x)f](t) = f(t + x), \ [\rho_2(y)f](t) = e^{2\pi i\lambda yt} f(t).
\]

This finishes to prove that the unitary dual of \( H_1 \) is:

\[
\hat{H}_1 = \mathbb{R}^2 \cup \{ \pi_t : t \neq 0 \}.
\] (3.11)
Similarly, we can define the \(n\)-dimensional Heisenberg groups \(H_n\) for \(n \in \mathbb{N}\), by defining their Lie algebra \(\mathfrak{h}_n = \text{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}\), where the Lie brackets between them can be summarized by \([x_i, y_j] = \delta_{ij}\). \(H_n\) can then be identified with \(\mathbb{R}^{2n+1}\), endowed with the operation

\[
(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' - \frac{1}{2}(xy' - yx'))
\]

Its dual can be computed in a similar fashion and we obtain:

\[
\hat{H}_n = \mathbb{R}^{2n} \cup \{\pi_t : t \neq 0\},
\]

where again the one dimensional representations indexed by \(\mathbb{R}^{2n}\) are:

\[
\chi_{a, b}(x, y, z) = e^{2\pi i (ax + by)},
\]

and the remaining ones are realised on \(L^2(\mathbb{R}^n)\) in the following way:

\[
[\pi_t(x, y, z)\phi](\theta) = e^{2\pi i (z - y \cdot \theta + \frac{xy}{2}^t \phi(\theta - x)).}
\]

### 3.5.3 Step-two nilpotent Lie groups

The process of reducing the computation of the dual of a nilpotent group to the same problem for a lower dimensional nilpotent group is easiest to detail in the case of a nilpotent group of step-two.

Let \(G\) denote such a group. Denoting as usual by \(\mathfrak{g}\) the Lie algebra of \(G\), let us start by writing \(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2\), where \(\mathfrak{g}_2 = \mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}]\) and \(\mathfrak{g}_1\) is a subspace of \(\mathfrak{g}\) complementary to \(\mathfrak{g}_2\), hence the sum makes sense as vector spaces, and we can induce a Lie bracket on the sum: taking elements \(X = X_1 + X_2\) and \(Y = Y_1 + Y_2\) of \(\mathfrak{g}\), with \(X_1, Y_1 \in \mathfrak{g}_1\) and \(X_2, Y_2 \in \mathfrak{g}_2\), we have \([X, Y] = [X_1, Y_1] \in \mathfrak{g}_2\).

We start with the proposition:

**Proposition 5.** Let \(G\) and \(\mathfrak{g}\) be as defined previously. If \(V\) is a subspace of \(\mathfrak{g}_2\) of codimension 1, then there exist positive integers \(n\) and \(k\) such that:

\[
\mathfrak{g}/V \cong \mathfrak{h}^n \oplus \mathbb{R}^k,
\]

as a Lie algebra direct sum,

where \(\mathfrak{h}^n\) denotes the Lie algebra of the Heisenberg group \(H_n\).

**Proof.** We start by taking the quotient \(\mathfrak{g}' = \mathfrak{g}/V = \mathfrak{g}_1 \oplus \mathfrak{g}_2/V = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2\), which is a nilpotent Lie algebra, and by construction \(\mathfrak{g}'_2\) has dimension 1. Let us choose an element \(Z \neq 0\) in \(\mathfrak{g}'_2\). We can then construct a skew-symmetric bilinear form \(\sigma\) on \(\mathfrak{g}'_1\) in the following way: \([X, Y] = \sigma(X, Y)Z\), for \(X, Y \in \mathfrak{g}'_1\). Denote \(E = \{X \in \mathfrak{g}'_1 \mid \sigma(X, Y) = 0, \forall Y \in \mathfrak{g}'_1\}\) and \(F\) a complementary subspace.
to $E$ in $g'_1$. In the case where $E = 0$, $\sigma$ is a non-degenerate skew-symmetric bilinear form on $g'_1$, and there exists an $n \in \mathbb{N}$ such that $\dim g'_1 = 2n$, which yields $g' \cong \mathfrak{h}^n$. If instead $E \neq 0$, then $g' = E \oplus F \oplus \text{span}\{Z\}$, and there exists an $n$ such that $\dim F = 2n$ and $F \oplus \text{span}\{Z\} \cong \mathfrak{h}^n$. Moreover, since for elements $X, Y \in E$, $[X, Y] = 0$, $E$ is a commutative Lie algebra and is therefore isomorphic to $\mathbb{R}^k$ for a certain $k$, resulting in $g' \cong \mathbb{R}^k \oplus \mathfrak{h}^n$ as a direct sum of Lie algebras.

Take $(\pi, H)$ an irreducible unitary representation of $G$, yielding a skew-adjoint representation (which we still denote $\pi$) of its step-two Lie algebra $g$. Again, we decompose $g = g_1 \oplus g_2$, and recall that by definition $g_2$ is contained in the center of $g$. $\pi$ is therefore a scalar skew-adjoint representation of $g_2$, which means a homomorphism

$$\pi : g_2 \to i\mathbb{R}.$$ 

This leads to two possibilities: either $\pi|_{g_2} = 0$, or $\mathcal{V} := \{X \in g_2| \pi(X) = 0\}$ is a subalgebra of $g_2$ of codimension 1.

In the first case, the representation $\pi$ of $G$ is trivial on $G_2 := \exp(g_2)$ and factors through a representation $\hat{\pi}$ of $G/G_2$, i.e. $\pi = \hat{\pi} \circ \rho$, where $\rho$ is the projection from $G$ to $G/G_2$. Since $g/g_2$ is a commutative Lie algebra, $G/G_2$ is an abelian simply connected Lie group, and is therefore isomorphic to $\mathbb{R}^m$ for a positive integer $m$. Moreover, note that as $\hat{\pi}$ is a homomorphism from a quotient group of $G$ to the space of unitary operators on $H$, it is also a unitary representation. Also, supposing that $\hat{\pi}$ is reducible, i.e. that there exists a subspace $W$ of $H$ such that $\hat{\pi}(G/G_2)W \subseteq W$, we obtain that for every $g \in G$ and $w \in W$, $\pi(g)w = \hat{\pi}(\rho(g))w = \hat{\pi}([g])w \in W$. The subspace $W$ would therefore be invariant under $\pi$, which contradicts the irreducibility of $\pi$. We conclude that $\hat{\pi}$ is a unitary irreducible representation of $\mathbb{R}^m$.

In the second case, since $\pi$ is 0 on the subalgebra $\mathcal{V}$, it is trivial on the corresponding subgroup $V := \exp(\mathcal{V})$ and therefore there exists a representation $\hat{\pi}$ of $G/V$ such that $\pi = \hat{\pi} \circ \rho$, where $\rho : G \rightarrow G/V$ is the natural projection. However, notice that by Proposition 5, we have: $g/V \cong \mathfrak{h}^n \oplus \mathbb{R}^k$ as a Lie algebra direct sum, which yields $G/V \cong H_n \rtimes \mathbb{R}^k$. In a similar way as in the previous case, we can also prove that $\hat{\pi}$ is irreducible and unitary.

We have therefore reduced the problem of finding the dual of $G$ to computing the dual of $\mathbb{R}^m$ and of $H_n \rtimes \mathbb{R}^k$, which are both well-known.
3.5.4 Nilpotent groups of arbitrary step

Suppose now that $G$ is a nilpotent Lie group of step $n$, with Lie algebra $\mathfrak{g}$ having center $\mathfrak{z}$. Let us choose $(\pi, \mathcal{H}_\pi)$ an irreducible unitary representation of $G$. $\pi$ therefore becomes a skew-adjoint scalar representation of $\mathfrak{z}$, which means that there exists a linear functional $\lambda$ on $\mathfrak{z}$ verifying: $\pi(Z) = i\lambda(Z)I$, for $Z \in \mathfrak{z}$. Denoting by $\mathfrak{z}_1$ the kernel of $\lambda$, just as in the step-two case there are two possibilities: either $\mathfrak{z}_1 = \mathfrak{z}$, or $\mathfrak{z}_1$ has codimension 1 in $\mathfrak{z}$.

In the first case, $\pi$ is trivial on $\mathfrak{z}$ and therefore factors through a unitary irreducible representation of $\mathfrak{g}/\mathfrak{z}$, which is a lower-dimensional nilpotent Lie algebra of step smaller than $n$.

In the second case, $\pi$ factors through a unitary irreducible representation of $\mathfrak{g}/\mathfrak{z}_1$, which is a step-$n$ nilpotent Lie algebra with one-dimensional center. From now on we will therefore suppose that $\mathfrak{g}$ is an $n$-step nilpotent Lie algebra with one-dimensional center $\mathfrak{z}$, and we consider a unitary irreducible representation $\pi$ on a Hilbert space $\mathcal{H}_\pi$ which is not trivial on $\mathfrak{z}$. Let $Z \neq 0$ be a generator of $\mathfrak{z}$.

**Lemma 2.** Let $\mathfrak{g}$ be a Lie algebra with the properties listed above, and $Z$ as defined above. There exist elements $X, Y \in \mathfrak{g}$ such that $[X, Y] = Z$.

**Proof.** Using Theorem 12 with $\rho$ the adjoint representation on $\mathfrak{g}$, we can assume that we have a decomposition $\mathfrak{g} = \mathfrak{g}_n \supseteq \mathfrak{g}_{n-1} \supseteq \cdots \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_0 = 0$, where $\dim(\mathfrak{g}_j) = j$ and $[\mathfrak{g}_j, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$. We therefore have $\mathfrak{z} = \mathfrak{g}_1$, and hence $Z$ generates $\mathfrak{g}_1$. Let us choose an element $Y \in \mathfrak{g}_2$ that is not in $\mathfrak{g}_1$, $Y \neq 0$. Since $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_1$, there exists an element $X$ of $\mathfrak{g}$, but not in $\mathfrak{g}_1$ or $\mathfrak{g}_2$ such that $[X, Y] = Z$. \(\square\)

Denoting $\mathcal{U} = \span\langle X, Y, Z \rangle$, $\mathcal{U}$ is therefore a subalgebra of $\mathfrak{g}$, clearly isomorphic to the Heisenberg Lie algebra $\mathfrak{h}$. We define another subspace $\mathcal{L}$ of $\mathfrak{g}$ by

$$\mathcal{L} = \{ W \in \mathfrak{g} \mid [W, Y] = 0 \},$$

which, by the Jacobi identity, can be shown to also be a subalgebra. Since the center $\mathfrak{g}_1$ is one-dimensional, the inclusion $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_1$ means that the codimension of $\mathcal{L}$ is 0 or 1. However, since $X \neq 0$ and $X \notin \mathcal{L}$, then $\mathrm{codim}(\mathcal{L}) = 1$ and $\mathfrak{g} = \span\langle X \rangle \oplus \mathcal{L}$ as direct sum of Lie algebras.

Moreover, we can prove that any codimension 1 subalgebra $V$ of a step-$n$ nilpotent Lie algebra $\mathfrak{g}$ is in fact an ideal, meaning that $[\mathfrak{g}, V] \subseteq V$. Indeed, we already have $[V, V] \subseteq V$. Now suppose
by contradiction that there exists a non-zero element $X$ of $\mathfrak{g}/V$ for which we can find a non-zero element $v \in V$ such that $[X,v] \notin V$. By scaling $v$ if necessary, we can write $[v,X] = X + v_1$ for a $v_1 \in V$. Taking the Lie bracket again we get: $[v,[v,X]] = X + v_2$, where $v_2 = v_1 + [v,v_1] \in V$, and doing the procedure $k$ times we obtain the length $k$ Lie bracket $[v,\ldots[v,X]] = X + v_k$ which is non-zero, and therefore contradicts the nilpotency of $\mathfrak{g}$ for $k > n$.

To summarize, we have constructed, on an arbitrary step-$n$ nilpotent Lie algebra with one-dimensional center, two subalgebras: one isomorphic to the Heisenberg Lie algebra, and one of codimension 1, both containing the center of $\mathfrak{g}$. Recall that we were considering a unitary irreducible representation of $G$, non-trivial on $Z(G) = \exp(\mathfrak{z})$, and we will try to express it using a unitary irreducible representation of a lower dimensional nilpotent Lie group. Considering the restriction of $\pi$ to both subalgebras $U$ and $L$ will generate crucial information on such a representation.

Let us first restrict $\pi$ to the subalgebra $\mathfrak{u} \cong \mathfrak{h}$. Applying the Stone-von Neumann theorem as stated previously in the chapter, we can decompose the representation space $\mathcal{H}_\pi = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$, each of these being isomorphic to $L^2(\mathbb{R})$, and such that on each $\mathcal{H}_i$, $[\pi(\exp(pX)f)(s) = f(s + p)$ and $[\pi(\exp(qY)f)](s) = e^{2\pi i q s} f(s)$. Note that an element in $\mathcal{H}_\pi = \mathcal{H}_i$, for $i \in I \subseteq \mathbb{N}$, is a sequence of functions $(f_i)_{i \in I}$ in $L^2(\mathbb{R})$, and such that for each $t \in \mathbb{R}$, $\sum_{i \in I} |f_i(t)|^2 < \infty$. We can therefore naturally identify $\mathcal{H}_\pi$ with $L^2(\mathbb{R}, \tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is a Hilbert space of dimension equal to the cardinality of $I$ (i.e is $\mathbb{R}^n$ for $\text{card}(I) = n$, and is $\ell^2$ when $\text{card}(I) = \infty$). We then look at $\pi$ as acting on the vector-valued functions of $L^2(\mathbb{R}, \tilde{\mathcal{H}})$.

We have therefore established how the representation $\pi$ acts on the subgroup $\exp(\mathfrak{u})$ of $G$, and from this we want to extract information on the behaviour of $\pi$ on elements of $L = \exp(\mathfrak{L})$. By definition, elements of $L$ commute with elements of the form $\exp(tY)$, $\forall t \in \mathbb{R}$, and as $[\pi(\exp(tY)f)](s) = e^{2\pi i t s} f(s)$ are multiplication operators, the operators $\pi(l)$ must be multiplication operators as well, for all $l \in L$. For a function $f$ in $L^2(\mathbb{R}, \tilde{\mathcal{H}})$, we therefore have that there exist unitary operators $T(l,s)$ on $\tilde{\mathcal{H}}$ such that

$$[\pi(l)f](s) = T(l,s)f(s), \text{ for all } l \in L, s \in \mathbb{R}.$$ 

Let us now define the unitary operators $T_0(l) := T(l,0)$ on $\tilde{\mathcal{H}}$. $T_0$ is clearly a homomorphism, and it can be proved to be strongly continuous. It is therefore a representation of $L$ on $\tilde{\mathcal{H}}$. The fact that such representations in fact determine all unitary irreducible representations of $G$ is given by the
following proposition.

**Proposition 6.** The representation $\text{Ind}^G_{L,T_0}$, with $L$ and $T_0$ as defined above is irreducible, and is unitarily equivalent to $\pi$.

(For the brevity of our exposition, we refer to Chapter 6 of [Tay86] for a detailed proof). It is important to note that if $\pi$ is irreducible, then $T_0$ must be irreducible too. Indeed, suppose by contradiction that there exist a subspace $V \subseteq \hat{H}$ that is invariant under $T_0$. Recall that the representation $\sigma := \text{Ind}^G_{L,T_0}$ acts on the space $\mathcal{H}_\sigma$ of functions $f : G \to \hat{H}$, that in addition verify: $f(lg) = T_0(l)f(g)$, $\forall l \in L, g \in G$, and that are square summable: $\int_{L \setminus G} \|f(g)\|d\mu < \infty$, where $d\mu$ is a right-invariant measure on $L \setminus G$. It is then immediate from the definition of an induced representation, that the subspace of $\mathcal{H}_\sigma$ of functions with image contained in $V$, is invariant under $\sigma$.

Note that we have decomposed $g$ in the Lie algebra direct sum $g = L \oplus \text{span}\{X\}$, which enables us to write $G$ as the semi-direct product: $G = L \rtimes \exp(\mathbb{R}X)$. Taking a function $f$ in the Hilbert space $\mathcal{H}_\sigma$, we find: $f(g) = f(l \exp(tX)) = T_0(l)f(\exp(tX))$ for an $l \in L$ and $t \in \mathbb{R}$. We can therefore consider $f$ as a function on $\exp(\mathbb{R}X)$, and furthermore on $\mathbb{R}$. Moreover, the square integrability condition on the functions of $\mathcal{H}_\sigma$ means that we in fact identify this Hilbert space with $L^2(\mathbb{R}, \hat{H})$.

To conclude, we have proved that every irreducible unitary representation of $G$ is induced by an irreducible unitary representation of the subgroup $L$. The fact that the converse is also true, i.e that every irreducible unitary representation of $L$ leads to one of $G$, is given by the following Proposition (see for example in [Tay86]).

**Proposition 7.** Let $G$ be a simply connected nilpotent Lie group with one-dimensional center. Let $L$ be a connected subgroup of codimension one. Then $L$ is normal, and if $Z(G)$ is the center of $G$ then $Z(G) \subseteq L$. Let $U_0$ be an irreducible unitary representation of $L$ on a Hilbert space $H_1$, such that $U_0|Z$ is not trivial. If the center of $L$ has dimension at least 2, then $\text{Ind}^G_{L,U_0}$ is irreducible.

This induction process we constructed may be confusing to the reader, and in order to clarify it we will illustrate it on the Engel group in the next section.
3.5.5 Example: The Engel group

The Engel group is at the heart of the analysis in Chapter 5, and we here compute its irreducible unitary representations, after first describing its structure.

The Engel group is the unique simply connected Lie group with Lie algebra generated by elements $l_1$, $l_2$, $l_3$ and $l_4$ verifying the only non-zero Lie bracket relations: $[l_1, l_2] = l_3$, and $[l_1, l_3] = l_4$. We denote $\mathfrak{L}_4$ this Lie algebra, and $\mathfrak{G}_4$ the Engel group. We will show in Chapter 5 that $\mathfrak{G}_4$ can be identified with $\mathbb{R}^4$ by:

$$\exp(a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4) \sim (x_1, x_2, x_3, x_4),$$

with $x_1 = a_1$, $x_2 = a_2$, $x_3 = a_3 + a_1a_2/2$ and $x_4 = a_4 + a_1a_3/2 + a_2a_4^2/6$. Therefore $\mathfrak{G}_4$ is isomorphic to $\mathbb{R}^4$ endowed with the operation

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2, x_4 + y_4 + x_1y_3 + y_2x_4^2/2).$$

The center of $\mathfrak{L}_4$ is clearly the span of $l_4$, and in the notations of the preceding section, we can choose $X = l_1$, $Y = l_3$, and construct the subalgebra $\mathcal{U} = \text{span}\{l_1, l_3, l_4\}$, isomorphic to the Heisenberg Lie algebra. We then have that the codimension 1 subalgebra of $\mathfrak{L}_4$ which will be of interest is $\mathcal{L} = \text{span}\{l_2, l_3, l_4\}$, which contains the center, and is abelian. Proposition 7 therefore applies, and we can use the theory developed in the previous section to compute all irreducible unitary representations of $\mathfrak{G}_4$.

Let $(\pi, \mathcal{H}_\pi)$ denote one of them, and let us suppose first that $\pi$ has a non-trivial central character $\chi_\pi$: there exists a $\lambda \neq 0$ such that $\chi_\pi(\exp(zl_4)) = e^{2\pi i z \lambda}$. First, we consider this representation restricted to the subgroup $U = \exp(\mathcal{U})$, isomorphic to the Heisenberg group. Since $\exp(xl_1) \sim (x, 0, 0, 0)$ and $\exp(yl_3) = (0, 0, y, 0)$, we can define two unitary representations of $\mathbb{R}$ by $\rho_1(x) = \pi(\exp(xl_1))$ and $\rho_2(y) = \pi(\exp(yl_2))$. They verify the commutation relation

$$\rho_1(x)\rho_2(y)\rho_1(x)^{-1}\rho_2(y)^{-1} = \pi((x, 0, 0, 0) \cdot (0, 0, y, 0) \cdot (-x, 0, 0, 0) \cdot (0, 0, -y, 0))$$

$$= \pi((x, 0, y, xy)(-x, 0, -y, xy)) = \pi((0, 0, 0, xy))$$

$$= e^{2\pi i \lambda xy},$$

and therefore, by the Stone-von Neumann theorem, $\mathcal{H}_\pi$ is isomorphic to $L^2(\mathbb{R}, \tilde{\mathcal{H}})$, where $\mathcal{H} = \mathbb{R}^n$ or $\ell^2$. For a vector valued function in $L^2(\mathbb{R}, \tilde{\mathcal{H}})$, $\pi$ acts through $U$ on each component in the
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classical way:
\[ [\pi(\exp(xl_1))]f(t) = f(t + x), \quad [\pi(\exp(yl_3))]f(t) = e^{2\pi i \lambda y} f(t), \quad [\pi(\exp(zl_4))]f(t) = e^{2\pi i \lambda z} f(t). \]

We now want to consider the representation \( \pi \) on the codimension 1 subgroup \( L = \{ \exp(\alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4) \mid \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \} \). By the previous section, \( L \) will act by multiplication on \( L^2(\mathbb{R}, \mathcal{H}) \), i.e. for \( l \in L \) and \( s \in \mathbb{R} \) there exist operators \( T(l, s) \) on \( \mathcal{H} \) such that \( \pi(l)f(s) = T(l, s)f(s) \). We already have

\[ T(\exp(yl_3), s) = e^{2\pi i \lambda y} \text{Id}, \quad T(\exp(zl_4), s) = e^{2\pi i \lambda z} \text{Id}, \]

and we define a unitary representation of \( L \) on \( \mathcal{H} \) by: \( T_0(l) := T(l, s) \). We have:

\[ T_0(\exp(\alpha_3 l_3)) = \text{Id}, \quad T_0(\exp(\alpha_4 l_4)) = e^{2\pi i \lambda \alpha_4} \text{Id}, \]

and since elements \( \exp(xl_2) \) commute with all elements of \( L \), by Schur’s Lemma \( T_0(\exp(\alpha_2 l_2)) \) is a scalar multiple of the identity operator. There therefore exists a real number \( \delta \) such that \( T_0(\exp(\alpha_2 l_2)) = e^{2\pi i \alpha_2 \delta} \). To summarize we have obtained a scalar representation parametrized by two parameters, \( \delta \in \mathbb{R} \) and \( \lambda \in \mathbb{R}^* \):

\[ T_0(\exp(\alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4)) = e^{2\pi i (\delta \alpha_2 + \lambda \alpha_4)} \text{Id}. \]

As \( T_0 \) acts as scalar multiples of the identity on \( \mathcal{H} \), we can in fact consider it as a character of \( L \), i.e as a representation of \( L \) on \( \mathbb{R} \). By Proposition 6, \( \pi \) is equivalent to \( \sigma := \text{Ind}_{L}^{G} T_0 \), and we can therefore denote it \( \pi_{\delta, \lambda} \). It acts on the space \( \mathcal{H}_{\sigma} \), which can be identified with \( L^2(\mathbb{R}, \mathbb{R}) = L^2(\mathbb{R}) \), in the following way:

\[ [\pi_{\delta, \lambda}(x_1, x_2, x_3, x_4)]f(t) = [\pi_{\delta, \lambda}(x_1, x_2, x_3, x_4)](\exp(t l_1)) = [\pi_{\delta, \lambda}(x_1, x_2, x_3, x_4)](t, 0, 0, 0) \]
\[ = f((t, 0, 0, 0) \cdot (x_1, x_2, x_3, x_4)) \]
\[ = f((0, x_2, x_3 + tx_2, x_4 + tx_3 + x_2 t^2 / 2) \cdot (t + x_1, 0, 0, 0)) \]
\[ = T_0(\exp(x_2 l_2 + (x_3 + tx_2) l_3 + (x_4 + tx_3 + x_2 t^2 / 2) l_4))f(\exp(t + x_1) X) \]
\[ = \exp(2\pi i (\delta x_2 + \lambda (x_4 + tx_3 + x_2 t^2 / 2)))f(t + x_1), \]

where we have used the identification of \( \mathfrak{G}_4 \) with \( \mathbb{R}^4 \) as detailed in the beginning of the section.

Redefining \( \delta = -\frac{\delta}{\mathfrak{G}_4} \) (since \( \lambda \neq 0 \)), we find a formula equivalent to the one computed by Dixmier
in [Dix58]:
\[ [\pi_{\delta,\lambda}(x_1, x_2, x_3, x_4) f](t) = \exp \left( 2\pi i \left( -\frac{\delta}{2\lambda} x_2 + \lambda(x_4 + tx_3 + x_2t^2/2) \right) \right) f(t + x_1) \] (3.13)

These elements of the dual are the only ones that will be of interest for our analysis of the Engel group in Chapter 5, since it can be shown (see [CG90]) that the Plancherel measure has support in the subset of \( \hat{G}_4 \) consisting of the equivalence classes of these representations \( \pi_{\delta,\lambda} \). The remaining unitary irreducible representations are the ones that are trivial on the center \( Z(G_4) = \{ \exp(tl_4) | t \in \mathbb{R} \} \), and these consequently factor through unitary irreducible representations of the group \( G_4/Z(G_4) = \{ \exp(\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 | \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \} \), which is clearly isomorphic to the Heisenberg group. This way we have obtained all the elements of the dual of the Engel group.
Chapter 4

Short-time behaviour of the hypoelliptic heat kernel

4.1 Some functional analysis

In this section we recall the definitions of certain types of operators that we will need later on, and review some of their basic properties. A detailed exposition of these notions can be found in most Functional Analysis textbooks (see for example Part I Chapter II of [HP57], Appendix A of [EN06], [Bre99], [Dav80]). We will consider a linear and generally unbounded transformation $A$ from a Banach space $X$ to a Banach space $Y$, and let us denote $D(A)$ the domain of $A$, i.e. the elements of $X$ on which the operator $A$ acts. In this thesis we will always find ourselves in the case $Y = X$.

**Definition 20.** An operator $A$ is bounded if there exists a constant $M$ such that, for all $x \in D(A)$,

$$\|Ax\| \leq M\|x\|.$$  \hspace{1cm} (4.1)

We call unitary operators the bounded operators that have norm equal to 1 (and, therefore, are isometries in $X$) and contraction operators the ones with norm less or equal to 1.

A compact operator is an operator that maps a bounded set to a relatively compact set.

**Definition 21.** We say that a linear operator $(A, D(A))$ on a Banach space $X$ is closed if one of
the following equivalent properties is verified:

1. For any sequence of elements \((x_n)_{n \in \mathbb{N}}\) in \(D(A)\) such that \(\lim_{n \to \infty} x_n = x \in X\) and \(\lim Ax_n = y\), then in fact \(x \in D(A)\) and \(y = Ax\).

2. The graph of the operator \(A\), i.e., the set \(\mathcal{G}(A) = \{(x, Ax) | x \in D(A)\}\) is closed in \(X \times X\).

3. The normed space \((D(A), \| \cdot \|_A)\), with the graph norm: \(\|x\|_1 = \|x\| + \|Ax\|\) for \(x \in D(A)\), is a Banach space.

Note that if it is true that the class of all linear bounded operators on \(X\) forms a Banach space, the same does not hold for the class of closed linear operators on \(X\), and in fact that class is in general not even a linear system. Indeed, the sum of two closed linear operators \(A\) and \(B\), considered on the intersection of their respective domains, is in general not closed. We also note that by definition the domain of a closed operator is closed in \(X\), and so is its range.

**Definition 22.** We say that an operator \((B, D(B))\) is an extension of \((A, D(A))\) if \(D(A) \subset D(B)\) and if \(B = A\) on \(D(A)\). We denote this using the notation: \(A \subset B\). The smallest closed extension of \(A\), if it exists, is called the closure of \(A\), and is denoted \(\bar{A}\). We then say that \(A\) is closable.

Another way to say this is that \(A\) is closable if the closure of its graph \(\mathcal{G}(A)\) defined above is the graph of another operator.

Let us now suppose that \(A\) is an operator on a Hilbert space \(\mathcal{H}\), endowed with an inner product \((\cdot, \cdot)\), and that \(D(A)\) is dense in \(\mathcal{H}\). Let us define the functionals \(L_y : X \to \mathbb{C}\) where \(L_y(x) = (Ax, y)\). We want to define the adjoint operator \(A^*\) of \(A\). We start by defining its domain: \(D(A^*) = \{y \in \mathcal{H} | L_y \text{ is continuous} \}\). For each \(y \in D(A^*)\), \(L_y\) is a continuous linear functional defined on a dense set \((D(A))\), and can therefore be extended to a continuous linear functional on all of \(X\). By the Riesz Representation theorem, there exists an element \(z_y\) such that \(L_y(x) = (x, z_y) = (Ax, y)\). We then define \(A^*y := z_y\). Note that the adjoint operator is always closed, as is clear from its definition.

**Definition 23.** We say that an operator \(A\) with dense domain is symmetric if:

\[
(Au, v) = (u, Av), \quad \forall u, v \in D(A),
\]

(4.2)
which means that $D(A) \subset D(A^*)$ and that $A$ and $A^*$ agree on $D(A)$. When $D(A) = D(A^*)$ (and therefore $A = A^*$), we will say that $A$ is self-adjoint. This implies that self-adjoint operators are always closed.

We say that a symmetric operator $A$ is positive (or non-negative) if

$$
(Au,u) \geq 0, \forall u \in D(A).
$$

(4.3)

If $\Re(Au,u) \leq 0, \forall u \in D(A)$, we say that $A$ is dissipative.

Note that the condition for $A$ being symmetric is equivalent to the condition $(Ax,x) \in \mathbb{R}, \forall x \in D(A)$. Indeed if condition 4.2 is verified, then for $x \in D(A)$ we have $(x, Ax) = (Ax, x) = (Ax, x)$ hence $(Ax, x) \in \mathbb{R}$. Conversely, if $(Ax, x) \in \mathbb{R}, \forall x \in D(A)$, then $(A(x+iy), x+iy)) \in \mathbb{R}, \forall x, y \in D(A)$, and when developing we recover equation 4.2.

A useful result is that a densely defined operator is closable if and only if its adjoint also has dense domain. This results yields that all symmetric operators are closable, since they have dense domain and verify $D(A) \subset D(A^*)$.

Finally we say that $A$ is essentially self-adjoint if $A$ is closable and $\bar{A} = A^*$, which is equivalent to $A$ having a unique self-adjoint extension, or to $A$ being symmetric and having self-adjoint closure. It is not always easy to show that an operator is essentially self-adjoint. However, for certain operators it is possible to define a canonical (but maybe not unique) self-adjoint extension. In particular this is true for non-negative symmetric operators, whose canonical self-adjoint extension is called “Friedrich’s extension” (for a detailed exposition of this notion, we refer to section II.5 of [VSCC92]). The definition of this new operator is based on the theory of quadratic forms associated to operators, which we will not develop here.

Much study has been done on self-adjoint operators. Note that by definition, these operators are densely defined, symmetric and closed and that the converse is true for bounded operators, but not in general. A very strong result in the theory of self-adjoint operators is the well-known spectral theorem (for details see for example Chapter 4 of [Dav80]).

**Theorem 14.** If $A$ is an (unbounded) self-adjoint operator on a Hilbert space $\mathcal{H}$, there exists a measure space $(M, dm)$, a unitary identification of $\mathcal{H}$ with $L^2(M, dm)$ and a real measurable function
h on $\mathcal{M}$ such that:

$$D(H) = \left\{ f \in L^2(\mathcal{M}, dm) \left| \int_{\mathcal{M}} (1 + h(m)^2) |f(m)|^2 dm < \infty \right\}$$

and:

$$(Hf)(m) = h(m)f(m), \forall m \in \mathcal{M} \text{ and } f \in D(H).$$

### 4.2 Notions of one-parameter semigroup theory

We first recall the definition of a strongly continuous semigroup ([EN06]).

**Definition 24.** Let $(T(t))_{t \geq 0}$ be a family of bounded linear operators on a Banach space $X$, indexed by $t \in \mathbb{R}^+$. We will say that $(T(t))_{t \geq 0}$ is a strongly continuous semigroup of operators if the following conditions are verified:

1. $T(t)T(s) = T(t+s), \forall t, s \in \mathbb{R}^+$;
2. $T(0) = Id$;
3. The map: $\mathbb{R}^+ \to X$, where: $t \mapsto T(t)x$, is continuous $\forall x \in X$.

It is well-known that, for strongly continuous semigroups $(T(t))_{t \geq 0}$, there exist constants $M \geq 1$ and $w \in \mathbb{R}$ such that for all $t \geq 0$, $\|T(t)\| \leq Me^{wt}$. We call growth bound of $(T(t))_{t \geq 0}$ the following number:

$$w_0 := \inf\{w \in \mathbb{R} | \exists M_w \geq 1 \text{ such that } \|T(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0\}.$$  

We will say that a semigroup is bounded if we can take $w = 0$, and a contraction semigroup if we can take $w = 0$ and $M = 1$. In the special case where $\|T(t)x\| = \|x\|$ for all $x \in X$, we will say that $(T(t))_{t \geq 0}$ is a semigroup of isometries.

The generator of a semigroup $T(t)$ is the operator $A$ defined by

$$Ax := \lim_{t \to 0} \frac{T(t)x - x}{t}$$

for all $x \in X$ for which this limit exists; the domain of this operator is therefore defined as

$$D(A) = \left\{ x \in X \left| \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$
Consistent with the notation adopted for Theorem 3 and Corollary 1, we will denote the semigroup \((T(t))_{t \geq 0}\) generated by the operator \(A\) by \(e^{tA}\). Using the definition of \(A\) and of its domain, it is possible to show that \(A\) is a closed operator and that its domain is a dense subspace of \(X\).

It is natural to ask if every closed densely defined operator is the generator of a strongly continuous semigroup, and what properties of the generator translate to properties of the semigroup. The answer to these two questions are the subject of the following theorems (often called the generation theorems). It is not the purpose of this thesis to give a detailed exposition of the theory of one-parameter semigroups, so we will state these theorems without proof, except the last one.

**Theorem 15.** (Hille–Yosida, 1948 - Contraction case) Let \((A, D(A))\) a linear operator on a Banach space \(X\). The operator \(A\) is the generator of a strongly continuous contraction semigroup if and only if it is closed and densely defined, and for every \(\lambda > 0\), \(\lambda\) is in the resolvent set of \(A\), \(\rho(A)\), and \(\|\lambda(\lambda - A)^{-1}\| \leq 1\).

**Theorem 16.** (Lumer–Phillips, 1961) A closed, densely defined and dissipative operator \((A, D(A))\) on a Banach space \(X\) is the generator of a contraction semigroup if and only if \(\text{rg}(\lambda - A)\) is dense in \(X\) for some (hence all) \(\lambda > 0\).

**Theorem 17.** (Feller–Miyadera–Phillips, 1952 - General case) A linear operator \((A, D(A))\) is the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) verifying \(\|T(t)\| \leq Me^{w t}\), for \(t \geq 0\), if and only if \((A, D(A))\) is closed, densely defined, and for every \(\lambda > w\) one has \(\lambda \in \rho(A)\) and \(\|[\lambda - w)(\lambda - A)^{-1}]^n\| \leq M, \forall n \in \mathbb{N}\).

For self-adjoint contraction semigroups (i.e. for semigroups \((T(t))_{t \geq 0}\) where for all \(t\), \(T(t)\) is a self-adjoint contraction operator), we have the following generation theorem.

**Theorem 18.** The generators of self-adjoint contraction semigroups \((T(t))_{t \geq 0}\) are precisely the operators \((-A)\), where \(A\) are positive self-adjoint operators.

**Theorem 19.** (Stone’s theorem, 1932) Let \((A, D(A))\) be a densely defined operator on a Hilbert space \(\mathcal{H}\). Then \(A\) generates a unitary group \((T(t))_{t \in \mathbb{R}}\) on \(\mathcal{H}\) if and only if \(A\) is skew-adjoint, i.e: \(A^* = -A\).

**Proof.** Suppose first that \(A\) generates a unitary group \((T(t))_{t \in \mathbb{R}}\). For every \(t \in \mathbb{R}\), we therefore have that \(T(t)^* = T(t)^{-1} = T(-t)\). We can therefore compute for elements \(u, v\) of \(H\), and \((\cdot, \cdot)\) the inner
product on this Hilbert space:

\[
(Au, v) = \lim_{t \to 0} \frac{1}{t} [(T(t) - I)u, v] = \lim_{t \to 0} \frac{1}{t} [(T(t)u, v) - (u, v)]
= \lim_{t \to 0} \frac{1}{t} [(u, T(-t)v) - (u, v)] = \lim_{t \to 0} -\frac{1}{t} (u, (T(t) - I)v)
= -(u, Av),
\]

and therefore \(A^* = -A\). Conversely, if \(A\) is skew-adjoint, then

\[
(Au, u) = -(u, Au) = -(Au, u),
\]

and hence \((Au, u)\) is purely imaginary. By definition we know that \(A\) and its adjoint \(-A\) are dissipative and closed, and by the Lumer–Phillips theorem stated above all that is left to do is to show that \(rg(\lambda - A)\) is dense of for some \(\lambda > 0\). By contradiction assume the \(rg(I - A) \neq H\). There exists an element \(y \in H, y \neq 0\), such that for all \(x \in D(A): (\langle I_A \rangle x, y) = 0\). This is equivalent to \((x, (I + A)y) = 0\), which means that \((I + A)y\) is orthogonal to the dense subset \(D(A)\), and must therefore be 0. However, recall that we had said that on a reflexive Banach space (and therefore on a Hilbert space), the dissipative operator \(B\) verifies \(||(\lambda - B)x|| \geq \lambda ||x||, \forall x \in D(B), \forall \lambda > 0\). Taking \(B = -A\), one gets \(||(I + A)y|| \geq ||y|| > 0\), which is a contradiction. \(\square\)

4.3 Notation - Asymptotic expansions

The aim of this section is to introduce a few notions of asymptotic development. For a complete exposition of this subject, we refer to Chapter 2 of Dieudonné [Die68]. The idea to keep in mind for this theory is that we choose certain functions whose behaviour at a certain point is considered well-known and understood, and consequently we want to compare the behaviour of a given function at a point with the behaviour of these functions. In this thesis, we will only be interested in the asymptotic behaviour at the origin, and we will define the notions accordingly.

**Definition 25.** Consider functions \(f, g : \mathbb{R} \to \mathbb{R}\), such that \(g \neq 0\) in a neighbourhood \(U\) of the origin. We say that a function \(c \cdot g\) is the principal part of \(f\) if the quotient \(f/g\) defined in \(U\) has a finite limit \(c \neq 0\) at zero, and we denote this \(f \sim c \cdot g\). In the case where \(f \sim g\) (i.e. the quotient tends to 1), we say that \(f\) is equivalent to \(g\) in a neighbourhood of 0.

As the requirements for this concept are very strong, it is not always possible to find a principal
part. We therefore introduce a weaker notion of comparison of two functions.

**Definition 26.** Let $f$, $g$ and $U$ be as defined in the previous definition. If a function $|f|/g$ is bounded in a neighbourhood $V \subset U$ of the origin, we write:

$$f = O(g).$$  \hspace{1cm} (4.4)

This notation is called the Landau notation (and sometimes the big O notation).

To say that $f = O(g)$ is therefore equivalent to saying that $f$ and $g$ have the same growth rate as the origin. From their definition, we deduce some calculus rules on the Landau symbols. For any functions $g_1$, $g_2$, and for any constants $c$, $\lambda > 0$, we have the following:

1. $O(g) + O(g) = O(g)$;
2. $c \cdot O(g) = O(c \cdot g)$;
3. $O(g_1)O(g_2) = O(g_1g_2)$;
4. $|O(g)|^\lambda = O(g^\lambda)$.

The Landau notation is what we will use to give bounds on the error terms for the asymptotic approximations we find in Section 4.5 and in Chapter 5, and we will make constant use of the above rules. Most often our expressions will look like $f(t) = \tilde{f}(t) + O(t^k)$.

### 4.4 The Trotter product formula

Let us consider an arbitrary operator $C$ that can be expressed as the sum of two operators:

$$C := A + B.$$

One way to compute the semigroup generated by $C$, knowing the semigroups associated to $A$ and $B$, is to use the following well-known theorem.

**Theorem 20. (Trotter product formula)**

Let $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be strongly continuous semigroups on a Banach space $X$ satisfying the stability condition:

$$\| [T(t/n)S(t/n)]^n \| \leq Me^{wt}, \; \forall t \geq 0, n \in \mathbb{N},$$  \hspace{1cm} (4.5)
and for constants $M \geq 1$, $w \in \mathbb{R}$. Consider the sum $A+B$ on $D := D(A) \cap D(B)$ of the generators $(A,D(A))$ of $(T(t))_{t \geq 0}$ and $(B,D(B))$ of $(S(t))_{t \geq 0}$, and assume that $D$ and $(\lambda_0 - A - B)D$ are dense in $X$ for some $\lambda_0 \geq w$. Then the closure of the sum of these two operators $C := A + B$ generates a strongly continuous semigroup $(U(t))_{t \geq 0}$ given by the Trotter product formula:

$$U(t)x = \lim_{n \to \infty} [T(t/n)S(t/n)]^n x,$$

with uniform convergence for $t$ in compact intervals.

There exists another very useful version of the Trotter product formula, for the case of nonnegative self-adjoint operators. It is often referred to as Kato’s strong Trotter product formula, and is proved in [Kat78].

**Theorem 21.** (Kato,1978) Let $A$, $B$ be non-negative self-adjoint operators on a Hilbert space $H$. Let $D' = D(A^{1/2}) \cap D(B^{1/2})$, let $H'$ be the closure of $D'$, and $P'$ denote the orthogonal projection of $H$ onto $H'$. We define the form sum of $A$, $B$, $C' = A + B$

as the self-adjoint operator in $H'$ associated with the non-negative, closed quadratic form

$$u \mapsto \|A^{1/2}u\|^2 + \|B^{1/2}u\|^2$$

which is densely defined in $H'$. Then

$$s - \lim_{n \to \infty} \left[ e^{-(t/n)A} e^{-(t/n)B} \right]_n = e^{-tC'}P', \ t > 0,$$

(4.7)

the convergence being uniform in $t \in [0,T]$ for any $T > 0$ when applied to $u \in H'$, and in $t \in [T_0,T]$ for any $0 < T_0 < T$ when applied to $u \perp H'$.

Here, by $s-\lim$ we mean the limit in the strong operator topology. Also, note that if $D(A^{1/2}) \cap D(B^{1/2})$ is dense in $H$, then $H' = H$ and $P'$ is the identity operator. We therefore obtain that Equation (4.7) is equivalent to Equation (4.6).

### 4.5 Short-time behaviour of the heat kernel

We now give an exposition of the approach we suggest for the computation of the short-time asymptotic behaviour of the hypoelliptic heat kernel.
4.5.1 Approximation in small-time of the semigroup $e^{t\hat{\Delta}_r}$

We consider the case of a left-invariant manifold $(G,\mathcal{H},g)$, where the family of operator equations obtained by Generalized Fourier Transform $\frac{d}{dt}u^\lambda(t,x) = \hat{\Delta}_r^{\lambda}u^\lambda(t,x)$ (where we considered the operators $\hat{\Delta}_r^{\lambda}$ as operators on $H^\lambda$) is not solvable, but where the transformed sub-Laplacian is of the form: $\hat{\Delta}_r^{\lambda} = A^\lambda + B^\lambda$, where the semigroups generated by the operators $A^\lambda$ and $B^\lambda$ are known.

Temporarily we will put aside the sub-Riemannian context, and solely focus on the functional analysis aspect of the problem. We will consider an arbitrary semigroup generated by an operator $C$ which can be written as the sum of two operators $A$ and $B$. As suggested by the previous section, one can try to apply the Trotter product formula (or Kato’s self-adjoint positive corollary), as was done for example in [Bea98] for the Schrödinger equation with quadratic potential, but in most cases, the expression of $[T(t/n)S(t/n)]^n$ is too complicated to be computable. We will show a way to simplify this calculation, when we are only interested in the small-time behaviour of the semigroup generated by $C$. We start by the following proposition.

**Proposition 8.** Let $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be two strongly continuous contraction semigroups on a Banach space $L^p(\Omega)$ or $C_0(\Omega)$, for $\Omega$ a locally compact space, and $1 \leq p < \infty$. Suppose that these semigroups verify the conditions of Theorem 20, or of Theorem 21 with the additional assumption that $H' = H$. Denote by $(A, D(A))$ and $(B, D(B))$ the respective infinitesimal generators of the two semigroups, and by $C$ the formal sum of those operators, which we consider on $D = D(A) \cap D(B)$. Assume that for the operators $B$, $S(t)$ and $T(t)$ and any compact set $K \subset \Omega$, there exists a constant $\alpha_K$ such that for all $f \in C_0^\infty(\Omega)$ with support in $K$, and every operator $E$ that is a composition of the operators $B, T(t)$ and $S(t)$: $\|Ef\|_1 \leq \alpha_K \|f\|_1$, where $k$ is the number of times the operator $B$ appears in $E$. Then $C$ generates a strongly continuous semigroup $(U(t))_{t \geq 0}$, which has the following short-time asymptotic:

$$U(t)f = T(t)f + \lim_{n \to \infty} \frac{1}{n} \left[ T(t/n)^n B + T(t/n)^{n-1} BT(t/n) + \ldots + T(t/n) BT(t/n)^{n-1} \right] f + \mathcal{O}(t^2)f,$$

(4.8)

where by $\mathcal{O}(t^2)f$ we mean an operator $D_t$ acting on $f$, such that $\|D_t f\|_1 \leq M t^2 \|f\|_1$ for a constant $M$ and for $t$ small enough.

It will be sufficient to prove formula (4.8) for functions $f \in C_0^\infty(\Omega)$, since $C_0^\infty(\Omega)$ is dense in both
$C_0(\Omega)$ and $L^p(\Omega)$, $1 \leq p < \infty$. First, we will need a version of Taylor’s theorem with remainder for strongly continuous semigroups of operators (for more details see sections 11.6-11.8, in [HP57]).

**Theorem 22.** Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space $X$, with corresponding infinitesimal generator $A$. Then, for all $x$ in $D(A^m)$, and for all $t > 0$:

$$T(t)x = \sum_{k=0}^{m-1} \frac{t^k}{k!} A^k x + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} T(s) A^m x ds, \quad \forall \ m \geq 1, \ \forall \ x \in X. \quad (4.9)$$

The class of semigroups for which such an expansion is true is in fact more general than the class of strongly continuous semigroups, as one can relax the hypothesis of strong continuity of the semigroup at $t = 0$. However, in our analysis we will never consider such a larger class, which justifies the fact that we decided to modify the theorem from its original version (see [HP57]).

Notice that if $(T(t))_{t \geq 0}$ is a multiplication semigroup on $L^p(\mathbb{R})$, i.e there is a function $q$ such that $[T(t)f](s) = e^{tq(s)} f(s)$, with infinitesimal generator $A$: $[Af](s) = q(s)f(s)$, then Theorem 22 is nothing other than the classic Taylor’s theorem for $m$-times differentiable functions, with remainder in integral form.

We can use the mean value theorem to transform the remainder to a more convenient form. We have, $\forall f \in C_0^\infty(\Omega)$, and $\forall f \in D(A^m)$:

$$\frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} T(s) A^m f(x) ds = \frac{1}{m!} \int_0^t (t-s)^{m-1} T(s) A^m f(x) ds = \frac{1}{m!} \int_0^t (t-s)^{m-1} T(s) A^m f(x) ds = \frac{1}{m!} \int_0^t (t-s)^{m-1} T(s) A^m f(x) ds, \quad \forall \ x \in X.$$

Note that $\xi$ is actually a function of $x$.

For $m = 2$, we therefore obtain $\forall f \in C_0^\infty(\Omega)$, and $\forall f \in D(A^2)$:

$$[T(t)f]x = f(x) + tAf(x) + \frac{t^2}{2} T(\xi) A^2 f(x), \quad \forall \ x \in X.$$

We are now ready to prove Proposition 8.

**Proof.** Consider strongly continuous semigroups $(T(t))$ and $(S(t))$ that verify the assumptions of the theorem. Applying Trotter’s Product formula (Theorem 20), we obtain that the semigroup $(U(t))$ generated by the sum of the operators $A$ and $B$ verifies:

$$U(t)f = \lim_{n \to \infty} [T(t/n)S(t/n)]^n f, \quad \forall \ f \in C_0^\infty(\Omega).$$
Applying Theorem 22 to the second semigroup $S(t)$, with $m = 2$, yields:

$$S(t/n)f = f + \frac{t}{n}Bf + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 f,$$
for a $\xi_n \in [0, t/n]$.

Therefore, combining the two expressions, we obtain:

$$U(t)f = \lim_{n \to \infty} \left[ T(t/n) \left[ I + \frac{t}{n}B + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 \right] \right]^n f.$$

Notice that the main idea is to approximate the semigroup $S(t)$ by the two first terms given in its Taylor expansion, while keeping $T(t)$ untouched. This comes down to considering the operator $A$ as the main approximation of the sum operator $C$, and $B$ as a perturbation of $A$.

Denote by $K$ the compact support of a function $f \in C^\infty_c(\Omega)$. Let us take a closer look at the composition of operators:

$$\left( T(t/n) \left[ I + \frac{t}{n}B + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 \right] \right)^n$$

We wish to express the resulting operator in the following form:

$$D_0 + tD_1 + t^2D_2 + \ldots + t^{2n}D_{2n},$$

where the $D_i$, $i \in \{0, \ldots, 2n\}$, are operators on the space of functions on which the original semigroups act. It is straightforward that the first two operators are given by

$$D_0 = T(t/n)^n$$
$$D_1 = (1/n)[T(t/n)BT(t/n)^{n-1} + \ldots + T(t/n)^nB].$$

Notice that by using the Trotter product formula with $S(t) \equiv Id$, $\forall t \geq 0$, we get that

$$\lim_{n \to \infty} T(t/n)^n f = T(t)f.$$

Taking the limit $n \to \infty$ in the expressions therefore gives us the two first terms of formula 4.8.

We are now left with the non-trivial problem of proving that we can control the remaining terms, i.e. that they can be bounded in the following way:

$$\| \lim_{n \to \infty} [t^2D_2 + \ldots + t^{2n}D_{2n}]f \|_1 \leq t^2M\|f\|_1,$$

for a constant $M$, with $t$ small enough and $f \in C^\infty_c(\Omega)$. With this aim, we will only need an upper
bound on the growth of the norm of the operators $D_k$, and we prove in Appendix A.1 that
\[\|D_k f\|_1 \leq \alpha_k^k \|f\|_1, \quad k \in \{0, 1, 2, \ldots, 2n\},\]
for $f \in C_0^\infty(\Omega)$ with support in a compact set $K$. Using this we obtain
\[
\| \lim_{n \to \infty} \left[ t^2 D_2 + \ldots + t^{2n} D_{2n} \right] f \|_1 \leq \lim_{n \to \infty} \left[ t^2 \alpha_K^2 + \ldots + t^{2n} \alpha_K^{2n} \right] \|f\|_1
\]
\[= t^2 \sum_{k=0}^{\infty} (\alpha_K^k) \|f\|_1
\]
\[= t^2 M \|f\|_1
\]
when $t$ is chosen small enough so that $t\alpha_K \leq 1$, and where $M$ is the value of the infinite series. This concludes the proof of the Proposition.

Let us summarize the main ideas behind Proposition 8. Using the semigroup version of Taylor’s theorem, instead of considering the operator $[T(t/n)S(t/n)]^n$, one is in fact led to the simpler $[T(t/n) \left[ 1 + \frac{t}{n} B + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 \right]]^n$, with $\xi_n$ as defined in the proof. Separating this into two parts, and then using the binomial theorem we get
\[
\left[ T(t/n) \left[ 1 + \frac{t}{n} B + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 \right] \right]^n = \left[ \left( T(t/n) \left[ 1 + \frac{t}{n} B + \frac{1}{2} \left( \frac{t}{n} \right)^2 S(\xi_n)B^2 \right] \right)^{C_1(t)} \right]^{C_2(t)}
\]
\[= C_1(t)^n + \sum_{k=1}^{n} \binom{n}{k} C_1(t)^{n-k} C_2(t)^k.
\]
We then prove that this last sum of operators was in fact an $O(t^2)$ operator, (where the definition of such operators is given in the statement of the Proposition), and that the operator $C_1(t)^n$ is itself equal to $T(t/n)^n + [T(t/n)BT(t/n)^{n-1} + \ldots + T(t/n)^nB]$ modulo an $O(t^2)$ operator. Moreover, we show that these operators remain $O(t^2)$ when taking the limit as $n$ goes to infinity.

It is important to note that the control of the remaining terms is due to the last assumption in the statement of the proposition: “Assume that for the operators $B$, $S(t)$ and $T(t)$ and any compact set $K \subset \Omega$, there exists a constant $\alpha_K$ such that for all $f \in C_0^\infty(\Omega)$ with support in $K$, and every operator $E$ that is a composition of the operators $B$, $T(t)$ and $S(t)$: $\|Ef\|_1 \leq \alpha_K^k \|f\|_1$, where $k$ is the number of times the operator $B$ appears in $E$.” A sufficient condition for this to be verified is to have both $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ contraction semigroups, where every operator $T(t)$ and $S(t)$ does
not enlarge the support of the function it acts on, and to have $B$ bounded on the set of all functions having support in a compact set $K$. However, it seems to the author that the condition of $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ being contraction semigroups is necessary, otherwise, $\|E\|$ would be dependent of the number of times that these operators appear in $E$.

To conclude, we obtain the following approximation for the operators $U(t) = e^{tC}$:

$$
\tilde{U}(t)f = T(t)f + \lim_{n \to \infty} \frac{t}{n} [T(t/n)BT(t/n)^{n-1} + \ldots + T(t/n)^n B]f.
$$

(4.12)

for any function $f$ in the domain of $C$.

We now wish to bring these operator theory results back into the context of sub-Riemannian geometry. Consider a left-invariant sub-Riemannian manifold $(G, \mathcal{H}, g)$, and as usual denote by $\Delta_{sr}$ the associated hypoelliptic Laplacian and by $\hat{\Delta}_\lambda_{sr}$ its Generalized Fourier Transform, where $\lambda$ indexes the elements of the dual $\hat{G}$. We define a list $(L)$ of assumptions on this left-invariant sub-Riemannian manifold.

Assumption 1: For each $\lambda$, the transformed hypoelliptic Laplacian decomposes as a sum of two operators:

$$
\hat{\Delta}_\lambda_{sr} = A^\lambda + B^\lambda.
$$

Assumption 2: The operators $A^\lambda$ and $B^\lambda$ (or canonical, closed extensions of them) generate strongly semigroups $(T^\lambda(t))_{t \geq 0}$ and $(S^\lambda(t))_{t \geq 0}$ that verify the conditions of Proposition 8.

Assumption 3: For each $t \geq 0$, $T^\lambda(t)$ is an integral operator with kernel $k^\lambda_t(s, r)$:

$$
[T^\lambda(t)f](s) = \int_{\Omega_\lambda} k^\lambda_t(s, r)f(r)dr.
$$

Assumption 4: There exists an integrable function $K^\lambda_{t/n}(s, r)$, uniformly bounded in $s$ by an integrable function $G^\lambda_{t/n}(s, r)$ for all $n \geq 1$, such that

$$
(t/n) [T(t/n)^n B^\lambda + T^\lambda(t/n)^{n-1} B^\lambda T^\lambda(t/n) + \ldots
+ T^\lambda(t/n)^n B^\lambda T^\lambda(t/n)^{n-1}] f(s) = t \int_{\Omega_\lambda} K^\lambda_{t/n}(s, r)f(r)dr
$$

(4.13)

Our main result is the following theorem.
Chapter 4. Short-Time Behaviour of the Hypoelliptic Heat Kernel

Theorem 23. For $G$ a unimodular Lie group of type I, let $(G, \mathcal{H}, g)$ be a left-invariant sub-Riemannian manifold satisfying the list of assumptions detailed above. Denote by $\Delta_{sr}$ the associated hypoelliptic Laplacian and by $\hat{\Delta}_\lambda$ its Generalised Fourier Transform, with $\lambda$ indexing the elements of the dual $\hat{G}$, and suppose that each acts on the Hilbert space $H^\lambda = L^2(\Omega_\lambda)$. Then, the operators of the semigroup $e^{t\hat{\Delta}_\lambda}$ are integral operators having the following expression in small-time:

$$[e^{t\hat{\Delta}_\lambda} f](s) = \int_{\Omega_\lambda} [k^\lambda_t(s,r) + t \lim_{n \to \infty} K^\lambda_{t/n}(s,r)] f(r) dr + [O(t^2) f](s), \quad (4.14)$$

where the functions $k^\lambda_t$ and $K^\lambda_{t/n}$ are as defined in (L), and where by $O(t^2) f$ we mean an operator $D_t$ acting on $f$, such that $\|D_t f\|_1 \leq Mt^2\|f\|_1$ for a constant $M$ and for $t$ small enough.

Proof. Let $G$ be a unimodular Lie group of type I, and $(G, \mathcal{H}, g)$ a left-invariant sub-Riemannian manifold verifying the conditions stated above as well as all the assumptions above. From the assumptions 1 and 2, we deduce that for every $\lambda \in \hat{G}$, the short-time asymptotic of the semigroup generated by the operator $\hat{\Delta}_{sr}$ is given by equation 4.8 of Proposition 8. The first term of equation 4.14 is a direct consequence of assumption 3, whereas assumption 4 enables us to use the Dominated Convergence Theorem in order to obtain the second term. 

We now specialize our main theorem to the case where our Lie group $G$ is such that, for each $\lambda$ in the dual $\hat{G}$, $H^\lambda$ is the Hilbert space $L^2(\mathbb{R})$, $A^\lambda$ is the one dimensional Laplacian

$$[A^\lambda f](x) = |\Delta f|(x) = \frac{d^2}{dx^2} f(x), \quad (4.15)$$

and $B^\lambda$ is the multiplication operator

$$[B^\lambda f](x) = q^\lambda(x) f(x), \quad (4.16)$$

where $q^\lambda$ is a polynomial depending on the parameter $\lambda$ for which there exists a polynomial $p^\lambda$ such that we can write

$$q^\lambda(x) = -p^\lambda(x)^2.$$ 

Writing $q^\lambda(x)$ in expanded form yields

$$q^\lambda(x) = -[a^\lambda_{2m} x^{2m} + a^\lambda_{2m-1} x^{2m-1} + \ldots + a^\lambda_2 x^2 + a^\lambda_1 x + a^\lambda_0],$$

where $m$ is the degree of the polynomial $p^\lambda(x)$, and $a^\lambda_{2m}$ and $a^\lambda_0$ are positive constants. We have the following.
Corollary 2. Let \((G, \mathcal{H}, g)\) be a left-invariant sub-Riemannian manifold, with \(G\) a unimodular Lie group of type I. Denote by \(\Delta_{sr}\) the associated hypoelliptic Laplacian, and by \(\hat{\Delta}_\lambda\) its Generalized Fourier Transform, where \(\lambda\) indexes the elements of the dual \(\hat{G}\). Assume that for all \(\lambda\), \(\hat{\Delta}_\lambda = A^\lambda + B^\lambda\), for operators \(A^\lambda\) and \(B^\lambda\) in the form described above. Then, the conditions of Theorem 23 are verified, and the semigroup generated by the transformed hypoelliptic Laplacian has the following expression in small time:

\[
[e^{t\hat{\Delta}_\lambda} f](s) = \int_\mathbb{R} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{1}{4t}(s-r)^2\right) \left(1 - \frac{a^\lambda_0 t}{k+1} \sum_{i=0}^{2m} \frac{a^\lambda_k}{s^i r^{k-i}}\right) f(r)dr + \mathcal{O}(t^{3/2}) f(s), \quad \forall f \in C_c^\infty(\mathbb{R}).
\] (4.17)

Proof. As our proof is independent of the element \(\lambda\) of \(\hat{G}\), as much as to simplify the notation we will consider \(\lambda\) fixed and we drop the indexing of the operators, functions and constants defined above. We first want to show that the operators \(-A\) and \(-B\) verify the assumptions of Theorem 21. To this end, we define the operators \(A' := -A = -\Delta\) and \(B' := -B\) on \(L^2(\mathbb{R})\). Notice that the operators \(A'\) and \(B'\) are non-negative and symmetric. Indeed, for \(f, g \in C_c^\infty(\mathbb{R})\),

\[
(-\Delta f, g) = \int_\mathbb{R} -\Delta f(x)g(x)dx = \int_\mathbb{R} \nabla f(x)\overline{\nabla g(x)}dx = -\int_\mathbb{R} f(x)\Delta g(x)dx = (f, -\Delta g)
\]

\[
(-\Delta f, f) = \int_\mathbb{R} -\Delta f(x)\overline{f(x)}dx = \int_\mathbb{R} |\nabla f(x)|^2dx \geq 0,
\]

and the same is clearly true for the multiplication operator \(B'\) by a positive function. As a result, both \(A'\) and \(B'\) have a canonical self-adjoint extension, called the Friedrichs extension (introduced in Section 4.1). We can identify the two operators with their Friedrichs extension, and therefore assume that they are self-adjoint. It follows that the corresponding semigroups verify the conditions of Theorem 21. We now compute the domains \(D(A'^{1/2})\) and \(D(B'^{1/2})\). We have that, for \(f \in C_c^\infty(\mathbb{R})\),

\[
(A'^{1/2} f, A'^{1/2} f) = (A' f, f) = \int_\mathbb{R} |\nabla f(x)|^2dx
\]

\[
(B'^{1/2} f, B'^{1/2} f) = (B' f, f) = \int_\mathbb{R} p(x)^2 f(x)^2dx.
\]

Hence

\[
D(A'^{1/2}) = \{f \in L^2(\mathbb{R})| \nabla f \in L^2(\mathbb{R})\} = H^1(\mathbb{R}) = W^{1,2}(\mathbb{R}),
\]

and similarly we find that \(D(B'^{1/2}) = \{f \in L^2(\mathbb{R})| pf \in L^2(\mathbb{R})\}\). Hence we have \(C_c^\infty(\mathbb{R}) \subseteq D(A'^{1/2}) \cap \)
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$D(B^{1/2})$, and as $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ so is $D' = D(A^{1/2}) \cap D(B^{1/2})$. Therefore, the closure $H'$ of $D'$ is equal to $L^2(\mathbb{R})$, and $P'$ is just the identity operator.

The semigroup associated to the 1-dimensional Euclidean Laplacian is the convolution semigroup given by

$$ T(t)f(s) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{(s-r)^2}{4t}} f(r) dr $$

and the semigroup associated with the multiplication operator $B$ is given by

$$ S(t)f(s) = e^{-t\rho(s)^2} f(s). $$

We therefore have that $(S(t))$ and $(T(t))$ are contraction semigroups, and, as $B$ is a multiplication operator, and considering any function $f \in C_c^\infty(\mathbb{R})$ with support in a compact $K$, there exists a constant $\alpha_K$ such that any operator $E$ which is a composition of operators $S(t)$, $T(t)$ and $B$ verifies $\|Ef\|_1 \leq \alpha_K^k \|f\|_1$, where $k$ is the number of times the operator $B$ appears in $E$. Indeed, the operators $S(t)$ and $T(t)$ have norm less then 1, and don’t change the support of a function when acting on it, and in addition the function $q = -p^2$ is bounded on all compacts. Hence, we can in fact choose $\alpha_K$ to be the maximum of the function $q$ on the compact set $K$. All the hypotheses of Proposition 8 are verified, and we only have to verify the additional conditions of Theorem 23.

We need to compute the sum

$$ S_{t/n}(s_0) := \frac{t}{n} \left[ T(t/n)^n B + T(t/n)^{n-1} BT(t/n) + \ldots + T(t/n)BT(t/n)^{n-1} \right] f(s_0), $$

which is in our case

$$ S_{t/n}(s_0) = \frac{t}{n} \left( \frac{4\pi t}{n} \right)^{-n/2} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \exp \left[ -\frac{n}{4t} \left( (s_0 - s_1)^2 + (s_1 - s_2)^2 + \ldots + (s_{n-1} - s_n)^2 \right) \right]$$

$$ \left[ -p(s_1)^2 - p(s_2)^2 - \ldots - p(s_n)^2 \right] f(s_n) ds_n ds_{n-1} \ldots ds_1 $$

$$ = -\int_{-\infty}^{\infty} K_{n, t/n}(s_0, s_n) f(s_n) ds_n, $$

where

$$ K_{n, t/n}(s_0, s_n) = \frac{t}{n} \left( \frac{4\pi t}{n} \right)^{-n/2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \exp \left[ -\frac{n}{4t} \left( (s_0 - s_1)^2 + (s_1 - s_2)^2 + \ldots + (s_{n-1} - s_n)^2 \right) \right]$$

$$ \left[ p(s_1)^2 + p(s_2)^2 + \ldots + p(s_n)^2 \right] ds_{n-1} \ldots ds_1. $$

In this last expression of the function $S_{t/n}$, Fubini’s theorem can be used to exchange the order of
integration, which is justified as the integrand is the pointwise multiplication of a Schwartz function and a function in $L^2$. By the Cauchy–Schwarz inequality, the resulting function is in $L^1$.

The following lemma on the computation of $n$-dimensional Gaussian integral with linear term will be used in most of the calculations that will follow:

**Lemma 3.** If $A$ is a symmetric positive-definite $n \times n$ matrix, and $\vec{b}$ an $n \times 1$ vector, the following equality is true:

$$
\int \cdots \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \vec{x}^T A \vec{x} + \vec{b}^T \vec{x} \right) d\vec{x} = \sqrt{\frac{(2\pi)^n}{\det A}} \exp (\vec{b}^T A^{-1} \vec{b}).
$$

Using the previous lemma, we can find a bound on the integral kernel; more precisely, $\forall n \in \mathbb{N},$

$$
0 \leq K_{n,t/n}(s_0, s_n) \leq \left( \frac{4\pi t}{n} \right)^{-n/2} \int \cdots \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{n}{4t} \left( (s_0 - s_2)^2 + (s_1 - s_2)^2 + \ldots + (s_{n-1} - s_n)^2 \right) \right] ds_{n-1} \ldots ds_1 = \frac{1}{4\pi t} \exp \left[ \frac{1}{4t} (s_0 - s_n)^2 \right]
$$

(see Appendix A.2.1 for the explicit calculations). Therefore, we can use the Dominated Convergence Theorem and exchange the limit and the integral, to finally get

$$
\lim_{n \to \infty} S_{t/n}(s_0) = \int_{-\infty}^{\infty} \left[ \lim_{n \to \infty} K_{n,t/n}(s_0, s_n) \right] f(s_n) ds_n. \tag{4.20}
$$

We also have verified all the conditions of Theorem 23, and all that remains is the computation of the limit of the integral kernel $K_{n,t/n}(s_0, s_n)$ as $n \to \infty$.

$$
K_{n,t/n}(s_0, s_n) = \frac{t}{n} \left( \frac{4\pi t}{n} \right)^{-n/2} \exp \left[ -\frac{n}{4t} (s_0^2 + s_n^2) \right] \int_{\mathbb{R}^{n-1}} \cdots \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{n}{2t} \left( s_1^2 + s_2^2 + \ldots + s_{n-1}^2 - s_0 s_1 - s_1 s_2 - \ldots - s_{n-1} s_n \right) \right] ds_{n-1} \ldots ds_1.
$$

Since the terms in $s_n$ or in $a_0$ can be taken out of the integral, we only need to compute the following
integrals:
\[ I_0(s_0, s_n) = \int_{\mathbb{R}^{n-1}} \cdots \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{n}{2t} \left( \sum_{j=1}^{n-1} s_j^2 - s_0 s_1 - \ldots - s_{n-1} s_n \right) \right] ds_{n-1} \ldots ds_1 \]
\[ I_{i,k}(s_0, s_n) = \int_{\mathbb{R}^{n-1}} \cdots \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{n}{2t} \left( \sum_{j=1}^{n-1} s_j^2 - s_0 s_1 - \ldots - s_{n-1} s_n \right) \right] s_k^i ds_{n-1} \ldots ds_1, \]
where \( i \in \{1, \ldots, n-1\}, \ k \in \{1, \ldots, 2m\} \).

**Proposition 9.** For \( i \in \{1, 2, \ldots, n-1\} \), the previous integrals are equal to
\[
I_0(s_0, s_n) = \left( \frac{t}{n} \right)^{(n-1)/2} \frac{2^{n-1} \pi^{(n-1)/2}}{\sqrt{n}} \exp \left( \frac{1}{4t} \left( (n-1)s_0^2 + (n-1)s_n^2 + 2s_0 s_n \right) \right) \tag{4.21}
\]
\[
I_{i,k}(s_0, s_n) = I_0(s_0, s_n) \cdot \sum_{\substack{j=0 \atop j \text{ even}}}^k \frac{k!}{(k-j)! (j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{t}{n} \right)^{j/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j} \tag{4.22}
\]

**Proof.** See Appendix A.2. \( \square \)

All that is left to obtain formula (4.17) is to compute the sums over \( i \) of the integrals \( I_{i,k}(s_0, s_n) \); for these calculations, see Appendix A.3.

This completes the proof of the Corollary. \( \square \)

### 4.5.2 Remarks on the decomposition

As stated previously, our approach comes down to considering the operator \( A \) as the main approximation of \( C \), and \( B \) as a perturbation of \( A \), in order to recover some information about the semigroup \( e^{tC} \) when it is not directly computable. Letting, as usual, \( U(t) \), \( T(t) \) and \( S(t) \) denote the strongly continuous semigroups generated by \( C \), \( A \) and \( B \) respectively, our first result consisted in showing that one can approximate the operators
\[ U(t)x = \lim_{n \to \infty} [T(t/n)S(t/n)]^n x \]
by the operators
\[ \hat{U}(t)x = T(t)x + \lim_{n \to \infty} \frac{t}{n} [T(t/n)BT(t/n)]^{n-1} + \ldots + T(t/n)^n B]x \]
for all \( x \) in the domain of \( C \).

As one operator can possibly admit many decompositions as a sum of two operators, it is natural to ask which one leads to the best approximation of the semigroup \( e^{tC} \). We will not try to give a
formal definition of “better approximation”, but intuitively, from the remarks above, the more the behaviour of the operator $C$ is reflected in its component $A$, then the better is the resulting estimate.

There are two extremes of such a decomposition: $A = C$, $B = 0$ and $A = 0$, $B = C$. Clearly, the first one corresponds to doing no approximation, and leads to the trivial and useless equality $\tilde{U}(t) = U(t)$. For the second one, recall that the semigroup corresponding to the operator $A = 0$ is $T(t) \equiv \text{Id}, \ \forall t \geq 0$. For any element $x$ in the domain of $C$ we obtain the approximation

$$\tilde{U}(t)x = x + \lim_{n \to \infty} \frac{t}{n} Bx = x + tBx = x + tCx,$$

and we recognise the two first terms of the Taylor series expansion of the semigroup $U(t)$. This approximation is, in fact, often too rough.

Between these two extremes of not approximating at all and maybe approximating too much, there may exist an array of possibilities. However, we have to keep in mind that the assumptions from the statement of the Proposition induce strong restrictions on the operators we could wish to choose as our $A$ and $B$. Indeed, as we had pointed out after proving the Proposition, it seems to the author that the last assumption in its statement implies that both operators have to be generators of contraction semigroups, and furthermore, the operator $B$, which is the one that we approximate, has to be an operator which is bounded for all functions having support in the same compact set. This ensures (see Appendix A.1), that we keep a certain control on the error term. It is not obvious to us what kind of operators, outside of multiplication operators, verify this condition. Finally, many choices of $A$ and $B$ lead to intractable calculations for $\tilde{U}(t)$, or have semigroups that are difficult to compute and hence should be excluded.

As an example, let us consider an operator of the form considered in Corollary 2: $[Cf](x) = [\frac{d^2}{dx^2} - p(x)^2]f(x)$, where $f \in L^2(\mathbb{R})$. Our choice of decomposition $A = \frac{d^2}{dx^2}$, $B = -p(x)^2$ (where we mean that $B$ is the multiplication operator by the function $-p^2$) is clearly the most natural one, and leads to an operator $\tilde{U}(t)$ containing more information than the Taylor series approximation $1 + tC$.

Moreover, notice that the two resulting operators $-A$ and $-B$ enjoy the particularly nice properties of being non-negative and essentially self-adjoint, and hence from Theorem 18 we know that $A$ and $B$ are generators of contraction semigroups. In the case where $a_2 > 0$, a second option that one could wish to consider is the choice $A = \frac{d^2}{dx^2} - a_2x^2$ and $B = -p(x)^2 + a_2x^2$, as we recognise in the first operator the Schrödinger-type operator with quadratic potential, otherwise known as the
harmonic oscillator. Unfortunately, the semigroups generated by $A$ and $B$ will not be contraction semigroups, which will prevent us from using Proposition 8. There doesn't seem to be, in this case, any other natural decomposition to which the theory we developed above could be applied, but one shouldn't exclude the fact that such a decomposition could exist.

### 4.5.3 Recovering the original heat kernel

We have obtained, under certain assumptions on the left-invariant sub-Riemannian manifold, expressions for the short-time behaviour of the semigroups generated by the transformed sub-Laplacian $\hat{\Delta}^\lambda_{sr}$. We would now like to combine these together with the classical harmonic analysis methods used by Agrachev et al. that we reviewed in Theorem 3 of Chapter 2. Explicitly, we want to perform an inverse Fourier transform on the short-time approximations of the semigroups $e^{t\hat{\Delta}^\lambda_{sr}}$, in order to recover the short-time behaviour of the hypoelliptic heat kernel $p_t(g)$.

Using the notation of Chapter 2, recall that if for every $t \geq 0$ and $\lambda \in \hat{G}$, $e^{t\hat{\Delta}^\lambda_{sr}}$ is an integral operator with integral kernel $Q^\lambda_t(\theta, \bar{\theta})$, the heat kernel takes the form

$$p_t(g) = \int_{\hat{G}} \int_{X^\lambda} X^\lambda(g)Q^\lambda_t(\theta, \bar{\theta})|_{\theta = \bar{\theta}} d\bar{\theta} dP(\lambda). \tag{4.23}$$

Also recall that for the left-invariant sub-Riemannian manifolds that we had considered in Theorem 23 (and therefore Corollary 2 as well), we had obtained the following expression for the short-time behaviour of the semigroup generated by the transformed sub-Laplacian:

$$[e^{t\hat{\Delta}^\lambda_{sr}}f](s) = \int_{\Omega^\lambda} [k^\lambda_t(s, r) + t \lim_{n \to \infty} K^\lambda_{t/n}(s, r)]f(r)dr + [O(t^2)f](s). \tag{4.24}$$

For each $t \geq 0$, by the Schwartz kernel theorem, these operators are in fact integral operators. To compute the corresponding integral kernel, consider a sequence of functions $f_n$ with $L^1$ norm constant and equal to 1, and such that $\lim_{n \to \infty} f_n = \delta_{\bar{s}}$ where $\delta_{\bar{s}}$ denotes the Dirac distribution at the point $\bar{s}$. Letting the operator $e^{t\hat{\Delta}^\lambda_{sr}}$ act on this sequence and letting $n$ go to $\infty$, we obtain the value of the integral kernel of this operator at the point $\bar{s}$:

$$Q^\lambda_t(s, \bar{s}) = k^\lambda_t(s, \bar{s}) + t \lim_{n \to \infty} K^\lambda_{t/n}(s, \bar{s}) + O(t^2). \tag{4.25}$$

In the last expression, $O(t^2)$ denotes a function or distribution such that when integrated against a test function, it is an operator of order $O(t^2)$ as defined in Theorem 23. Putting this together with
expression (4.23), we obtain the following expression for the heat kernel:

\[
p_t(g) = \int_G \int_{X^\lambda} \mathcal{X}^\lambda(g) \left[ \hat{Q}_t^\lambda(s, \bar{s}) + \mathcal{O}(t^2) \right] |_{s=\bar{s}} d\bar{s} dP(\lambda).
\] (4.26)

One would now be tempted to separate this integral into two, in order to have an expression of the form:

\[
p_t(g) = \int_G \int_{X^\lambda} \mathcal{X}^\lambda(g) \hat{Q}_t^\lambda(s, \bar{s}) |_{s=\bar{s}} d\bar{s} dP(\lambda) + \int_G \int_{X^\lambda} \mathcal{X}^\lambda(g) \mathcal{O}(t^2) |_{s=\bar{s}} d\bar{s} dP(\lambda).
\]

However, the two resulting integrands will in general not be integrable. Let us consider the following example on \( G = \mathbb{R} \): suppose that starting from a heat equation corresponding to a hypoelliptic operator, we found an expression in short-time for the fundamental solution of the transformed heat equation of the form \( \hat{p}_t(y) = 1 - ty^2 + \mathcal{O}(t^2) \). Taking the inverse Fourier transform, we recover the original heat kernel: \( p_t(x) = \int_\mathbb{R} e^{ixy} (1 - ty^2 + \mathcal{O}(t^2)) dy \). By the hypoellipticity of the original operator, this is an honest smooth function on \( \mathbb{R} \), but notice that this is not the case of the integral \( \int_\mathbb{R} e^{-ixy} (1 - ty^2) dy \), which does not exist as a function. \( \int_\mathbb{R} e^{-ixy} dy \) is in fact interpreted as the Dirac distribution at 0. We can summarize this by saying that whatever convenient properties \( \hat{p}_t(y) \) is endowed with, those same properties don’t translate to the first terms of an asymptotic development for \( \hat{p}_t(y) \).

To work around this obstacle, a first possibility is to define, in a formal way,

\[
\tilde{p}_t(g) = \int_G \int_{X^\lambda} \mathcal{X}^\lambda(g) \tilde{Q}_t^\lambda(s, \bar{s}) |_{s=\bar{s}} d\bar{s} dP(\lambda),
\]

and to give a meaning to this expression in the sense of distributions by applying it to a function \( f \in C_c^\infty(G) \): \( \int_G \tilde{p}_t(g) f(g) dg \). The drawback of this approach is that it might not give much insight into the behaviour of the heat kernel.

An alternative is to choose a new function \( \tilde{Q}_t^\lambda(s, \bar{s}) \) that verifies

\[
\tilde{Q}_t^\lambda(s, \bar{s}) = \hat{Q}_t^\lambda(s, \bar{s}) + \mathcal{O}(t^2),
\]

which will then imply \( Q_t^\lambda(s, \bar{s}) = \tilde{Q}_t^\lambda(s, \bar{s}) + \mathcal{O}(t^2) \), and to choose it in such a way that the function \( \mathcal{X}^\lambda(g) \tilde{Q}_t^\lambda(s, \bar{s}) |_{s=\bar{s}} \) will be integrable, and that the resulting approximation of the heat kernel,

\[
\tilde{p}_t(g) = \int_G \int_{X^\lambda} \mathcal{X}^\lambda(g) \tilde{Q}_t^\lambda(s, \bar{s}) |_{s=\bar{s}} d\bar{s} dP(\lambda),
\]

will be a \( C^\infty \) function on \( G \). If such conditions are verified, since from the hypoellipticity of the
CHAPTER 4. SHORT-TIME BEHAVIOUR OF THE HYPOELLIPTIC HEAT KERNEL

Sub-Laplacian we had that $p_t$ is also a $C^\infty$ function, we obtain that the resulting error term is a well defined function (and is $C^\infty$). To clarify this idea, let us go back to the example we used above. Suppose we had obtained, on $G = \mathbb{R}$, an expression for the hypoelliptic heat kernel of the form: $p_t(x) = \int_{\mathbb{R}} e^{-ixy}(1 - ty^2 + \mathcal{O}(t^2))dy$. Now using that $1 - ty^2 = e^{-ty^2} + \mathcal{O}(t^2)$, we obtain the more useful: $p_t(x) = \int_{\mathbb{R}} e^{-ixy}(e^{-ty^2} + \mathcal{O}(t^2))dy$, which we can now separate:

$$p_t(x) = \int_{\mathbb{R}} e^{-ixy}e^{-ty^2}dy + \int_{\mathbb{R}} e^{ixy}\mathcal{O}(t^2)dy.$$  

By construction, the first integral exists and is a $C^\infty$ function, and, therefore, so is the second one since they add up to a $C^\infty$ function. Moreover, we now obtain a true approximation of the heat kernel: $\tilde{p}_t(x) = \int_{\mathbb{R}} e^{-ixy}e^{-ty^2}dy$, which is easily computable.

In this example, the key idea is that we chose a function $f_t(y)$ under certain constraints, the first one being that it verifies:

$$1 - ty^2 + \mathcal{O}(t^2) = f_t(y) + \mathcal{O}(t^2),$$

i.e that it had a power series expansion at $t = 0$ with two first terms $1 - ty^2$ and the other terms being of order $\mathcal{O}(t^2)$, and the second one being that the function $\int_{\mathbb{R}} e^{-ixy}f_t(y)dy$ be a $C^\infty$ function of $x$, which is equivalent to $f_t$ being a rapidly decreasing function of $y$, i.e.

$$\sup_{y \in \mathbb{R}} |y^k f_t(y)| < \infty, \forall k \in \mathbb{N}.$$  

We therefore chose the obvious function which verified both conditions, but one should keep in mind that infinitely many others would have worked.

In the case of an arbitrary group $G$, it is more difficult to formulate the exact constraints that we have on the function $\tilde{Q}_t^\lambda(s, \bar{s})$ in order to obtain two separate $C^\infty$ functions on $G$. However, keeping in mind the example above, there probably exists an infinite number of possibilities to choose from, each leading to a different approximation of the heat kernel. It is therefore natural to wonder which choice leads to a better estimate.

In the next chapter, especially in the first section, we have tried to illustrate which criteria come into account when making such a decision, and we show how all the theory developed in this chapter can be used to find a final approximation of the hypoelliptic heat kernel on different left-invariant sub-Riemannian manifolds.
Chapter 5

Applications

5.1 Illustration of our approach: the Heisenberg group

In this section we illustrate the main steps of the analysis developed in Chapter 4 by computing the short-time behaviour of the heat kernel on a canonical example: the Heisenberg group. This step-two, 3-dimensional nilpotent Lie group is the simplest non-trivial (i.e not Euclidean nor Riemannian) left-invariant sub-Riemannian manifold, and has been studied extensively. Also, its dual is well-known, and as noted in Chapter 1, its heat kernel has been computed explicitly multiple times. These properties will lead to simple calculations, as well as to a final approximation which can be compared to the exact formula. This will enable us to evaluate the quality of our method. The first four parts of this section follow the work of [ABGR09], whereas the last ones are original.

5.1.1 The Heisenberg group as a left-invariant subRiemannian manifold

The Heisenberg group consists of the $3 \times 3$ upper-triangular matrices of the form

$$H_1 = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$

(5.1)
with the usual matrix multiplication. We can identify the Heisenberg group with $\mathbb{R}^3$ in the following way:

$$(x, y, z) \sim \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$ (5.2)

with the resulting multiplication law:

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + \frac{1}{2}(ay - bx)).$$ (5.3)

From this it is straightforward that the center of the Heisenberg group $H_1$ is:

$$Z(H_1) = \{(0, 0, z) | z \in \mathbb{R}\}$$ (5.4)

By the identification of $H_1$ with $\mathbb{R}^3$, we construct a measure on $H_1$ by considering the Lebesgue measure on $\mathbb{R}^3$:

$$\int_{H_1} f(h) dh = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(a, b, c) da \, db \, dc, \ \forall f \in C_c(H_1).$$ (5.5)

It is possible to show (by a direct calculation) that this is a Haar measure on $H_1$ (i.e. is both right and left invariant). Therefore, the Heisenberg group is unimodular.

### 5.1.2 Sub-Riemannian structure and hypoelliptic Laplacian

Let

$$\gamma(t) = \begin{pmatrix} 1 & \gamma_1(t) & \gamma_3(t) + \frac{1}{2}\gamma_1(t)\gamma_2(t) \\ 0 & 1 & \gamma_2(t) \\ 0 & 0 & 1 \end{pmatrix}$$

denote a curve in $H_1$ defined on an interval $I \subset \mathbb{R}$ such that $0 \in I$ and $\gamma(0) = I_3$. The derivatives of such curves at $0$ are, by definition, elements of the Lie algebra $\mathfrak{g}$ of $H_1$:

$$\frac{d}{dt} \bigg|_{t=0} \begin{pmatrix} 1 & \gamma_1(t) & \gamma_3(t) + \frac{1}{2}\gamma_1(t)\gamma_2(t) \\ 0 & 1 & \gamma_2(t) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \gamma'_1(0) & \gamma'_3(0) \\ 0 & 0 & \gamma'_2(0) \\ 0 & 0 & 0 \end{pmatrix}.$$
We have therefore shown that $g$ can be presented in matrix form by

$$
g = \begin{cases} 
\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 
\end{pmatrix}, & x, y, z \in \mathbb{R} 
\end{cases}.
$$

A natural basis for $g$ is the set $(p_1, p_2, k)$, where

$$
p_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{pmatrix},
$$

where the only non-zero Lie bracket is $[p_1, p_2] = k$. The Lie group $H_1$ is therefore step-two nilpotent (note that it is the free step-2 nilpotent group on two generators). Let us define the vector fields $X_1$ and $X_2$ on $G$ by translating the vectors $p_1$ and $p_2$ on $g$, i.e. $X_i(g) = gp_i$, for $g \in G$, $i = 1, 2$.

Now, we endow $H_1$ with a left-invariant structure $(G, \mathcal{H}, g)$, defined in the following way: $\mathcal{H}(g) = \text{span}\{X_1(g), X_2(g)\}$, and $g$ is such that at every point $g$, $X_1(g)$ and $X_2(g)$ are orthonormal. In matrix notation, with $g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 
\end{pmatrix}$, and $\gamma_1(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 
\end{pmatrix}$ a representative of the equivalence class of curves corresponding to the vector $p_1$, we have

$$
X_1(g) = \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 
\end{pmatrix} \cdot \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 
\end{pmatrix} = \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix} 1 & t + x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 
\end{pmatrix} \sim \left. \frac{d}{dt} \right|_{t=0} (t + x, y, z - \frac{y}{2}t),
$$

which means that in $\mathbb{R}^3$ coordinates, $X_1$ corresponds to the vector field $\partial/\partial x - \frac{y}{2}\partial/\partial z$. Similarly, we find that the vector field $X_2$ corresponds to $\partial/\partial y + \frac{x}{2}\partial/\partial z$. By construction, $X_1$ and $X_2$ are clearly bracket generating.

Because $H_1$ is unimodular and from the results of section 2.2, we know that the intrinsic hypoelliptic Laplacian is in fact the following second order differential operator:

$$
\Delta_{sr} = X_1^2 + X_2^2, \quad (5.6)
$$

where we identified the vector fields $X_i$ with their Lie derivatives $L_{X_i}$.
5.1.3 Dual of $H_1$

The dual $\hat{H}_1$ of $H_1$ was constructed in Section 3.5.2 of Chapter 3. Here we don’t give the same
description of it but rather an equivalent one, in order to follow the work of [ABGR09]. We have

$$\hat{H}_1 = \mathbb{R}^2 \cup \{\pi_t : t \neq 0\},$$

(5.7)

where the one-dimensional representations indexed by $\mathbb{R}^2$ are: $\chi_{a,b}(x,y,z) = e^{i(ax+by)}$, and the representations $\pi_t, t \neq 0$, act on the Hilbert space $L^2(\mathbb{R})$ in the following way:

$$[\pi_t(x,y,z)\phi](\theta) = e^{2\pi it(z+xy/2-y\theta)}\phi(\theta - x).$$

We also need the expression of the Plancherel measure on the Heisenberg group. We refer to
Chapter 12 of [Dei05] for the proof of the following theorem.

**Theorem 24.** The Plancherel measure is zero on the set of one-dimensional representations in $\hat{H}_1$,
and for the resulting ones, indexed by $t \in \mathbb{R}^*$, it is equal to $|t|dt$. Moreover, for every $t \in \mathbb{R}^*$ and
$f \in S(H_1)$, we have that $\pi_t(f)$ is a Hilbert-Schmidt operator:

$$\int_{\mathbb{R}^*} \|\pi_t(f)\|_{HS}^2 t|dt = \int_{H_1} |f(h)|^2 dh.$$ (5.8)

5.1.4 General Fourier Transform

We want to compute the General Fourier Transform of the differential operator $\Delta_{sr}$. As we just
saw, the Plancherel measure on the Heisenberg group is zero on the set of equivalence classes of
one-dimensional representations $\chi_{(a,b)}$, for $(a, b) \in \mathbb{R}^2$, and hence these will vanish when taking the
inverse Generalized Fourier Transform. We therefore compute only the operators $\hat{X}_i^\lambda$, where $\lambda \in \mathbb{R}^*$
indexes the representations $\pi_\lambda$.

Note that from Proposition 4, we can identity $\hat{X}_i^\lambda$ with the operator $d\pi_\lambda(X_i)$ on the representa-
tion space $L^2(\mathbb{R})$ of $\pi_\lambda$, for $i = 1, 2$. We compute, for $\psi \in L^2(\mathbb{R})$,

$$[\hat{X}_1^\lambda \psi](\theta) = \left. \frac{d}{dt} \right|_{t=0} \pi_\lambda(e^{t\mu_1})\psi(\theta),$$

and, as $e^{t\mu_1} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim (t,0,0)$, we have

$$[\hat{X}_1^\lambda \psi](\theta) = \left. \frac{d}{dt} \right|_{t=0} \pi_\lambda(t,0,0)\psi(\theta) = \left. \frac{d}{d\theta} \right|_{\theta=0} \psi(\theta-t) = \frac{d}{d\theta} \psi(\theta).$$
Similarly, $e^{t\psi_2} \sim (0, t, 0)$, and hence
\[ [X^2_2\psi](\theta) = \frac{d}{dt} \bigg|_{t=0} [\pi(0, t, 0)\psi](\theta) = e^{i\lambda(-t\theta)}\psi(\theta) = -i\lambda\theta\psi(\theta). \]

We have obtained the GFT of the hypoelliptic Laplacian:
\[ [\hat{\Delta}^\lambda_{sr}\psi](\theta) = \left[ \frac{d^2}{d\theta^2} - \lambda^2\theta^2 \right] \psi(\theta). \] (5.9)

5.1.5 Approximation

Let us now follow the procedures developed in Section 4.5 to find an approximation in small-time of the semigroups $e^{t\Delta^\lambda_{sr}}$. We start by separating the operator $\hat{\Delta}^\lambda_{sr}$ in the most natural way:
\[ \hat{\Delta}^\lambda_{sr} = A^\lambda + B^\lambda, \]
where $A^\lambda\psi(\theta) = \frac{d^2}{d\theta^2}\psi(\theta)$ and $B^\lambda\psi(\theta) = -\lambda^2\theta^2\psi(\theta)$. We have therefore obtained an operator of the form of Corollary 2, where in the same notation we have $a_0^\lambda = a_1^\lambda = 0$, and $a_2^\lambda = \lambda^2$. Applying it yields the following expression for the operators $e^{t\Delta^\lambda_{sr}}$:
\[ [e^{t\Delta^\lambda_{sr}}f](s) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t}(s-r)^2 \right) \left( 1 - t\frac{\lambda^2}{3}(s^2 + sr + r^2) \right) f(r)dr + [O(t^{3/2})](s), \] (5.10)
and we, therefore, have shown that the operators $e^{t\Delta^\lambda_{sr}}$ are integral operators with integral kernels:
\[ Q^\lambda(t, s, r) = \frac{1}{\sqrt{4\pi it}} \exp \left( -\frac{1}{4t}(s-r)^2 \right) \left( 1 - t\frac{\lambda^2}{3}(s^2 + sr + r^2) \right) + O(t^{3/2}), \]

which, since $\frac{1}{\sqrt{4\pi it}} \exp \left(-\frac{1}{4t}(s-r)^2\right) = O(t^{-1/2})$ and $O(t^{-1/2})O(t^2) = O(t^{3/2})$ we can equivalently write as
\[ Q^\lambda(t, s, r) = \frac{1}{\sqrt{4\pi it}} \exp \left( -\frac{1}{4t}(s-r)^2 \right) \left( 1 - t\frac{\lambda^2}{3}(s^2 + sr + r^2) + O(t^2) \right). \] (5.11)
5.1.6 Inverse Fourier transform

Combining now this expression with Corollary 1, we can formally compute for an element \( g \sim (x, y, z) \in H_1 \):

\[
p_t(g) = \int_R \int_{R^*} \pi_\lambda(g) Q_\lambda^- s, r) \cdot \lambda \exp \left( -\frac{1}{4t} (s - x - r)^2 \right) \left( 1 - t \frac{x^2}{3} (s - x)^2 + (s - x) r + r^2 + O(t^2) \right) \left| \lambda \right| dr d\lambda
\]

\[
= \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) \int_R \int_{R^*} e^{i\lambda (z - y)} \left( 1 - t \frac{x^2}{3} (r - x)^2 + (r - x) r + r^2 + O(t^2) \right) \left| \lambda \right| dr d\lambda
\]

Notice that the expression \( e^{-i\lambda yr} \left( 1 - t \frac{x^2}{3} (3r^2 - 3xr + x^2) \right) \) is not integrable in the variable \( r \), and so we cannot at this point separate the integrals.

As \( 1 - t \frac{x^2}{3} (3r^2 - 3xr + x^2) = 1 - t \frac{x^2}{12} \left( r - \frac{1}{2} x \right)^2 \), let us make the change of variables \( (r - \frac{1}{2} x) \mapsto r \), to obtain the simpler expression

\[
p_t(g) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) \int_R \int_{R^*} e^{i\lambda x} \int_R e^{-i\lambda yr} \left( 1 - t \frac{x^2}{12} \left( r - \frac{1}{2} x \right)^2 + O(t^2) \right) \left| \lambda \right| dr d\lambda.
\]

Here, following the ideas introduced in Section 4.5.3, the most natural decision is to use the equality:

\[
1 - t \frac{x^2}{12} \left( r - \frac{1}{2} x \right)^2 + O(t^2) = \exp \left( -t\lambda^2 \left( \frac{x^2}{12} + r^2 \right) \right) + O(t^2),
\]

which yields

\[
p_t(g) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) \int_R \int_{R^*} e^{i\lambda x} \left[ \exp \left( -t\lambda^2 \left( \frac{x^2}{12} + r^2 \right) \right) + O(t^2) \right] \left| \lambda \right| dr d\lambda,
\]

and this integral can finally be separated into a sum of two. The first one will be

\[
\tilde{p}_t(g) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) \int_R \int_{R^*} e^{i\lambda x} \exp \left( -t\lambda^2 \left( \frac{x^2}{12} + r^2 \right) \right) \left| \lambda \right| dr d\lambda
\]

\[
= \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right) \int_{R^*} e^{i\lambda x} \int_R \exp \left( -t\lambda^2 \frac{x^2}{12} + \frac{3}{2} r^2 \right) \exp \left( -\frac{1}{4t} y^2 \right) \left| \lambda \right| d\lambda
\]

\[
= \frac{1}{2t} \exp \left( -\frac{x^2 + y^2}{4t} \right) \int_{R^*} e^{i\lambda x} \exp \left( -t\lambda^2 \frac{x^2}{12} \right) d\lambda
\]

\[
= \frac{\sqrt{3\pi}}{t^{3/2} |x|} \exp \left( -\frac{x^2 + y^2}{4t} \right) \exp \left( -\frac{3\lambda^2}{t^2} \right),
\]
whereas the second term can be simplified, using the equalities \( t^2 \mathcal{O}(1) = \mathcal{O}(t^2) \) and \( \mathcal{O}(t^{-1/2}) \mathcal{O}(t^2) = \mathcal{O}(t^{3/2}) \):

\[
\text{Err}(g) = \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t} \right) \int_{\mathbb{R}^*} e^{i\lambda z} \exp \left(-t\lambda^2 \frac{x^2}{12} \right) \int_{\mathbb{R}} e^{-i\lambda y} \mathcal{O}(t^2) |\lambda| d\lambda d\lambda
\]

\[
= \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t} \right) t^2 \int_{\mathbb{R}^*} e^{i\lambda z} \exp \left(-t\lambda^2 \frac{x^2}{12} \right) \int_{\mathbb{R}} e^{-i\lambda y} \mathcal{O}(1) |\lambda| d\lambda d\lambda
\]

\[
= \mathcal{O}(t^{3/2}).
\]

We therefore obtained that the small time behaviour of the hypoelliptic heat kernel on the Heisenberg group was given by the expression \( \tilde{p}_t(g) = \tilde{p}_t(g) + \mathcal{O}(t^{3/2}) \), with \( \tilde{p}_t(g) \) given by (5.14).

However, notice that this estimate of the heat kernel is not symmetric in \( x \) and \( y \), which doesn’t reflect the \( SO(2) \) symmetry of the Heisenberg group in \((x, y)\). Also, this expression is defined only for \( x \neq 0 \).

Let us therefore ask ourselves what other choices could have been made in Equation (5.12), and compare the corresponding expressions of \( \tilde{p}_t(g) \). Following the previous remark, we want to choose a new function with the additional constraint that the resulting expression of \( \tilde{p}_t(g) \) is symmetric in \( x \) and \( y \), and is, therefore, a better reflection of the real behaviour of the heat kernel.

We start with a function of the form \( \exp \left( g_1(t) \frac{x^2}{12} + g_2(t) r^2 \right) \), where \( g_1 \) and \( g_2 \) remain to be chosen. The first constraint is of course that the two first terms of its power series expansion at \( t = 0 \) are given by

\[
\exp \left( g_1(t) \frac{x^2}{12} + g_2(t) r^2 \right) = 1 - t\lambda^2 \left( \frac{x^2}{12} + r^2 \right) + \mathcal{O}(t^2),
\]

which translates into \( g_1(0) = g_2(0) = 0 \), and \( g_1'(0) = g_2'(0) = -\lambda^2 \). Let us choose right away the simple \( g_1(t) = -t\lambda^2 \). Then, assuming that \( g_2 \) is a negative function (and strictly negative away from 0), the Inverse Generalized Fourier transform yields:

\[
\tilde{p}_t(g) = \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t} \right) \int_{\mathbb{R}^*} e^{i\lambda z} \exp \left(-t\lambda^2 \frac{x^2}{12} \right) \int_{\mathbb{R}} e^{-i\lambda y} \exp(g_2(t) r^2) |\lambda| d\lambda d\lambda
\]

\[
= \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{x^2}{4t} \right) \int_{\mathbb{R}^*} e^{i\lambda z} \frac{1}{\sqrt{g_2(t)}} \exp \left(-t\lambda^2 \frac{x^2}{12} \right) \exp \left( \frac{\lambda^2 y^2}{4g_2(t)} \right) |\lambda| d\lambda.
\]

Therefore, in order to recover the symmetry in \( x \) and \( y \) in this expression, we have to choose \( g_2 \) such that

\[
\frac{\lambda^2 y^2}{4g_2(t)} = -t\lambda^2 \frac{y^2}{12} - \frac{y^2}{4t},
\]
which results in \( g_2(t) = -\frac{3t\lambda^2}{t^2\lambda^2 + 3} \) (which is indeed negative, and strictly negative away from 0), and gives a final approximation of the heat kernel:

\[
\hat{p}_t(g) = \frac{1}{\sqrt{4t}} \exp\left(-\frac{x^2}{4t}\right) \int_{\mathbb{R}^*} e^{i\lambda x} \frac{1}{\sqrt{3 + t^2\lambda^2}} \exp\left(-\frac{t\lambda^2 x^2}{12t}\right) \frac{1}{|\lambda|} \exp\left(-\frac{y^2(3 + t^2\lambda^2)}{12t}\right) |\lambda| d\lambda
\]

This last expression is a \( C^\infty \) function when \( \sqrt{1 + \frac{t^2\lambda^2}{3}} \exp\left(-\frac{t\lambda^2 x^2 + y^2}{12t}\right) \) is rapidly decreasing, i.e. when \( x^2 + y^2 \neq 0 \). This estimate is therefore valid in a much larger domain than the previous one, and, moreover, this domain is in fact all the points outside of the cut locus of the origin (see for example [Gav77]).

### 5.1.7 Remarks

The evolution equation associated to the transformed subLaplacian given in 5.9 is in fact a Schrödinger equation with quadratic potential, whose fundamental solution is known (see for example [Bea98]). We therefore know that the operators \( e^{t\hat{\Delta}_s} \) are integral operators with integral kernel:

\[
P^\lambda_t(s,r) = \frac{\lambda}{2\pi \sinh(2\lambda t)} \exp\left(-\frac{1}{2} \frac{\lambda \cosh(2\lambda t)}{\sinh(2\lambda t)} (s^2 + r^2) + \frac{\lambda sr}{\sinh(2\lambda t)}\right).
\]

(this is called the “Mehler kernel”). Let us try to compare this exact kernel \( P^\lambda_t(s,r) \) and the short-time expression \( Q^\lambda_t \) we obtained. More precisely, let us try to see if it is possible to obtain \( Q^\lambda_t \) by doing various approximations on \( P^\lambda_t \). For \( t \) small, we have the expressions: \( \cosh(2\lambda t) = 1 + \frac{(2\lambda t)^2}{2} + \mathcal{O}(t^4) \) and \( \sinh(2\lambda t) = 2\lambda t + \frac{(2\lambda t)^3}{3} + \mathcal{O}(t^5) \), which leads to:

\[
\frac{1}{\sinh(2\lambda t)} = \frac{1}{2\lambda t + \frac{(2\lambda t)^3}{3} + \mathcal{O}(t^5)} = \frac{1}{2\lambda t} \left(1 - \frac{(2\lambda t)^2}{3} + \mathcal{O}(t^4)\right) = \frac{1}{2\lambda t} - \frac{2\lambda t}{3} + \mathcal{O}(t^3).
\]

From this we compute the terms

\[
\sqrt{\frac{\lambda}{2\pi \sinh(2\lambda t)}} = \sqrt{\frac{1}{4\pi t} + \mathcal{O}(t)},
\]

\[
-\frac{1}{2} \frac{\lambda \cosh(2\lambda t)}{\sinh(2\lambda t)} (s^2 + r^2) = -\frac{s^2 + r^2}{4t} - \frac{\lambda^2 t(s^2 + r^2)}{3} + \mathcal{O}(t),
\]

\[
\frac{\lambda sr}{\sinh(2\lambda t)} = \frac{sr}{2t} - \frac{2\lambda^2 t sr}{3} + \mathcal{O}(t^3),
\]
and obtain the short-time behaviour
\[ P_\lambda^t(s, r) = \sqrt{\frac{1}{4\pi t} + O(t)} \exp \left( -\frac{1}{4t}(s - r)^2 - \frac{\lambda^2 t (s^2 + r^2)}{3} - \frac{2\lambda^2 t}{3}sr + O(t) \right) \]
\[ = \sqrt{\frac{1}{4\pi t} + O(t)} \exp \left( -\frac{1}{4t}(s - r)^2 \right) \left( 1 - t\lambda^2 (s^2 + 2sr + r^2) + O(t) \right). \] (5.17)

Note that this is very similar to the expression \( Q_\lambda^t(s, r) \) computed in Section 5.1.5:
\[ Q_\lambda^t(s, r) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t}(s - r)^2 \right) \left( 1 - t\lambda^2 (s^2 + sr + r^2) + O(t^2) \right). \]

However, we would mostly like to compare the approximation in small time of the heat kernel found in the previous section
\[ \tilde{p}_t(g) = \frac{1}{2t} \exp \left( -\frac{(x^2 + y^2)}{4t} \right) \int_{\mathbb{R}^*} e^{i\lambda z} \exp \left( -t\lambda^2 x^2 + y^2 \frac{12}{t^2} + t^2 \lambda^2 \right) d\lambda, \]
to the exact formula. (We chose the second approximation, as it also reflects the symmetry in \( x \) and \( y \)). By doing the change of variables: \( \lambda \mapsto \frac{2\tau}{t} \), and noticing that all the terms in the integrand, except \( e^{i\lambda z} \), are even functions of \( \lambda \), we obtain the new expression:
\[ \tilde{p}_t(g) = \frac{1}{t^2} \exp \left( -\frac{(x^2 + y^2)}{4t} \right) \int_{\mathbb{R}^*} \exp \left( -\frac{(x^2 + y^2)}{3t} \tau^2 \right) \sqrt{1 + \frac{4\tau^2}{3}} \cos \left( \frac{2\tau z}{t} \right) d\tau, \] (5.18)
which is highly comparable to the exact formula for the heat kernel (again see for example [Bea98]):
\[ p_t(g) = \frac{1}{(2\pi t)^2} \exp \left( -\frac{x^2 + y^2}{2t \sinh(2\tau)} \right) \cos \left( \frac{2\tau z}{t} \right) d\tau. \] (5.19)

These comparisons, despite not being very rigorous, enable us to conclude that the analysis developed in Chapter 4 can lead to useful approximations in short-time of the hypoelliptic heat kernel.

In the next sections we will present an application of our main theorem and of its corollary to two step-3 nilpotent Lie groups, the Engel and the Cartan groups. For these groups, there exist no formula, exact or not, for the hypoelliptic heat kernel. The first parts of the two next sections follow the work of [BGRar], and we then carry out the method developed in this thesis to compute on each of the groups, to our knowledge, the first explicit short-time expressions for their heat kernels.
5.2 Application to the Engel group

5.2.1 Motivation

Denote by $L_4$ the Lie algebra generated by the four elements $l_1$, $l_2$, $l_3$, and $l_4$, where these generators verify the Lie bracket relations

$$[l_1, l_2] = l_3, [l_1, l_3] = l_4, [l_2, l_3] = [l_2, l_4] = 0.$$  

This Lie algebra is clearly step-3 nilpotent with growth vector $(2, 3, 4)$, and we denote by $G_4$ the unique simply connected Lie group having $L_4$ as its Lie algebra. We will call it the Engel group.

According to [Mon02], this name is due to Cartan, who in a 1901 paper wrote that Engel was the first to classify Pfaffian systems of rank two on four variables (roughly speaking, a Pfaffian system is equivalent to a system of differential equations on a manifold). It is worth noticing that the Engel Lie algebra admits the decomposition

$$L_4 = V_1 \oplus V_2 \oplus V_3,$$

where $V_1 = \mathbb{R}\text{-span}\{l_1, l_2\}$, $V_2 = \mathbb{R}\text{-span}\{l_3\}$ and $V_3 = \mathbb{R}\text{-span}\{l_4\}$, and these spaces satisfy $[V_1, V_2] = V_3$, $[V_1, V_3] = V_4$, all the other brackets being 0. Moreover, $V_1$ Lie-generates all of $L_4$, hence by definition $\mathfrak{g}_4$ is a Carnot group.

5.2.2 The Engel group as a sub-Riemannian manifold

Following the exposition of [BGRar], we can present the Lie algebra $L_4$ as the space of matrices

$$L_4 = \left\{ \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| a_1, \ldots, a_4 \in \mathbb{R} \right\},$$

and define the element $l_i$ as the matrix with components $a_j = \delta_{ij}$, for $i, j \in \{1, 2, 3, 4\}$. These matrices are clearly linearly independent and generate $L_4$, and simple calculations show that they verify the bracket relations described above.

Now we can find the matrix presentation for the elements of the Lie group $\mathfrak{g}_4$ by using the fact that the exponential map for matrix Lie algebras is just the ordinary matrix exponential. Simple
Chapter 2, the intrinsic hypoelliptic Laplacian is the operator:

\[
\exp \begin{pmatrix}
  0 & a_1 & 0 & a_4 \\
  0 & 0 & a_1 & a_3 \\
  0 & 0 & 0 & a_2 \\
  0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
  1 & x_1 & x_1^2/2 & x_4 \\
  0 & 1 & x_1 & x_3 \\
  0 & 0 & 1 & x_2 \\
  0 & 0 & 0 & 1
\end{pmatrix} \sim (x_1, x_2, x_3, x_4), \quad (5.21)
\]

with \(x_1 = a_1, x_2 = a_2, x_3 = a_3 + \frac{1}{2}a_1a_2, \) and \(x_4 = a_4 + \frac{1}{2}a_1a_3 + \frac{1}{4}a_2a_2^2. \) From this we deduce that \( \mathfrak{g}_4 \) is isomorphic to \( \mathbb{R}^4, \) when \( \mathbb{R}^4 \) is endowed with the following operation:

\[
(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2, x_4 + y_4 + x_1y_3 + y_2x_1^2/2). \quad (5.22)
\]

We now define the left-invariant vector fields \( X_1 \) and \( X_2 \) on \( \mathfrak{g}_4, \) by \( X_i(g) = gI_i. \) Since \( I_1 \) corresponds to the equivalence class of the curve \( \gamma_1(t) = \left( \frac{1}{2}t^2, 0, 0, 1 \right), \) we can compute:

\[
X_1(g) = \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix}
  1 & x_1 & x_1^2/2 & x_4 \\
  0 & 1 & x_1 & x_3 \\
  0 & 0 & 1 & x_2 \\
  0 & 0 & 0 & 1
\end{pmatrix}, \quad \left. \frac{d}{dt} \right|_{t=0} \begin{pmatrix}
  1 & t & x_1 & x_4 \\
  0 & 1 & t + x_1 & x_3 \\
  0 & 0 & 1 & x_2 \\
  0 & 0 & 0 & 1
\end{pmatrix}.
\]

Using the isomorphism \( \mathfrak{g}_4 \approx \mathbb{R}^4, \) we obtain an isomorphism of the tangent spaces \( T_g \mathfrak{g}_4 \approx T_{(x_1, x_2, x_3, x_4)} \mathbb{R}^4, \) and using the same notation for vector fields in both we can write explicitly:

\[
X_1(x_1, x_2, x_3, x_4) = \frac{\partial}{\partial x_1}. \quad \text{In the same way, we obtain } X_2(x_1, x_2, x_3, x_4) = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{x_1^2}{2} \frac{\partial}{\partial x_4}.
\]

We construct as usual a left-invariant sub-Riemannian structure on \( \mathfrak{g}_4 \) by defining the distribution \( \mathcal{H} \) such that \( X_1(g) \) and \( X_2(g) \) form an orthonormal basis of \( \mathcal{H}(g), \) for all \( g \in \mathfrak{g}_4. \) Moreover, we clearly have that \( X_1 \) and \( X_2 \) are bracket generating at every point of \( \mathfrak{g}_4. \) Therefore, by the results of Chapter 2, the intrinsic hypoelliptic Laplacian is the operator:

\[
\Delta_{sr} = X_1^2 + X_2^2. \quad (5.23)
\]

5.2.3 Generalized Fourier transform

Before taking the General Fourier Transform of the sub-Laplacian, we start with a study of the harmonic analysis of the Engel group. We computed its dual explicitly in Section 3.5.5 of Chapter 3, but to follow the work of [BGRar] we will use an equivalent description, given by Dixmier ([Dix58]).

**Proposition 10.** (Dixmier, 1958) The dual space of the group \( \mathfrak{g}_4 \) is \( \hat{\mathfrak{g}}_4 = \{ \mathfrak{X}^\lambda \mu | \lambda \neq 0, \mu \in \mathbb{R} \} \)
where each representation $X_{\lambda,\mu}$ acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ endowed with the standard inner product $(f_1, f_2) := \int_{\mathbb{R}} f_1(\theta)\overline{f_2(\theta)} d\theta$ and acts in the following way:

$$\mathcal{H} \rightarrow \mathcal{H}$$

$$X_{\lambda,\mu}(x_1, x_2, x_3, x_4) : \phi(\theta) \mapsto \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 - \lambda x_3 \theta + \frac{\lambda}{2} x_2 \theta^2 \right) \right) \phi(\theta + x_1)$$

The Plancherel measure on $\mathfrak{G}_4$ is the Lebesgue measure on $\mathbb{R}^2$: $dP(\lambda, \mu) = d\lambda d\mu$.

Again, from Chapter 2, recall that we can consider the operators $\hat{X}_1^{\lambda,\mu}$ as operators on $L^2(\mathbb{R})$, the representation space of the $X_{\lambda,\mu}$, by identifying $\hat{X}_1^{\lambda,\mu}(X_i)$. Using the fact that $e^{t_1} \sim (t, 0, 0, 0)$ and $e^{t_2} \sim (0, t, 0, 0)$, for an arbitrary function $\phi \in L^2(\mathbb{R})$, the computations yield:

$$\left[ dX_1^{\lambda,\mu} \phi \right](\theta) = \left[ \frac{d}{dt} \right]_{t=0} X_{\lambda,\mu}(t, 0, 0, 0) \phi(\theta) = \frac{d}{d\theta} \phi(\theta)$$

$$\left[ dX_2^{\lambda,\mu} \phi \right](\theta) = \left[ \frac{d}{dt} \right]_{t=0} X_{\lambda,\mu}(0, t, 0, 0) \phi(\theta) = \frac{d}{d\theta} \exp \left( i \left( -\frac{\mu}{2\lambda} t + \frac{\lambda}{2} t \theta^2 \right) \right) \phi(\theta) = \left( -\frac{i}{2} \lambda + \frac{i}{2} \lambda \theta^2 \right) \phi(\theta)$$

From these we obtain the General Fourier Transform of our sub-Laplacian:

$$\hat{\Delta}_{sr}^{\lambda,\mu} f(\theta) = \left( \frac{d^2}{d\theta^2} - \frac{1}{4} \left( \lambda \theta^2 - \frac{\mu}{\lambda} \right)^2 \right) f(\theta)$$

(5.26)

This operator is the Laplacian with quartic potential (also called the quartic oscilator), and its associated evolution equation is not solvable. However, in the next section we will compute an expression describing the short-time behaviour of the associated fundamental solution by carrying out the theory developed in Chapter 4.

### 5.2.4 Heat kernel on the Engel group

Our main result for the Engel group is the following:

**Theorem 25.** The expression in small time of the hypoelliptic heat kernel $p_t(x_1, x_2, x_3, x_4)$ on the Engel group is given by

$$p_t(x_1, x_2, x_3, x_4) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} x_1^2 \right) \int_{\mathbb{R}^2} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 \right) \right)$$

$$\left[ \int_{\mathbb{R}} \exp \left( i \left( -\lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) e^{-tP_{\lambda}^{(r,x_1)}} dr \right] d\lambda d\mu + O(t^{3/2}),$$

(5.27)
where
\[ P_4(r, x_1) = \frac{\mu^2}{4\lambda^2} - \frac{\mu}{6}((r+x_1)^2+r^2+(r+x_1)r) + \frac{\lambda^2}{20}((r+x_1)^2+r^2+(r+x_1)^2r+(r+x_1)3r+(r+x_1)r^3). \]

Proof. The Engel group endowed with the sub-Riemannian structure we defined is a left-invariant sub-Riemannian manifold verifying the conditions of Corollary 2, with \((\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}\) indexing the elements of the dual and
\[ m = 2, \quad a_0^{\lambda,\mu} = \left( \frac{\mu}{2\lambda} \right)^2, \quad a_2^{\lambda,\mu} = -\frac{\mu}{2}, \quad a_4^{\lambda,\mu} = \frac{\lambda^2}{4}, \quad a_1^{\lambda,\mu} = a_3^{\lambda,\mu} = 0. \]

Let us denote by \(Q_t^{\lambda,\mu}(s, r)\) the integral kernel of the operator \(e^{t\hat{\Delta}_{\lambda,\mu}}\), for which as we said no expression is known, but of which Corollary 2 gives the behaviour in short time:
\[
\left[ e^{t\hat{\Delta}_{\lambda,\mu}} f \right](s) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} (s - r)^2 \right) \left( 1 - a_0^{\lambda,\mu} t - t \sum_{k=1}^{2m} \frac{a_k^{\lambda,\mu}}{k+1} \sum_{i=0}^{k} s^i r^{k-i} \right) f(r) dr + [\mathcal{O}(t^{3/2}) f](s),
\]
\[
= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} (s - r)^2 \right) \left( 1 - \frac{t}{2} \left( \frac{\mu^2}{2\lambda} \right)^2 + \frac{t}{3} \frac{\mu}{2} (s^2 + sr + r^2) - \frac{t}{5} \frac{\lambda^2}{4} (s^4 + s^3 r + s^2 r^2 + sr^3 + r^4) \right) f(r) dr + [\mathcal{O}(t^{3/2}) f](s)
\]
From this, (or directly from formula (4.25)), we find the value of the integral kernel at the point \(s\):
\[
Q_t^{\lambda,\mu}(s, \bar{s}) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} (s - \bar{s})^2 \right) \left( 1 - \frac{t}{2} \left( \frac{\mu^2}{2\lambda} \right)^2 + \frac{t}{6} \frac{\mu}{2} (s^2 + s\bar{s} + \bar{s}^2) - \frac{t}{20} \frac{\lambda^2}{4} (s^4 + s^3 \bar{s} + s^2 \bar{s}^2 + s\bar{s}^3 + \bar{s}^4) + \mathcal{O}(t^{3/2}) \right)
\]
\[
= \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} (s - \bar{s})^2 \right) \left( 1 - \frac{t}{2} \left( \frac{\mu^2}{2\lambda} \right)^2 + \frac{t}{6} \frac{\mu}{2} (s^2 + s\bar{s} + \bar{s}^2) - \frac{t}{20} \frac{\lambda^2}{4} (s^4 + s^3 \bar{s} + s^2 \bar{s}^2 + s\bar{s}^3 + \bar{s}^4) + \mathcal{O}(t^2) \right),
\]
where we used that \(\mathcal{O}(t^{-1/2})\mathcal{O}(t^2) = \mathcal{O}(t^{3/2})\). We can now combine this expression with Corollary 1.
to obtain

\[ p_t(x_1, x_2, x_3, x_4) = \int_{\mathbb{R}^4} \mathcal{X}^{\lambda,\mu}(x_1, x_2, x_3, x_4) Q_t^{\lambda,\mu}(s, r)|_{s=r} \, dr \, d\lambda d\mu \]

\[ = \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 - \lambda x_3 s + \frac{\lambda}{2} x_2 s^2 \right) \right) Q_t^{\lambda,\mu}(s + x_1, r)|_{s=r} \, dr \, d\lambda d\mu \]

\[ = \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 - \lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) \left[ \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} x_1^2 \right) \right] \left( 1 - t \frac{\mu^2}{4\lambda^2} + \frac{\mu}{6} t ((r + x_1)^2 + r^2 + (r + x_1)r) - \frac{\lambda^2}{20} t ((r + x_1)^4 + r^4 + (r + x_1)^2 r^2 \right.

\[ \left. + (r + x_1)^2 r + (r + x_1) r^3) + O(t^2) \right] \right) \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 - \lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) \]

\[ (1 - tP_4(r, x_1) + O(t^2)) \, dr \, d\lambda d\mu, \]

where we denoted \( P_4(r, x_1) \) the degree 4 polynomial

\[ \frac{\mu^2}{4\lambda^2} - \frac{\mu}{6} t ((r + x_1)^2 + r^2 + (r + x_1)r) + \frac{\lambda^2}{20} t ((r + x_1)^4 + r^4 + (r + x_1)^2 r^2 + (r + x_1)^2 r + (r + x_1)^3 r + (r + x_1)r^3). \]

It will be important to notice that the coefficient of \( r^4 \) in the expansion of \( P_4(r, x_1) \) is always positive.

Choosing the standard substitution

\[ 1 - tP_4(r, x_1) + O(t^2) = e^{-tP_4(r, x_1)} + O(t^2), \quad (5.28) \]

we can now separate the heat kernel in the following way:

\[ p_t(x_1, x_2, x_3, x_4) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} x_1^2 \right) \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 \right) \right) \]

\[ \left[ \int_{\mathbb{R}} \exp \left( i \left( -\lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) e^{-tP_4(r, x_1)} \, dr \right] \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 \right) \right) \]

\[ \left[ \int_{\mathbb{R}} \exp \left( i \left( -\lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) O(t^2) \, dr \right] \int_{\mathbb{R}^4} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 \right) \right) \]

By construction the first part is a well-defined \( C^\infty \) function, but as its computation necessitates the use of special functions, we will leave it in this form. Moreover, the second part is by default a well-defined \( C^\infty \) function, and using the same tools as in the case of the Heisenberg group we can show that it is in fact a function of order \( O(t^{3/2}) \).
The estimate we obtain is:
\[
\tilde{p}_t(x_1, x_2, x_3, x_4) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} x_1^2 \right) \int_{\mathbb{R} \times \mathbb{R}} \exp \left( i \left( -\frac{\mu}{2\lambda} x_2 + \lambda x_4 \right) \right) \left[ \int_{\mathbb{R}} \exp \left( i \left( -\lambda x_3 r + \frac{\lambda}{2} x_2 r^2 \right) \right) e^{-tP_4(r, x_1)} dr \right] d\lambda d\mu.
\]

\[\square\]

\section{5.3 Application to the Cartan group}

\subsection{5.3.1 Motivation}

Cartan, in his paper “Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre” (1910), was one of the first to study regular rank-two distributions on 5-dimensional manifolds with growth vector \((2, 3, 5)\), and he described complete local invariants for such distributions. There are infinitely many non-equivalent distributions of this kind, and the most intuitive one is given by the problem of rolling a ball on a table. In fact, one obtains a distribution of growth vector \((2, 3, 5)\) when considering a Riemannian surface rolling over another, in the case where their Gaussian curvatures are different (for more details we refer to the paper by Montgomery and Bor [BM06]).

In this thesis, the Cartan group \(G_5\) will be defined as the unique simply connected Lie group associated to the Lie algebra \(\mathfrak{g}_5\) generated by the elements \(l_1, l_2, l_3, l_4\) and \(l_5\) verifying the Lie brackets relations
\[
[l_1, l_2] = l_3, \quad [l_1, l_3] = l_4, \quad [l_2, l_3] = l_5,
\]
all the other brackets being equal to 0. The Cartan Lie algebra is, therefore, the free nilpotent Lie algebra of step 3 on 2 generators, sometimes denoted \(n_{2,3}\), and it has growth vector \((2, 3, 5)\).
5.3.2 The Cartan group as a sub-Riemannian manifold

Again following the exposition of [BGRar], we can describe the Lie algebra \( \mathfrak{L}_5 \) by the space of \( 8 \times 8 \) matrices of the form

\[
\mathfrak{L}_5 = \left\{ \begin{pmatrix} M_1(a_1, a_2, a_3, a_4) & 0_{4 \times 4} \\ 0_{4 \times 4} & M_2(a_1, a_2, a_3, a_5) \end{pmatrix} \right\},
\]

where

\[
M_1(a_1, a_2, a_3, a_4) = \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad M_2(a_1, a_2, a_3, a_5) = \begin{pmatrix} 0 & a_2 & 0 & a_5 \\ 0 & 0 & a_2 & -a_3 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

From this we can define in a natural way the elements \( l_i \) as the matrices with components \( a_j = \delta_{ij} \), and it is straightforward to check that they generate \( \mathfrak{L}_5 \) and verify the bracket relations described above.

We then find the matrix presentation of the Cartan group by taking the usual matrix exponential of the elements of \( \mathfrak{L}_5 \), and we obtain matrices of the form

\[
\exp \begin{pmatrix} M_1(a_1, a_2, a_3, a_4) & 0_{4 \times 4} \\ 0_{4 \times 4} & M_2(a_1, a_2, a_3, a_5) \end{pmatrix} = \begin{pmatrix} \exp(M_1(a_1, a_2, a_3, a_4)) & 0_{4 \times 4} \\ 0_{4 \times 4} & \exp(M_2(a_1, a_2, a_3, a_5)) \end{pmatrix} = \begin{pmatrix} N_1(x_1, x_2, x_3, x_4) & 0_{4 \times 4} \\ 0_{4 \times 4} & N_2(x_1, x_2, x_3, x_5) \end{pmatrix},
\]

where

\[
N_1(x_1, x_2, x_3, x_4) = \begin{pmatrix} 1 & x_1 & x_1^2/2 & x_4 \\ 0 & 1 & x_3 & x_2 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad N_2(x_1, x_2, x_3, x_5) = \begin{pmatrix} 1 & x_2 & x_2^2/2 & x_5 \\ 0 & 1 & x_2 & -x_3 + x_1 x_2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

with

\[
x_1 = a_1, \quad x_2 = a_2, \quad x_3 = a_3 + \frac{1}{2} a_1 a_2, \quad x_4 = a_4 + \frac{1}{2} a_1 a_3 + \frac{1}{6} a_2 a_1^2, \quad \text{and} \quad x_5 = a_5 + \frac{1}{6} a_1 a_2^2 - \frac{1}{2} a_2 a_3.
\]
CHAPTER 5. APPLICATIONS

We can then define an isomorphism from $\mathfrak{g}_5$ to $\mathbb{R}^5$ by

\[
\begin{pmatrix}
N_1(x_1, x_2, x_3, x_4) & 0_{4 \times 4} \\
0_{4 \times 4} & N_2(x_1, x_2, x_3, x_5)
\end{pmatrix}
\sim
\begin{pmatrix}
x_1, x_2, x_3, x_4, x_5, \\
0, 0, 0, 0, 0
\end{pmatrix},
\]

and endowing $\mathbb{R}^5$ with the operation resulting from the matrix multiplication

\[
(x_1, x_2, x_3, x_4, x_5) \cdot (y_1, y_2, y_3, y_4, y_5)
= (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2, x_4 + y_4 + x_1y_3 + \frac{1}{2}y_2x_1^2, x_5 + y_5 - x_2y_3 + y_1y_2x_2 + \frac{1}{2}y_1x_2^2).
\]

Let us construct the left-invariant vector fields $X_1$ and $X_2$ in the usual way: $X_i(g) = g t_i$, $i = 1, 2$. Choose $\gamma_1(t)$ a representative of the equivalence class of curves in $\mathfrak{g}_5$ that corresponds to the vector $t_1$:

\[
\gamma_1(t) = \begin{pmatrix}
N_1(t) & 0_{4 \times 4} \\
0_{4 \times 4} & N_2(t)
\end{pmatrix}, \text{ where } N_1(t) = \begin{pmatrix}
1 & t & t^2/2 & 0 \\
0 & 1 & t & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \text{ and } N_2(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Since

\[
X_1(g) = \frac{d}{dt} \bigg|_{t=0} \begin{pmatrix}
N_1(x_1, x_2, x_3, x_4) \cdot N_1(t) & 0_{4 \times 4} \\
0_{4 \times 4} & N_2(x_1, x_2, x_3, x_5) \cdot N_2(t)
\end{pmatrix},
\]

we compute the components:

\[
N_1(x_1, x_2, x_3, x_4) \cdot N_1(t) = \begin{pmatrix}
1 & t + x_1 & \frac{(t + x_1)^2}{2} & x_4 \\
0 & 1 & t + x_1 & x_3 \\
0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and conclude that $X_1$ can be identified with the vector field $X_1(x) = \frac{\partial}{\partial x_1} + \frac{x_2^2}{2} \frac{\partial}{\partial x_5}$ on $\mathbb{R}^5$ (with $x = (x_1, x_2, x_3, x_4, x_5)$). In the same way, we can show that $X_2$ corresponds on $\mathbb{R}^5$ to the vector field $X_2(x) = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{x_2^2}{2} \frac{\partial}{\partial x_4}$. 
5.3.3 General Fourier Transform

Again, we refer to Dixmier [Dix58] for the description of the dual of the Cartan group and of its Plancherel measure.

**Proposition 11.** (Dixmier, 1958) The dual space of the group $G$ is $\hat{G} = \{ X_{\lambda,\mu,\nu} | \lambda^2 + \mu^2 \neq 0, \nu \in \mathbb{R} \}$, where the representations $X_{\lambda,\mu,\nu}$ all have representation space $H = L^2(\mathbb{R})$, and act in the following way:

$$H \to H, \quad X_{\lambda,\mu,\nu}(x_1, x_2, x_3, x_4, x_5) : \phi(\theta) \mapsto \exp \left( i A_{x_1, x_2, x_3, x_4, x_5}^{\lambda,\mu,\nu} (\theta) \right) \phi \left( \theta + \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \right),$$

with

$$A_{x_1, x_2, x_3, x_4, x_5}^{\lambda,\mu,\nu}(\theta) = -\frac{1}{2} \frac{\nu}{\lambda^2 + \mu^2} (\mu x_1 - \lambda x_2) + \lambda x_4 + \mu x_5$$

$$-\frac{1}{6} \frac{\mu}{\lambda^2 + \mu^2} (\lambda^2 x_1^3 + 3\lambda \mu x_1^2 x_2 + 3\mu^2 x_1 x_2^2 - \lambda \mu x_2^3)$$

$$+ (\lambda^2 + \mu^2) x_3 \theta + \mu^2 x_1 x_2 \theta + \lambda \mu (x_1^2 - x_2^2) \theta - \frac{1}{2} (\lambda^2 + \mu^2) (\mu x_1 - \lambda x_2) \theta^2.$$ 

The Plancherel measure on $G$ is the Lebesgue measure on $\mathbb{R}^3$: $dP(\lambda, \mu, \nu) = d\lambda d\mu d\nu$.

We can now compute the General Fourier Transform of the vector fields $X_{1}^{\lambda,\mu,\nu}$ and $X_{2}^{\lambda,\mu,\nu}$ by identifying them to the differential of the representations $X_{1,2}^{\lambda,\mu,\nu}$ at $X_1$ and $X_2$. We use the fact that $e^{tX_1} \sim (t, 0, 0, 0, 0)$ and $e^{tX_2} \sim (0, t, 0, 0, 0)$, and for all $\psi \in L^2(\mathbb{R})$, we obtain

$$[\hat{X}_1^{\lambda,\mu,\nu}] \psi(\theta) = [dX_{1,2}^{\lambda,\mu,\nu}(X_1) \psi](\theta)$$

$$= \left. \frac{d}{dt} \right|_{t=0} X_{1,2}^{\lambda,\mu,\nu}(t, 0, 0, 0, 0) \psi(\theta)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \exp \left( i \left( -\frac{1}{2} \frac{\nu}{\lambda^2 + \mu^2} \frac{\mu t}{\lambda^2 + \mu^2} x_1^3 + \lambda \mu t^2 \theta - \frac{1}{2} (\lambda^2 + \mu^2) \mu t^2 \theta \right) \right) \psi \left( \theta + \frac{\lambda t}{\lambda^2 + \mu^2} \right)$$

$$= \left( -i \frac{\nu \mu}{2 \lambda^2 + \mu^2} - i \frac{\lambda^2 + \mu^2}{2 \lambda^2 + \mu^2} \frac{\mu t}{\lambda^2 + \mu^2} \frac{d}{d\theta} \right) \psi(\theta).$$
Using the change of variables $\tau$

Theorem 26. The expression in small time of the hypoelliptic heat kernel $p_t(x_1, x_2, x_3, x_4, x_5)$ on the Cartan group is given by:

$$p_t(x_1, x_2, x_3, x_4, x_5) = \frac{\sqrt{\lambda^2 + \mu^2}}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} \frac{(\lambda x_1 + \mu x_2)^2}{\lambda^2 + \mu^2} \right) \int_{\mathbb{R}} \int_{\lambda^2 + \mu^2 \neq 0} \int_{\mathbb{R}} \exp(iA_{x_1, x_2}(r)) \exp(\lambda P_1(r, x_1, x_2))d\lambda d\mu d\nu + O\left(t^{3/2}\right),$$

where $\lambda = (x_1, x_2, x_3, x_4, x_5)$, and where the expression for $A_{x_1, x_2}$ is given in Proposition 11, and

$$P_1(r, x_1, x_2) = \frac{\nu^2}{4(\lambda^2 + \mu^2)} + \frac{\mu^2}{6(\lambda^2 + \mu^2)}(r^2 + r^2 + \bar{r}r) + \frac{(\lambda^2 + \mu^2)^3}{20}(\bar{r}^4 + r^4 + r^2 + \bar{r}^2 + \bar{r}^2 + \bar{r}^3 + \bar{r}^3).$$
with \( \hat{r} = r + \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \).

**Proof.** The proof of the theorem is very similar to the one of Theorem 25. Using Corollary 2, this time with:

\[
m = 2, \quad a_0^{\lambda,\mu,\nu} = \frac{\nu^2}{4}, \quad a_2^{\lambda,\mu,\nu} = \frac{\nu(\lambda^2 + \mu^2)^2}{2}, \quad a_4^{\lambda,\mu,\nu} = \frac{(\lambda^2 + \mu^2)^4}{4}, \quad a_1^{\lambda,\mu,\nu} = a_3^{\lambda,\mu,\nu} = 0,
\]

and again, using equation 4.25, we compute the small-time behaviour of the integral kernel \( Q^{\lambda,\mu,\nu}_r(\theta, r) \) of the operator \( e^{\tau \hat{\Delta}^{\lambda,\mu,\nu}} \):

\[
Q^{\lambda,\mu,\nu}_r(s, r) = \frac{1}{\sqrt{4\pi \tau}} \exp \left( -\frac{1}{4\tau} (s - r)^2 \right) \left( 1 - \frac{\nu^2}{4} - \frac{\nu(\lambda^2 + \mu^2)^2}{6} (s^2 + r^2 + sr) - \frac{\tau}{5} \frac{(\lambda^2 + \mu^2)^4}{4} (s^4 + r^4 + s^2r^2 + s^3r + sr^3) + O(\tau^{3/2}) \right)
\]

Putting this together with Corollary 1, and using the notation \( x = (x_1, x_2, x_3, x_4, x_5) \), we obtain

\[
p_t(x) = \int_{\mathbb{R}} \int_{\lambda^2 + \mu^2 \neq 0} \int_{\mathbb{R}} \exp \left( i A_x^{\lambda,\mu,\nu}(s) \right) Q^{\lambda,\mu,\nu}_{\frac{s + \lambda x_1 + \mu x_2}{\lambda^2 + \mu^2}, r} \left| s = r \right| \, dr \, d\lambda \, d\mu \, d\nu \]

\[
= \frac{\sqrt{\lambda^2 + \mu^2}}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t} \frac{(\lambda x_1 + \mu x_2)^2}{\lambda^2 + \mu^2} \right) \int_{\mathbb{R}} \int_{\lambda^2 + \mu^2 \neq 0} \int_{\mathbb{R}} \exp \left( i A_x^{\lambda,\mu,\nu}(r) \right) (1 - t P_4(r, x_1, x_2) + O(t^2)) \, dr \, d\lambda \, d\mu \, d\nu,
\]

where we define

\[
P_4(r, x_1, x_2) = \frac{\nu^2}{4(\lambda^2 + \mu^2)} + \frac{\nu}{6} (\lambda^2 + \mu^2) (\hat{r}^2 + r^2 + \hat{r}r) + \frac{(\lambda^2 + \mu^2)^3}{20} (\hat{r}^4 + r^4 + r^2 \hat{r}^2 + \hat{r}^3 r + \hat{r}r^3),
\]

with \( \hat{r} = r + \frac{\lambda x_1 + \mu x_2}{\lambda^2 + \mu^2} \). Finally, we use the equality \( 1 - t P_4(r, x_1, x_2) = \exp(-t P_4(r, x_1, x_2)) + O(t^2) \).

Using the fact that the coefficient of \( r^4 \) in \( P_4(r, x_1, x_2) \) is always positive, we obtain a separable expression for \( p_t(x) \), where the estimate is the one given in Equation 5.30, and the error term is of order \( O(t^{3/2}) \). This concludes the proof of the theorem.

\( \square \)
Chapter 6

Summary and Conclusions

6.1 Summary

In this thesis we have suggested a new approach to compute the short-time behaviour of the hypoelliptic heat kernel corresponding to left-invariant sub-Riemannian structures on type I unimodular Lie groups. Our starting point is a classical method of non-commutative harmonic analysis, used for the first time in a hypoelliptic context in [ABGR09], and combined here with the theory of perturbation of operators.

Our method, summarized in three steps, consists in taking the Generalized Fourier Transform of the hypoelliptic heat equation, using the Trotter-product formula on the resulting family of operator equations to obtain an expression in small-time of their fundamental solutions, and finally carefully applying the Inverse Generalized Fourier Transform to recover the short-time behaviour for the hypoelliptic heat kernel.

Our approach distinguishes itself from the existing short-time behaviour results in the sense that it does not rely on an existing closed form expression of the heat kernel, but only on the harmonic analysis of the underlying Lie group. It can, therefore, lead to short-time estimates for a much larger class of Lie groups than the ones for which we have a closed form expression.

Indeed, we have used our results to compute the short-time behaviour of the hypoelliptic heat kernel on the Engel and Cartan groups, two step-3 nilpotent Lie groups for which there exist, to
our knowledge, no closed-form expression of the heat kernel. This thesis, therefore, contains a first expression in short-time of the hypoelliptic heat kernel on both groups.

6.2 Future Work

The first question that arises from our work is whether there exists a class of Lie groups for which the conditions of Theorem 23 are verified, and to which, therefore, all of our method applies. Indeed, the assumptions within our results in Chapter 4 are conditions on the corresponding transformed subLaplacian $\hat{\Delta}^\lambda_{sr}$, but we would like to translate these to conditions on the underlying Lie group of the left-invariant sub-Riemannian manifold.

Unlike Theorem 23, Corollary 2 gives an explicit behaviour in short-time of the semigroup corresponding to the transformed subLaplacian $\hat{\Delta}^\lambda_{sr}$, and is therefore the most useful result of this thesis. The assumptions for its use are however very strong: the transformed subLaplacian has to be of the very specific form $\Delta + M_q^\lambda$, where $\Delta$ denotes the classic Laplacian on $\mathbb{R}$, and $M_q^\lambda$ is the multiplication operator by a negative polynomial $-(p^\lambda)^2$. As we have seen in Chapter 5, the Heisenberg, Engel and Cartan groups all satisfy this condition. As they are all nilpotent Lie groups, and have a natural sub-Riemannian structure generated by a rank-two distribution, one could venture the hypothesis that these are sufficient conditions to obtain a transformed subLaplacian of the desired form. However, a closer look at Dixmier’s paper [Dix58] forces us to reject it, as it contains counterexamples.

Explicit conditions on the Lie group for it to be eligible for our approach therefore remain to be determined. In future work, we would also like to weaken the assumptions of Theorem 23, in order to ensure a wide class of groups to which our method applies. Furthermore, we would like to work towards obtaining a higher order approximation for the heat kernel on the Engel and Cartan group, as well as apply our method to other Lie groups for which there exist, to this day, very little information on the hypoelliptic heat kernel. Finally, we wish to study if the approach described in this thesis could lead to the construction of parametrices.
Bibliography


BIBLIOGRAPHY


Appendix A

Calculations

A.1 Upper bound on the operators $D_k$

We will want to study the operators $D_k$, $k \geq 2$, in a combinatorial way, and for this purpose let us go back to the composition of operators

$$T(t/n) \cdot \left[ I + \frac{1}{2} (t/n)^2 S(\xi_n)B^2 \right] \cdot \ldots \cdot T(t/n) \cdot \left[ I + \frac{1}{2} (t/n)^2 S(\xi_n)B^2 \right]$$

and recall that we defined the operator $D_k$ as the factor in $t^k$ obtained when expanding this expression. Let us start with the term $D_2$. This term is obtained by summing the operators that result either from choosing the term in $t^2$ in one of the $n$ factors and the identity operator in the remaining ones, or by choosing the term in $t$ in two factors and the identity operator for the remaining ones. The result is the operator

$$D_2 = \frac{1}{2n^2} [T(t/n)S(\xi_n)B^2T(t/n)^{n-1} + T(t/n)^2 S(\xi_n)B^2T(t/n)^{n-2} + \ldots + T(t/n)^n S(\xi_n)B^2]$$

$$+ \frac{1}{n^2} [T(t/n)BT(t/n)BT(t/n)^n - 2 + \ldots + T(t/n)^{n-2} BT(t/n)B]$$

We can see that there are $n$ operators of the first kind, corresponding to the $n$ choices of the factor in which we pick the term of degree 2, and $\binom{n}{2}$ of the second kind. Moreover, notice that each one of these operators is a composition of the operators $S(t)$, $T(t)$ and $B$, where the operator $B$ appears
twice. Using the hypothesis of Proposition 8, we obtain:

\[ \| D_2 f \|_1 \leq \frac{1}{n^2} N(2) \alpha_K^2 \| f \|_1, \]

where \( N(2) = n + \binom{n}{2} \). \( N(k) \) can in fact be viewed as the number of ways to obtain a final sum of \( k \) when picking a number in \( n \) different jars, each containing the numbers 0, 1 and 2. It is important to note that \( N(2) \leq n^2 \). Continuing in the same fashion, we find that

\[ \| D_k f \|_1 \leq \frac{1}{n^k} N(k) \alpha_K^k \| f \|_1, \quad k \in \{0, 1, 2, \ldots, 2n\}. \]

The general expression for \( N(k) \) varies for \( k \) even or odd. For \( k = 2s \), and supposing first that \( k \leq n \), we have

\[ N(k) = \binom{n}{s} + \left( \frac{n}{s-1} \right) \binom{n-1}{2} + \ldots + \left( \frac{n}{1} \right) \binom{n-1}{2s-2} + \left( \frac{n}{2s} \right). \]

Similarly, for \( k = 2s + 1 \) and also supposing that \( k \leq n \), we have the expression

\[ N(k) = \binom{n}{s} \binom{n-1}{1} + \left( \frac{n}{s-1} \right) \binom{n-1}{3} + \ldots + \left( \frac{n}{1} \right) \binom{n-1}{2s-1} + \left( \frac{n}{2s+1} \right). \]

In both cases, if \( k > n \), \( N(k) \) will consist only of some of the first terms of the expression.

We want to prove using induction that \( N(k) \leq n^k \). We have shown that it is true for \( k = 2 \), and let us now assume that it is true for \( k = 2s \). If \( k+1 \leq n \), using the identity: \( \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} \), we have

\[ N(k + 1) = \binom{n}{s} \binom{n-1}{1} + \left( \frac{n}{s-1} \right) \binom{n-1}{3} + \ldots + \left( \frac{n}{1} \right) \binom{n-1}{2s-1} + \left( \frac{n}{2s+1} \right) \]

\[ = \frac{n}{s} (n-1) + \left( \frac{n}{s-1} \right) \binom{n-2}{2} \binom{n-1}{3} + \ldots + \left( \frac{n}{2} \right) \binom{n-1}{2s-1} \binom{n}{2s+1} \]

\[ \leq n \left[ \binom{n}{s} + \left( \frac{n}{s-1} \right) \binom{n-2}{1} + \ldots + \left( \frac{n}{1} \right) \binom{n-2}{2s-1} + \left( \frac{n}{2s+1} \right) \right] \]

\[ \leq n N(k) \leq n^{k+1}. \]

If we now assume that it is true for \( k = 2s + 1 \) and that \( k+1 \leq n \), we obtain that

\[ N(k + 1) = \left( \frac{n}{s+1} \right) + \left( \frac{n}{s} \right) \binom{n}{2} + \ldots + \left( \frac{n}{1} \right) \binom{n}{2s} + \left( \frac{n}{2s+2} \right) \]

\[ = \left( \frac{n}{s+1} \right) + \left( \frac{n}{s} \right) \binom{n-2}{1} \binom{n-1}{2} + \ldots + \left( \frac{n}{2s+1} \right) \binom{n}{2s+2} \]

\[ \leq \left( \frac{n}{(k+1)/2} \right) + \frac{n}{2} N(k) \leq \frac{n^{(k+1)/2}}{2} + \frac{n^{k+1}}{2} \leq n^{k+1}, \]

for \( k \geq 3 \). In the case where \( k+1 > n \), then \( N(k+1) \) will have either the same number of terms as \( N(k) \), or will have one less term. In both scenarios, using the same identity as above together with
the assumption that $N(k) \leq n^k$ will lead to $N(k+1) \leq n^{k+1}$.

This finishes to prove that for all $k \in \{0, 1, \ldots, 2n\}$, $N(k) \leq n^k$, and hence that $\|D(k)f\|_1 \leq \alpha_k^N \|f\|_1$.

### A.2 Proof of Proposition 9

#### A.2.1 Calculation of $I_0$

Recall that we defined

$$I_0(s_0, s_n) = \int_{\mathbb{R}^{n-1}} \cdots \int \exp \left[ -\frac{n}{2t} \left( \sum_{i=1}^{n-1} s_i^2 - s_0 s_1 - s_1 s_2 - \ldots - s_{n-1} s_n \right) \right] ds_{n-1} \ldots ds_1$$

Using a change of variables $\sqrt{\frac{t}{n}} s_i = u_i$, $i = 0, \ldots, n$, we obtain an integral independent of $\frac{t}{n}$:

$$I_0(u_0, u_n) = \left( \frac{t}{n} \right)^{(n-1)/2} \int_{\mathbb{R}^{n-1}} \cdots \int \exp \left[ -\frac{1}{2} \left( \sum_{i=1}^{n-1} u_i^2 - u_0 u_1 - u_1 u_2 - \ldots - u_{n-1} u_n \right) \right] du_{n-1} \ldots du_1.$$ 

This $(n-1)$-dimensional integral is of the form $\int \cdots \int \exp \left( -\frac{1}{2} \bar{x}^T A \bar{x} + \bar{b}^T \bar{x} \right) d\bar{x}$, with $k = n-1$, $\bar{x} = \bar{u}$. Let $A$ be the following clearly positive-definite $(n-1) \times (n-1)$ matrix:

$$A = \begin{bmatrix}
1 & -\frac{1}{2} & 0 & \cdots & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\
0 & -\frac{1}{2} & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & -\frac{1}{2} & 1 \end{bmatrix}$$

and $\bar{b}$ the following vector of length $(n-1)$:

$$\bar{b}^T = \frac{1}{2}[u_0, 0, \ldots, 0, u_n].$$

We can therefore use Lemma 3, for which we first need to compute the determinant of $A$ and its inverse. Denote by $A_k$ the $k \times k$ matrices in the same form as $A$, by $D_k$ its determinant and by $A_k^{-1}$
its inverse. Using induction we find that $D_k = (k+1)/k^k$, and that
\[
A_k^{-1} = \frac{2}{k+1} \begin{bmatrix}
  k & k-1 & k-2 & k-3 & \cdots & 2 & 1 \\
  k-1 & 2(k-1) & 2(k-2) & 2(k-3) & \cdots & 4 & 2 \\
  k-2 & 2(k-2) & 3(k-2) & 3(k-3) & \cdots & 6 & 3 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  2 & 4 & 6 & 8 & \cdots & (k-1)2 & k-1 \\
  1 & 2 & 3 & 4 & \cdots & k-1 & k
\end{bmatrix}.
\]
We therefore have all we need to use Lemma 3. We obtain
\[
I_0(u_0, u_n) = \left( \frac{t}{n} \right)^{(n-1)/2} \sqrt{\frac{(2\pi)^{n-1}}{\det(A)}} \exp \left( \frac{1}{2} \vec{b}^T A^{-1} \vec{b} \right)
= \left( \frac{t}{n} \right)^{(n-1)/2} \sqrt{\frac{(2\pi)^{n-1}}{\det(A_{n-1})}} \exp \left( \frac{1}{2} \vec{b}^T A_{n-1}^{-1} \vec{b} \right)
= \left( \frac{t}{n} \right)^{(n-1)/2} 2^{n-1} \pi^{(n-1)/2} \sqrt{n} \exp \left( \frac{1}{4t} [(n-1)u_0^2 + (n-1)u_n^2 + 2u_0u_n] \right),
\]
and hence by substituting in our original variables we obtain the desired result:
\[
I_0(s_0, s_n) = \left( \frac{t}{n} \right)^{(n-1)/2} 2^{n-1} \pi^{(n-1)/2} \sqrt{n} \exp \left( \frac{1}{4t} [(n-1)s_0^2 + (n-1)s_n^2 + 2s_0s_n] \right).
\]

A.2.2 Calculation of the integrals $I_{i,k}$

Recall that we had defined those integrals as:
\[
I_{i,k}(s_0, s_n) = \int \cdots \int_{\mathbb{R}^{n-1}} \exp \left[ \frac{-n}{2t} \sum_{j=1}^{n-1} s_j^2 - s_0 s_1 - \cdots - s_{n-1} s_n \right] s_i ds_{n-1} \cdots ds_1,
\]
for $i \in \{1,2,\ldots,n-1\}, k \in \{1,2,\ldots,2m\}$, and that we want to prove that they are equal to
\[
I_{i,k}(s_0, s_n) = I_0 \cdot \sum_{j=0}^{k} \frac{k!}{(k-j)!} \frac{1}{(j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{t}{n} \right)^{j/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j}.
\]

Those integrals will involve an integration by parts step, which will always be of the form of the following lemma:

**Lemma 4.** For $d$, $e \in \mathbb{R}^+$,
\[
\int_{-\infty}^{\infty} \exp(-dw^2 + ew) w^m dw = \exp \left( \frac{e^2}{4d} \right) \sqrt{\pi d} \sum_{k=0}^{m} \frac{(m)!}{(m-k)!} \frac{1}{(k/2)!} \frac{1}{2^k} \frac{d^{-k/2}}{m-k} \left( \frac{e}{2d} \right)^{m-k}. \tag{A.1}
\]

**Proof.** Straightforward by using a change of variable and the binomial theorem. \qed
We can separate this integral to get:

\[
I_{i,k}(u_0, u_n) = \left( \frac{t}{n} \right)^{\frac{n-1}{2}} \left( \frac{t}{n} \right)^{\frac{k}{2}} \int_{\mathbb{R}^{n-1}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{n-1} u_i^2 - u_0 u_1 - \ldots - u_{n-1} u_n \right] u^k_i du_{n-1} \ldots du_1
\]

We can see that the \((n-2)\) dimensional integral \(I_{i,k}\) defined above resembles \(I_0\) computed above. Indeed, it is also of the form \(\int \cdots \int \exp \left( -\frac{1}{2} \sum_{j=1}^{n-1} u_j^2 - u_0 u_1 - \ldots - u_{n-1} u_n \right) \) \(du_{n-1} \ldots du_{i+1} du_{i-1} \ldots du_1\) \(du_i\).

We can now use Lemma 3. Notice that our matrix \(A\) is block-diagonal with first block \(A_{i-1}\) and second block \(A_{n-i-1}\), using the same notation for the matrices \(A_k\) as in section A.2.1. By using our previous work on those matrices we get

\[
\det(A) = \det(A_{i-1}) \cdot \det(A_{n-i-1}) = \frac{2^{i-1}}{i} \cdot \frac{2^{n-i-1}}{n-i} = \frac{2^{n-2}}{i(n-i)}
\]

and

\[
A^{-1} = \begin{bmatrix} A_{i-1}^{-1} & 0 \\ 0 & A_{n-i-1}^{-1} \end{bmatrix}.
\]

We are ready to apply Lemma 3. We obtain

\[
J_{i,k} = \frac{2^{n-2} \pi^{(n-2)/2}}{\sqrt{i(n-i)}} \exp \left[ \frac{i-1}{4i} u_0^2 + \frac{n-i-1}{4(n-i)} u_n^2 + \left( \frac{u_0}{2i} + \frac{u_n}{2(n-i)} \right) u_i + \left( \frac{i-1}{4i} + \frac{n-i-1}{4(n-i)} \right) u_i^2 \right].
\]
Using this in our expression for \( I_{i,k} \)
\[
I_{i,k}(u_0, u_n) = \left( \frac{t}{n} \right)^{(n-1)/2} \cdot \left( \frac{t}{n} \right)^{k/2} \cdot \frac{2^{n-2} \pi^{(n-2)/2}}{\sqrt{i(n-i)}} \cdot \exp \left[ \frac{i - 1}{4i} u_0^2 + \frac{n - i - 1}{4(n-i)} u_k^2 \right] 
\]
\[
\int_{\mathbb{R}} \exp \left[ -u_i^2 \left( 1 - \frac{i - 1}{4i} + \frac{n - i - 1}{4(n-i)} \right) + \frac{(n-i)u_0 + iu_n}{2i(n-i)} \right] u_i^k du_i.
\]

Now this last integral is exactly of the form of Lemma 4, with, in the same notation:
\[
d = \frac{1}{2} - \left( \frac{i - 1}{4i} + \frac{n - i - 1}{4(n-i)} \right) = \frac{n}{4i(n-i)},
\]
\[
e = \frac{(n-i)u_0 + iu_n}{2i(n-i)}.
\]

Therefore, applying Lemma 4 and rearranging we get the formula for \( I_{i,k} \):
\[
I_{i,k}(u_0, u_n) = \left( \frac{t}{n} \right)^{(n-1)/2} \cdot \frac{2^{n-1} \pi^{(n-1)/2}}{\sqrt{n}} \cdot \left( \frac{t}{n} \right)^{k/2} \cdot \exp \left[ \frac{1}{4n} [(n-1)u_0^2 + (n-1)u_k^2 + 2u_0u_n] \right]
\]
\[
\sum_{j=0}^{k} \frac{k!}{(k-j)!} \frac{1}{(j/2)!} \frac{1}{2^j} \left( \frac{4i(n-i)}{n} \right)^{j/2} \left( \frac{(n-i)u_0 + iu_n}{n} \right)^{k-j}.
\]

And substituting our original variables we obtain the expression given in proposition 9:
\[
I_{i,k}(s_0, s_n) = I_0 \cdot \left( \frac{t}{n} \right)^{k/2} \cdot \sum_{j=0}^{k} \frac{(k)!}{(k-j)!} \frac{1}{(j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{n}{n} \right)^{(k-j)/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j}
\]
\[
= I_0 \cdot \sum_{j=0}^{k} \frac{(k)!}{(k-j)!} \frac{1}{(j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{t}{n} \right)^{j/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j}.
\]

### A.3 End of the proof of Corollary 2

Recall that we obtained the expression
\[
K_{n,t/n}(s_0, s_n) = \frac{t}{n} \left( \frac{4\pi t}{n} \right)^{-n/2} \exp \left[ -\frac{n}{4t} (s_0^2 + s_n^2) \right]
\]
\[
\left[ (p(s_n)^2 + (n-1)a_0)I_0(s_0, s_n) + \sum_{k=1}^{2m} a_k \sum_{i=1}^{n-1} I_{i,k}(s_0, s_n) \right].
\]
Hence, putting this together with the results from Proposition 9, we find

\[
\lim_{n \to \infty} K_{n,t/n}(s_0, s_n) = \lim_{n \to \infty} \frac{t}{n} \left( \frac{4\pi}{n} \right)^{n/2} \exp \left[ -\frac{n}{4t} (s_0^2 + s_n^2) \right] \\
\left( \frac{1}{n} \right)^{(n-1)/2} 2^{n-1} \pi^{(n-1)/2} \sqrt{n} \exp \left( \frac{1}{4t} [(n-1)s_0^2 + (n-1)s_n^2 + 2s_0s_n] \right) \\
\left[ (n-1)a_0 + p(s_n)^2 + \sum_{k=1}^{2m} a_k \sum_{i=1}^{n-1} \left[ \left( \frac{(n-i)s_0 + is_n}{n} \right)^k + g_k(i,n,s_0,s_n) \right] \right],
\]

where this function \(g\) is defined by

\[
g_k(i,n,s_0,s_n) = \sum_{j=2}^{k} \frac{(k)!}{(k-j)! (j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{t}{n} \right)^{j/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j},
\]

and groups together all the terms in \(I_{i,k}(s_0,s_n)/I_0(s_0,s_n)\) that can be factored by \(t\). Therefore, we have obtained

\[
\lim_{n \to \infty} K_{n,t/n}(s_0, s_n) = \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{1}{4t} (s_0 - s_n)^2 \right] \\
\left[ a_0 + \sum_{k=1}^{2m} a_k \lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k} \right] \\
+ \sum_{k=1}^{2m} a_k \lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} g_k(i,n,s_0,s_n),
\]

and all that is left is to compute are the sums in this expression. We will need the following proposition.

**Proposition 12. (Faulhaber’s formula)** We have for any \(p \in \mathbb{N}\):

\[
\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{i+p} B_{p+1-i} n^i,
\]

where \(B_i\) is the \(i\)th Bernoulli number.

Notice that as \(B_0 = 1\), \(\sum_{k=1}^{n} k^p\) is a polynomial in \(n\) of degree \(p + 1\), where the coefficient of the
highest order term is \( \frac{1}{p+1} \). We have

\[
\sum_{i=1}^{n-1} [(n - i)s_0 + is_n]^k = \sum_{i=1}^{n-1} \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} (n - i)^h (n - i)^{k-h}
\]

\[
= \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} \sum_{i=1}^{n-1} (n - i)^h (n - i)^{k-h}
\]

\[
= \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} \sum_{i=1}^{n-1} \sum_{l=0}^{h} \binom{h}{l} (-1)^{h-l} n^l t^{h-l} (n - i)^{k-h}
\]

\[
= \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} \sum_{l=0}^{h} \binom{h}{l} (-1)^{h-l} n^l \sum_{i=1}^{n-1} t^{h-l}.
\]

This term \( n! \sum_{i=1}^{n-1} t^{k-l} \) is, by Faulhaber's formula, equal to a polynomial in \( n - 1 \) (hence in \( n \)) of degree \( k + 1 \), where the term of degree \( k + 1 \) has a coefficient \( \frac{1}{k - l + 1} \). Notice that we had a factor of \( \frac{1}{n^{k+1}} \) multiplying this term, so when taking the limit the only term that remains is \( \frac{1}{k - l + 1} \).

Explicitly

\[
\lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} \left( \frac{(n - i)s_0 + is_n}{n} \right)^k = \frac{t}{n} \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} \sum_{l=0}^{h} \binom{h}{l} (-1)^{h-l} \frac{1}{k - l + 1}
\]

\[
= \frac{t}{n} \sum_{h=0}^{k} \binom{k}{h} s_0^h s_n^{k-h} \frac{1}{k - l + 1}.
\]

In this last expression we used the equality

\[
\sum_{l=0}^{h} \binom{h}{l} (-1)^{h-l} \frac{1}{k - l + 1} = \frac{1}{k + 1},
\]

which can be deduced from

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{i - \beta} = \frac{k! \Gamma(-\beta)}{\Gamma(k + 1 - \beta)}
\]

(see [Gar07]).

The computation of the remaining terms can be done in a similar fashion, but this time we are not interested in an exact expression but only in an upper bound. We will start by proving that

\[
\sum_{k=1}^{2m} a_k \lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} g_k(i, n, s_0, s_n) = t^2 \sum_{k=1}^{2m} a_k P_k(t, s_0, s_n),
\]
where \( P_k(t, s_0, s_n) \) is a polynomial in \( t, s_0 \) and \( s_n \). Indeed, for each \( j \) even, \( 2 \leq j \leq k \), we have

\[
\lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} \frac{(k)!}{(k-j)!} \frac{1}{(j/2)!} \left( \frac{i(n-i)}{n} \right)^{j/2} \left( \frac{t}{n} \right)^{j/2} \left( \frac{(n-i)s_0 + is_n}{n} \right)^{k-j} = t^{(j+2)/2} \frac{(k)!}{(k-j)!} \frac{1}{(j/2)!} \lim_{n \to \infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n-1} (i(n-i))^{j/2}((n-i)s_0 + is_n)^{k-j} = t^{(j+2)/2} P_{k,j}(s_0, s_n),
\]

where this \( P_{k,j} \) is a polynomial in \( s_0 \) and \( s_n \), independent of \( t \), and is what remains after taking the limit. To show that this limit exists, we detail the computation of \( P_{k,2} \):

\[
\sum_{i=1}^{n-1} i(n-i)((n-i)s_0 + is_n)^{k-2} = \sum_{i=1}^{n-1} i(n-i) \sum_{h=0}^{k-2} \binom{k-2}{h} s_0^h s_n^{k-2-h} (n-i)^h t^{k-2-h} = \sum_{h=0}^{k-2} \binom{k-2}{h} s_0^h s_n^{k-2-h} \sum_{i=1}^{n-1} i^{k-1-h} (n-i)^{h+1} = \sum_{h=0}^{k-2} \binom{k-2}{h} s_0^h s_n^{k-2-h} \sum_{i=1}^{n-1} i^{k-1-h} \sum_{l=0}^{h+1} \binom{h+1}{l} (-1)^{h+1-l} i^{h+1-l} = \sum_{h=0}^{k-2} \binom{k-2}{h} s_0^h s_n^{k-2-h} \sum_{l=0}^{h+1} \binom{h+1}{l} (-1)^{h+1-l} \sum_{i=1}^{n-1} i^{k-l}.
\]

Again, we can use Faulhaber’s formula to conclude that \( n^{l} \sum_{i=1}^{n-1} i^{k-l} \) is a polynomial in \((n-1)/(h+1)\) of degree \( k+1 \), where the coefficient of \( n^{k+1} \) is \( 1/(k-l+1) \). Therefore we obtain the expression for \( P_{k,2} \):

\[
P_{k,2}(s_0, s_n) = k(k-1) \sum_{h=0}^{k-2} \binom{k-2}{h} s_0^h s_n^{k-2-h} \sum_{l=0}^{h+1} \binom{h+1}{l} (-1)^{h+1-l} \frac{1}{k-l+1},
\]

which is a polynomial in \( s_0 \) and \( s_n \). From this we can conclude:

\[
\sum_{k=1}^{2m} a_k \lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^{n-1} g_k(i, n, s_0, s_n) = \sum_{k=1}^{2m} a_k \sum_{j=2}^{k} t^{(j+2)/2} P_{k,j}(s_0, s_n) = t^2 \sum_{k=1}^{2m} a_k \sum_{j=2}^{k} t^{(j-2)/2} P_{k,j}(s_0, s_n)
\]

where this expression is what remains after taking the limit.

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We have therefore shown that
\[
\lim_{n \to \infty} K_{n, t/n}(s_0, s_n) = \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{1}{4t} (s_0 - s_n)^2 \right] \left[ a_0 t + \sum_{k=1}^{2m} a_k t \sum_{h=0}^{k} s_0^h s_n^{k-h} \frac{1}{k+1} + t^2 \sum_{k=1}^{2m} a_k P_k(t, s_0, s_n) \right].
\]

However notice that for every \( k \), the function \( \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{1}{4t} (s_0 - s_n)^2 \right] t^2 P_k(t, s_0, s_n) \) is in \( L^2(\mathbb{R}^2) \), and that the operator obtained by integrating this function against a test function in the variable \( s_n \) would be an operator of order \( \mathcal{O}(t^{3/2}) \) as defined in Theorem 23. We will say that this function \( \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{1}{4t} (s_0 - s_n)^2 \right] t^2 P_k(t, s_0, s_n) \) is of order \( \mathcal{O}(t^{3/2}) \). We have therefore obtained another expression:
\[
\lim_{n \to \infty} K_{n, t/n}(s_0, s_n) = \frac{1}{\sqrt{4\pi t}} \exp \left[ -\frac{1}{4t} (s_0 - s_n)^2 \right] \left[ a_0 t + \sum_{k=1}^{2m} a_k t \sum_{h=0}^{k} s_0^h s_n^{k-h} \frac{1}{k+1} \right] + \mathcal{O}(t^{3/2}).
\]

Thus, we can combine the error term \( \mathcal{O}(t^{3/2}) \) obtained here to the \( \mathcal{O}(t^2) \) resulting from the application of Theorem 23, and use the fact that \( \mathcal{O}(t^{3/2}) + \mathcal{O}(t^2) = \mathcal{O}(t^{3/2}) \). This concludes the proof of Corollary 2.