Stochastic Dynamic Optimization of Cut-off Grade in Open Pit Mines

By

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Abstract

Mining operations exploit mineral deposits, processing a portion of the extracted material to produce salable products. The concentration of valuable commodities within these deposits, or the grade, is heterogeneous. Not all material has sufficiently high grades to economically justify processing. Cut-off grade is the lowest grade at which material is considered ore and is processed to create a concentrated commodity product. The choice of cut-off grade at a mining project can be varied over time and dramatically impacts both the operation of the mine and the economics of the project.

The majority of literature and the accepted industry practices focus on optimizing cut-off grade under known commodity prices. However, most mining operations sell their products into highly competitive global markets, which exhibit volatile commodity prices. Making planning decisions assuming that a given commodity price prediction is accurate can lead to sub-optimal cut-off grade strategies and inaccurate valuations.

Some academic investigations have been conducted to optimize cut-off grade under stochastic or uncertain price conditions. These works made large simplifications in order to facilitate the computation of a solution. These simplifications mean that detailed mine planning data cannot be used and the complexities involved in many real world projects cannot be considered.

A new method for optimizing cut-off grade under stochastic or uncertain prices is outlined and demonstrated. The model presented makes use of theory from the field of Real Options and is designed to incorporate real mine planning data. The model introduces two key innovations. The first is the method in which it handles the cut-off grade determination. The second innovation is
the use of a stochastic price model of the entire futures curve and not simply a stochastic spot price model. The model is applied to two cases. The first uses public data from a National Instrument 43-101 report. The second case uses highly detailed, confidential data, provided by a mining company from one of their operating mines.
Acknowledgements

This thesis, though technically authored by myself, in reality represents the combined contributions of many. I would like to take this opportunity to thank my supervisor, Prof. Jim Martin for his support and encouragement throughout my Master’s program, as well as the freedom he provided me in exploring my thesis topic. I would also like to thank Dr. Matt Thompson who provided invaluable technical input. Without his contributions and guidance this thesis would not have been nearly as engaging or as complete.

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# Contents

Abstract ................................................................................................................................. i  
Acknowledgements ................................................................................................................ iii  
Table of Figures ...................................................................................................................... vi  
List of Tables ......................................................................................................................... viii  
Glossary of Terms and Symbols ........................................................................................... ix  
Chapter 1 Introduction ......................................................................................................... 1  
  1.1 Objective statement ....................................................................................................... 1  
  1.2 Context ......................................................................................................................... 1  
  1.3 Problem Statement ...................................................................................................... 3  
  1.4 Construction of Thesis ............................................................................................... 7  
Chapter 2 Literature Review ................................................................................................. 9  
  2.1 Cutoff Grade Optimization – Deterministic Prices ...................................................... 9  
  2.2 Real Options in Mining .......................................................................................... 11  
  2.3 Cut-Off Optimization – Stochastic Prices ................................................................ 14  
Chapter 3 Description of Model and Solution Technique ............................................... 18  
  3.1 A Summary of the Open Pit Mine Design Process .................................................. 18  
  3.2 Operational Model .................................................................................................. 23  
  3.3 Modeling Price Dynamics: ....................................................................................... 30  
    3.3.1 Selecting a Price Model ...................................................................................... 34  
    3.3.2 Determining Price Model Parameter Values ..................................................... 35  
    3.3.3 Combining the Price Model with the Operating Model .................................... 37  
  3.4 Derivation of the Partial Differential Equations: ...................................................... 37  
  3.5 Implementation of the Numerical Solution: .............................................................. 41  
  3.6 Potential modifications to the operational model ...................................................... 45  
  3.7 Model Assumptions and Limitations ....................................................................... 47  
Chapter 4 Worked Example – Detour Lake ..................................................................... 49  
  4.1 Project Description ..................................................................................................... 49  
  4.2 Data .......................................................................................................................... 50  
  4.3 Method ....................................................................................................................... 56  
  4.4 Results ....................................................................................................................... 57  
    4.4.1 Single Simulation in Detail ............................................................................... 60  
  4.5 Conclusions and Discussion ...................................................................................... 63
# Table of Figures

Table 1: Steps of the traditional open pit mine design process (Dagdelen, 2001)........................ 18
Table 2: A 3-D block model representation of a copper deposit (Dagdelen, 2001) ..................... 19
Table 3: Typical cross section of internal pit phases or pushbacks (Bohnet, 1990) ..................... 21
Table 4: Tonnage histogram of a copper deposit from (Hustruulid & Kuchta, 2006) ................. 25
Table 5: Grade tonnage plot of copper deposit from Table 4...................................................... 27
Table 6: An explanatory illustration of the material distribution function f(x) ............................ 28
Table 7: A plot illustrating three simulations of gold price modeled using GBM. Parameter values used were an initial expectations curve starting at $1,100, a constant $\gamma(t)$ of 0.1 and $\sigma = 0.25$. ............................................................................................................................................... 33
Table 8: A plot illustrating three simulations of gold price modeled using a mean reverting model where $\eta = 1$, the initial expectations curve $F$ is a flat $1,100 and $\sigma = 0.25$. .................. 34
Table 9: Flow chart outlining steps in the numerical solution...................................................... 44
Table 10: A graph illustrating the total reserves and average grade of the Detour Lake project at various cut-off grades .................................................................................................................... 51
Table 11: A plan view of the ultimate pit design at Detour Lake.................................................... 53
Table 12: Histogram showing the results of the 2,000 before tax simulations............................. 58
Table 13: Histogram showing the results of the 2,000 after tax simulations................................ 58
Table 14: Cut-off strategy at one day ........................................................................................... 59
Table 15: Cut-off strategy at 24 years ........................................................................................ 60
Table 16: Cut-off strategy for a single price simulation............................................................... 61
Table 17: A single price simulation and corresponding daily cash flows .................................... 62
Table 18: A detailed view of Table 17, for the final six years of the simulation............................ 62
Table 19: The grouping of blocks from the Pit 1 block model corresponding to Phase 5............ 69
Table 20: The grouping of blocks from the Pit 2 block model corresponding to Phase 5............ 69
Table 21: Grade-tonnage histogram for phase 5 of Pit 1............................................................ 70
Table 22: Grade-tonnage histogram for phase 5 of Pit 2............................................................ 70
Table 23: Phase 5, combined grade-tonnage histogram for both pits......................................... 71
Table 24: Distribution of simulated RO valuations ...................................................................... 72
Table 25: Cash flow details of a single price simulation. Banding indicates the change from one phase to the next ............................................................................................................................ 74
Table 26: Cut-off grade strategy for Pit 1 for a single simulation................................................. 74
Figure 27: Cut-off grade strategy for Pit 2 for a single simulation.................................75
Figure 28: Comparison of un-hedged and hedged simulation results.................................78
Figure 29: Histograms of NPV for the hedged and un-hedged simulations .........................78
Figure 30: A single simulation showing the size of the dynamic hedge over time...............79
List of Tables

Table 1: Historically estimated volatilities of the spot and futures price 27 months forward and the correlation of these two values ................................................................. 35
Table 2: The total reserve and average grade of the Detour Lake project at various cut-off grades ........................................................................................................... 51
Table 3: List of parameters required for RO analysis and the values used .................................. 52
Table 4: Details of cut-off grade calculation from the BBA reserve report ................................ 54
Table 5: Operating parameters at all cut-off grades, assumes a gold price of $935/oz .................. 55
Table 6: Summary of valuation results ..................................................................................... 57
Table 7: Explanatory statistics for simulation populations .......................................................... 57
Table 8: Optimization parameters for the six material types ....................................................... 66
Table 9: Summary of input values ............................................................................................ 66
Table 10: Example of current practice cut-off grade calculation .............................................. 67
Table 11: Schedule showing fraction of material in each pit for all phases ................................. 69
Table 12: Valuation results ....................................................................................................... 72
Table 13: Summary statistics for the simulation population ...................................................... 72
Table 14: Derivatives for project at initial conditions ................................................................. 76
Table 15: Summary statistic of unhedged and hedged simulation populations ............................. 77
Table 16: Summary of Detour Lake valuation results ................................................................. 85
Table 17: Summary of valuation results, confidential case study .............................................. 86
Table 18: Reasons for not using real options (Block, 2007) ...................................................... 99
Glossary of Terms and Symbols

What follows is a summary of important terms used in this document along with any notation used to represent them.

<table>
<thead>
<tr>
<th>Term</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contained Metal</td>
<td>( M )</td>
<td>The quantity of a commodity contained in phase.</td>
</tr>
<tr>
<td>Cut-off grade</td>
<td>( c(I,X,t) )</td>
<td>The grade value that distinguishes material ore from waste.</td>
</tr>
<tr>
<td>Expected Price</td>
<td>( F(0,T) )</td>
<td>The initial expected commodity price for all points in the future.</td>
</tr>
<tr>
<td>Extraction Rate</td>
<td>( Ex(I,c) )</td>
<td>The rate at which material is excavated from the current phase.</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>( \delta )</td>
<td>The exponential rate at which future cash flows are discounted to determine their equivalent present value. Expressed as a percentage.</td>
</tr>
<tr>
<td>Fixed Costs</td>
<td>( C_{\text{fixed}} )</td>
<td>Costs that are incurred solely based on the passage of time. Includes costs such as G&amp;A and sustaining capital.</td>
</tr>
<tr>
<td>Ore Tonnage</td>
<td>( Q_o )</td>
<td>The fraction of material that has a grade equal to or above the cut-off grade.</td>
</tr>
<tr>
<td>Waste Tonnage</td>
<td>( Q_w )</td>
<td>The fraction of material that has a grade below the cut-off grade.</td>
</tr>
<tr>
<td>Grade</td>
<td>( x )</td>
<td>The concentration of a commodity within a given quantity of material. Expressed in various units.</td>
</tr>
<tr>
<td>Grade-tonnage distribution</td>
<td>( F_{j0}(x) )</td>
<td>A function describing the quantity of material in phase ( j ) with grade of ( x ).</td>
</tr>
<tr>
<td>Mining Limit</td>
<td>( K_{\text{mine}} )</td>
<td>The maximum tonnage of material that can be excavated in a given interval of time. Expressed in units of mass per unit of time.</td>
</tr>
<tr>
<td>Net cash flow</td>
<td>( P(t) )</td>
<td>Revenues minus costs. Also referred to as operating profit.</td>
</tr>
<tr>
<td>Processing Limit</td>
<td>( K_{\text{proc}} )</td>
<td>The maximum tonnage of material that can be treated as ore in a given time interval. Expressed in units of mass per unit of time.</td>
</tr>
<tr>
<td>Phase</td>
<td>( j(I) )</td>
<td>A fraction of the material within the planned mine reserve. All of the phases are sequenced to define a mine schedule.</td>
</tr>
<tr>
<td>Selling Price</td>
<td>( s(t) )</td>
<td>The price which the mine will receive for each unit of salable product it sells.</td>
</tr>
<tr>
<td>Term</td>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Stripping ratio</td>
<td>$Sr(I,c)$</td>
<td>The ratio of waste over ore. $Sr = \frac{Q_w}{Q_o}$</td>
</tr>
<tr>
<td>Tonnage</td>
<td>$Q$</td>
<td>A quantity of material within a phase of the mine plan. Expressed in units of mass.</td>
</tr>
<tr>
<td>Tonnage Remaining</td>
<td>$I$</td>
<td>The tonnage of reserve remaining within the mine plan.</td>
</tr>
<tr>
<td>Time</td>
<td>$t$</td>
<td>The time passed since the initial time of the analysis. Expressed in years.</td>
</tr>
<tr>
<td>Variable Ore Costs</td>
<td>$C_{ore}$</td>
<td>The costs associated with excavating a unit of material and treating it as ore.</td>
</tr>
<tr>
<td>Variable Waste Costs</td>
<td>$C_{waste}$</td>
<td>The costs associated with excavating a unit of material and treating it as waste</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma(t)$</td>
<td>The expected degree of annualized “randomness” in the commodity price. Expressed as a percentage.</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Objective statement
The objective of this document is to outline and demonstrate a method for optimizing cut-off grade strategy, to maximize present value of an open pit mine, under uncertain future economic conditions. The method simultaneously produces optimal cut-off grade strategies, valuations and sensitivities. These outputs are critical in mine design, operation, valuation and hedging.

1.2 Context
Due to the competitive and diversified nature of the global resources industry, mining companies are often price takers and are exposed to highly volatile commodity prices. As price volatility rises it becomes increasingly difficult to accurately value and manage mineral assets using traditional methods. The value of a mining asset clearly depends on the underlying commodity price(s), however this relationship is more complicated than it first appears.

Changes in commodity price directly impact the value of the reserves/resources in the ground as well as the ongoing revenue the mining project generates. The nature of this impact is complicated by the mine management’s ability to react to changes in commodity price. For instance, if commodity prices rise the mine may expand production, conversely if prices fall the management may temporarily close the mine. Optimizing these decisions in the face of volatile prices is critically important to correctly operating and valuing mining assets and many have investigated these questions. Another option uniquely available to mining projects is the ability to alter cut-off grade, which affects the rate of commodity production, the size of the reserves and
the unit operating costs. Understanding the role of cut-off grade and how to optimize it in conditions of uncertain future prices is the focus of this discussion.

Mines extract material from natural concentrations of minerals or other commodities in the Earth’s crust, called deposits. These deposits are formed by natural processes and the distribution of a commodity throughout a deposit is heterogeneous. There are areas within a deposit of higher and lower concentration of various commodities. The degree of concentration of a commodity in a volume of material is referred to as its grade. When material has a sufficiently high enough grade to economically justify processing it to produce a salable product, it is referred to as ore. Material which needs to be excavated from a deposit but is not processed is called waste and is disposed of. Cut-off grade is the criterion employed at mines to distinguish between ore and waste. Material which has a grade below the cut-off grade is classified as waste, while material with a grade equal to or greater than the cut-off grade is classified as ore.

Mine operators are free to set cut-off grade within technical constraints and are able to change the cut-off grade throughout the life of a mine. Varying cut-off grade has profound impacts on the economics and operation of a project. Increasing it will reduce the fraction of a deposit that is considered ore while the average grade of the ore will increase. Decreasing cut-off grade will have the opposite impact. Altering cut-off grade will impact not only the quantity of product produced by an operation and hence the revenue, but it will also impact the operating cost per unit of product. It is paramount to optimize cut-off grade to determine the maximum value of a given mine design in order to facilitate better design, valuation and operation of mining assets.
1.3 Problem Statement

The objective of mine operators are varied, however often the primary objective is to maximize value. Lane offers three possible definitions that match this objective (Lane K., 1964):

1. Maximize total profits
2. Maximize present value of all future profits
3. Maximize short term profits

As Lane points out, objectives one and three are specific cases of objective 2. Objective one is the case of optimizing with a zero discount rate and while objective three is the case of an arbitrarily high discount rate. By focusing on objective two we can consider all three. The net present value (NPV) of a continuous series of cash flows is defined by equation 1-1. Readers may be more familiar with the discrete time approximation shown by equation 1-2.

\[ NPV = \int_{t}^{T} \frac{P}{e^{\delta t}} dt \quad 1-1 \]

\[ NPV = \sum_{t}^{T} \frac{P_{t}}{(1 + \delta)^{t}} \quad 1-2 \]

Where:

P = net cash flow

t = time of cash flow

\( \delta \) = discount rate

Mining operations generate revenue by excavating material and further processing it to create a salable product. However, not all of the material a mine excavates is processed; only material that has a sufficiently high concentration of metal to economically justify the cost of processing is considered ore. Material that does not have a high enough concentration is considered waste and
is (for the most part) irreversibly dumped. The level of concentration or grade that distinguishes between ore and waste is referred to as the cut-off grade.

Variable costs for treating material as waste are often significantly different from those associated with treating it as ore. Typical ore costs may include drilling, blasting, hauling, crushing, concentration, tailings disposal among others. Waste costs may include drilling, blasting, dumping, remediation and others. Mines also incur fixed costs, which include overhead expenditures required to keep the mine in operation.

Incorporating both revenues and the aforementioned three types of operating costs equation 1-3 is a representative formula for a mine’s instantaneous operating cash flow.

\[
P = \left( s(t) \times G(g, I) \times \frac{Q_o(I, g)}{Q} - C_{\text{mine}}(I) \times \frac{Q_o(I, g)}{Q} - C_{\text{waste}}(I) \times \frac{Q_w(I, g)}{Q} \right) dI
\]

\[\quad - C_{\text{fixed}} \times dt \]

Where:

- \(s\) = Selling price of output
- \(g\) = cut-off grade
- \(G\) = Recoverable ore grade
- \(Q_o/Q\) = Fraction of material treated as ore
- \(Q_w/Q\) = Fraction of material treated as waste
- \(C_{\text{mine}}\) = Cost of treating material as ore
- \(C_{\text{waste}}\) = Cost of treating material as waste
- \(I\) = Fraction of resource remaining (1 = un-mined, 0 = completely mined.)
- \(C_{\text{fixed}}\) = Fixed costs per unit time

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1 A full list of all symbols used in the document is provided in Appendix A.
As seen in equation 1-3, cut-off grade $g$, influences the operating cash flow in three ways; it impacts the quantity of ore, the quantity of waste and the grade of the ore. Raising the cut-off grade will decrease the proportion of the deposit treated as ore while the average grade of the ore will rise. In a given open pit mine design all material within the pit limits must be excavated. Thus any material that is not treated as ore is treated as waste. By increasing cut-off grade the fraction of material defined as waste is increased. For a given underground mine design the impact of changes to cut-off grade on the quantity of waste is less straightforward and is not considered in this thesis. A more complete mathematical explanation of the impact of cut-off grade on these three parameters is provided in Chapter 3.

Combining equations 1-1 and 1-3 the objective function, equation 1-4 is derived.

$$NPV = \max_c \left[ \int_t^T \left( (s(t) \times G(g, I) \times Q_o(I, g) - C_{mine}(I) \times Q_o(I, g) \ight. \\
- C_{waste}(I) \times Q_w(I, g)) \times dI - C_{fixed} \times dt \left. \right) e^{-\delta t} dt \right]$$  1-4

Optimizations of equation 1-4 (and variations thereof) have been conducted by many, including (Lane K., 1964), (Xiaojwei, Wang, Chu, & Zhang, 2010), among others. A thorough review of these and other works is provided in Chapter 2. However these authors make one large assumption; that all future prices $s(t)$ are known with certainty. This assumption could not be farther from reality. As mine operators sell commodities into global markets they are by and large price takers and are exposed to volatile prices for their products. Incorporating this price volatility is critical to both properly optimize cut-off grade and to correctly value mineral assets. Failure to consider price volatility leads to suboptimal mine operation and inaccurate valuations.
Some authors have attempted to incorporate price uncertainty into the optimization of cut-off grade including (Krautkraemer, 1988) and (Johnson, Evatt, Duck, & Howell, 2010). Unfortunately in order to facilitate solving the optimization they have simplified the description of the mining operation, functions $G(g)$, $Q_o(g)$ and $Q_w(g)$ to forms which fail to reflect the geological and engineering realities of a mine. These simplifications can lead not only to suboptimal choices of cut-off grade but can result in completely incorrect operating choices, i.e. increasing cut-off grade when it should in fact be decreased (Shinkuma, 1999). Further discussion regarding these works is provided in Chapter 2.

The optimization method and mine model presented in Chapter 3 incorporates both price uncertainty and a representation of the mine (functions $G(g)$, $Q_o(g)$ and $Q_w(g)$) that is consistent with current mining engineering practices and accepts data generated from commercial mine design software. This model is presented in detail in Chapter 3 along with a more detailed description of the cut-off grade optimization problem.

The model is then demonstrated in two case studies in Chapter 4 and Chapter 5. The first case was conducted using public information on the Detour Lake gold mine. The second case was completed using detailed, confidential data from an operating mine. In both cases, optimized cut-off grade strategies are determined and corresponding real option values are computed.
1.4 Construction of Thesis

Chapter 2: Literature Review
Literature on cut-off grade optimization under deterministic conditions is first reviewed. Then a discussion on Real Options work in mining is provided. Finally, a summary of attempts to optimize cut-off grade under stochastic (uncertain) price conditions is conducted and their shortcomings discussed.

Chapter 3: Mathematical Model
A detailed description of the mathematical problem of cut-off grade optimization is provided along with the construction of the model proposed by Thompson and Barr. This innovative model allows for the direct use of industry standard mine planning data and makes use of a stochastic futures price model. The derivation of the system of partial differential equations (PDEs) is provided along with the techniques to solve them is described in detail.

Chapter 4: Worked Example – Public Data
The model and methodology described in Chapter 3 is applied to the Detour Lake project in Northern Ontario. The data used for the example is taken directly from a publicly disclosed NI-43-101 report. Results from the analysis are compared and contrasted against results provided by other valuation and optimization methods.

Chapter 5: Worked Example – Confidential Data
A second, more detailed case is examined. Confidential data was provided by large mining company from one of their operating mines. The full dataset allowed for the inclusion of a more detailed mining model. Some of the additional complexities considered include; a fully detailed mine plan, multiple material classifications and the presence of multiple pits. A full cut-off grade strategy is determined and is tested against current cut-off practices at the mine. The real options value of the project is compared against existing traditional value estimates.
Chapter 6: Summary, Conclusions & Recommendations

The content from the earlier chapters is summarised and the results discussed. A discussion of potential applications and adaptations for the model is provided as well as some closing remarks.
Chapter 2

Literature Review

The problem of cut-off grade selection has been the subject of much research. Since Lane’s seminal work in 1964 the subject has been explored by numerous authors. The works presented here have been collected into three categories. First, works pertaining to the optimization of cut-off grade under known, or deterministic prices are reviewed. Following that, a summary of works pertaining to the application of Real Options in mining is provided. Finally works that incorporate both of these topics are discussed, in the third section which focuses on cut-off grade optimization under stochastic prices.

2.1 Cutoff Grade Optimization – Deterministic Prices

Lane’s 1964 paper entitled “Choosing the Optimum Cut-off Grade” (Lane K., 1964) is considered the pioneering, formal investigation into the topic of cut-off grade optimization. In this work Lane models a mining operation as three distinct stages: mining, concentrating and refining each of which has its own limiting capacity. Lane then derives formulas for six critical values for cut-off grade; three limiting cut-off grades and three balanced cut-off grades. If the mine operates at one of the limiting cut-off grades, it will reach the capacity of one of the individual three stages. Operating at a balanced cut-off, would ensure the capacities of two of the three stages are balanced. Lane then goes on to show that, under deterministic prices, the optimum cut-off grade is always one of these six possible choices.

Lane further develops his ideas in his book: The Economic Definition of Ore (Lane K., 1988). Here he describes the problem of cut-off grade optimization as dependent on the opportunity cost of deferring the processing of higher grade ore. He explains if low quality material is processed
today we are in essence deferring the processing of any available higher quality ore to some later date. Since the longer these potential profits are deferred into the future the more they are discounted, then the act of processing marginal ore imposes an opportunity cost due to the time-
value of money. This opportunity cost can be minimized by operating at higher cut-off grades and processing higher quality material sooner. With this understanding, the cut-off grade optimization problem becomes a balancing act between reducing the opportunity cost against the wasting of the resource. In order to solve such a problem, Lane outlines a dynamic programming approach that is able to ensure an optimal cut-off grade is chosen for all points in time, assuming future prices are known (Lane K., 1988).

Cut-off grade under deterministic price environments continues to be an area of active research. Minnett applies Lane’s theories on a simulated Witwatersrand-type gold deposit to illustrate the increase in the project’s NPV (Minnett, 2004) that can be achieved. Others have developed different methods for optimizing net present value when future prices are known. Yi and Surgul developed a method for solving the problem using Optimum Control Theory (Yi & Sturgul, 1988). Their method involved using a Hamiltonian to facilitate the determination of a solution, a popular technique for solving continuous time problems. Asad presented an algorithm for optimizing cut-off grade for an open pit mine with multiple metals and stockpiles (Asad, 2005) (Asad, 2007). Xiaojwei et al. used Dynamic Programming to optimize cut-off grade in an underground African copper mine (Xiaojwei, Wang, Chu, & Zhang, 2010).

Although the papers discussed above demonstrate a number of approaches to optimizing cut-off grade, they all depend on the significant assumption that future commodity prices are known with

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2 For further information on the use Hamiltonians readers are direct to: Control Theory - Applications to Management 11. NILSSON, D. and BENGTT AARO. Cutoff grade op- Science. 1981.
certainty for the duration of the mine life. This assumption is highly unrealistic and is recognized as a serious shortcoming in these solutions. Since the beginning of cut-off optimization research, the need to incorporate price uncertainty was seen as paramount. In his original paper Lane admits:

"Prices and costs are assumed to be stable throughout. This is a severe deficiency of present theory, but much more research is necessary before adequate general theory for cut-off grades in a fluctuating market can be developed."

In order to consider stochastic or unknown future prices, theory from the field of real options (RO)\(^3\) is required. A discussion of literature pertaining to applications of RO in mining is presented, followed by a discussion of cut-off optimization under stochastic prices.

### 2.2 Real Options in Mining

Real Options provides an alternative valuation framework to traditional, deterministic, discounted cash flow (DCF) methods. Like DCF, RO borrows its underlying theory from the pricing of financial securities. However, unlike DCF which is based on theory intended for assets which have linear dependence on uncertainty (Fisher, 1930), RO is based on theory applied to pricing financial derivatives known as options (Kester, 2004). Financial options offer the holder the right, but not the obligation to either purchase or sell some underlying asset at a future date. Financial options inherently represent the flexibility to act, should economic conditions in the future warrant (Wilmott, 2007).

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\(^3\) Real Options is also referred to as Modern Asset Pricing Theory (MAP) or Contingent Claims Analysis in some literature
The first complete solution for valuing a financial option was published by Black and Scholes in 1973 (Black & Scholes, 1973). This landmark and Nobel Prize winning paper, applied stochastic calculus to develop a partial differential equation (PDE) which described the dynamics of a European financial option. They then go on to provide an analytical solution to this PDE, creating the now famous Black-Scholes equation, which is still used today.

Following the publication of the Black-Scholes equation there was an explosion of research in the area of financial derivatives pricing. Others realized that many of the strategic options available to businesses, such as the option to expand production capacity or invest in research and development, behave similarly to financial options. Strategic decisions are options available to management which they will exercise if the economic environment warrants spending the required capital investment. The application of option pricing theory to non-financial options is referred to as Real Options (Schwartz & Trigeorgis, 2004).

Real Options theory has been extensively applied to the area of natural resource project valuation and has considered a variety of managerial flexibilities inherent to mineral assets. Mining assets are considered ideal candidates for Real Options valuation as many of the commodities produced by mines have active futures markets with options available on those commodity futures. The presence of these exchange traded instruments, allows analysts to use market observed values for key inputs, such as commodity price volatility.

With the financial support of the Canadian federal taxation authorities, the first paper to incorporate Real Options in natural resource project valuation was Brennan and Schwartz (1985). The authors derive a system of PDEs to represent a mining operation with the option to change

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4 European options are those which can only be exercised at expiry
production rates. In order to produce a model that was analytically tractable, they limit the mine to two operating states: closed (zero production) and open (a fixed production capacity) and also assume the mineral reserves to be infinite (Brennan & Schwartz, 1985). This model allowed the authors to investigate the option of temporary closure of the mine. Since this pioneering work’s publication, the concepts proposed in this paper have subsequently been extended to numerous mining applications.

Authors have since developed methods to address some of the shortcomings of Brennan and Schwartz’s model, including eliminating the need to make the (unrealistic) assumptions regarding an infinite resource and expand the types of options that are considered. Most authors have now abandoned attempts to develop analytical solutions and have instead, opted to make use of numerical methods for determining solutions to the otherwise intractable partial differential equations. Several different numerical methods have been employed including binomial lattices, finite difference and simulation.

With the exponential rise in computational power, simulation methods have become an increasingly popular choice for solving complex RO valuations. Sabour and Poulin make use of Longstaff and Schwartz’s Least Squared Monte Carlo (LSM) technique for solving real options problems to value a copper mine (Sabour & Poulin, 2006). Dimitrakopoulos and Sabour followed this example and have written extensively on using real options to optimize mine design to ensure a robust mine plan is selected (Dimitrakopoulos & Sabour, 2007), (Sabour, Dimitrakopoulos, & Kunak, 2008), (Dimitrakopoulos R., 2011). These simulation solutions are only capable of incorporating simple options with limited choices such as closure and re-opening.

5 One notable exception is (Shafiee, Topal, & Nehring, 2009) who extend Brennan and Schwartz and determine analytical solutions.
Other recent mining related examples include: (Samis & Poulin, 2001), (Trigeorgis, 2004), Dessureault et al. (2007), Dogbe et al. (2007), Guj and Garzon (2007), and Shafiee and Topal (2007). None of these citations however consider stochastic dynamic cut-off grade optimization but focus on such real optionality as capacity expansion, temporary shut-down or abandonment decisions.

Due to the sensitive nature of strategic decisions made by mining companies, it is difficult to assess the adoption of RO by the mining industry. Some real world applications of RO in mining are listed below. S. Kelly used a binomial lattice model to value the IPO of the Lihir gold project (Kelly, 1998). In 1999 Rio Tinto publicly stated it had been using Real Option models as a tool for project valuation for 10 years (Monkhouse, 1999). They first began by modeling the asset itself as the stochastic variable and later moved to modeling commodity price as the stochastic variable. In 2010 Rio Tinto retained Ernst and Young to conduct a RO valuation of the Oyu Tolgoi project in Mongolia, some of the results from this analysis were published in a publicly released AMEC technical report (AMEC, 2010) commissioned by Rio Tinto’s partner Ivanhoe Mines. Jane McCarthy and Peter Monkhouse published an article on behalf of BHP Billiton attempting to determine when to exercise the options to open and close a copper mine (McCarthy & Monkhouse, 2002).

2.3 Cut-Off Optimization – Stochastic Prices

As outlined previously in this chapter there exists a substantial body of theory on optimizing cut-off grade under known prices. There are two fundamental concerns with the assumption of known prices. Foremost is that future prices are highly uncertain. The decision to waste material when all future prices and consequences are known with absolute certainty, is not the same as the decision
when these prices are uncertain. When the decision is made to waste ore it is important to include
the extrinsic value derived from the material’s potential to become economic in the future, should
prices increase.

The second issue is that cut-off grade’s greatest contribution lies not in its ability to optimize the
value under a single assumed price path, but rather the flexibility it provides to adapt the mine
operation in the face of evolving market conditions. The ability to dynamically change the
definition of ore, in response to changing market conditions, is an extremely valuable tool for the
mining industry and optimization of cut-off grade is a problem of paramount importance.

One of the first papers to incorporate price uncertainty in cut-off grade optimization was
Krautkraemer (1988). In this ground-breaking paper, the mineral deposit was represented by a
cylinder with the highest grade at its axis decreasing outwards towards the cylinder’s
circumference. This simple geometry and grade distribution facilitated the development of an
analytical solution. Unfortunately the method could not be applied to the more complex
geometries and grade distributions found in actual mines.

Mardones took a different approach and attempted to extend Lane’s work into the contingent
claims framework (Mardones, 1993), however as Sagi points out Mardones’ treatment is
inconsistent as the cut-off decision is not optimized along with the objective function. Sagi in turn
proposed his own model which used a log-normal distribution to represent the deposit (Sagi,
2000). Although Sagi’s model is an improvement to Krautkraemer’s representation of a
geological deposit, his assumption of a single log-normal distribution is still too limiting to
accurately depict real mineral deposits. Cairns and Shinkuma investigated the problem of cut-off
grade under both deterministic prices and stochastic prices (Cairns & Van Quygen, 1998)
(Shinkuma, 2000). Although their work revealed some interesting insights, they fail to provide a method for determining the actual optimal cut-off grade strategy.

More recently Johnson et al. (2010) incorporated both stochastic price and a more detailed geological model into the cut-off grade optimization problem. In their model, the mine was divided into 60,000 individual blocks. The order of extraction of these blocks was assumed to be sequential, given a priori and the precise mineral content of each block was assumed to be known with certainty. Based on a no-arbitrage argument, a partial differential equation (PDE) was derived for the optimal value of the mine, from which the optimal strategy for processing the blocks could be determined. There are two main ways in which this description of the problem differs from reality. First, at any time there can be hundreds of exposed blocks that can be extracted in any order and multiple pieces of machinery can simultaneously extract multiple blocks. As such there are virtually an infinite number of possible extraction orderings. The valuation and optimal strategy produced by the Johnson et al. (2010) paper depends on the initial assumed block ordering, if this order is changed the value and the strategy are altered. Secondly, at this level of granularity, substantial uncertainty exists in the mineral content of each block as only a minuscule fraction of these blocks will have been sampled a priori. It is only once a given block has been exposed that its mineral content can be determined with sufficient certainty.

The decision to waste ore, when the precise location of all the highest quality portions of the deposit are known with certainty, is not the same as when they are not known with certainty. Moreover, any stochastic optimization algorithm that is provided with such detailed knowledge will inherently use this (unattainable) knowledge to produce a higher valuation. The paper also assumed that the mine did not have the option to shut-down production in the event that operation was no longer profitable. In such unprofitable scenarios, when presented with no other
alternatives, a cut-off grade optimization will always attempt to minimize the loss by increasing cut-off grade until nothing is processed and the entire deposit is wasted, mining production costs. Complete cut-off grade optimization requires that the very real closure optionality be incorporated into the analysis.

The cut-off grade optimization method proposed by Thompson and Barr considers both stochastic prices and a realistic representation of an open pit mine’s geology and engineering. The model they proposed divides the open pit mine into a series of phases, which must be mined sequentially. Each phase has its own grade distribution, which can be any arbitrary function. This description of a mine is consistent with current engineering practices and the required inputs can be easily generated from commercial mining software such as Surpac. Thompson and Barr then solve the resulting system of PDE’s using a finite difference method. This optimization method is described in detail in Chapter 3 and is demonstrated in Chapters 4 and 5.
Chapter 3

Description of Model and Solution Technique

3.1 A Summary of the Open Pit Mine Design Process

Prior to developing a mathematical representation of an open pit mine, it is important that the reader is familiar with the basic engineering and design of such an operation. For those readers who are familiar with these topics they are encouraged to skip this sub-section and move on to section 3.2 which describes the operational model.

Open pit mine design is a multistep process which involves the determination of a number of technical parameters and the design of a number of aspects of the operation. Complicating the process is the fact that many of the decisions from one step impact all subsequent steps and the entire process becomes circular. Figure 1 illustrates the various stages in open pit mine design and their interactions.

Figure 1: Steps of the traditional open pit mine design process (Dagdelen, 2001)
Due to the interdependence of the various stages of open pit mine design, the process is currently conducted by solving discrete sub-problems. A brief description of the current practice for solving each of these sub-problems follows.

The primary input into the entire design process is a geological model of the orebody (orebodies). Although these geological models can be created in variety of forms, by far the most common format is a three dimensional array of blocks, referred to as block model, see Figure 2 for an example. When building the block model, a resource geologist assigns properties to each block in the array, such as its metal grade(s), density and geological characteristics.

![Figure 2: A 3-D block model representation of a copper deposit (Dagdelen, 2001)](image)

Based on the size of the deposit, its grade distribution and other physical characteristics, mining engineers can estimate the scale of mining operation that is feasible and thus determine the size and number of the mining equipment and overall excavating capacities. With basic knowledge
regarding the mining fleet, the mining engineer can then determine rough operating costs per
tonne of material excavated. Similarly rough processing costs per tonne of ore can be determined.

Once operating costs have been determined and geotechnical characteristics have been
investigated, the engineering team can design the ultimate pit. The ultimate pit is the excavation
such that no combination of blocks could be subtracted or added to the outline such that the total
(undiscounted) value increases (Whittle, 1990). Research in determining the ultimate pit has been
extensive, notable papers include: (Lerchs & Grossman, 1965) , (Johnson & Barnes, 1988) ,
(Whittle, 1990). The description of the algorithms used to determine the ultimate pit is beyond the
scope of this document. It suffices to say that several commercial software packages, including
Whittle™, are accepted by industry to determine the ultimate pit based on all the required
technical parameters. For a more detailed discussion of ultimate pit algorithms readers are
directed to (Osanloo, Gholamnejad, & Karimi, 2008).

Once the ultimate pit has been outlined the design engineers begin to schedule the material within
this excavation limit. This scheduling is usually done in successively more detailed stages. The
first stage of scheduling is the design of pushbacks. The design of pushbacks is the development
of intermediate excavations which facilitate the realization, of the optimal production schedule
through physical design (Thompson J., 2010). Pushbacks are essentially a set of nested
excavation limits, set within the ultimate pit limit. Figure 3 illustrates a set of three pushbacks.
During operation the pushbacks are mined in sequence, with some overlap when material is
excavated from multiple pushbacks simultaneously. The design of pushbacks helps ensure
sufficient ore feed to the processing facilities at all points in the mine life while attempting to
bring forward cash flows as much as possible. Due to physical constraints, the pushbacks must be
mined in order, however there can be transitions in the scheduling, during which two or more adjacent pushbacks are mined simultaneously.

Figure 3: Typical cross section of internal pit phases or pushbacks (Bohnet, 1990)

Next a cut-off grade strategy can be selected. Current practice for choosing cut-off grade relies on either selecting the marginal grade under a long-term price forecast, or determining an optimized cut-off grade under an assumed long-term price forecast. The marginal cut-off grade is the grade required to cover all the costs associated with treating a quantity of material as ore, \( C_{\text{proc}} \).

Equation 3-1 illustrates a calculation for determining marginal cut-off grade, \( c_{\text{marginal}} \).

\[
c_{\text{marginal}} = \frac{C_{\text{proc}}}{\text{Price} \times \text{Recovery}}
\]

Assuming the long-term price forecast is correct and simply choosing the marginal cut-off grade is sub-optimal. Although choosing marginal cut-off grade ensures the maximum total profit under a given price forecast, it does not generate the maximum net present value. This was first discussed formally by Kenneth Lane (Lane K., 1964). He observed that by processing all material that produced even a miniscule profit, the processing of more profitable ore is deferred incurring a opportunity cost, caused by the time value of money. Lane further developed his
theories in his book *The Economic Definition of Ore* (Lane K., 1988) and provides detailed discussion on optimizing cut-off grade choice under known or deterministic prices.

After a cut-off strategy has been chosen, a more detailed schedule is created. This schedule further divides the pushbacks, into smaller sequenced excavations. The degree of granularity of this division depends on the nature of the schedule being produced\(^6\). Roughly speaking a long-term schedule may only divide the ultimate pit into pushbacks and stop there. A medium term schedule may further divide the pushbacks into major areas and/or benches. Finally a short-term schedule may further divide benches and areas into individual blasts. Sometimes a single schedule may contain varying degrees of detail. For example the first portion of schedule may be very detailed and defined by many individual blasts. Then after these initial blasts the remaining material in the ultimate pit is only scheduled by pushback.

Note in Figure 1 that the scheduling and cut-off grade steps are circular. An individual production schedule is paired with a given cut-off strategy. If the result is unsatisfactory either the production schedule or the cut-off strategy can be altered. This is done until the result is economically acceptable. The benefit of developing a method for optimizing cut-off grade, is that for a given mine design and schedule there is only a single optimal cutoff strategy. This means that mine planning engineers do not need to try different cut-off grade strategies as the optimal one can be computed directly.

\(^6\) The expressions long, medium and short term schedule are not rigorously defined and may have somewhat varying interpretations among mining engineers. The principal however of increasingly detailed schedules based on further sub-division of the ultimate pit is universally understood and practiced.
3.2 Operational Model

The objective of cut-off optimization is to optimize the total net present value (NPV) of a given mine design. The net present value of a discrete series of cash flows is given by equation 3-2. For a continuous cash flow series NPV is calculated using equation 3-3. What follows is the development of a generalized function for \( P(t) \) for an open pit mine. Some discussion is provided on how to modify the model for an underground mine later in this chapter.

\[
NPV = \sum_{t}^{T} \frac{P(t)}{(1 + \delta)^t} \tag{3-2}
\]

\[
NPV = \int_{t}^{T} P(t)e^{-\delta t}dt \tag{3-3}
\]

Where:

\( P = \) net cash flow

\( t = \) time of cash flow

\( \delta = \) discount rate

Mining operations generate revenue by excavating and processing material to create a salable product. Revenue is calculated as the product of recoverable ore grade\(^7\), tonnes of ore processed and the selling price of the product. During operation mines incur both fixed and variable costs. Variable costs can be further broken down into those associated with the excavating and treating waste and those associated with excavating and treating ore. Typical variable ore costs may include: drilling and blasting, hauling, crushing, flotation, tailings disposal among others. Variable waste costs may include drilling and blasting, dumping, remediation and others. Fixed costs include sustaining capital, general and administrative costs and others. Incorporating both

---

\(^7\) Recoverable ore grade considers both mining and processing recoveries, which may themselves be functions of the resource grade or other variables.
revenues and the aforementioned three types of operating costs, equation 3-4 is a representative formula for a mine’s before tax operating cash flow. Readers should note that \( dl \) is necessarily negative, as material can only be removed from the reserves and not added\(^8\). Non-operating costs such as capital costs and working capital can simply be discounted and subtracted from the discounted operating cash flows to determine NPV, as they are considered deterministic.

\[
P(t) = -1 \times \left( s(t) \times \bar{X}_{ore} \times Q_{ore} - C_{ore} \times Q_{ore} \right) \times dl - C_{fixed} \times dt \quad 3-4
\]

Where:

- \( s \) = Selling price
- \( \bar{X}_{ore} \) = Average ore grade
- \( Q_{ore} \) = Fraction of material treated as ore
- \( Q_{waste} \) = Fraction of material treated as waste
- \( C_{ore} \) = cost of treating material as ore
- \( C_{waste} \) = cost of treating material as waste
- \( I \) = Tonnes of resource remaining
- \( C_{fixed} \) = Fixed costs per unit time

Cut-off grade impacts three of the variables in equation 3-4, the quantity of ore, the quantity of waste and ore grade. Before further defining the role of cut-off grade the concept of phases is introduced.

As described in section 3.1, open pit mine plans divide an ultimate pit into a series of smaller excavations. The number and size of these excavations varies depending on the detail of the mine plan.

\(^8\) Although reserves are often revised upwards throughout a mine life this requires the definition of an entirely new mine plan/
plan. We define a phase as a sequenced excavation within the ultimate pit. By defining a phase in this manner, we can introduce the function \( j(I) \) where \( j \) is the current phase dictated by \( I \), the reserves remaining. When demarking the phases, designers must ensure that at all points in the mine plan, only one phase is being excavated. This means that pushbacks may not be an appropriate phase definition, as there may be points in the schedule when multiple pushbacks are mined simultaneously. To account for this, a transition phase should be defined that is comprised of the appropriate material from both pushbacks. Similarly individual blasts are generally not good candidates for phases, as often multiple blasts are mined simultaneously. In this case, sets of blasts that will be exploited simultaneously, should be aggregated into phases.

Consider an open pit mine design comprised of \( n \) phases. Each phase has a tonnage \( Q_j \) with any arbitrary distribution \( f_{ij}(x) \), of material of grade \( x \) within the phase. \( F_{ij}(x) \) can be determined by first aggregating all of the blocks in the phase and creating a histogram, which represents a discrete approximation of \( f_{ij}(x) \). Figure 4 is an example of one such histogram adapted from (Hustruulid & Kuchta, 2006).

![Tonnage histogram of a copper deposit from (Hustruulid & Kuchta, 2006).](image)

**Figure 4:** Tonnage histogram of a copper deposit from (Hustruulid & Kuchta, 2006).
Understanding how the function $f_{j(I)}(x)$ is determined from a geological model and corresponding mine plan, we continue with the mathematical model. Equation 3-5 is the total tonnage of the phase.

$$Q = \int_{0}^{\infty} f_{j(I)}(x) dx \quad 3-5$$

Following the above, equation 3-6 represents the total contained metal in the phase

$$M = \int_{0}^{\infty} x f_{j(I)}(x) dx \quad 3-6$$

Finally dividing contained metal, equation 3-5, by total tonnage, equation 3-6, we get the average grade of the phase, equation 3-7.

$$\bar{x} = \frac{M}{Q} = \frac{\int_{0}^{\infty} x f_{j(I)}(x) dx}{\int_{0}^{\infty} f_{j(I)}(x) dx} \quad 3-7$$

Once a cut-off grade $c$ is selected, only material with a grade equal to or above this criterion is treated as ore. Equation 3-8 is the fraction of the deposit considered ore, while equation 3-9 is the average grade of the ore material.

$$Q_o(c) = \int_{c}^{\infty} f_{j(I)}(x) dx \quad 3-8$$

$$\bar{x}_{ore}(c) = \frac{\int_{c}^{\infty} x f_{j(I)}(x) dx}{\int_{c}^{\infty} f_{j(I)}(x) dx} \quad 3-9$$

Plotting $Q_o(c)$ and $\bar{x}_{ore}(c)$ for a range of cut-offs is referred to as a grade-tonnage curve, see Figure 5 for an example.
For an open pit mine, it follows that all material within the excavation not classified as ore is classified as waste. As such the tonnage of waste is determined using equation 3-10.

\[ Q_w(g) = \int_0^c f_{j(t)}(x) \, dx = 1 - Q_o(c) \tag{3-10} \]

The stripping ratio is defined as the ratio of waste material to ore material and is expressed by equation 3-11.

\[ Sr(g) = \frac{Q_w(c)}{Q_o(c)} = \frac{\int_0^c f_{j(t)}(x) \, dx}{\int_c^\infty f_{j(t)}(x) \, dx} \tag{3-11} \]
To assist readers visualize the functions described so far, Equations 3-8 and 3-10 are represented graphically in Figure 6

Figure 6: An explanatory illustration of the material distribution function \( f(x) \)

Mine operations are subject to two primary physical production constraints; the mining limit and the processing limit. The mining limit is the total amount of material which can be excavated from the deposit in year. The processing limit is the maximum amount of material that can be treated as ore in a year. If \( Ex \) is the total tonnage excavated in a year, these constraints are defined using equations 3-12 and 3-13 respectively.

\[
\left( \frac{Q_o + Q_w}{Q} \right) \times Ex \leq K_{mine} \tag{3-12}
\]

\[
\left( \frac{Q_o}{Q} \right) \times Ex \leq K_{proc} \tag{3-13}
\]

Mines will remove material until either the mining constraint or the process constraint is reached, thus the extraction rate, \( Ex \), is defined by equation 3-14. In practice mines are almost always operated so that the processing constraint is binding, fully utilizing the production capacity.
To understand equation 3.14, suppose the stripping ratio, \( Sr \), was 3. Three units of waste are extracted for every one unit of ore so the maximum rate that we could extract material from the deposit before hitting the processing constraint would be \( 4K_{proc} \). If this value was greater than the mining constraint \( K_{mine} \) then \( K_{mine} \) would determine the extraction rate. The rate of change of \( I \), the tonnes of resource remaining, is therefore given by equation 3.15.

\[
dI = -Ex(I, c)\, dt
\]  

Once the extraction rate, ore grade, and stripping ratio have been determined the rate of metal production can be calculated using equation 3.16, which is simply the product of annual tonnes of ore processed and the average recoverable grade of this ore:

\[
M(I, c) = Ex(I, c)Q_{ore}\bar{x}_{ore} = \frac{Ex(I, c)}{1 + Sr(I, c)} \times \frac{\int_{x}^{\infty}x f_{j}(x)\, dx}{\int_{x}^{\infty} f_{j}(x)\, dx} \text{ tonnes/year}
\]  

Finally substituting equations 3.16 and 3.15 into 3.4 we can derive the formula for the mine’s instantaneous cash flow:

\[
P(t) = \left( \frac{s(t) \times \bar{x}(I, c)}{1 + Sr} - \frac{C_{ore}}{1 + Sr} - C_{waste} \right) \times Ex(I, c) - C_{fixed} \times dt
\]  

In the following section we discuss the functional form of the stochastic price model, \( s(t) \).
3.3 Modeling Price Dynamics:

In economic terms, many components of the global commodities industry can be considered a near perfect competitive industry. Producers sell a indistinguishable product into open markets, where pricing information is readily available. Due to this, most mine operators are price takers and must accept prices determined by international commodity markets. The complexity of these markets and the global economic considerations that impact them, make it extremely difficult to accurately predict future price trends. Assuming any predictions are certain will produce suboptimal selection of cut-off grade and poor estimates of value. What’s more, for many commodities there exist large markets for futures and options. These derivative markets provide a wealth of information regarding the market’s price expectations and the degree of expected randomness or volatility. The efficient market hypothesis, a cornerstone of financial theory, requires that assets are priced in a manner that does not create any arbitrage opportunities, for these derivatives.

In order to model the uncertain behavior of commodity prices, a stochastic differential equation is used. Stochastic models are a collection of functions that involve a random or uncertain component. This approach is consistent with current practices used in the pricing of financial derivatives, which are valued based on the random behavior of underlying commodity prices. Several stochastic equations exist that describe different types of probabilistic behavior. Among the most widely used stochastic models for financial applications are Geometric Brownian Motion (GBM) and Mean Reversion.
The stochastic model used, begins with an expectations curve \( F(0, T) \). This function represents the initial best estimate as to the future price of the commodity. The expectations curve can be derived from a proprietary commodity forecast or, if risk-neutral valuation is being conducted, \( F(0, T) \) could be taken directly from a market observed futures curve. The futures price of the commodity follows the stochastic function 3-18:

\[
\frac{dF}{F(t, T)} = \sigma_t e^{\eta(T-t)} dX_t
\]

This function may be unfamiliar to those accustomed to the stochastic spot price models used in (Brennan & Schwartz, 1985), (Samis & Poulin, 2001) and elsewhere. However, the above stochastic futures price is in fact equivalent to the traditional spot price models. In Appendix C it is shown that the stochastic spot price model derived from equation 3-18 is given by equation 3-19 when mean reversion is present. This can be further simplified to equation 3-20, when reversion is not present.

\[
\frac{dS}{S} = \left[ y(t) + \eta \left( \ln(F(0, t)) - \ln(S) \right) \right] dt + \sigma_t dX
\]

\[
\frac{dS}{S} = y(t) dt + \sigma_t dX
\]

Equation 3-20 matches exactly Geometric Brownian Motion (GBM) spot price models, used in (Brennan & Schwartz, 1985). The advantage in starting with a full stochastic futures curve is that \( y(t) \) can take any form. This means that any initial expectations curve can be used, allowing analysts to directly input either proprietary forecasts or a market derived futures curve. By contrast, in (Brennan & Schwartz, 1985) a simple fixed yield curve is used out of necessity.
A spot price following GBM, is described by equation 3-20, where $S$ is the price of the commodity\(^9\) being modeled, $y(t)$ is the price drift, $\sigma$ is the volatility and $dX$ is an increment of a Wiener process. To better understand equation 3-20 the terms of the equation will be discussed individually.

The left-hand side of the equation, $\frac{dS}{S}$, is the instantaneous percentage change in the commodity price or its return. The first term on the right hand side is the deterministic component of the change in commodity price. For example a drift ($y(T)$) of 0.1 would indicate that the expected value of the commodity price is to rise by 10% annually. The second term on the right hand side represents the random or stochastic component of price change. The Wiener process essentially generates a random value drawn from a normal distribution with mean zero and standard deviation of $\sqrt{\Delta t}$. This random value is then scaled by the volatility, $\sigma$. The larger the value of volatility, the greater the impact is of randomness on the evolution of price. To assist readers in visualizing equation 3-20, Figure 7 illustrates three simulations of gold price modeled using a GBM model. Note how the simulations tend to drift upwards due to the positive value of $\mu$, however they fluctuate up and down due to the impact of the stochastic term, $\sigma dX$.

---

\(^9\) Stochastic models are used to model a variety of asset classes including commodities, equity securities and interest rates. As the model presented in this section is concerned with mining projects the underlying asset being modeled is assumed to be a commodity.
Mean reversion is described by equation 3-19 and includes the parameter, \( F(0, t) \), to the GBM model and replaces the drift parameter with \( \eta \), the speed of reversion. Remembering that \( F(0, t) \) is the initial value of the expectations curve at the current time or in other terms the initial estimate of the long term equilibrium price of the commodity price. Note that when the current spot price is above the equilibrium price, \( S > F(0, t) \), the deterministic term is negative. This will cause the price to drift, down towards the equilibrium price. The larger the difference between the spot price and equilibrium price the “stronger the pull” is back toward the equilibrium. When spot price is below the equilibrium price, \( S < F(0, t) \), then the deterministic component is positive and the price will tend to drift up, towards the equilibrium price.

Figure 7 illustrates three simulations of gold price modeled using a mean reverting model. In contrast to Figure 8, note how the mean reverting model causes all the simulations to fluctuate around the equilibrium price of $1,100.
Figure 8: A plot illustrating three simulations of gold price modeled using a mean reverting model where $\eta = 1$, the initial expectations curve $F$ is a flat $\$1,100$ and $\sigma = 0.25$.

For a more thorough discussion on stochastic processes readers are directed to (Wilmott, 2007) or for a more complete reference (Bass, 2011).

3.3.1 Selecting a Price Model

By observing market data, the type of price model to use and the necessary parameters can be determined. The relative movements of the front and back end of the futures curve can determine if the price model requires reversion. If the market is pricing commodity futures that incorporate a degree of mean reversion then it would be expected that the volatility in the front end (near term) of the futures curve would be greater than that of the backend (long term). Conversely, if movements in the front end of the curve have the same volatility as of the backend, then the market is not pricing in any reversion. To put it another way, if changes in short-term expectations are accompanied by equal changes in long term expectations then no-reversion is being considered. Table 1 provides the standard deviations of the prompt and 27 month futures contract for several commodities. The standard deviations of the prompt price and long term
futures price are extremely similar for the four base metal commodities, suggesting the market is not pricing in any reversion, however the same could not be said for WTI crude. The high correlations of the near and long term contracts indicate that a single factor model is sufficient to capture the observable behavior of the futures curves of the base metals.

Table 1: Historically estimated volatilities of the spot and futures price 27 months forward and the correlation of these two values

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Prompt Std. Dev.</th>
<th>27 Month Futures Std. Dev.</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>0.297</td>
<td>0.268</td>
<td>0.97</td>
</tr>
<tr>
<td>Zinc</td>
<td>0.284</td>
<td>0.285</td>
<td>0.95</td>
</tr>
<tr>
<td>Lead</td>
<td>0.394</td>
<td>0.397</td>
<td>0.96</td>
</tr>
<tr>
<td>Nickel</td>
<td>0.345</td>
<td>0.312</td>
<td>0.97</td>
</tr>
<tr>
<td>WTI Crude</td>
<td>0.344</td>
<td>0.268</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Going forward, it is assumed that the commodity price follows a stochastic model which does not contain reversion. Interested readers could use the mean reverting model defined by equation 3-19, and derive the corresponding partial differential equations if a mean reverting price model is required.

3.3.2 Determining Price Model Parameter Values

As shown in Appendix C, given an initial expectations curve, the stochastic process for spot price, when there is no reversion present, is equation 3-20. Equation 3-20 contains two components; a deterministic component $y(t)dt$ and a stochastic component $\sigma dX$. The deterministic component is controlled by the drift or yield $y(t)$ and the stochastic component is controlled by the volatility. There are a number of ways to estimate these parameters. However, if the evaluator is concerned with producing a true risk-neutral valuation, then these parameters must be taken from the pricing of market observed instruments where available. For further discussion on the reasons for using market derived parameter values see Appendix D.
The deterministic component represents the behavior of price irrespective of random fluctuations. If there was no randomness in price, that is volatility ($\sigma$) equaled zero, than the initial expectations curve would be 100% accurate and spot price would follow this curve. Using the definition of $y(t)$ from Appendix C and setting $\sigma = 0$, we find that the drift is simply the first derivative of the expectations curve with respect to time. Since the $y(t)$ is completely defined by the initial expectations curve and volatility no estimates of drift or yield are required for the deterministic term behavior of price.

Most mined commodities have active futures markets. For instance the NYMEX set a record for outstanding metals futures contracts on March 20$^{th}$ 2008. At the time there were a total of 293,615 gold contract outstanding, each for 100 troy ounces (Holdings, 2008). This represented a total of over 29 million ounces under contract or more than 40% of the previous year’s global production (USGS, 2009). Markets such as the NYMEX offer a price today to transact on the future delivery of a commodity.

The stochastic component of the price model is controlled by volatility. Volatility can be estimated a number of ways. Historical price data could be used to determine historical trends in volatility which could be applied forward. Proprietary estimates of volatility could be derived from confidence intervals on future price. However, just as with the expectations curve, the current best estimate of future price volatility can also be derived from market traded financial instruments.

The market’s view on volatility $\sigma$, can be inferred from the price of options on commodity futures. Most types of simple financial options can be priced using either analytical methods or
numerical methods. For instance European puts and calls can be priced using the Black Scholes
equation. By substituting in the market price of these derivatives into the appropriate formula one
can solve for the implied volatility, as all other parameter values are known. This implied
volatility is sometimes referred to as the Black’s volatility.

3.3.3 Combining the Price Model with the Operating Model

Given the initial expectations curve or futures curve $F(0,T)$ we know that at any time $t$:

$$F(t, T) = F(0, T)e^{\left(\sigma X(t) - \sigma(t)^2 t\right)}$$  (3-21)

For a detailed proof of the above see Appendix B. Since the spot price $S(t)$ at time $t$ is always $F(t, T)$ we also know:

$$s(t) = F(0, t)e^{\left(\sigma X(t) - \sigma(t)^2 t\right)}$$  (3-22)

Rather than deriving a stochastic differential equation for $S$, it is easier to use $X$ as the random
variable and use equation 3-22 to convert from one to another. Substituting equation 3-22 into the
mine cash flow equation 3-17 the equation for cash flow under stochastic prices 3-23 is derived.

$$P(X, I, t; c) = \left(\frac{F(0, t)e^{(\sigma X(t) - \sigma(t)^2 t)} \times \bar{x}(I, c)}{1 + S_r} - \frac{C_{ore}}{1 + S_r} - C_{waste}\right) \times Ex(I, c)$$

$$- C_{fixed} \times dt$$  (3-23)

3.4 Derivation of the Partial Differential Equations:

Having developed expressions for the dynamic process which govern variables $I$ and $X$, it is
possible to derive equations for the optimal cut-off grade strategy and the corresponding value.

We begin by defining the optimal cut-off grade strategy $c(x, I, t)$ and the corresponding optimal
value $V(X, I, t, c)$
\[ V(X, I, t, c) = \max_c E \left[ \int_t^T e^{-r(t-s)} P(X, I, t) \, ds \right] \] 3-24

This equation can be rewritten as:

\[
V(X, I, t, c) = \max_c E \left[ \int_t^T e^{-r(t-s)} P(X, I, t) \, ds + \int_t^T e^{-r(t-s)} P(X, I, t) \, ds \right]
\]

\[
V(X, I, t, c) = \max E \left[ \int_t^T e^{-r(t-s)} P(X, I, t) \, ds + e^{-r(t-s)} V(X', I', t') \right]
\]

Where \( X' \) and \( I' \) are the (unknown) values of \( X \) and \( I \) at time \( t' > t \). We let \( t' \) equal a small increment greater than \( t \) (i.e. \( t' = t + dt \)). This allows for the substitution of the second term in the square brackets with \( dV/dt \). We can now apply Ito’s Lemma which states: if \( f \) is a function of an Ito drift-diffusion process:

\[ dS_t = \mu_t \, dt + \sigma_t \, dX_t \] 3-25

Then:

\[ df(t, S_t) = \left( \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial s^2} \right) dt + \sigma_t \frac{\partial f}{\partial s} dX_t \] 3-26

For a more complete reference on Ito’s Lemma readers are directed to (Neftci, 2008). Note that by choosing \( \mu = 0 \) equation 3-25 matches the price model described by 3-20. By expanding the right hand side of equation 3-24 in a Taylor’s series substituting equation 3-26 we find that:

\[
V = \max_c E \left[ P(X, I, t; c) \, dt + (1 - rt) V \right.
\]

\[ + (1 - rt) \left( V_t + \frac{1}{2} V_{xx} - Ex(I, c) V_t \right) \, dt \]

\[ + (1 - dt) V_X dX \] 3-27

Eliminating all terms that go to zero faster than \( dt \) and simplifying shows that:

\[ 0 = \max_c E \left[ \left( V_t + \frac{1}{2} V_{xx} - Ex(I, c) V_t - rV + P(X, I, t; c) \right) \, dt + V_X dX \right] \]
Taking expectations and dividing through by $dt$ gives

$$\max_c E \left[ \left( V_t + \frac{1}{2} V_{xx} - Ex(I, c)V_t - rV + P(X, I, t; c) \right) \right] = 0$$

Only two terms in the above equation involve $c$, so the optimal value for $c$ maximizes:

$$\max_c E \left[ \left( -Ex(I, c)V_t + P(X, I, t; c) \right) \right] = 0 \quad 3-28$$

This implies that when $c(X,I,t)$ is chosen to maximize equation 3-28 then:

$$V_t + \frac{1}{2} V_{xx} - Ex(I, c)V_t - rV + P(X, T, t; c) = 0 \quad 3-29$$

We now have two conditions which will allow us to simultaneously determine the expected discounted cash flows $V(X,I,t)$ and the optimal strategy $c(X,I,t)$. It remains only to define boundary conditions.

The terminal condition at time $t = T$ is given by

$$V(X, I, T; c) = 0 \quad 3-30$$

The boundary condition for $I=0$, i.e., once the resource has been exhausted, the value is zero

$$V(X, 0, t; c) = 0 \quad 3-31$$

We now need to define an additional PDE and corresponding boundary conditions in order to consider the option for temporary shut-down. Let $W(X,I,t)$ be the value of the mine given that it is temporarily shut down. Let $C_{\text{offline}}$ be the total fixed costs associated with maintaining the mine in this state. Let $C_{\text{turnoff}}$ be the cost of transitioning into the shutdown state and let $C_{\text{turnon}}$ be the cost of transitioning from the shutdown state to the on-line state. By repeating the steps for the derivation of $V$, we find that since $Ex \equiv 0$ when off-line $W$ must obey the PDE given by equation 3-32.
\[ W_t + \frac{1}{2} W_{XX} - rW - C_{offline} = 0 \]  \hspace{1cm} 3-32

With the corresponding boundary conditions

\[ W(X, I, T) = 0 \quad 3-33 \]

\[ W(X, 0, t) = 0 \quad 3-34 \]

The values of \( W \) and \( V \) are linked to one another. Given the options to temporarily shut down and to reopen by paying the appropriate transition fee, the conditions to maintain optimality are given by:

\[ W(X, I, t) \geq V(X, I, t) - C_{turnon} \quad 3-35 \]

\[ V(X, I, t) \geq W(X, I, t) - C_{turnoff} \quad 3-36 \]

Equation 3-35 is an upper free boundary condition on \( W \) with respect to \( X \) while equation 3-36 provides a lower free boundary condition on \( W \) with respect to \( X \). One final boundary condition is required for the PDEs that govern \( V \) and \( W \). As \( X \) approaches \( \infty \) so too does spot price and the likelihood of the temporary closure option being exercised declines towards zero. Additionally all mineralized material is considered ore when \( X \) approaches \( \infty \) and the cut-off strategy becomes constant. In these regions \( V \) is therefore linear in \( S \) (log-linear in \( X \)). As numerical solution is used to solve the system of PDEs, it is impossible to solve for values \( \pm \infty \) so a far field boundary condition must be applied.
3.5 Implementation of the Numerical Solution:

What follows is a description of the standard, finite difference solution to the mining valuation system of PDEs outlined above. This is similar to the method successfully used in (Thompson, Davidson, & Ramussen, Real options valuation and optimal operation of electrical power plants in competitive markets., 2004) for power plant valuation and was described in (Thompson, Davidson, & Rausmussen, Natural gas storage valuation and optimization: A real options application, 2009) for use in gas storage valuation.

The solution begins with a pair of grids of values. Let \( V_{i,j}^k \) and \( W_{i,j}^k \) correspond to the value \( V \) and \( W \) at the ith value of \( X \) and the jth value of \( I \) at time step k. The first step is to calculate the partial derivatives with respect to \( X \) and \( I \) at each grid point. The second partial derivative with respect to \( X \) at an interior grid point is calculated using a second-order central difference approximation:

\[
\frac{\partial^2 V_{i,j}^k}{\partial X^2} = \frac{V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k}{(\partial X)^2}
\]

where \( \partial X \) is the distance between neighboring grid points in the \( X \) axis. Clearly an analogous equation can be used to determine \( \frac{\partial^2 W_{i,j}^k}{\partial X^2} \). For the derivatives at the boundary, where either \( V_{i+1,j}^k \) or \( V_{i-1,j}^k \) do not exist, a value is extrapolated using a log-linear extrapolation shown in equation 3-38.

\[
V_{i+1,j}^k = \frac{V_{i,j}^k - V_{i-1,j}^k}{e^{x_i} - e^{x_{i-1}}} \times (e^{x_{i+1}} - e^{x_i}) + V_{i,j}^k
\]

Determining the partial derivative with respect to \( I \) in equation 3-37 is not as straightforward. As equation 3-29 only depends on the first derivative with respect to \( I \), it behaves as a hyperbolic equation in the \( I \) dimension. This creates issues of tractability. One approach to overcoming these concerns is to use a first order upwind differencing scheme. Essentially a second-order accurate
scheme is used everywhere except regions where the slope is changing rapidly. Where this occurs a small amount of artificial diffusion is introduced to prevent instability. For a complete discussion on these topics readers are referred to (LeVeque, 1992).

The minmod slope limiter finite difference formula is given by:

\[
\frac{\partial V_{i,j}^k}{\partial I} = \frac{-(V_{i,j-1}^k - V_{i,j}^k)}{\delta I} + \frac{\varphi^+}{2\delta I} (V_{i,j}^k - V_{i,j-1}^k) - \frac{\varphi^-}{2\delta I} (V_{i,j+1}^k - V_{i,j}^k)
\]  

3-39

Where

\[
\varphi^+ = \max \left(0, \min \left(1, \frac{V_{i,j-1}^k - V_{i,j}^k}{V_{i,j+1}^k - V_{i,j}^k}\right)\right)
\]  

3-40

and

\[
\varphi^- = \max \left(0, \min \left(1, \frac{V_{i,j}^k - V_{i,j-1}^k}{V_{i,j+1}^k - V_{i,j}^k}\right)\right)
\]  

3-41

In the explicit finite difference case the time derivative can be approximated by a one-sided finite difference approximation given by:

\[
\frac{\partial V_{i,j}^k}{\partial t} = \frac{V_{i,j}^{k+1} - V_{i,j}^k}{\delta t}
\]  

3-42

It should be noted that the condition for stability is that \(\delta t \leq \delta X\).

Given the initial conditions defined earlier, \(V_{i,j}^0 = 0\) and \(W_{i,j}^0 = 0\) for all \(i\) and \(j\). Given knowledge of \(V\) and \(W\) for all values of \(i\) and \(j\), the partial derivatives with respect to \(I\) and \(X\) can be computed for every grid point for time step \(t = 0\). Using the now determined values of the derivatives, it is possible to determine \(c_{i,j}^0\) by maximizing equation 3-28. We can then substitute \(c\) along with the derivative estimates from equations 3-37 and 3-39 into equation 3-29 to solve for the partial derivative of value with respect to time, \(V_t\). Finally by using a one-sided finite difference approximation for \(V_t\) outlined in equation 3-42 we can solve for \(V_{i,j}^{k+1}\). Similarly we can
determine $W_{i,j}^{k+1}$ by first substituting the value of $W_{x}$ into 3-32 to solve for $W$, and again using a one-sided finite difference approximation for $W$, and solve for $W_{i,j}^{k+1}$.

To apply the boundary conditions from equations 3-35 and 3-36 at time step $k+1$, we test to see if $V_{i,j}^{k+1} < W_{i,j}^{k+1} - C_{turnoff}$ and if it is we set $V_{i,j}^{k+1} = W_{i,j}^{k+1} - C_{turnoff}$. Likewise we test $W_{i,j}^{k+1} < V_{i,j}^{k+1} - C_{turnon}$ and if it is we set $W_{i,j}^{k+1} = V_{i,j}^{k+1} - C_{turnon}$. For a more complete reference on using this approach to solving free boundary problems in this manner see (Tavella & Randall, 2000).

Once the values of $V_{i,j}^{k+1}$ and $W_{i,j}^{k+1}$ have been determined for all $i$ and $j$ then the entire process can be repeated at the time step $t = k + 1$. The process continues to work backwards in a recursive manner until the values of $V_{i,j}^{0}$ and $W_{i,j}^{0}$ are determined for all $i$ and $j$. The value of $V_{i,j}^{0}$ is the real options value of the mining asset, where $\bar{i}$ represents the value of $i$ where $X = 0$ and $\bar{j}$ represents the value of $j$ where $i = I$.

To assist readers in following the process described in this section, Figure 9 is a flowchart illustrating the connection between the steps above.
Figure 9: Flow chart outlining steps in the numerical solution

1. Set time step, $k = T$
2. Set $V_{i,j}^0$ and $W_{i,j}^0 = 0$ for all values of $i$ and $j$.
3. Calculate $\frac{\partial^2 V_{i,j}^k}{\partial x^2}$ for all values of $i$ and $j$.
4. Calculate $\frac{\partial V_{i,j}^k}{\partial t}$ for all values of $i$ and $j$.
5. Determine optimal cut-off, $c_{i,j}$ for all $i$ and $j$.
6. Calculate $V_{i,j}^{k+1}$ for all values of $i$ and $j$ using:
   - If $W_{i,j}^{k+1} - C_{\text{turnoff}} \geq V_{i,j}^{k+1}$ set $V_{i,j}^{k+1} = W_{i,j}^{k+1} - C_{\text{turnoff}}$
   - If $V_{i,j}^{k+1} - C_{\text{turnon}} \geq W_{i,j}^{k+1}$ set $W_{i,j}^{k+1} = V_{i,j}^{k+1} - C_{\text{turnon}}$
7. Set time step, $k = k - 1$
8. Repeat until $k = 0$.
3.6 Potential modifications to the operational model

The operational model outlined in section 3.2 is meant to represent a generic, single commodity, open pit mining operation. Several modifications can be made to the model in order to cover specific case requirements. The following section provides a brief discussion on the types of modifications that can be made to the model.

Many mines incur selling costs associated with the sale of their product. For instance mines that produce a concentrate that requires further processing, will not receive the full value of the contained commodity. In other cases royalties and mining taxes are levied directly off the gross revenue from commodity sale or insurance and brokerage fees are paid on the gross value. There are several methods for incorporating these costs into the analysis, the simplest of which is to modify equation 3-17. Equation 3-43 incorporates one such modification and includes per unit of product charges and revenue based royalties. The modifications in equation 3-43 would need to carried through the system of PDE’s defined in section 3.4.

\[
P = \left( \frac{(s(t) - \text{selling costs}) \times \bar{x}(l,g)}{1 + Sr} \right) \left( 1 - \text{royalty} \right) - \frac{C_{\text{core}}}{1 + Sr} \\
- C_{\text{waste}} \times Ex(l,g) - C_{\text{fixed}} \right) \times dt
\]

3-43

It is common for mines to produce more than one salable commodity. For instance many base metal mines produce multiple base metal concentrates. In order to incorporate the revenue generated from the multiple products equation 3-17 needs to be modified to:
\[ P = \left( \sum_{p=0}^{P} s_p(t) \times \bar{x}_p(l, g_0, g_1, \ldots, g_P) \right) \times \frac{C_\text{ore}}{1 + Sr} - \frac{C_\text{waste}}{1 + Sr} \times Ex(l, g_0, g_1, \ldots, g_P) - C_{\text{fixed}} \times dt \]

where \( P \) represents the number of salable products and \( p \) represents each specific product. If the prices of the salable commodities are not perfectly correlated, then separate stochastic models are required for each commodity price. Introducing multiple stochastic models requires the system of PDEs in section 3.4 be adjusted to reflect the additional partial derivatives. Readers should also note with multiple salable constituents in the resource, multiple cut-offs need to be determined. From a practicality standpoint, no-more than two commodities can be modeled otherwise the problem becomes intractable.

There are some significant differences in how an underground mine operates as compared to an open pit mine. If valuing an underground mine, some fundamental changes are required to be made to the operational model in section 3.2.

In an underground mine the concept of cut-off grade is somewhat different from the definition applied in the open-pit case. Again the cut-off grade represents the minimum grade at which material is considered ore, however the total tonnage excavated \( Q \) is itself a function of cut-off grade. This is due to the fundamental physical differences of underground mining as compared to open-pit mining. Many underground mining techniques are selective, meaning that there is a high degree of control as to what material is excavated. The functional forms of \( Q \), \( Q_o \) and \( Q_w \) would need to be defined for the specific underground mine. For instance they could be discrete functions defined by a series of alternative stope designs. Also underground development costs would need to be considered.
Another adjustment to the model would be the inclusion of mean-reversion. This of course would alter the stochastic functions 3-20,3-21 and 3-22 used to model the evolution of price, as well as the subsequent system of PDEs.

### 3.7 Model Assumptions and Limitations

As with all models of complex real world systems, the one described above makes some assumptions about the behavior of the system it describes. What follows is a discussion of these assumptions and the corresponding limitations.

The model assumed the only options which management can exercise are the ability to alter cut-off grade and to temporarily close and re-open the mine. In reality there are number of other options that may be available to the mine management. These include production capacity expansion, altering the mine plan and possibly changes to the downstream mineral process. All of these options are major changes to the initial design of the mining operation and are generally considered outside the scope of the valuation of a single mine design. In a traditional, deterministic mine valuation the only way to consider any of the above modifications to the initial plan is to conduct a separate analysis. Of course this approach could be taken in a real options analysis by simply conducting a separate real options analysis. It should also be noted that if one needs to include a planned mine expansion or other such deterministic change, this can be included in the real options analysis by making the appropriate parameter, such as processing or mining limit, a function of time.

The model assumes there is no delay in the time when the decision to exercise an option is made and its execution. This is realistic for the option to alter cut-off as in practice this simply amounts
to informing ore-control geologists and possibly process plant managers about the change. In some cases changes in cut-off may require alterations to the process plant operation. Many of these concerns can however be captured by making the adjusting recoverable grade and making processing cost a function of cut-off grade. As for the option to shut-down and restart the mine there is likely some delay between the decision and the execution. This is not a great concern on the closure side as the mining fleet and processing plant can likely stop operating on a relatively short notice. The expenditure of the closure funds may take some to complete, however as long as it is assumed those costs are irrecoverable, then the model’s assumptions are not a concern. However when management decides to re-open a mine there is likely an appreciable delay before the operation is once again producing a salable product. By ignoring the delay the real options analysis is somewhat overvaluing the operation. To determine the magnitude of this overestimation a Monte Carlo simulation could be used.

The model assumes that management will always exercise the available options using the optimal strategy. The model computes the optimal strategy for cut-off grade and mine closure/reopening based on the maximization of expected value. There are many reasons why management would not follow the optimal strategy. For example they may be unwilling to close the mine as that may be viewed as a sign of failure. Management may also be reluctant to use the optimal cut-off grade for fear of “wasting material” or for short-term gains by high-grading. If management does not follow the optimal strategy then the expected value will be less.
Chapter 4

Worked Example – Detour Lake

The following section contains a worked example of the model described in Chapter 3, valuing the Detour Lake Gold property. The model optimizes the cut-off grade under stochastic or uncertain future prices. The analysis considers an infinite number of price scenarios and inserts a decision making process that can both alter cut-off grade and temporarily close\reopen the mine in order to optimize net present value. Applying this optimal cut-off grade strategy, determines the highest possible expected net present value for the project. All of the data used is public information and is provided to allow the reader to replicate the results.

4.1 Project Description

As of writing, the Detour Lake property is 100% owned\(^{10}\) by Detour Gold Corp. and is being developed into an open-pit gold mine. The Detour Lake property is located in northeastern Ontario, approximately 185 kilometers by road from Cochrane. The Property is located in the Abitibi Greenstone Belt in the Superior Province of the Canadian Shield. Two types of gold mineralization have been identified at the site; sulphide poor quartz stockworks and a lower sulphide hangingwall mineralization.

Exploration began at the site in 1974 when Amoco Canada Petroleum Company first identified a 2 km long anomaly. This was followed up with extensive drilling the following year. In 1979 Campbell Red Lake Mines took over the property and by 1982 the decision to commence open pit mining was made. In 1987, coinciding with the merger between Campbell and Dome Mines, forming Placer Dome, an underground mine was opened on the property. From 1987 to 1999, ...

\(^{10}\) The property which contains the reserves used in the following analysis are 100% owned by Detour however surrounding exploration properties are subject to a variety of agreements with other interests.
when Placer Dome had closed the operation, approximately 1.8 million ounces of gold were recovered from the property. In 2007 Detour acquired the project for $75M.

A resource model was built by BBA Engineering, a mining consulting firm, using ordinary krigging to populate a block model. Under the base case assumptions, the open pit would operate at a cut-off grade of 0.5 g/t resulting in 479M tonnes of reserves, highlighted in Table 2. The mine life was estimated to be approximately 21 years with an average production of 675 koz per year.

The primary mining fleet to be used consists of 46 ultra-class haul trucks, 2 electric shovels, 3 electric hydraulic shovels and six blasthole drills, giving the operation a combined annual mining rate of 120 Mt per annum. The plant design is standard Carbon in Pulp (CIP) flow sheet. Plant capacity is estimated at 55,000 tpd or 20.075Mt per year at an overall recovery of 91% with an expansion to 61,000 tpd or 22.26 Mt to be completed by the beginning of 2015.

4.2 Data

The majority of the necessary data was taken directly from the March 2011 Detour Lake - Mineral Resource and Mineral Reserve Update, prepared by BBA Engineering (BBA, 2011). Some parameters had to be estimated or extrapolated as they were not provided in the report. Since mining technical reports typically only include deterministic analysis, no information was provided for estimating the volatility required for the stochastic evaluation. This was inferred from historic gold prices.

The entire mine was modeled as a single phase shown in Figure 11, with one grade tonnage curve throughout the mine life. This simplification was applied due to limitations on the data available.
The report mentions a design comprised of four phases, however it does not provide sufficient details regarding the portion of the reserves in each phase. The tonnage and grade distribution used for the single phase is presented in Table 2 and graphically in Figure 10. The tonnage and grade values represent the sum of the Measured and Indicated resources as stated in the Mineral Resource and Mineral Reserve report and subsequently adjusted for mining loss and dilution to convert them to reserves.

**Table 2: The total reserve and average grade of the Detour Lake project at various cut-off grades**

<table>
<thead>
<tr>
<th>Cut-off (Au g/t)</th>
<th>Grade (Au g/t)</th>
<th>Tonnage ('000s)</th>
<th>Contained Gold ('000s oz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.81</td>
<td>675,542</td>
<td>17,565</td>
</tr>
<tr>
<td>0.4</td>
<td>0.92</td>
<td>550,258</td>
<td>16,273</td>
</tr>
<tr>
<td><strong>0.5</strong></td>
<td><strong>1.03</strong></td>
<td><strong>450,612</strong></td>
<td><strong>14,931</strong></td>
</tr>
<tr>
<td>0.6</td>
<td>1.14</td>
<td>370,792</td>
<td>13,603</td>
</tr>
<tr>
<td>0.7</td>
<td>1.25</td>
<td>306,650</td>
<td>12,331</td>
</tr>
<tr>
<td>0.8</td>
<td>1.36</td>
<td>254,960</td>
<td>11,144</td>
</tr>
<tr>
<td>0.9</td>
<td>1.47</td>
<td>212,656</td>
<td>10,038</td>
</tr>
<tr>
<td>1.0</td>
<td>1.58</td>
<td>178,187</td>
<td>9,027</td>
</tr>
</tbody>
</table>

**Figure 10: A graph illustrating the total reserves and average grade of the Detour Lake project at various cut-off grades**
The basic parameters used in the analysis are presented in Table 3. Some discussion on the source and calculation of these values is provided below.

**Table 3: List of parameters required for RO analysis and the values used**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Maximum Rates</strong></td>
<td></td>
</tr>
<tr>
<td>Mining</td>
<td>120.00 M.T/y</td>
</tr>
<tr>
<td>Processing</td>
<td>22.34 M.T/y</td>
</tr>
<tr>
<td><strong>Operating Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>Recovery</td>
<td>91.0%</td>
</tr>
<tr>
<td>Total Contained Tonnage</td>
<td>2,230 M.T</td>
</tr>
<tr>
<td>Pre-production time remaining</td>
<td>20 Months</td>
</tr>
<tr>
<td><strong>Economic Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>Exchange Rate</td>
<td>1.10 C$/US$</td>
</tr>
<tr>
<td>Mining Cost</td>
<td>$1.74 C$/T</td>
</tr>
<tr>
<td>Processing Cost</td>
<td>$6.09 C$/T</td>
</tr>
<tr>
<td>Fixed cost</td>
<td>$68 M C$/y</td>
</tr>
<tr>
<td>Silver Credit</td>
<td>$0.14 C$/T</td>
</tr>
<tr>
<td>Royalty</td>
<td>2% Gross Value</td>
</tr>
<tr>
<td>Transportation and Refining</td>
<td>$5.50 C$/oz</td>
</tr>
<tr>
<td>Expected Gold Price</td>
<td>$935 C$/oz</td>
</tr>
<tr>
<td>Pre-Production Capital Costs</td>
<td>$1,192 M C$</td>
</tr>
<tr>
<td>Reclamation</td>
<td>$12 M C$</td>
</tr>
<tr>
<td>Discount Rate</td>
<td>5%</td>
</tr>
<tr>
<td><strong>Estimated Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>Closure cost</td>
<td>$11.95 M C$</td>
</tr>
<tr>
<td>Care and Maintenance cost</td>
<td>$5.50 M C$/y</td>
</tr>
<tr>
<td>Re-Open Cost</td>
<td>$11.95 M C$</td>
</tr>
<tr>
<td>Gold Price Volatility</td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>25 years</td>
</tr>
<tr>
<td>Volatility</td>
<td>20%</td>
</tr>
</tbody>
</table>

11 Processing rate for the first 32 months was limited to 20.08 M tpy (55,000 tpd) as this is prior to the planned pant expansion.
Figure 11: A plan view of the ultimate pit design at Detour Lake

Total contained tonnage was calculated as, contained waste, 1,751 M.T., plus the reserves at 0.5 g/t cut-off of 479 M.T. It should be noted that overburden material totaling 96.5 M. T. was included in the waste figure. Based on these values, the LOM average stripping ratio was 3.88.

Mining rate of 120 MT per year was taken as the maximum annual combined ore and waste excavated in the mining schedule provided in the report. Processing rate for the first 32 months was limited to 20.08 M tpy (55,000 tpd) and then increased to the 22.34 M tpy (61,000 tpd) to reflect the planned expansion.

The mining cost listed in Table 3, was labeled in the report as the cost of mining waste. The cost for mining ore was stated as $2.12 C$/T. The difference between the ore and waste mining cost, $0.38, was added to the reported processing of $5.71 C$/T to give a total operating cost per tonne of $6.09 C/T. This treatment of additional costs associated with mining ore is consistent with current practices for mine planning and open pit optimization (Lane K. , 1988).
Fixed costs were approximated by taking the reported General and Administrative cost per tonne of ore, $1.22 C$/T, and multiplying by the 61,000 tonnes per day capacity of the mill multiplied by 365 days per year and adding the average sustaining capital per year of C$40.59M.

In their report BBA used a constant cut-off grade of 0.5 g/t over the life of the mine. This was calculated as a modified marginal cut-off grade. The details of this calculation are provided in Table 4. Readers should note that the true marginal cut-off grade, that is using the actual processing cost of $6.09 and ignoring the minimum profit, is 0.27 g/t.

Table 4: Details of cut-off grade calculation from the BBA reserve report

<table>
<thead>
<tr>
<th>Reported Cut-off Grade Calculation</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gold Price</td>
<td>850 [$US/oz]</td>
</tr>
<tr>
<td>Exchange Rate</td>
<td>1.1 C$/US$</td>
</tr>
<tr>
<td>Milling Cost (Artificially Increased)</td>
<td>$9.47 /t milled</td>
</tr>
<tr>
<td>General and Administration</td>
<td>$1.23 /t milled</td>
</tr>
<tr>
<td>Total Operating Cost</td>
<td>$11.65 /t milled</td>
</tr>
<tr>
<td>Gold Price</td>
<td>$935 $/oz</td>
</tr>
<tr>
<td>Gold Price</td>
<td>30.06 $/g</td>
</tr>
<tr>
<td>Recovery</td>
<td>91.5 %</td>
</tr>
<tr>
<td>Gold Gross Value</td>
<td>27.51 $/g</td>
</tr>
<tr>
<td>Royalty (2%)</td>
<td>0.55 $/g</td>
</tr>
<tr>
<td>Refining</td>
<td>0.161 $/g</td>
</tr>
<tr>
<td>Gold payment</td>
<td>99.935 %</td>
</tr>
<tr>
<td>Minimum Profit (5%)</td>
<td>1.38 $/g</td>
</tr>
<tr>
<td>Net gold value</td>
<td>25.4 $/g</td>
</tr>
<tr>
<td>Cut-off grade</td>
<td>0.42 g/t</td>
</tr>
<tr>
<td>Cut-off grade USED</td>
<td>0.5 g/t</td>
</tr>
</tbody>
</table>

Rather than modeling silver as a separate metal with its own price model and grade distribution, the reported average silver credit per tonne of ore was used. It was determined that this simplification had a negligible impact on the valuation as silver content was very small and thus under all but the most extreme price scenarios, it contributes an insignificantly small portion to
the value of ore. This treatment of silver is also consistent with the methods applied by BBA in the pit optimization.

A 2% royalty was deducted on the gross revenue generated from the sale of gold. In addition a refining and transportation charge of $5.00 US$ /oz was included for the cost of selling the mine doré.

Volatility was calculated using a historical price data. Daily closing price for gold bullion on the COMEX, spanning from 01/01/2001 through 10/28/2011 were used. Historical volatility was calculated using equation 4-1, where 252 is the assumed number of trading days per year. The calculated volatility was 0.1925 which was rounded to 0.2.

\[
Volatility = STDEV\left(\ln\left(\frac{Price_t}{Price_{t-1}}\right)\right) \times \sqrt{252}
\]

4-1

When combining all of the parameters values from Table 3, and the reserves from Table 2 with the operating model from Chapter 3, \(Ex\), \(C_{mine}\) and \(C_{processing}\) can be determined at all cut-off grades. These values are presented in Table 5 along with the revenue and free cash flow assuming a gold price of $935/oz.

**Table 5: Operating parameters at all cut-off grades, assumes a gold price of $935/oz**

<table>
<thead>
<tr>
<th>Cut-off</th>
<th>Strip Ratio</th>
<th>Ex</th>
<th>Grade</th>
<th>Revenue</th>
<th>Mine Cost</th>
<th>Proc. Cost</th>
<th>Fixed Cost</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g/t)</td>
<td>(Mt /y)</td>
<td>(g/t)</td>
<td>(M C$/y)</td>
<td>(M C$/y)</td>
<td>(M C$/y)</td>
<td>(M C$/y)</td>
<td>(M C$/y)</td>
<td>(M C$/y)</td>
</tr>
<tr>
<td>0.3</td>
<td>2.2 X</td>
<td>65</td>
<td>0.81</td>
<td>444</td>
<td>120</td>
<td>112</td>
<td>68</td>
<td>145</td>
</tr>
<tr>
<td>0.4</td>
<td>2.9 X</td>
<td>79</td>
<td>0.92</td>
<td>505</td>
<td>145</td>
<td>112</td>
<td>68</td>
<td>180</td>
</tr>
<tr>
<td>0.5</td>
<td>3.8 X</td>
<td>97</td>
<td>1.03</td>
<td>566</td>
<td>176</td>
<td>112</td>
<td>68</td>
<td>211</td>
</tr>
<tr>
<td>0.6</td>
<td>4.9 X</td>
<td>118</td>
<td>1.14</td>
<td>627</td>
<td>212</td>
<td>112</td>
<td>68</td>
<td>235</td>
</tr>
<tr>
<td>0.7</td>
<td>6.1 X</td>
<td>120</td>
<td>1.25</td>
<td>580</td>
<td>215</td>
<td>94</td>
<td>68</td>
<td>202</td>
</tr>
<tr>
<td>0.8</td>
<td>7.5 X</td>
<td>120</td>
<td>1.36</td>
<td>524</td>
<td>214</td>
<td>78</td>
<td>68</td>
<td>164</td>
</tr>
<tr>
<td>0.9</td>
<td>9.2 X</td>
<td>120</td>
<td>1.47</td>
<td>472</td>
<td>213</td>
<td>65</td>
<td>68</td>
<td>126</td>
</tr>
<tr>
<td>1.0</td>
<td>11.2 X</td>
<td>120</td>
<td>1.58</td>
<td>424</td>
<td>213</td>
<td>55</td>
<td>68</td>
<td>89</td>
</tr>
</tbody>
</table>
4.3 Method

Prior to conducting the real options analysis a series of traditional discounted cash flow (DCF) models were constructed. This was done to validate the choice of operating parameter values as well as to assess the impact of necessary simplifications.

The first DCF model combined the input values outlined in section 4.2, Table 3 with an annual production schedule taken directly from the report. The NPV at a 5% discount, according to this model was $1,307 M.C$ or around 2.7% higher than the reported before tax NPV$_{5\%}$ of $1,273$. This small discrepancy is likely due to the timing of some cash flows such as sustaining capital, where assumptions had to be made due to lack of information from the report.

A second DCF model was constructed that replaced the reported schedule with one dictated by the reserves stated in Table 2 and production constraints defined by equations 3-12 and 3-13 using the aforementioned values for $K_{\text{mine}}$ and $K_{\text{process}}$. This model produced a NPV$_{5\%}$ of $1,397M$ or 9.7% above the reported value. On an after tax basis the model had a NPV$_{5\%}$ $944M$ or 9.1% above the reported estimate. This greater discrepancy is due to the simplified schedule. If further details regarding the pushbacks had been provided in the report this discrepancy could have been reduced.

The real options analysis conducted was consistent with the model presented in Chapter 3. To allow for the inclusion of previously mentioned 2% royalty and the $5.00 US/oz selling charge the equations for the instantaneous cash flow of the mine were modified in accordance to equation 3-43.
The initial RO valuation was conducted on a before tax basis and verified using a Monte Carlo simulation. Then a full tax schedule was added to the simulation and an after tax value was determined. For an explanation on why taxes were treated in this manner readers are directed to Appendix D.

4.4 Results

The real options model produced a before tax value of $2.08B roughly 46% above the comparable DCF model. A summary of the valuations in presented in Table 6. For comparison the RO value without the option to adjust cut-off grade was $2.03B suggesting that in this case, the option to adjust cut-off grade increased value by 2.1%.

Table 6: Summary of valuation results

<table>
<thead>
<tr>
<th></th>
<th>Before Tax</th>
<th>After Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'000 C$</td>
<td>'000 C$</td>
</tr>
<tr>
<td>DCF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reported DCF</td>
<td>$1,273,000</td>
<td>$865,307</td>
</tr>
<tr>
<td>Recreated DCF</td>
<td>$1,397,742</td>
<td>$944,828</td>
</tr>
<tr>
<td>RO</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PDE</td>
<td>2,079,054</td>
<td>N\A</td>
</tr>
<tr>
<td>Simulation</td>
<td>2,043,443</td>
<td>1,443,633</td>
</tr>
</tbody>
</table>

The simulations confirmed the value as determined by the PDE, since the PDE valuation was within two standard errors of the simulation. Details on the simulation sample population are presented in Table 7.

Table 7: Explanatory statistics for simulation populations

<table>
<thead>
<tr>
<th></th>
<th>Before Tax</th>
<th>After Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'000 C$</td>
<td>'000 C$</td>
</tr>
<tr>
<td>Number of Simulations</td>
<td>2,000</td>
<td>2,000</td>
</tr>
<tr>
<td>Mean</td>
<td>2,043,443</td>
<td>1,260,349</td>
</tr>
<tr>
<td>Minimum</td>
<td>-1,242,231</td>
<td>-1,725,389</td>
</tr>
<tr>
<td>Maximum</td>
<td>10,682,951</td>
<td>29,341,336</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3,485,494</td>
<td>2,750,792</td>
</tr>
<tr>
<td>Standard Error</td>
<td>77,938</td>
<td>61,510</td>
</tr>
</tbody>
</table>
Figure 12 is a histogram showing the distribution of before tax values as determined by the simulations, Figure 13 is the after tax histogram. Of the 2,000 after tax trials, 30% had negative project value while 37% had a value greater than the reported estimate.

Figure 12: Histogram showing the results of the 2,000 before tax simulations

Figure 13: Histogram showing the results of the 2,000 after tax simulations
The model also produced a complete cut-off grade strategy, consisting of optimum cut-off grade choices for all points in time, at all locations in the mine plan, under all price conditions. Although this massive amount of information cannot be published or plotted in any reasonable manner some of it has been presented in Figure 14 and Figure 15. Figure 14 shows the optimal cut-off grade at day one for a range of prices and reserve sizes. At high prices and larger reserves the optimal cut-off choice is elevated to 0.6 g/t, resulting in the highest possible rate of metal production (see Table 5). When prices drop below $626 the mine closes. As tonnage decreases so too does the optimal cut-off grade. This is due to the higher option value of the remaining, limited reserves.

![Figure 14: Cut-off strategy at day one](image)

Contrasting the strategy in Figure 14 is the optimal strategy shown in Figure 15, at the 24th year. With only one year remaining preserving reserves has little option value. It is more valuable to operate at the highest metal output at a cut-off grade of 0.6 g/t and maximize cashflow for the remaining one year of the time horizon regardless of the quantity of reserves remaining.
Figure 15: Cut-off strategy at 24 years

It should be noted that under no circumstances was the optimal strategy at a cut-off grade above 0.6 g/t. Operating at a higher cut-off grade would result in insufficient ore feed to completely utilize the installed processing capacity due to prohibitively high stripping ratios causing the mining constraint to become binding, see Table 5. Since the optimal strategy always “filled the mill” it is consistent with both Kenneth Lane’s observations (Lane K. , 1988) as well as the intuition of mine planners.

4.4.1 Single Simulation in Detail

For readers to better understand and visualize the results, a single simulation was selected to show the details of how the model determines cut-off and calculates cash flows. Figure 16 shows a simulated gold price path and the corresponding cut-off grade strategy as dictated by the PDE solution. Note how the cut-off grade starts above marginal cut-off of 0.27 g/t and generally declines over time, this is consistent with Lane’s observations for deterministic optimization of
cut-off (Lane K., 1988). Where the cut-off drops to zero in year 20 indicates a temporary closure of the mine followed by a reopening in the middle of year 22. The small oscillations in cut-off grade in year 12 and 19 are likely artificial. These could be smoothed out by using a more detailed grade-tonnage curve and/or by using a more finely spaced PDE solution grid.

Figure 16: Cut-off strategy for a single price simulation

Figure 17 illustrates the same price simulation as Figure 16 but with the daily cash flows plotted on the secondary axis. As expected, the cash flows closely follow the price path. Further observation reveals that as time passes and cut-off grade declines, daily cash flows decline relative to price. Again this is expected as a lower cut-off grade produces lower operating profits, see Table 5. In year 20, when the cash flows ‘flat line’ the mine is temporarily closed. Finally, late in year 23 the mine’s reserves are exhausted and the operation is closed. Figure 18 provides a closer view of the final six years of the simulation. Note that the locations where the cash flow

---

12 Only the stochastic operating cash flows are shown. Deterministic cash flows such as the initial capital cost and final reclamation are not included in the plot.
line drops below the limits of the plot, are days in which the closure and re-opening costs of $11.95M are paid.

![Figure 17: A single price simulation and corresponding daily cash flows](image1)

![Figure 18: A detailed view of Figure 17, for the final six years of the simulation](image2)
4.5 Conclusions and Discussion

Before and after tax, real options valuations were calculated for the Detour Mining project, based on the details provided in the 2010 reserve update (BBA, 2011). The real options model presented in Chapter 3 estimated a before tax value of $2.08B or approximately 46% greater than the value as predicted by the comparable DCF model. On an after tax basis these values are $1.44B and 53% respectively. These results were in line with expectations, as the addition of the options to adjust cut-off grade and to temporarily close the mine, when correctly exercised, will increase expected project value. The after tax RO analysis benefits from carried losses associated with any closure and re-opening costs. This explains why the after tax RO analysis attains greater gains over a deterministic analysis relative to the RO before tax case over the before tax DCF.

Reviewing a number of simulations in detail, the strategy appears to be consistent with existing cut-off theory (Lane K., 1988). When comparing the RO optimal strategy to the marginal value computed, it is interesting that BBA did not calculate a true marginal cut-off grade and instead artificially increased the cut-off grade. It is clear that BBA understood that the NPV of the project would be unacceptably low if the marginal grade was used but did not have a method for optimizing cut-off grade. It is unclear why they chose to use a single elevated cut-off grade rather than performing a deterministic optimization.

The computed value of $2.08B should not be considered the exact value of the Detour Mine at the time of the report’s publication. Due to lack of information contained in the report, some assumptions had to be made that likely impacted the valuation. First the simplification of the entire deposit to a single phase is likely insufficient for an accurate valuation. Second the use of a single value for the mining rate is likely inaccurate. Readers should also be aware that the ‘valuation’ provided in this section is in actuality a speculative value; based on the reports
forecasted long term price average and the author’s estimate of volatility. For a true risk-neutral valuation the gold futures curve and the Black’s volatilities from the time should be used.
Chapter 5

Worked Example – Confidential Data

A second case was studied using the model described in Chapter 3. A confidential dataset was obtained for an operating mine from a mining company. The data included two complete geological block models and other planning data. This highly detailed dataset allowed for the incorporation of additional complexities that could not be included in the previous example. Specifically these were:

- A mine plan divided into 18 phases
- High resolution grade-tonnage curves for each phase
- Multiple material types in each phase

5.1 Project Description

Due to the confidential nature of the data, no identifying details are provided. The project is a large open pit gold heap leach mining operation. The operation is comprised of two open pits and a heap leaching processing facility. Mineralization at the project has been described as a typical porphyry style system.

5.2 Data

The dataset provided consisted of two block models, two ultimate pit designs and a number of spreadsheets containing a variety of technical and economic data. As the intermediate pushback designs for Pit 1 were not provided a set of designs were prepared to closely approximate the provided schedule.

The mine plan was divided into 18 distinct phases. The phases were modeled from the existing 17 year mine plan. All of the material that was scheduled to be excavated from both pits, in a given
year was grouped into a phase, with a small remainder falling into phase 18. By defining the phases in this way the mine plan remains relatively rigid, ensuring that the optimal strategy can be feasibly executed. The disadvantage in including this many phases, is that some degree of flexibility is removed from the optimization. Also processing time of the numerical solution increases with the number of phases modeled.

Each block within the model was classified as one of three material types: oxide, mixed and sulphide. Each of these three material types had differing recoveries and processing costs associated with them. Table 8 summarizes these values.

**Table 8: Optimization parameters for the six material types**

<table>
<thead>
<tr>
<th>Ore Type</th>
<th>Pit 1</th>
<th>Pit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Oxide</td>
<td>Mixed</td>
</tr>
<tr>
<td>Processing Cost (US$/t)</td>
<td>4.63</td>
<td>4.63</td>
</tr>
<tr>
<td>Au Recovery (%)</td>
<td>85%</td>
<td>75%</td>
</tr>
</tbody>
</table>

The main inputs used in the analysis are summarized in Table 9.

**Table 9: Summary of input values**

<table>
<thead>
<tr>
<th>Parameter Values</th>
<th>Parameter Values</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Maximum Rates</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mining</td>
<td>34 M.T/y</td>
<td></td>
</tr>
<tr>
<td>Processing</td>
<td>16 M.T/y</td>
<td></td>
</tr>
<tr>
<td><strong>Operating Parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recovery</td>
<td>Varies</td>
<td></td>
</tr>
<tr>
<td>Total Contained Tonnage</td>
<td>479 M. T</td>
<td></td>
</tr>
<tr>
<td><strong>Economic Parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mining Cost</td>
<td>$1.89 US$/T</td>
<td></td>
</tr>
<tr>
<td>Processing Cost</td>
<td>Varies US$/ T</td>
<td></td>
</tr>
<tr>
<td>Fixed cost</td>
<td>27.36 M C$/y</td>
<td></td>
</tr>
<tr>
<td>Royalty</td>
<td>1.55% Gross Value</td>
<td></td>
</tr>
<tr>
<td>Transportation and Refining</td>
<td>$5.66 US$/ oz</td>
<td></td>
</tr>
</tbody>
</table>
Expected Gold Price $900 US$/ oz
Total Discounted Capital Costs $309 M. US$
Discount Rate 5%

**Estimated Parameters**
- Closure cost $20 M. US$
- Care and Maintenance cost $14 M. US$/y
- Re-Open Cost $20 M. US$
- Gold Price Volatility 20%
- Time 20 years

Closure and re-opening costs were estimated as the average of one year’s worth of sustaining capital. The care and maintenance costs were estimated as half the annual G&A costs. All capital costs, including sustaining capital, were considered deterministic and unavoidable. The deposit also contained trace amounts of copper and silver. As under the provided valuation model these metals combined produced 3% of the revenue they were ignored for the RO valuation.

The current practice for determining cut-off grade at the mine is to use a marginal cut-off grade. A sample calculation for cut-off grade determination is presented in table Table 10.

**Table 10: Example of current practice cut-off grade calculation**

<table>
<thead>
<tr>
<th>Pit 1</th>
<th>Oxide</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gold Price Used to Establish Cut-off Grade</strong>&lt;br&gt;Gold Price</td>
<td>US$/g</td>
</tr>
<tr>
<td>Recovery (at cut-off grade)</td>
<td>%</td>
</tr>
<tr>
<td>Effective Revenue</td>
<td>US$/g</td>
</tr>
<tr>
<td>Less Royalty</td>
<td>0% of sales</td>
</tr>
<tr>
<td>Less Per Gram Costs</td>
<td>US$/g</td>
</tr>
<tr>
<td>Realized Revenue</td>
<td>US$/g</td>
</tr>
<tr>
<td>Costs to Produce (Processing + G&amp;A)</td>
<td>US$/t</td>
</tr>
<tr>
<td>Cut-off (in place)</td>
<td>g/t</td>
</tr>
<tr>
<td>Dilution</td>
<td>%</td>
</tr>
<tr>
<td>Reserve Cut-off Grade</td>
<td>g/t</td>
</tr>
</tbody>
</table>
5.3 Method

5.3.1 Model Modifications
The overall method for determining the optimum cut-off grade strategy and corresponding real options value is generally consistent with the model presented in Chapter 3. A few modifications to the model were required in order to address the specifics of the project.

The first modification to the model was to incorporate the optimization of multiple cut-off grades. A total of six different material types were identified in original data: oxide, mixed and sulphide for pits one and two. Ordinarily six cut-off grades would need to be determined for all points in the strategy space. However, noting that the parameters for pit one shows that all three material types have the same processing cost, only recovery differs. Since the geological models used were already adjusted for recovery, from the optimization standpoint all three materials from pit 1 are the same.

Looking at the material classes for Pit 2 we see that the processing costs are similar. Again all three types of material were considered as a single class and assigned only one optimal cut-off grade. This simplification results in a slightly sub-optimal cut-off grade strategy for Pit 2, however, at the level of accuracy of the grade-tonnage data it is almost unperceivable. The optimal cut-off grade strategy is thus defined as two post recovery cut-off grades, one for each pit. Actual insitu cutoff grades for all materials can then be determined by dividing by the respective recoveries.

5.3.2 Data Preparation
Prior to conducting the Real Options analysis, the necessary grade-tonnage curves for all 18 of the phases were prepared. The two geological block models, one for each pit, were loaded into Surpac™. A total of 29 groupings of blocks, were created, 18 for Pit 1 and 11 for Pit 2. This
reflects the 18 year schedule provided for the combined operation, illustrated in Table 11. Figure 19 and Figure 20 show the constraints corresponding to phase 5 for Pits 1 and 2 respectively.

Table 11: Schedule showing fraction of material in each pit for all phases

<table>
<thead>
<tr>
<th>Phase</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pit 1</td>
<td>100%</td>
<td>100%</td>
<td>10%</td>
<td>6%</td>
<td>5%</td>
<td>33%</td>
<td>33%</td>
<td>36%</td>
<td>33%</td>
<td>62%</td>
<td>0%</td>
<td>3%</td>
<td>14%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td></td>
</tr>
<tr>
<td>Pit 2</td>
<td>0%</td>
<td>0%</td>
<td>90%</td>
<td>94%</td>
<td>95%</td>
<td>67%</td>
<td>67%</td>
<td>64%</td>
<td>67%</td>
<td>38%</td>
<td>100%</td>
<td>97%</td>
<td>86%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Figure 19: The grouping of blocks from the Pit 1 block model corresponding to Phase 5

Figure 20: The grouping of blocks from the Pit 2 block model corresponding to Phase 5
Using the block groupings, 29 corresponding grade-tonnage histograms were exported from Surpac. Figure 21 and Figure 22 show the histograms corresponding to the block groupings in Figure 19 and Figure 20. Finally the 18 histograms from Pit 1 were merged with the 11 histograms from Pit 2 to create 18 distinct phase histograms, according to the schedule illustrated in Table 11. Figure 23 shows the combined histogram created from Figure 21 and Figure 22.

**Figure 21: Grade-tonnage histogram for phase 5 of Pit 1**

**Figure 22: Grade-tonnage histogram for phase 5 of Pit 2**
Figure 23: Phase 5, combined grade-tonnage histogram for both pits

In addition to the grade tonnage characteristics, the average processing cost was determined for all cut-off grade combinations in each phase. This was done using a weighted average cost based on the relative proportions of the six material classes.

5.4 Results

5.4.1 Overall results

The real options analysis provided an expected value of $509M. This is roughly 33% higher than the discounted cash flow value determined by a DCF model provided by the company. The RO results were validated using a Monte Carlo simulation (RO simulation in Table 12) of the optimal strategy. If the option to adjust cut-off grade is removed the expected value drops to $477M, roughly 6% less. A regular Monte Carlo simulation was conducted which followed the current cut-off grade strategy of choosing the marginal cut-off grades and rule to close the mine should prices fall below $600/oz. This strategy produced a value of only $407M. The summary statistics for the simulation population are provided in Table 13. All results were conducted on a before tax basis to remain consistent with the valuation provided by the company.
Table 12: Valuation results

<table>
<thead>
<tr>
<th>Results</th>
<th>'000 US$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DCF</td>
<td></td>
</tr>
<tr>
<td>NPV&lt;sub&gt;5%&lt;/sub&gt;</td>
<td>$383,770</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>$407,463</td>
</tr>
<tr>
<td>RO</td>
<td></td>
</tr>
<tr>
<td>PDE</td>
<td>$509,180</td>
</tr>
<tr>
<td>RO Simulation</td>
<td>$482,129</td>
</tr>
</tbody>
</table>

Table 13: Summary statistics for the simulation population

<table>
<thead>
<tr>
<th>RO Simulation Statistics</th>
<th>'000 C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Simulations</td>
<td>2,000</td>
</tr>
<tr>
<td>Mean</td>
<td>$482,129</td>
</tr>
<tr>
<td>Minimum</td>
<td>-$528,744</td>
</tr>
<tr>
<td>Maximum</td>
<td>$7,456,188</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>$1,050,673</td>
</tr>
<tr>
<td>Standard Error</td>
<td>$23,494</td>
</tr>
</tbody>
</table>

Plotting the distribution of values from the RO simulation in Figure 24, we find that 36% of the simulations had a negative NPV while 23% of the simulations had NPVs greater than what was predicted by the DCF model. For comparison, in the regular Monte Carlo simulation which followed the current cut-off grade strategy 53% of the simulations had negative values and again 23% had values above the standard DCF.

Figure 24: Distribution of simulated RO valuations
5.4.2 Investigation of a single simulation

In order to evaluate the optimal cut-off grade strategy determined by the RO solution, a number of individual simulations were investigated. The exercise of the cut-off grade and closure options appeared reasonable and produced acceptable results. To illustrate the optimal strategy determined by the model a single simulation has been shown in detail.

Figure 25 shows the simulated price path in grey and the resulting cash flows in black. As expected, cash flows generally follow the same trend as the price path. The exception being at the transition from one phase to the next, indicated by the banding, which results in a change in grade tonnage curve and hence cash flow behavior.

Another notable trend begins in the middle of year eight when cash flows begin to fall into the negatives. The mine continues to operate at a loss through year nine. Then in year ten the combination of improving gold prices and the starting of a new phase result in positive cash flows. Prices then begin to decline and the mine is closed at the beginning of year eleven. This results in a onetime closure of $20M (which falls below the axis limits) and the flat line cashflows from the care and maintenance costs. The mine re-opens at the end of year 13 and again a $20M charge is paid.

For the above simulation, the cut-off grade strategy for pits one and two are shown in Figure 26 and Figure 27 respectively. Note how in both pits all three material types follow the same cut-off grade trend. This is a result of the manner in which the cut-off grade strategy was conducted on grades adjusted for recovery. The cut-off grades shown have been adjusted back and should be considered insitu cut-offs grades. The areas in which the cut-off grades drop below the limits of the axis indicate a mine closure.
Figure 25: Cash flow details of a single price simulation. Banding indicates the change from one phase to the next.

Figure 26: Cut-off grade strategy for Pit 1 for a single simulation.
Figure 27: Cut-off grade strategy for Pit 2 for a single simulation
5.4.3 Hedging strategy results

One advantage in using a finite difference numerical technique is that all of the derivatives of the asset value are computed. These derivatives, sometimes referred to as the Greeks, are summarized in Table 14, with values for the mining project at the initial conditions\(^{13}\). Since the finite difference calculation was conducted using \(X(t)\) instead of \(S(t)\), the first derivative with respect to \(S\) was computed using the equation 5-1. A similar equation was used for the determination of the second derivative with respect to \(S\).

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
<th>Value at Initial Conditions</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta</td>
<td>(\Delta = \frac{\partial V}{\partial S})</td>
<td>2.5M.</td>
<td>Sensitivity with respect to metal price</td>
</tr>
<tr>
<td>Gamma</td>
<td>(\Gamma = \frac{\partial^2 V}{\partial S^2})</td>
<td>3,834</td>
<td>Second order derivative with respect to metal price. Indication of how often the hedge needs to be re-adjusted</td>
</tr>
<tr>
<td>Theta</td>
<td>(\theta = \frac{\partial V}{\partial t})</td>
<td>51.23 M.</td>
<td>Sensitivity with respect to the passage of time</td>
</tr>
</tbody>
</table>

\[
Delta = \frac{\partial V}{dS} = \frac{\partial V}{dX} \cdot \frac{\partial S}{\partial X} = \frac{\partial V}{dX} \cdot \frac{\partial V}{\partial t} \cdot e^{\sigma(t)\sigma X(t) - 0.5\sigma^2(t)}
\]

To understand delta, consider a $1 increasing in initial gold price from $900/oz to $901/oz. The delta value suggests an increase in value of $2.5M. Of course since value does not necessarily change linearly with an increase in price, the delta value is not reflective for larger shifts in price. For example a $100 increase in the price of gold would not yield a $250M increase in value. Now consider the initial price of gold to be $900 but our time horizon decreased by one year, from 20 to 19. The value of theta predicts a decrease in value of $51.23M.

\(^{13}\) The initial conditions are: \(X = 0, I = 479\) MT and \(t = 0\).
By computing these derivatives, dynamic hedging strategies can be designed. The simplest dynamic strategy is a delta-neutral hedge. The owner of the project is essentially long gold price, thus a delta-neutral hedge involves assuming a short position in gold. The size of the hedge is equal to delta. In this example initially the hedge position is shorting 2.5Moz. Since the delta value changes with time, commodity price and reserves remaining, the hedge is readjusted at various points in the future. Theoretically if the readjustments could be done continuously, the value of the hedged project would be unaffected by any changes in gold price.

To test the effectiveness of a dynamic hedging strategy, 1,000 simulations were conducted with an accompanying dynamic hedging portfolio. The values of these simulations were calculated for both hedged and un-hedged conditions. The results from these simulations are summarized in Table 15. In these simulations the hedge was adjusted every day and no transaction costs were associated with the hedging activity. Readers should note, a static hedge of forward selling all production, would yield a value close to the DCF value of $384M as quoted in Table 12.

Table 15: Summary statistic of un-hedged and hedged simulation populations

<table>
<thead>
<tr>
<th></th>
<th>Un-hedged</th>
<th>Hedged</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Simulations</td>
<td>1,000</td>
<td>1,000</td>
</tr>
<tr>
<td>Mean</td>
<td>500,296</td>
<td>524,737</td>
</tr>
<tr>
<td>Minimum</td>
<td>-526,666</td>
<td>416,312</td>
</tr>
<tr>
<td>Maximum</td>
<td>8,697,024</td>
<td>1,001,652</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>988,743</td>
<td>45,076</td>
</tr>
<tr>
<td>Standard Error</td>
<td>31,267</td>
<td>1,425</td>
</tr>
</tbody>
</table>

The hedging results can also be visualized in Figure 28. The light grey line represents the simulated un-hedged project values for each of the 1,000 simulations in particular order. The dark grey line is the value of the same 1,000 simulations but with an accompanying dynamic hedge.
Note how the hedged values are much more stable and none of them have negative values. Figure 29 shows the histograms for the hedged and un-hedged simulations. Note how the hedged values are clustered close around $525M area while the un-hedged values are scattered across a wide range of values.

![Comparison of un-hedged and hedged simulation results](image)

**Figure 28: Comparison of un-hedged and hedged simulation results**

![Histograms of NPV for the hedged and un-hedged simulations](image)

**Figure 29: Histograms of NPV for the hedged and un-hedged simulations**
To assist readers in understanding how the dynamic hedging strategy works, a single simulation is illustrated in Figure 30. Note how over time, as reserves are depleted and the 20 year time window diminishes, the size of the hedge decreases. The hedge size is also affected by changes in prices as well as changing of phases.

![Figure 30: A single simulation showing the size of the dynamic hedge over time](image)

The ability to create the above hedge in the real world has some concerns. First, a firm would unlikely re-adjust the hedge every day, due to the associated transaction costs. It would be more realistic to assume that a firm re-hedged every month or quarter. By re-hedging less often, the hedge’s ability to reduce the impact of commodity price change is diminished. To mitigate this, a gamma-neutral hedge could be constructed by purchasing the correct amount and type of options on gold price. A discussion of this type of hedge is beyond the scope of this analysis.

Another issue is the size of the initial hedge. Many would consider a 2.5Moz short position too big. Such a large hedge can create large positive and negative cash flows from relatively small changes in commodity price. For example, if the price of gold rose by $5 an ounce in the first day of the above project, the mining firm would need to pay approximately $12.5 M to cover the...
hedging losses. These losses are technically offset by an equal increase in the value of the mining asset, however these gains will not be realized until sometime in the future. One approach to reducing the size of hedge required, is to only hedge a portion of the value of the mine. This would not fully protect the value of the mine, but would greatly decrease the probability of an economic loss on the project.

5.5 Conclusions and Discussion

The above case study illustrates how the model presented in Chapter 3 can be successfully applied to a real world mining project using accurate industry data. The analysis incorporated a pair of geological models in conjunction with a complete multi-phase schedule. The model was able to accept data output from Surpac™, a commercial mine design software. Several real world complexities were introduced, including multiple material types and multiple pits exploited simultaneously. Both of these were successfully incorporated into the analysis.

The before tax RO valuation, determined by the cut-off grade optimization model, was $509M US. This represents a 33% premium to the standard DCF model provided by the Company. The valuation was confirmed using a Monte Carlo simulation. The RO simulations were also compared to a set of simulations which followed the current strategy of using the marginal cut-off grade. The simulations which followed the real options optimized cut-off grade strategy had a 17% lower probability of a negative economic outcome.

Finally accurate derivatives or ‘Greeks’ were also computed for the project. These allowed for the development of a dynamic hedging, which was tested by a series of simulations. For the unhedged simulations 95% of the values fell between ± $1,977M of the mean. By contrast for the hedged simulations 95% of the values fell between ±$90M, a much narrower range.
Chapter 6

Summary, Conclusions and Recommendations

6.1 Summary
The determination of cut-off grade for mining projects is a fundamental problem in mine planning and requires both technical and economic consideration. The choice of cut-off grade determines the stripping ratio and ore grade of an operation. These in turn have a wide range of impacts, affecting operating parameters such as ore and waste tonnage, rate of metal production and ultimately, the overall economics of the project.

Traditional theory and currently accepted practices, optimize cut-off grade under deterministic or known prices. Most commodity prices are subject to volatile global markets and are difficult to predict. Assuming that price forecasts are correct will result in sub-optimal cut-off grade selection and inaccurate valuations.

An innovative method for optimizing cut-off grade under probabilistic or stochastic prices was outlined and demonstrated. The method presented is the first to directly incorporate industry standard mine planning information in a manner consistent with current planning practices. The mine plan is divided into a series of phases each represented by a grade distribution defined from block model data. A stochastic futures price model was used rather than simply a spot price model. The Geometric Brownian Motion model requires two inputs: an expected price path and a volatility curve. The two sets of inputs, technical and financial, are combined to define a system of stochastic partial differential equations (PDEs). This system can then be solved numerically using a finite difference technique. The model was successfully demonstrated on two mining projects with differing degrees of complexity. In both cases higher values were achieved when
compared to traditional DCF analysis through the optimal exercise of the open/closure and cut-off grade options.

6.1.1 Literature Review Summary
The problem of cut-off grade optimization was first formally considered by Kenneth Lane in 1964. Following his pioneering paper the body of literature on the subject has grown prolifically. The recognized industry standard text is Kenneth Lane’s 1988 book, *The Economic Definition of Ore*, in which he addresses cut-off grade optimization under known or deterministic price forecasts. Lane provides a variety of techniques for optimizing cut-off grade under different circumstances but defines the problem as ultimately one which can be solved by dynamic programming. Other authors expanded Lane’s work in deterministic optimization such as Asad and Xiao-Wei who addressed considerations such as stockpiling.

Real options (RO) is the application of stochastic optimization to real economic decisions. RO has been applied to mining projects since its beginnings in 1985, with Brennan and Schwartz’s seminal work. Although the paper made some unrealistic assumptions regarding mine operation and financial modeling, it pioneered the way for others to apply option theory to mining problems. Numerous authors have published many works applying RO to mining problems including Sabour and Samis. These works however, have focused on options such as closure, abandonment and capacity expansions.

Two notable works have attempted to apply RO theory to cut-off grade optimization: (Krautkraemer, 1988) and (Johnson, Evatt, Duck, & Howell, 2010). Krautkramer modeled the orebody and mine excavation as cylinders, to facilitate an analytical solution. This assumption however is too restricting for all but the most specific cases. Johnson approached the problem by first sequencing all of the blocks within a block model. This simplified the cut-off decision to a
binary waste/process decision for each block in succession. Unfortunately real mine plans are not
designed with such granularity and to assume such an apriori ordering exists and will be
followed, creates inaccurate decisions. More critically, there exists a high degree of error in the
estimation of grade of a single block. The combination of these concerns results produces a model
that cannot be applied in practice.

6.1.2 Summary of model description

The optimal cut-off grade strategy is the one which maximize net present (NPV) defined by
equation 6-1.

$$NPV = \int_{t}^{T} P(t)e^{-\alpha t} dt \quad 6-1$$

The instantaneous cash flow of an open pit mine can be described by equation 6-2.

$$P(t) = \left( \frac{s(t) \times \bar{x}(I, c)}{1 + Sr(I, c)} - \frac{C_{ore}}{1 + Sr(I, c)} - C_{waste} \right) \times Ex(I, c) - C_{fixed} \right) \times dt \quad 6-2$$

The mine plan is divided into a series of phases each defined by a grade-tonnage distribution. The
average grade $\bar{x}$, the stripping ratio $Sr$, and extraction rate $Ex$ are all dictated by the grade-tonnage
distribution of the current phase of the mine plan and the cut-off chosen. The phase distributions
can be directly defined by aggregating all of the blocks from a block model that fall within each
phase. The other inputs into equation 3-17 are standard mining engineering parameters with the
exception of price $s(t)$.

Price $s(t)$ is modeled using the stochastic equation 6-3. The financial inputs needed to define this
model are an initial price expectation curve $F(0, \forall t)$ and a volatility curve $\sigma(t)$. These two
functions can take any form.
Combining equations 6-1, 6-2 and 6-3 and applying Ito’s Lemma we find that the NPV of the mine, $V$, is given by equation 6-4.

$$V = \max_c E \left[ C_{\text{flow}}(X, l, t; c) dt + (1 - r dt)V \right.$$
$$+ (1 - r dt) \left( V_t + \frac{1}{2} V_{xx} - Ey(l, c) V_t \right) dt$$
$$\left. + (1 - dt)V_X dX \right]$$ 6-4

The operating mine can temporarily close by paying the transition cost of $C_{\text{turnoff}}$ and can re-open by paying the transition cost of $C_{\text{turnon}}$. While the mine is closed the value of the mine is defined by equation 6-5.

$$W_t + \frac{1}{2} W_{xx} - r W - C_{\text{off line}} = 0$$ 6-5

The mine will thus close when $W(X, l, t) \geq V(X, l, t) + C_{\text{turnoff}}$ and will re-open if $V(X, l, t) \geq W(X, l, t) + C_{\text{turnon}}$.

Finally boundary conditions for the PDEs 6-4 and 6-5 are:

$$V(X, l, T; c) = 0$$
$$V(X, 0, t; c) = 0$$
$$W(X, l, T) = 0$$
$$W(X, 0, t) = 0$$

The above system of PDEs, with the corresponding boundary conditions can be solved numerically using finite difference.
6.1.3 Summary of Public Data Case

The above cut-off optimization and valuation technique was applied to the Detour Lake gold mine project in northern Ontario. The data was primarily sourced from the Reserve and Resource Update prepared by BBA Engineering and published in 2010.

The model produced valuations substantially higher as compared to a standard discounted cash flow (DCF) valuation. Table 6 summarizes the results. This increase in value was due to the ability of the RO model to mitigate the downside by temporarily closing the mine and to capture greater upside through the optimizing of cut-off grade. The value of these two options is the difference between the RO models and their corresponding DCF valuations. The RO model was also validated using a Monte Carlo simulation.

Table 16: Summary of Detour Lake valuation results

<table>
<thead>
<tr>
<th></th>
<th>Before Tax</th>
<th>After Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>'000 C$</td>
<td>'000 C$</td>
</tr>
<tr>
<td><strong>DCF</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reported DCF</td>
<td>$1,273,000</td>
<td>$865,307</td>
</tr>
<tr>
<td>Recreated DCF</td>
<td>$1,397,742</td>
<td>$944,828</td>
</tr>
<tr>
<td><strong>RO</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PDE</td>
<td>2,079,054</td>
<td>N\A</td>
</tr>
<tr>
<td>Simulation</td>
<td>2,043,443</td>
<td>1,443,633</td>
</tr>
</tbody>
</table>

6.1.1 Summary of Confidential Data Case

Confidential mine planning data was provided by a mining company and used as the basis for an RO cut-off grade optimization. The data included two geological block models, two ultimate pit designs as well as a wealth of other technical and financial data. All of this information was incorporated into the RO analysis which valued an 18 phase, multi-pit, multiple material type, cut-off grade optimization.
The results of the optimization produced a real options valuation that was approximately 33% higher than the standard DCF valuation. A simulation of the current, non-optimized strategy of selecting the marginal cut-off grade, was also conducted. This produced a value roughly 20% lower as compared to the optimized RO cut-off grade strategy. What’s more, following the optimized strategy decreased the probability of economic loss by 17%, making for a more robust project. The results from the various valuations are summarized in Table 17.

**Table 17: Summary of valuation results, confidential case study**

<table>
<thead>
<tr>
<th>Results</th>
<th>$'000 US$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DCF</td>
<td></td>
</tr>
<tr>
<td>NPV&lt;sub&gt;5%&lt;/sub&gt;</td>
<td>$383,770</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>$407,463</td>
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<tr>
<td>RO</td>
<td></td>
</tr>
<tr>
<td>PDE</td>
<td>$509,180</td>
</tr>
<tr>
<td>RO Simulation</td>
<td>$482,129</td>
</tr>
</tbody>
</table>

**6.2 Conclusions**

The ability to adjust cut-off grade is an important strategic tool for mining operations. It provides the flexibility to adapt a mining operation in the face of changing economic conditions. Current practices for determining a cut-off grade strategy provide sub-optimal results in the face of volatile commodity prices.

A new model for optimizing cut-off grade under stochastic price conditions has been presented. This model divides the mine plan into a sequence of phases each defined by a grade tonnage curve taken from commercial mine design software. The technical engineering data is then combined with a stochastic price model to produce a system of partial differential equations, which can be solved numerically using a finite difference method.
The model was applied to two case studies. In the two examples conducted, significantly higher expected net present values were achieved by using the model to optimize cut-off grade and closure decisions. The probability of economic loss was also decreased by following the optimized strategy. Between the two cases, various real world complexities were incorporated in the RO analysis including: a full tax schedule, multiple pits mined simultaneously and multiple material classifications. In both cases higher expected values were determined by stochastically optimizing cut-off grade.

### 6.3 Recommendations

The development of a new technique for optimizing cut-off grade and determining value opens a range of possible applications. Comparing the valuations predicted by the model to historical trading and transaction valuations may provide insight to how the market ascribes value. Contrasting historical pre-mature mine closures against the optimal strategy as determined by the model may reveal trends in closure decisions. Comparing actual historical cut-off grade data to the optimal strategy could also be illuminating.

One issue with the ultimate strategies produced by the optimization is the presence of small, short term fluctuations in the cut-off grade. In practice mines would be unlikely to make such small and short lived changes to cut-off. In order to smooth out these oscillations a cut-off grade changing cost could be added. Another approach would be to change the detail of the grade tonnage distribution to remove changes that are too small.

There is ample room for the expansion and modification of the model to cover additional scenarios. Modifying the model to include multiple commodity products would be necessary to evaluate many mining operations. Adapting the operating model to underground mines would also expand the possible applications. Other features to consider include: mean-reverting price
models, additional switching options such as capacity expansions and the option to stockpile material.

There are also other real world considerations that could be added to the model. Some mines stockpile some of the material that below cut-off material to be processed after the mine is closed. Another issue is that many mines are not paid spot price for their production. Often payments are received some time after the commodity is produced and the actual amount paid may be an average of the price during the interim. Both of these considerations warrant further investigation and would make for interesting additions to the model.

6.4 Closing Remarks

Global commodity prices exhibit a high degree of volatility, exposing mine operators to significant commodity price risk. In most technical studies, a sensitivity analysis is conducted and almost always commodity price is the variable that has the greatest influence on value. Determining operating strategies and asset valuations by assuming future prices are known with certainty, not only leads to poor estimates of value but also exposes mines to greater risk of economic failure.

It behooves mine planners and management to understand methods for incorporating stochastic prices into their analysis and planning. As demonstrated in the above two case studies, performing a stochastic optimization of cut-off grade not only produces higher expected values but also creates a more economically robust project. Adding a more complete understanding of the financial markets in which the commodities are traded, allows mine operators to use the stochastic optimization results to build effective dynamic hedging strategies.
The stochastic cut-off grade optimization technique presented is both practical and flexible. In the two case studies data of varying degrees was successfully applied and a number of real world complexities were considered.

Despite the demonstrated benefits of stochastic optimization and valuation, gaining industry acceptance will be a challenge. When ultimate pit algorithms were being developed many industry professionals were reluctant to adopt them. Over time the algorithms achieved results and commercial software was developed. Today, many mine planning engineers make use of these techniques without questioning or even fully understanding them. Hopefully stochastic cut-off grade optimization will follow this path and be established as a standard industry practice.

As large producing mines are depleted, mining companies are forced to consider developing more economically marginal projects. These companies are reluctant to invest capital into these assets since volatile commodity prices may quickly move against them, forcing the mine to close. By correctly optimizing, valuing and hedging these marginal mines, they would become more economically robust. This could in turn bring greater stability to the overall commodity supply and help reduce the highly cyclical nature of the mining industry.
Works Cited


Thompson, J. (2010). *TEST OF AN INNOVATIVE STOCHASTIC DESIGN SYSTEM ON AN OPEN PIT.* Kingston, Ontario, Canada: Queen’s University.


Appendix A: Futures price derivation

Given:

\[ dF(t, T) = \sigma(T)F(t, T)dX \]

Show

\[ F(t, T) = s(t)e^{(\sigma X(t) - \sigma(t)^2t)} \]

(The functional dependencies of \( F \) and \( \sigma \) have been suppressed until the end of the proof.)

Define:

\[ \sigma = \ln F \]

Thus:

\[ \frac{dY}{dF} = \frac{1}{F} \]

and

\[ \frac{d^2Y}{dF^2} = -\frac{1}{F^2} \]

By Ito’s Lemma we know:

\[ dY = F\sigma \frac{dY}{dF}dX + \frac{d^2Y}{dF^2}(F\sigma)^2 dt \]

Substituting for the first and second derivatives of \( Y \) and the definition of \( dF \) into the above

\[ dY = \frac{1}{F} F\sigma dX - \frac{1}{F^2} (F\sigma)^2 dt \]

Simplifying:

\[ dY = \sigma dX - \sigma^2 dt \]

Integrating both sides:

\[ Y = \int_0^t \sigma dX - \int_0^t \sigma^2 dt + Y_0 \]

\[ Y = \sigma X(t) - \sigma(t)^2 t + Y_0 \]
Finally we take $F = e^Y$

$$F(t, T) = F(t, t)e^{(\sigma_X(t) - \sigma(t)^2)t}$$

By definition

$$s(t) = F(t, t)$$

Thus

$$F(t, T) = s(t)e^{(\sigma_X(t) - \sigma(t)^2)t} \blacksquare$$
Appendix B: Stochastic Spot Price Derivation

In this appendix a proof is provided to show the link between the mean-reverting stochastic futures curve model:

\[
\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\eta(t-T)}dX
\]

and the corresponding stochastic spot price model:

\[
\frac{dS}{S} = \left[ y(t) + \eta (\ln(F(0,t)) - \ln(S)) \right] dt + \sigma dX
\]

It is also shown that when reversion is not present the GBM spot price model below is obtained:

\[
\frac{dS}{S} = y(t)dt + \sigma dX
\]

**Proof:**

Given:

\[
\frac{dF}{F(t,T)} = \sigma e^{-\eta(T-t)}dX
\]

Then:

\[
F(t,T) = F(0,T)\exp\left(-\frac{1}{2} \int_0^t \sigma^2 e^{-2\eta(T-S)}dS + \int_0^t \sigma e^{-\eta(T-S)}dX(S)\right)
\]

By definition the spot price, \(S(t)\), is the immediately expiring futures price when \(t = T\) thus:

\[
S(t) = F(t,T) = F(0,t)\exp\left(-\frac{1}{2} \int_0^t \sigma^2 e^{-2\eta(t-S)}ds + \int_0^t \sigma e^{-\eta(t-S)}dX(S)\right)
\]

Taking logs of both sides:

\[
\ln S = \ln F(0,t) - \frac{1}{2} \int_0^t \sigma^2 e^{-2\eta(t-S)}ds + \int_0^t \sigma e^{-\eta(t-S)}dX(S)
\]

Differentiating:

\[
\frac{dS}{S} = \left( \frac{\partial \ln F(0,t)}{\partial t} - \eta \int_0^t \sigma^2 e^{-2\eta(t-S)}ds - \eta \int_0^t \sigma e^{-\eta(t-S)}dX(S) \right) dt + \sigma dX(S)
\]
Therefore:

\[
\frac{dS}{S} = \left( \frac{\partial \ln F(0,t)}{\partial t} + \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 e^{-2\eta t} - \eta \int_0^t \sigma e^{-\eta(t-s)} dX(S) \right) dt + \sigma \partial X(S) \tag{2}
\]

From (1) we know that:

\[
\int_0^t \sigma e^{-\eta(t-s)} dX(S) = \ln S - \ln F(0,t) + \frac{1}{2} \int_0^t \sigma^2 e^{-2\eta(t-s)} dt
\]

\[
= \ln S - \ln F(0,t) + \frac{\sigma^2 e^{-2\eta t}}{4\eta} \bigg|_0^t
\]

\[
= \ln S - \ln F(0,t) + \frac{\sigma^2}{4\eta} - \frac{\sigma^2 e^{-2\eta t}}{4\eta}
\]

Substituting the above into (2):

\[
\frac{dS}{S} = \left( \frac{\partial \ln F(0,t)}{\partial t} + \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 e^{-2\eta t} - \eta \left( \ln S - \ln F(0,t) \right) - \frac{\sigma^2}{4} + \frac{\sigma^2 e^{-2\eta t}}{4} \right) dt + \sigma \partial X(S)
\]

Define:

\[
y(t) = \frac{\partial \ln F(0,t)}{\partial t} + \frac{1}{4} \sigma^2 (1 - e^{-2\eta t})
\]

Then:

\[
\frac{dS}{S} = \left( y(t) + \eta \left( \ln F(0,t) - \ln S \right) \right) dt + \sigma \partial X(S)
\]

∴ When \( \eta > 0 \) spot price follows the standard mean reversion model

When \( \eta = 0 \):

\[
\frac{dS}{S} = y(t) dt + \sigma \partial X(S)
\]

∴ When \( \eta = 0 \) spot price follows the standard Geometric Brownian motion (GBM) model ■
Appendix C: A note on taxation

The cut-off grade optimization and real-options valuation in Chapter 3 is done on a pre-tax basis. This is necessary as the numerical method used to solve the system of PDE’s is an application of Dynamic Programming. This technique is recursive, working from the final time period backwards towards the first. Due to this, calculations of the project value at time period t cannot be influenced by what has occurred from time periods 0 through t -1, except for those occurrences that solely impact the state variables.

There are two state variables in the model described in Chapter 3; $X$ and $I$. $X$ represents the stochastic component of the commodity price while $I$ represents the current location within the mine plan, that is, how much has been excavated and what is the current phase. Decisions such as prior cut-off grade choices and previous closures can be fully captured in their impacts on $I$ and associated instantaneous costs.

Detailed taxation modeling introduces several complexities that cannot be captured using the existing state variables. For instance, depreciation available in a given time period is highly dependent on decisions made in prior periods. For example if the mine was temporarily closed in the prior year, it would likely have built operating losses which it could use to offset taxes in the current period. Even regular depreciation, associated with known capital expenditures is highly dependent on what happened in prior periods. The commodity price realized in previous periods will have impacted profits which may in-turn impact the amount of carried loss available for use in the current period. The reader can likely imagine a number of other such issues associated with various tax schemes in different jurisdictions.
This inability to account for the complexities of tax in the optimization does not mean that they cannot be incorporated in the valuation. An after tax valuation can be conducted using a Monte Carlo simulation which makes use of the already determined optimized cut-off grades and closure/reopening decisions, while including a fully modeled tax schedule. The value determined from this simulation can be considered an after tax Real Option valuation.

Readers should be aware that optimizing cut-off grade before tax is consistent with current deterministic practices, see for instance the Detour Lake technical report (BBA, 2011). By optimizing cut-off grade and closure decisions on a before tax bases, the objective function maximizes value to both private owners and public stakeholders. If the cut-off optimization were conducted on an after tax basis it may attempt to game the system and move production around in order to minimize taxes paid and maximize profits to private owners. This could be viewed as inefficient use of the resource from the perspective of public stakeholders.
Appendix D: Reflections on Real Options

The following is a discussion of some of the existing concerns with real options analysis. In order to address these concerns in the context of mining asset valuation, discussions are provided on the topics of parameter estimation and the differences between valuation and speculation.

Criticisms of Real Options

It is difficult to accurately assess the degree of utilization of the Real Options but it is clear from industry surveys that it is not a leading technique. In a 2001 study conducted by Bain Company 451 senior executives were polled regarding their views on management techniques. Only 9% reported the using RO while a third of the managers had stated they had abandoned the technique (Rigby, 2001).

To better understand why the technique was so poorly adopted, Block solicited his own survey in 2007 (Block, 2007). Block surveyed 279 executives from Fortune 1,000 companies of which only 40 were using real options. For the 239 respondents who reported not using real options the questionnaire provided five choices to explain why not. The results from this particular question are provided in Table 18.

Table 18: Reasons for not using real options (Block, 2007)

<table>
<thead>
<tr>
<th>Reason</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lack of top management support</td>
<td>42.7%</td>
</tr>
<tr>
<td>Discounted cash flow is a proven method</td>
<td>25.6%</td>
</tr>
<tr>
<td>Requires too much sophistication</td>
<td>19.5%</td>
</tr>
<tr>
<td>Encourages too much risk taking</td>
<td>12.2%</td>
</tr>
<tr>
<td></td>
<td>100.0%</td>
</tr>
</tbody>
</table>

Due to the significance of the first concern Block used the additional comments from the participants to understand why management did not support the technique. From the comments, it
appeared that most managers were not comfortable using methodologies that they could not follow step by step. Block suggested that managers may fear being removed from the decision making loop.

As for the second explanation it is difficult to determine if in fact DCF analysis does produce correct decisions. Many managers are aware of projects such as research and development initiatives that had a negative NPV in initial analysis but were approved and provided significant gains for the company. Conversely many projects that appeared to have a positive net present value ended up losing money. What is very concerning is that DCF analysis provides no information on the probabilities of success or failure or any information on how to hedge the various project risks.

When considering concerns relating to sophistication, it appears that the greatest utilization of real options is in highly technical industries such as technology, energy, utilities, etc. For instance in Triantis and Borison’s 2001 survey of 34 companies that were using RO, 15 were in the energy sector and another four were in the Transportation sector (Trantis & Borison, 2001). Block suggested that management in these highly technical industries are more likely to have engineering or technology backgrounds. These managers may be more mathematically sophisticated. Testing this hypothesis showed that managers with technical backgrounds are more likely to use RO with a 0.05 level of significance.

The final reason for rejecting RO is one of either application or perception. It is unclear how DCF analysis accurately identifies which projects are overly risky. DCF provides only a single deterministic view of a project and does not consider other possible outcomes. Capital budgeting theory states that if access to capital is not scarce, then management should select all projects
with a positive NPV, however few managers are willing to accept projects that don’t at least pass a minimum threshold NPV. Similar requirements could be placed on the results from real options analysis. In fact, since real options models generate more information, more sophisticated requirements can be applied to project decisions. For instance, a maximum acceptable probability of loss could be implemented or a 95% probability minimum return requirement. By implementing these types of considerations on real options analysis management could avoid “overly risky” projects, allowing for almost any definition of “overly risky”. Of course management must first understand real options before they can determine appropriate requirements.

It is the opinion of the author that the most valid and important criticism of Real Options is absent from the aforementioned management surveys. In most cases real options requires the estimation of parameters that are very difficult to determine. These parameters are related to the stochastic processes that drive the uncertain factor(s). For instance it is difficult to predict the probabilities associated with progress and success rates of research into new pharmaceutical products or the adoption rates of new consumer products in foreign markets. Determining the parameters for stochastic models of these inputs would require a great deal of data analysis and subjective input.

However in the context of a project that produces an exchange traded commodity or commodities, which includes the vast majority of mining projects, these parameters do not need to be estimated. Futures and options markets exist for the underlying commodities and provide a wealth of information on the market’s current estimation of these parameters. The existence of these markets completely eliminates the need to estimate these otherwise difficult to determine parameters. A discussion on determining values for input parameters follows.
Parameter Determination

The vast majority of the parameters required for the real option model presented in Chapter 3 are the same factors required for a DCF model of a mining asset. These include parameters such as operating costs, geological reserves, ore grades, processing recovery, mining and processing rates, discount rate, etc.. There are a number of sources that provide methods for determining values for these inputs which are not the primary concern of this discussion. As compared to a standard DCF analysis, the real options model introduces a stochastic commodity price model which is defined by two parameters, expected price $F(0,t)$ and volatility $\sigma(t)$. What follows is a discussion on assigning values to these two inputs.

Expected future price is the mean outcome price for any given point in time. Currently expected price is estimated by mine valuators for use in DCF models and is often it is referred to as long term average price. The methods used to determine expected price vary, however any of these methods could be used to estimate an expected price for a stochastic price model. Readers should be warned that using any of the current methods for determining expected price result in an analysis being a speculation and not a valuation. Further discussion on this distinction is provided later in the chapter. In order to ensure that valuation of a mining project is consistent with the complete market theory, the futures curve of the commodity should be used to determine the expected future price.

Volatility represents the degree of uncertainty or randomness in the commodity price. Volatility can be determined a number of ways. One method would be to look at historical volatility of a commodity price and infer future volatility. Another method would be to use confidence intervals on expected price to imply volatility. For example if management estimates the commodity price

14 Interested readers are directed to (Hustruulid & Kuchta, 2006)
in years’ time to be $1000 +/- 20%, 19 times out of 20 this suggests a standard deviation of 10% or a volatility of 1%. However the reader should be aware that again the above methods for determining volatility result in any analysis being speculation. In order to ensure that an estimate of volatility is consistent with existing market information, volatility should be implied from the price of options on commodity futures.

**Valuation versus Speculation**

In the literature on mining finance and economics, it is clear there exists a deal of confusion regarding the difference between valuation and speculation. Valuation is the determination of a fair price to pay for an asset today, given all current market information. This is also referred to as a risk neutral valuation. Speculation is the determination of a value of an asset, given a certain perspective and set of assumptions that may differ from the current market perspective.

For an asset to be fairly valued in a complete market, it must be priced so that no opportunities for arbitrage exist. The principle of no-arbitrage is central to the efficient market hypothesis and general market equilibrium theory. An arbitrage opportunity is one in which an asset or portfolio of assets can be replicated with another asset or portfolio of assets and the market value of each of these portfolios is different.

Imagine I possess a 1 oz gold bar that I must sell today, what is the value of the gold bar? Few people would argue that the fair value is today’s spot price. If the spot price is $1,000 the only amount of money I can reasonable expect to receive for selling my gold bar is $1,000\(^{15}\). I may personally feel that price is too low, but if I must transact today then I have to be willing to accept the market price. If I was able to sell the gold bar for more than the spot price say $1,050, then I

---

\(^{15}\) This example assumes no transaction costs as a simplification. The argument still holds if transaction costs are added however the numbers will change.
could purchase a large amount of gold on the open market at $1,000 and sell it right away at $1,050, making a risk free profit. This pricing discrepancy is referred to as an arbitrage opportunity. If I started exploiting this price differential other market participants would catch on and start buying and selling gold bars as well. This activity would start to drive up the purchase price of $1,000 and drive down the selling price of $1,050 until they were equal and the arbitrage opportunity disappeared.

Now assume instead of having a gold bar that I will sell today I now have a gold bar that is difficult to access and it will take me a year to get the gold bar from storage to market. What is the value of this gold bar today? Using the traditional approach of a mining company, we would first estimate what the spot price will be in a year, say $900 and then discount that revenue at an appropriate discount rate, say 5%, which would be $857. Is this the fair value of the gold bar today? No, this is a speculative value based on my estimate of future spot price.

Gold along with most other commodities have highly active futures markets. A commodity futures contract is simply an obligation to deliver a set quantity of a commodity on a set day for a set price. Returning to our example, suppose the price of a one-year gold, futures contract is $1,100 per oz.\(^{16}\) This means that the market is willing to commit to pay $1,100 for an oz of gold in a year’s time. That is, the market value today of my difficult to access gold bar is $1,100 discounted at the correct discount rate. Even if I personally believe the spot price will be $900 in a year, I would be irrational to commit to $900 in a year’s time, if the open market is offering a price of $1,100. What if the opposite were true, what if I believed the spot price a year from today will be $1,500? If I value you my gold at $1,500 I am simply speculating. I would be unable to

\(^{16}\) Readers should be aware that a futures price can either be higher than the current spot, contango, or lower than current spot, backwardation.
find a buyer to commit to $1,500 if the open market is only willing to commit to a price of $1,100. In fact, if I believed that the future spot price of gold will be higher than the current futures price, I should buy futures contracts today at $1,100 and wait a year's time and sell the forthcoming gold for $1,500. It should be clear that if I were sign a contract to deliver a year from now for any price other than the futures price, an arbitrage opportunity would be created and market participants would exploit this.

Valuing the revenue from the sale of a commodity at anything other than the current futures price amounts to speculation as it is incongruent with current market information. If a transaction were to be agreed upon today at any price other than the futures price, an arbitrage opportunity would be created, allowing market participants to make a risk free profit. Despite this simple and fundamental economic principal, mining companies continue to “value” the future revenue from mining operations at estimated future spot prices that are generally lower than the current futures price. This practice amounts to speculating on these commodity revenues but is generally accepted because the use of a lower long-term commodity price, provides a conservative speculation on what future revenues are worth today. It is likely clear to readers that if mining executives actually believed their future spot price predictions were at all accurate, they would be active traders in the futures market in order to exploit their superior knowledge to the market regarding future commodity prices.