Abstract

The present thesis is devoted to an analysis of the possibility of Bose condensates supporting persistent currents in the ring geometry. Our analysis is based on an approach developed by F. Bloch [1] which focuses on the ground state energy of the condensate as a function of its angular momentum $L$. According to this approach, persistent currents are stable if the energy exhibits a local minimum at some non-zero angular momentum. We have used this approach for a single-species gas within a mean-field approximation to show that persistent currents are stable at integral multiples of $N\hbar$, where $N$ is the number of atoms in the system, provided a certain interaction parameter exceeds some critical value. These results are extended to a binary mixture of bosonic atoms and we show that the system is still capable of supporting persistent currents under certain conditions. Some of our conclusions contradict those appearing in the earlier literature.
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Chapter 1

Introduction

Bose-Einstein condensation, commonly abbreviated to BEC, is a phenomenon occurring in gases obeying Bose statistics at temperatures close to absolute zero. Below a certain critical temperature, a finite fraction of particles in a gas find themselves in the lowest quantum state. As a result, the gas effectively behaves as a coherent entity and quantum phenomena become observable on a macroscopic scale [2]. Due to its unique properties, a BEC is sometimes regarded as a fifth state of matter alongside the conventional states of matter of gases, liquids, solids and plasmas [3].

The conceptual origin of BEC dates back to 1924. That year S.N. Bose published a paper on statistics of photons [4]. A. Einstein then generalized Bose’s ideas and applied them to massive particles. He then showed that below a certain temperature a gas of non-interacting particles undergoes a phase transition, as a result of which, atoms condense into the lowest quantum state. From a modern perspective, this phase transition is a consequence of the effects of quantum statistics which had not even been developed at the time of Einstein’s publication. Nor was there an understanding that all particles in nature belong to one of two groups, fermions or bosons, which
determine the kinds of quantum states accessible to a system of identical particles.

Even though Einstein’s prediction was made for non-interacting gases, BEC also occurs in systems with strong interparticle interactions, for instance, liquid $^4$He. After the discovery of superfluidity in liquid $^4$He in 1938, F. London suggested that BEC was responsible for this strange phenomenon [5]. However, strong interactions in liquid $^4$He obscure the properties one would associate with BEC. For example, these interactions reduce the occupancy of the lowest quantum state, the $p = 0$ state, which is found experimentally to contain only 10 percent of all particles [6]. Nevertheless, later theoretical work made it clear that BEC and superfluidity are intimately related. This deeper understanding arose in the 1950s and 60s following the development of many-body theories of strongly interacting Bose liquids [7].

Interest in achieving BEC in systems other than liquid $^4$He was the motivation for studying trapped atomic gases. However many technical challenges stood in the way of reaching the extremely low temperatures that were needed to achieve BEC in these dilute systems. Finally, in 1995 using modern trapping and cooling techniques two independent experimental teams observed BEC in $^{87}$Rb [8] and $^{23}$Na [9]. In the same year existence of BEC in $^7$Li was also reported [10], but due to the attractive interactions in $^7$Li it was not quite clear that BEC was actually occurring. Without question, these experimental achievements were a significant breakthrough and have led to a whole new field of both theoretical and experimental research.

### 1.1 Ideal Bose gas

To introduce the basic ideas of BEC it is useful to consider a non-interacting Bose gas in a box of volume $V$. In the absence of an external potential the Hamiltonian is
simply
\[ \hat{H} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2M}, \tag{1.1.1} \]
which accounts for the kinetic energy of the particles. The total number of particles in the gas is \( N \) and the mass of each particle is \( M \). The quantum states of the system are the stationary state solutions of the Schrödinger equation
\[ \hat{H}\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N), \tag{1.1.2} \]
where \( \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \) is the many-particle wave functions. For the Hamiltonian in (1.1.1) this wave function is a symmetrized product of single-particle states which are the momentum eigenstates \( \phi_k(\mathbf{r}) = e^{\mathbf{k} \cdot \mathbf{r}}/\sqrt{V} \) with energy \( \varepsilon_k = \hbar^2 k^2/2M \). Such a many-body state can be described by a set of occupation numbers \( \{n_k\} \) in terms of which the energy of the state is given by \( E = \sum_k n_k\varepsilon_k \).

In thermal equilibrium, the mean occupation number of the \( k \)-th state is given by the Bose distribution \[ \bar{n}_k = \frac{1}{e^{(\varepsilon_k - \mu)/kT} - 1}, \tag{1.1.3} \]
where \( T \) is the temperature and \( \mu \) is chemical potential. The latter parameter must be chosen to ensure that the total number of particles is given by \( N = \sum_k \bar{n}_k \).

For a given temperature \( T \), the maximum value of \( \sum_k \bar{n}_k \) occurs for \( \mu = 0 \). The temperature at which this maximum value equals \( N \) defines the critical temperature \( T_c \) given by
\[ kT_c = \frac{2\pi\hbar^2}{M} \left( \frac{N}{V g_{3/2}(1)} \right)^{2/3}, \tag{1.1.4} \]
where \( g_{3/2} \) is the Bose function
\[ g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} dx \, x^{1/2} \frac{1}{z^{-1} e^x - 1}. \tag{1.1.5} \]
Below this temperature a finite fraction of the particles reside in the $p = 0$ state. This fraction is given by

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}.$$  (1.1.6)

A similar calculation \cite{12} can be carried out for an ideal Bose gas confined in a harmonic trap of the form

$$V(r) = \frac{1}{2}M\omega_x x^2 + \frac{1}{2}M\omega_y y^2 + \frac{1}{2}M\omega_z z^2,$$  (1.1.7)

where $\omega_i$ are the oscillation frequencies for each of the directions. The single-particle energies in this case are given by

$$\epsilon_{n_x n_y n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z,$$  (1.1.8)

where the numbers $n_i = 0, 1, 2...$ are the harmonic oscillator quantum numbers. Using these energies in the Bose-Einstein distribution, and following the same arguments for the uniform case, one finds that the fraction of particles in the lowest quantum state is given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3,$$  (1.1.9)

where the transition temperature in this case is given by

$$kT_c = 0.94\hbar\omega_{ho} N^{1/3},$$  (1.1.10)

where $\omega_{ho} = (\omega_x\omega_y\omega_z)^{1/3}$ is the geometric mean of oscillation frequencies \cite{12}. We observe that the dependence on the ratio $T/T_c$ is different from that of the uniform gas. A comparison of the two dependences is shown in Fig. 1.1.1. In spite of this difference, the phenomenon of condensing into the lowest quantum state is essentially the same.
1.2 The Inclusion of interactions

The interaction between particles modifies the properties of Bose gas from those described in the previous section. One of the earliest treatments of this problem was provided by N.N. Bogoliubov in 1947 who considered a weakly-interacting Bose gas \[13\]. Introducing the idea of Bose-broken symmetry, he was able to develop a new perturbation technique to determine the properties of the system.

When the gas is dilute, the effect of interactions arises as a result of binary collisions. At low energies, the collision cross-section is characterized by the s-wave scattering length \(a_s\). In this situation it is possible to represent the interactions in terms of a pseudopotential of the form

\[
v_p(r) = U\delta(r),
\]

(1.2.1)
where the strength of interactions is given by

$$U = \frac{4\pi a_s \hbar^2}{M}. \quad (1.2.2)$$

This dilute gas of particles can then be represented by the Hamiltonian

$$\hat{H} = \int d^3 r \frac{\hbar^2}{2M} \nabla \hat{\psi} \nabla \hat{\psi} + V(r) \hat{\psi}^\dagger(r) \hat{\psi}(r) + \frac{U}{2} \int d^3 r \ \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r), \quad (1.2.3)$$

where \( V(r) \) is the confining potential. Here \( \hat{\psi}(r)(\hat{\psi}^\dagger(r)) \) is the field operator destroying (creating) a particle at the position \( r \).

For a Bose-condensed gas the field operator \( \hat{\psi}(r) \) in \((1.2.3)\) can effectively be replaced by a classical field \( \Phi(r) \) which is called the order parameter or the condensate wave function. The argument for this replacement is based on a variational estimate of the ground state energy of the many-particle system. The wave function

$$\Psi(r_1, ... r_N) = \prod_{i=1}^{N} \phi(r_i) \quad (1.2.4)$$

represents a state in which all the particles are in the same quantum state \( \phi(r) \). The expectation value of \((1.2.3)\) for this state gives the energy

$$E = N \int dr \left[ \frac{\hbar^2}{2M} |\nabla \phi(r)|^2 + V(r)|\phi(r)|^2 + \frac{(N-1)}{2} U|\phi(r)|^4 \right]. \quad (1.2.5)$$

If we now introduce the macroscopic wave function

$$\Phi(r) = \sqrt{N} \phi(r), \quad (1.2.6)$$

the energy becomes

$$E[\Phi] = \int dr \left[ \frac{\hbar^2}{2M} |\nabla \Phi(r)|^2 + V(r)|\Phi(r)|^2 + \frac{1}{2} U|\Phi(r)|^4 \right], \quad (1.2.7)$$

where we have replaced \( N - 1 \) by \( N \) which is valid for large \( N \).
We see that this energy functional follows from (1.2.3) with the replacement of the field operator \( \hat{\psi}(r) \) by the c-number function \( \Phi(r) \). It is for this reason that \( \Phi(r) \) is referred to as a classical field.

To this point, the single-particle state \( \phi(r) \) has not been specified. To determine it, and hence \( \Phi(r) \), we minimize (1.2.7) with respect to \( \Phi(r) \) subject to the constraint

\[
\int dr |\Phi(r)|^2 = N. \tag{1.2.8}
\]

This minimization can be carried out by introducing a Lagrange multiplier \( \mu \) which plays the role of the chemical potential. We thus obtain the equation

\[
-\frac{\hbar^2}{2M} \nabla^2 \Phi(r) + V(r)\Phi(r) + U|\Phi(r)|^2\Phi(r) = \mu \Phi(r), \tag{1.2.9}
\]

which is known as the Gross-Pitaevskii (GP) equation. As we can see, (1.2.9) is similar to the Schrödinger equation for a single particle but is non-linear due to interactions between particles. The GP equation plays an important role in the theory of BEC since it accurately describes the properties of a dilute gas of bosons at zero temperature [6].

### 1.3 Superfluidity and persistent currents

As mentioned earlier, the phenomenon of superfluidity is associated with Bose-Einstein condensation. It was first observed in liquid \(^4\text{He}\) in 1938 by P. Kapitza [14] and J. Allen, and D. Misener [15]. One dramatic manifestation of superfluidity is the observation that a sample of boiling liquid helium suddenly becomes quiescent when its temperature is reduced below 2.17K, the so-called lambda-point. It is now understood that this behaviour is due to the fact that the thermal conductivity of the liquid becomes very large when the superfluid phase is entered.
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Another, more direct, manifestation of superfluidity is the observation that a superfluid can flow without dissipation through very narrow capillaries [16]; in effect, the superfluid exhibits zero viscosity. This phenomenon can be understood on the basis of Landau’s two-fluid model of liquid helium. In the superfluid phase, the liquid is postulated to consist of two-components, a normal liquid which is viscous, and a superfluid component which has zero viscosity. When flowing through a narrow capillary, the normal fluid ‘sticks’ to, and remains at rest relative to the walls. The superfluid, however, moves past the walls with zero resistance provided its velocity does not exceed a critical velocity $v_{cr}$. Landau also derived a criterion determining this critical velocity which is associated with the elementary excitations of the superfluid. This criterion revealed that the interaction between the atoms in the liquid is essential for the existence of superfluidity; an ideal Bose gas, even when Bose-condensed, cannot exhibit superfluidity.

One can imagine that, if a capillary through which a superfluid is flowing is bent into a torus or ring, the flow will continue indefinitely. This is what one means
by persistent currents. Recently, stable persistent flow of $^{23}\text{Na}$ atoms for about 40 seconds has been observed [17] in a toroidal all-optical trap using Laguerre-Gauss beams. The possibility of creating a toroidal trap in this way was first suggested theoretically [18, 19] and was subsequently realized experimentally [20, 21]. These kinds of experiments allow one, in principle, to investigate the properties of persistent currents in detail.

A central question regarding persistent currents concerns the conditions under which they are stable. F. Bloch was one of the first to address this question theoretically [1]. He considered an idealized model in which an interacting Bose gas is confined in a ring geometry. The confinement of the atoms in the transverse direction is so strong that the motion of the atoms is effectively one-dimensional. Using quite general arguments, Bloch was able to show that the eigenenergies of the system take the form

\[ E_n(L) = \frac{L^2}{2MR^2} + \epsilon_n(L), \]

where $L$ is the angular momentum of the eigenstate $n$ and $R$ is the radius of the ring. The first term on the right-hand side is the kinetic energy of the collective flow around the ring and the second term includes the effect of interactions. He proved that $\epsilon_n(L)$ has the following important properties: (i) that it is an even function of $L$ and (ii) that it is a periodic function of $L$ with period $N\hbar$, where $N$ is the number of atoms in the ring. He then formulated a criterion for the stability of persistent currents in terms of the behaviour $E_n(L)$ as a function of $L$: persistent currents are stable if $E_n(L)$ exhibits a local minimum at some finite value of $L$.

The issue of stability in the ring geometry actually takes several forms. Most of the discussions appearing in the literature are based on the Gross-Pitaevskii energy
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For repulsive interactions, the ground state in the ring geometry has a uniform density. One can show that any arbitrary nonuniform perturbation of this density leads to an increase in energy. This system is then said to be energetically stable. The situation for attractive interactions, however, is different. If the interactions are sufficiently attractive, one finds that the uniform state is energetically unstable; the ground state in this case is nonuniform [22].

The question about energetic stability can also be investigated in a frame of reference rotating with an angular frequency $\Omega$. The ground state of the system in the rotating frame can now have a non-zero angular momentum which takes on values which are integral multiples of $N\hbar$ for different values of the angular momentum. These ground states carry a current but are still spatially uniform when the interactions are again sufficiently attractive. However, there is a critical value of the attractive interaction strength beyond which the system becomes nonuniform. This critical value is a function of $\Omega$ [22, 23].

Another kind of stability is referred to as dynamic stability. Here one starts with a uniform ground state and considers the collective excitations of the system. If these excitations have an imaginary frequency, the amplitude of the excitations grows with time, signalling an instability. As we explain in more detail in Chapter 2, the criteria for energetic and dynamic instabilities are the same in the ring geometry [24].

Following Bloch’s argument, the stability of persistent currents is based on the energy of the excited state having a definite, but arbitrary value of the angular momentum. As such, these states cannot be determined by calculating the ground state energy in the rotating frame. Rather, one must minimize the GP energy functional with the constraint that the angular momentum has some prescribed value. This
is a different kind of minimization procedure which was first performed by G.M. Kavoulakis [25]. Such a calculation determines the critical value of the strength of the repulsive interactions above which persistent currents are possible.

The possibility of persistent currents in mixtures of two different species is also of interest. This problem was addressed in [24], generalizing the earlier calculation of Kavoulakis [25]. This paper is the motivation for the work done in the present thesis. To understand this paper required far more effort than was originally anticipated. Many of the results in [24] are stated without explanation and required considerable work to verify. In the process, several new results have been obtained which extend this earlier work. Most importantly, some of the conclusions in [24] turn out to be erroneous.

1.4 Overview

In this thesis we explore the single-species and two-species systems in parallel. In Chapter 2 we study the energetic and dynamic stability of these systems in detail. In the case of the mixture, we generalize the earlier work by considering two species with different masses, $M_A$ and $M_B$. Chapter 3 is devoted to a formal discussion of the stability of persistent currents following Bloch’s approach. His arguments are extended to the two-species case and we find that the behaviour of the energy as a function of $L$ takes a form different from the one found by Bloch, except when $M_A = M_B$. This latter case is analyzed in detail in Chapter 4. As stated earlier, some of the results presented here can be found in [24], but many of the details are original and extend this earlier work. Most importantly, we find that persistent currents in mixtures are stable over a considerably broader range of parameters than is claimed.
in [24]. In particular, we do not confirm the sensitivity of persistent currents found in [24] to the admixture of even small amounts of the minority component. All of our results are summarized in the concluding chapter.
Chapter 2

General Theoretical Background

2.1 Reduction of the Gross-Pitaevskii energy functional to a 1D form

We start by considering a cylindrical trap of length $L$ with harmonic confinement in the transverse direction defined by the potential $V_{\text{trap}}(\rho) = \frac{1}{2} M \omega_{\perp}^2 \rho^2$. For the case of a single-species Bose gas, the energy functional has the following form

$$E[\Phi] = \int d^3r \frac{\hbar^2}{2M} |\nabla \Phi|^2 + \int d^3r V_{\text{trap}}(\rho)|\Phi|^2 + \frac{1}{2} U \int d^3r |\Phi|^4, \quad (2.1.1)$$

where the interaction strength $U$ is given by

$$U = \frac{4\pi \hbar^2 a_s}{M}. \quad (2.1.2)$$

Here, $a_s$ is the s-wave scattering length and $M$ is the particle mass. The first two terms in (2.1.1) are, respectively, the kinetic energy and the potential energy due to the confinement in the transverse direction with frequency $\omega_{\perp}$. The last term represents the interaction between the particles.
The minimization of (2.1.1) with respect to $\Phi$, with the normalization constraint

$$\int d^3r |\Phi|^2 = N, \quad (2.1.3)$$

leads to the Gross-Pitaevskii (GP) equation

$$-\frac{\hbar^2}{2M} \nabla^2 \Phi + \frac{1}{2}M\omega_\perp^2 \rho^2 \Phi + U|\Phi|^2 \Phi = \mu \Phi. \quad (2.1.4)$$

Since the potential does not vary along the length $L$ of the cylinder, the ground state condensate wave function depends only on the transverse coordinate $\rho$. When the transverse confinement is sufficiently strong the interactions between particles in the gas may be neglected and the wave function is given to a good approximation by the solution of the equation

$$-\frac{\hbar^2}{2M} \nabla^2 \Phi + \frac{1}{2}M\omega_\perp^2 \rho^2 \Phi = \mu \Phi. \quad (2.1.5)$$

With the normalization given by (2.1.3), the condensate wave function is

$$\Phi = \sqrt{\frac{N}{L}} \chi(\rho), \quad (2.1.6)$$

where $\chi(\rho)$ is the normalized Gaussian

$$\chi(\rho) = \frac{1}{\sigma \sqrt{\pi}} e^{-\rho^2/2\sigma^2}. \quad (2.1.7)$$

Here $\sigma$ is the harmonic oscillator length defined by

$$\sigma = \sqrt{\frac{\hbar}{M\omega_\perp}}. \quad (2.1.8)$$

In this limit of strong transverse confinement, the energy functional in (2.1.1) can be reduced to a one-dimensional form by assuming that

$$\Phi(\mathbf{r}) \simeq \sqrt{N} \chi(\rho) \psi(x), \quad (2.1.9)$$
where \( \chi(\rho) \) is the normalized Gaussian in (2.1.7). With the explicit factor of \( \sqrt{N} \) in (2.1.9), \( \psi(x) \) is normalized as

\[
\int_0^L dx |\psi(x)|^2 = 1. \tag{2.1.10}
\]

Substituting (2.1.9) into (2.1.1) and performing the integration over the transverse coordinate, we obtain

\[
E^{1D}[\psi] = N\hbar\omega_\perp + \int_0^L dx \frac{\hbar^2}{2M} \left| \frac{d\psi}{dx} \right|^2 + \frac{N^2}{2} U^{1D} \int_0^L dx |\psi(x)|^4, \tag{2.1.11}
\]

where

\[
U^{1D} = U \int d^2 \rho |\chi(\rho)|^4 = \frac{U}{2\pi \sigma^2}. \tag{2.1.12}
\]

Using (2.1.2), we have

\[
U^{1D} = \frac{a\hbar^2}{M\sigma^2}. \tag{2.1.13}
\]

We thus see that the strong confinement in the transverse direction leads to an effective interaction parameter \( U^{1D} \) for this one-dimensional system. The functional in (2.1.11) can be generalized to include a potential variation \( V_{\text{ext}}(x) \) along the length of the cylinder. In this case, we have

\[
E^{1D}[\psi] = N\hbar\omega_\perp + \int_0^L dx \left( \frac{\hbar^2}{2M} \left| \frac{d\psi}{dx} \right|^2 + V_{\text{ext}}(x)|\psi(x)|^2 \right) + \frac{N^2}{2} U^{1D} \int_0^L dx |\psi(x)|^4. \tag{2.1.14}
\]

The one-dimensional GP equation corresponding to this energy functional is

\[
-\frac{\hbar^2}{2M} \frac{d^2 \psi(x)}{dx^2} + V_{\text{ext}}(x)\psi(x) + NU^{1D}|\psi(x)|^2\psi(x) = (\mu - \hbar\omega_\perp)\psi(x). \tag{2.1.15}
\]

One final generalization is to consider a wave-guide which follows some smooth curve in space \( \mathbf{r}(s) \) parametrized by a coordinate \( s \) as shown in Fig 2.1.1. If one
assumes harmonic confinement in directions perpendicular to the curve with frequency \( \omega_\perp \), it can be shown [26] that the effective 1D GP equation is given by

\[
- \frac{\hbar^2}{2M} \frac{d^2\psi(s)}{ds^2} + V_{\text{ext}}(s)\psi(s) - \frac{\hbar^2}{8MR^2(s)} \psi(s)
+ NU^{1D}|\psi(s)|^2\psi(s) = (\mu - \hbar\omega_\perp)\psi(s).
\] (2.1.16)

Here \( R(s) \) is the local radius of curvature which is seen to give rise to an effective potential \( -\frac{\hbar^2}{8MR^2(s)} \). This shows that positions that have a smaller local radius of curvature are attractive relative to positions which have a larger radius of curvature. In the case of the ring geometry, the curvature potential becomes independent of the coordinate \( s \). As a result, it has no effect on the condensate wave function and can therefore be neglected.

We now consider the situation of a mixture of two different Bose gases trapped in the ring potential. The energy functional in this case is a straightforward generalization of (2.1.1). We have
CHAPTER 2. GENERAL THEORETICAL BACKGROUND

\[ E[\Phi_1, \Phi_2] = \int d^3r \left( \frac{\hbar^2}{2M_1} |\nabla \Phi_1|^2 + \frac{\hbar^2}{2M_2} |\nabla \Phi_2|^2 \right) \]
\[ + \int d^3r V_{\text{trap}, 1}(\rho) |\Phi_1|^2 + \int d^3r V_{\text{trap}, 2}(\rho) |\Phi_2|^2 \]
\[ + \frac{1}{2} U_{11} \int d^3r |\Phi_1|^4 + \frac{1}{2} U_{22} \int d^3r |\Phi_2|^4 \]
\[ + U_{12} \int d^3r |\Phi_1|^2 |\Phi_2|^2, \]  
(2.1.17)

where each of the condensate wave functions \( \Phi_i \) is normalized to \( N_i \)

\[ \int d^3r |\Phi_i|^2 = N_i, \]  
(2.1.18)

the total number of particles of species \( i \). For generality, we have allowed for different confining potentials for the two species

\[ V_{\text{trap}, i} = \frac{1}{2} M_i \omega_{\perp i}^2 \rho^2. \]  
(2.1.19)

The interaction parameters appearing in (2.1.17) in general also depend upon the pair of atoms being considered. These parameters are given by

\[ U_{ij} = \frac{2\pi \hbar^2 a_{s,ij}}{\mu_{ij}}, \]  
(2.1.20)

where \( a_{s,ij} \) is the s-wave scattering length for the \( ij \)-pair and \( \mu_{ij} \) is the reduced mass

\[ \mu_{ij} = \frac{M_i M_j}{M_i + M_j}. \]  
(2.1.21)

Eq. (2.1.20) reduces to (2.1.2) when \( M_i = M_j \).

We again assume that, due to the tight confinement, the wave functions take the form

\[ \Phi_i(\mathbf{r}) \simeq \sqrt{N_i} \chi_i(\rho) \psi_i(x). \]  
(2.1.22)
The radial functions are the normalized Gaussians

\[ \chi_i(\rho) = \frac{1}{\sigma_i \sqrt{\pi}} e^{-\rho^2 / 2\sigma_i^2} \]  

(2.1.23)

with

\[ \sigma_i = \sqrt{\frac{\hbar}{M_i \omega_{i}}} \]  

(2.1.24)

Analogous to (2.1.10) the functions \( \psi_i(x) \) are normalized as

\[ \int_0^L dx |\psi_i(x)|^2 = 1. \]  

(2.1.25)

Following the procedure for the single-species case, we substitute (2.1.22) into (2.1.17) to obtain the one-dimensional energy functional

\[
E[\psi_1, \psi_2] = N_1 \hbar \omega_{11} + N_2 \hbar \omega_{12} + \int_0^L dx \left( \frac{N_1 \hbar^2}{2M_1} \left| \frac{d\psi_1}{dx} \right|^2 + \frac{N_2 \hbar^2}{2M_2} \left| \frac{d\psi_2}{dx} \right|^2 \right) \\
+ \frac{N_1^2}{2} U_{11}^{1D} \int_0^L dx |\psi_1|^4 + \frac{N_2^2}{2} U_{22}^{1D} \int_0^L dx |\psi_2|^4 \\
+ N_1 N_2 U_{12}^{1D} \int_0^L dx |\psi_1|^2 |\psi_2|^2,
\]

(2.1.26)

where the interaction parameters are given by

\[ U_{11}^{1D} = \frac{U_{11}}{2\pi \sigma_1^2} \]  

(2.1.27)

\[ U_{22}^{1D} = \frac{U_{22}}{2\pi \sigma_2^2} \]  

(2.1.28)

\[ U_{12}^{1D} = \frac{U_{12}}{\pi \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)} \]  

(2.1.29)

We can write (2.1.27), (2.1.28) and (2.1.29) in the common form

\[ U_{ij}^{1D} = \frac{U_{ij}}{2\pi \sigma_{ij}^2} \]  

(2.1.30)

by defining \( \sigma_{ij}^2 = \frac{1}{2}(\sigma_i^2 + \sigma_j^2) \). Using (2.1.20) we have

\[ U_{ij}^{1D} = \frac{a_{ij} \hbar^2}{\mu_{ij} \sigma_{ij}^2} \]  

(2.1.31)
The energy functional \((2.1.26)\) can be used to derive coupled non-linear GP equations
\[
-\frac{\hbar^2}{2M_k} \frac{d^2\psi_k}{dx^2} + 2 \sum_{l=1}^{2} N_k U_{kl}^{1D} |\psi_l|^2 \psi_k = (\mu_k - \hbar \omega_{\perp k}) \psi_k, \tag{2.1.32}
\]
where \(k = 1, 2\).

In the ring geometry it is more convenient to work in terms of the angular variable \(\theta\) related to \(x\) by \(x = R\theta\), where \(R\) is the radius of the ring. The integrals in the energy functional \((2.1.26)\) are transformed as follows
\[
\int_{0}^{L} dx \rightarrow R \int_{0}^{2\pi} d\theta. \tag{2.1.33}
\]
The normalization in \((2.1.10)\) then becomes
\[
R \int_{0}^{2\pi} d\theta |\psi(R\theta)|^2 = 1. \tag{2.1.34}
\]
Defining \(\tilde{\psi}(\theta) \equiv \sqrt{R} \psi(R\theta)\) this normalization condition becomes
\[
\int_{0}^{2\pi} d\theta |\tilde{\psi}(\theta)|^2 = 1. \tag{2.1.35}
\]
Also, derivatives with respect to \(x\) must be replaced by ones with respect to \(\theta\)
\[
\frac{d\psi_k}{dx} \rightarrow \frac{1}{R} \frac{d\psi_k}{d\theta}. \tag{2.1.36}
\]
With these replacements and dropping the constant terms in \((2.1.26)\), the energy functional becomes
\[
E[\tilde{\psi}_1, \tilde{\psi}_2] = \int_{0}^{2\pi} d\theta \left( \frac{N_1 h^2}{2M_1 R^2} \left| \frac{d\tilde{\psi}_1}{d\theta} \right|^2 + \frac{N_2 h^2}{2M_2 R^2} \left| \frac{d\tilde{\psi}_2}{d\theta} \right|^2 \right)
+ \frac{1}{2} N_1 U_{11}^{1D} \int_{0}^{2\pi} d\theta |\tilde{\psi}_1|^4 + \frac{1}{2} N_2 U_{22}^{1D} \int_{0}^{2\pi} d\theta |\tilde{\psi}_2|^4
+ U_{12}^{1D} N_1 N_2 \int_{0}^{2\pi} d\theta |\tilde{\psi}_1|^2 |\tilde{\psi}_2|^2, \tag{2.1.37}
\]
where $\bar{U}_{ij}^{1D} = U_{ij}^{1D}/R$. For this energy functional the coupled non-linear GP equations are

$$-\frac{\hbar^2}{2M_k R^2} \frac{d^2 \bar{\psi}_k}{d\theta^2} + \sum_{l=1}^{2} N_l \bar{U}_{lk}^{1D} |\bar{\psi}_l|^2 \bar{\psi}_k = \mu_k \bar{\psi}_k. \quad (2.1.38)$$

For simplicity, we will henceforth drop the bars which appear on the quantities in (2.1.37) and (2.1.38). In particular, we will write (2.1.35) simply as

$$\int_0^{2\pi} d\theta |\psi(\theta)|^2 = 1. \quad (2.1.39)$$

### 2.2 Energetic stability

#### 2.2.1 The single-species case

In this section we investigate the energetic stability of a single-species Bose gas. From (2.1.11) and using the change of variables shown by (2.1.31-2.1.36), the single-species energy functional is given by (note that $U^{1D}$ was previously denoted as $\bar{U}^{1D}$)

$$E[\psi] = \int_0^{2\pi} d\theta \left( \frac{N \hbar^2}{2MR^2} \left| \frac{d\psi}{d\theta} \right|^2 \right) + \frac{1}{2} N^2 U^{1D} \int_0^{2\pi} d\theta |\psi|^4, \quad (2.2.1)$$

and the corresponding GP equation is

$$-\frac{\hbar^2}{2MR^2} \frac{d^2 \psi}{d\theta^2} + NU^{1D} |\psi|^2 \psi = \mu \psi. \quad (2.2.2)$$

The equilibrium state $\psi_0(\theta)$ is determined by minimizing $E(\psi)$ with respect to $\psi$ subject to the normalization constraint in (2.1.39). From the GP equation it is clear that a constant wave function $\psi_0 = 1/\sqrt{2\pi}$ is a solution which gives the energy $E_0 = N^2 U^{1D}/(4\pi)$. To determine whether this energy is an absolute minimum we consider the effect of small deviations away from $\psi_0$. We thus represent the wave function as

$$\psi = \psi_0 + \delta \psi, \quad (2.2.3)$$
where $\delta \psi$ is the deviation from $\psi_0$. Inserting this wave function into (2.2.1) and expanding each term to second order in the fluctuation gives

$$E[\psi] = E_0 + \delta E[\delta \psi],$$

(2.2.4)

where $\delta E[\delta \psi]$

$$\delta E[\delta \psi] = \int_0^{2\pi} d\theta \left( \frac{N\hbar^2}{2MR^2} \left| \frac{\partial \delta \psi}{\partial \theta} \right|^2 \right)$$

(2.2.5)

$$+ \frac{1}{2} N^2 U^{1D} \int_0^{2\pi} d\theta \left( \psi_0^2 (\delta \psi + \delta \psi^*)^2 + 2 \psi_0^2 \psi_0 (\delta \psi + \delta \psi^*) + \delta \psi \delta \psi^* \right).$$

We first show that the integral of the terms in the square brackets in (2.2.5) vanishes because of the normalization of $\psi$. Substituting (2.2.3) into (2.1.39), we have

$$\int_0^{2\pi} d\theta \psi_0^2 + \int_0^{2\pi} d\theta \psi_0 (\delta \psi + \delta \psi^*) + \int_0^{2\pi} d\theta |\delta \psi|^2 = 1.$$  

(2.2.6)

Since $\psi_0$ is normalized, this implies

$$\int_0^{2\pi} d\theta \psi_0 (\delta \psi + \delta \psi^*) = - \int_0^{2\pi} d\theta |\delta \psi|^2.$$

(2.2.7)

We thus see that the square bracketed term in (2.2.5) does not contribute to $\delta E(\delta \psi)$.

To further simplify the calculations it is convenient to define the energy fluctuation per particle in units of $\hbar^2/(2MR^2)$. We then have

$$\delta \bar{E}[\delta \psi] \equiv \frac{\delta E[\delta \psi]}{Nh^2/(2MR^2)}$$

$$= \int_0^{2\pi} d\theta \left| \frac{\partial \delta \psi}{\partial \theta} \right|^2 + \frac{1}{2} \gamma \int_0^{2\pi} d\theta (\delta \psi^2 + 2 \delta \psi \delta \psi^* + \delta \psi^* \delta \psi)^2,$$

(2.2.8)

where the dimensionless interaction parameter $\gamma$ is defined as

$$\gamma = \frac{N^2 U^{1D}}{Nh^2/(2MR^2)} \frac{1}{2\pi} \left( \frac{NU^{1D}}{2\pi} \right) \left( \frac{2MR^2}{\hbar^2} \right).$$

(2.2.9)
Since the wave function $\psi(\theta)$, and hence $\delta\psi(\theta)$, is periodic with period $2\pi$ the arbitrary fluctuation $\delta\psi$ can be expressed as the complex Fourier expansion

$$\delta\psi = \sum_n c_n \phi_n(\theta), \quad (2.2.10)$$

where

$$\phi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}. \quad (2.2.11)$$

The sum in (2.2.10) extends over all integers $n$. The Fourier components $c_n$ are given by

$$c_n = \int_0^{2\pi} d\theta \phi_n^*(\theta) \delta\psi(\theta). \quad (2.2.12)$$

With the above representation each term in the expression for energy perturbation $\delta\bar{E}$ can be examined separately. The kinetic energy term in (2.2.8) is

$$KE = \int_0^{2\pi} d\theta \sum_n c_n i n \phi_n(\theta) \bigg| \sum_n c_n^* \phi_n(\theta) \bigg|^2 = \sum_n |c_n|^2 n^2. \quad (2.2.13)$$

The first term in the interaction integral in (2.2.8) becomes

$$\int_0^{2\pi} d\theta \delta\psi^2 = \int_0^{2\pi} d\theta \left( \sum_n c_n \phi_n \right) \left( \sum_{n'} c_{n'} \phi_{n'} \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \sum_{nn'} c_n c_{n'} e^{i\theta(n+n')}$$

$$= \sum_{nn'} c_n c_{n'} \delta_{n,-n'}$$

$$= \sum_n c_n c_{-n}. \quad (2.2.14)$$

Similarly, we have

$$2 \int_0^{2\pi} d\theta \delta\psi \delta\psi^* = 2 \sum_n |c_n|^2, \quad (2.2.15)$$

and

$$\int_0^{2\pi} d\theta \delta\psi^2 = \sum_n c_n^* c_{-n}. \quad (2.2.16)$$
The total energy fluctuation then becomes
\[
\delta \bar{E} = \sum_n |c_n|^2 n^2 + \frac{1}{2} \gamma \sum_n (c_n c_{-n} + 2|c_n|^2 + c_n^* c_{-n}^*) . \tag{2.2.17}
\]

We must now determine the conditions for which this energy fluctuation is positive for arbitrary Fourier coefficients \(c_n\). Since (2.2.17) is not a quadratic form, it is not obvious what the required condition is. However (2.2.17) can be put in a quadratic form by representing the fluctuation \(\delta \psi\) as a sum of real and imaginary parts
\[
\delta \psi(\theta) = a(\theta) + ib(\theta), \tag{2.2.18}
\]
where \(a(\theta)\) and \(b(\theta)\) are real functions of \(\theta\). Taking the Fourier transform of (2.2.18) we obtain
\[
c_n = a_n + ib_n. \tag{2.2.19}
\]
We emphasize here that \(a_n\) and \(b_n\) are complex quantities and therefore are not the real and imaginary parts of \(c_n\). Furthermore, since \(a(\theta)\) and \(b(\theta)\) are real we see that \(a_{-n} = a_n^*\) and \(b_{-n} = b_n^*\). We now consider each term in (2.2.17) separately. For example,
\[
|c_n|^2 = (a_n + ib_n)(a_n^* - ib_n^*)
= |a_n|^2 + i(a_n b_n - a_n b_n^*) + |b_n|^2. \tag{2.2.20}
\]
Multiplying this by \(n^2\) and summing over \(n\) we obtain
\[
\sum_n n^2 |c_n|^2 = \sum_n n^2 \left( |a_n|^2 + |b_n|^2 \right), \tag{2.2.21}
\]
where the cross terms in (2.2.20) cancel as a result of
\[
\sum_n n^2 a_n b_n^* = \sum_n n^2 a_{-n} b_{-n}^*
= \sum_n n^2 a_n^* b_n. \tag{2.2.22}
\]
We also have
\[ c_n c_{-n} = (a_n + i b_n)(a_{-n} + i b_{-n}) \]
\[ = (a_n + i b_n)(a_n^* + i b_n^*) \]
\[ = |a_n|^2 - i(a_n^* b_n + a_n b_n^*) - |b_n|^2, \quad (2.2.23) \]
and
\[ c_n^* c_{-n}^* = |a_n|^2 + i(a_n^* b_n + a_n b_n^*) - |b_n|^2. \quad (2.2.24) \]

We thus see that
\[ c_n c_{-n} + c_n^* c_{-n}^* = 2 \left( |a_n|^2 - |b_n|^2 \right). \quad (2.2.25) \]

The sum of the interaction terms in (2.2.17) then becomes
\[ \sum_n (c_n c_{-n} + 2|c_n|^2 + c_n^* c_{-n}^*) = 4 \sum_n |a_n|^2. \quad (2.2.26) \]

Putting these results together we find
\[ \delta \bar{E} = \sum_n n^2 \left( |a_n|^2 + |b_n|^2 \right) + 2\gamma \sum_n |a_n|^2 \]
\[ = \sum_n \left[ (n^2 + 2\gamma) |a_n|^2 + n^2 |b_n|^2 \right]. \quad (2.2.27) \]

We have thus reduced the energy fluctuation to a sum of quadratic terms. For \( \delta \bar{E} \) to be positive for arbitrary \( |a_n| \) and \( |b_n| \) we require
\[ n^2 + 2\gamma > 0 \quad (2.2.28) \]
for all \( |n| > 0 \). Taking \( n = 1 \), we thus see that the condition for energetic stability is
\[ \gamma > -\frac{1}{2}, \quad (2.2.29) \]
that is
\[ \frac{NU^{1D}}{\pi} > -\frac{\hbar^2}{2MR^2}. \quad (2.2.30) \]
This shows that the homogeneous state is the true ground state as long as this inequality is satisfied. Conversely, if $\gamma < -1/2$ the ground state of the system will be inhomogeneous \[22\]. We thus see that if the interaction is sufficiently attractive, the ground state is no longer homogeneous.

For $\gamma$ just below $-1/2$, the $n = \pm 1$ components are the dominant Fourier components. We can then determine the wave function $\psi(\theta)$ by using the ansatz

$$\psi(\theta) = c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1,$$  \hspace{1cm} \text{(2.2.31)}

where the coefficients $c_l$ obey the normalization condition

$$|c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1.$$  \hspace{1cm} \text{(2.2.32)}

These coefficients must be determined by minimizing the energy functional in (2.2.1). Inserting (2.2.31) into the dimensionless form of (2.2.1) we have

$$\bar{E}[\psi] = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left( c_{-1}^*e^{i\theta} + c_0^* + c_1^*e^{-i\theta} \right) \left( c_{-1}e^{-i\theta} + c_1e^{i\theta} \right)$$

$$+ \frac{\gamma}{2(2\pi)} \int_0^{2\pi} d\theta \left( c_{-1}^*e^{i\theta} + c_0^* + c_1^*e^{-i\theta} \right)^2 \left( c_{-1}e^{-i\theta} + c_0 + c_1e^{i\theta} \right)^2$$

$$= |c_{-1}|^2 + |c_1|^2$$

$$+ \frac{\gamma}{2} \left[ |c_{-1}|^4 + |c_0|^4 + |c_1|^4 + 4|c_{-1}|^2|c_0|^2 + 4|c_{-1}|^2|c_1|^2 + 4|c_0|^2|c_1|^2 \right]$$

$$+ \frac{\gamma}{2} \left[ 2c_{-1} (c_0^*)^2 c_1 + 2c_{-1}^* (c_0)^2 c_1^* \right],$$ \hspace{1cm} \text{(2.2.33)}

The objective is now to minimize this expression with respect to the complex amplitudes $c_{-1}$, $c_0$ and $c_1$ subject to the normalization constraint (2.2.32).

We first note that the phases of the complex coefficients only appear explicitly in the last term of (2.2.33). Writing

$$c_j = |c_j|e^{i\alpha_j},$$ \hspace{1cm} \text{(2.2.34)}
this term becomes

$$2c_{-1}(c_0^*)^2c_1 + 2c_{-1}c_0^*c_1^* = 4|c_0|^2|c_{-1}||c_1|\cos(2\alpha_0 - \alpha_{-1} - \alpha_1). \quad (2.2.35)$$

Our objective now is now to minimize (2.2.33) with respect to the phase angle $(2\alpha_0 - \alpha_{-1} - \alpha_1)$. If $\gamma > 0$, the minimum is achieved by taking

$$2\alpha_0 - \alpha_{-1} - \alpha_1 = \pi \quad (2.2.36)$$

since the cosine in (2.2.35) is equal to -1. On the other hand, if $\gamma < 0$, the minimum is achieved by taking

$$2\alpha_0 - \alpha_{-1} - \alpha_1 = 0, \quad (2.2.37)$$

which makes the cosine equal to +1. Since we are exploring the region $\gamma \lesssim -1/2$, the appropriate choice of the phase difference is $2\alpha_0 - \alpha_{-1} - \alpha_1 = 0$. With this choice of the phases, the energy functional takes the form

$$\bar{E} = |c_{-1}|^2 + |c_1|^2$$

$$+ \frac{\gamma}{2} \left[ |c_{-1}|^4 + |c_0|^4 + |c_1|^4 + 4|c_{-1}|^2|c_0|^2 + 4|c_{-1}|^2|c_1|^2 + 4|c_0|^2|c_1|^2 \right]$$

$$+ \frac{\gamma}{2} 4|c_{-1}||c_0|^2|c_1|, \quad (2.2.38)$$

with the understanding that we are considering $\gamma < 0$.

We now use the normalization condition (2.2.32) in order to eliminate $|c_0|^2$ and obtain

$$\bar{E} = |c_{-1}|^2 + |c_1|^2$$

$$+ \frac{\gamma}{2} \left[ 1 + 2|c_{-1}|^2(1 - |c_{-1}|^2 - |c_1|^2) + 2|c_{-1}|^2|c_1|^2 + 2(1 - |c_{-1}|^2 - |c_1|^2)|c_1|^2 \right]$$

$$+ \frac{\gamma}{2} 4|c_{-1}||c_1|^2|c_1|$$

$$= \frac{\gamma}{2} + |c_{-1}|^2 + |c_1|^2 + \gamma (-|c_{-1}|^4 + |c_1|^4 - |c_{-1}|^2c_1|^2 + |c_{-1}|^2 + |c_1|^2)$$

$$+ \gamma \left( 2|c_{-1}||c_1| - 2|c_{-1}|^3|c_1| - 2|c_{-1}||c_1|^3 \right). \quad (2.2.39)$$
The minimum value of the energy functional is now determined by the conditions
\[
\frac{\partial \bar{E}}{\partial |c_{-1}|} = 0 \quad , \quad \frac{\partial \bar{E}}{\partial |c_1|} = 0. \tag{2.2.40}
\]
This leads to the equations
\[
2|c_{-1}| + \gamma(-4|c_{-1}|^3 - 2|c_{-1}||c_1|^2 + 2|c_{-1}| + 2|c_1| - 6|c_{-1}|^2|c_1| - 2|c_1|^3) = 0 \tag{2.2.41}
\]
\[
2|c_1| + \gamma(-4|c_1|^3 - 2|c_{-1}|^2|c_1| + 2|c_{-1}| + 2|c_1| - 6|c_{-1}||c_1|^2 - 2|c_{-1}|^3) = 0. \tag{2.2.42}
\]
If we now divide (2.2.41) by \(2|c_{-1}|\) and (2.2.42) by \(2|c_1|\) and introduce a variable
\[
\beta = \frac{|c_1|}{|c_{-1}|}
\]
we then obtain the two equations
\[
\gamma \left[ (\beta + 1) (1 - |c_1|^2) - 3|c_{-1}||c_1| - 2|c_{-1}|^2 \right] = -1 \tag{2.2.43}
\]
\[
\gamma \left[ (\beta^{-1} + 1) (1 - |c_{-1}|^2) - 3|c_{-1}||c_1| - 2|c_{-1}|^2 \right] = -1. \tag{2.2.44}
\]
Taking the ratio of these equations we obtain
\[
\beta - \beta|c_1|^2 + |c_1|^2 = \frac{1}{\beta} - \frac{1}{\beta^3}|c_1|^2 + \frac{1}{\beta^2}|c_1|^2. \tag{2.2.45}
\]
By inspection, we see that \(\beta = 1\) is a solution of this equation for any \(|c_1|^2\). This implies that \(|c_{-1}| = |c_1|\), which is to be expected since the interchange of \(|c_{-1}|\) and \(|c_1|\) in (2.2.43 - 2.2.44) leaves the equations invariant.

Having determined that \(\beta = 1\), (2.2.43) can be rewritten as
\[
\gamma \left[ 2(1 - |c_1|^2) - 3|c_1|^2 - 2|c_1|^2 \right] = -1, \tag{2.2.46}
\]
which implies
\[
|c_1|^2 = \frac{1 + 2\gamma}{7\gamma}. \tag{2.2.47}
\]
We now observe that \(|c_1|^2 = 0\) when \(\gamma = -1/2\). This defines the critical value of \(\gamma\) which leads to a nonuniform ground state. For \(\gamma = -1/2 - \delta\), where \(\delta\) is the deviation
from the critical value of \( \gamma \), it follows that

\[
|c_1|^2 = \frac{4}{7} \delta + O(\delta^2). \tag{2.2.48}
\]

This shows that the non-homogeneous state is stable for \( \delta > 0 \), that is \( \gamma < -1/2 \).

It should be noted that (2.2.47) is only valid for \( \gamma \) sufficiently close to \(-1/2\). If \( \gamma \) is much less than \(-1/2\) our original ansatz in (2.2.31) will cease to be valid since higher Fourier components will become non-negligible and \( \psi(\theta) \) will have to be represented as an infinite Fourier series.

### 2.2.2 The two-species case

We now extend our discussion to the two-species Bose gas and determine the criterion for the energetic stability in this case. We express the condensate wave function for each species as we did for the single-species case, namely

\[
\psi_i = \psi^0_i + \delta \psi_i, \tag{2.2.49}
\]
where
\[ \psi_i^0 = \frac{1}{\sqrt{2\pi}}. \] (2.2.50)

In terms of these wave functions, the energy functional in (2.1.37) takes the form (for simplicity, we drop the bars on all quantities)
\[
E[\psi_1, \psi_2] = \int_0^{2\pi} d\theta \left( \frac{N_1 \hbar^2}{2M_1 R_1^2} \left| \frac{\partial (\psi_1^0 + \delta \psi_1)}{\partial \theta} \right|^2 + \frac{N_2 \hbar^2}{2M_2 R_2^2} \left| \frac{\partial (\psi_2^0 + \delta \psi_2)}{\partial \theta} \right|^2 \right) + \frac{1}{2}N_1 U_{11} \int_0^{2\pi} d\theta |\psi_1^0 + \delta \psi_1|^4 + \frac{1}{2}N_2 U_{22} \int_0^{2\pi} d\theta |\psi_2^0 + \delta \psi_2|^4 + U_{12} N_1 N_2 \int_0^{2\pi} d\theta |\psi_1^0 + \delta \psi_1|^2 |\psi_2^0 + \delta \psi_2|^2. \] (2.2.51)

To simplify this expression we follow the same sequence of steps used in the single-species case and retain only terms up to the second order in the fluctuations. The terms involving a single species in (2.2.51) are dealt with as in the single-species case, making use of the normalization constraint of the form (2.2.7). The last term in (2.2.51) yields to the second order
\[
\int_0^{2\pi} d\theta |\psi_1^0 + \delta \psi_1|^2 |\psi_2^0 + \delta \psi_2|^2 \\
= \int_0^{2\pi} d\theta \left( |\psi_1^0|^2 + |\psi_1^0(\delta \psi_1 + \delta \psi_1^*) + |\delta \psi_1|^2 \right) \left( |\psi_2^0|^2 + |\psi_2^0(\delta \psi_2 + \delta \psi_2^*) + |\delta \psi_2|^2 \right) \\
= \int_0^{2\pi} d\theta \left\{ |\psi_1^0|^2 \left[ |\psi_1^0(\delta \psi_1 + \delta \psi_1^*) + |\delta \psi_1|^2 \right] + |\psi_1^0|^2 \left[ |\psi_2^0(\delta \psi_2 + \delta \psi_2^*) + |\delta \psi_2|^2 \right] \\
+ |\psi_1^0|^2 |\psi_2^0|^2 + |\psi_1^0| |\psi_2^0| (\delta \psi_1 + \delta \psi_1^*) (\delta \psi_2 + \delta \psi_2^*) \right\}. \] (2.2.52)

The square-bracketed terms vanish on using the normalization constraints analogous to (2.2.7). We thus find that \( E(\psi_1, \psi_2) = E_0 + \delta E(\delta \psi_1, \delta \psi_2) \), where
\[
E_0 = \frac{1}{4\pi} N_1^2 U_{11}^{1D} + \frac{1}{4\pi} N_2^2 U_{22}^{1D} + \frac{1}{2\pi} N_1 N_2 U_{12}^{1D}, \] (2.2.53)
and

$$
\delta E[\delta \psi_1, \delta \psi_2] = \int_0^{2\pi} \! d\theta \left[ \frac{N_1 \hbar^2}{2 M_1 R^2} \left| \frac{\partial (\delta \psi_1)}{\partial \theta} \right|^2 + \frac{1}{2} N_1^2 U_{11}^{1D} \left( \psi_1^0 \right)^2 \left( \delta \psi_1^2 + 2 \delta \psi_1 \delta \psi_1^* + \delta \psi_1^{*2} \right) \right]
$$

$$
+ \int_0^{2\pi} \! d\theta \left[ \frac{N_2 \hbar^2}{2 M_2 R^2} \left| \frac{\partial (\delta \psi_2)}{\partial \theta} \right|^2 + \frac{1}{2} N_2^2 U_{22}^{1D} \left( \psi_2^0 \right)^2 \left( \delta \psi_2^2 + 2 \delta \psi_2 \delta \psi_2^* + \delta \psi_2^{*2} \right) \right]
$$

$$
+ N_1 N_2 U_{12}^{1D} \psi_1^0 \psi_2^0 \int_0^{2\pi} \! d\theta (\delta \psi_1 + \delta \psi_1^*) (\delta \psi_2 + \delta \psi_2^*). \tag{2.2.54}
$$

The similarity of this expression for $\delta E(\delta \psi_1, \delta \psi_2)$ with (2.2.8) for the single-species case is evident. We now express the fluctuations $\delta \psi_k$ as the complex Fourier series

$$
\delta \psi_k = \sum_n c_{n,k} \phi_n, \tag{2.2.55}
$$

where

$$
c_{n,k} = \int_0^{2\pi} \! d\theta \phi_n^* \delta \psi_k. \tag{2.2.56}
$$

All the terms in (2.2.54) may be rewritten immediately by analogy with the single-species case

$$
\delta E[\delta \psi_1, \delta \psi_2] = \sum_n \left( \frac{N_1 \hbar^2}{2 M_1 R^2} |c_{n,1}|^2 + \frac{N_2 \hbar^2}{2 M_2 R^2} |c_{n,2}|^2 \right) n^2
$$

$$
+ \frac{1}{2(2\pi)} N_1^2 U_{11}^{1D} \sum_n \left( c_{n,1} c_{-n,1} + 2 |c_{n,1}|^2 + c_{n,1}^* c_{-n,1}^* \right)
$$

$$
+ \frac{1}{2(2\pi)} N_2^2 U_{22}^{1D} \sum_n \left( c_{n,2} c_{-n,2} + 2 |c_{n,2}|^2 + c_{n,2}^* c_{-n,2}^* \right)
$$

$$
+ \frac{1}{2\pi} N_1 N_2 U_{12}^{1D} \sum_n \left( c_{n,1} c_{-n,2} + c_{n,1}^* c_{-n,2}^* + c_{n,1}^* c_{n,2} + c_{n,1} c_{n,2}^* \right). \tag{2.2.57}
$$
The above expression for the energy fluctuation is not a quadratic form. To put it in such a form it is again helpful to represent the fluctuations $\delta\psi_1$ and $\delta\psi_2$ as a sum of real and imaginary parts

$$\delta\psi_1(\theta) = a(\theta) + ib(\theta)$$

$$\delta\psi_2(\theta) = u(\theta) + iv(\theta).$$

(2.2.58)

(2.2.59)

Taking the Fourier transform of (2.2.58) and (2.2.59) we obtain

$$c_{n,1} = a_n + ib_n$$

(2.2.60)

$$c_{n,2} = u_n + iv_n.$$  

(2.2.61)

By analogy with the single-species case the energy fluctuation may be rewritten as

$$\delta E = \sum_n \left[ \frac{N_1\hbar^2}{2M_1R^2} (|a_n|^2 + |b_n|^2) + \frac{N_2\hbar^2}{2M_2R^2} (|u_n|^2 + |v_n|^2) \right] n^2$$

$$+ \frac{1}{\pi} \sum_n \left[ N_1^2 U_{11}^{1D} |a_n|^2 + N_2^2 U_{22}^{1D} |u_n|^2 \right]$$

$$+ \frac{1}{\pi} N_1 N_2 U_{12}^{1D} \sum_n (a_n u_n^* + a_n^* u_n)$$

$$\equiv \sum_n \delta E_n.$$  

(2.2.62)

The term $\delta E_n$ in this sum can be presented in the matrix form

$$\delta E_n = \begin{pmatrix} a_n & b_n & u_n & v_n \end{pmatrix} \begin{pmatrix} C + A_n & 0 & E & 0 \\ 0 & A_n & 0 & 0 \\ E & 0 & D + B_n & 0 \\ 0 & 0 & 0 & B_n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ u_n \\ v_n \end{pmatrix},$$

(2.2.63)

where the constants are defined as follows

$$A_n = \frac{N_1\hbar^2 n^2}{2M_1R^2}, \quad B_n = \frac{N_2\hbar^2 n^2}{2M_2R^2}$$

$$C = \frac{1}{\pi} N_1^2 U_{11}^{1D}, \quad D = \frac{1}{\pi} N_2^2 U_{22}^{1D}, \quad E = \frac{1}{\pi} N_1 N_2 U_{12}^{1D}.$$  

(2.2.64)

(2.2.65)
With the definition of the symmetric matrix $M_n$

$$M_n = \begin{pmatrix} C + A_n & 0 & E & 0 \\ 0 & A_n & 0 & 0 \\ E & 0 & D + B_n & 0 \\ 0 & 0 & 0 & B_n \end{pmatrix} \quad (2.2.66)$$

and the column vector

$$w_n = \begin{pmatrix} a_n \\ b_n \\ u_n \\ v_n \end{pmatrix} \quad (2.2.67)$$

(2.2.63) can be written as

$$\delta E_n = w_n^\dagger M_n w_n. \quad (2.2.68)$$

We now determine the eigenvectors and eigenvalues of the $M_n$ matrix which satisfy

$$M_n v_n^{(j)} = \lambda_n^{(j)} v_n^{(j)}. \quad (2.2.69)$$

Since $M_n$ is a real symmetric matrix, the eigenvalues $\lambda_n^{(j)}$ are real and the eigenvectors $v_n^{(j)}$ can be chosen to be an orthonormal set, that is

$$v_n^{(i)\dagger} v_n^{(j)} = \delta_{ij}. \quad (2.2.70)$$

Expanding the vector $w_n$ as

$$w_n = \sum_i \alpha_n^{(i)} v_n^{(i)}, \quad (2.2.71)$$

we then obtain

$$\delta E_n = \sum_i |\alpha_n^{(i)}|^2 \lambda_n^{(i)}. \quad (2.2.72)$$
From this we see that for $\delta E_n$ to be positive, all the eigenvalues $\lambda_n^{(i)}$ must be positive.

The eigenvalues are determined by the characteristic equation

$$\det \left( M_n - \lambda \mathbb{1} \right) = 0. \quad (2.2.73)$$

This equation yields

$$[(C + A_n - \lambda)(D + B_n - \lambda) - E^2] (A_n - \lambda)(B_n - \lambda) = 0, \quad (2.2.74)$$

whose roots are $\lambda_n^{(1)} = A_n$, $\lambda_n^{(2)} = B_n$ and

$$\lambda_n^{(3,4)} = \frac{1}{2} (C + D + A_n + B_n) \pm \frac{1}{2} \sqrt{(C - D + A_n - B_n)^2 + 4E^2}. \quad (2.2.75)$$

Clearly, $\lambda_n^{(1)}$, $\lambda_n^{(2)}$ and $\lambda_n^{(3)}$ (obtained with the plus sign in (2.2.75)) are positive for all $n$. Thus the critical eigenvalue determining the positiveness of $\delta E_n$ is $\lambda_n^{(4)}$. This eigenvalue is greater than zero when

$$CD + A_nB_n + DA_n + CB_n > E^2. \quad (2.2.76)$$

Inserting the definition of the various quantities we have

$$\frac{1}{\pi^2} N_1^2 N_2^2 U_{11}^{1D} U_{22}^{1D} + \frac{1}{\pi} N_1^2 N_2 U_{11}^{1D} \frac{\hbar^2 n^2}{2M_2 R^2} + \frac{1}{\pi} N_2^2 N_1 U_{22}^{1D} \frac{\hbar^2 n^2}{2M_1 R^2}$$

$$+ \frac{(\hbar^2 n^2)^2}{4M_1 M_2 R^4} N_1 N_2 > \frac{1}{\pi^2} N_1^2 N_2^2 (U_{12}^{1D})^2. \quad (2.2.77)$$

We now introduce the new variables $\gamma_{ij}$

$$\gamma_{ij} = \frac{U_{ij}^{1D}}{2\pi} \sqrt{\frac{N_i N_j}{\left( \frac{\hbar^2}{2M_i R^2} \right) \left( \frac{\hbar^2}{2M_j R^2} \right)}}, \quad (2.2.78)$$

where $i, j = 1, 2$. By using these variables, (2.2.77) can be written in the simpler form

$$(2\gamma_{11} + n^2)(2\gamma_{22} + n^2) > 4\gamma_{12}^2 \quad (2.2.79)$$
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If this is true for \( n = 1 \), it is true for all \( n \). Thus the condition for energetic stability is

\[
\frac{1}{4} + \frac{1}{2}(\gamma_{11} + \gamma_{22}) > \gamma_{12}^2 - \gamma_{11}\gamma_{22}.
\] (2.2.80)

This condition is a generalization of the stability criterion for the single-species case given in (2.2.29). In fact, it is straightforward to show that it reduces to this case in the appropriate limit of taking \( \psi_1 = \psi_2 \) in (2.2.51) with \( M_1 = M_2, \ N_1 = N_2 = N/2, \ U_{11}^{1D} = U_{22}^{1D} = U_{12}^{1D} = U^{1D} \). For this choice of parameters, the two-species problem is effectively a single-species system with \( N \) particles. In particular, we have \( \gamma_{ij} = \gamma/2 \) where \( \gamma \) is defined in (2.2.9). Replacing \( \gamma_{ij} \) in (2.2.80) by this value, we find the condition \( \gamma > -1/2 \), which is precisely (2.2.29). Eq. (2.2.80) is a generalization of the result quoted in [24] for the case \( M_1 = M_2 \).

2.3 Dynamic stability

2.3.1 The single-species case

We now consider the stability of the system from a different point of view. The dynamics of the single-species Bose gas is described by means of a time-dependent version of (2.2.2). This equation is known as the time-dependent GP equation and is given by

\[
i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \psi}{\partial \theta^2} + NU^{1D} |\psi|^2 \psi.
\] (2.3.1)

The solution of this equation determines the possible dynamical evolution of the system.

One class of solutions corresponds to situations in which the wave function deviates
only slightly from its equilibrium value. We thus define
\[ \psi = (\psi_0 + \delta \psi)e^{-i\mu t/\hbar}, \]  
(2.3.2)

where \( \psi_0 \) is the equilibrium state, \( \psi_0 = 1/\sqrt{2\pi} \) and \( \delta \psi(\theta, t) \) is considered to be a small fluctuation. Plugging this form of the wave function into (2.3.1) allows us to derive an equation for \( \delta \psi \). We have (after cancelling a common factor of \( e^{-i\mu t/\hbar} \))
\[ i\hbar \frac{\partial \delta \psi}{\partial t} + (\psi_0 + \delta \psi)\mu = -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \delta \psi}{\partial \theta^2} + NU^{1D}|\psi_0 + \delta \psi|^2(\psi_0 + \delta \psi). \]  
(2.3.3)

Expanding to first order in the fluctuations we obtain
\[ i\hbar \frac{\partial \delta \psi}{\partial t} + (\psi_0 + \delta \psi)\mu = -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \delta \psi}{\partial \theta^2} + NU^{1D} \left(|\psi_0|^2 \psi_0 + \psi_0^2 \delta \psi + 2\psi_0^2 \delta \psi^* \right). \]  
(2.3.4)

We now note that \( \mu = NU^{1D}|\psi_0|^2 \). Thus,
\[ i\hbar \frac{\partial \delta \psi}{\partial t} = -\frac{\hbar^2}{2MR^2} \frac{\partial^2 \delta \psi}{\partial \theta^2} + \frac{NU^{1D}}{2\pi} (\delta \psi + \delta \psi^*). \]  
(2.3.5)

If we are to look for harmonic solutions of (2.3.5), the structure of this equation requires that the solution takes the form
\[ \delta \psi = ue^{-i\omega t} - v^* e^{i\omega t}, \]  
(2.3.6)

where \( u \) and \( v \) are complex functions of the angular variable \( \theta \). Substituting this into (2.3.5) and equating the terms proportional to \( e^{i\omega t} \) and \( e^{-i\omega t} \) on either side of the equation, we obtain the pair of equations
\[ -\frac{\hbar^2}{2MR^2} \frac{\partial^2 u}{\partial \theta^2} + \mu u - \mu v = \hbar \omega u \]  
(2.3.7)
\[ -\frac{\hbar^2}{2MR^2} \frac{\partial^2 v}{\partial \theta^2} + \mu v - \mu u = -\hbar \omega v. \]  
(2.3.8)
We now divide both sides of the above two equations by \( \hbar^2/(2MR^2) \) and observe that 
\[ \mu(2MR^2)/\hbar^2 = \gamma \]
which we defined earlier in (2.2.9). Equations (2.3.7) and (2.3.8) have solutions of the form

\[
\begin{align*}
    u(\theta) &= u_0 e^{in\theta} \\
    v(\theta) &= v_0 e^{in\theta},
\end{align*}
\]

(2.3.9) \hspace{1cm} (2.3.10)

where \( u_0 \) and \( v_0 \) satisfy

\[
\begin{align*}
    n^2 u_0 + \gamma (u_0 - v_0) &= \bar{\omega} u_0 \\
    n^2 v_0 + \gamma (v_0 - u_0) &= -\bar{\omega} v_0,
\end{align*}
\]

(2.3.11) \hspace{1cm} (2.3.12)

where \( \bar{\omega} = \omega(2MR^2)/\hbar \). We may now represent this system of equations in the matrix form

\[
M \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} n^2 + \gamma & -\gamma \\ \gamma & -n^2 - \gamma \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \bar{\omega} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.
\]

(2.3.13)

The frequencies of the various modes are then obtained from the equation

\[
\det \left( M - \bar{\omega} 1 \right) = 0,
\]

(2.3.14)

which yields

\[
\bar{\omega}_n^2 = n^4 + 2n^2 \gamma.
\]

(2.3.15)

If the right-hand side of (2.3.15) is negative, the frequencies are imaginary and the time dependence of the modes varies as \( e^{\pm |\bar{\omega}|t} \). This exponential dependence signals a dynamic instability. Thus, the condition for the instability is

\[
n^4 + 2n^2 \gamma < 0,
\]

(2.3.16)
which implies
\[ \gamma < -\frac{n^2}{2}. \]  
(2.3.17)

The critical value of \( \gamma \) is obtained for \( n = 1 \), giving \( \gamma_{cr} < -1/2 \), a result obtained previously by the authors of [22]. This condition is exactly the same as the condition for the energetic stability (2.2.80) obtained earlier. Thus, from either point of view, the system becomes unstable when the interactions are sufficiently attractive.

2.3.2 The two-species case

We now consider the extension of the previous analysis to the two-species case. Here the dynamics is described by the coupled time-dependent Gross-Pitaevskii equations

\[
\begin{align*}
\frac{i\hbar}{\partial t}\psi_1 &= -\frac{\hbar^2}{2M_1R^2}\frac{\partial^2\psi_1}{\partial \theta^2} + N_1U_{11}^{1D}|\psi_1|^2\psi_1 + N_2U_{12}^{1D}|\psi_2|^2\psi_1, \\
\frac{i\hbar}{\partial t}\psi_2 &= -\frac{\hbar^2}{2M_2R^2}\frac{\partial^2\psi_2}{\partial \theta^2} + N_2U_{22}^{1D}|\psi_2|^2\psi_2 + N_1U_{12}^{1D}|\psi_1|^2\psi_2,
\end{align*}
\]

which are an obvious generalization of (2.3.1). As before, we represent each of the wave functions in the time-dependent form

\[
\begin{align*}
\psi_1 &= (\psi_0^1 + \delta\psi_1) \cdot e^{-i\mu_1 t/\hbar} \\
\psi_2 &= (\psi_0^2 + \delta\psi_2) \cdot e^{-i\mu_2 t/\hbar},
\end{align*}
\]

where the chemical potentials are determined by the equilibrium solutions and are given by \( \mu_i = N_iU_{i1}^{1D}/(2\pi) + N_jU_{ij}^{1D}/(2\pi) \). We now substitute these wave functions in (2.3.18) and (2.3.19) and to first order in \( \delta\psi \) we obtain

\[
\begin{align*}
\frac{i\hbar}{\partial t}(\delta\psi_1) &= -\frac{\hbar^2}{2M_1R^2}\frac{\partial^2\delta\psi_1}{\partial \theta^2} + \frac{N_1U_{11}^{1D}}{2\pi}(\delta\psi_1 + \delta\psi_1^*) + \frac{N_2U_{12}^{1D}}{2\pi}(\delta\psi_2 + \delta\psi_2^*) \\
\frac{i\hbar}{\partial t}(\delta\psi_2) &= -\frac{\hbar^2}{2M_2R^2}\frac{\partial^2\delta\psi_2}{\partial \theta^2} + \frac{N_2U_{22}^{1D}}{2\pi}(\delta\psi_2 + \delta\psi_2^*) + \frac{N_1U_{12}^{1D}}{2\pi}(\delta\psi_1 + \delta\psi_1^*). 
\end{align*}
\]
As before, we represent the fluctuations $\delta\psi_i$ as

$$
\begin{align*}
\delta\psi_1 &= u_1 e^{-i\omega t} - v_1^* e^{i\omega t} \\
\delta\psi_2 &= u_2 e^{-i\omega t} - v_2^* e^{i\omega t}.
\end{align*}
$$

Substituting these expressions into (2.3.21) and (2.3.22) and equating terms with $e^{-i\omega t}$ and $e^{i\omega t}$, we obtain the system of four coupled equations

\begin{align}
-\hbar^2 \frac{\partial^2 u_1}{M_1 R^2 \frac{\partial}{\partial t^2}} + \frac{N_{11} U_1^{1D}}{2\pi} (u_1 - v_1) + \frac{N_{21} U_2^{1D}}{2\pi} (u_2 - v_2) &= \hbar \omega u_1 \\
-\hbar^2 \frac{\partial^2 v_1}{M_1 R^2 \frac{\partial}{\partial t^2}} + \frac{N_{11} U_1^{1D}}{2\pi} (v_1 - u_1) + \frac{N_{21} U_2^{1D}}{2\pi} (v_2 - u_2) &= -\hbar \omega v_1 \\
-\hbar^2 \frac{\partial^2 u_2}{M_2 R^2 \frac{\partial}{\partial t^2}} + \frac{N_{22} U_1^{1D}}{2\pi} (u_2 - v_2) + \frac{N_{12} U_2^{1D}}{2\pi} (u_1 - v_1) &= \hbar \omega u_2 \\
-\hbar^2 \frac{\partial^2 v_2}{M_2 R^2 \frac{\partial}{\partial t^2}} + \frac{N_{22} U_1^{1D}}{2\pi} (v_2 - u_2) + \frac{N_{12} U_2^{1D}}{2\pi} (v_1 - u_1) &= -\hbar \omega v_2.
\end{align}

From these equations we see that $u_j$ and $v_j$ will have solutions of the form

$$
\begin{align*}
u_j &= u_j^0 e^{in\theta} \\
v_j &= v_j^0 e^{in\theta}.
\end{align*}
$$

Dividing the left- and right-hand sides of (2.3.24 - 2.3.27) by $\sqrt{\frac{\hbar^2}{2M_1 R^2} \frac{\hbar^2}{2M_2 R^2}}$, and introducing the parameters $\gamma_{ij}$ as in (2.2.78) with $\tilde{\omega} = \omega(2\sqrt{M_1 M_2 R^2})/\hbar$, we have

\begin{align}
\alpha n^2 u_1^0 + \alpha \gamma_{11} (u_1^0 - v_1^0) + \beta \gamma_{12} (u_2^0 - v_2^0) &= \tilde{\omega} u_1^0 \\
\alpha n^2 v_1^0 + \alpha \gamma_{11} (v_1^0 - u_1^0) + \beta \gamma_{12} (v_2^0 - u_2^0) &= -\tilde{\omega} v_1^0 \\
\alpha^{-1} n^2 u_2^0 + \alpha^{-1} \gamma_{22} (u_2^0 - v_2^0) + \beta^{-1} \gamma_{12} (u_1^0 - v_1^0) &= \tilde{\omega} u_2^0 \\
\alpha^{-1} n^2 v_2^0 + \alpha^{-1} \gamma_{22} (v_2^0 - u_2^0) + \beta^{-1} \gamma_{12} (v_1^0 - u_1^0) &= -\tilde{\omega} v_2^0.
\end{align}
where \( \alpha = \sqrt{M_2/M_1} \) and \( \beta = \sqrt{N_2/N_1} \). The above system of four equations can now be written in the matrix form

\[
\begin{pmatrix}
\alpha(n^2 + \gamma_{11}) & -\alpha\gamma_{11} & \beta\gamma_{12} & -\beta\gamma_{12} \\
-\alpha\gamma_{11} & \alpha(n^2 + \gamma_{11}) & -\beta\gamma_{12} & \beta\gamma_{12} \\
\beta^{-1}\gamma_{12} & -\beta^{-1}\gamma_{12} & \alpha^{-1}(n^2 + \gamma_{22}) & -\alpha^{-1}\gamma_{22} \\
-\beta^{-1}\gamma_{12} & \beta^{-1}\gamma_{12} & -\alpha^{-1}\gamma_{22} & \alpha^{-1}(n^2 + \gamma_{22}) \\
\end{pmatrix}
\begin{pmatrix}
u_1^0 \\
v_1^0 \\
u_2^0 \\
v_2^0 \\
\end{pmatrix}
= \bar{\omega}
\begin{pmatrix}
u_1^0 \\
v_1^0 \\
u_2^0 \\
v_2^0 \\
\end{pmatrix}.
\]

(2.3.34)

The frequencies \( \bar{\omega} \) are then determined by setting the determinant

\[
D(\bar{\omega}) \equiv \begin{vmatrix}
\alpha(n^2 + \gamma_{11}) - \bar{\omega} & -\alpha\gamma_{11} & \beta\gamma_{12} & -\beta\gamma_{12} \\
-\alpha\gamma_{11} & \alpha(n^2 + \gamma_{11}) + \bar{\omega} & -\beta\gamma_{12} & \beta\gamma_{12} \\
\beta^{-1}\gamma_{12} & -\beta^{-1}\gamma_{12} & \alpha^{-1}(n^2 + \gamma_{22}) - \bar{\omega} & -\alpha^{-1}\gamma_{22} \\
-\beta^{-1}\gamma_{12} & \beta^{-1}\gamma_{12} & -\alpha^{-1}\gamma_{22} & \alpha^{-1}(n^2 + \gamma_{22}) + \bar{\omega} \\
\end{vmatrix} = 0
\]

(2.3.35)

equal to zero. By interchanging rows and columns of the above determinant it is straightforward to show that it is an even function of \( \bar{\omega} \): \( D(-\bar{\omega}) = D(\bar{\omega}) \). This means that the characteristic equation contains only even powers of \( \bar{\omega} \). We obtain

\[
n^8 + 2n^6(\gamma_{11} + \gamma_{22}) + 4n^4\gamma_{11}\gamma_{22} - 4n^4\gamma_{12}^2 - n^4\alpha^2\bar{\omega}^2 - n^4\alpha^{-2}\bar{\omega}^2 - 2n^2\alpha^2\gamma_{11}\bar{\omega}^2 - 2n^2\alpha^{-2}\gamma_{22}\bar{\omega}^2 + \bar{\omega}^4 = 0,
\]

(2.3.36)

which can be written in the compact form

\[
[\alpha^{-2}\bar{\omega}^2 - \bar{\omega}_1^2] [\alpha^2\bar{\omega}^2 - \bar{\omega}_2^2] - 4n^4\gamma_{12}^2 = 0,
\]

(2.3.37)

where \( \bar{\omega}_1^2 = n^4 + 2n^2\gamma_{11} \) and \( \bar{\omega}_2^2 = n^4 + 2n^2\gamma_{22} \) are the frequencies (see (2.3.15)) obtained when the coupling \( \gamma_{12} \) between the two species is set equal to zero. The
above equation is quadratic in $\tilde{\omega}^2$ and has two roots

$$\tilde{\omega}_{\pm}^2 = \frac{1}{2} \left[ \alpha^2 \tilde{\omega}_1^2 + \alpha^{-2} \tilde{\omega}_2^2 \right] \pm \frac{1}{2} \sqrt{(\alpha^2 \tilde{\omega}_1^2 + \alpha^{-2} \tilde{\omega}_2^2)^2 - 4 \left( 4\gamma_{12}^2 - \tilde{\omega}_1^2 \tilde{\omega}_2^2 \right)}. \quad (2.3.38)$$

The dynamic stability criterion requires $\tilde{\omega}^2 > 0$, so we consider only $\tilde{\omega}_-^2$ as this is the only case when $\tilde{\omega}^2$ can be negative. We therefore obtain

$$\tilde{\omega}_1^2 \tilde{\omega}_2^2 - 4n^4 \gamma_{12}^2 > 0, \quad (2.3.39)$$

or equivalently, using the definitions of $\tilde{\omega}_1^2$ and $\tilde{\omega}_2^2$,

$$\frac{1}{4}n^4 + \frac{1}{2}n^2(\gamma_{11} + \gamma_{22}) > \gamma_{12}^2 - \gamma_{11} \gamma_{22}. \quad (2.3.40)$$

If we set $n = 1$, this condition is identical to (2.2.80) which determined the energetic stability of the two-species system. We thus have established quite generally that the criteria of energetic and dynamic stability are equivalent for both the single- and two-species situations. The criteria we have obtained for $M_1 \neq M_2$ is a generalization of the criteria found earlier [24] for $M_1 = M_2$. 
Chapter 3

Persistent currents

In this chapter we review Bloch’s arguments regarding the possibility of persistent currents in a superfluid confined in a circular ring of radius $R$. These arguments are developed for a situation in which there is tight confinement in directions perpendicular to the ring. As we have seen in the previous chapter, in this limit the system can effectively be thought of as one-dimensional.

3.1 Bloch’s analysis for a single-species gas

Bloch considers $N$ bosonic atoms on a ring described by the many-particle wave function $\Psi(x_1, ..., x_N)$. Here the variable $x_i = R \theta_i$ gives the position of the $i$-th particle on the ring illustrated in the Fig. 3.1.1.

The momentum conjugate to $x_i$ is given by

$$\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i} = \frac{\hbar}{iR} \frac{\partial}{\partial \theta_i} \quad (3.1.1)$$

in the coordinate representation. Alternatively, one can think about the angular
CHAPTER 3. PERSISTENT CURRENTS

Figure 3.1.1: Illustration of a ring geometry.

momentum of the particle about the centre of the ring defined by

\[
\hat{l}_i \equiv R \hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial \theta_i}.
\]  (3.1.2)

The total angular momentum operator is then

\[
\hat{L} = \sum_{i=1}^{N} \hat{l}_i = \sum_{i=1}^{N} \frac{\hbar}{i} \frac{\partial}{\partial \theta_i}.
\]  (3.1.3)

The Hamiltonian of the system of \(N\) interacting atoms can be written as follows

\[
\hat{H} = \sum_{i=1}^{N} \frac{\hat{l}_i^2}{2MR^2} + \sum_{i<j} \upsilon(\theta_i - \theta_j).
\]  (3.1.4)

Since the total angular momentum and the Hamiltonian commute

\[
[\hat{L}, \hat{H}] = 0,
\]  (3.1.5)

one can find simultaneous eigenstates of these two operators.

We begin by looking for the eigenfunctions \(\Phi\) of the operator \(\hat{L}\). The eigenvalue problem in this case is

\[
\hat{L}\Phi = L\Phi,
\]  (3.1.6)
where the $\hat{L}$ operator that is given by (3.1.3). Using the method of separation of variables, the wave function $\Phi$ can be expressed in the product form

$$\Phi(\theta_1, ..., \theta_N) = A \phi_1(\theta_1) \phi_2(\theta_2) \cdots \phi_N(\theta_N),$$  

(3.1.7)

where $A$ is some normalization constant. Each of the factors $\phi_i(\theta)$ is given by

$$\phi_i(\theta_i) = \frac{1}{\sqrt{2\pi}} \exp (il_i \theta_i),$$  

(3.1.8)

where the separation constants $l_i$ must take the integer values $0, \pm 1, \pm 2, ...$ in order for $\phi_i(\theta_i)$ to be single-valued. The total angular momentum of the system is

$$L = \hbar \sum_{i=1}^{N} l_i.$$  

(3.1.9)

From this we see that the allowed values of $L$ are $\hbar \nu$, where $\nu$ is an integer. It will be convenient to write (3.1.9) as

$$L = \hbar N l,$$  

(3.1.10)

where $l$ is the angular momentum per particle in units of $\hbar$. We see that $l = \frac{1}{N} \sum_{i=1}^{N} l_i$.

The product wave function in (3.1.7) is a valid solution of (3.1.6) but it is not a proper bosonic wave function since it is not symmetric under the interchange of a pair of coordinates $\theta_i$ and $\theta_j$. However a totally symmetric function can be constructed by applying the symmetrization operator

$$\hat{S} = \frac{1}{N!} \sum_{P} \hat{P}$$  

(3.1.11)

to (3.1.7). Here $\hat{P}$ is a permutation operator which permutes the coordinates in the product appearing in (3.1.7) and the sum extends over the $N!$ distinct permutations.
of the particles. We then obtain

\[
\Phi_{\{l_i\}}(\theta_1, ..., \theta_N) = A \sum_{P} \hat{P} \exp \left[ i \sum_{i=1}^{N} l_i \theta_i \right] = A \sum_{P} \exp \left[ i \sum_{i=1}^{N} l_i \theta_{P_i} \right],
\]

(3.1.12)

where \(\theta_{P_i}\) represents the \(i\)-th angular coordinate after the permutation.

We first note that \(\Phi_{\{l_i\}}\) is still an angular momentum eigenfunction with angular momentum \(\hbar \sum_{i=1}^{N} l_i = \hbar N l\), since

\[
\hat{L} \exp \left[ i \sum_{i=1}^{N} l_i \theta_{P_i} \right] = \hat{L} \hat{P} \exp \left[ i \sum_{i=1}^{N} l_i \theta_i \right] = \hat{P} \hat{L} \exp \left[ i \sum_{i=1}^{N} l_i \theta_i \right] = \left( \hbar \sum_{i=1}^{N} l_i \right) \exp \left[ i \sum_{i=1}^{N} l_i \theta_{P_i} \right].
\]

(3.1.13)

To obtain this result, we have used the fact that \(\hat{L}\) and \(\hat{P}\) commute.

We next note that the exponential in (3.1.12) can be written as

\[
\exp \left[ i \sum_{i=1}^{N} l_i \theta_{P_i} \right] = \exp [il\Theta] \exp \left[ i \sum_{i=1}^{N} (l_i - l) \theta_{P_i} \right],
\]

(3.1.14)

where \(\Theta = \sum_{i=1}^{N} \theta_i\). The first exponential on the right-hand side is itself an eigenfunction of \(\hat{L}\) with eigenvalue \(\hbar Nl\) and by definition, is a symmetric function of the angular variables \(\theta_i\). Applying \(\hat{L}\) to the second exponential and using the same argument given in (3.1.13), we see it in fact is an eigenfunction of \(\hat{L}\) with eigenvalue 0. In addition, it is a function of the relative angular coordinates \(\theta_i - \theta_j\) which can be seen by noting that under the transformation \(\theta_{P_i} \to \theta_{P_i} + \Delta \theta\), this factor remains unchanged since \(\sum_{i=1}^{N} (l_i - l) \Delta \theta = 0\).
The symmetrized function in (3.1.12) can thus be written as

$$\Phi_{\{l_i\}}(\theta_1, ..., \theta_N) = \exp [il\Theta] \chi_{\{l_i\}}(\theta_1, ..., \theta_N),$$

(3.1.15)

where $\chi_{\{l_i\}}$ is a function of $(\theta_i - \theta_j)$. The functions in (3.1.15) provide a basis of symmetric $N$-particle states. A state with a total angular momentum $L = \hbar N l$ can thus be constructed from a subset of these states satisfying the constraint $\sum_{i=1}^{N} l_i = N l$. In particular, an eigenfunction of $\hat{H}$ with total angular momentum $L$ can be written as

$$\Psi_L(\theta_1, ..., \theta_N) = \sum_{\{l_i\}} C(\{l_i\}) \Phi_{\{l_i\}}(\theta_1, ..., \theta_N)$$

$$= \exp [il\Theta] \sum_{\{l_i\}} C(\{l_i\}) \chi_{\{l_i\}}(\theta_1, ..., \theta_N)$$

$$\equiv \exp [il\Theta] \chi_L(\theta_1, ..., \theta_N).$$

(3.1.16)

By construction, the function $\chi_L(\theta_1, ..., \theta_N)$ is also a function of $(\theta_i - \theta_j)$ and is determined by requiring $\Psi_L$ to be a solution of the Schrödinger equation

$$\hat{H} \Psi_L = E \Psi_L,$$

(3.1.17)

where the Hamiltonian is given in (3.1.4). The action of $\hat{H}$ on $\Psi_L$ yields

$$\hat{H} \Psi_L = \left( \sum_{i=1}^{N} \frac{l_i^2}{2MR^2} + \sum_{i<j} v(\theta_i - \theta_j) \right) \Psi_L$$

$$= \sum_{i=1}^{N} \frac{l_i^2}{2MR^2} \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \chi_L + \sum_{i<j} v(\theta_i - \theta_j) \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \chi_L.$$

(3.1.18)

Let us now investigate the effect of the operator $\hat{l}_i^2$ on $\Psi_L$. Using (3.1.16) we observe
that

\[ \hat{l}_i \Psi_L = \frac{\hbar}{i} \frac{\partial \Psi_L}{\partial \theta_i} \]

\[ = \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \hbar \chi_L + \frac{\hbar}{i} \frac{\partial \chi_L}{\partial \theta_i} \right) \tag{3.1.19} \]

and similarly,

\[ \hat{l}_i^2 \Psi_L = \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \hbar^2 l^2 \chi_L + 2 \hbar l \frac{\hbar}{i} \frac{\partial \chi_L}{\partial \theta_i} - \hbar^2 \frac{\partial^2 \chi_L}{\partial \theta_i^2} \right) \tag{3.1.20} \]

We thus see that

\[ \sum_{i=1}^{N} \frac{\hat{l}_i^2}{2MR^2} \Psi_L = \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \frac{1}{2MR^2} \hbar^2 l^2 N \chi_L + \frac{2\hbar l}{2MR^2} \hat{L}_L = \frac{\hbar^2}{2MR^2} \sum_{i=1}^{N} \frac{\partial^2 \chi_L}{\partial \theta_i^2} \right) \]

\[ = \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \frac{N\hbar^2 l^2}{2MR^2} \chi_L + \sum_{i=1}^{N} \frac{\hat{l}_i^2}{2MR^2} \chi_L \right) , \tag{3.1.21} \]

where we have used \( \hat{L}_L = 0 \). Inserting this result into (3.1.18), we see that (3.1.17) becomes

\[ \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \hat{H}_L + \frac{N\hbar^2 l^2}{2MR^2} \right) = E \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \chi_L. \tag{3.1.22} \]

That is, \( \chi_L \) satisfies the Schrödinger equation

\[ \hat{H}_L \chi_L = \epsilon(L) \chi_L, \tag{3.1.23} \]

where

\[ \epsilon(L) = E - \frac{N\hbar^2 l^2}{2MR^2} = E - \frac{L^2}{2MNR^2}. \tag{3.1.24} \]

We now investigate the implications of the wave function \( \Psi_L \) being single-valued around the ring. This requires \( \Psi_L \) to be unchanged when any one of its coordinates \( \theta_i \) is increased by \( 2\pi \), that is,

\[ \Psi_L(..., \theta_i + 2\pi, ...) = \Psi_L(..., \theta_i, ...). \tag{3.1.25} \]
Using (3.1.16), this implies

\[ \chi_L(\theta_1, ..., \theta_i + 2\pi, ..., \theta_N) = \exp[-i2\pi l] \chi_L(\theta_1, ..., \theta_i, ..., \theta_N). \tag{3.1.26} \]

These conditions for \( i = 1, ..., N \) are the boundary conditions that the solutions of (3.1.23) must satisfy. Since \( L = h\nu = Nh\), the phase factor in (3.1.26) is \( \exp(-2\pi i\nu/N) \). This factor is unchanged if \( \nu \) is increased by any multiple of \( N \). In other words, the boundary conditions for \( L' = h\nu' = h(\nu + N) = L + hN \) are the same as for \( L \). Thus the function \( \chi_{L'} \) must necessarily belong to the set of functions \( \chi_{L\alpha} \) with eigenvalue \( \epsilon_{L\alpha} \), where the index \( \alpha \) distinguishes the various possible quantum states. Thus, we conclude that \( \epsilon_{L\alpha} \) is a periodic function of \( L \) with period \( Nh \). This is true of any of the eigenvalues of (3.1.23) including the lowest energy eigenvalue, which we denote as \( \epsilon_0(L) \).

Finally we note that the energy cannot depend on the sense of the angular momentum. If \( \Psi_L \) is the solution of (3.1.17) then

\[ \hat{H}\Psi_L^* = E\Psi_L^*, \tag{3.1.27} \]

since \( \hat{H} \) is a Hermitian operator. The state \( \Psi_L^* \) thus has the same energy as \( \Psi_L \). However, the state \( \Psi_L^* \) is an eigenstate of the total angular momentum operator with eigenvalue of \(-L\). We thus conclude that \( \epsilon_0(-L) = \epsilon_0(L) \).

To illustrate these general considerations, it is helpful to consider the simpler case of an ideal Bose gas made up of \( N \) particles. As before, the total angular momentum of the system is given by \( L = h\nu \), where the integer \( \nu \) lies in the range between 0 and \( N \). The particular state in (3.1.12) with \( \sum_{i=1}^{N} l_i = \nu \) has the total energy given by

\[ E = \frac{\hbar^2}{2MR^2} \sum_{i=1}^{N} l_i^2. \tag{3.1.28} \]
We now ask: what set of angular momenta, \( \{ l_i \} \), subject to the constraint \( \sum_{i=1}^{N} l_i = \nu \), gives the lowest possible energy? We assert that this state is defined by placing \( \nu \) particles into the \( l_i = 1 \) state and the remaining \( N - \nu \) particles into the \( l_i = 0 \) state.

The energy of this state

\[
E_0(\nu) = \frac{\hbar^2}{2MR^2}(N - \nu) \cdot 0^2 + \frac{\hbar^2}{2MR^2}(\nu) \cdot 1^2 = \frac{\hbar^2}{2MR^2}(\nu). \tag{3.1.29}
\]

To verify this assertion we consider the change of configuration of promoting one \( l_i = 1 \) particle into the \( l_i = 2 \) state and transferring two other \( l_i = 1 \) particles into \( l_i = 0 \) state to preserve the total angular momentum. The energy of this new configuration is

\[
E(\nu) = \frac{\hbar^2}{2MR^2} \left[ (N - \nu + 2) \cdot 0^2 + (\nu - 3) \cdot 1^2 + 1 \cdot 2^2 \right]
= \frac{\hbar^2}{2MR^2} (\nu + 1)
> E_0(\nu). \tag{3.1.30}
\]

Any other change of configuration will necessarily give a state with even higher energy.

According to (3.1.24) the function \( \epsilon_0(\nu) \) is given by

\[
\epsilon_0(\nu) = E_0(\nu) - \frac{L^2}{2MR^2N}
= \frac{\hbar^2}{2MR^2} \left( \nu - \frac{\nu^2}{N} \right)
= \frac{\hbar^2N}{2MR^2N} \nu \left( 1 - \frac{\nu}{N} \right). \tag{3.1.31}
\]

Thus, the function \( \epsilon_0(L) \) has minima at \( \nu = 0 \) and \( \nu = N \), or equivalently, \( L = 0 \) and \( L = \hbar N \) respectively.

Earlier we proved that \( \epsilon(L) \) is periodic with period \( N\hbar \). The behaviour for \( 0 \leq \nu \leq N \) would then suggest that \( \epsilon_0(L) \) would have minima at every integral multiple
of \( Nh \). To confirm this, let us consider another range of \( \nu \), \( mN \leq \nu \leq (m+1)N \), where \( m \) is an integer number. The total energy when \( [(m+1)N - \nu] \) particles are placed into the \( l_i = m \) state while other \( (\nu - mN) \) particles occupy the \( l_i = m + 1 \) state is

\[
E_0(\nu) = \frac{\hbar^2}{2MR^2} [(m+1)N - \nu] \cdot m^2 + \frac{\hbar^2}{2MR^2} (\nu - mN) \cdot (m+1)^2
\]

\[
= \frac{\hbar^2 N}{2MR^2} \left[ -m(m+1) + \frac{\nu}{N}(2m+1) \right].
\]  (3.1.32)

Thus the function \( \epsilon_0(\nu) \) takes the following form

\[
\epsilon_0(\nu) = \frac{\hbar^2}{2MR^2} \left[ -m(m+1) N + \nu(2m+1) \right] - \frac{\hbar^2}{2MR^2} \frac{\nu^2}{N}
\]

\[
= \frac{\hbar^2 N}{2MR^2} \left( \frac{\nu}{N} - m \right) \left( m + 1 - \frac{\nu}{N} \right).
\]  (3.1.33)

For \( \nu \) in the given range we define the variable \( \nu' = \nu - mN \). Eq. (3.1.33) can thus be written as

\[
\epsilon_0(\nu) = \frac{\hbar^2 N}{2MR^2} \frac{\nu'}{N} \left( 1 - \frac{\nu'}{N} \right),
\]  (3.1.34)

which is precisely of the form given in (3.1.31) for \( m = 0 \). In other words, the variation of \( \epsilon_0(\nu) \) with \( \nu \) in each interval \( mN \leq \nu \leq (m+1)N \) is the same, confirming that the function is indeed periodic with period \( N \).

In Fig. 3.1.2 we show a plot of \( E_0(\nu) \), as given by (3.1.32), as a function of \( \nu/N \). It consists of straight line sections for each interval \( mN \leq \nu \leq (m+1)N \). The dashed curve in this figure is \( (\nu/N)^2 = [E_0(\nu) - \epsilon_0(\nu)]/[Nh^2/(2MR^2)] \). Since \( \epsilon_0(mN) = 0 \), the solid line touches this curve at the points \( \nu = mN \). The vertical offset between the solid and dashed curves is just \( \epsilon_0(\nu) \) as given by (3.1.34). We see that \( E_0 \) has an absolute minimum at \( \nu = 0 \) and is a monotonically increasing function of \( \nu \).

We now argue, following Bloch [1] that the behaviour of \( E_0(\nu) \) is incompatible with the possibility of persistent flow. We consider the point A in Fig. 3.1.2(a) which
Figure 3.1.2: The solid line in (a) is a plot of the energy given in (3.1.33) as a function of $\nu/N$ for non-interacting particles in units of $N\hbar^2/(2MR^2)$. The dashed line is the curve $\frac{\hbar^2\nu^2}{2MNR^2}$. The solid line in (b) is a schematic of the excited behaviour for interacting particles.

corresponds to an excited state of the ideal gas. We now ask: is it possible for the gas to lower its energy by transferring energy to the “container”, which we imagine to be the vessel holding the gas in the ring? It is clear that if the container absorbs a small amount of energy and momentum, the point A can move continuously towards $\nu = 0$. According to Bloch, this possibility leads to the continuous degradation of the flow until the gas is finally in equilibrium with the container (at zero temperature). This argument indicates that an ideal gas cannot exhibit persistent currents when in contact with an imperfect (i.e. not perfectly smooth) container.

In view of this, one can imagine a $\epsilon_0(\nu)$ function which results in $E_0(\nu)$ shown in Fig. 3.1.2(b). This curve is seen to exhibit a local minimum at $\nu = N$. If we start with the state A, the system can lower its energy by moving towards $\nu = 0$ in a continuous fashion, i.e. the flow decays in this case. However, if we consider point B or C, a continuous decrease in energy moves the points towards $\nu = N$. This point is not an absolute minimum, which occurs at $\nu = 0$, but to arrive at the latter would require a transition with a macroscopically large change in angular momentum. The
improbability of such a large angular momentum transfer results in the system being “stuck” in the $\nu = N$ state. In other words, there is persistent flow which is not disrupted by imperfections in the walls of the container.

To obtain a local minimum at $\nu = N$ we require $\left. \frac{dE}{d\nu} \right|_{\nu = N} < 0$, i.e. $\left. \frac{d\epsilon}{d\nu} \right|_{\nu = N} + \frac{\hbar^2 \nu}{MNR^2} \left|_{\nu = N} < 0$. Since the periodicity of $\epsilon(\nu)$ implies $\epsilon'(N^-) = -\epsilon'(0^+)$, this inequality can be written as $\epsilon'(0^+) > \frac{\hbar^2}{2MR^2}$. For the ideal gas, we have $\epsilon'(0^+) = \frac{\hbar^2}{2MR^2}$, and clearly this inequality cannot be satisfied in this case. As concluded earlier, we cannot have persistent currents for an ideal gas. However, if $E(\nu)$ were to behave as in Fig. 3.1.2(b), persistent currents would be possible. As we shall see, the inclusion of interactions does in fact lead to a $E(\nu)$ curve having the behaviour shown in Fig. 3.1.2(b).

### 3.2 Extension of Bloch’s argument to a two-species gas

We now extend the analysis for a single species to the case where two types of particles $A$ and $B$ are confined to the ring. The total angular momentum operator is now

$$\hat{L} = \hat{L}_A + \hat{L}_B,$$

where the index $i$ runs from 1 to $N_A$ for the particles of the type $A$ type and from $N_A + 1$ to $N_A + N_B = N$ for the particles of type $B$. It is clear from its definition that $\hat{L} = \hat{L}_A + \hat{L}_B$, where $\hat{L}_A(\hat{L}_B)$ is the angular momentum operator of the $A(B)$ particles. A possible eigenfunction of the total angular momentum operator is the
product wave function

\[ \Phi(\theta_1, \ldots, \theta_N) = \phi_1(\theta_1) \times \cdots \times \phi_N(\theta_N), \quad (3.2.2) \]

where \( \phi_i(\theta_i) \propto \exp(il_i\theta_i) \). This state has total angular momentum

\[ L = \hbar \sum_{i=1}^{N} l_i + \hbar \sum_{i=N_A+1}^{N_A+N_B} l_i = L_A + L_B. \quad (3.2.3) \]

If each of the \( A \) and \( B \) species consists of indistinguishable bosons, the wave function \( \Phi \) in (3.2.2) must be symmetrized for it to be a suitable bosonic state. This can be achieved by applying the symmetrization operator \( \hat{S} \)

\[ \hat{S} = \hat{S}_A \hat{S}_B \quad (3.2.4) \]

to (3.2.2), where

\[ \hat{S}_A = \frac{1}{N_A!} \sum_{P_A} \hat{P}_A \quad (3.2.5) \]
\[ \hat{S}_B = \frac{1}{N_B!} \sum_{P_B} \hat{P}_B. \quad (3.2.6) \]

The operator \( \hat{P}_A \) permutes the coordinates of the \( A \) particles, whereas \( \hat{P}_B \) does the same for \( B \) particles. Applying the symmetrization operator to the wave function \( \Phi(\theta_1, \ldots, \theta_N) \) yields

\[ \hat{S} \Phi(\theta_1, \ldots, \theta_N) = \frac{1}{N_A!} \sum_{P_A} \hat{P}_A \exp \left( i \sum_{i=1}^{N_A} l_i \theta_i \right) \frac{1}{N_B!} \sum_{P_B} \hat{P}_B \exp \left( i \sum_{i=N_A+1}^{N_A+N_B} l_i \theta_i \right) \]
\[ = \frac{1}{N_A!} \frac{1}{N_B!} \sum_{P_A P_B} \exp \left( i \sum_{i=1}^{N_A} l_i \theta_{i,A} \right) \exp \left( i \sum_{i=N_A+1}^{N_A+N_B} l_i \theta_{i,B} \right). \quad (3.2.7) \]

The total angular momentum operator commutes with both \( \hat{P}_A \) and \( \hat{P}_B \). We may
now rewrite the exponentials in (3.2.7) in the following way

\[
\exp \left( i \sum_{i=1}^{N_A} l_i \theta_{PA}^i \right) \exp \left( i \sum_{i=N_A+1}^{N_A+N_B} l_i \theta_{PB}^i \right) = \exp[il\Theta] \exp \left[ i \sum_{i=1}^{N_A} (l_i - l) \theta_{PA}^i \right] \exp \left[ i \sum_{i=N_A+1}^{N_A+N_B} (l_i - l) \theta_{PB}^i \right],
\]

(3.2.8)

where \( \Theta = \sum_{i=1}^{N} \theta_i \). If we define \( l \) such that \( Nl = \sum_{i=1}^{N} l_i \), the last two exponentials in (3.2.8) are functions of the relative angular coordinates \( (\theta_i - \theta_j) \) regardless of what species an angular coordinate belongs to. This can again be checked by noting that the replacement \( \theta_{PA}^i \to \theta_{PA}^i + \Delta \theta \) and \( \theta_{PB}^i \to \theta_{PB}^i + \Delta \theta \) leaves the product of exponentials unchanged. With these results we see that (3.2.7) can be written as

\[
\Phi_{\{l_i\}} = \exp[il\Theta] \chi_{\{l_i\}}(\theta_1, \ldots, \theta_N),
\]

(3.2.9)

as was the case for the single-species situation. These functions form a basis of states in terms of which an arbitrary two-species wave function \( \Psi(\theta_1, \ldots, \theta_N) \) can be expanded.

The Hamiltonian \( \hat{H} \) for the two-species system is taken to be

\[
\hat{H} = \sum_{i=1}^{N_A} \frac{\hat{p}_i^2}{2MA^2} + \sum_{i=N_A+1}^{N_A+N_B} \frac{\hat{p}_i^2}{2MB^2} + \sum_{i<j} v_{ij}(\theta_i - \theta_j),
\]

(3.2.10)

where we allow for the masses of the \( A \) and \( B \) particles to be different. In addition, the interactions between the masses of the particles will in general be species-dependent (for example \( v_{ij}(\theta_i - \theta_j) = v_{AB}(\theta_i - \theta_j) \) if \( i \in A \) and \( j \in B \)). The stationary states are solutions of the equation \( \hat{H}\Psi = E\Psi \). Since \( \hat{H} \) commutes with the total angular momentum operator \( \hat{L} \) in (3.2.1), we can look for stationary states \( \Psi_L \) that are simultaneous eigenstates of \( \hat{H} \) and \( \hat{L} \). For a total angular momentum \( L \), the eigenstates can be
CHAPTER 3. PERSISTENT CURRENTS

expanded in terms of the basis functions (3.2.9) with $\hbar \sum_{i=1}^{N} l_i = L$, in other words

$$\Psi_L(\theta_1, ..., \theta_N) = \sum_{\{l_i\}} C(\{l_i\}) \Phi_{\{l_i\}}(\theta_1, ..., \theta_N) \equiv \exp \{il\Theta\} \chi_L(\theta_1, ..., \theta_N). \quad (3.2.11)$$

As explained before, $\chi_L(\theta_1, ..., \theta_N)$ is a function of the relative angular coordinates $\theta_i - \theta_j$.

Applying $\hat{H}$ to $\Psi_L$, we obtain

$$\hat{H}\Psi_L = \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \frac{\hbar^2 l^2 N_A}{2M_AR^2} \chi_L + \frac{\hbar l}{M_AR^2} \hat{L}_A \chi_L - \frac{\hbar^2}{2M_AR^2} \sum_{i=1}^{N_A} \frac{\partial^2 \chi_L}{\partial \theta_i^2} \right) \chi_L + \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \left( \frac{\hbar^2 l^2 N_B}{2M_BR^2} \chi_L + \frac{\hbar l}{M_BR^2} \hat{L}_B \chi_L - \frac{\hbar^2}{2M_BR^2} \sum_{i=N_A+1}^{N_A+N_B} \frac{\partial^2 \chi_L}{\partial \theta_i^2} \right) \chi_L + \exp \left[ il \sum_{i=1}^{N} \theta_i \right] \sum_{i<j} v_{ij}(\theta_i - \theta_j) \chi_L. \quad (3.2.12)$$

We thus see that the function $\chi_L$ satisfies the Schrödinger equation

$$\hat{H}_L \chi_L = \epsilon_L \chi_L, \quad (3.2.13)$$

where

$$\hat{H}_L = \sum_{N=1}^{N_A} \frac{\hat{L}_i^2}{2M_AR^2} + \sum_{N=N_A+1}^{N_A+N_B} \frac{\hat{L}_i^2}{2M_BR^2} + \sum_{i<j} v_{ij}(\theta_i - \theta_j) + \frac{L}{NR^2} \left( \frac{\hat{L}_A}{M_A} + \frac{\hat{L}_B}{M_B} \right) \quad (3.2.14)$$

and

$$\epsilon_L = E - \left( \frac{\hbar^2 l^2 N_A}{2M_AR^2} + \frac{\hbar^2 l^2 N_B}{2M_BR^2} \right) = E - \frac{L^2}{2NR^2} \left( \frac{N_A}{M_A} + \frac{N_B}{M_B} \right). \quad (3.2.15)$$
As compared to the single-species case, we see that the Hamiltonian depends explicitly on $L$ through the last term in (3.2.14). Since $(\hat{L}_A + \hat{L}_B)\chi_L = 0$, we see

$$
\left( \frac{\hat{L}_A}{M_A} + \frac{\hat{L}_B}{M_B} \right) \chi_L = \left( \frac{1}{M_A} - \frac{1}{M_B} \right) \hat{L}_A \chi_L. \tag{3.2.16}
$$

This shows that the last term in (3.2.15) has no effect if $M_A = M_B$. However, if $M_A \neq M_B$, this last term spoils the property that $\epsilon_L$ is a periodic function of $L$. Nonetheless, the periodic boundary condition in (3.1.25) still applies and implies

$$
\chi_L(\ldots, \theta_i + 2\pi, \ldots) = \exp \left[ -i2\pi \frac{\nu}{N} \right] \chi_L(\ldots, \theta_i, \ldots). \tag{3.2.17}
$$

If we consider $L' = L + N\hbar$ (i.e. $\nu' = \nu + N$), we have

$$
\chi_{L'}(\ldots, \theta_i + 2\pi, \ldots) = \exp \left[ -i2\pi \frac{\nu}{N} \right] \chi_{L'}(\ldots, \theta_i, \ldots), \tag{3.2.18}
$$

so that $\chi_{L'}$ obeys the same boundary condition as $\chi_L$. However, $\chi_{L'}$ satisfies the equation

$$
\left[ \hat{H} + \frac{\hbar}{R^2} \left( \frac{\hat{L}_A}{M_A} + \frac{\hat{L}_B}{M_B} \right) \right] \chi_{L'} = \epsilon_{L'} \chi_{L'}, \tag{3.2.19}
$$

which is different from the equation satisfied by $\chi_L$. As a result, $\epsilon_{L'} \neq \epsilon_L$, and we thus conclude that $\epsilon_L$ is no longer a periodic function of $L$ with period $N\hbar$. In the following, we restrict ourselves to the case $M_A = M_B$ for which $\epsilon_L$ is periodic as in the single-species case.
Chapter 4

Mean-field theory of persistent currents

In the previous chapter we obtained an expression for the condensate energy using quite general arguments. In this chapter we shall perform calculations in the context of mean-field theory which allows us to obtain an expression for the condensate energy with good accuracy and to make conclusions regarding the possibility of persistent currents.

4.1 The single-species case

In this section we shall shed light on how the energy behaves as a function of angular momentum. In particular, we show that if the interactions are sufficiently strong, the energy behaves as shown in Fig. 3.1.2(b). We derive the ground state energy of a condensate within mean-field approximation by using the formula for the GP energy.
obtained in (2.2.1) calculated per particle in units of $\hbar^2/(2MR^2)$

$$E_0[\psi] = \int_0^{2\pi} d\theta \left| \frac{d\psi}{d\theta} \right|^2 + \gamma \pi \int_0^{2\pi} d\theta |\psi|^4,$$  \hspace{1cm} (4.1.1)

where the interaction parameter $\gamma$ was introduced in (2.2.9). Since all particles of the condensate are in the same quantum state, the many-body wave function $\Psi(\theta_1, ..., \theta_N)$ may be presented in the product form

$$\Psi(\theta_1, ..., \theta_N) = \psi(\theta_1) ... \psi(\theta_N),$$  \hspace{1cm} (4.1.2)

where the wave functions $\psi(\theta)$ are normalized to 1 around the ring

$$\int_0^{2\pi} d\theta |\psi(\theta)|^2 = 1.$$  \hspace{1cm} (4.1.3)

The expectation value of the angular momentum operator $\hat{L}$ is given by

$$\langle \hat{L} \rangle = \langle \Psi | \hat{\mathbf{L}} | \Psi \rangle = N \int_0^{2\pi} d\theta \psi^*(\theta) \frac{\hbar}{i} \frac{\partial}{\partial \theta} \psi(\theta) = N \hbar l,$$  \hspace{1cm} (4.1.4)

where $l$ is the total angular momentum per particle expressed in units of $\hbar$.

We now allow the wave functions $\psi(\theta)$ to have an arbitrary angular momentum dependence and expand them in a basis of the angular wave functions

$$\psi(\theta) = \sum_m c_m \phi_m(\theta),$$  \hspace{1cm} (4.1.5)

where $\phi_m = e^{im\theta}/\sqrt{2\pi}$. For (4.1.3) to be satisfied we require

$$\sum_m |c_m|^2 = 1.$$  \hspace{1cm} (4.1.6)
In addition, we impose the constraint that the angular momentum per particle is

\[ l = \sum_m m|c_m|^2. \]  

(4.1.7)

Clearly, if \( l \) is not integral, \( \psi(\theta) \) must be a superposition of different angular momentum states. Using the definition in (3.1.24), we can write the full energy as

\[ \bar{E}_0(l) = l^2 + \bar{\epsilon}_0(l). \]  

(4.1.8)

Substituting (4.1.5) into (4.1.1), we obtain

\[ \bar{E}_0(l) = \sum_m m^2|c_m(l)|^2 + \gamma \pi \int_0^{2\pi} \left| \sum_m c_m(l)\phi_m(\theta) \right|^4. \]  

(4.1.9)

and using (4.1.8) we have

\[ \bar{\epsilon}_0(l) = \sum_m (m^2 - l^2)|c_m(l)|^2 + \gamma \pi \int_0^{2\pi} \left| \sum_m c_m(l)\phi_m(\theta) \right|^4. \]  

(4.1.10)

The general problem is to minimize \( \bar{E}_0(l) \) or \( \bar{\epsilon}_0(l) \) with respect to the \( c_m(l) \) coefficients subject to the constraints given by (4.1.6) and (4.1.7). The dependence of the coefficients on the variable \( l \) is shown explicitly.

Equations (4.1.9) and (4.1.10) define the energy for arbitrary values of \( l \). We now consider

\[ \bar{\epsilon}_0(l + 1) = \sum_m \left[ m^2 - (l + 1)^2 \right] |c_m(l + 1)|^2 \]

\[ + \gamma \pi \int_0^{2\pi} \left| \sum_m c_m(l + 1)\phi_m(\theta) \right|^4. \]  

(4.1.11)

By redefining the summation index \( m \), (4.1.11) can be rewritten

\[ \bar{\epsilon}_0(l + 1) = \sum_m \left[ (m^2 - l^2) + 2(m - l) \right] |c_{m+1}(l + 1)|^2 \]

\[ + \gamma \pi \int_0^{2\pi} \left| \sum_m c_{m+1}(l + 1)\phi_{m+1}(\theta) \right|^4. \]  

(4.1.12)
Since \(\phi_{m+1}(\theta) = \phi_m e^{i\theta}\), we see that
\[
\left| \sum_m c_{m+1}(l+1)\phi_{m+1}(\theta) \right|^4 = \left| \sum_m c_{m+1}(l+1)\phi_m(\theta) \right|^4.
\]
We now observe that if
\[
c_{m+1}(l+1) = c_m(l)
\]
we have
\[
\sum_m (m-l)|c_{m+1}(l+1)|^2 = \sum_m (m-l)|c_m(l)|^2 = 0
\]
and
\[
\int_0^{2\pi} d\theta \left| \sum_m c_{m+1}(l+1)\phi_{m+1}(\theta) \right|^4 = \int_0^{2\pi} d\theta \left| \sum_m c_{m+1}(l+1)\phi_m(\theta) \right|^4 = \int_0^{2\pi} d\theta \left| \sum_m c_m(l)\phi_m(\theta) \right|^4.
\]
We thus see that \(\bar{\epsilon}_0(l+1)\) in (4.1.12) reduces to \(\bar{\epsilon}_0(l)\) in (4.1.10). In other words, the assumed dependence in (4.1.13) ensures the periodicity of \(\bar{\epsilon}_0(l)\) that Bloch’s argument requires [1]. Because of this periodicity we can restrict our attention to \(0 \leq l \leq 1\).

In addition to the periodicity property in (4.1.13), we also have the property
\[
c_m(-l) = c^*_m(l),
\]
which follows from the observation that \(\psi(\theta; -l) = \psi^*(\theta; l)\). In other words, the state with angular momentum \(-l\) is just the complex conjugate of the state with angular momentum \(l\). This property implies that
\[
\bar{\epsilon}_0(-l) = \bar{\epsilon}_0(l),
\]
which is analogous to the general result stated after (3.1.27).

The \(l\)-dependence of the coefficients can, in fact, be verified by explicit calculations of the kind we now describe. The minimization of (4.1.9) with respect to \(c_m(l)\) subject to the constraints (4.1.6) and (4.1.7) is in general very complex because of...
the nonlinearity of the interactions. We will therefore approach the problem in a step-wise fashion by starting with the simplest possible wave function which we then systematically improve. If we take the wave function to take the form

$$\psi(\theta) = c_0 \phi_0 + c_1 \phi_1,$$  \hspace{1cm} (4.1.18)

the coefficients $c_0$ and $c_1$ must satisfy the constraints

$$|c_0|^2 + |c_1|^2 = 1$$  \hspace{1cm} (4.1.19)

and

$$|c_1|^2 = l.$$  \hspace{1cm} (4.1.20)

As a result, there is no variational freedom and the coefficients are given by $|c_1|^2 = l$ and $|c_0|^2 = 1 - l$.

For the assumed wave function, the interaction energy is

$$\gamma \pi \int_0^{2\pi} d\theta \left| \sum_m c_m(l) \phi_m(\theta) \right|^4 = \frac{\gamma}{4\pi} \int_0^{2\pi} d\theta \left( c_0^* + c_1^* e^{-i\theta} \right)^2 \left( c_0 + c_1 e^{i\theta} \right)^2$$

$$= \frac{\gamma}{4\pi} \int_0^{2\pi} d\theta \left[ (c_0^*)^2 + 2c_0^* c_1^* e^{-i\theta} + (c_1^*)^2 e^{-2i\theta} \right]$$

$$\times \left[ (c_0)^2 + 2c_0 c_1 e^{i\theta} + (c_1)^2 e^{2i\theta} \right]$$

$$= \frac{\gamma}{2} \left[ (|c_0|^2 + |c_1|^2)^2 + 2|c_0|^2 |c_1|^2 \right]$$

$$= \frac{\gamma}{2} \left[ 1 + 2l(1 - l) \right].$$  \hspace{1cm} (4.1.21)

From (4.1.9), the total energy is therefore

$$\bar{E}_0(l) = l + \frac{\gamma}{2} \left[ 1 + 2l(1 - l) \right]$$  \hspace{1cm} (4.1.22)

and from (4.1.8)

$$\bar{\epsilon}_0(l) = l(1 - l) + \frac{\gamma}{2} \left[ 1 + 2l(1 - l) \right].$$  \hspace{1cm} (4.1.23)
CHAPTER 4. MEAN-FIELD THEORY OF PERSISTENT CURRENTS

Figure 4.1.1: (a) The full energy (4.1.8) and (b) The periodic function $\bar{\epsilon}_0(l)$ (4.1.23) plotted in the interval $0 \leq l \leq 1$ for the case $\gamma = 2$.

The functions in (4.1.8) and (4.1.23) are plotted in Fig. 4.1.1 for $\gamma = 2$. Using the fact that (4.1.23) is periodic, $\bar{E}_0(l)$ in (4.1.8) can be plotted for any $l$; the range $0 \leq l \leq 2$ is shown in Fig. 4.1.1(a). The graph shows that $\bar{E}_0(l)$ has a local minimum at $l = 1$.

As explained in Section 3.1, this implies that persistent currents are possible. The condition for their existence at $l = 1$ is $d\bar{E}_0(l)/dl \bigg|_{l=1} < 0$; the limiting condition is

$$
\frac{d}{dl} \bar{E}_0(l) \bigg|_{l=1} = 0.
$$

Using the expression for the full energy in (4.1.22) and setting its derivative to zero at $l = 1$, we have

$$
\frac{d}{dl} E_0(l) \bigg|_{l=1} = 1 - \gamma_{cr} = 0,
$$

which implies $\gamma_{cr} = 1$. Within the approximations made, $\gamma$ must exceed this critical value of $\gamma$ for persistent currents to exist.

To improve this estimate, we now consider the calculation of $\bar{\epsilon}_0(l)$ retaining additional terms in the wave function. Since $\bar{\epsilon}_0'(0^+) = -\bar{\epsilon}_0'(1^-)$, the required behaviour near $l = 1$ can be determined by analyzing the behaviour of $\bar{\epsilon}_0(l)$ near $l = 0$. For this
reason we take $l = \varepsilon$, where $\varepsilon$ is a small quantity and consider the three-component condensate wave function

$$
\psi(\theta) = c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1. \tag{4.1.26}
$$

For $\varepsilon \to 0$, $|c_0|^2 \to 1$ and $|c_{-1}|^2$ and $|c_1|^2$ tend to zero. We thus expect $|c_{-1}|^2$ and $|c_1|^2$ to be of order $\varepsilon$ and we will systematically retain contributions to the energy to this order. Later we show that other angular momentum components are of higher order in $\varepsilon$. It is therefore appropriate to restrict the analysis to the three-component wave function in (4.1.26).

The normalization condition requires

$$
|c_{-1}|^2 + |c_0|^2 + |c_1|^2 = 1 \tag{4.1.27}
$$

and the angular momentum constraint now reads

$$
-|c_{-1}|^2 + |c_1|^2 = \varepsilon. \tag{4.1.28}
$$

This equation indicates why we expect these coefficients to be of order $\varepsilon$.

As compared to the two-species ansatz in (4.1.18), the three-component wave function provides some variational freedom in the minimization of the energy. The coefficients $c_l$ are in general complex, and we write them in the following form

$$
c_l = |c_l|e^{i\alpha_l}. \tag{4.1.29}
$$

In this case we find

$$
\bar{E}_0 = |c_{-1}|^2 + |c_1|^2
+ \frac{\gamma}{2} \left[ 1 + 2|c_{-1}|^2|c_0|^2 + 2|c_0|^2|c_1|^2 + 2|c_{-1}|^2|c_1|^2 
+ 4|c_{-1}||c_0|^2|c_1| \cos(\alpha_{-1} - 2\alpha_0 + \alpha_1) \right]. \tag{4.1.30}
$$
We see that the phases appear only in the combination \((\alpha - 1 - 2\alpha_0 + \alpha_1)\) and to minimize \(\bar{E}_0(l)\) for positive \(\gamma\) we must choose this combination to be \(\pi\). Using the normalization condition to express \(|c_0|^2\) in favour of \(|c_{-1}|^2\) and \(|c_1|^2\), and retaining only terms of order \(\varepsilon\), we obtain

\[
\bar{E}_0(l) - \frac{\gamma}{2} \simeq -\varepsilon + 2|c_1|^2 + \gamma (|c_{-1}| - |c_1|)^2. \tag{4.1.31}
\]

Our task is now to minimize this expression subject to the angular momentum constraint. Although one can express the energy in terms of one variable, say \(|c_1|\), it is difficult to minimize the resulting expression. We have therefore used the Lagrange method in which the angular momentum constraint is introduced by means of a Lagrange multiplier \(\lambda\). We thus consider the minimization of the function

\[
F(|c_{-1}|, |c_1|) = -\varepsilon + 2|c_1|^2 + \gamma (|c_{-1}| - |c_1|)^2 + \lambda (-|c_{-1}|^2 + |c_1|^2). \tag{4.1.32}
\]

\(|c_{-1}|\) and \(|c_1|\) can now be treated as independent variables. Minimizing this function with respect to \(|c_{-1}|\) and \(|c_1|\) we obtain

\[
\frac{\partial F}{\partial |c_{-1}|} = 2\gamma (|c_{-1}| - |c_1|) - 2\lambda |c_{-1}| = 0 \tag{4.1.33}
\]

\[
\frac{\partial F}{\partial |c_1|} = 4|c_1| - 2\gamma (|c_{-1}| - |c_1|) + 2\lambda |c_1| = 0. \tag{4.1.34}
\]

Adding (4.1.33) and (4.1.34) we obtain

\[
|c_{-1}| = |c_1| \left(\frac{\lambda + 2}{\lambda}\right). \tag{4.1.35}
\]

The substitution of this expression for \(|c_{-1}|\) into either (4.1.33) or (4.1.34) determines the Lagrange parameter \(\lambda\) as a solution of

\[
2\gamma = \lambda(\lambda + 2). \tag{4.1.36}
\]
Since this is a quadratic equation, there are two roots

\[ \lambda_{\pm} = -1 \pm \sqrt{1 + 2\gamma}. \]  

(4.1.37)

Using (4.1.35) in (4.1.28), we obtain

\[ |c_1|^2 = \frac{-\lambda^2}{4(\lambda + 1)} \varepsilon = \varepsilon \left[ \frac{(1 \pm \sqrt{1 + 2\gamma})^2}{\pm 4\sqrt{1 + 2\gamma}} \right], \]  

(4.1.38)

where the upper(lower) sign corresponds to \( \lambda_+ (\lambda_-) \). If \( \varepsilon > 0 \), we see that we must choose \( \lambda_- \) (lower sign) to obtain a positive quantity on the right-hand side of (4.1.38). Conversely, if \( \varepsilon < 0 \) we must choose \( \lambda_+ \). Using (4.1.28) again as well as the normalization (4.1.27) one may obtain expressions for all three coefficients:

\[ |c_{-1}|^2 = \varepsilon \left[ \frac{(1 - \text{sgn}(\varepsilon)\sqrt{1 + 2\gamma})^2}{4\sqrt{1 + 2\gamma}} \right], \]  

(4.1.39)

\[ |c_0|^2 = 1 - \varepsilon \left[ \frac{1 + \gamma}{\sqrt{1 + 2\gamma}} \right], \]  

(4.1.40)

\[ |c_1|^2 = \varepsilon \left[ \frac{(1 + \text{sgn}(\varepsilon)\sqrt{1 + 2\gamma})^2}{4\sqrt{1 + 2\gamma}} \right]. \]  

(4.1.41)

If we now substitute these expressions into (4.1.31), we obtain the energy

\[ \bar{E}_0(l) = \frac{\gamma}{2} + \varepsilon \sqrt{1 + 2\gamma}. \]  

(4.1.42)

The corresponding periodic function is then

\[ \bar{\epsilon}_0(l) = \bar{E}_0(l) - l^2 \simeq \frac{\gamma}{2} + \varepsilon \sqrt{1 + 2\gamma} \]  

(4.1.43)

to lowest order in \( \varepsilon \).

As we discussed earlier, the condensate can support persistent currents when \( \bar{E}_0(l) \) exhibits a local minimum at integer values of \( l \). The limiting condition for this minimum to exist at \( l = 1 \) is given in (4.1.24). Using \( \bar{E}_0(l) = \bar{\epsilon}_0(l) + l^2 \) and the fact
that $\epsilon'_0(1^-) = -\epsilon'_0(0^+)$, this condition becomes

$$\left. \frac{\partial \tilde{E}_0(l)}{\partial l} \right|_{l=1^-} = 0 = -\sqrt{1 + 2\gamma} + 2. \quad (4.1.44)$$

From this the critical value of the interaction parameter is found to be $\gamma_{cr} = 3/2$. This value improves on the value $\gamma_{cr} = 1$ obtained earlier using the two-component wave function. For $l = n + l'$, where $n = 1, 2\ldots$ and $0 \leq l' \leq 1$ we obtain

$$\tilde{E}_0(l) = (n + l')^2 + \bar{\epsilon}_0(n + l')$$

$$\left. \frac{d\tilde{E}_0(l)}{dl} \right|_{l=n^-} = 2n + \left. \frac{d\bar{\epsilon}_0(l)}{dl} \right|_{l=0^-} = 2n - \sqrt{1 + 2\gamma}. \quad (4.1.45)$$

Thus

$$\gamma_{cr,n} = \frac{(2n - 1)(2n + 1)}{2}. \quad (4.1.46)$$

The critical value increases with $n$; $\gamma_{cr,1} = 3/2$.

The remaining question is whether the value $\gamma_{cr} = 3/2$ is changed if more angular momentum terms are retained in the wave function, for example $c_{-2}$ and $c_2$. The following argument indicates that these coefficients are of higher order in $\epsilon$ when $l$ is small. The variation of $\tilde{E}_0$ with respect to $c_2^*$ gives the equation

$$\frac{\partial \tilde{E}_0}{\partial c_2^*} = 8c_2 + \frac{\gamma}{4\pi} \int_0^{2\pi} d\theta \ 2e^{-2i\theta} \left( c_{-2}e^{2i\theta} + c_{-1}e^{i\theta} + c_0^* + c_1^*e^{-i\theta} + c_2^*e^{-2i\theta} \right) \times \left( c_{-2}e^{-2i\theta} + c_{-1}e^{-i\theta} + c_0 + c_1e^{i\theta} + c_2e^{2i\theta} \right)^2 = 0. \quad (4.1.47)$$

Assuming $c_{\pm 2}$ are negligible compared to the other terms in the integral, we find

$$8c_2 + \gamma \left( c_0^*c_1^2 + 2c_{-1}^*c_0c_1 \right) = 0. \quad (4.1.48)$$

Since $c_0 \sim O(1)$ and $|c_{\pm 1}| \sim O(\epsilon^{1/2})$, this implies that $|c_2| \sim O(\epsilon)$, i.e. $|c_2|^2 \sim O(\epsilon^2)$. As a result these higher order coefficients do not affect the $l$-dependence of $E_0(l)$.
given in (4.1.42) for small $l$. Thus, the critical value $\gamma_{cr} = 3/2$ obtained with the three-component wave function is a rigorous result. This confirms the statements made in [23] and [22] where no details of the calculation were provided.

It is also possible to determine the value of the energy at $l = 1/2$. In order to do this it is necessary to include at least 4 coefficients. We thus introduce a four-component wave function

$$\psi(\theta) = c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1 + c_2\phi_2,$$

(4.1.49)

where the constraints are

$$|c_{-1}|^2 + |c_0|^2 + |c_1|^2 + |c_2|^2 = 1$$

(4.1.50)

and

$$-|c_{-1}|^2 + |c_1|^2 + 2|c_2|^2 = l.$$  

(4.1.51)

We start by calculating the kinetic energy which is found to be

$$\int_0^{2\pi} d\theta \left| \frac{d\psi}{d\theta} \right|^2 = |c_{-1}|^2 + |c_1|^2 + 4|c_2|^2.$$  

(4.1.52)

The interaction energy is given by

$$\gamma\pi \int_0^{2\pi} d\theta |\psi|^4 = \frac{\gamma}{2} \left[ |c_{-1}|^4 + |c_0|^4 + |c_1|^4 + |c_2|^4 + 4|c_{-1}|^2|c_1|^2 + 4|c_{-1}|^2|c_0|^2 ight.
+ 4|c_{-1}|^2|c_2|^2 + 4|c_0|^2|c_2|^2 + 4|c_1|^2|c_2|^2 + 4|c_0|^2|c_1|^2 + 2c_0^2c_{-1}c_1 + 2c_1^2c_0c_2 + 2c_1^2c_0^*c_2^* + 4c_{-1}^*c_0c_1c_2 + 4c_{-1}^*c_0^*c_1c_2^* \left. \right].$$  

(4.1.53)

As before, we represent each of the coefficients in the form (4.1.29). Thus, the complex-valued terms (4.1.53) become

$$2c_{-1}^*c_1c_1 = 4|c_0|^2|c_{-1}|^2 |c_1| \cos(2\alpha_0 - \alpha_1 - \alpha_2)$$

(4.1.54)

$$2c_0^2c_0c_1c_2 + 2c_1^2c_0^*c_2^* = 4|c_1|^2|c_0||c_2| \cos(2\alpha_1 - \alpha_0 - \alpha_2)$$

(4.1.55)

$$4c_{-1}^*c_0c_1c_2 + 4c_{-1}^*c_0^*c_1c_2 = 8|c_{-1}||c_0||c_1||c_2| \cos(-\alpha_2 + \alpha_0 + \alpha_1 - \alpha_2).$$

(4.1.56)
At this point we note that we are exploring the behaviour of the full energy at just one point \( l = 1/2 \). We then have \( |c_0| = |c_1| \) and \( |c_{-1}| = |c_2| \) which for simplicity we denote by \( x \) and \( y \) respectively. The demonstration of these equalities is given later.

The normalization constraint in (4.1.50) then becomes

\[
x^2 + y^2 = 1/2 \tag{4.1.57}
\]

and the angular momentum constraint (4.1.51) is given by

\[
x^2 + y^2 = l. \tag{4.1.58}
\]

At \( l = 1/2 \), these two constraints are in fact equivalent.

If we define \( \beta_1 = 2\alpha_0 - \alpha_{-1} - \alpha_1 \) and \( \beta_2 = 2\alpha_{-1} - \alpha_0 - \alpha_2 \), we have \(-\alpha_1 + \alpha_0 + \alpha_1 - \alpha_2 = \beta_1 + \beta_2 \). Thus, there are only two independent phases in (4.1.54 - 4.1.56).

Using these definitions and the normalization constraint (4.1.57), the energy becomes

\[
\bar{E}_0(x, y) = x^2 + 5y^2 + \gamma \left[ \frac{1}{2} + x^4 + y^4 + 4x^2y^2 + 2x^3y \cos \beta_1 + 2x^3y \cos \beta_2 + 4x^2y^2 \cos(\beta_1 + \beta_2) \right]. \tag{4.1.59}
\]

Minimizing this expression with respect to \( \beta_1 \) and \( \beta_2 \), we obtain the equations

\[
\frac{\partial \bar{E}_0}{\partial \beta_1} = \gamma \left[ -2x^3y \sin \beta_1 - 4x^2y^2 \sin(\beta_1 + \beta_2) \right] = 0 \tag{4.1.60}
\]

\[
\frac{\partial \bar{E}_0}{\partial \beta_2} = \gamma \left[ -2x^3y \sin \beta_2 - 4x^2y^2 \sin(\beta_1 + \beta_2) \right] = 0. \tag{4.1.61}
\]

These equations imply \( \beta_1 = \beta_2 = \beta \). Using this result in (4.1.60) we obtain

\[
\sin \beta \left( x + 4y \cos \beta \right) = 0. \tag{4.1.62}
\]

This equation can be satisfied in two different ways: either for

\[
\beta = \pi n, \tag{4.1.63}
\]
Figure 4.1.2: Plots of energy functionals (4.1.65) and (4.1.66) for $\gamma = 2$ for different choices of the phase $\beta$. The upper solid curve corresponds to $\beta = 0$, the middle dashed curve to $\beta = \pi$, and the lower solid curve to $\cos \beta = -x/4y$.

where $n = 0, \pm 1, \pm 2, \ldots$, or for

$$\cos \beta = -\frac{x}{4y}.$$  \hfill (4.1.64)

If we choose $\beta = \pi n$, the energy takes the form

$$\tilde{E}_0(x, y) = x^2 + 5y^2 + \gamma \left(\frac{1}{2} + x^4 + y^4 + 8x^2y^2 \pm 4x^3y\right),$$  \hfill (4.1.65)

whereas for $\cos \beta = -x/(4y)$, we have

$$\tilde{E}_0(x, y) = x^2 + 5y^2 + \gamma \left(\frac{1}{2} + \frac{1}{2}x^4 + y^4\right).$$  \hfill (4.1.66)

The normalization condition (4.1.57) allows one to plot these expressions as functions of $x$. In the case of (4.1.65), $x$ is restricted to the range $x \leq 1/\sqrt{2} \approx 0.707$. However, for (4.1.66) we must have $\cos \beta = -x/4y$ which imposes the restriction $x/\sqrt{1/2 - x^2} \leq 1$, that is

$$x \leq \sqrt{\frac{8}{17}} \approx 0.686.$$  \hfill (4.1.67)
Figure 4.1.3: The upper solid curve is the periodic function $\tilde{\epsilon}_0(l)$ for the two-component wave function in (4.1.23). The dashed curve is the fit $\tilde{\epsilon}_0(l) \simeq \gamma/2 + al(1 - l)$, where $a$ is equal to the slope at $l = 0$ as determined by the three-component wave function analysis. The point at $l = 1/2$ is the value obtained using the four-component wave function. The calculations are for $\gamma = 2$.

The results plotted in Fig. 4.1.2 are obtained for $\gamma = 2$. The curve corresponding to $\beta = \pi$ is seen to provide the lowest possible value of the energy $\tilde{E}_0(x)$. This occurs at $x \simeq 0.696$ where $\tilde{E}_0 = 1.816$. Using $\tilde{\epsilon}_0 = \tilde{E}_0 - l^2$, we have $\tilde{\epsilon}_0 = 1.566$ for $l = 1/2$. Interestingly, the $\cos \beta = -x/4y$ curve lies below the $\beta = \pi$ curve for all values of $x$ up to the point where it touches the $\beta = \pi$ curve at $x = \sqrt{8/17}$. The $\beta = \pi$ curve continues beyond this point to reach its absolute minimum at $x \simeq 0.696$.

These results could actually have been obtained more directly by using the general properties in (4.1.13) and (4.1.16). For example, these properties imply

$$c_2 \left( \frac{1}{2} \right) = c^*_2 \left( -\frac{1}{2} \right) = c^*_1 \left( \frac{1}{2} \right).$$

In other words, we have $|c_2 \left( \frac{1}{2} \right)| = |c_1 \left( \frac{1}{2} \right)|$, as stated earlier, and $\alpha_2 = -\alpha_1$. Similarly, we have $|c_0 \left( \frac{1}{2} \right)| = |c_1 \left( \frac{1}{2} \right)|$ and $\alpha_1 = -\alpha_0$. With these relations, the phases...
appearing in (4.1.65) are \( \beta_1 = 3\alpha_0 + \alpha_2 \), \( \beta_2 = -3\alpha_0 - \alpha_2 \) and \( \beta_1 + \beta_2 = 0 \). This shows that there is indeed only one independent phase and that the phase that minimizes the energy is \( \beta = \beta_1 = -\beta_2 = \pi \).

Fig. 4.1.3 summarizes the results for \( \bar{\epsilon}_0(l) \) that we have obtained to this point. The solid curve gives the two-component result in (4.1.23) with \( \gamma/2 \) subtracted. It reaches a maximum value of 0.75 at \( l = 1/2 \). The three-component analysis gave the behaviour of \( \bar{\epsilon}_0(l) \) near integral values of \( l \). For \( l \) close to zero we have \( \bar{\epsilon}_0(l) \simeq \frac{\gamma}{2} + |l|\sqrt{1 + 2\gamma} \). This result as well as \( \bar{\epsilon}_0(1 - l) = \bar{\epsilon}_0(-l) \), are shown by the short lines which are seen to lie below the two-component curve. Finally, the four-component wave function gives the single point \( \bar{\epsilon}_0 \simeq 1.566 \) at \( l = 1/2 \). As expected this point also lies below the two-component result. The dashed curve is the approximation to the behaviour for \( 0 \leq l \leq 1 \) assuming \( \bar{\epsilon}_0(l) \simeq \frac{\gamma}{2} + a(l - 1) \), where \( a \) is set to \( \sqrt{1 + 2\gamma} \) to reproduce the slope of \( \bar{\epsilon}_0(l) \) at \( l = 0 \). For \( \gamma = 2 \) this approximation takes the value 1.559 at \( l = 1/2 \) which is very close to the four-component value of 1.566. This close agreement suggests that the parabolic dependence assumed is a good approximation.

We can also use the information obtained for the behaviour of the coefficients \( |c_i|^2 \) to infer their approximate \( l \)-dependence. For \( l \) close to zero \( |c_{-1}|^2 \), \( |c_0|^2 \) and \( |c_1|^2 \) behave according to (4.1.39 - 4.1.41), whereas \( |c_{\pm 2}|^2 \sim O(\epsilon^2) \). Furthermore, we have \( |c_{l+1}(l+1)|^2 = |c_i(l)|^2 \). In Fig. 4.1.4, we show the lines passing through the values of \( |c_0|^2 \) and \( |c_1|^2 \) at \( l = 0 \) and \( l = 1 \) with the known values of the slopes at these points. The behaviour of \( |c_0|^2 \) between \( l = 0 \) and \( l = 1 \) can be fit by the function

\[
    f_0(l) = a + bl + cl^2 + dl^3, \quad (4.1.69)
\]
Figure 4.1.4: Slopes of the coefficients $|c_0|^2$, $|c_1|^2$ (fine lines) and approximating functions (bold lines) plotted for $\gamma = 2$. The point is the value of $|c_0|^2$ and $|c_1|^2$ at $l = 1/2$.

where the parameters are determined by the known limiting behaviour, namely,

$$
\begin{align*}
    f_0(0) &= 1 \\
    f_0(1) &= 0 \\
    f_0'(0) &= -\frac{1+\gamma}{\sqrt{1+2\gamma}} \\
    f_0'(1) &= -\frac{(1+\sqrt{1+2\gamma})^2}{4\sqrt{1+2\gamma}}.
\end{align*}
$$

(4.1.70)

For $\gamma = 2$ we have $f_0'(0) = -1.342$ and $f_0'(1) = -1.171$. We thus find that $f_0(l)$ takes the form

$$
f_0(l) = 1 - 1.342l + 0.857l^2 - 0.513l^3.
$$

(4.1.71)

Since $|c_1(l)|^2 = |c_0(l-1)|^2$ and $|c_0(-l)|^2 = |c_0(l)|^2$ we see that $f_1(l) = f_0(1-l)$. The solid curves in Fig. 4.1.4 show the functions $f_0(l)$ and $f_1(l)$. A similar analysis can be carried out for the $|c_{-1}|^2$ and $|c_1|^2$ coefficients. The simplest fits reproducing the
Figure 4.1.5: Slopes of the coefficients $|c_{-1}|^2$, $|c_2|^2$ (fine lines) and approximating functions (bold lines) plotted for $\gamma = 2$. The point is the value of $|c_{-1}|^2$ and $|c_2|^2$ at $l = 1/2$.

values and slopes at $l = 0$ and $l = 1$ are

$$f_{-1}(l) = 0.171l(1 - l)^2$$  \hspace{1cm} (4.1.72)

$$f_2(l) = 0.171l^2(1 - l).$$  \hspace{1cm} (4.1.73)

These functions are plotted in Fig. 4.1.5 for $0 \leq l \leq 1$.

To check these fits we can evaluate the fitting functions at $l = 1/2$ and compare these values with the known values obtained using the four-component wave function. We find $f_0(1/2) = f_1(1/2) = 0.479$ while $|c_0|^2(1/2) = |c_1|^2(1/2) = 0.484$ and $f_{-1}(1/2) = f_2(1/2) = 0.021$ while $|c_{-1}|^2(1/2) = |c_2|^2(1/2) = 0.016$. From this comparison one can see that the approximating functions give reasonably good results. In Fig. 4.1.6 we have used the periodicity of the coefficients to extend the plots to a larger range of $l$. 
Figure 4.1.6: The approximating functions for the coefficients $|c_l|^2$ plotted for $\gamma = 2$ for $-1 \leq l \leq 2$. 
4.2 The two-species case

In the previous section we investigated the behaviour of the condensate energy for the single-species gas. It is now of interest to examine the behaviour of a condensate containing \( N_A \) and \( N_B \) particles of two distinguishable species. We start by considering the condensate energy, which for the two-species gas has the form (2.1.37)

\[
E[\psi_A, \psi_B] = \int_0^{2\pi} d\theta \left( \frac{N_A h^2}{2M_A R^2} \left| \frac{d\psi_A}{d\theta} \right|^2 + \frac{N_B h^2}{2M_B R^2} \left| \frac{d\psi_B}{d\theta} \right|^2 \right) + \frac{1}{2} N_A^2 U_{AA} \int_0^{2\pi} d\theta |\psi_A|^4 + \frac{1}{2} N_B^2 U_{BB} \int_0^{2\pi} d\theta |\psi_B|^4 + U_{AB} N_A N_B \int_0^{2\pi} d\theta |\psi_A|^2 |\psi_B|^2.
\]

(4.2.1)

Here we have used \( A \) and \( B \) to distinguish the two species rather than ‘1’ and ‘2’ as in (2.1.37). This notation in fact conforms to that used in Section 3.2. As stated at the end of Section 3.2, we take \( M_A = M_B = M \) since the periodicity of \( \epsilon_L \) only applies in this case. Furthermore, we set \( U_{AA} = U_{BB} = U_{AB} = U \) for simplicity.

It is now useful to express the energy in units of \( Nh^2/(2MR^2) \), where \( N = N_A + N_B \) is the total number of particles. Dividing (4.2.1) by this factor gives

\[
\bar{E}[\psi_A, \psi_B] = \int_0^{2\pi} d\theta \left( x_A \left| \frac{d\psi_A}{d\theta} \right|^2 + x_B \left| \frac{d\psi_B}{d\theta} \right|^2 \right) + x_A^2 \pi \gamma \int_0^{2\pi} d\theta |\psi_A|^4 + x_B^2 \pi \gamma \int_0^{2\pi} d\theta |\psi_B|^4 + 2x_A x_B \pi \gamma \int_0^{2\pi} d\theta |\psi_A|^2 |\psi_B|^2,
\]

(4.2.2)

where \( x_A = N_A/N, x_B = N_B/N \) are the relative fractions of the two species in the system. The interaction parameter \( \gamma \) is introduced according to (2.2.9).

Following the procedure used in Section 4.1, we expand the two condensate wave
functions as

\[ \psi_A(\theta) = \sum_m c_m \phi_m(\theta) \]  \hspace{1cm} (4.2.3) \\
\[ \psi_B(\theta) = \sum_m d_m \phi_m(\theta), \]  \hspace{1cm} (4.2.4)

where normalization requires

\[ \sum_m |c_m|^2 = 1, \quad \sum_m |d_m|^2 = 1. \]  \hspace{1cm} (4.2.5)

The angular momentum for each of the species is

\[ L_A = N_A \hbar \sum_m m|c_m|^2 \]  \hspace{1cm} (4.2.6) \\
\[ L_B = N_B \hbar \sum_m m|d_m|^2 \]  \hspace{1cm} (4.2.7)

and the total angular momentum is given by \( L = L_A + L_B \). We now define the total angular momentum per particle

\[ l = \frac{L}{N \hbar} = x_A l_A + x_B l_B, \]  \hspace{1cm} (4.2.8)

where \( l_A = \sum_m m|c_m|^2 \) and \( l_B = \sum_m m|d_m|^2 \). If we now substitute the wave functions (4.2.3, 4.2.4) into the energy (4.2.2) we obtain

\[ \bar{E}_0(l) = x_A \sum m^2 |c_m(l)|^2 + x_B \sum m^2 |d_m(l)|^2 + x_A^2 \pi \gamma \int_0^{2\pi} d\theta \left| \sum_m c_m(l) \phi_m(\theta) \right|^4 \]
\[ + x_B^2 \pi \gamma \int_0^{2\pi} d\theta \left| \sum_m d_m(l) \phi_m(\theta) \right|^4 \]
\[ + 2 x_A x_B \pi \gamma \int_0^{2\pi} \left| \sum_m c_m(l) \phi_m(\theta) \right| \left| \sum_m d_m(l) \phi_m(\theta) \right|^2. \]  \hspace{1cm} (4.2.9)
Using (3.2.15) we now write the periodic function $\bar{\epsilon}_0(l)$ as

$$
\bar{\epsilon}_0(l) = x_A \sum_m (m^2 - l^2) |c_m(l)|^2 + x_B \sum_m (m^2 - l^2) |d_m(l)|^2 \\
+ x^2_A \pi \gamma \int_0^{2\pi} d\theta \left| \sum_m c_m(l)\phi_m(\theta) \right|^4 \\
+ x^2_B \pi \gamma \int_0^{2\pi} d\theta \left| \sum_m d_m(l)\phi_m(\theta) \right|^4 \\
+ 2x_Ax_B \pi \gamma \int_0^{2\pi} \left| \sum_m c_m(l)\phi_m(\theta) \right|^2 \left| \sum_m d_m(l)\phi_m(\theta) \right|^2.
$$

(4.2.10)

With the assumption that $c_{m+n}(l+n) = c_m(l)$ and $d_{m+n}(l+n) = d_m(l)$, we find that $\bar{\epsilon}_0(l+n) = \bar{\epsilon}_0(l)$, which is the periodicity condition derived in Section 3.2. Because of this periodicity we need only consider $l$ in the range $0 \leq l \leq 1$.

To begin, we consider wave functions $\psi_A$ and $\psi_B$ containing only two components, as we did for the single-species case, that is,

$$
\psi_A = c_0 \phi_0 + c_1 \phi_1
$$

(4.2.11)

$$
\psi_B = d_0 \phi_0 + d_1 \phi_1,
$$

(4.2.12)

where the coefficients $c_i$ and $d_i$ are normalized according to (4.2.5) and the angular momentum constraints are

$$
l_A = |c_1|^2
$$

(4.2.13)

$$
l_B = |d_1|^2,
$$

(4.2.14)

that is

$$
x_A|c_1|^2 + x_B|d_1|^2 = l.
$$

(4.2.15)

For the wave functions in (4.2.11, 4.2.12) the kinetic energy term and the first two interaction terms in (4.2.9) are similar to those in the single-species case. The cross
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The term is given by

\[
2x_Ax_B\pi\gamma \int_0^{2\pi} d\theta \left| \sum_m c_m(l) \phi_m(\theta) \right|^2 \left| \sum_m d_m(l) \phi_m(\theta) \right|^2 = x_Ax_B\gamma \left( 1 + c_0^*d_0^*d_1^* + c_0^*c_1d_0^*d_1^* \right),
\]

(4.2.16)

where we have used the normalization conditions \(|c_0|^2 + |c_1|^2 = 1\) and \(|d_0|^2 + |d_1|^2 = 1\).

If we express the complex coefficients in the form (4.1.26), namely,

\[
c_l = |c_l|e^{i\alpha_l},
\]

(4.2.17)

\[
d_l = |d_l|e^{i\beta_l},
\]

(4.2.18)

then the complex terms in (4.2.16) give

\[
c_0^*c_1^*d_0^*d_1^* + c_0^*d_0^*d_1^* = 2|c_0||c_1||d_0||d_1| \cos(\alpha_0 - \alpha_1 - \beta_0 + \beta_1).
\]

The phase difference is chosen to be \(\pi\) since this minimizes the energy for any positive \(\gamma\). The condensate energy now takes the form

\[
E_0 = x_Al_A + x_Bl_B + x_A^2\gamma \left[ \frac{1}{2} + |c_0|^2|c_1|^2 \right] + x_B^2\gamma \left[ \frac{1}{2} + |d_0|^2|d_1|^2 \right] + x_Ax_B\gamma(1 - 2|c_0||c_1||d_0||d_1|)
\]

\[
= l + \gamma \left[ \frac{1}{2} + (x_A|c_0||c_1| - x_B|d_0||d_1|)^2 \right].
\]

(4.2.19)

It is obvious from the above equation that the energy takes on its minimum value when

\[
x_A|c_0||c_1| = x_B|d_0||d_1|.
\]

(4.2.20)

If this condition is satisfied, the energy behaves simply as

\[
E_0(l) = l + \gamma/2.
\]

(4.2.21)

We argue later that this is in fact the exact dependence of \(E_0\) on \(l\) when \(0 \leq l \leq x_B\) and \(x_A \leq l \leq 1\). Using (4.2.20) together with the normalization constraints and
Figure 4.2.1: The dependence of the coefficients in (4.2.22 - 4.2.25) on the total angular momentum $l$ plotted for $x_A = 0.6$. 

(4.2.15), this condition leads to the coefficients

$$|c_0|^2 = \frac{(x_A - l)(1 - l)}{x_A(1 - 2l)}$$  \hspace{1cm} (4.2.22)

$$|c_1|^2 = \frac{l(x_B - l)}{x_A(1 - 2l)}$$  \hspace{1cm} (4.2.23)

$$|d_0|^2 = \frac{(x_B - l)(1 - l)}{x_B(1 - 2l)}$$  \hspace{1cm} (4.2.24)

$$|d_1|^2 = \frac{l(x_A - l)}{x_B(1 - 2l)}$$  \hspace{1cm} (4.2.25)

To be concrete, we take $x_B < x_A$. Since $x_A + x_B = 1$, we have $x_A > 1/2$ and $x_B < 1/2$. One can check that the coefficients $|c_i|^2$ and $|d_i|^2$ are positive provided $l$ is in the range $0 \leq l \leq x_B$ or $x_A \leq l \leq 1$. This is confirmed in Fig. 4.2.1 where we plot the coefficients $|c_i|^2$ and $|d_i|^2$ as functions of $l$. In the range where $|c_1|^2$ and $|d_1|^2$ are positive $|c_0|^2$ and $|d_0|^2$ are less than 1.

We can also calculate the density $n(\theta)$ corresponding to the coefficients in (4.2.22...
We have
\[ n(\theta) = N_A |\psi_A|^2 + N_B |\psi_B|^2 \]
\[ = \frac{N_A}{2\pi} (|c_0|^2 + |c_1|^2 + c_0^* c_1 \phi_0^* \phi_1 + c_0 c_1 \phi_0 \phi_1^*) \]
\[ + \frac{N_B}{2\pi} (|d_0|^2 + |d_1|^2 + d_0^* d_1 \phi_0^* \phi_1 + d_0 d_1 \phi_0 \phi_1^*). \]  
(4.2.26)

Using the normalization of the coefficients and the polar representation (4.2.17 - 4.2.18), we obtain
\[ n(\theta) = N \frac{2}{\pi} + \frac{2}{\pi N} \left[ x_A |c_0||c_1| \cos(\theta - \alpha_0 + \alpha_1) + x_B |d_0||d_1| \cos(\theta - \beta_0 + \beta_1) \right]. \]  
(4.2.27)

Since \( x_A |c_0||c_1| = x_B |d_0||d_1| \), this equation simplifies to
\[ n(\theta) = N \frac{2}{\pi} \left[ x_A \cos(\theta - \alpha_0 + \alpha_1) + x_B \cos(\theta - \beta_0 + \beta_1) \right] \]
\[ \cdot \cos \left[ \frac{1}{2} (-\alpha_0 + \alpha_1 - \beta_0 + \beta_1) \right]. \]  
(4.2.28)

The argument of the second cosine function is \( \pi/2 \) since the phase difference \( -\alpha_0 + \alpha_1 - \beta_0 + \beta_1 \) was previously chosen to be \( \pi \). The second term in (4.2.28) is then identically zero and the density becomes \( n(\theta) = N/(2\pi) \equiv n_0 \). Thus we find that the density described by (4.2.22 - 4.2.25) is homogeneous when the total angular momentum lies in the range \( 0 \leq l \leq x_B \) or \( x_A \leq l \leq 1 \).

We will now demonstrate that the solution defined by (4.2.22 - 4.2.25) is in fact exact. To do so we consider wave functions of the form
\[ \psi_A(\theta) = \psi_A^0(\theta) + \delta \psi_A, \quad \psi_B(\theta) = \psi_B^0(\theta) + \delta \psi_B, \]  
(4.2.29)

where
\[ \psi_A^0(\theta) = c_0 \phi_0 + c_1 \phi_1, \quad \psi_B^0(\theta) = d_0 \phi_0 + d_1 \phi_1, \]  
(4.2.30)
with the coefficients given by (4.2.22 - 4.2.25). The deviations are expressed in the form

\[ \delta \psi_A = \sum_m \delta c_m \phi_m, \quad \delta \psi_B = \sum_m \delta d_m \phi_m. \quad (4.2.31) \]

With these definitions the total density takes the form

\[ n(\theta) = n_0 + \delta n(\theta), \quad (4.2.32) \]

where the fluctuation \( \delta n(\theta) \) depends on \( \delta \psi_A \) and \( \delta \psi_B \).

The fluctuations \( \delta \psi_A \) and \( \delta \psi_B \) are arbitrary but must be consistent with the constraint that the total number of particles of each species remains unchanged. This is ensured with the normalizations

\[ \int_0^{2\pi} d\theta |\psi_A(\theta)|^2 = 1, \quad \int_0^{2\pi} d\theta |\psi_B(\theta)|^2 = 1. \quad (4.2.33) \]

This of course also implies that the total number of particles \( N \) is unchanged. From (4.2.32) we therefore have

\[ \int_0^{2\pi} \delta n(\theta) d\theta = 0. \quad (4.2.34) \]

The second constraint is that the total angular momentum per particle is \( l \), that is,

\[ x_A \sum_m m |c_m + \delta c_m|^2 + x_B \sum_m m |d_m + \delta d_m|^2 = l. \quad (4.2.35) \]

We now evaluate (4.2.2) with these wave functions. The interaction energy can be written as

\[ \bar{E}_{INT} = \pi \gamma \int_0^{2\pi} d\theta \left( x_A |\psi_A|^2 + x_B |\psi_B|^2 \right)^2 = \frac{\pi \gamma}{N^2} \int_0^{2\pi} d\theta \left[ n(\theta) \right]^2. \quad (4.2.36) \]

Using (4.2.32) and (4.2.34) we obtain

\[ \bar{E}_{INT} = \frac{\gamma}{2} + \frac{\pi \gamma}{N^2} \int_0^{2\pi} d\theta |\delta n(\theta)|^2. \quad (4.2.37) \]
It is now obvious that the interaction energy is greater than that found for the homogeneous density $\gamma/2$.

Similarly, the kinetic energy term is given by

$$\bar{E}_{\text{KIN}} = x_A \int_0^{2\pi} d\theta \left( \left| \frac{\partial \psi_A^0}{\partial \theta} \right|^2 + \frac{\partial \psi_A^0}{\partial \theta} \frac{\partial \delta \psi_A^*}{\partial \theta} + \frac{\partial \delta \psi_A^*}{\partial \theta} \frac{\partial \psi_A^0}{\partial \theta} + \left| \frac{\partial \psi_A^0}{\partial \theta} \right|^2 \right)$$

$$+ x_B \int_0^{2\pi} d\theta \left( \left| \frac{\partial \psi_B^0}{\partial \theta} \right|^2 + \frac{\partial \psi_B^0}{\partial \theta} \frac{\partial \delta \psi_B^*}{\partial \theta} + \frac{\partial \delta \psi_B^*}{\partial \theta} \frac{\partial \psi_B^0}{\partial \theta} + \left| \frac{\partial \psi_B^0}{\partial \theta} \right|^2 \right). \quad (4.2.38)$$

The two terms not involving the deviations give the kinetic energy $\bar{E}_{\text{KIN}}^0$ for the homogeneous case. The remaining terms can be simplified using (4.2.30) and (4.2.31), and we find

$$\bar{E}_{\text{KIN}} = \bar{E}_{\text{KIN}}^0 + x_A \sum_m m^2 |\delta c_m|^2 + x_B \sum_m m^2 |\delta d_m|^2. \quad (4.2.39)$$

The total angular momentum constraint in (4.2.35) reduces to

$$x_A |c_1|^2 + x_A c_1 (\delta c_1 + \delta c_1^*) + x_A \sum_m m |\delta c_m|^2$$

$$+ x_B |d_1|^2 + x_B d_1 (\delta d_1 + \delta d_1^*) + x_B \sum_m m |\delta d_m|^2 = l. \quad (4.2.40)$$

Since $x_A |c_1|^2 + x_B |d_1|^2 = l$, this implies that

$$x_A c_1 (\delta c_1 + \delta c_1^*) + x_B d_1 (\delta d_1 + \delta d_1^*) = -x_A \sum_m m |\delta c_m|^2 - x_B \sum_m m |\delta d_m|^2. \quad (4.2.41)$$

Substituting this result into (4.2.39), we have

$$\bar{E}_{\text{KIN}} = \bar{E}_{\text{KIN}}^0 + x_A \sum_m (m^2 - m) |\delta c_m|^2 + x_B \sum_m (m^2 - m) |\delta d_m|^2. \quad (4.2.42)$$

Since $(m^2 - m) \geq 0$, we see that any deviation from the zeroth order wave functions leads to a higher kinetic energy. Combining this result with (4.2.36), we conclude
that the solution defined by (4.2.22 - 4.2.25) gives the lowest possible energy. In other words, this solution, when valid, is the exact solution that minimizes the energy functional. This should be contrasted with the single-species case where the two-term ansatz in (4.1.18) does not minimize the energy functional. The reason for this is that the density corresponding to (4.1.18) is not homogeneous as we are finding here for the two-species problem.

We must next analyze the situation when $x_B \leq l \leq x_A$. In particular we consider the situation when $l$ is close to $x_A$, that is $l - x_A = \varepsilon$, where $\varepsilon$ is a small quantity. When $x_A = l$ we see from (4.2.22 - 4.2.25) that

$$|c_0|^2 = 0, \quad |c_1|^2 = 1 \quad (4.2.43)$$
$$|d_0|^2 = 1, \quad |d_1|^2 = 0. \quad (4.2.44)$$

As $\varepsilon$ increases from zero, we therefore expect deviations from these limiting values and additional components in the expansion of the $\psi_A$ and $\psi_B$ wave functions. To be specific, we consider the three-component wave functions

$$\psi_A = c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2 \quad (4.2.45)$$
$$\psi_B = d_{-1} \phi_{-1} + d_0 \phi_0 + d_1 \phi_1. \quad (4.2.46)$$

We anticipate that $|c_0|^2$, $|c_2|^2$, $|d_{-1}|^2$ and $|d_1|^2$ are all of order $\varepsilon$. With this assumption, we can retain contributions to the energy to the lowest order in $\varepsilon$. Our final results confirm that this procedure is consistent.

Substituting (4.2.45) and (4.2.46) into (4.2.2) we obtain the expression for the condensate energy. For the kinetic energy terms we obtain

$$\int_0^{2\pi} d\theta \left( x_A \left| \frac{d\psi_A}{d\theta} \right|^2 + x_B \left| \frac{d\psi_B}{d\theta} \right|^2 \right) = x_A \left( |c_1|^2 + 4 |c_2|^2 \right) + x_B \left( |d_{-1}|^2 + |d_1|^2 \right). \quad (4.2.47)$$
The interaction terms have the following form

\[ x_A^2 \pi \gamma \int_0^{2\pi} d\theta |\psi_A|^4 = x_A^2 \gamma \left[ \frac{1}{2} + |c_0|^2 |c_1|^2 + \frac{|c_0|^2 |c_2|^2}{|c_1|^2 |c_2|^2} + 2|c_0||c_1|^2 |c_2| \cos(\alpha_0 - 2\alpha_1 + \alpha_2) \right] \]  

(4.2.48)

\[ x_B^2 \pi \gamma \int_0^{2\pi} d\theta |\psi_B|^4 = x_B^2 \gamma \left[ \frac{1}{2} + |d_{-1}|^2 |d_0|^2 + \frac{|d_{-1}|^2 |d_1|^2}{|d_0|^2 |d_1|^2} + 2|d_{-1}||d_0|^2 |d_1| \cos(\beta_{-1} - 2\beta_0 + \beta_1) \right] \]  

(4.2.49)

The cross term in (4.2.2) is given by

\[ 2x_A x_B \pi \gamma \int_0^{2\pi} d\theta |\psi_A|^2 |\psi_B|^2 = x_A x_B \gamma \left[ 1 + c_0^* d_{-1}^* d_0 + c_0^* c_1 d_{-1} d_0^* 
+ c_0^* d_0^* d_1 + c_0^* c_1 d_0 d_1^* + c_0^* c_2 d_{-1} d_1 + c_0^* c_2 d_{-1} d_1^* 
+ c_1^* d_{-1}^* d_0 + c_1^* c_2 d_{-1} d_0^* + c_1^* c_2 d_{-1} d_0^* 
+ c_1^* c_2 d_{-1} d_1 + c_1^* c_2 d_{-1} d_1^* \right] \]

\[ = 2x_A x_B \gamma \left[ \frac{1}{2} + |c_0||c_1||d_{-1}||d_0| \cos(\alpha_0 - \alpha_1 - \beta_{-1} + \beta_0) 
+ |c_0||c_1||d_0||d_1| \cos(\alpha_0 - \alpha_1 - \beta_0 + \beta_1) 
+ |c_0||c_2||d_{-1}||d_1| \cos(\alpha_0 - \alpha_2 - \beta_{-1} + \beta_1) 
+ |c_1||c_2||d_{-1}||d_0| \cos(\alpha_1 - \alpha_2 - \beta_{-1} + \beta_0) 
+ |c_1||c_2||d_0||d_1| \cos(\alpha_1 - \alpha_2 - \beta_0 + \beta_1) \right] . \]  

(4.2.50)

Recalling the analysis of the three-component wave function in Section 4.1, when the angular momentum approaches an integral value \( m \), the coefficient \( |c_m|^2 \) corresponding to this value is effectively equal to 1. Other coefficients \( |c_{m+1}|^2 \) and \( |c_{m-1}|^2 \) are of the order of \( \varepsilon \), where \( \varepsilon \) is a small number. In this case \( l_4 \) is close to 1, therefore we expect \( |c_1|^2 \) to be of order 1, and \( |c_0|^2 \) and \( |c_2|^2 \) to be of order \( \varepsilon \). Likewise, when
\( l_B \) is close to 0, \(|d_0|^2\) is effectively 1, while \(|d_{-1}|^2\) and \(|d_1|^2\) are of order \( \varepsilon \). We now retain terms to first order in \( \varepsilon \) and neglect higher order terms; the underlined terms in \([4.2.48 - 4.2.50]\) are of order \( \varepsilon^2 \) and can be neglected.

At this point one may notice that the phases in the above three equations are not independent of each other. Defining the phases

\[
\chi_1 \equiv \alpha_0 - 2\alpha_1 + \alpha_2 \\
\chi_2 \equiv \beta_{-1} - 2\beta_0 + \beta_1 \\
\theta_1 \equiv \alpha_0 - \alpha_1 - \beta_{-1} + \beta_0,
\]

the remaining three phases are given by

\[
\alpha_0 - \alpha_1 - \beta_0 + \beta_1 = \theta_1 - \chi_2 \\
\alpha_1 - \alpha_2 - \beta_{-1} + \beta_0 = \theta_1 - \chi_1 \tag{4.2.52} \\
\alpha_1 - \alpha_2 - \beta_0 + \beta_1 = \theta_1 - \chi_1 - \chi_2.
\]

We thus see that there are actually only three independent phases.

With these definitions, the energy becomes

\[
E = x_A \left( |c_1|^2 + 4|c_2|^2 \right) + x_B \left( |d_{-1}|^2 + |d_1|^2 \right) \\
+ x_A^2 \gamma \left( \frac{1}{2} + |c_0|^2 + |c_2|^2 + 2|c_0||c_2| \cos \chi_1 \right) \\
+ x_B^2 \gamma \left( \frac{1}{2} + |d_{-1}|^2 + |d_1|^2 + 2|d_{-1}||d_1| \cos \chi_2 \right) \\
+ 2x_A x_B \gamma \left[ \frac{1}{2} + |c_0||d_{-1}| \cos \theta_1 + |c_0||d_1| \cos(\theta_1 - \chi_2) \right. \\
+ |c_2||d_{-1}| \cos(\theta_1 - \chi_1) + |c_2||d_1| \cos(\theta_1 - \chi_1 - \chi_2) \bigg]. \tag{4.2.53}
\]

If we now minimize the above energy with respect to all three phases we obtain the
system of three equations

\[
\frac{\partial E}{\partial \chi_1} = 0 = -|c_0| |c_2| \sin \chi_1 + x_B |c_2| |d_{-1}| \sin(\theta_1 - \chi_1) \\
+ x_B |c_2| |d_1| \sin(\theta_1 - \chi_1 - \chi_2) 
\] (4.2.54)

\[
\frac{\partial E}{\partial \chi_2} = 0 = -|d_{-1}| |d_1| \sin \chi_2 + x_A |c_0| |d_1| \sin(\theta_1 - \chi_2) \\
+ x_A |c_2| |d_1| \sin(\theta_1 - \chi_1 - \chi_2) 
\] (4.2.55)

\[
\frac{\partial E}{\partial \theta_1} = 0 = -|c_0| |d_{-1}| \sin \theta_1 - |c_0| |d_1| \sin(\theta_1 - \chi_2) - |c_2| |d_{-1}| \sin(\theta_1 - \chi_1) \\
- |c_2| |d_1| \sin(\theta_1 - \chi_1 - \chi_2). 
\] (4.2.56)

The above three equations can be satisfied if all sine functions are zero, which will be true if the phases \( \chi_1, \chi_2 \) and \( \theta_1 \) are either 0 or \( \pi \). If we arbitrarily set them to 0, the energy is then given by

\[
\bar{E}_0 \simeq x_A \left( |c_1|^2 + 4|c_2|^2 \right) + x_B \left( |d_{-1}|^2 + |d_1|^2 \right) \\
+ x_A^2 \gamma \left( \frac{1}{2} + |c_0|^2 + |c_2|^2 + 2|c_0||c_2| \right) \\
+ x_B^2 \gamma \left( \frac{1}{2} + |d_{-1}|^2 + |d_1|^2 + 2|d_{-1}| |d_1| \right) \\
+ 2x_A x_B \gamma \left( \frac{1}{2} + |c_0||d_{-1}| + |c_0||d_1| + |c_2||d_{-1}| + |c_2||d_1| \right), 
\] (4.2.57)

which must now be minimized with respect to the coefficients \( |c_0|, |c_2|, |d_{-1}| \) and \( |d_1| \).

If this minimization in the end leads to coefficients that are negative, we would know that our initial arbitrary choice of phases is incorrect. They would then have to be adjusted to ensure that the minimum occurs at positive values of the coefficients. This will be explained in more detail later.

For \( \psi_A \) and \( \psi_B \) given by (4.2.45) and (4.2.46) respectively, the angular momentum constraint is given by

\[
l = x_A - \varepsilon = x_A \left( 1 - |c_0|^2 + |c_2|^2 \right) + x_B \left( -|d_{-1}|^2 + |d_1|^2 \right), 
\] (4.2.58)
where the normalization condition $|c_0|^2 + |c_1|^2 + |c_2|^2 = 1$ has been used to eliminate $|c_1|^2$. Using (4.2.58), (4.2.57) simplifies to

$$
\bar{E}_0 - \frac{\gamma}{2} \simeq l + 2x_A |c_2|^2 + 2x_B |d_{-1}|^2 \\
+ \gamma \left[ x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|) \right]^2.
$$

(4.2.59)

We now wish to minimize this with respect to the constraint that the angular momentum is $l = x_A - \epsilon$. The minimization of the energy subject to this constraint is performed using the Lagrange method. Introducing the constraint via a Lagrange multiplier $\lambda$, we must now minimize the function

$$
F(|c_0|, |c_2|, |d_{-1}|, |d_1|) = \frac{\gamma}{2} + l + 2x_A |c_2|^2 + 2x_B |d_{-1}|^2 \\
+ \gamma \left[ x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|) \right]^2 \\
+ \lambda \left[ x_A (1 - |c_0|^2 + |c_2|^2) + x_B (-|d_{-1}|^2 + |d_1|^2) \right],
$$

(4.2.60)

where the variations of the coefficients are now unconstrained. Taking derivatives of the function $F$ with respect to each of the coefficients, we obtain

$$
\frac{\partial F}{\partial |c_0|} = 0 = 2x_A \gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] - 2|c_0|x_A \lambda
$$

(4.2.61)

$$
\frac{\partial F}{\partial |c_2|} = 0 = 4x_A |c_2| + 2x_A \gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] + 2|c_2|x_A \lambda
$$

(4.2.62)

$$
\frac{\partial F}{\partial |d_{-1}|} = 0 = 4x_B |d_{-1}| + 2x_B \gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] \\
- 2|d_{-1}|x_B \lambda
$$

(4.2.63)

$$
\frac{\partial F}{\partial |d_1|} = 0 = 2x_B \gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] + 2|d_1|x_B \lambda.
$$

(4.2.64)
Eliminating common factors we have

\[
\gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] - \lambda |c_0| = 0 \quad (4.2.65)
\]

\[
\gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] + (\lambda + 2)|c_2| = 0 \quad (4.2.66)
\]

\[
\gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] + (2 - \lambda)|d_{-1}| = 0 \quad (4.2.67)
\]

\[
\gamma [x_A (|c_0| + |c_2|) + x_B (|d_{-1}| + |d_1|)] + \lambda |d_1| = 0. \quad (4.2.68)
\]

We now subtract (4.2.66) from (4.2.65), and (4.2.68) from (4.2.67), which leads to

\[
\lambda (|c_0| + |c_2|) + 2|c_2| = 0 \quad (4.2.69)
\]

\[
\lambda (|d_{-1}| + |d_1|) - 2|d_{-1}| = 0. \quad (4.2.70)
\]

These equations give the coefficient ratios

\[
\frac{|c_2|}{|c_0|} = -\frac{\lambda}{\lambda + 2} \quad (4.2.71)
\]

\[
\frac{|d_{-1}|}{|d_1|} = \frac{\lambda}{2 - \lambda}. \quad (4.2.72)
\]

Using these relations in (4.2.68) we find that

\[
|d_1| = \frac{2\gamma x_A}{(\lambda + 2)[2\gamma x_B - \lambda(\lambda - 2)]} |c_0| = (\lambda - 2)\zeta |c_0|, \quad (4.2.73)
\]

where \(\zeta\) is given by

\[
\zeta = \frac{2\gamma x_A}{(\lambda + 2)[2\gamma x_B - \lambda(\lambda - 2)]}. \quad (4.2.74)
\]

We now use (4.2.71 - 4.2.73) to eliminate \(|c_2|, |d_{-1}|\) and \(|d_1|\) from (4.2.65) to obtain

\[
\gamma \frac{2x_A |c_0|}{\lambda + 2} \left[ 1 + \frac{2\gamma x_B}{\lambda(\lambda - 2) - 2\gamma x_B} \right] - \lambda |c_0| = 0, \quad (4.2.75)
\]

which gives us the equation

\[
2\gamma = \frac{\lambda (\lambda^2 - 4)}{\lambda + 2(x_B - x_A)}. \quad (4.2.76)
\]
This equation determines the Lagrange multiplier \( \lambda \) in terms of \( \gamma \) and \( x_A \) (recall \( x_B = 1 - x_A \)). Since it is a cubic equation, there in general are three roots, which are the solutions of the equation

\[
f(\lambda) \equiv \lambda(\lambda^2 - 4) - 2\gamma\lambda + 4\gamma(x_A - x_B) = 0. \tag{4.2.77}
\]

The roots are to be determined for \( \gamma > 0 \) and \( 0 \leq (x_A - x_B) \leq 1 \). In Fig. 4.2.2, \( f(\lambda) \) is plotted for the limiting cases \((x_A - x_B) = 0\) and \((x_A - x_B) = 1\) and for one intermediate value. For \((x_A - x_B) = 0\), \( f(\lambda) = \lambda(\lambda^2 - 4 - 2\gamma) \), which has the roots \( \lambda = 0 \) and \( \lambda = \pm \sqrt{4 + 2\gamma} \). For \((x_A - x_B) = 1\), \( f(\lambda) = (\lambda - 2)[\lambda(\lambda + 2) - 2\gamma] \), which has the roots \( \lambda = 2 \) and \( \lambda = -1 \pm \sqrt{1 + 2\gamma} \). The latter two are the roots for the single-species case as shown in (4.1.37). Since the term \( 4\gamma(x_A - x_B) \) in \( f(\lambda) \) simply shifts the curves in Fig. 4.2.2 vertically, it is clear that there are always three real roots for the physical range of \((x_A - x_B)\) values.

Bounds on the location of the real roots can be determined more quantitatively by noting that the right-hand side of (4.2.76) must be positive, which implies

\[
\begin{cases}
\lambda(\lambda^2 - 4) > 0 \\
\lambda - 2(x_A - x_B) > 0
\end{cases}
\tag{4.2.78}
\]

or

\[
\begin{cases}
\lambda(\lambda^2 - 4) < 0 \\
\lambda - 2(x_A - x_B) < 0
\end{cases}
\tag{4.2.79}
\]

Condition (4.2.78) requires \( \lambda > 2 \), while (4.2.79) implies \( \lambda < -2 \) or \( 0 < \lambda < 2(x_A - x_B) \). The curves in Fig. 4.2.2 are consistent with these bounds on the values of the roots. Which of these roots is physically relevant will be explained later.
CHAPTER 4. MEAN-FIELD THEORY OF PERSISTENT CURRENTS

Figure 4.2.2: The plot of the function \( f(\lambda) \) in (4.2.77) for (a) \( \gamma = 2.0 \) and (b) \( \gamma = 8.0 \). The bold solid curves represent the functions when \( (x_A - x_B) = 0 \), the upper dashed curves correspond to the case \( (x_A - x_B) = 1 \). The middle fine curves are plotted for \( (x_A - x_B) = 0.5 \).

The denominator in (4.2.74) is given by

\[
(\lambda + 2)[2\gamma x_B - \lambda(\lambda - 2)] = 2\gamma x_B(\lambda + 2) - \lambda(\lambda^2 - 4)
\]

\[
= 2\gamma(1 - x_A)(\lambda + 2) - 2\gamma[\lambda + 2(1 - 2x_A)]
\]

\[
= 2\gamma x_A(2 - \lambda). \quad (4.2.80)
\]

Thus, the value of the constant \( \zeta \) in (4.2.74) is simply

\[
\zeta = \frac{1}{2 - \lambda}, \quad (4.2.81)
\]

which from (4.2.73) implies \( |d_1| = -|c_0| \). If we assume \( |c_0| \) to be positive, this indicates that \( |d_1| \) is negative, which of course is incorrect. We would have obtained a positive result for \( |d_1| \) if we had used \(-|d_1|\) instead of \(|d_1|\) in (4.2.57) which amounts to a particular choice of phases in (4.2.51). We return to the proper choice of phases later.

Using \( |d_1| = -|c_0| \) together with (4.2.71) and (4.2.72) in the angular momentum
constraint (4.2.58), we find
\[ x_A - \varepsilon = x_A \left( 1 - |c_0|^2 + \frac{\lambda^2}{(\lambda + 2)^2} |c_0|^2 \right) + x_B \left( |c_0|^2 - \frac{\lambda^2}{(2 - \lambda)^2} |c_0|^2 \right). \]  
(4.2.82)

This gives
\[ |c_0|^2 = \frac{\varepsilon}{4 \left[ x_A \frac{\lambda + 1}{(\lambda + 2)^2} + x_B \frac{\lambda - 1}{(\lambda - 2)^2} \right]} = \frac{\varepsilon (\lambda^2 - 4)^2}{4[x_A(\lambda + 1)(\lambda - 2)^2 + x_B(\lambda - 1)(\lambda + 2)^2]} \]  
(4.2.83)

It is clear from the above expression that the \( \lambda < -2 \) root makes \( |c_0|^2 \) negative which is incorrect. This root is therefore physically inadmissible and only the positive roots are relevant. This conclusion and (4.2.71) indicates that the sign of \( |c_2| \) in (4.2.57) also turns out to be negative.

We now return to the energy functional (4.2.59) and rewrite it using (4.2.71 - 4.2.72)
\[ \bar{E}_0 - \frac{\gamma}{2} = x_A - \varepsilon + 2x_A \frac{\lambda^2|c_0|^2}{(\lambda + 2)^2} + 2x_B \frac{\lambda^2|c_0|^2}{(\lambda - 2)^2} + \gamma \left( \frac{x_A}{\lambda + 2} + \frac{x_B}{\lambda - 2} \right)^2 \]
\[ = x_A - \varepsilon + 2|c_0|^2 \lambda^2 \left[ \frac{x_A}{(\lambda + 2)^2} + \frac{x_B}{(\lambda - 2)^2} \right] + 4\gamma |c_0|^2 \left( \frac{x_A}{\lambda + 2} + \frac{x_B}{\lambda - 2} \right)^2. \]  
(4.2.84)

Using (4.2.76) to eliminate \( \gamma \), we find
\[ \bar{E}_0 - \frac{\gamma}{2} = x_A - \varepsilon + 2\lambda|c_0|^2 \left[ \frac{x_A \lambda}{(\lambda + 2)^2} + \frac{x_B \lambda}{(\lambda - 2)^2} + \frac{\lambda - 2(x_A - x_B)}{(\lambda^2 - 4)} \right] \]
\[ = x_A - \varepsilon + \frac{|c_0|^2 \lambda}{(\lambda^2 - 4)^2} \left[ \lambda(\lambda - 2)^2 x_A + \lambda(\lambda + 2)^2 x_B \right. \]
\[ \left. + (\lambda^2 - 4)(\lambda - 2x_A + 2x_B) \right] \]
\[ = x_A - \varepsilon + \frac{2|c_0|^2 \lambda}{(\lambda^2 - 4)^2} \left[ \lambda(\lambda - 2)^2 x_A + \lambda(\lambda + 2)^2 x_B \right. \]
\[ \left. + (\lambda^2 - 4)(\lambda x_A + x_B) - 2x_A + 2x_B \right] \]  
(4.2.85)
\[ = x_A - \varepsilon + 4|c_0|^2 \frac{x_A \lambda}{(\lambda^2 - 4)^2} \left[ (\lambda + 1)(\lambda - 2)^2 x_A + (\lambda - 1)(\lambda + 2)^2 x_B \right]. \]
Substituting the expression (4.2.83) for $|c_0|^2$ into the above equation we obtain the remarkably simple result

$$\bar{E}_0 - \frac{\gamma}{2} = x_A - \varepsilon + \lambda \varepsilon,$$

(4.2.86)

and

$$\bar{\varepsilon}_0 - \frac{\gamma}{2} = x_A - \varepsilon + \lambda \varepsilon - l^2.$$

(4.2.87)

We now see that the smaller of the two positive $\lambda$ roots gives the lowest possible energy. This thus identifies the root in the range $0 < \lambda < 2(x_A - x_B)$ as the one that minimizes the energy.

It is useful at this point to summarize in Fig. 4.2.3 the dependence of $\bar{E}_0$ on $l$ that we have found. For $l$ in the ranges $0 \leq l \leq x_B$ and $x_A \leq l \leq 1$ we found that $\bar{E}_0 - \gamma/2 = l$; this behaviour is indicated in the figure. For $l$ just below $x_A$ the energy

![Figure 4.2.3: Plot of the dependence of $\bar{E}_0$ on the angular momentum $l$ for $x_A = 0.8$ and $\gamma = 2$.](image-url)
Figure 4.2.4: Plot of the periodic functions $\bar{\epsilon}_0(l) - \gamma/2$ for $0 \leq l \leq 1$ for $\gamma = 2$. The dashed curve is the function $-\gamma/2 + l(1 - l)$ which is $\bar{\epsilon}_0(l)$ in the range $0 \leq l \leq x_B$ and $x_A \leq l \leq 1$. The solid curve indicates the approximate behaviour in the range $x_B \leq l \leq x_A$ according to (4.2.93) with $x_A = 0.8$. Is given by

$$E_0 - \frac{\gamma}{2} = \lambda x_A + (1 - \lambda)l,$$

whose slope is $(1 - \lambda)$. This slope is indicated in Fig. 4.2.3 for the case $\gamma = 2$ and $x_A = 0.8$. For these values, (4.2.76) gives $\lambda \simeq 0.6315$. We note that $E_0$ exhibits a kink as a function of $l$ at $l = x_A$. From this behaviour, it is now clear that a local minimum will arise at $l = x_A$ if $\lambda > 1$, leading to stability of persistent currents at this value of $l$. This analysis leads to the conclusion that $\lambda = 1$ defines the critical condition for persistent currents. From (4.2.76) we see that the critical value of the interaction parameter $\gamma$ is

$$\gamma_{cr} = \frac{3}{4(x_A - x_B) - 2} = \frac{3}{2(4x_A - 3)},$$

(4.2.89)
For \( x_A = 1 \) this reduces to the value \( \gamma_{cr} = 3/2 \) that we found earlier in the single-species case. Since \( \gamma_{cr} \) diverges when \( x_A \) is reduced to \( 3/4 \), (4.2.89) indicates that persistent currents are only possible when \( 0.75 < x_A \leq 1 \).

To extend the behaviour of \( \bar{E}_0 \) outside the range \( 0 \leq l \leq 1 \), we make use of the periodicity of \( \bar{\epsilon}_0(l) = \bar{E}_0(l) - l^2 \). Using (4.2.21), we see that for \( 0 \leq l \leq x_B \) and \( x_A \leq l \leq 1 \), \( \bar{\epsilon}_0(l) = \gamma/2 + l(1-l) \). This behaviour is indicated in Fig. 4.2.4 by the dashed curve. The slope of \( \bar{\epsilon}_0(l) \) at \( l = x_A^- \) is \((1 - 2x_A) - \lambda\), which is more negative, for positive \( \lambda \), than the slope at \( l = x_A^+ \) which is \((1 - 2x_A)\). This means that \( \bar{\epsilon}_0(l) - \gamma/2 \) will lie above the curve \( l(1-l) \) when \( x_B \leq l \leq x_A \). As an approximate representation of this function for this range of \( l \), we will write

\[
\bar{\epsilon}_0(l) - \frac{\gamma}{2} = l(1-l) + C(l - x_B)(x_A - l),
\]

where the factor \( C \) is determined by the condition

\[
\left. \frac{d\bar{\epsilon}_0}{dl} \right|_{l=x_A^-} = 1 - 2x_A - \lambda = 1 - 2x_A + C(1 - 2x_A). \quad (4.2.91)
\]

This equation gives

\[
C = \frac{\lambda}{2x_A - 1}. \quad (4.2.92)
\]

The function \( \bar{\epsilon}_0(l) \) thus takes the approximate form

\[
\bar{\epsilon}_0(l) - \frac{\gamma}{2} \simeq l(1-l) + \frac{\lambda}{2x_A - 1}(l - x_B)(x_A - l), \quad (4.2.93)
\]

for \( x_B \leq l \leq x_A \). The expected behaviour of \( \bar{\epsilon}_0(l) \) is shown by the solid curve in Fig. 4.2.4.

We now return to the question of the phases appearing in (4.2.53). These phases were all chosen to be zero to obtain (4.2.57). Our subsequent analysis starting from this equation showed that \(|d_1| < 0\), \(|c_2| < 0\) and \(|d_{-1}| < 0\) (using (4.2.72)). All of
Figure 4.2.5: The condensate energy plotted for $\gamma = 10$ and $x_A = 0.8$ (a) and $x_A = 0.95$ (b) as a function of $l$. The corresponding values of $\lambda$ are $\lambda \simeq 1.4641$ and $\lambda \simeq 1.7073$, respectively. We see that in both cases persistent currents are possible at $l = x_A$, but not at $l = x_A + 1$.

these coefficients would turn out to be positive if the signs of each of them in (4.2.57) is reversed. If this is done, (4.2.57) becomes

\[
\bar{E} = x_A \left( |c_1|^2 + 4|c_2|^2 \right) + x_B \left( |d_{-1}|^2 + |d_1|^2 \right) \\
+ x_A^2 \gamma \left( \frac{1}{2} + |c_0|^2 + |c_2|^2 - 2|c_0||c_2| \right) + x_B^2 \gamma \left( \frac{1}{2} + |d_{-1}|^2 + |d_1|^2 + 2|d_{-1}||d_1| \right) \\
+ 2x_A x_B \gamma \left[ \frac{1}{2} - |c_0||d_{-1}| - |c_0||d_1| + |c_2||d_{-1}| + |c_2||d_1| \right].
\]

Comparing this with (4.2.53) we see that the correct phases are $\chi_1 = \pi$, $\chi_2 = 0$ and $\theta_1 = \pi$. Had we chosen to minimize this function we would have found

\[
\frac{|c_2|}{|c_0|} = \frac{\lambda}{\lambda + 2},
\]

instead of (4.2.71), which gives a positive value for $|c_2|$ as it should be. Using the property of periodicity of $\bar{e}_0(l)$ we may now plot the energy for any range of $l$. Since the energy is given by $\bar{E}_0(l) = l^2 + \bar{e}_0(l)$, then for $l = n + l'$, where $n = 0, 1, ...$ and
0 \leq l' \leq 1 the energy is written as

\[ \tilde{E}_0(l) = (n + l')^2 + \bar{\epsilon}_0(n + l') \]

\[ = n^2 + 2nl' + l'^2 + \bar{\epsilon}_0(l'). \quad (4.2.96) \]

The slope is then given by

\[ \left. \frac{d\tilde{E}_0(l)}{dl} \right|_{l=(n+x_A)^-} = \left. 2n + 2x_A + \frac{d\bar{\epsilon}_0(l')}{dl'} \right|_{l'=x_A^-} = 2n + 1 - \lambda. \quad (4.2.97) \]

Since 0 < \lambda < 2(x_A - x_B), the slope cannot be zero for any n \geq 0 which implies the impossibility of persistent currents except for the case l = x_A. This should be compared to the single-species case where we found \( \gamma_{cr,n} = (2n - 1)(2n + 1)/2 \). In Fig. 4.2.5 we plot the energy in the range 0 \leq l \leq 2 for a case where \( \tilde{E}_0 \) exhibits a negative slope at l = x_A^- . We see that the slope at l = (1 + x_A)^- is positive, confirming the general conclusion made above. This result demonstrates the sensitivity of the persistent currents at higher angular momenta to the inclusion of even the smallest amount of the second species. This is the conclusion that the authors of [24] came to.

To expand on this result it is of interest to consider the limit of taking \( x_B \to 0 \). One might reasonably expect that the single-species results are recovered in this limit. Based on the above analysis, however, this is not the case. The explanation of this is provided by the detailed examination of the roots of the function \( f(\lambda) \) in (4.2.77).

For \( x_B \to 0 \), we have \( f(\lambda) = (\lambda - 2) [\lambda(\lambda + 2) - 2\gamma] \) which has the roots \( \lambda = 2 \) and \( \lambda = -1 \pm \sqrt{1 + 2\gamma} \). The latter two are the ones found in the single-species case. The smallest positive root is the one that minimizes the energy. If \( \gamma < 4 \), this root is \( \lambda = -1 + \sqrt{1 + 2\gamma} \) and in this case, the value of the energy in (4.2.86) reduces to the single-species result. This situation is illustrated in Fig. 4.2.2(a) which is shown...
for $\gamma = 2$. The root shown there in the range $0 < \lambda < 2$ in the $x_B \to 0$ limit is the $\lambda = -1 + \sqrt{1 + 2\gamma}$ root referred to above.

The situation for $\gamma > 4$ is different. This is illustrated in Fig. 4.2.2(b) which shows $f(\lambda)$ for $\gamma = 8$. In this case, the $\lambda = -1 + \sqrt{1 + 2\gamma}$ root lies above $\lambda = 2$ and the root at $\lambda = 2$ is the one that is relevant for the minimization of the energy. This root only appears in the two-species analysis and is the limiting value for $x_B \to 0$ of the root found in the range $0 < \lambda < 2$. Because of this root, one does not recover the single-species energy that would be obtained for $\lambda = -1 + \sqrt{1 + 2\gamma}$. As a result, the $x_B \to 0$ limit of the two-species analysis for $\gamma > 4$ cannot recover the results in the single-species limit, and leads to the conclusion that persistent currents are not possible at higher angular momenta.

Further insight into the $x_B \to 0$ limit is provided by examining the behaviour of the wave function coefficients $c_0$, $c_2$, $d_{-1}$ and $d_1$ as functions of $x_B$. These coefficients contribute to the angular momentum of each species according to

$$l_A = x_A \left( |c_1|^2 + 2|c_2|^2 \right) = x_A + x_A \left( |c_2|^2 - |c_0|^2 \right)$$  \hspace{1cm} (4.2.98)

and

$$l_B = x_B \left( -|d_{-1}|^2 + |d_1|^2 \right).$$  \hspace{1cm} (4.2.99)

Since $l = l_A + l_B = x_A - \varepsilon$, we have

$$x_A \left( |c_2|^2 - |c_0|^2 \right) + x_B \left( -|d_{-1}|^2 + |d_1|^2 \right) = -\varepsilon. \hspace{1cm} (4.2.100)$$

The square of each of these coefficients is proportional to $\varepsilon$. Defining the quantities

$$\frac{\Delta l_A}{\varepsilon} = x_A \left( \frac{|c_2|^2}{\varepsilon} - \frac{|c_0|^2}{\varepsilon} \right)$$ \hspace{1cm} (4.2.101)

and

$$\frac{\Delta l_B}{\varepsilon} = x_B \left( -\frac{|d_{-1}|^2}{\varepsilon} + \frac{|d_1|^2}{\varepsilon} \right), \hspace{1cm} (4.2.102)$$
Figure 4.2.6: (a,b) Contribution of the coefficients $|c_0|^2$ and $|c_2|^2$ and (c,d) $|d_{-1}|^2$ and $|d_1|^2$ to the angular momentum plotted for $\gamma = 2$. 
Figure 4.2.7: (a,b) Contribution of the coefficients $|c_0|^2$ and $|c_2|^2$ and (c,d) $|d_{-1}|^2$ and $|d_1|^2$ to the angular momentum plotted for $\gamma = 8$. 
Figure 4.2.8: The angular momenta plotted for $\gamma = 2$ as a function of $x_B$.

Figure 4.2.9: The angular momenta plotted for $\gamma = 8$ as a function of $x_B$. 
we see that \((\Delta l_A/\varepsilon + \Delta l_B/\varepsilon) = -1\). These ratios thus represent the relative contribution that each species makes to the change in angular momentum. In Fig. 4.2.6 - 4.2.8 we plot \(|c_0|^2/\varepsilon\), \(|c_2|^2/\varepsilon\), \(|d_{-1}|^2/\varepsilon\) and \(|d_1|^2/\varepsilon\) together with \(\Delta l_A/\varepsilon\) and \(\Delta l_B/\varepsilon\) as functions of \(x_B\) for \(\gamma < 4\) and \(\gamma > 4\). For \(\gamma < 4\), we see that species \(B\) makes a relatively small contribution to \(\Delta l\) which vanishes in the \(x_B \to 0\) limit. This species therefore has no effect in this limit. For \(\gamma > 4\) on the other hand we see that the angular momentum change is carried entirely by the \(B\) species in the \(x_B \to 0\) limit.

The reason for this surprising result is that \(|d_{-1}|^2 \propto x_B^{-1}\) so that \(x_B|d_{-1}|^2\) has a finite limit for \(x_B \to 0\). The divergence of \(|d_{-1}|^2\) as \(x_B \to 0\) is indicating that the result can only be valid for a decreasingly smaller range of \(\varepsilon\) since the normalization \(1 = |d_{-1}|^2 + |d_0|^2 + |d_1|^2\) must be preserved. In other words, for a fixed value of \(\varepsilon\) our theory must be breaking down when \(x_B\) approaches zero. This observation calls into question our conclusions on the behaviour of \(\bar{\varepsilon}_0(l)\) for \(x_B \leq l \leq x_A\).

To check this behaviour, we now investigate the energy at \(l = 1/2\) as we did in the single-species case. To do so, one has to include four components in each of the wave functions

\[
\psi_A = c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1 + c_2\phi_2 \\
\psi_B = d_{-1}\phi_{-1} + d_0\phi_0 + d_1\phi_1 + d_2\phi_2.
\]

(4.2.103)

(4.2.104)
Substituting these wave functions into (4.2.2), we have

\[
\bar{E} = x_A \left( |c_{-1}|^2 + |c_1|^2 + 4|c_2|^2 \right) + x_B \left( |d_{-1}|^2 + |d_1|^2 + 4|d_2|^2 \right)
\]

\[
+ x_A^2 \pi \gamma \int_0^{2\pi} d\theta |c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1 + c_2\phi_2|^4
\]

\[
+ x_B^2 \pi \gamma \int_0^{2\pi} d\theta |d_{-1}\phi_{-1} + d_0\phi_0 + d_1\phi_1 + d_2\phi_2|^4
\]

\[
+ 2x_A x_B \pi \gamma \int_0^{2\pi} d\theta |c_{-1}\phi_{-1} + c_0\phi_0 + c_1\phi_1 + c_2\phi_2|^2
\]

\[
\times |d_{-1}\phi_{-1} + d_0\phi_0 + d_1\phi_1 + d_2\phi_2|^2.
\]

(4.2.105)

We now use the properties established earlier for the complex coefficients \(c_l = |c_l|e^{i\alpha_l}\) and \(d_l = |d_l|e^{i\chi_l}\) at \(l = 1/2\). We have (see (4.1.68) for example)

\[
|c_0| = |c_1| \equiv x
\]

\[
|c_{-1}| = |c_2| \equiv y
\]

\[
\alpha_1 = -\alpha_0
\]

\[
\alpha_{-1} = -\alpha_2
\]

\[
|d_0| = |d_1| \equiv u
\]

\[
|d_{-1}| = |d_2| \equiv v
\]

\[
\chi_1 = -\chi_0
\]

\[
\chi_{-1} = -\chi_2.
\]

Using these relations, the normalization constraints reduce to

\[
x^2 + y^2 = \frac{1}{2}
\]

(4.2.106)

and

\[
u^2 + v^2 = \frac{1}{2}.
\]

(4.2.107)
Furthermore, the angular momentum of each species is given by

\[
l_A = x_A (\frac{-|c_{-1}|^2 + |c_1|^2 + 2|c_2|^2}{x^2 + y^2}) = x_A (x^2 + y^2) \tag{4.2.108}
\]

\[
l_B = x_B (\frac{-|d_{-1}|^2 + |d_1|^2 + 2|d_2|^2}{u^2 + v^2}) = x_B (u^2 + v^2). \tag{4.2.109}
\]

Using (4.2.106) and (4.2.107) we see that the total angular momentum is

\[
l_A + l_B = \frac{1}{2} (x_A + x_B) = \frac{1}{2} \text{ as required.}
\]

With the variables introduced above, each of the terms in (4.2.105) can be simplified. The first two integrals in (4.2.105) are similar to the one found in the single-species case, that is

\[
x^2 A \pi \gamma \int_0^{2\pi} d\theta |c_{-1} \phi_{-1} + c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2|^4 \\
= x^2 A \gamma \left[ \frac{1}{2} + x^4 + y^4 + 8x^2 y^2 + 4x^3 y \cos \beta \right],
\]

where \( \beta = 3\alpha_0 + \alpha_2 \). Similarly,

\[
x^2 B \pi \gamma \int_0^{2\pi} d\theta |d_{-1} \phi_{-1} + d_0 \phi_0 + d_1 \phi_1 + d_2 \phi_2|^4 \\
= x^2 B \gamma \left[ \frac{1}{2} + u^4 + v^4 + 8u^2 v^2 + 2u^3 v \cos \xi \right],
\]

where \( \xi = 3\chi_0 + \chi_2 \).

The cross term in (4.2.105) is new and is given by

\[
2x_A x_B \pi \gamma \int_0^{2\pi} d\theta |c_{-1} \phi_{-1} + c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2|^2 |d_{-1} \phi_{-1} + d_0 \phi_0 + d_1 \phi_1 + d_2 \phi_2|^2 \\
= x_A x_B \gamma \left[ 1 + 8xyuv \cos(\alpha_0 + \alpha_2 - \chi_0 - \chi_2) \right. \\
+ 8xyuv \cos(-\alpha_0 + \alpha_2 + \chi_0 - \chi_2) \\
+ 4xyu^2 \cos(\alpha_0 + \alpha_2 + 2\chi_0) + 4x^2 uv \cos(2\alpha_0 + \chi_0 + \chi_2) \\
+ 2x^2 u^2 \cos(2\alpha_0 - 2\chi_0) + 2y^2 v^2 \cos(2\alpha_2 - 2\chi_2) \right]. \tag{4.2.110}
\]
Using $\alpha_2 = \beta - 3\alpha_0$ and $\chi_2 = \xi - 3\chi_0$ to eliminate $\alpha_2$ and $\chi_2$, we obtain

$$2x_A x_B \pi \gamma \int_0^{2\pi} d\theta |c_{-1} \phi_{-1} + c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2|^2 |d_{-1} \phi_{-1} + d_0 \phi_0 + d_1 \phi_1 + d_2 \phi_2|^2$$

$$= x_A x_B \gamma \left[ 1 + 8xyuv \cos(\theta - \beta + \xi) + 8xyuv \cos(2\theta - \beta + \xi) 
+ 4xyu^2 \cos(\theta - \beta) + 4x^2 uv \cos(\theta + \xi) 
+ 2x^2 u^2 \cos \theta + 2y^2 v^2 \cos(3\theta - 2\beta + 2\xi) \right],$$

where we have defined the phase $\theta = 2(\alpha_0 - \chi_0)$. The final expression for the energy is

$$\bar{E}_0 = \frac{1}{2} + 4x_A y^2 + 4x_B v^2 + \frac{1}{2}\gamma 
+ x_A^2 \gamma \left[ x^4 + y^4 + 8x^2 y^2 + 4x^3 y \cos \beta \right] 
+ x_B^2 \gamma \left[ u^4 + v^4 + 8u^2 v^2 + 4u^3 v \cos \xi \right] 
+ x_A x_B \gamma \left[ 8xyuv \cos(\theta - \beta + \xi) + 8xyuv \cos(2\theta - \beta + \xi) 
+ 4xyu^2 \cos(\theta - \beta) + 4x^2 uv \cos(\theta + \xi) 
+ 2x^2 u^2 \cos \theta + 2y^2 v^2 \cos(3\theta - 2\beta + 2\xi) \right]. \quad (4.2.111)$$

We see that the energy depends on three independent phases $\beta$, $\chi$ and $\theta$ and two amplitudes $x \left( y = \sqrt{1/2 - x^2} \right)$ and $u \left( v = \sqrt{1/2 - u^2} \right)$. From this expression for $E_0$ we see that the energy must reduce to the single-species result in the $x_A \to 1$ limit, which the analysis leading to Fig. 4.2.7 did not do.

If we set $x_A = 1 \ (x_B = 0)$ in (4.2.111), the energy reduces to that of the single-species case. We found earlier that this energy is minimized for $\beta = \pi$. We do not expect this conclusion to change when $x_A$ is close to, but not exactly equal to 1. For these values of $x_A$, the term in (4.2.111) proportional to $x_B^2$ can be neglected and the
energy takes the form

\[ \bar{E}_0 \simeq \frac{1}{2} + 4xAy^2 + 4xBv^2 + \frac{1}{2} \gamma \]
\[ + x_A^2 \gamma \left[ x^4 + y^4 + 8x^2y^2 - 4x^3y \right] \]
\[ + x_A x_B \gamma \left[ -8xyuv \cos(\theta + \xi) - 8xyuv \cos(2\theta + \xi) \right. \]
\[ - 4xyu^2 \cos \theta + 4x^2uv \cos(\theta + \xi) \]
\[ \left. + 2x^2u^2 \cos \theta + 2y^2v^2 \cos(3\theta + 2\xi) \right]. \] (4.2.112)

From this we see that the phases \( \theta \) and \( \xi \) only appear in the last term. The minimization of \( \bar{E} \) with respect to these phases is in general a difficult problem which leads to equations for \( \theta \) and \( \xi \) that are coupled to all the other variables. Rather than following this path, we will make the assumption that \( \theta \) and \( \xi \) are restricted to 0 and \( \pi \) and explore the various possibilities. We define the function

\[ f(x, u, \xi, \theta) = -8xyuv \cos(\theta + \xi) - 8xyuv \cos(2\theta + \xi) - 4xyu^2 \cos \theta \]
\[ + 4x^2uv \cos(\theta + \xi) + 2x^2u^2 \cos \theta + 2y^2v^2 \cos(3\theta + 2\xi), \] (4.2.113)

which is the quantity multiplying \( x_A x_B \gamma \) in (4.2.112). This function is tabulated in Table 4.2.1 for various values of \( \xi \) and \( \theta \). From this table it is clear that \( \xi = 0, \theta = \pi \) will give a lower energy than \( \xi = \pi, \theta = \pi \). For \( \xi = 0, \theta = 0 \) we have

\[ f(x, u, 0, 0) - f(x, -u, 0, 0) = 8xuv(x - 4y). \] (4.2.114)
CHAPTER 4. MEAN-FIELD THEORY OF PERSISTENT CURRENTS

Figure 4.2.10: Contour plot of the energy in (4.2.111) for $x_A = 0.95$, $\xi = 0$, $\theta = \pi$, $\gamma = 2$. Horizontal axis indicates values of $x$, vertical axis values of $u$.

Since $x_A$ is close to 1, we know that $x$ is much larger than $y$. This implies that the function $f(x, u, 0, 0)$ will have its minimum at negative values of $u$. But $u$ much be positive (recall $u = |d_0|$), so this case must be rejected.

Finally, for $\xi = \pi$, $\theta = 0$, we have

$$f(x, u, \pi, 0) - f(x, -u, \pi, 0) = -8xuv(x - 4y).$$  \hfill (4.2.115)

The same argument implies that $f(x, u, \pi, 0)$ will have its minimum at positive $u$. A comparison of the contour plots of $f(x, u, \pi, 0)$ and $f(x, u, 0, \pi)$ shows that the latter is the one that provides the lowest energy. In Fig. 4.2.10 we show a contour plot of $E(x, u, 0, \pi)$ for $x_A = 0.95$ and $\gamma = 2$ in the vicinity of the minimum. We find that $x_{\text{min}} \simeq 0.697$ and $u_{\text{min}} \simeq 0.675$. The value of $x_{\text{min}}$ found here is close to the value of 0.696 found for the single-species given in Section 4.1. In the limit $x_A \to 1$, the $u$
variable becomes irrelevant and the minimum shown in Fig. 4.2.10 (with respect to \(x\)) becomes the minimum of the single-species situation.

In the single-species analysis we found that persistent currents could occur at higher angular momenta for \(\gamma\) sufficiently large (e.g. \(\gamma_{cr} = 15/2\) for \(n = 2\) in (4.1.46)). The analysis of the two-species case, however, indicated that this is not possible. We will show now that this conclusion is erroneous.

To be specific, we take \(x_A = 0.95\) and \(\gamma = 32\). The value of the energy at \(l = 1 + l'\) with \(0 \leq l' \leq 1\) is given by

\[
\bar{E}_0(1 + l') = (1 + l')^2 + \bar{\epsilon}_0(1 + l')
= (1 + l')^2 + \bar{\epsilon}_0(l')
= (1 + l')^2 + \bar{E}_0(l') - l'^2
= 1 + 2l' + \bar{E}_0(l').
\]

At \(l' = x_A\) (4.2.21) gives \(\bar{E}_0(x_A) = x_A + \gamma/2\). We then find that \(\bar{E}_0(1 + x_A) - \gamma/2 = 1 + 3x_A = 3.85\) at \(x_A = 0.95\). This value is exact.

To obtain an approximation to \(\bar{E}_0(1 + l')\) for \(x_B \leq l' \leq x_A\) we can use the expression for \(\bar{\epsilon}_0(l)\) given in (4.2.93). For \(x_A = 0.95\) and \(\gamma = 32\), the value of \(\lambda\) is \(\lambda = 1.78\). With \(\bar{\epsilon}_0(l)\) determined in this way, we can calculate \(\bar{E}_0(1 + l')\) which is shown in Fig. 4.2.11 by the dashed curve. Evidently, we do not have a local minimum at \(x_A = 0.95\). However, we can use (4.2.111) to evaluate \(\bar{E}_0(l = 3/2)\) which from (4.2.116) is given by

\[
\bar{E}_0\left(\frac{3}{2}\right) = 2 + \bar{E}_0\left(\frac{1}{2}\right).
\]

The value we find is \(\bar{E}_0(3/2) - \gamma/2 = 5.26\) which is plotted as the single point at \(l = 3/2\) in Fig. 4.2.11. Since \(\bar{E}_0(3/2) > \bar{E}_0(1.95)\) we conclude that there must be a
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Figure 4.2.11: The energy plotted in the range $1 \leq l \leq 2$ for $x_A = 0.95$ and $\gamma = 32$. The single point is the value at $l = 3/2$ as determined by the four-component wave function.

Figure 4.2.12: Minimum energy plotted as a function of $\gamma$ for $x_A = 0.95$, $l = 3/2$. The horizontal solid line is the value $E_0(1 + 0.95) - \gamma/2$. 
local minimum between \( l = 3/2 \) and \( l = 1.95 \). Thus the approximate representation of \( \bar{E}_0(1 + l') \) shown by the dashed curve in Fig. 4.2.11 cannot give accurate values of the energy near \( l = 3/2 \). In other words, the slope of \( \bar{E}_0(l) \) that we calculated at \( l = x_A^- \), although correct, is not sufficient to determine the global behaviour of \( \bar{E}_0(l) \) and cannot be used to provide a criterion for the existence of persistent currents. This conclusion contradicts the claims made in [24]. We have checked that the same conclusion is valid for even smaller values of \( x_A \), but increasingly larger values of \( \gamma \) are required to achieve a local minimum between \( l = 3/2 \) and \( l = 1 + x_A \). For completeness, we show in Fig. 4.2.12 the behaviour of \( \bar{E}_0(l = 3/2, x_A = 0.95) - \gamma/2 \) as a function of \( \gamma \). From this we see that the critical value of \( \gamma \) is approximately \( \gamma_{cr}(x_A = 0.95) \simeq 16 \).

We finally discuss the behaviour of \( \bar{E}_0(l = 1/2) \) when \( x_A = 1/2 \). In this limit the expression for \( \bar{E}_0(l) \) given in (4.2.21) is correct for all \( l \) and gives in particular \( \bar{E}_0(l = 1/2, x_A = 1/2) = 1/2 + \gamma/2 \). This value is reproduced by (4.2.111) at \( x_A = 1/2 \) irrespective of the phases \( \beta \) and \( \xi \) since the minimum occurs for \( \theta = \pi \) and \( x = u = 1/\sqrt{2} \), where all the \( \beta \) and \( \xi \) dependent terms are zero since \( y = v = 0 \). For intermediate values of \( x_A \) between \( x_A = 1/2 \) and \( x_A = 1 \), a more careful analysis of (4.2.111) is required to determine the appropriate phases \( \beta, \xi \) and \( \theta \) is general. This is a project for the future.
Chapter 5

Conclusions

In order to make predictions regarding the possibility of persistent currents in the ring geometry a thorough understanding of the behaviour of the condensate energy must be acquired. In Section 2.1 we showed how the three-dimensional Gross-Pitaevskii energy functional could be reduced to a 1D form suitable to describe a condensate in the ring geometry. This energy functional was then used throughout the subsequent chapters in order to determine expressions for the condensate energy for various cases. To obtain the criterion for energetic stability, a small perturbation to the condensate wave functions was introduced. If the perturbation in energy is found to be positive, the ground state is in an absolute minimum and the system is energetically stable. To determine dynamic stability we considered small-amplitude oscillations of the condensate about the ground state. The condensate becomes dynamically unstable if the frequencies $\omega$ are imaginary which leads to an increase of the amplitude with time. As shown in Sections 2.2 and 2.3, the energetic and dynamic stability criteria are, in fact, equivalent.

In Chapter 3 we reviewed the general arguments used by Bloch [1] to determine the
dependence of the energy on the angular momentum for a single-species system. This argument was extended to the two-species gas. We concluded that the periodicity of the function $\epsilon_0(l)$ held for the two-species situation only when $M_A = M_B$.

In Section 4.1 a detailed analysis of the possibility of persistent currents was provided for the case of a single-species condensate. We started by using a simple two-term wave function to obtain the ground state energy in the range $0 \leq l \leq 1$. This provided an approximate critical value of the interaction parameter for which persistent currents are stable. To improve on this estimate we next considered a three-term wave function which is suitable to analyze the energy in the vicinity of $l = 1$. These results allowed us to determine the critical value of the interaction parameter $\gamma_{cr}$ at integral values of $l$. Our results confirm the statements made in [24, 25, 23]. In order to obtain a better understanding of the $l$-dependence of the energy and coefficients in the wave function expansion, we also performed calculations at $l = 1/2$. The idea of calculating the energy at $l = 1/2$ also turned out to be useful in the two-species calculation.

Section 4.2 explained what happens when atoms of a second species are introduced. The two-species situation is quite different from the single-species case. It was found that the energy depends linearly on $l$ outside the interval $x_B + n \leq l \leq x_A + n$ ($n = 0, 1, 2, ...$). This implied that persistent currents were not possible at integral values of the angular momentum, but rather at angular momenta close to $l = x_A + n$. The behaviour of the energy was then investigated in the vicinity of these points. This analysis led to the conclusion that persistent currents could only occur in the interval $0 \leq l \leq 1$. However, the analysis of the results in the limit $x_A \rightarrow 1$ indicated that this conclusion could not be correct. To check this, we calculated the energy
at the midpoint $l = 1/2$ as we did in the single-species case. These results allowed us to conclude that persistent currents are possible for sufficiently large interaction parameter $\gamma$ outside the interval $0 \leq l \leq 1$, contrary to the claims of [24]. Thus an analysis of the energy based on the slopes at $l = x_A^-$ is not sufficient to determine the existence of persistent currents in a mixture of two species. Furthermore, we found that the results based on the slope analysis did not recover the single-species situation in the $x_A \to 1$ limit, as it should. In other words, the presence of a small admixture of a second species cannot have the profound effect on the possibility of persistent currents that the authors of [24] suggest.
Bibliography


