The Volume of Black Holes

by

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Abstract

The invariant four-volume ($\mathcal{V}$) of a complete four-dimensional black hole (the volume of the spacetime at and interior to the horizon) diverges. However, if one considers the black hole resulting from the gravitational collapse of an object and integrates only a finite time to the future of the collapse, the resultant volume is well-defined and finite. We show that for non-degenerate black holes, the volume in this case can be written as $\mathcal{V} \propto \ln|\lambda|$, where lambda is the affine generator of the horizon and we define our volume $\mathcal{V}^*$ to be the constant of proportionality. In spherical symmetry, this is the Euclidean volume divided by the surface gravity (κ).

More generally, it turns out that $\mathcal{V}^*$ is the Parikh volume (${}^3\mathcal{V}^*$), as defined in [1], divided by κ. This allows us to define an alternative local and invariant definition of the surface gravity of a stationary black hole. It also encourages us to find a generalization of the Parikh volume (which depends on the existence of an asymptotically timelike Killing vector) to any region of space or spacetime of arbitrary dimension, provided that this space or spacetime contains a Killing vector. We find some properties of this generalized "Killing volume" and rewrite our volume as a Killing volume for a particular Killing vector.

We revisit the laws of black hole mechanics, considering them in terms of volumes rather than areas, by writing out our volume and the Parikh volume of Kerr-Newman black holes and then considering their variation with respect to the parameters M, J and Q to find a modified BH mechanics first law. We also use our new definition of κ to develop an alternate demonstration of the BH mechanics third law. We note that the Parikh volume of a Kerr-Newman black hole is equal to $Ar_+/3$, where A is the horizon surface area and r_+ the value of the radius at the horizon, and we offer some interpretations of this relationship. We review some other relevant work by Parikh as well as some by Cvetič et al. [2] and by Hayward [3]. We point out some possible next steps to follow up on the work in this thesis.

Co-Authorship

My supervisor, Dr. Kayll Lake, co-authored some of this work, in particular content posted to the arXiv as [4].

Acknowledgments

I'd like to thank my supervisor, Dr. Kayll Lake, for his insight, encouragement and patience, starting with showing an interest in me (with an Engineering Physics background) in the first place and continuing through the suggestion of this project and support throughout it. The other members of my research group, Cédric Grenon, Majd Abdelqader, and especially my current officemate Dmitri Lebedev, have been a great boon to my work. Many thanks also to the faculty, especially Dr. Stéphane Courteau and Dr. Jordan Morelli for their advice and friendship.

Thanks also to my mother Anne Shannon and finally to Rosemary Nolan. I don't want to know what the process of writing this would have been without you.

Portions of this work were completed using GRTensor [5].

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Nomenclature

We adopt many of the conventions from Poisson [6]. In some cases we keep the dimensionality D of the spacetime general; sometimes we work in specific 4-dimensional spacetimes, in which obviously $D = 4$. Greek indices $(\alpha, \beta, ...)$ run from 0 to $D - 1$. Lower-case Latin indices $(i, j, ...)$ run from 1 to $D - 1$. Upper-case Latin indices $(A, B, ...)$ run from 2 to $D - 1$. Geometrized units, in which $G = c = 1$ (where G is Newton's gravitational constant and c is the speed of light) are employed. Throughout this thesis we use abstract index notation, wherein if a given symbol appears as both a covariant and contravariant index we sum over all possible values for that symbol. For example, the quantity $T^{\alpha\beta}T_{\alpha\gamma}$ is interpreted as $\sum_{\alpha=0..D-1} T^{\alpha\beta}T_{\alpha\gamma}$.

The following should be taken as a general guide only. Sometimes the symbols that follow have slightly different meanings than the most common ones given here, and not every symbol used in the thesis is included here; let the text of each chapter or section be your guide when there is any confusion. The most common symbols are included in Table 1. The next table, Table 2, is split into three sections (one identifying notation for different types of volume, one identifying common Roman symbols, and one including common Greek and other symbols) and lists less common symbols as well as their first section.

Symbol	Common Meaning
$x^\alpha, \tilde{x}^\alpha$	Coordinates in the spacetime.
D	Number of dimensions in spacetime (or space). We often work in $D = 4$.
$g_{\alpha\beta}$	Metric tensor (covariant).
g, g_D	Determinant of $g_{\alpha\beta}$.
g_{D-1}	Determinant of g_{ij}.
$\psi_{,\alpha} = \partial_\alpha \psi = \frac{\partial\psi}{\partial x^\alpha}$	Partial differentiation of ψ with respect to coordinate x^α.
$\psi_{,y} = \partial_y \psi = \frac{\partial\psi}{\partial y}$	Partial differentiation of ψ with respect to variable y.
$A^\alpha_{;\beta} = \nabla_\beta A^\alpha$	Covariant differentiation of A^α with respect to x^β
$\mathcal{L}_u A^\alpha$	Lie differentiation of A^α with respect to vector u^α
$\mathrm{d}\Omega_2^2$	Line element of unit two-sphere ($= \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$).
δ^α_β	The Kronecker delta symbol, which is +1 if $\alpha = \beta$ and 0 otherwise.

Table 1. Some of the most common symbols used throughout this thesis.

Symbol	Common Meaning	Ch./Sect. of First Use
	Volume Notation	
$\mathcal{V}, \mathcal{V}_\mathcal{R}$	Full D-dimensional volume of a region (region $\mathcal{R}$ if subscript specified).	3

Table 2. Miscellaneous symbols.

Symbol	Common Meaning	Ch./Sect. of First Use
$\mathcal{V}^* = \mathrm{d}\mathcal{V}/\mathrm{d}\ln\vert\lambda\vert$	Our volume: the derivative of the volume of the black hole with respect to the logarithm of the affine parameter on the null generators of the horizon.	3
${}^3\mathcal{V}_E$	Euclidean 3-volume of a region.	3
$\mathcal{V}_s, \mathcal{V}_s^*$, etc.	The subscript s corresponds to a radial "shell" s for which $r_1 < r < r_+$.	4.3
$\mathcal{V}_{s'}, \mathcal{V}_{s'}^*$, etc.	The subscript s' corresponds to a "shell" s' bounded by outgoing radial null geodesics $0 < u < u_0$.	4.5
${}^3\mathcal{V}'$	Constant of proportionality between $\mathcal{V}^*$ and the surface gravity inverse in Kerr-Newman case.	5
${}^{D-1}\mathcal{V}^* = \mathrm{d}\mathcal{V}/\mathrm{d}T$	The Parikh volume: the derivative of the volume of the black hole with respect to the time coordinate.	6.1
${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}$	The generalized Parikh or Killing volume of a region $\mathcal{R}$ w.r.t. Killing vector ξ^α.	7
${}^{D-1}\mathcal{V}$	$(D-1)$-dimensional volume of a $(D-1)$ dimensional hypersurface region.	7.3.1

Roman Character Symbols

A	Black hole surface area.	11

Table 2. Miscellaneous symbols.

Symbol	Common Meaning	Ch./Sect. of First Use
a	Depending on context: 1. The value of radius r at event horizon on a Schwarzschild black hole. 2. J/M rotational parameter in Kerr black hole.	3; 5
$C_{\alpha\beta\gamma\delta}$	Weyl conformal tensor.	5.3
E	Energy.	12
$f(r)$	$-g_{tt}$ in spherical symmetry.	4
$\hbar$	Planck's reduced constant.	12
J	Depending on context: 1. Jacobian. 2. Black hole angular momentum.	7.3.1; 12
k^α	The vector field with $k^\beta k^\alpha_{;\beta} = k^\alpha$ on a BH horizon.	9.1
M	Black hole mass.	3
n_α	Unit normal (if timelike or spacelike), unnormalized normal (if null) of a hypersurface.	7.1
Q	Black hole charge.	5.3
$\mathcal{Q}$	Heat.	12
$\mathcal{Q}_\Phi(\lambda)$	A region useful in defining a Killing volume.	7.2.3
$\mathcal{R}$	A D-dimensional region, often the black hole region.	3
r	Radial coordinate. Aerial radius in spherical symmetry.	3
r_+	The value of r at the outer (event) horizon of a black hole.	4
S	Entropy.	12

Table 2. Miscellaneous symbols.

Symbol	Common Meaning	Ch./Sect. of First Use
$t, T, t_s \ldots$	Time coordinates.	3
T	Temperature.	12
u, v	Set of double-null coordinates.	3
V	Ingoing null coordinate.	5
W	Work.	12
$\mathcal{W}(x)$	Lambert-W function.	5

Table 2. Miscellaneous symbols.

Symbol	Common Meaning	Ch./Sect. of First Use
	Greek Symbols	
$[\alpha\,\beta\,\gamma\,\delta]$	Permutation symbol.	5.3
Γ	Used to represent $(D-1)$-dimensional hypersurfaces.	3
$\Gamma^{\alpha}_{\beta\gamma}$	Christoffel symbols constructed from $g_{\alpha\beta}$	7.2.2
Δ	$r^2 + a^2 - 2Mr + Q^2$ in Kerr-Newman.	5.3
$\epsilon_{\alpha\beta\gamma\delta}$	Levi-Civita tensor ($= \sqrt{-g}[\alpha\,\beta\,\gamma\,\delta]$).	5.3
θ	Inclination angle.	3
κ	Surface gravity of a black hole.	3
Λ	Cosmological constant.	4.4
λ	Parametrization of a curve.	3
μ	Alternate parameter.	4.7
$\xi^{\alpha}, \zeta^{\alpha}, \psi^{\alpha}, \omega^{\alpha}...$	Commonly used symbols for Killing vectors.	7.1
ρ	$r^2 + a^2\cos^2\theta$ in Kerr-Newman.	5.3
Σ	$(D-1)$-dimensional region.	6.1
$\mathrm{d}\Sigma_{\alpha}$	Directed surface element.	7.2.2
τ	Proper time.	6.1
ϕ	One azimuthal angle in space with azimuthal symmetry.	3
ψ	An azimuthal angle in Kerr-Newman.	5.3
Ω_H	Black hole angular speed in Kerr-Newman.	8.4

Table 2. Miscellaneous symbols.

Chapter 1

Introduction

The purpose of this thesis is to discuss the importance of the rate of growth of the invariant volume ($\mathcal{V}$) of stationary non-degenerate black holes and its relationship to black hole surface gravity and black hole mechanics. For the most part we work in four-dimensional spacetime, for which this volume is a four-volume, though in some sections we include generalizations to higher dimensions. It is well known (see, for example, Poisson [6]) that, following Bardeen, Carter and Hawking [7], the "surface area" of black holes (A) is of physical interest, as it is non-decreasing through classical (non-quantum) processes and is thus generally associated with the black hole entropy. In this thesis we attempt to determine whether there is a *meaningful, invariant* definition of volume which might be of comparable physical interest. The naive volume one might associate with black holes, the Euclidean three-volume, does not take into account the non-Euclidean geometry of space. The three-volume of a black hole, correctly calculated, depends on the choice of slicing [8]. Whereas these three-volumes are in general finite, the full four-volume ($\mathcal{V}$) is almost never discussed because it is formally infinite. For example, consider the four-volume of the region on and below the future horizons in the Kruskal-Szekeres plane. Even if we consider the black hole formed by the gravitational collapse of an object and so consider only the part of the Kruskal-Szekeres plane on and below the future horizon and to the future of the boundary surface of the collapsing object, the four-volume still diverges as we integrate to the infinite future. However, we can integrate to a finite future instead of an infinite future and thus consider the evolution of $\mathcal{V}$.

To begin, we examine explicitly in regular coordinates the four-volume bounded by the horizon, the central singularity and two distinct ingoing null cones in the Schwarzschild spacetime. This situation is shown in Figure 1.1. This introductory calculation points out a relation which we eventually show is a universal feature of stationary, non-degenerate black holes: that the invariant 4-volume $\mathcal{V}$ of a black hole grows as $\mathcal{V} \propto \ln|\lambda|$ where λ is the affine generator of the horizon. We identify the constant of proportionality as "our volume," $\mathcal{V}^*$. In spherically symmetric cases we note that $\mathcal{V}^*$ is the Euclidean three-volume divided by the surface gravity of the horizon.

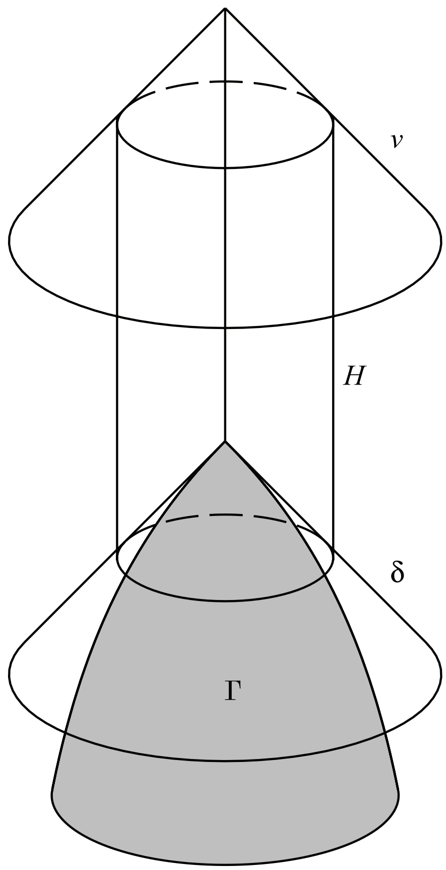

Figure 1.1. The collapse of a timelike boundary surface Γ that terminates at the central singularity simultaneously with the null cone δ and produces a black hole with horizon $\mathcal{H}$. The null cone v is any null cone to the future of δ. The invariant four-volume $\mathcal{V}$ calculated here is bounded by δ and v and is to the interior of $\mathcal{H}$.

More investigation finds that the constant of proportionality is the volume developed by M. Parikh [1] (which we identify as ${}^3\mathcal{V}^*$), divided by the surface gravity. This is consistent with the fact that the Parikh volume in spherical symmetry is simply the Euclidean 3-volume. This encourages us to examine the Parikh volume in greater detail. We generalize the Parikh volume to any space or spacetime with a Killing vector, so that we can write a more general "Killing volume" as a function of a specific region in a space or spacetime and a specific Killing vector. We then consider how the Killing volume is affected by multiplying the Killing vector by a constant and by taking the sum of two linearly independent Killing vectors. This allows us to rewrite both our volume and Parikh's volume as specific cases of a more general Killing volume of black holes, with different Killing vectors.

We use the relationship between our volume and the Parikh volume to propose an alternative definition of the surface gravity κ. In the four-dimensional case, defining $\mathcal{V}^* \equiv \mathrm{d}\mathcal{V}/\mathrm{d}\ln|\lambda|$, we have an alternative definition of the surface gravity: $\kappa = {}^3\mathcal{V}^*/\mathcal{V}$. We additionally develop expressions for the 4-volume growth rate of shells which terminate at the horizon, which show a similar relationship with the Parikh volume of the shell and the surface gravity, revealing that the relationship of these volume rates to the surface gravity is a property of the horizon alone.

Armed with an invariant definition of the volume of black holes, we revisit the laws of black hole mechanics. These laws, first formulated in the 1970's by Hawking, Bardeen and Carter, express the relationship between the "surface area," mass, charge and angular momentum of the black holes, and imply an equivalence between the surface area and entropy of the black holes, up to a multiplicative constant. By analogy with these area relations, we consider the relationship between first our volume and then the Parikh volume and the mass, charge and angular momentum of a Kerr-Newman black hole to reformulate the first law of black hole mechanics and to speculate on alternative analogies with classical thermodynamics. We show that the Parikh volume is equivalent to both a geometrical volume defined by Cvetič et al. [2], which in some cases has interesting thermodynamic properties, and a volume used by Hayward [3].

In addition to this, we point out several curious properties of the black hole volume(s) and their relationship with the radial coordinate, the surface area and the surface gravity in stationary black holes. Finally, we point out several lines of inquiry that could be interesting for further exploration.

Chapter 2

Literature Review

The volume of black holes, or of regions of spacetime generally, is a field that remains relatively ill-studied. Very little background is required to read this thesis, and the handful of very important results in the literature are discussed in detail within the body of the thesis.

The basic tools for calculating volumes of regions in spacetime are well known, and come from differential geometry. The important results for our work are summarized in Poisson [6] (chapters 1 and 3). The N-dimensional volume element of an N-dimensional region or hypersurface which can be expressed with coordinates x^α and a metric tensor $g_{\alpha\beta}$ (such that $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$) is $\mathrm{d}^N\mathcal{V} = \sqrt{|g|}\mathrm{d}^N x$, or the square root of the determinant of the metric multiplied by the product of the coordinate differentials. In a full D-dimensional Riemannian manifold, the D-dimensional volume of some region is invariant under a change of coordinates. The $(D-1)$-dimensional volume of a given D-dimensional region, which in four-dimensional spacetime corresponds to the three-dimensional volume, on the other hand, is calculated by measuring the volume of a hypersurface—and so depends on the choice of hypersurface.

Very little work has been done on the volume of black holes because the three-volume of a black hole depends on the choice of slicing and the full four-volume is formally infinite. As a result, few have ventured to attempt to ascribe meaning to black hole volumes. DiNunno and Matzner worked out explicitly the three-volume for the Schwarzschild black hole for many different choices of slicings in [8]. This was written in 2008, and included this comment on the state of the literature of (three-dimensional) volumes for the Schwarzschild black hole: "Elucidating these results for the volume provides a new pedagogical resource of facts already known in principle to the relativity community, but rarely worked out."

The main result from the literature—and one of the main goals of attempting to work out a meaningful definition for the volumes of black holes—is the laws of black hole mechanics, as (first) formulated by Bardeen, Carter and Hawking [7]. An introduction of the essential features of these laws, which are formulated in terms of the black hole surface area A, is included in Chapter 12. One of the results of the laws of black hole mechanics is that the entropy S of a black hole is proportional to the area and not to the black hole volume.

There has been an increase of activity in the field of black hole volumes in the past decade. The most significant result in the literature is from Maulik Parikh [1] in 2005 (on the arXiv) and 2006 (published), who develops a method for defining an invariant volume definition for a stationary spacetime. This method is examined in detail in Chapter 6, which reviews Parikh's method and connects it to our results. Chapter 7 generalizes Parikh's result further. In summary, Parikh defined a constant volume for stationary black hole regions which depends on the (asymptotically timelike) Killing vector. After having defined this, Parikh attempted to find cases of black holes which have infinite Parikh volume and finite area, and found no such cases. Later that same year, Daniel Grumiller calculated the Parikh volume for 2-dimensional black holes (a case Parikh did not examine) [9] and similarly found no cases of black holes with infinite Parikh volume and finite area.

Another series of recent developments involves considering Kerr-Newman black holes embedded in anti-de Sitter spacetimes. In particular, the work of M. Cvetič, G.W. Gibbons, D. Kubizňàk and C.N. Pope [2] is discussed at length in Section 14.2; some other papers along similar lines are two papers by Dolan [10, 11]. The suggestion by Cvetič et al. is the following. It is possible in (four-dimensional) Kerr-Newman-de Sitter to write the mass M in terms of the area A of the event horizon as well as the angular momentum J, the electrical charge Q, and the cosmological constant Λ. (Cvetič et al. also work in higher dimensions.) The "conjugate" Θ to Λ is defined by the partial derivative of M with respect to Λ, with A, J and Q held constant:

$$\Theta = \left. \frac{\partial M}{\partial \Lambda} \right|_{A,J,Q} . \tag{2.0.1}$$

In black hole mechanics, the mass M is often interpreted as a thermodynamic energy and Λ (up to a constant) as a pressure. As a consequence, Cvetič suggests interpreting Θ as a multiplicative constant times a volume term. This volume turns out in certain cases (though Cvetič doesn't note this) to be the Parikh volume.

It may be, however, that the apparent lack of activity in black hole volumes until recently is simply because ideas about black hole volumes were sometimes examined but only as an aside as part of a broader work. For example, a paper by S. Hayward from 1998 [3], and some of his subsequent work (such as [12] among many others) includes, somewhat incidentally to the central argument, a definition of a particular volume that also turns out to be equivalent to the Parikh volume in certain situations. Hayward develops laws of black hole dynamics, valid in dynamic spherical symmetry, noting a variation law along the trapping horizon of a dynamic black hole which relates the Misner-Sharp energy to the black hole's area and the Euclidean volume of the black hole, which is defined in a way that makes its connection to the Parikh volume clear. This is discussed further in Section 14.3.

The most useful resources as general reference for the line elements of spacetimes, for Gauss' law, for information about Killing vectors, and other general information are Eric Poisson's *The Relativist's Toolkit* [6], Misner, Thorne and Wheeler's *Gravitation* [13], and Wald's *General Relativity* [14].

Chapter 3

The Four-Volume of a Schwarzschild Black Hole

For definiteness, in the first several sections we will work with four-dimensional spacetime only. The first volume calculation was done for the black hole of the Schwarzschild spacetime in four dimensions. In conventional coordinates, (t, r, θ, ϕ), the line element is

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2, \tag{3.0.1}$$

where $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the metric for the two-sphere. These coordinates are unfortunately not regular on the horizon. It has been shown elsewhere [15, 16, 17] that the Kruskal-Szekeres form of the Schwarzschild metric can be written as:

$$\mathrm{d}s^2 = (2M)^2\mathrm{d}\tilde{s}^2 \tag{3.0.2}$$

where

$$\mathrm{d}\tilde{s}^2 = \frac{-4}{(1+\mathcal{W})e^{1+\mathcal{W}}}\mathrm{d}u\mathrm{d}v + (1+\mathcal{W})^2\mathrm{d}\Omega_2^2 \tag{3.0.3}$$

with $\mathcal{W} \equiv \mathcal{W}(-uv/e)$ where $\mathcal{W}$ is the Lambert-W function, which is defined by $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$ [18]. We note immediately the correspondence between the variables in (3.0.1) and (3.0.2):

$$r = 2M(1 + \mathcal{W}) \tag{3.0.4}$$

and

$$t = 2M \ln \left| \frac{v}{u} \right| . \tag{3.0.5}$$

As a result, we find that

$$uv = -\left(\frac{r - 2M}{2M}\right) \exp\left(\frac{r}{2M}\right) \tag{3.0.6}$$

and

$$\left| \frac{v}{u} \right| = \exp\left(\frac{t}{2M}\right) \tag{3.0.7}$$

We review briefly the features of the full Kruskal-Szekeres plane which is shown in Figure 3.1. We choose u and v both increasing to the future. The event horizon $r = 2M$ lies along the axes of the $v - u$ plane (where $uv = 0$). The complete Schwarzschild manifold represented by these coordinates actually contains both a black hole and a white hole region ($r < 2M$), which correspond to quadrants I and III of the $v - u$ plane ($uv > 0$), respectively, as well as two separate regions outside the holes in quadrants II and IV. However, the complete Schwarzschild manifold is not representative of a real black hole formed by gravitational collapse. In this case, the black hole only exists to the future of some timelike boundary surface representing the collapsing star. If we call this timelike boundary surface Γ, we can then interpret the Kruskal-Szekeres plane as being meaningful only to the right (to the future) of Γ. (See, for example, [6] section 5.1.2.) A representation of this situation is found in Figure 3.2.

The metric (3.0.3) is regular across all horizons. It is singular only on $\mathcal{W} = -1$, which corresponds to the black hole essential singularity at $r = 0$ in the usual form of the metric (3.0.1).

Before discussing the volume calculation, we review a few details about the Kruskal-Szekeres form of the metric. Trajectories with tangents $\mathcal{K}^\alpha = e^{\mathcal{W}}(1 + \mathcal{W})\delta^\alpha_v$ (constant $u = u_0$, θ and ϕ) are

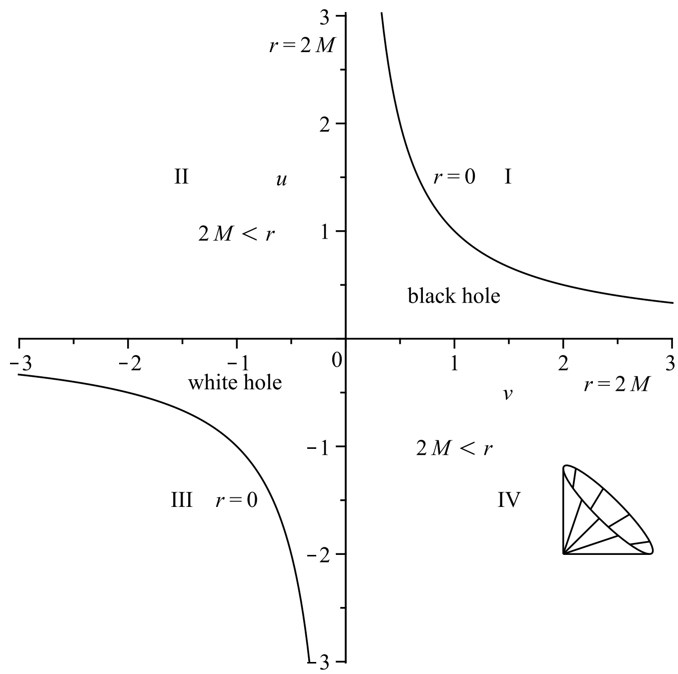

Figure 3.1. This shows the full Kruskal-Szekeres plane. The quadrants are marked. $uv = 0$ corresponds to $r = 2M$. Quadrants II and IV represent distinct regions outside the black hole. A real Schwarzschild black hole formed by spherical collapse will not have quadrants II and III at all, but will look more like that of Figure 3.2.

radial null geodesics given by

$$v(\lambda) = \lambda e^{-u_0 \lambda / e} \tag{3.0.8}$$

where λ is an affine parameter, well-defined only up to a linear transformation. The expansion is

$$\nabla_\alpha \mathcal{K}^\alpha = -\frac{-2u_0}{e(1+\mathcal{W})}. \tag{3.0.9}$$

Outside the black hole (i.e. where $r > 2M$, or where $u < 0$), r increases along these trajectories. Thus we can call these outgoing radial null geodesics. Trajectories with tangents $\mathcal{M}^\alpha = e^{\mathcal{W}}(1+\mathcal{W})\delta^\alpha_u$ (constant $v = v_0$, θ and ϕ) are radial null geodesics given by

$$u(\lambda) = \lambda e^{-v_0 \lambda / e}, \tag{3.0.10}$$

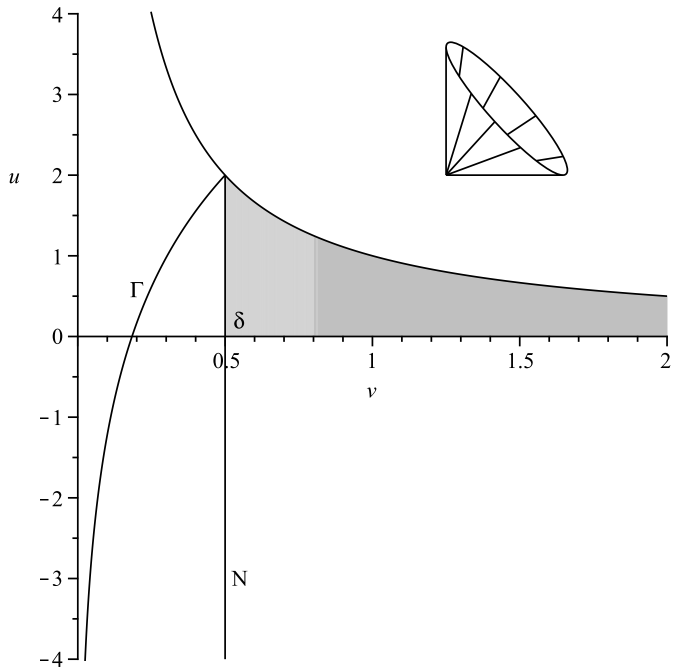

Figure 3.2. $\mathcal{R}$ for the calculation of $\mathcal{V}$. The black hole is created by the collapse of some timelike boundary Γ as in Figure 1.1. (The diagram to the left of Γ is irrelevant.) The first ingoing null geodesic to hit the vacuum singularity is labeled N and crosses the horizon at $v = \delta$.

with expansion

$$\nabla_\alpha \mathcal{M}^\alpha = -\frac{-2v_0}{e(1+\mathcal{W})}. \tag{3.0.11}$$

Both inside and outside the black hole, r decreases along these trajectories, so we can call these ingoing radial null geodesics. The result of (3.0.9) and (3.0.11) is that on the horizons $u = 0$ and $v = 0$, v and u (respectively) are affine parameters.

We now turn to calculating the volume. It is well known that the invariant four-volume over a region $\mathcal{R}$ of spacetime is given by

$$\mathcal{V} = \int_{\mathcal{R}} \sqrt{|g|}\mathrm{d}^4x \tag{3.0.12}$$

where g is the determinant of the metric, d^4x is the product of the coordinate differentials, and the integration is over $\mathcal{R}$. For the metric (3.0.2), this becomes

$$\mathcal{V} = \int_{\mathcal{R}} (2M)^4 \frac{1+\mathcal{W}}{e^{1+\mathcal{W}}} \sin\theta \mathrm{d}u\mathrm{d}v\mathrm{d}\theta\mathrm{d}\phi. \tag{3.0.13}$$

The question then becomes: what region $\mathcal{R}$ should we define? First, in our black hole represented by Figures 1.1 and 3.2, let ingoing radial null geodesics labelled by $v = \delta$ strike the black hole singularity right at the black hole formation (so that they intersect with Γ). This provides a useful “starting” surface for the region $\mathcal{R}$ which is independent of the nature of the collapse (and thus the shape of Γ) and which, given the point of intersection between Γ and the singularity, is uniquely defined. If we then let $\mathcal{R}$ be the entire region bounded by the singularity, the horizon, and $v = \delta$ (as in Figure 3.2), we have a problem: when we integrate to the infinite future, we find

$$\mathcal{V} = (2M)^4 \int_0^{2\pi} \int_0^{\pi} \int_{\delta}^{\infty} \int_0^{1/v} \frac{1+\mathcal{W}}{e^{1+\mathcal{W}}} \mathrm{d}u\mathrm{d}v \sin\theta \mathrm{d}\theta\mathrm{d}\phi, \tag{3.0.14}$$

which diverges. This means the four-volume is formally infinite. In order to define a finite quantity representing the black hole volume, we have to terminate at some point in the future. The most sensible surface would be that constructed from the set of ingoing null geodesics which intersect each other at the black hole singularity. These will be defined by some $v = v_{max}$, a constant. The four-volume of the region bounded by the singularity, the horizon, $v = \delta$ and $v = v_{max}$, is

$$\mathcal{V} = (2M)^4 \int_0^{2\pi} \int_0^{\pi} \int_{\delta}^{v_{max}} \int_0^{1/v} \frac{1+\mathcal{W}}{e^{1+\mathcal{W}}} \mathrm{d}u\mathrm{d}v \sin\theta \mathrm{d}\theta\mathrm{d}\phi, \tag{3.0.15}$$

which becomes

$$\mathcal{V} = \frac{8\pi}{3}(2M)^4 \ln\left(\frac{v_{max}}{\delta}\right). \tag{3.0.16}$$

If we make the substitution $v_{max} \to v$, then we have the expression for the four-volume of the black hole region between the surface $v = \delta$ and the surface of constant v for some future v:

$$\mathcal{V}(v) = \frac{8\pi}{3}(2M)^4 \ln\left(\frac{v}{\delta}\right), \tag{3.0.17}$$

which indicates that the four-volume grows logarithmically with v. On the horizon, $u = 0$ and so v is affine along trajectories of constant u, θ, ϕ from (3.0.9). Differentiating with respect to v and then substituting $v \to \lambda$ to make clear the affine nature of the coordinate, we have the manifestly invariant statement,

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\lambda} = \frac{8\pi}{3}(2M)^4 \frac{1}{\lambda}. \tag{3.0.18}$$

which is independent of δ. We should be careful how we interpret this derivative, as will be discussed at length in Section 4.7. λ here is an affine parameter for the null generators of the horizon. Equation (3.0.18) is equivalent to

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\lambda} = \frac{{}^3\mathcal{V}_E}{\kappa\lambda} \tag{3.0.19}$$

where κ $(= \frac{1}{4M})$ is the surface gravity and ${}^3\mathcal{V}_E$ $(= \frac{4\pi}{3}(2M)^3)$ is the Euclidean three-volume of the sphere of radius $r = 2M$. Defining

$$\mathcal{V}^* \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda} \tag{3.0.20}$$

we can rewrite (3.0.19) as

$$\kappa = \frac{{}^3\mathcal{V}_E}{\mathcal{V}^*}. \tag{3.0.21}$$

Chapter 4

General Spherical Symmetry

We can generalize this concept of volume to general spherically symmetric static metrics. The first calculation we did is presented in this section. This is method involves constructing double-null coordinates similar to the Kruskal-Szekeres coordinates. Because this construction is valid about one root at a time, it is necessary to take special care in dealing with black holes with multiple horizons. This is leads us to construct shells terminating at the horizon, as discussed in detail in Section 4.3. An alternate way to make the calculation which does not require the same care in dealing with shells is detailed in Chapter 5. The latter, which uses Eddington-Finkelstein-type coordinates, gives a smooth transition to our discussion of the Parikh volume in Chapter 6.

In coordinates (t, r, θ, ϕ), the line element for a static, spherically symmetry spacetime is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_2^2, \tag{4.0.1}$$

where $f(r)$ is a polynomial function with simple root(s) which locate the horizon(s). The metric (4.0.1) is well-known in classical general relativity since it includes the Reissner-Nordström-de Sitter class of black holes, for which [19]

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{4.0.2}$$

where M corresponds to the mass of the black hole, Q the electrical charge and Λ the cosmological constant. This form of the metric is also of interest for regular black holes (those without internal singularities), which is a subject that goes back several years. (For an extensive list of references on these topics see [20].) The interesting feature in (4.0.1), that $g_{tt}g_{rr} = -1$, has been discussed by Jacobson [21], who showed among other things that this implies the vanishing radial null-null component of the Ricci tensor.

By contrast, consider a similar metric, with line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{j(r)} + r^2\mathrm{d}\Omega_2^2. \tag{4.0.3}$$

Since f is a polynomial with simple root(s), write

$$f(r) = (r-a)h(r) \tag{4.0.4}$$

with $h(a) \neq 0$. In (4.0.1), all scalars constructed from the Riemann tensor are finite for $r > 0$ and $h \in C^2$. In (4.0.3) these scalars all diverge at $r = a$ unless $j(a) = 0$. This helps to explain the prevalence of (4.0.1). On the other hand, if $j(r)$ can be written in the form

$$j(r) = (r-a)k(r) \tag{4.0.5}$$

where $k(a)$ is finite and non-zero, then the scalars derived from the Riemann tensor do not diverge at $r = a$, indicating that metrics of the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{h}{k}\frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_2^2 \tag{4.0.6}$$

may not be so easily dismissed as unphysical, though $g_{rr}g_{tt} \neq -1$ in this case. At any rate, the main metrics under study can be written in the form (4.0.1).

Unfortunately the form (4.0.1) is defective on the horizon. It is necessary to choose new coordinates.

4.1 Regular Double-Null Coordinate Construction

As we did with the Schwarzschild black hole, we find double-null coordinates corresponding to (4.0.1). This is not the only set of coordinates which are regular, but it was in these that our first calculations were performed, and it flows smoothly from the discussion of the Kruskal-Szekeres coordinates. The construction follows [19]. We begin by defining new coordinates u and v according to:

$$2a(u)\mathrm{d}u = \frac{\mathrm{d}r}{f(r)} - \mathrm{d}t \tag{4.1.1}$$

and

$$2b(v)\mathrm{d}v = \frac{\mathrm{d}r}{f(r)} + \mathrm{d}t. \tag{4.1.2}$$

Here, $a(u)$ and $b(v)$ are as-yet unspecified functions. This leads to an implicit differential relationship between the new coordinates (u, v) and r:

$$a(u)\mathrm{d}u + b(v)\mathrm{d}v = \frac{\mathrm{d}r}{f(r)}. \tag{4.1.3}$$

This implies that r is a function of (u, v) alone. The line element in terms of (u, v, θ, ϕ) is

$$\mathrm{d}s^2 = 4f(r(u, v))a(u)b(v)\mathrm{d}u\mathrm{d}v + r(u, v)^2\mathrm{d}\Omega_2^2. \tag{4.1.4}$$

Since $g_{uu} = g_{vv} = 0$, we confirm that u and v are indeed null coordinates. Trajectories of constant (u, θ, ϕ) are radial null geodesics which we shall call "outgoing," and trajectories of constant (v, θ, ϕ) are radial null geodesics which we shall call "ingoing."

In principle we can choose any functions $a(u)$ and $b(v)$. It turns out that in the case where $f(r)$ has a simple root at $r = a$, which corresponds to a non-degenerate horizon, regular coordinates can be achieved with the choice $a(x) = b(x) = \mathcal{C}/x$, where $\mathcal{C}$ is a constant which we will later specify. We can rewrite the line element (4.1.4) as

$$\mathrm{d}s^2 = \frac{4\mathcal{C}^2 f(r)\mathrm{d}u\mathrm{d}v}{uv} + r(u, v)^2\mathrm{d}\Omega_2^2. \tag{4.1.5}$$

Equation (4.1.3) becomes

$$\mathcal{C}\left(\frac{\mathrm{d}u}{u}+\frac{\mathrm{d}v}{v}\right)=\frac{\mathrm{d}r}{f(r)}, \tag{4.1.6}$$

which when integrated gives

$$\ln|uv|+\mathcal{D}=\int\frac{\mathrm{d}r}{\mathcal{C}f(r)} \tag{4.1.7}$$

where $\mathcal{D}$ is a constant. We shall leave this integral indefinite for now.

We write $f(r)=(r-a)h(r)\ni h(a)\neq 0$. Then $1/f(r)$ has the unique partial fraction decomposition

$$\frac{1}{f(r)}=\frac{1}{(r-a)h(r)}=\frac{1}{(r-a)h(a)}+\frac{g(r)}{h(r)} \tag{4.1.8}$$

$\ni g(a)/h(a)\neq 0$. (Technically, if we chose a function $f(r)=h(r)(r-a)$ where $h'(a)\equiv \mathrm{d}h/\mathrm{d}r|_{r=a}=0$, then we would then find that $g(a)/h(a)=0$. However, these solutions are unphysical due to the behaviour of the Einstein tensor G_{α}^{β}, so we can safely ignore them.) Now we can write

$$\int\frac{\mathrm{d}r}{\mathcal{C}f(r)}=\int\frac{\mathrm{d}r}{(r-a)\mathcal{C}h(a)}+\int\frac{g(r)}{h(r)\mathcal{C}}\mathrm{d}r \tag{4.1.9}$$

$$\therefore \ln|uv|+\mathcal{D}=\frac{1}{h(a)\mathcal{C}}\ln|r-a|+\int\frac{g(r)}{h(r)\mathcal{C}}\mathrm{d}r \tag{4.1.10}$$

$$\therefore |uv|=|r-a|^{\frac{1}{h(a)\mathcal{C}}}\exp\left(\int\frac{g(r)}{h(r)\mathcal{C}}\mathrm{d}r-\mathcal{D}\right). \tag{4.1.11}$$

Now substitute (4.1.11) and $f(r)=(r-a)h(r)$ back into (4.1.5).

$$\mathrm{d}s^2=\frac{4\mathcal{C}^2(r-a)h(r)}{\pm|r-a|^{\frac{1}{\mathcal{C}h(a)}}}\exp\left(-\int\frac{g(r)\mathrm{d}r}{h(r)\mathcal{C}}+\mathcal{D}\right)\mathrm{d}u\mathrm{d}v+r^2\mathrm{d}\Omega_2^2, \tag{4.1.12}$$

where of course the $\pm$ sign comes from the removal of the absolute value signs of $|uv|$ in (4.1.11). For the metric to be regular, g_{uv} must be finite and non-zero across the horizon. All terms in (4.1.12) are finite and non-zero at $r=a$ except for the $(r-a)$ terms in the numerator and denominator. They cancel exactly only in the case where $1/(\mathcal{C}h(a))=1$, or $\mathcal{C}=1/h(a)$. Define

$$\kappa\equiv\frac{1}{2}h(a)=\frac{1}{2}f'(a). \tag{4.1.13}$$

This is, as is well known, the surface gravity of the black hole. Now scaling the coordinates by setting $\mathrm{e}^{\mathcal{D}} = a$, (4.1.12) becomes

$$\mathrm{d}s^2 = K(r)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega_2^2 \tag{4.1.14}$$

where

$$K(r) = \pm\frac{ah(r)}{\kappa^2}\exp\left(-2\kappa\int\frac{g(r)\mathrm{d}r}{h(r)}\right), \tag{4.1.15}$$

and the product uv is

$$uv = \pm\frac{r-a}{a}\exp\left(2\kappa\int\frac{g(r)\mathrm{d}r}{h(r)}\right) = \frac{f(r)}{\kappa^2 K(r)}. \tag{4.1.16}$$

The choice of sign depends on how we choose to orient the axes. Note that $r = r(uv)$.

These coordinates then have $u = 0$ or $v = 0$ at $r = a$; surfaces of constant r are hyperbolae of constant uv. From (4.1.1) and (4.1.2) we see that time is related to v/u, as

$$-\frac{\mathrm{d}u}{2\kappa u} + \frac{\mathrm{d}v}{2\kappa v} = \mathrm{d}t \tag{4.1.17}$$

$$\therefore \frac{1}{2\kappa}\ln\left|\frac{v}{u}\right| = t \tag{4.1.18}$$

$$\therefore \left|\frac{v}{u}\right| = \exp\left(2\kappa t\right) \tag{4.1.19}$$

taking the integration constant to be zero.

If f permits two horizons with simple roots, about (say) $r = a$ and $r = b \neq a$, this procedure, done about $r = a$, will yield coordinates that are regular only about that root. If we wish to find coordinates that are regular about the second root, $r = b$, it is necessary to perform the same procedure with b replacing a, which will yield coordinates regular across $r = b$ but not across $r = a$.

Now let us fix the sign in (4.1.16) in a way that is consistent with the Kruskal-Szekeres coordinates we used in Chapter 3. So we shall choose the negative sign. This implies that in the region $v > 0$, the horizon $r = a$ is represented by the axis $u = 0$, the inside of the black hole $r < a$ is represented by $u > 0$, and the outside of the black hole $r > a$ is represented by $u < 0$. Nevertheless, the choice is arbitrary, and in some situations (such as dealing with cosmological horizons) the opposite sign choice may be more useful.

Trajectories with tangents $\mathcal{K}^\alpha = -\frac{4}{K(r)}\delta^\alpha_v$ (constant $u = u_0, \theta, \phi$) are radial null geodesics with expansion

$$\nabla_\alpha \mathcal{K}^\alpha = -\frac{8u_0}{K(r)r}\frac{\mathrm{d}r}{\mathrm{d}(uv)}. \tag{4.1.20}$$

We must be careful since $K(a) = 0$. We can calculate $\mathrm{d}r/\mathrm{d}(uv)$ by noting that

$$\frac{\mathrm{d}(uv)}{uv} = \mathrm{d}\ln(uv) = \mathrm{d}(\ln u + \ln v) = \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}v}{v} \tag{4.1.21}$$

so that we can rewrite (4.1.6) (with $\mathcal{C} = 1/2\kappa$) as

$$\frac{\mathrm{d}(uv)}{2\kappa(uv)} = \frac{\mathrm{d}r}{f(r)} \tag{4.1.22}$$

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}(uv)} = \frac{f(r)}{2\kappa(uv)} = \frac{\kappa K(r)}{2} \tag{4.1.23}$$

where the second equality makes use of (4.1.16). Substituting this into (4.1.20) we find

$$\nabla_\alpha \mathcal{K}^\alpha = -\frac{8u_0}{K(r)r}\frac{\kappa K(r)}{2} = -\frac{4\kappa u_0}{r}, \tag{4.1.24}$$

which is finite provided that $r \neq 0$.

Trajectories with tangents $\mathcal{M}^\alpha = -\frac{4}{K(r)}\delta^\alpha_u$ (constant $v = v_0, \theta, \phi$) are radial null geodesics with expansion

$$\nabla_\alpha \mathcal{M}^\alpha = -\frac{8v_0}{K(r)r}\frac{\mathrm{d}r}{\mathrm{d}(uv)} = -\frac{4\kappa v_0}{r} \tag{4.1.25}$$

where the last equality is obtained by substituting in (4.1.23).

As a result, we find that on the horizons $u = 0$ and $v = 0$, v and u (respectively) are affine parameters.

The above procedure can be applied to the Schwarzschild case, and we recover Kruskal-Szekeres coordinates (up to the choice of constant in (4.1.7)).

4.2 Calculation of Volume

The determinant of the metric can be calculated from (4.1.6) (recalling $\mathcal{C} = 1/2\kappa$). We find

$$-g = \frac{f(r)^2 r^4 \sin^2\theta}{4\kappa^4 u^2 v^2} \tag{4.2.1}$$

so that our volume integral becomes

$$\mathcal{V} = \int \sqrt{|g|}\mathrm{d}^4x = \int \left|\frac{f(r)r^2}{2\kappa^2 uv}\right| \mathrm{d}u\mathrm{d}v \sin\theta \mathrm{d}\theta \mathrm{d}\phi. \tag{4.2.2}$$

We can integrate over the 2-sphere to find

$$\mathcal{V} = 4\pi \int \left(\int \left|\frac{fr^2}{2\kappa u}\right| \mathrm{d}u \right) \frac{\mathrm{d}v}{|\kappa v|}. \tag{4.2.3}$$

In his section we shall consider the case where $f(r)$ has only one root, and thus the coordinates are regular for all $r > 0$. Then, in Section 4.3 we shall consider a shell construction that allows us to deal with with black holes with more than one horizon.

Assume that the one horizon is at $r = a$ and that it is a black hole horizon. Across black hole horizons $r = a$, $f(r)$ switches sign from negative in $r < a$ to positive in $r > a$, and $\kappa = f'(a)/2 > 0$. As in the Kruskal-Szekeres calculation, the black hole region of interest lies in the first quadrant, i.e. with u, v both positive. This allows us to remove the absolute value signs in (4.2.2), recalling that $f(r) < 0$. Now we apply boundary conditions on the integral. As in the Kruskal-Szekeres calculation, we can integrate v from some initial δ to some final v_{max}. Then if we want to integrate from $r = a$ to $r = 0$ we choose the limits on u accordingly. Now (4.2.3) becomes

$$\mathcal{V} = 4\pi \int_{\delta}^{v_{max}} \left(\int_{r=a}^{r=0} \frac{-fr^2}{2\kappa u} \mathrm{d}u \right) \frac{\mathrm{d}v}{\kappa v}. \tag{4.2.4}$$

We now make use of (4.1.6) to calculate the inner integral. Since the inner integral is evaluated at $v = const.$, we find

$$\left(\int_{r=a}^{r=0} \frac{-fr^2 \mathrm{d}u}{2\kappa u} \right) \bigg|_{v=const.} = \int_{r=a}^{r=0} \left(-fr^2 \frac{\mathrm{d}r}{f} \right) = \int_0^a r^2 \mathrm{d}r = \frac{a^3}{3}. \tag{4.2.5}$$

So then (4.2.4) yields, making the substitution $v_{max} \to v$ so that we can view the growth of the volume:

$$\mathcal{V}(v) = \frac{4\pi a^3}{3} \frac{\ln(v/\delta)}{\kappa} = \frac{\ln(v/\delta)}{\kappa} {}^3\mathcal{V}_E \tag{4.2.6}$$

where ${}^3\mathcal{V}_E$ is the Euclidean three-volume of a sphere of radius a. Differentiating we can write

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}v} = \frac{{}^3\mathcal{V}_E}{\kappa v}. \tag{4.2.7}$$

To interpret v, we note our previous result that v is affine on the horizon. As a result, we can replace v with λ and so write:

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\lambda} = \frac{{}^3\mathcal{V}_E}{\kappa \lambda}. \tag{4.2.8}$$

If we define

$$\mathcal{V}^* \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda}, \tag{4.2.9}$$

then we can write explicitly

$$\kappa = \frac{{}^3\mathcal{V}_E}{\mathcal{V}^*}. \tag{4.2.10}$$

This generalizes our discussion in Chapter 3 to general spherical symmetry.

4.3 Radial shells

Of course, our coordinates are not regular across multiple horizons, so we have to be more careful in these cases. Additionally, there are some situations in which it is not possible to integrate all the way to a singularity. Because of the instabilities below the Cauchy horizons in the Reissner-Nordström-de-Sitter class of spacetimes (see, for example, [6, 22, 23]), one could argue that integrating below the inner horizon would be including unphysical parts of the spacetime. The solution proposed here is to calculate the volume of shells extending from the horizon to some $r = r_1$, which lies within the range covered by the coordinates. When we use the word "shell" here, it should be clear that we are only referring to a spherical shell, and not to a new construction involving thin shells, boundary conditions and the like.

For definiteness, consider the situation where we are integrating below a horizon at $r = a$ to some $r = r_1 < a$. (We can, and will below in Section 4.4, calculate the volume from $r = a$ to some $r = r_1 > a$ as well.) We assume that $f(r)$ is regular, finite and non-zero for $r_1 \le r < a$, so that our double-null coordinates are regular throughout. Since our situation is now more general, we do not make *a priori* assumptions about the sign of κ or $f(r)$. That said, since $2\kappa = f'(a)$, $f(r)$ in the region $r_1 < r < a$, wherein $f(r)$ is continuous and non-zero, should be the opposite sign of κ, so that again we can set $|f(r)/\kappa| \to -f(r)/\kappa$. This means that we recover (4.2.4), only changing the limit $r = 0$ to $r = r_1$ and adding absolute value signs to the κ multiplying v.

For boundary conditions, we again terminate the integral on two sets of ingoing shells $v = \delta$ and $v = v_{max}$. Our integral then is (using the subscript s to indicate that this is a shell):

$$\mathcal{V}_s = 4\pi \int_{\delta}^{v_{max}} \left(\int_{r=a}^{r=r_1} \frac{-fr^2}{2\kappa u} \mathrm{d}u \right) \frac{\mathrm{d}v}{|\kappa| v}. \tag{4.3.1}$$

Clearly then we can repeat the substitution in (4.2.5) with r_1 replacing 0. We find

$$\mathcal{V}_s(v) = \frac{4\pi(a^3 - r_1^3)}{3} \frac{\ln(v/\delta)}{|\kappa|} = \frac{\ln(v/\delta)}{|\kappa|} {}^3\mathcal{V}_{E,s}, \tag{4.3.2}$$

where of course ${}^3\mathcal{V}_{E,s}$ is the Euclidean 3-volume of a spherical shell with radius $r_1 \le r \le a$. Again noting that v is affine, and writing $\mathcal{V}_s^* \equiv \mathrm{d}\mathcal{V}_s/\mathrm{d}\ln\lambda$ we find (just as in (4.2.10)) that

$$|\kappa| = \frac{{}^3\mathcal{V}_{E,s}}{\mathcal{V}_s^*}. \tag{4.3.3}$$

Since the depth of the shell is irrelevant, we can view this ratio as a property of the horizon alone. We can also write this as

$$\mathcal{V}_s^* = \frac{4\pi}{3}(a^3 - r_1^3). \tag{4.3.4}$$

4.4 Cosmological horizons

An example of how to apply the foregoing radial shell calculation in a case where $r_1 > a$ is to consider horizons in de Sitter space. We emphasize this point by calculating the volume explicitly in regular coordinates.

A complete covering of de Sitter space is given by [24]

$$\mathrm{d}s^2 = \frac{3}{\Lambda}\mathrm{d}\bar{s}^2 \tag{4.4.1}$$

where

$$\mathrm{d}\bar{s}^2 = \frac{4}{(1-uv)^2}\mathrm{d}u\mathrm{d}v + \left(\frac{1+uv}{1-uv}\right)^2 \mathrm{d}\Omega_2^2. \tag{4.4.2}$$

Consider the region $0 < r \equiv \sqrt{\frac{3}{\Lambda}}(1+uv)/(1-uv) < \infty$. Trajectories with tangents $\mathcal{K}^\alpha = (1-u_0v)^2\delta_v^\alpha$ (constant $u = u_0$, θ and ϕ) are radial null geodesics given by

$$v(\lambda)u_0 = 1 - \frac{1}{u_0\lambda} \tag{4.4.3}$$

where λ is an affine parameter, provided $u_0 \neq 0$. If $u_0 = 0$ then the geodesic is affinely parametrized by v. We note the expansion

$$\nabla_\alpha \mathcal{K}^\alpha = 4u_0\left(\frac{1-u_0v}{1+u_0v}\right). \tag{4.4.4}$$

Trajectories with tangents $\mathcal{M}^\alpha = (1-uv_0)^2\delta_u^\alpha$ (constant $v = v_0$, θ, and ϕ) are radial null geodesics given by

$$u(\lambda)v_0 = 1 - \frac{1}{v_0\lambda} \tag{4.4.5}$$

for $v_0 \neq 0$. If $v_0 = 0$ then the geodesic is affinely parameterized by u. We note the expansion

$$\nabla_\alpha \mathcal{M}^\alpha = 4v_0\left(\frac{1-uv_0}{1+uv_0}\right). \tag{4.4.6}$$

On the cosmological horizons, $r = \sqrt{\frac{3}{\Lambda}}$ and either $u = 0$ or $v = 0$, with v and u (respectively) affine parameters. The metric (4.4.2) is regular across the cosmological horizon. We note that $r = 0$

for $uv = -1$ and $r = \infty$ for $uv = 1$. To calculate a finite volume for a cosmological horizon, we integrate from $r = \sqrt{\frac{3}{\Lambda}}$ out to (say) some $\epsilon\sqrt{\frac{3}{\Lambda}}$ where $\epsilon > 1$. The situation is qualitatively similar to Figure 3.2, but r is now increasing along N. We now have a shell volume $\mathcal{V}_s$ which we can integrate:

$$\mathcal{V}_s = 4\pi \left(\frac{3}{\Lambda}\right)^2 (\epsilon^3 - 1) \ln\left(\frac{v}{\delta}\right) \tag{4.4.7}$$

so that we arrive at the manifestly invariant statement

$$\frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\lambda} = \frac{4\pi}{3} \left(\frac{3}{\Lambda}\right)^2 (\epsilon^3 - 1) \frac{1}{\lambda} \tag{4.4.8}$$

irrespective of initial conditions for v.

We note that the Euclidean 3-volume here is

$${}^3\mathcal{V}_{E,s} = \frac{4\pi}{3}(r_{max}^3 - r_{horizon}^3) = \frac{4\pi}{3} \left(\frac{3}{\Lambda}\right)^{3/2} \left(\epsilon^3 - 1\right) \tag{4.4.9}$$

and the surface gravity is well known to be

$$|\kappa| = \sqrt{\frac{\Lambda}{3}}, \tag{4.4.10}$$

so that we recover (with the asterisk, as usual, denoting $\mathrm{d}/\mathrm{d}\ln\lambda$)

$$|\kappa| = \frac{{}^3\mathcal{V}_{E,s}}{\mathcal{V}_s^*}. \tag{4.4.11}$$

4.5 A word on the choice of shells

The shells as defined here allow us to calculate the four-volume of a region beneath the horizon but that do not extend past any inner horizons (at constant r). Our initial work used the shells as written here. That said, it is not *a priori* obvious why we should use $r = const.$ as our termination point for the shell. We can find coordinates which are regular everywhere below the horizon and so dispense with the necessity for shell calculations, as we will do in Chapter 5. Alternatively, we can find another boundary for our four-volume region, one which permits a simple interpretation. This is what we shall do here. We shall use s' to identify these shells. The situation is shown schematically in Figure 4.1.

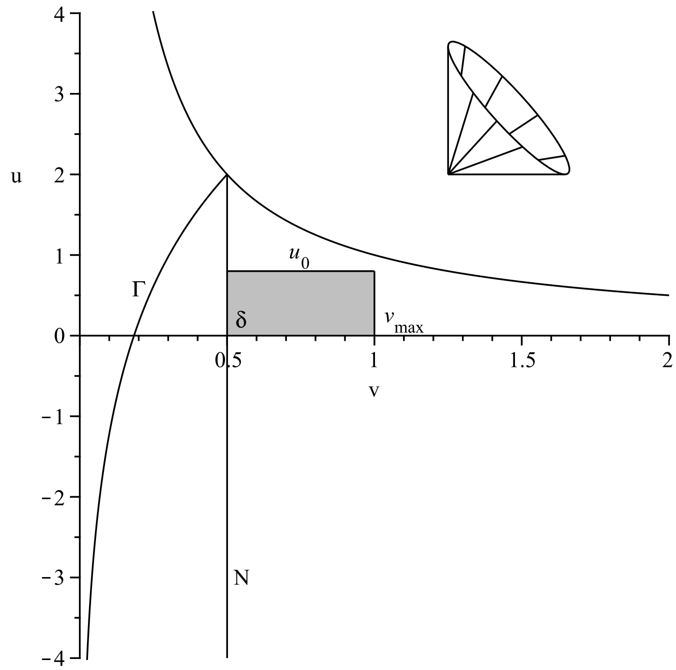

Figure 4.1. Similar to Figure 3.2. This shows the region $\mathcal{R}$ for the calculation of $\mathcal{V}_{s'}$. Here, instead of the boundaries for the region $\mathcal{R}$ being $\delta < v < v_{max}$ and u chosen so that it lies inside the black hole region, we have used a box $0 < u < u_0$, $\delta < v < v_{max}$.

Now the volume becomes:

$$\mathcal{V}_{s'} = \int_{\delta}^{v_{max}} \int_{0}^{u_0} \int_{0}^{\pi} \int_{0}^{2\pi} \sqrt{|g|} \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \tag{4.5.1}$$

As in Section 4.3, we assume that $f/\kappa < 0$ in the region of interest so that $|f/\kappa| = -f/\kappa$.

$$\therefore \mathcal{V}_{s'} = 4\pi \int_{\delta}^{v_{max}} \int_{0}^{u_0} -\frac{fr^2 \mathrm{d}u}{2\kappa u} \frac{\mathrm{d}v}{|\kappa| v}. \tag{4.5.2}$$

We can convert the inner integral over u to one over r as we did in (4.2.5). Now instead of integrating from $r = 0$ to a, we integrate from $r(uv) = r(u_0 v)$ to a.

$$\int_0^{u_0} \frac{-fr^2 \mathrm{d}u}{2\kappa u} = \int_{r(u_0 v)}^{a} r^2 \mathrm{d}r = \frac{1}{3}\left(a^3 - r(u_0 v)^3\right). \tag{4.5.3}$$

This is the Euclidean three-volume of spherical shell from $r = r(u_0 v)$ to $r = a$. Of course, this volume varies as a function of v. One can then write, letting ${}^3\mathcal{V}_{E,s'}(v)$ be the Euclidean volume for the shell calculated at v,

$$\mathcal{V}_{s'} = \int_{\delta}^{v_{max}} \frac{{}^3\mathcal{V}_{E,s'}(v)\mathrm{d}v}{|\kappa| v}. \tag{4.5.4}$$

As usual, make the substitution $v_{max} \to v$; since we have not calculated the integral explicitly, we will replace the dummy variable v with v'.

$$\mathcal{V}_{s'}(v) = \int_{\delta}^{v} \frac{{}^3\mathcal{V}_{E,s'}(v')\mathrm{d}v'}{|\kappa| v'}. \tag{4.5.5}$$

Then by the Fundamental Theorem of Calculus, we can differentiate the above to get:

$$\frac{\mathrm{d}\mathcal{V}_{s'}}{\mathrm{d}v} = \frac{{}^3\mathcal{V}_{E,s'}(v)}{|\kappa| v}, \tag{4.5.6}$$

which we can rewrite to find

$$\frac{\mathrm{d}\mathcal{V}_{s'}}{\mathrm{d}\ln v} = \frac{\mathrm{d}\mathcal{V}_{s'}}{\mathrm{d}v/v} = \frac{{}^3\mathcal{V}_{E,s'}(v)}{|\kappa| v}. \tag{4.5.7}$$

Now since v is affine on the horizon, we substitute v with λ everywhere. (Remember that λ is only an affine measure on the horizon itself.) The leftmost quantity in the equation directly above is, of course, $\mathcal{V}^* = \mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$. We now have

$$|\kappa| = \frac{{}^3\mathcal{V}_{E,s'}(\lambda)}{\mathcal{V}^*_{s'}(\lambda)}. \tag{4.5.8}$$

Significantly, both the Euclidean shell volume and the growth rate with respect to the affine parameter are functions of λ. They are *not* constants as we had before. The Euclidean volume is

$$\mathcal{V}_{s'}(\lambda) = \frac{4\pi}{3}(a^3 - r(u_0\lambda)^3) \tag{4.5.9}$$

which depends on the implicit relationship between r and $uv \equiv u_0\lambda$, which is given by (4.1.16). Obviously, then $\mathcal{V}^*$ is this times a factor of one over the magnitude of the surface gravity. It makes sense that since these quantities are constant when we define shells at constant radius, they will not be constant when we define shells of changing radius.

If we try to evaluate (4.5.5) explicitly, we find:

$$\mathcal{V}_{s'}(\lambda) = \frac{4\pi}{3|\kappa|}\left(\ln(v/\delta)a^3 - \int_\delta^\lambda \frac{r(u_0v')^3\mathrm{d}v'}{v'}\right), \tag{4.5.10}$$

which indicates that while the ratio of the Euclidean volume to the logarithmic growth rate of the volume is the surface gravity (magnitude) as usual, the usual form of the 4-volume itself, which is generally logarithmic, is changed, and now has an extra factor. If we note that the integral is over constant u, we can make the substitution using (4.1.6) (where we recall that v' here is v in (4.1.6))

$$\int_\delta^\lambda \frac{r(u_0v')^3\mathrm{d}v'}{v'} = 2\kappa\int_{r(u_0\delta)}^{r(u_0\lambda)} \frac{r^3\mathrm{d}r}{f}, \tag{4.5.11}$$

which implies that (4.5.2) becomes

$$\mathcal{V}_{s'} = \frac{4\pi a^3}{3|\kappa|}\ln(\lambda/\delta)a^3 + \frac{8\pi}{3}\mathrm{sgn}(\kappa)\int_{r(u_0\delta)}^{r(u_0\lambda)} \frac{r^3\mathrm{d}r}{f}. \tag{4.5.12}$$

So, for example, when we have Schwarzschild, wherein $f = 1 - 2M/r$, $a = 2M$, $\kappa = 1/4M$, this becomes

$$\mathcal{V}_{s'} = \frac{8\pi(2M)^4}{3}\ln(v/\delta) + \frac{8\pi}{3}\left(\frac{r^4}{4} + \frac{2Mr^3}{3} + 2M^2r^2 + 8M^3r + 16M^4\ln(-r+2M)\right)\Bigg|_{r(u_0\delta)}^{r(u_0\lambda)} \tag{4.5.13}$$

where we note

$$r(uv) = 2M(1 + \mathcal{W}(-uv/e)). \tag{4.5.14}$$

Our conclusion from this section is that there is something of particular interest in radial-type shells which leads to very simple logarithmic expansion of the four-volume. The relationship between the logarithmic growth rate and the Euclidean three-volume seems to be ironclad, at least when we have shells which terminate at constant v.

4.6 A word on signs

It is interesting that the sign of κ does not come into the relationship between ${}^3\mathcal{V}_E$ and $\mathcal{V}^*$. Why is this the case? In all our calculations above, we have assumed that the volume should always be positive, and thus that the volume increases as we increase each coordinate. We further assumed that the coordinate v was positive. Since the Euclidean volume is, similarly, always positive, it makes sense that the ratio between the Euclidean volume and the logarithmic growth rate of the volume is always positive.

It is worth noting that if we *do not* take these assumptions, and instead either allowed $v < 0$ or allowed $\mathrm{d}\mathcal{V}/\mathrm{d}v < 0$, then we would have $\mathrm{d}\mathcal{V}/\mathrm{d}\ln|v| = v\mathrm{d}\mathcal{V}/\mathrm{d}v < 0$, and thus we would recover the opposite sign for the ratio between ${}^3\mathcal{V}_E$ and $\mathcal{V}^*$. We will return to the question of the sign of κ in Chapter 5 and later on.

4.7 A few notes on interpretation

It is useful to pause here to talk about the quantity $\mathcal{V}^* \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda}$, which derives from the expression $\mathrm{d}\mathcal{V}/\mathrm{d}v$. It is tempting to view $\mathcal{V}$ as some sort of scalar function of the coordinates, $\mathcal{V}(x^\alpha)$, so that if we differentiate with respect to, say, v, we are in fact writing

$$\frac{\mathrm{d}\mathcal{V}(x^\alpha)}{\mathrm{d}v} \equiv \frac{\partial\mathcal{V}(x^\alpha)}{\partial v}. \tag{4.7.1}$$

or perhaps (along similar lines) $\mathrm{d}\mathcal{V}/\mathrm{d}v = \nabla_v\mathcal{V}(x^\alpha)$, where ∇_α is the covariant differentiation operator. This is incorrect, because $\mathcal{V}$ is *not* a local quantity. It is calculated by integrating the differential volume element $\mathrm{d}^4\mathcal{V} = \sqrt{|g|}\mathrm{d}^4x$—which *is* a local quantity—over a region, say $\mathcal{R}$. The volume *el-*

ement is invariant and constant. Consequently, if we have a non-constant volume, on which it is possible to define a derivative, we must have a non-constant region of integration.

Explicitly, let the region over which we calculate a four-volume be $\mathcal{R}(\mu)$, where μ is a parameter which is allowed to vary. We will further stipulate that the region grows with increasing μ, so that every spacetime point in $\mathcal{R}(\mu)$ will be in $\mathcal{R}(\nu)$ iff $\mu < \nu$. Then let $\mathcal{V}_{\mathcal{R}(\mu)}$ be the four-volume of the region $\mathcal{R}(\mu)$, which is of course

$$\mathcal{V}_{\mathcal{R}(\mu)} = \int_{\mathcal{R}(\mu)} \sqrt{|g|}\mathrm{d}^4 x. \tag{4.7.2}$$

Then we can define the derivative of $\mathcal{V}_{\mathcal{R}(\mu)}$ with respect to our parameter μ. We do so in the usual way derivatives are defined. For a very small (but positive) increment $\delta\mu$ we define

$$\frac{\mathrm{d}\mathcal{V}_{\mathcal{R}(\mu)}}{\mathrm{d}\mu} \equiv \lim_{\delta\mu\to 0} \frac{\mathcal{V}_{\mathcal{R}(\mu+\delta\mu)} - \mathcal{V}_{\mathcal{R}(\mu)}}{\delta\mu}. \tag{4.7.3}$$

Even more generally, we may wish to define a derivative of the volume with respect to a function of μ, say $f(\mu)$. Then we have from the chain rule,

$$\frac{\mathrm{d}\mathcal{V}_{\mathcal{R}(\mu)}}{\mathrm{d}(f(\mu))} \equiv \frac{1}{f'(\mu)} \left(\lim_{\delta\mu\to 0} \frac{\mathcal{V}_{\mathcal{R}(\mu+\delta\mu)} - \mathcal{V}_{\mathcal{R}(\mu)}}{\delta\mu} \right). \tag{4.7.4}$$

This should give the reader an appreciation for the meaning behind our quantity $\mathcal{V}^* \equiv \mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$. Here, we have $\mu \equiv \lambda$, and $\mathcal{V}^*$ depends explicitly, and *only*, on how we define our region $\mathcal{R}(\lambda)$, and how it changes with respect to λ. In the simplest case we have studied, the Schwarzschild case in Kruskal-Szekeres coordinates, the region $\mathcal{R}(\lambda)$ corresponds to the region $0 \leq u \leq 1/v$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and finally $\delta \leq v \leq \lambda$. On the *specific* trajectories $u = 0, \theta, \phi = const.$ which correspond to the null generators of the horizon, and *only* there, λ is an affine parameter.

This may seem obvious, but it is important to emphasize that our result is still dependent on how we choose our region and how the boundary of the region depends on λ. As an illustrative example, say we have a different set of limits on v: say some new region $\tilde{\mathcal{R}}(\lambda)$ has the same limits on u, θ, ϕ as $\mathcal{R}(\lambda)$, but here $\delta \leq v \leq \lambda(1 + u)$. This would obviously be a different region whose volume grows at a different rate with respect to $\ln\lambda$. However, on the horizon itself, $\delta \leq v \leq \lambda$, and so we recover that λ is an affine generator on the horizon. This gives another apparently manifestly invariant $\mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$ for a different region $\tilde{\mathcal{R}}$. So we must remember exactly what $\mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$ means in our context—that it is the rate of growth of the volume of a region bounded by the sets of *ingoing geodesics* ($v = const.$).

The reason we reject three-dimensional volumes as a useful measure of the volume of a black hole is that they are dependent on the choice of slicing. Our volume, at present, is defined for a very specific choice of boundary for the region of integration of the volume. Have we shown this choice of region of integration to be any less arbitrary than the choice of slicing for the three-dimensional volume?

At least, we can make one statement: if we tried to calculate the three-dimensional volume of a slice of constant $v \equiv v_0$, i.e. along one congruence of ingoing null geodesics, by analogy with the boundary chosen for the region of integration for the four-volume, we would find a three-volume of zero, since this slice is null. More explicitly, in the general spherically symmetric double-null coordinate construction, with metric given by (4.1.14), on the three-surface $v = const. \equiv v_0$, the metric becomes

$$\mathrm{d}s^2 = r(uv_0)^2 \mathrm{d}\Omega_2^2. \tag{4.7.5}$$

Thus the three-metric g_{ij} is obviously $g_{ij} = \mathrm{diag}(g_{uu}, g_{\theta,\theta}, g_{\phi,\phi}) = \mathrm{diag}(0, r^2, r^2 \sin^2\theta)$, which has determinant 0. This means that, insofar as the set of ingoing null geodesics which intersect the horizon at the same spacetime point is "special," it is possible to define a non-zero four-volume growth rate based on this boundary but not a non-zero three-volume.

This is still somewhat unsatisfying. We will use work by Maulik Parikh to help resolve this, but first we will summarize our results from this section, and then transition into a review of his work by calculating our volume in Eddington-Finkelstein-type coordinates.

4.8 Summary

We have shown that the invariant four-dimensional volume $\mathcal{V}$ of a region is proportional to the logarithm of the affine generators of the horizon in the non-degenerate case. We have shown that the ratio of the Euclidean three-volume ${}^3\mathcal{V}_E$ of the black hole region to the rate $\mathcal{V}^* = \mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$ is equal to the magnitude of the surface gravity $|\kappa|$. We have further shown that this proportionality still holds if we take the Euclidean three-volume and volume rate $\mathcal{V}^*$ of a shell terminating on the horizon. We note as well that we have not been able to find an expression for the invariant four-dimensional volume $\mathcal{V}$ when the black hole horizon is degenerate, i.e. when $\kappa = 0$.

Chapter 5

Alternate approach: Eddington-Finkelstein-like Coordinates

In this chapter we present a set of coordinates which allow for easier calculation. The reason these are presented second is because they were discovered after the initial work detailed in the previous few chapters. The presentation in this thesis reflects the process of our calculations as well as the calculations themselves.

After we found regular coordinates in Section 4.1, we used those same coordinates throughout Chapter 4. It should be obvious, however, that the results are coordinate independent—the volume rates depend only on the way our region is defined. This means that any set of coordinates which are regular and which refer to the same region will yield the same results.

We can dispense with the double-null coordinate construction entirely. Since the region which yielded $\mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda = const. = {}^3\mathcal{V}_E/|\kappa|$ was one defined by radially bounded shells, bounded by shells of constant ingoing null geodesics, it makes sense to choose a set of coordinates which includes a coordinate r (aerial radius) and one, say V, which parameterizes sets of ingoing null geodesics. The most well-known of these is the Eddington-Finkelstein coordinates for the Schwarzschild geometry.

They use coordinates (r, θ, ϕ, V) instead of (r, θ, ϕ, t), and have

$$\mathrm{d}V = \frac{\mathrm{d}r}{1 - \frac{2M}{r}} + \mathrm{d}t. \tag{5.0.1}$$

These can be generalized to include all spherically symmetric spacetimes by substituting $f(r)$ for $1 - 2M/r$.

There are three advantages to these coordinates: they are regular across degenerate horizons (a horizon that is not a simple root); one single set of coordinates can be regular across more than one horizon; and they permit a simple generalization to certain other spacetimes which are not spherically symmetric, in particular the Kerr-Newman geometry.

5.1 Spherical Symmetry

We can write the spherically symmetric metric (4.0.1) in ingoing (Eddington-Finkelstein-like) coordinates by setting

$$\mathrm{d}V = \frac{\mathrm{d}r}{f(r)} + \mathrm{d}t. \tag{5.1.1}$$

The line element is now

$$\mathrm{d}s^2 = -f(r)\mathrm{d}V^2 + 2\mathrm{d}r\mathrm{d}V + r^2\mathrm{d}\Omega_2^2. \tag{5.1.2}$$

We can imagine a black hole formed by some spherical collapse of matter along a timelike geodesic Γ as before. Here, the singularity is formed at some initial value of V, say V_1. The situation is shown in Figure 5.1.

Ingoing radial null geodesics (as defined in Section 4.1) travel along paths of constant (V, θ, ϕ). Say we have a black hole with an event horizon at $r = a$. Then the black hole region is $0 \leq r \leq a$, and ingoing null geodesics (as defined in Section 4.1) are labelled by constant (V, θ, ϕ). As before consider a finite region bounded by sets of ingoing null geodesics. Set the limits as $V_1 \leq V \leq V_2$. As in Section 4.3 consider a radial shell defined by $r_1 < r < a$. The determinant g of the metric is

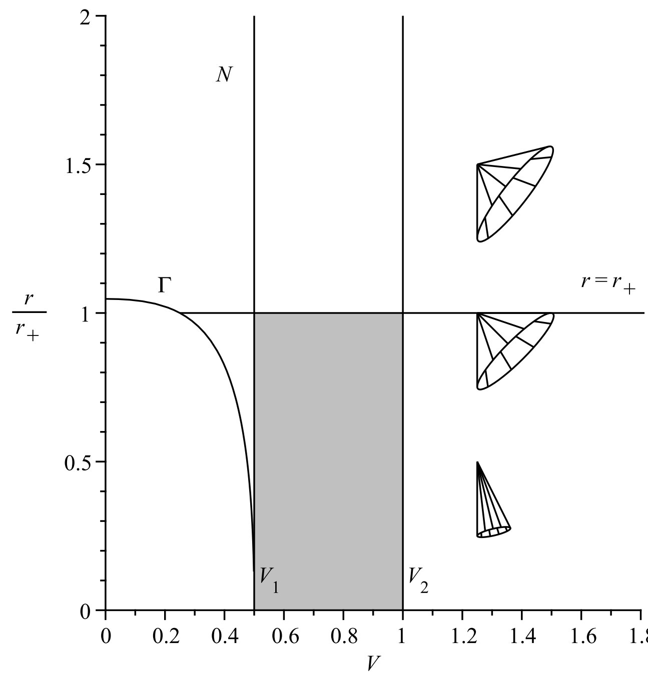

Figure 5.1. Similar to Figure 3.2. This shows the region for the calculation of $\mathcal{V}$. Once again, the black hole is created by the collapse of a timelike boundary Γ. The first ingoing null cone to hit the singularity is labelled by N and corresponds to $V = V_1$. The horizon here is located by $r = r_+$. The null cone V_2 is any null cone to the future of V_1. The invariant four-volume $\mathcal{V}_s$ calculated here is bounded by V_1 and V_2 and is to the interior of the horizon.

equal to $-r^4 \sin^2\theta$—which is entirely $f(r)$-independent. Then the four-volume of our shell, $\mathcal{V}_s$, is:

$$\mathcal{V}_s = \int_{V_1}^{V_2}\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{a} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}V = \frac{4\pi(a^3 - r_1^3)}{3}(V_2 - V_1). \tag{5.1.3}$$

We can substitute $V_2 \to V$ so that we get $\mathcal{V}_s$ as a function of V:

$$\mathcal{V}_s = \frac{4\pi(a^3 - r_1^3)}{3}(V - V_1) = {}^3\mathcal{V}_{E,s}(V - V_1). \tag{5.1.4}$$

Here we have used the three volume ${}^3\mathcal{V}_{E,s}$ of the spherical shell from $r = r_1$ to $r = a$. We find then that $\mathcal{V}_s$ grows linearly with V, and the constant of proportionality is ${}^3\mathcal{V}_{E,s}$. Taking the derivative, we find

$$\frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}V} = {}^3\mathcal{V}_{E,s}, \tag{5.1.5}$$

which is independent of V_1. This calculation has saved us several headaches. These coordinates are regular across not just the horizon at $r = a$ but across all horizons, so that the shell can cover inner horizons as well as outer ones. In fact, it is trivial to show that for any region $r_1 \leq r \leq r_2$ without singularities in $f(r)$ the relationship (5.1.5) holds, even if r_2 is not a horizon. It was not necessary to keep track of signs due to various coordinate transformations.

Further, we can now dispense with the shell construction entirely, and so avoid the reliance on the (somewhat arbitrary) choice of surfaces of constant r. These coordinates are not singular on any of the horizons. We can thus calculate directly the volume from the essential singularity at $r = 0$ to the outer horizon (say at $r = r_+$), or, if there are two horizons (say at $r = r_\pm$), one could equally take the volume of the region between the two horizons, $r_- \leq r \leq r_+$. Because of the instabilities below the Cauchy horizon in Reissner-Nordström spacetime, it remains possible that the region $r < r_-$ is unphysical in some way; but in the idealized, perfectly spherically symmetric metric, we don't have to worry about this.

We revisit the ingoing and outgoing geodesics as defined in Section 4.1. In (r, θ, ϕ, V) coordinates, the ingoing null geodesics lie on constant V, θ, ϕ. The outgoing null geodesics, which include the null generators of the horizon, can be written (affinely) as

$$m^\alpha = \left(\frac{f(r)}{2}\exp\left(-\frac{1}{2}f'(r)V\right), 0, 0, \exp\left(-\frac{1}{2}f'(r)V\right)\right). \tag{5.1.6}$$

On the horizon $r = a$, $f(a) = 0$ and $f'(a) = 2\kappa$, so the null generators of the horizon are $p^\alpha = (0, 0, 0, \exp(-\kappa V))$. For non-degenerate horizons, which is what we have worked with up to this point, $\kappa \neq 0$. Then we have:

$$\frac{\mathrm{d}V}{\mathrm{d}\lambda} = \exp(-\kappa V). \tag{5.1.7}$$

We can rearrange this equation and integrate it, to find

$$V = \frac{\ln|\lambda|}{\kappa} + V_0, \tag{5.1.8}$$

where the V_0 is some constant of integration, which we can set to V_1 by redefining λ by a linear transformation. Substituting into (5.1.4) we find

$$\mathcal{V}_s = \frac{{}^3\mathcal{V}_{E,s}}{\kappa} \ln|\lambda|, \tag{5.1.9}$$

from which, defining $\mathcal{V}_s^* \equiv \mathrm{d}\mathcal{V}_s/\mathrm{d}\ln|\lambda|$,

$$\kappa = \frac{{}^3\mathcal{V}_{E,s}}{\mathcal{V}_s^*} \tag{5.1.10}$$

which is the same as (4.3.3) with the absolute value sign removed. Why is there a difference in sign when $\kappa < 0$? The difference lies in our choice of parameter along which $\mathcal{V}_s$ is expanding. In (4.3.3) we had defined $\lambda = v$ to be positive, and stipulated that $\mathcal{V}_s$ increased along λ. In this calculation, we chose to set $\mathrm{d}\mathcal{V}_s/\mathrm{d}V > 0$ here. From (5.1.8) we have

$$\frac{\mathrm{d}V}{\mathrm{d}\lambda} = \frac{1}{\kappa\lambda}. \tag{5.1.11}$$

If $\kappa < 0$, it is impossible for λ to be both positive, and to increase in the same direction as V. So having our volume increase with increasing V necessarily means that either $\lambda < 0$ or that the volume decreases with increasing λ. It is another reminder to be careful about signs.

5.2 Degenerate Horizons

A further advantage to these coordinates is that we can deal with horizons which are degenerate. We made no assumption about the nature of the horizon at $r = a$ up to (5.1.5). In a spacetime with a degenerate horizon, the outgoing radial null geodesics still take the form (5.1.6). In these spacetimes, however, $f'(a) = 0$, where $r = a$ locates the horizon, so that the null generators of the horizon have (affinely parametrized) form $p^\alpha = (0, 0, 0, 1)$. So we have $\mathrm{d}V/\mathrm{d}\lambda = 1$, or $V = \lambda + V_0$.

We can always redefine the affine parameter λ by a linear transformation. For the moment, choose λ such that we have

$$V = \mathcal{C}\lambda + V_1 \tag{5.2.1}$$

where $\mathcal{C}$ is (as yet) an arbitrary non-zero constant. Then we find that the volume from (5.1.4) has the form

$$\mathcal{V}_s = {}^3\mathcal{V}_{E,s}\mathcal{C}\lambda. \tag{5.2.2}$$

So then we can conclude

$$\frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\lambda} = {}^3\mathcal{V}_{E,s}\mathcal{C}. \tag{5.2.3}$$

We note

$$\mathcal{V}_s^* = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\ln|\lambda|} = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\ln|\lambda|} = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\lambda}\lambda = {}^3\mathcal{V}_{E,s}\mathcal{C}\lambda = \mathcal{V}_s. \tag{5.2.4}$$

This indicates that the ratio

$$\frac{{}^3\mathcal{V}_{E,s}}{\mathcal{V}_s^*} = \frac{{}^3\mathcal{V}_{E,s}}{\mathcal{V}_s} = \frac{1}{\mathcal{C}\lambda} \neq |\kappa|. \tag{5.2.5}$$

So in this case the relation (4.3.3) fails. We can conclude that the ratio between ${}^3\mathcal{V}_{E,s}$ and $\mathcal{V}_s^*$ is the surface gravity (or possibly its negative) in spherical symmetry only when $\kappa \neq 0$.

5.3 Kerr-Newman

One particularly useful feature of the technique of calculating volume using Eddington-Finkelstein-type coordinates is that it generalizes to (at least) one non-spherically symmetric case—that of the Kerr-Newman geometry. A Kerr-Newman black hole [13] has a central mass M, angular momentum $J \equiv aM$, and charge Q. There is a notation change for this section: the outer horizon of the Kerr-Newman geometry is at $r = r_+$ so that we can use a for the rotational parameter. We will vary r

from $r_1 < r_+$ to r_+. In Kerr coordinates,

$$\begin{aligned} \mathrm{d}s^2 = &-[1-\rho^{-2}(2Mr-Q^2)]\mathrm{d}V^2 + 2\mathrm{d}r\mathrm{d}V + \rho^2\mathrm{d}\theta^2 \\ &+ \rho^{-2}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta\mathrm{d}\psi^2 - 2a\sin^2\theta\mathrm{d}\psi\mathrm{d}r \\ &- 2a\rho^{-2}(2Mr-Q^2)\sin^2\theta\mathrm{d}\psi\mathrm{d}V, \end{aligned} \tag{5.3.1}$$

where $\Delta = r^2 + a^2 - 2Mr + Q^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$. θ corresponds to the θ in spherically symmetric metrics; it varies from 0 to π with $\theta = \pi/2$ as the equatorial plane. ψ is an angular coordinate with period 2π and the black hole spins in the ψ direction. The horizons are located by $\Delta = 0$, which leads to two (positive) solutions, $r = r_\pm$; the outermost horizon $r = r_+$ is the black hole horizon. The metric determinant is

$$g = -\sin^2\theta\rho^4. \tag{5.3.2}$$

There are two principal null geodesic congruences in Kerr-Newman. These are defined as follows. Given that $C_{\alpha\beta\gamma\delta}$ is the Weyl conformal tensor, its dual ${}^*C_{\alpha\beta\gamma\delta}$ is defined by [13]

$${}^*C_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\mu\nu}C^{\mu\nu}{}_{\gamma\delta}. \tag{5.3.3}$$

$\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor, defined by

$$\epsilon_{\alpha\beta\mu\nu} = \sqrt{|g|}[\alpha\;\beta\;\mu\;\nu]. \tag{5.3.4}$$

$[\alpha\;\beta\;\mu\;\nu]$ is the antisymmetric symbol defined as +1 if $(\alpha\;\beta\;\mu\;\nu)$ is an even permutation of $(0\,1\,2\,3)$, -1 if it is an odd permutation of $(0\,1\,2\,3)$, and 0 otherwise. In this case, the trajectories m^α on the principal null congruences have the properties,

$$C_{\alpha\beta\gamma[\delta}m_{\epsilon]}m^\beta m^\gamma = 0, \qquad {}^*C_{\alpha\beta\gamma[\delta}m_{\epsilon]}m^\beta m^\gamma = 0 \tag{5.3.5}$$

where the square brackets here represent antisymmetrization, such that, for example, $C_{\alpha\beta\gamma[\delta}m_{\epsilon]} =$

$C_{\alpha\beta\gamma\delta}m_\epsilon - C_{\alpha\beta\gamma\epsilon}m_\delta$.

As a result, we can view these null geodesics as "special" in the same way we can view the radial null geodesics in spherical symmetry as special. The two congruences are given in (r, θ, ψ, V) coordinates by

$$m^\alpha_{in} = (-E, 0, 0, 0) \qquad m^\alpha_{out} = \left(E, 0, \frac{2aE}{\Delta}, \frac{2(r^2+a^2)E}{\Delta}\right). \tag{5.3.6}$$

The E here is an arbitrary constant. The m^α_{in} trajectories are called "ingoing" because for any value of r, r decreases to the future along the trajectory; conversely, the m^α_{out} trajectories are called "outgoing" because, for $r > r_+$, r increases along these trajectories to the future. We note the similarity of these coordinates to Eddington-Finkelstein coordinates, in which, just like the ingoing principal null geodesics here, ingoing radial null geodesics lie on trajectories of constant (θ, ϕ, V) (where ϕ is an analogue to ψ).

By analogy with our work in spherical symmetry, consider the volume of some region bounded by sets of ingoing principal null congruences, $V_1 < V < V_2$, integrated over the equivalent of the two-sphere, $0 \le \theta \le \pi$, $0 \le \psi \le 2\pi$. Take radial shells from r_1 to r_+. Then we find

$$\mathcal{V}_s = \int_{V_1}^{V_2}\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_+}(r^2 + a^2\cos^2\theta)\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\psi\mathrm{d}V. \tag{5.3.7}$$

Integrating, and making the substitution $V_2 \to V$, we have:

$$\mathcal{V}_s = \frac{4\pi}{3}\left(r_+(r_+^2 + a^2) - r_1(r_1^2 + a^2)\right)(V - V_1). \tag{5.3.8}$$

Once again, we find that the result is proportional to V.

$$\therefore \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}V} = \frac{4\pi}{3}\left(r_+(r_+^2 + a^2) - r_1(r_1^2 + a^2)\right) \equiv {}^3\mathcal{V}'_s, \tag{5.3.9}$$

where the newly-defined constant ${}^3\mathcal{V}'_s$ is analogous to ${}^3\mathcal{V}_{E,s}$ in the static, spherically symmetric $f(r)$ case. It is *not*, however, the Euclidean three-volume of the black hole.

The null generators of the horizon are outgoing principal null geodesics with $r = r_+$ (or $r = r_-$ on the inner horizon). However, we note that since $\Delta = 0$ on the horizon(s), m^α_{out} as parametrized here diverges. As established in [13], the null generators of the horizon have the form

$$\theta = \text{const.}, \qquad r = r_+ = \text{const.}, \qquad \psi = 2a\mu, \qquad V = 2(r_+^2 + a^2)\mu \tag{5.3.10}$$

where μ (not affine) parametrizes the curve. We can solve for the proper affine parametrization by setting

$$p^\alpha = (0, 0, 2ah(V), 2(r_+^2 + a^2)h(V)) \tag{5.3.11}$$

as the null generator and solving for the function $h(V)$ for which $p^\beta \nabla_\beta p^\alpha = 0$ on the horizon. The solution is $h(V) = \mathcal{C}\exp(-\kappa V)$, where $\mathcal{C}$ is an arbitrary (non-zero) constant and

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} \tag{5.3.12}$$

is the surface gravity of the horizon at $r = r_+$. If we choose $\mathcal{C} = \left(2(r_+^2 + a^2)\right)^{-1}$, we have

$$p^\alpha = \left(0, 0, \frac{a}{r_+^2 + a^2}\exp(-\kappa V), \exp(-\kappa V)\right). \tag{5.3.13}$$

As in the spherically symmetric case, provided $\kappa \neq 0$, $\mathrm{d}V/\mathrm{d}\lambda = \exp(-\kappa V)$ implies

$$\frac{\mathrm{d}V}{\mathrm{d}\lambda} = \frac{1}{\kappa\lambda}, \tag{5.3.14}$$

so that

$$\mathcal{V}_s^* = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\ln|\lambda|} = \frac{{}^3\mathcal{V}_s'}{\kappa}, \tag{5.3.15}$$

so that (4.3.3) holds, with ${}^3\mathcal{V}_s'$ in place of ${}^3\mathcal{V}_{E,s}$. In other words,

$$\mathcal{V}_s^* = \frac{4\pi r_+(r_+^2 + a^2)}{3\kappa} - \frac{4\pi r_1(r_1^2 + a^2)}{3\kappa}. \tag{5.3.16}$$

If $\kappa = 0$, then we have a degenerate black hole and have $p^\alpha = \left(0, 0, \frac{a}{r_+^2+a^2}, 1\right)$. As in the case of a degenerate black hole in spherical symmetry, we have $\mathrm{d}V/\mathrm{d}\lambda = 1$. We can redefine λ by a linear transformation. We find, as we did in degenerate black holes with spherical symmetry, that $\mathcal{V}_s \propto \lambda$, and that

$$\frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\lambda} = \mathcal{C} \tag{5.3.17}$$

where $\mathcal{C}$ is any (non-zero) constant. Once again, we find that $\mathcal{V}_s^*$ is not a constant in this case, but is itself proportional to λ.

5.4 Summary and Transition to Parikh Volume

Our results for the Kerr-Newman spacetime show that it is not a feature of spherical symmetry alone that the volume $\mathcal{V}_s$ of a shell bounded by sets of null geodesics grows proportionally to the logarithm of the affine generator of the horizon when the surface gravity $\kappa \neq 0$, or that $\mathcal{V}_s$ grows linearly with the affine generator when $\kappa = 0$. Regardless of the value of κ, $\mathcal{V}_s$ grows linearly with the ingoing coordinate V in both spherical symmetry and Kerr-Newman. This suggests there is something fundamental at work here that does not require spherical symmetry.

One thing that the $f(r)$ metrics and the Kerr-Newman metric have in common is that they are stationary, i.e. they permit a Killing vector η^α, which is (assuming that $f(r) > 0$ for large r) is asymptotically timelike. In the Eddington-Finkelstein-like coordinates of the previous section, (r, θ, ϕ, V), $\eta^\alpha = (0, 0, 0, 1)$ (up to a multiplicative constant) in $f(r)$ spherically symmetric metrics, and in Kerr coordinates (r, θ, ψ, V) in Kerr-Newman, we again have a timelike Killing vector $\eta^\alpha = (0, 0, 0, 1)$. V is thus a stationary time coordinate, insofar as we can write $\eta^\alpha = \delta^\alpha_V$ for this Killing vector representing stationarity. This helps motivate our turning to the work of Maulik Parikh, who defines a volume specifically in terms of Killing vectors of stationary spacetimes.

Chapter 6

Parikh Volume of Black Holes

In his paper [1], Parikh defines a volume for stationary black hole spacetimes. Like our volume, it is defined as a rate, but his is a rate of volume growth with respect to a stationary time coordinate T, which satisfies $\eta^\alpha \nabla_\alpha T = 1$, where η^α is the Killing vector for stationarity, or a Killing vector which is asymptotically timelike. We will repeat Parikh's analysis as he does in his paper (with only some notation and wording changes) first in the section that follows. Afterward we generalize his volume to the volume of any space with a Killing vector, and after this is done we will return to our volume and state explicitly the connection between the two.

6.1 Parikh's development

Parikh's definition of volume requires only stationarity. For definiteness, however, he begins by examining the line element for a static, spherically symmetric metric in D dimensions,

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t_s^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega_{D-2}^2 \tag{6.1.1}$$

where $f(r)$ is some function of r and $\mathrm{d}\Omega_{D-2}^2$ is the line element of the $(D-2)$-sphere, with coordinates θ^A. The coordinate t_s is static because the metric is invariant under both a translation $t_s \to t_s + c$ and reversal $t_s \to -t_s$. We note then that $\delta_{t_s}^\alpha$ is a Killing vector. We assume that this is a black hole metric, so that there is a horizon somewhere where $f(r)=0$ outside of which $f>0$. Both g_{rr} and g^{tt} diverge at $f(r)=0$, so the metric is nonregular in these coordinates. We wish to introduce a new coordinate T for which the metric is regular across the horizon.

Let us switch to a new coordinate T. We wish δ^α_T to be a Killing vector, just as $\delta^\alpha_{t_s}$ is. The requirement that $\delta^\alpha_T = K\delta^\alpha_{t_s}$ (that the T-Killing vector is parallel to the t_s-Killing vector) is

$$t_s = KT + h(r, \theta^A) \tag{6.1.2}$$

where K is a constant and h is an arbitrary function of the other coordinates. The constant K is set to $+1$; the sign is chosen to preserve the orientation of time, and setting $|K|=1$ is done to demand a "fixed asymptotic form of the metric," or to fix the asymptotic normalization of the Killing vector. To reduce clutter, Parikh sets h to be a function of r only. The T-r part of the line element then becomes

$$\mathrm{d}s^2 = -f(r)\mathrm{d}T^2 - 2f(r)h'(r)\mathrm{d}T\mathrm{d}r + \mathrm{d}r^2\left(\frac{1}{f(r)} - f(r)h'(r)^2\right) \tag{6.1.3}$$

where $h'(r) = \mathrm{d}h/\mathrm{d}r$. Parikh states that we can choose $h(r)$ such that h' is real and $g_{rr} = (1 - f^2h'^2)/f$ remains positive and finite. Such a choice would ensure metric regularity through the horizon, due to the finiteness of g_{rr} and thus g^{TT}; it would also, though this is not as essential to Parikh's argument, guarantee that the normal to a surface of constant T would be everywhere timelike. The Killing vector δ^α_T still becomes spacelike inside the horizon.

What is an invariant measure of the slices of $T = const.$? As we did, Parikh notes that the $(D-1)$-volume of these sections would manifestly depend on the choice of slicing. On the other hand, he observes that the determinant of the spacetime has no dependence on time slicing, as $h'(r)$ drops out when taking the determinant. As a result, the full D-dimensional volume will be invariant. Further, if it is possible to find a slicing that extends through the horizon, as is clearly the case here (in spherical symmetry), then it is not necessary to use a slicing that extends through the horizon in calculating the volume—since the volume is slicing-independent. He defines the differential spacetime volume $\mathrm{d}\mathcal{V}_D$ by

$$\mathrm{d}\mathcal{V}_D(T) = \int_T^{T+\mathrm{d}T} \mathrm{d}T' \int_0^{r_+} \mathrm{d}r \int \mathrm{d}^{D-2}\theta\sqrt{-g_D} \tag{6.1.4}$$

where of course $r = r_+$ is the location of the horizon and g_D is the determinant of the D-dimensional metric $g_{\alpha\beta}$. If the time coordinate is of the form (6.1.2), then δ^α_T is Killing, and so the the metric is T-independent. As a result, T appears in the final "spacetime volume element" only through the multiplicative factor $\mathrm{d}T$. To note that his volume is a rate, we will use an asterisk; and to indicate that it has $(D-1)$-dimensionality we will use a $(D-1)$ to identify it. So his volume (written in our notation) is defined as

$$^{D-1}\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}_D}{\mathrm{d}T} = \int \mathrm{d}^{D-1}x\sqrt{-g_D}. \tag{6.1.5}$$

Thus, the volume is constant in time for all choices of Killing time, since the integrand is time-independent.

In the particular case of the spherically symmetric metric in 4 dimensions, this becomes

$$^3\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}T} = \int_0^{r_+} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\sqrt{r^4\sin^2\theta} = \frac{4\pi}{3}r_+^3, \tag{6.1.6}$$

which, remarkably, is the Euclidean three-volume of a sphere. This generalizes to higher dimensional spherically symmetric objects. Another case of great interest is the Kerr metric, which in Kerr coordinates (which we used in the previous chapter), which are asymptotically flat ($g_{VV} \to -1$, $g_{V\psi} \to 0$), has

$$^3\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}V} = \int_0^{r_+} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi(r^2 + a^2\cos^2\theta)\sin\theta = \frac{4\pi r_+(r_+^2 + a^2)}{3}. \tag{6.1.7}$$

One significant property of the Parikh volume is that this volume is invariant under the choice of stationary time slicing. Here is Parikh's justification, with square brackets representing minor changes for our notation:

> ...the integral is now invariant under stationary time slices.

> Here is why. Imagine the spacetime integral [written here as (6.1.5)] as a Riemann sum of little strips, each of coordinate length $\mathrm{d}t$, lined up side by side from $r = 0$ to $r = r_+$. According to $[t_s = T + h(r, \theta^A)]$, a particular constant-time slice merely shifts these strips up or down along the orbit of the Killing vector in an $[h]$-dependent manner. But the metric is unchanged under such shifts. Hence, the spacetime integral is invariant even though different time slicings correspond to an integration over different spacetime regions. After dividing out by $\mathrm{d}t$, we therefore obtain an invariant spatial volume. Indeed, when [the constant of proportionality, here called K] is fixed to 1, it is the unique invariant volume. Nor is the construction affected by the nature—timelike, spacelike, or null—of the Killing vector.

In Section 7.3.2, we express a related idea in some detail. A very quick demonstration of Parikh's principle, making use of his "differential spacetime volume" idea, follows; while based on Parikh's claims, the demonstration is ours. Use coordinates t_s as the static time and x^i as the spatial coordinates. We can define the volume by making reference to Parikh's "differential spacetime volume" $\mathrm{d}\mathcal{V}_D$, but by changing the limits on t_s to running from $h(x^i)$ to $h(x^i)+\mathrm{d}t_s$. (If the reader is uncomfortable with the idea of a differential volume being defined directly, he or she could imagine substituting a small volume section $\Delta\mathcal{V}_D$ defined by taking the four-volume of a small increment of time Δt_s, and then after calculations considering the limiting case where Δt_s is extremely small.) Let Σ be the region to which x^i belong within the black hole: for example, in Schwarzschild Σ would correspond to $0 \leq r \leq 2M, 0 \leq 0 \leq \pi, 0 \leq \phi < 2\pi$. Then we can write:

$$\mathrm{d}\mathcal{V}_D = \int_\Sigma \mathrm{d}^{D-1}x \int_{h(x^i)}^{h(x^i)+\mathrm{d}t_s} \mathrm{d}t'_s \sqrt{|g_D(x^i)|} = \mathrm{d}t_s \int_\Sigma \mathrm{d}^{D-1}x \sqrt{g_D(x^i)}. \tag{6.1.8}$$

This implies,

$$^{D-1}\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}_D}{\mathrm{d}t_s}. \tag{6.1.9}$$

Now say we want to make a slice at constant $T = t_s + h(x^i)$. If we change the inner integral over

$\mathrm{d}t_s$ to one over $\mathrm{d}T$, it obviously has T varying between 0 and $\mathrm{d}t_s$ so that

$$\mathrm{d}\mathcal{V}_D = \int_\Sigma \mathrm{d}^{D-1}x \int_0^{\mathrm{d}t_s} \mathrm{d}T \sqrt{|g_D(x^i)|} = \mathrm{d}t_s \int_\Sigma \mathrm{d}^{D-1}x \sqrt{g_D(x^i)} \tag{6.1.10}$$

as required. This confirms that $\mathrm{d}\mathcal{V}_D$ is invariant. This formulation is very explicit in demonstrating that the "starting" point of t_s for the integration and the spatial coordinates we use to describe Σ are irrelevant to $\mathrm{d}\mathcal{V}_D$. We will give another argument for the irrelevance of starting point t_s and spatial coordinates later on.

Parikh also mentions that the volume is the volume that would be measured by a stationary observer at infinity. It is unclear exactly what he means. The central point seems to be that the choice of stationary time is made so that the difference in coordinate time between constant T (or constant t_s) slices will be the same as the proper time difference between the two slices for an observer at infinity:

$$\left.\frac{\mathrm{d}T}{\mathrm{d}\tau}\right|_{r\to\infty} = 1. \tag{6.1.11}$$

This however is only true if $g_{TT} \to -1$ as $r \to \infty$, which is true in some metrics but not others (such as de Sitter space).

Parikh does not offer calculations to justify many of his claims, so we feel it is useful to include some direct verification of them in the chapter that follows. The major claim that we want to justify in more detail is that the determinant of the metric is unchanged under the choice of coordinate system (T, x^i) from a coordinate system (t_s, x^i) where $T = t_s + h(x^i)$. We will also generalize Parikh's volume to non-black hole spaces and then examine a few properties that this Killing-vector-based volume has.

First, however, we will note the relationship between our volume and Parikh's.

6.2 Connection between Our Volume and Parikh's

The coordinate V was used in Chapter 5 to calculate volumes as an intermediate step to calculating $\mathcal{V}^*$. In spherical symmetry with a horizon at $r = r_+$, we found

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}V} = {}^3\mathcal{V}_E = \frac{4\pi}{3} r_+^3 \tag{6.2.1}$$

or the Euclidean three-volume of a sphere of radius r_+, and in Kerr-Newman with a horizon at $r = r_+$ we found

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}V} = {}^3\mathcal{V}' = \frac{4\pi}{3} r_+(r_+^2 + a^2). \tag{6.2.2}$$

We now recognize these as the Parikh volumes for spherical symmetry and Kerr-Newman, respectively, so that we have in these particular cases ${}^3\mathcal{V}^* = {}^3\mathcal{V}_E$ (spherical symmetry) and ${}^3\mathcal{V}^* = {}^3\mathcal{V}'$ (Kerr-Newman). Why does this make sense? Reexamining the metrics for spherical symmetry (5.1.2) and Kerr-Newman (5.3.1), we note that in both cases V is a coordinate for which:

1. δ^α_V is a Killing vector; and
2. $g_{VV} \to -1$ asymptotically as $r \to \infty$ (Kerr-Newman) OR $g_{VV} = g_{tt}$ (spherical symmetry), where t is the static time coordinate in static spherical symmetry.

These are the conditions on the permissible coordinates T to calculate the Parikh volume

$$^3\mathcal{V}^* = \left.\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}T}\right|_{\text{evaluated in black hole}}. \tag{6.2.3}$$

Condition (1) refers to the fact that V is a Killing coordinate, and condition (2) refers to the fixed asymptotic normalization. So we have verified that ${}^3\mathcal{V}_E$ (spherical symmetry) and ${}^3\mathcal{V}'$ (in Kerr-Newman) are the Parikh volumes ${}^3\mathcal{V}^*$ for the respective spacetimes. We shall call a coordinate T which satisfies conditions (1) and (2) a *well-normalized Killing parameter*. So then V is a well-normalized Killing parameter.

Using this fact as well as (5.1.10) and (5.3.15), we have in these particular cases:

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln|\lambda|} \equiv \mathcal{V}^* = \frac{{}^3\mathcal{V}^*}{\kappa}. \tag{6.2.4}$$

Is this result true in some more general way, that our volume, the growth of the volume with respect to the affine parameter of the horizon, is proportional to the quotient of the Parikh volume and the surface gravity? The answer is yes.

The definition of our volume rate $\mathcal{V}^*$ is the volume growth rate of the black hole region, bounded by sets of ingoing radial null geodesics in spherical symmetry, or ingoing principal null geodesics in Kerr-Newman, with respect to the logarithm of the affine generator on the horizon. As in Chapter 5, V is a well-normalized Killing parameter for which $V = const.$ labels sets of ingoing null geodesics. Further, we let V be a function of an affine parameter λ on the horizon. Then we write:

$$\mathcal{V}^* \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln|\lambda|} \equiv \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}V}\frac{\lambda\mathrm{d}V}{\mathrm{d}\lambda}. \tag{6.2.5}$$

We have confirmed that $\mathrm{d}\mathcal{V}/\mathrm{d}V$ is equal to the Parikh volume ${}^3\mathcal{V}^*$. Further, following [14], any parameter which is proportional to the affine generator on the horizon and which permits the usual asymptotic normalization, whether V or T, has the property

$$\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}V} = \frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}T} = \kappa \tag{6.2.6}$$

where κ is the surface gravity. This implies that we have

$$\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}V}\frac{\lambda\mathrm{d}V}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}T}\frac{\lambda\mathrm{d}T}{\mathrm{d}\lambda} = \frac{{}^3\mathcal{V}^*}{\kappa} \tag{6.2.7}$$

This allows us, first of all, to note that just as the Parikh volume is independent of slicing, so is our volume: we do not require that our volume be bounded by sets of ingoing null geodesics at all. Any $T = const.$ surface will do just as well as the $V = const.$ surface we used to calculate the volume. Our volume can now be defined as the rate of growth of the volume below the black hole horizon with respect to the logarithm of the null generator of the horizon, as attached to surfaces of constancy of a well-normalized Killing time parameter. See also Chapter 9 for a further discussion of how our volume can be defined.

Second, we note that we have a simple relationship between our volume, the Parikh volume, and the surface gravity, which can supply a new definition for the surface gravity:

$$\kappa = \frac{{}^3\mathcal{V}^*}{\mathcal{V}^*}. \tag{6.2.8}$$

See Chapter 10 for further discussion of this remarkable point.

Before continuing, we note that by analogy with our own work, we can define the Parikh volume of a shell from $r_1 < r < r_+$. For definiteness, consider the Parikh volume of a shell in spherical symmetry, which we call ${}^3\mathcal{V}_s^*$. Then the value is, modifying (6.1.6),

$${}^3\mathcal{V}_s^* = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}T} = \int \sqrt{-g}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \int_{r_1}^{r_+} r^2\mathrm{d}r \int_0^\pi \sin\theta\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = \frac{4\pi}{3}(r_+^3 - r_1^3). \tag{6.2.9}$$

We define $\mathcal{V}_s^*$ by finding the growth rate of the volume for the region $r_1 < r < r_+$, rather than the full black hole region, with respect to the logarithm of the affine generator on the horizon, as attached to surfaces of constancy of a well-normalized Killing time parameter. It should be obvious that we find

$$\mathcal{V}_s^* = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}\ln|\lambda|} = \frac{\mathrm{d}\mathcal{V}_s}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\ln|\lambda|} = \frac{{}^3\mathcal{V}_s^*}{\kappa} \tag{6.2.10}$$

so we interpret this definition for the surface gravity as a property of the horizon alone.

Chapter 7

Generalization of Parikh Volume to Any Spaces with Killing Vector

Parikh defined his volume as the rate of change of the four-volume of a black hole region with respect to a stationary time coordinate. The essential features of the spacetime which Parikh uses in his definition of volume are the Killing vector corresponding to the stationary time coordinate, and the black hole region, whose boundaries do not change with time. We can thus generalize Parikh's development to define a generalized Parikh volume for any Killing vector, for any region whose boundaries do not change along the trajectory of this Killing vector. To distinguish between the specific case Parikh worked with and the more general case, we will use the term "Killing volume" to refer to this generalized Parikh volume, in order to emphasize the central role of the Killing vector. After some preliminaries in Section 7.1, we supply several equivalent definitions for the Killing volume in Section 7.2, and then show the equivalence of this volume under different slicings in Section 7.3.

7.1 Preliminaries

In this section we use a type of set-builder notation, in which a region (say) $\mathcal{A}$ of spacetime which represents all the points x^α which satisfy a certain condition p is written as

$$\mathcal{A} = \{x^\alpha | p\} \tag{7.1.1}$$

which is interpreted as the set of all points x^α for which p is true. Such a condition might be $p = x^3 > 0$, in which case $\mathcal{A}$ would be the set of all points for which $x^3 > 0$.

Similarly, if we wish to consider a $(D-1)$-dimensional region condition of points in the subspace with coordinates x^i, say Ξ, then we would write it as

$$\Xi = \{x^i | q\} \tag{7.1.2}$$

where q is some condition. We will usually use $\mathcal{R}$ to represent D-dimensional regions and Σ to represent $(D-1)$-dimensional regions.

Further, we use the intersection $\cap$ symbol, where $\mathcal{A} \cap \mathcal{B}$ represents the region corresponding to the set of points which lie in both $\mathcal{A}$ and $\mathcal{B}$. Finally, we use $\in$ to mean an element of, so that $x^\alpha \in A$ is true if and only if x^α is contained in $\mathcal{A}$.

Consider a D-dimensional spacetime (or space) with a Killing vector ξ^α. Any space with a Killing vector permits a (non-unique) adapted coordinate system in which $\xi^\alpha = \delta^\alpha_0$, and thus in which the metric $g_{\alpha\beta}$ is independent of x^0. We will work in such a coordinate system. Let $x^\alpha = (x^0, x^i)$, where Roman indices go over the $(D-1)$ coordinates which are not adapted to the Killing vector. In general the metric will depend on any of the other coordinates, so we can write the line element as:

$$\mathrm{d}s^2 = g_{\alpha\beta}(x^i)\mathrm{d}x^\alpha \mathrm{d}x^\beta. \tag{7.1.3}$$

We will define the volume in three different ways, all of which are equivalent. The Killing volume requires the following features:

1. The spacetime permits a Killing vector ξ^α. Its nature—timelike or spacelike—is not important for the generalized volume. In Parikh's case, this is the timelike vector for stationarity.

2. We can define a region $\mathcal{R}$ over which we calculate the volume, which can be written as

$$\mathcal{R} = \{x^\alpha | x^i \in \Sigma\} \tag{7.1.4}$$

   where Σ is a $(D-1)$-dimensional region for the coordinates x^i. This is the set of all points for which x^i lies in the region Σ. As we will show in Section 7.1.1, this has the property that the normal n_α to the boundary hypersurface $\partial\mathcal{R}$ of $\mathcal{R}$ has $n_\alpha \xi^\alpha = 0$.

   Alternatively, in some circumstances, it may be worthwhile to define a region $\mathcal{R}$ for which $x^i \in \Sigma$ is true only for some range of the variable x^0, say $a < x^0 < b$, in which case the Killing volume is only defined over the values $a < x^0 < b$. We refer to the case where $x^i \in \Sigma$ for all values of x^0 as the idealized case; whenever we work with this, it is assumed that any results determined are valid only in the range of x^0 values for which $x^i \in \Sigma$.

   In Parikh's case, the region $\mathcal{R}$ is the black hole region, usually $0 \leq r \leq r_+$ for some radial coordinate r and horizon radius r_+. In previous sections, we have often used a region where $r_1 \leq r \leq r_+$ for the same radial coordinate, where r_1 was an arbitrary inner termination point for a shell.

We will require both conditions to be true for our definitions in Section 7.2. In Section 7.3, we will show why the Parikh/Killing volume is independent of the coordinate system we choose. This is a generalization of Parikh's statement that the choice of time slicing is irrelevant to his volume. Before moving on, we will show the equivalence of the different requirements on the region $\mathcal{R}$.

7.1.1 The Region $\mathcal{R}$

Here we show the equivalence of the two requirements on $\mathcal{R}$. We begin in the adapted system and with the region $\mathcal{R}$ as described in the previous subsection. We note immediately that this (idealized) $\mathcal{R}$ imposes no restrictions on x^0: any possible value of x^0 is permissible. If some point P is in $\mathcal{R}$, with coordinates $x^\alpha = (x^0_P, x^i_P)$, then a point P' will also be in $\mathcal{R}$ if it has coordinates $x^\alpha = (x^0_{P'}, x^i_P)$ for any $x^0_{P'}$. In other words, there are no limits on x^0.

The boundary $\partial\mathcal{R}$ of $\mathcal{R}$ can be written as

$$\partial\mathcal{R} = \{x^\alpha | x^i \in \partial\Sigma\} \tag{7.1.5}$$

where $\partial\Sigma$ is the $(D-2)$-dimensional boundary of Σ. If we represent $\partial\mathcal{R}$ by some scalar function $\Theta(x^\alpha) = 0$, we note immediately that the function should depend only on the x^i, or $\Theta(x^i) = 0$. As a result, on $\partial\mathcal{R}$, $\mathrm{d}\Theta/\mathrm{d}x^0 = 0$. We note that this is exactly equal to the Lie derivative of Θ along ξ^α, in the adapted coordinate system. Thus we can conclude that if

$$\mathcal{L}_\xi\Theta \equiv \Theta,_\alpha \xi^\alpha \equiv 0 \tag{7.1.6}$$

on the region of interest, then throughout this region ξ^α will lie in $\partial\mathcal{R}$. If we let n_α be the normal to the surface $\partial\mathcal{R}$, which is proportional to $\Theta,_\alpha$—where n_α is normalized to a unit normal if $\partial\mathcal{R}$ is non-null, and if $\partial\mathcal{R}$ is null, then it remains unnormalized—then we can express this requirement as

$$n_\alpha \xi^\alpha = 0 \tag{7.1.7}$$

throughout $\partial\mathcal{R}$. This is our (coordinate-independent) requirement for the Killing volume of $\mathcal{R}$ to be defined with Killing vector ξ^α.

We can also see immediately that the converse is true. Say the normal n_α to the boundary $\partial\mathcal{R}$ has the property $n_\alpha\xi^\alpha = 0$. Then if we express boundary $\partial\mathcal{R}$ by a scalar equation $\Theta(x^\alpha) = 0$, then we can conclude that $\Theta,_\alpha \xi^\alpha = 0$, which, in adapted coordinates, suggests $\Theta,_0 = 0$ and thus that Θ is a function of x^i only. The boundary $\partial\mathcal{R}$ makes no stipulations on the value of x^0, and so, similarly, any value of x^0 will be allowed within $\mathcal{R}$.

7.2 Definitions

We use three methods to define the generalized Parikh/Killing volume, which are equivalent to each other. The first is direct, the second uses Gauss' law, and the third expresses the volume with fewer restrictions on our choice of coordinates.

7.2.1 Definition 1: Direct

The volume *element* in the space (7.1.3) is given by $\sqrt{|g_D(x^i)|}\mathrm{d}^D x$, where g_D is the determinant of $g_{\alpha\beta}$, which is (explicitly) written as a function of the x^i only as it is independent of x^0 in these adapted coordinates.

We begin by defining $\mathcal{V}_\mathcal{R}$ as the volume of the region $\mathcal{R}$. Let $\mathcal{R}$, in our adapted coordinates, be the region

$$\mathcal{R} = \{x^\alpha | x^0 \in (a,b) \text{ and } x^i \in \Sigma\}. \tag{7.2.1}$$

The boundaries on x^0 in $\mathcal{R}$ are useful for this particular definition. The D-volume of the region $\mathcal{R}$ can be written as

$$\mathcal{V}_\mathcal{R} = \int_\mathcal{R} \sqrt{|g_D(x^i)|}\mathrm{d}^D x = \int_a^b \mathrm{d}x^0 \int_\Sigma \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x = (b-a)\int_\Sigma \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x \tag{7.2.2}$$

where $\mathrm{d}^{D-1}x$ is the product of the differentials for the $\mathrm{d}x^i$. Now we make the substitution $b \to \bar{x}^0$, and thus allow the region $\mathcal{R}$ to vary with the variable $\bar{x}^0$. Now we have $\mathcal{R}(\bar{x}^0)$ defined by

$$\mathcal{R}(\bar{x}^0) = \{x^\alpha | x^0 \in (a,\bar{x}^0) \text{ and } x^i \in \Sigma\}, \tag{7.2.3}$$

or, alternatively, since this is the set of points which lie in $\mathcal{R}$ for which $x^0 \in (a,\bar{x}^0)$, by

$$\mathcal{R}(\bar{x}^0) = \mathcal{R} \cap \{x^\alpha | x^0 \in (a,\bar{x}^0)\}. \tag{7.2.4}$$

(This is the set of points which belong to both $\mathcal{R}$ and to the region for which $x^0 \in (a,\bar{x}^0)$.) The choice for the symbol $\bar{x}^0$ is because it represents the upper limit for the coordinate x^0 in $\mathcal{R}(\bar{x}^0)$. The volume will now vary with $\mathcal{R}(\bar{x}^0)$ and thus with $\bar{x}^0$. The volume can be written as

$$\mathcal{V}_{\mathcal{R}(\bar{x}^0)} = (\bar{x}^0 - a)\int_\Sigma \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x. \tag{7.2.5}$$

We now recognize $\mathcal{V}$ as a linear function of $\bar{x}^0$. We can take the derivative with respect to $\bar{x}^0$. In the case where $\bar{x}^0 = T$, a time coordinate as Parikh defined it, we recognize this as Parikh's volume from (6.1.4). So the derivative $\mathrm{d}\mathcal{V}_{\mathcal{R}(\bar{x}^0)}/\mathrm{d}\bar{x}^0$ is the *generalized Parikh/Killing volume.* By analogy with our notation for the Parikh volume, and noting that the volume depends both on the choice of Killing vector ξ^α and the region $\mathcal{R}$, we will call our new Parikh/Killing volume ${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}$. This volume is defined as:

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} \equiv \frac{\mathrm{d}\mathcal{V}_{\mathcal{R}(\bar{x}^0)}}{\mathrm{d}\bar{x}^0} \tag{7.2.6}$$

or, equivalently,

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} \equiv \int_\Sigma \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x, \tag{7.2.7}$$

where in the latter case of course $\mathcal{R}$ is defined by $x^i \in \Sigma$. We note immediately that it is a *constant* w.r.t. $\bar{x}^0$, which shows that the constancy of Parikh's volume (defined with $\bar{x}^0 = T$) is a particular case of a more general rule. We note too that the lower limit a for x^0 has dropped out. In Section 7.3, we will show explicitly that this volume is invariant under the particular choice of adapted coordinates.

In general when using the Parikh/Killing volume we will usually drop the $\mathcal{R}$ subscript when it is clear what the region for the volume is, but not the ξ subscript; note however that our definition requires both a Killing vector ξ^α and a well-defined region to be fully specified. When there is no possibility of confusion between $\bar{x}^0$, the largest possible value of x^0 in $\mathcal{R}(\bar{x}^0)$, and the coordinate x^0, we will drop the bar and simply write $\bar{x}^0$ as x^0, $\mathcal{R}(\bar{x}^0) \equiv \mathcal{R}(x^0)$.

7.2.2 Gauss' law

Another way to lead to (7.2.6) and (7.2.7) is the following. Consider a vector field $A^\alpha = x^0 \xi^\alpha = x^0 \delta^\alpha_0$. We note immediately from the definition of the covariant derivative that

$$A^\alpha_{;\alpha} = A^\alpha_{,\alpha} + \Gamma^\alpha_{\mu\alpha} A^\mu = 1 + \Gamma^\alpha_{0\alpha} x^0. \tag{7.2.8}$$

In general, Christoffel symbols can be written as

$$\Gamma^\alpha_{\alpha\beta} = \frac{g^{\alpha\mu}}{2} \frac{\partial g_{\alpha\mu}}{\partial x^\beta}. \tag{7.2.9}$$

This implies

$$\Gamma^\alpha_{0\alpha} = \Gamma^\alpha_{\alpha 0} = \frac{g^{\alpha\mu}}{2} \frac{\partial g_{\alpha\mu}}{\partial x^0} = 0, \tag{7.2.10}$$

where the first equality is due to the symmetry $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ and the third equality is due to the x^0-independence of the metric. As a result, we arrive at

$$A^\alpha_{;\alpha} = A^\alpha_{,\alpha} = 1. \tag{7.2.11}$$

The full D-volume of a region $\mathcal{R}$ is

$$\mathcal{V}_\mathcal{R} = \int_\mathcal{R} \sqrt{|g_D|} \mathrm{d}^D x = \int_\mathcal{R} A^\alpha_{;\alpha} \sqrt{|g_D|} \mathrm{d}^D x. \tag{7.2.12}$$

In the second equality we have inserted a factor of $A^\alpha_{;\alpha} = 1$. Now we recall Gauss' law (from Poisson section 3.22 [6]):

$$\int_\mathcal{R} A^\alpha_{;\alpha} \sqrt{|g_D|} \mathrm{d}^D x = \oint_{\partial\mathcal{R}} A^\alpha \mathrm{d}\Sigma_\alpha \tag{7.2.13}$$

where the latter integration is over the boundary of $\mathcal{R}$. The $\mathrm{d}\Sigma_\alpha$ term represents the directed surface element on $\partial\mathcal{R}$. (The Σ in $\lceil\Sigma_\alpha$ carries a different meaning than the Σ representing the $(D-1)$-dimensional set. We regret the confusion, but we are keeping our notation of the surface element consistent with Poisson.) This is proportional to the normal n_α of the surface. If we can define a hypersurface by some function $\Phi(x^\alpha) = 0$, then the normal to that hypersurface is proportional to $\Phi_{,\alpha}$.

We choose the same region $\mathcal{R}$ as in Section 7.2.1. In adapted coordinates where $\xi^\alpha = \delta^\alpha_0$, $\mathcal{R}$ is defined by $a < x^0 < b$ and $x^i \in \Sigma$. Then we can write this space as being the region *bounded by* the regions $(x^0 = a, x^i \in \Sigma)$, $(x^0 = b, x^i \in \Sigma)$, and $(a < x^0 < b, x^i \in \partial\Sigma)$, where $\partial\Sigma$ is the $(D-2)$-dimensional boundary of Σ. If we define $x^i \in \partial\Sigma$ by an equation $\Phi(x^i) = 0$ (which, clearly, has no dependence on x^0), the (contravariant) normal n_α to the hypersurface $x^i \in \partial\Sigma$ will be proportional to $\Phi_{,\alpha}$. We note that $\Phi_{,0} = 0$ since $\Phi(x^\alpha)$ has no x^0 dependence. This implies $n_0 = 0$. Recalling $A^\alpha = x^0\delta^\alpha_0$, we then have $A^\alpha n_\alpha = 0$ on this boundary surface. All that remains are the two boundary surfaces, $x^0 = a, x^i \in \Sigma$ and $x^0 = b, x^i \in \Sigma$.

The definition given in Poisson for $\mathrm{d}\Sigma_\alpha$, generalized to higher dimensions, is:

$$\mathrm{d}\Sigma_\mu = \epsilon_{\mu\alpha\beta\ldots\nu} e^\alpha_1 e^\beta_2 \ldots e^\nu_{D-1} \mathrm{d}^{D-1}y \tag{7.2.14}$$

where

$$e^\alpha_i = \frac{\partial x^\alpha}{\partial y^i} \tag{7.2.15}$$

where y^i are the coordinates intrinsic to the hypersurface, and where

$$\epsilon_{\mu\alpha\beta\ldots\nu} = \sqrt{|g|}[\mu\ \alpha\ \beta\ \ldots\ \nu] \tag{7.2.16}$$

is the Levi-Civita tensor, as defined in (5.3.4) but generalized to D dimensions. On the hypersurfaces $x^0 = \mathit{const.}$, we can choose coordinates $y^i = x^i$. For this choice, $e^\alpha_i = \delta^\alpha_i$ and so

$$\mathrm{d}\Sigma_\mu = \epsilon_{\mu 12\ldots(D-1)} \mathrm{d}^{D-1}x = (\partial_\mu x^0)\sqrt{|g|}\mathrm{d}^{D-1}x. \tag{7.2.17}$$

The normal should be pointing outward, so on $x^0 = b$ the normal should point in the $+\delta^\alpha_0$ direction, and on $x^0 = a$ the normal should point in the $-\delta^\alpha_0$ direction. Strictly speaking, on the $x^0 = a$ surface we want the sign of $\mathrm{d}\Sigma_\mu$ to be negative so that it remains in the direction of the normal. The way to achieve this while respecting the definition from the Levi-Civita tensor is to exchange the order of the coordinates (i.e. perhaps set $y^1 = x^2, y^2 = x^1, y^3 = x^3...$). We know what the signs should be for the $\mathrm{d}\Sigma_\mu$ on each surface, so there is no need to worry unduly about ensuring that the ordering of the coordinates is correct.

Now we can write the D-volume of some section of spacetime easily. We find from (7.2.12), (7.2.13), and (7.2.17),

$$\mathcal{V}_{\mathcal{R}} = \oint_{\partial\mathcal{R}} A^\alpha \mathrm{d}\Sigma_\alpha = \int_\Sigma x^0 \delta_0^\alpha (\partial_\alpha x^0) \sqrt{|g_D|} \mathrm{d}^{D-1}x \bigg|_{x^0=b} - \int_\Sigma x^0 \delta_0^\alpha (\partial_\alpha x^0) \sqrt{|g_D|} \mathrm{d}^{D-1}x \bigg|_{x^0=a} \quad (7.2.18)$$

$$\therefore \mathcal{V}_{\mathcal{R}} = (b-a) \int_\Sigma \sqrt{|g_D|} \mathrm{d}x^{D-1} = {}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}(b-a) \quad (7.2.19)$$

By making the substitution $b \to \bar{x}^0$ and taking the derivative with respect to $\bar{x}^0$ we recover (7.2.6) and (7.2.7). We can also conclude from the above that the Killing volume can also be written as

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_\Sigma \xi^\alpha \mathrm{d}\Sigma_\alpha. \quad (7.2.20)$$

Expressing the Killing volume in this particular form is inspired by equation 5.8a in Hayward [3]; the connection to Hayward's work is discussed in more detail in Section 14.3.

Obviously this method for developing the volume $\mathcal{V}_{\xi,\mathcal{R}}$ is equivalent to the previous one, but it aids our understanding to include it.

7.2.3 Alternate Definition

This method is equivalent to the previous two, but can be expressed with fewer restrictions on the coordinates. For this section, we introduce a coordinate system x^α which is partially adapted to ξ^α in that ξ^α's coordinate representation is constant throughout the spacetime:

$$\xi^\alpha{}_{,\beta} = 0. \quad (7.2.21)$$

We know that this can be done from the fully adapted coordinate system wherein $\xi^\alpha = \delta_0^\alpha$.

First of all, we note that the Killing vector ξ^α can be written as

$$\xi^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}, \tag{7.2.22}$$

where the derivative is along a Killing trajectory and λ parametrizes the curve. As a result, if a Killing trajectory passes through the point $x^\alpha = x^\alpha_A$, we can describe the entire trajectory by

$$x^\alpha = x^\alpha_A + \lambda \xi^\alpha \tag{7.2.23}$$

where λ is allowed to vary. This uses the fact that ξ^α's coordinate representation is constant throughout the spacetime. If we chose a set of coordinates for which ξ^α's coordinate representation changes at different points in the spacetime, we would instead have to write $x^\alpha = x^\alpha_A + \int \xi^\alpha \mathrm{d}\lambda$, where the integration is over the Killing trajectory. Our argument would be very similar from this point forward, but would be much less clear.

Now we have to choose a "slicing." Consider an unbounded hypersurface Γ which has the property that ξ^α does not lie in the hypersurface, such that for any point $x^\alpha_A \in \Gamma$, the point $x^\alpha_A + \xi^\alpha \lambda$ will lie in Γ for only $\lambda = 0$. In other words, if Γ is defined by some function $\Phi(x^\alpha_A) = 0$, then $\Phi(x^\alpha_A + \xi^\alpha \lambda) \neq 0$ for any non-zero value of λ. This will be satisfied if the derivative of Φ along the trajectory given by (7.2.23) is non-zero. Since the derivative of a scalar along a curve is also the Lie derivative $\mathcal{L}_\xi$ along that curve, we can write this stipulation as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} \equiv \mathcal{L}_\xi \Phi \equiv \Phi,_\alpha \xi^\alpha \neq 0. \tag{7.2.24}$$

We now assume that, at least in the neighbourhood of the region $\mathcal{R}$ over which we wish to calculate the Killing volume, there are no caustics or singularities, such that every Killing trajectory ξ^α intersects Γ once and only once. This means that if we let $x^\alpha(y^i)$ be the set of points which lie in Γ, parametrized by $(D-1)$ parameters y^i, we can in general write any point x^α in the spacetime as

$$x^\alpha = x^\alpha(y^i) + \xi^\alpha \lambda \tag{7.2.25}$$

for some value λ. Note immediately that we can use this to construct an adapted coordinate system $\tilde{x}^\alpha$ for our understanding by defining, at every point x^α defined according to (7.2.25), a set of coordinates $(\tilde{x}^0, \tilde{x}^i)$:

$$\tilde{x}^0 = \lambda, \qquad \tilde{x}^i = y^i, \tag{7.2.26}$$

or, in the particular decomposition from (7.2.25) based on Γ and ξ^α,

$$\tilde{x}^0 = \tilde{\xi}^\alpha \lambda, \qquad \tilde{x}^i = y^i. \tag{7.2.27}$$

It should be obvious from this that $\tilde{\xi}^\alpha = \mathrm{d}\tilde{x}^\alpha/\mathrm{d}\lambda = \tilde{\delta}^\alpha_0$ so that this is indeed an adapted coordinate system.

Now we define a hypersurface $\Gamma(\lambda)$ by

$$\Phi(x^\alpha - \lambda\xi^\alpha) = 0, \tag{7.2.28}$$

which represents the hypersurface at Γ shifted up along the Killing trajectories by λ. We can see this by noting that $\Phi(x^\alpha - \lambda\xi^\alpha) = 0$ has solutions $x^\alpha - \lambda\xi^\alpha = x^\alpha(y^i)$, where $x^\alpha(y^i)$ are the solutions to $\Phi(x^\alpha) = 0$. This corresponds to $x^\alpha = x^\alpha(y^i) + \xi^\alpha\lambda$, or, in our adapted coordinates, simply $\tilde{x}^0 = \lambda$. As a result, if we wanted to define the region defined by $0 < \tilde{x}^0 < \lambda$, we could do so by defining the region as $\mathcal{Q}_\Phi(\lambda)$ and writing it as

$$\mathcal{Q}_\Phi(\lambda) \equiv \{x^\alpha | \Phi(x^\alpha - \lambda'\xi^\alpha) = 0 \text{ for all } \lambda' \in (0, \lambda)\}. \tag{7.2.29}$$

We note immediately that the intersection of the region $\mathcal{R}$ with this new region $\mathcal{Q}_\Phi(\lambda)$ represents all the points which lie in $\mathcal{R}$ for which $0 < \tilde{x}^0 < \lambda$, which is exactly the definition used in equation (7.2.4) in Section 7.2.1 for $\mathcal{R}(\bar{x}^0)$, with λ in place of $\bar{x}^0$, $a = 0$. As a result, we can rewrite (7.2.6) as

$$\mathcal{V}^*_{\xi,\mathcal{R}} \equiv \frac{\mathrm{d}\mathcal{V}_{\mathcal{R}\cap\mathcal{Q}_\Phi(\lambda)}}{\mathrm{d}\lambda}. \tag{7.2.30}$$

This requires only a scalar function Φ for which $\Phi_{,\alpha}\,\xi^\alpha = 0$. We can construct such definition for any $\mathcal{R}$ satisfying $n_\alpha \xi^\alpha = 0$ throughout $\partial\mathcal{R}$, where n_α is the normal to $\partial\mathcal{R}$.

Note that this definition is invariant only if we confirm that the Killing volume is invariant under choice of adapted coordinate system—or, equivalently, the choice of initial slicing Γ. This will be done in Section 7.3. This definition is elegant but is perhaps not as useful for calculations.

7.2.4 An Aside on the Volume Derivative

Once again, defining a Parikh/Killing volume requires a space with a Killing vector and a region $\mathcal{R}$ for which (in adapted coordinates) $x^i \in \Sigma$. Before continuing, we want to show what happens when we relax some of these conditions. Say we have a D-dimensional metric $g_{\alpha\beta}$ and a general set of coordinates x^α. By analogy with the region $\mathcal{R}(\bar{x}^0)$ in the previous subsections, let $\mathcal{R}(\bar{x}^0)$ here be defined by allowing the coordinate x^0, which no longer has any particular meaning, vary between a and $\bar{x}^0$, and set $x^i \in \Sigma(x^0)$, a $(D-1)$-dimensional region whose boundaries depend on the hypersurface x^0 on which the region is evaluated. The volume of $\mathcal{R}(\bar{x}^0)$ is

$$\mathcal{V} = \int_{\mathcal{R}(\bar{x}^0)} \sqrt{|g(x^\alpha)|}\mathrm{d}^D x = \int_a^{\bar{x}^0} \left(\mathrm{d}x^0 \int_{\Sigma(x^0)} \sqrt{|g(x^\alpha)|}\mathrm{d}^{D-1}x \right). \tag{7.2.31}$$

By the fundamental theorem of calculus, we can differentiate this with respect to $\bar{x}^0$ to find:

$$\frac{\mathrm{d}\mathcal{V}_{\mathcal{R}(\bar{x}^0)}}{\mathrm{d}\bar{x}^0} = \int_{\Sigma(\bar{x}^0)} \sqrt{|g(x^\alpha)|}\mathrm{d}^{D-1}x. \tag{7.2.32}$$

This result shows that our ability to write the volume derivative as an integral over the other $(D-1)$-coordinates is not unique to spaces with a Killing vector, or to regions which have an x^0-independent Σ. We note that this volume derivative, unlike the Parikh/Killing volume, is not in general a constant with respect to $\bar{x}^0$. Further, the proof in Section 7.3 that the Killing volume is slicing-independent does not apply to this volume derivative.

7.3 The Equivalence of Different Slicings

In Parikh's paper, he claimed (correctly) that the choice of constant time T is irrelevant in calculating his volume, such that if t_s is the static time coordinate, any $T = t_s + h(x^i)$ will yield the same result:

$$ {}^{D-1}\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t_s} = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}T}, \text{ if } T = t_s + h(x^i). \tag{7.3.1} $$

Generalized to any space with a Killing vector, this corresponds to the idea that the Parikh/Killing volume is invariant under the choice of coordinates adapted to the Killing vector. This equivalence and the transformation properties of the $(D-1)$-dimensional coordinates x^0 will allow us to verify that ${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}$ is truly independent of coordinate choice.

We will demonstrate this in several new ways. We have already verified this by appealing to Parikh's notion of a differential spacetime volume, but these sections are more formal and demonstrate mathematically and conceptually why the invariance under choice of time slicing exists. First, we will work with (7.2.7) and show directly by transforming to new coordinates that the Parikh/Killing volume is unchanged. Then we will discuss the Lie Derivative and how it helps to show the invariance.

7.3.1 Direct Coordinate Substitution

Once again, we work in a D-dimensional spacetime with a Killing vector ξ^α. We begin with adapted coordinates such that $\xi^\alpha = \delta^\alpha_0$ and where Roman indices run over coordinates $1 \dots D-1$.

Now, consider a transformation to a new set of coordinates, which we will differentiate from the original set by the use of tildes. We will include tildes on all vectors, tensors, etc. to indicate that they are being expressed in the new coordinates. Begin with the most general transformation. We set

$$ \mathrm{d}\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\mathrm{d}x^\beta. \tag{7.3.2} $$

The Killing vector ξ^α transforms as (recalling that $\xi^\alpha = \delta_0^\alpha$):

$$\tilde{\xi}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\xi^\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^0}. \tag{7.3.3}$$

Say we wish to have $\tilde{\xi}^\alpha = \tilde{\delta}_0^\alpha$, so that the new coordinates are also adapted coordinates to this Killing vector. The necessary and sufficient condition for this is clearly

$$\frac{\partial \tilde{x}^\alpha}{\partial x^0} = \tilde{\delta}_0^\alpha \tag{7.3.4}$$

or, writing more explicitly,

$$\frac{\partial \tilde{x}^0}{\partial x^0} = 1, \frac{\partial \tilde{x}^i}{\partial x^0} = 0. \tag{7.3.5}$$

This is very significant. Since $\frac{\partial \tilde{x}^i}{\partial x^0} = 0$, the $\tilde{x}^i$ must be a function of x^j alone (using different indices to avoid confusion): $\tilde{x}^i \equiv \tilde{x}^i(x^j)$. So the ($x^0$-independent) region Σ of the $(D-1)$ dimensional subspace x^i in one adapted coordinate system x^α will still be a ($\tilde{x}^0$-independent) region of the $(D-1)$-dimensional subspace $\tilde{x}^i$ in another adapted coordinate system $\tilde{x}^\alpha$. Integrals over the $(D-1)$ coordinates, over a region Σ, will still be an integral over $(D-1)$ coordinates, over Σ, in a different set of adapted coordinates. It is the fact that the $\tilde{x}^i$ are a function of x^j alone, and not x^0, that allows this transformation from Σ to itself. If the $\tilde{x}^i$ were a function of x^j and x^0, in general Σ would be transformed to a new set $\tilde{\Sigma}$ which would depend on the $\tilde{x}^i$ and $\tilde{x}^0$.

We also note that (7.3.5) states that $\partial\tilde{x}^0/\partial x^0 = 1$, but that there are no other specifications for $\tilde{x}^0$. Thus, $\tilde{x}^0 = x^0 + h(x^i)$ is also an adapted coordinate, verifying that in Parikh's development, $T = t_s + h(x^i)$ is also adapted to the time Killing vector.

Now calculate the volume in these new coordinates, using (7.2.7). We will use ${}^{D-1}\tilde{\mathcal{V}}^*_{\xi,\mathcal{R}}$ to represent the volume calculated in these coordinates. Obviously we expect that it will be equal to ${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}$ (calculated in the original coordinates), if the volume is indeed independent of slicing.

$${}^{D-1}\tilde{\mathcal{V}}^*_{\xi,\mathcal{R}} = \int_\Sigma \sqrt{|\tilde{g}_D|}\mathrm{d}^{D-1}\tilde{x} \tag{7.3.6}$$

Now, note that

$$\tilde{g}_{\alpha\beta} = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}. \tag{7.3.7}$$

Since the determinant of the product of matrices is equal to the product of the determinants of the matrices, we write

$$\tilde{g}_D = \left|\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\right| \left|\frac{\partial x^\delta}{\partial \tilde{x}^\beta}\right| g_D = J_D^{-2} g_D \tag{7.3.8}$$

where we write J_D as the Jacobian for the transformation, defined as

$$J_D = \left|\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right|. \tag{7.3.9}$$

Writing out $\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}$, with the α index representing the rows and the β index representing the columns,

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \begin{bmatrix} 1 & 0 \\ \frac{\partial \tilde{x}^0}{\partial x^j} & \frac{\partial \tilde{x}^i}{\partial x^j} \end{bmatrix}. \tag{7.3.10}$$

Since there is only one non-zero term on the top row, the determinant will simply be 1 times the determinant of $\frac{\partial \tilde{x}^i}{\partial x^j}$ which we will call J_{D-1}—the Jacobian for the transformation from the x^j to the $\tilde{x}^i$. As a result, then we find $J_D = J_{D-1}$ in this case.

The integral becomes

$${}^{D-1}\tilde{\mathcal{V}}^*_{\xi,\mathcal{R}} = \int_\Sigma \sqrt{|\tilde{g}_D|}\mathrm{d}^{D-1}\tilde{x} = \int_\Sigma \frac{\sqrt{|g_D|}}{|J_{D-1}|}\mathrm{d}^{D-1}\tilde{x}. \tag{7.3.11}$$

We note that $\mathrm{d}^{D-1}\tilde{x} = |J_{D-1}|\mathrm{d}^{D-1}x$. We then conclude, writing out the coordinates explicitly for clarity,

$${}^{D-1}\tilde{\mathcal{V}}^*_{\xi,\mathcal{R}} = \int_{\tilde{x}^i \in \Sigma} \sqrt{|\tilde{g}_D(\tilde{x}^i)|}\mathrm{d}^{D-1}\tilde{x} = \int_{x^j \in \Sigma} \sqrt{|g_D(x^j)|}\mathrm{d}^{D-1}x = {}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} \tag{7.3.12}$$

which verifies that the volume rate is indeed invariant under (adapted) coordinate choice. The original definition (7.2.6) makes reference to sheets of constant x^0 (or $\tilde{x}^0$), but this is of no concern; the central features are the Killing vector and the region Σ of the subspace formed by other $D-1$

coordinates.

This is a good place to show explicitly that the $(D-1)$-volume of a hypersurface region ($x^0 = const., x^j \in \Sigma$) is not the same as the $(D-1)$-volume of the hypersurface region ($\tilde{x}^0 = const., \tilde{x}^j \in \tilde{\Sigma}$). Write the determinant of the $(D-1)$-surface of constant x^0 as g_{D-1} and the determinant of the metric of the $(D-1)$-surface of constant $\tilde{x}^0$ as $\tilde{g}_{D-1}$. Further, using the definition of the matrix inverse, we note that

$$g^{00} = \frac{\text{cofactor}(g_{00})}{g_D} = \frac{g_{D-1}}{g_D} \tag{7.3.13}$$

where cofactor$(g_{\mu\nu})$ is the determinant of the matrix $g_{\alpha\beta}$ when the μth row and the νth column are removed, which in this case is obviously the determinant of g_{ij} which we call here g_{D-1}. (The same relations apply to the transformed coordinates.) Then the $(D-1)$-volume of the hypersurface $\tilde{x}^0 = const., x^i \in \Sigma$ is given by

$$^{D-1}\tilde{\mathcal{V}} = \int_{\tilde{\Sigma}} \sqrt{|\tilde{g}_{D-1}|}\mathrm{d}^{D-1}\tilde{x} = \int_{\tilde{\Sigma}} \sqrt{\frac{|\tilde{g}_D|}{|\tilde{g}^{00}|}}\mathrm{d}^{D-1}\tilde{x} = \int_{\Sigma} \sqrt{\frac{1}{|\tilde{g}^{00}(\tilde{x}^i(x^j))|}}\sqrt{|g_D|}\mathrm{d}^{D-1}x \tag{7.3.14}$$

We can rewrite g_D as g_{D-1}/g^{00}. In general, $g^{00} \neq \tilde{g}^{00}$. Thus,

$$^{D-1}\tilde{\mathcal{V}} = \int_{\Sigma} \sqrt{\frac{|g^{00}|}{|\tilde{g}^{00}(\tilde{x}^i(x^j))|}}\sqrt{|g_{D-1}|}\mathrm{d}^{D-1}x \neq \int_{\Sigma} \sqrt{|g_{D-1}|} = {}^{D-1}\mathcal{V}, \tag{7.3.15}$$

so we can view the quantity $\sqrt{|g^{00}/\tilde{g}^{00}|}$ as a measure of the degree to which the volume of the hypersurface defined by Σ changes under coordinate transformation.

We do note that while $^{D-1}\mathcal{V}$ is coordinate-dependent, within a given adapted coordinate system, it is independent of the value of x^0 at which the slice is taken.

7.3.2 Lie Derivative

This section uses the intrinsic derivative, or Lie derivative. It is similar to Parikh's argument as quoted in Section 6.1.

Assume we have a D-dimensional spacetime which has a Killing vector ξ^α defined everywhere. One way to interpret this spacetime manifold is to consider it as a series of nonintersecting Killing trajectories. Each Killing trajectory is a curve for which ξ^α is always tangent to the curve. In adapted coordinates $\xi^\alpha = \delta_0^\alpha$, each curve has constant coordinates x^i and is parametrized by x^0. To keep our development clear and general, we introduce a series of labels y^i ($D-1$ labels), such that each curve is uniquely defined by a certain value of $y^1, y^2, ...y^{D-1}$. Further, along each trajectory labelled by y^i, we can introduce a variable μ which parametrizes the curve. The simplest case is the case in which $\mu = x^0$ and $y^i = x^i$ for some adapted coordinates. In general, each point in the spacetime can be described either by a set of (not necessarily adapted) coordinates x^α, *or* by the set of labels y^i and parameter μ—i.e. the labels for the Killing trajectory which intersects the point, and the value of the parameter along that trajectory. As a result, we can write $x^\alpha = x^\alpha(\mu, y^i)$, which is to say, the coordinates x^α are a function of μ and y^i. Let $\gamma(y^i)$ be the curve labelled by y^i; in adapted coordinates, we can let the label be x^i and so write the curve as $\gamma(x^i)$, where x^i is constant along the curve. A sketch of the situation is shown in Figure 7.1.

Now we supply a new interpretation of $\mathcal{V}_{\mathcal{R}(\bar{x}^0)}$. Here $\mathcal{R}(\bar{x}^0)$ is defined by $x^i \in \Sigma$, $a \leq x^0 \leq \bar{x}^0$. We can write the volume of $\mathcal{R}(\bar{x}^0)$ as:

$$\mathcal{V}_{\mathcal{R}(\bar{x}^0)} = \int_{x^i \in \Sigma} \left(\int_{x^0=a}^{x^0=\bar{x}^0} \mathrm{d}^D\mathcal{V}(x^0, x^i) \bigg|_{\gamma(x^i)} \right), \tag{7.3.16}$$

in which we first integrate all the (invariant) volume elements along each curve $\gamma(x^i)$ from $x^0 = a$ to $x^0 = \bar{x}^0$, and then second integrate over all curves $\gamma(x^i)$ which form $\mathcal{R}(\bar{x}^0)$. We now define a "curve volume element" $\delta\mathcal{V}(x^\alpha)$ for all $x^0 > a$ as follows:

$$\delta\mathcal{V}(x^\alpha) = \int_{x^{0\prime}=a}^{x^{0\prime}=x^0} \mathrm{d}^D\mathcal{V}(x^{0\prime}, x^i) \bigg|_{\gamma(x^i)}. \tag{7.3.17}$$

This quantity represents the sum of the contributions of all the volume elements along the curve $\gamma(x^i)$ from $x^0 = a$ to the maximum value at x^0. We can think of it as the volume contribution of a segment of the curve $\gamma(x^i)$. Since the curve $\gamma(x^i)$ is independent of coordinates, as are the invariant volume elements, we find that $\delta\mathcal{V}(x^\alpha)$ depends on coordinates only insofar as it depends

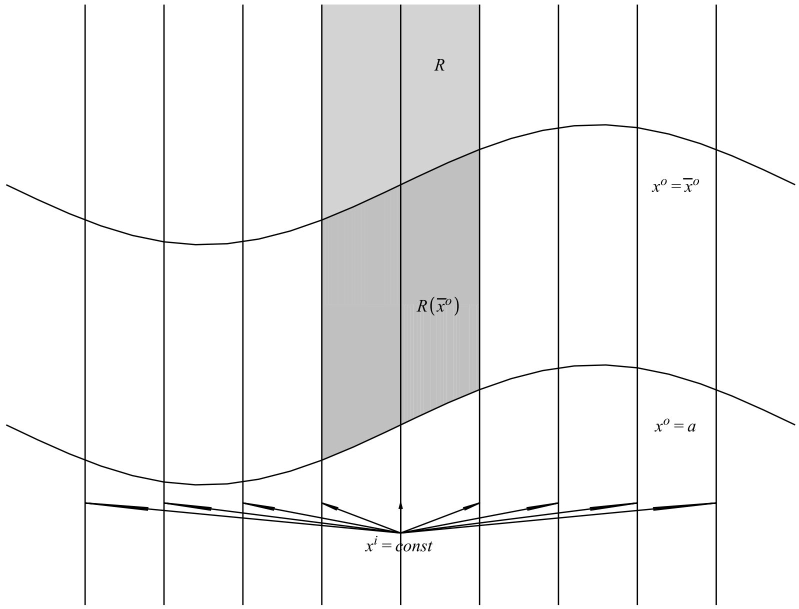

Figure 7.1. Schematic picture of spacetime. The horizontal axis represents different values of the coordinates x^i; motion along the vertical axis represents motion along the Killing vector, adapted to coordinate x^0. The hypersurfaces $x^0 = a$ and $x^0 = \bar{x}^0$ are shown. The light grey region (which includes the dark grey region) represents the full spacetime region $\mathcal{R}$, for which x^i belong to a certain subset and $x^0 \geq a$; the dark grey region represents $\mathcal{R}(\bar{x}^0)$ which has $a \leq x^0 \leq \bar{x}^0$. This diagram helps visualize the spacetime as being composed of Killing trajectories $x^i = const$.

on the "starting point" $x^0 = a$ of integration. Clearly we can then write

$$\mathcal{V}_{\mathcal{R}(\bar{x}^0)} = \int_{x^i \in \Sigma} \delta\mathcal{V}(\bar{x}^0, x^i). \tag{7.3.18}$$

Using the definition (7.2.6) for the Killing volume, we can write

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \frac{\mathrm{d}}{\mathrm{d}\bar{x}^0} \int_{x^i \in \Sigma} \delta\mathcal{V}(\bar{x}^0, x^i), \tag{7.3.19}$$

and, since x^i are independent of $\bar{x}^0$, we can rearrange the order of the integral and derivative like so:

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_{x^i \in \Sigma} \frac{\mathrm{d}}{\mathrm{d}\bar{x}^0} \delta\mathcal{V}(\bar{x}^0, x^i). \tag{7.3.20}$$

Now change variables from $\bar{x}^0$ to x^0 to see

$$^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_{x^i\in\Sigma} \frac{\mathrm{d}}{\mathrm{d}x^0}\delta\mathcal{V}(x^0, x^i). \tag{7.3.21}$$

Clearly differentiating along the curve removes the reference to $x^0 = a$ in the integral. In these adapted coordinates, $\mathrm{d}/\mathrm{d}x^0 = \partial/\partial x^0 = \xi^\alpha\partial_\alpha$. We can now write

$$^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_{x^i\in\Sigma} \xi^\alpha\partial_\alpha\delta\mathcal{V}(x^0, x^i). \tag{7.3.22}$$

It should be obvious that $\xi^\alpha\partial_\alpha\delta\mathcal{V}(x^0, x^i)$ is a manifestly invariant quantity: the only aspect of $\delta\mathcal{V}$ that was coordinate dependent was the "starting" point $x^0 = a$ for the integration, and this is removed by the (invariant) differentiation by $\xi^\alpha\partial_\alpha$, which can be interpreted as the intrinsic or Lie derivative along the curve. Thus $\xi^\alpha\partial_\alpha\delta\mathcal{V}(x^\alpha)$ is coordinate-invariant and defined at every point. We now note that if we evaluate this in adapted coordinates, it is equal to

$$\xi^\alpha\partial_\alpha\delta\mathcal{V} = \frac{\mathrm{d}}{\mathrm{d}x^0}\int_{x^{0\prime}=a}^{x^{0\prime}=x^0} \sqrt{|g_D(x^i)|}\mathrm{d}^D x = \frac{\mathrm{d}}{\mathrm{d}x^0}\left((x^0-a)\sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x\right) = \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x, \tag{7.3.23}$$

where $d^{D-1}x$ runs over the $(D-1)$ coordinates x^i. This is entirely independent of x^0, which means that it is an invariant along each curve $\gamma(x^i)$. Let us call this the *Killing volume element* which is associated with each curve $\gamma(x^i)$, which we will label $\delta\mathcal{V}^*_\xi(\gamma(x^i))$:

$$\delta\mathcal{V}^*_\xi(\gamma(x^i)) \equiv \xi^\alpha\partial_\alpha\delta\mathcal{V}(x^\beta). \tag{7.3.24}$$

Then the Killing volume can be interpreted as a Riemann sum over all Killing curves $\gamma(x^i)$ which intersect with $\mathcal{R}$. While it makes sense to use coordinates to label these Killing trajectories, any coordinate system we choose to label them will yield the same total set of Killing trajectories, and thus the same total Riemann sum of the (coordinate-invariant) Killing volume elements. To emphasize the coordinate-invariance, we reintroduce the labels y^i to label all the various Killing trajectories, which is not strictly necessary for our argument but we feel helps demonstrate the coordinate invariance. We still let the condition that $x^i \in \Sigma$ be written as $y^i \in \Sigma$, but it should be clear that the essential feature is that the integral is over all curves, labelled by y^i, which lie within $\mathcal{R}$.

$$ {}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_{y^i \in \Sigma} \delta\mathcal{V}^*_\xi(\gamma(y^i)). \tag{7.3.25}$$

This is clearly equal to previous definitions and is clearly invariant under change of coordinates.

Chapter 8

On the Choice of Killing Vector

8.1 Motivation

The development thus far has shown that once we have a well-defined region $\mathcal{R}$ and a Killing vector ξ^α, it is possible to develop a Parikh/Killing volume of this region. What we have not yet considered is the effect that comes from choosing a different Killing vector. What is the effect on the Killing volume of a region if we multiply the Killing vector by a multiplicative constant? What happens in spaces that permit multiple Killing vectors? It turns out that there is an important special case in which there is a set of Killing vectors of the form $\xi^\alpha + H\omega^\alpha$, where H can take on any value, that yield the same volume. This becomes particularly important in the Kerr-Newman spacetime.

In Section 8.2 we will show the effect on the Killing volume when we multiply its Killing vector by a multiplicative constant. In Section 8.3 we will examine what happens when a space or spacetime permits two independent Killing vectors, and what happens if we try to take the volume related to a linear combination of these two vectors. Then we show examples of this effect in Section 8.4 and in Section 8.5 we discuss an apparent contradiction, but one which we can resolve. Section 8.6 summarizes these results, and Section 8.7 reviews the choice of the t coordinate for the Parikh volume.

8.2 Killing vectors multiplied by a constant

Consider the situation in Chapter 7, where we begin with a D-dimensional spacetime with a Killing vector ξ^α, and we choose adapted coordinates $x^\alpha = (x^0, x^i)$ such that $\xi^\alpha_0 = \delta^\alpha_0$. Any Killing vector times a non-zero constant is also a Killing vector. Let us define a new coordinate system $\tilde{x}^\alpha$ which is adapted to the Killing vector $K\xi^\alpha$, where K is such a non-zero constant. Obviously these coordinates have

$$K\tilde{\xi}^\alpha = \tilde{\delta}^\alpha_0. \tag{8.2.1}$$

Using transformations between coordinate frames, we can write this as

$$K\tilde{\xi}^\alpha = K\xi^\beta \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = K\delta^\beta_0 \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = K\frac{\partial \tilde{x}^\alpha}{\partial x^0}. \tag{8.2.2}$$

This implies that $K\frac{\partial \tilde{x}^\alpha}{\partial x^0} = \tilde{\delta}^\alpha_0$. Let us say that we want all the coordinates x^i to be the same in both coordinate systems: $\tilde{x}^i = x^i$. As a result the only "new" coordinate is $\tilde{x}^0$. This implies $1 = K\frac{\partial \tilde{x}^0}{\partial x^0}$, or, up to an additive constant, $\tilde{x}^0 = x^0/K$. From (7.3.8), the determinant of the metric of the new coordinates $\tilde{g}_D$ is given by $\tilde{g}_D = J_D^{-2} g_D$. From (7.3.9), $J_D = \det\left|\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right|$. Since

$$\frac{\partial \tilde{x}^\alpha}{\partial x^0} = \frac{\delta^\alpha_0}{K}, \quad \frac{\partial \tilde{x}^\alpha}{\partial x^i} = \delta^\alpha_i, \tag{8.2.3}$$

the Jacobian for the transformation from x^α to $\tilde{x}^\alpha$ coordinates is $J_D = 1/K$, so that $\tilde{g}_D = K^2 g_D$. This implies that for any permissible region $\mathcal{R} = \{x^\alpha | x^i \in \Sigma\}$, we can write the volume ${}^{D-1}\mathcal{V}^*_{K\xi,\mathcal{R}}$ corresponding to the Killing vector $K\xi^\alpha$ as,

$${}^{D-1}\mathcal{V}^*_{K\xi,\mathcal{R}} = \int_\Sigma \sqrt{|\tilde{g}_D|}\mathrm{d}^{D-1}x = |K| \int_\Sigma \sqrt{|g_D|}\mathrm{d}^{D-1}x = |K|({}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}). \tag{8.2.4}$$

In other words, the Killing volume of a region is proportional to the magnitude of the Killing vector chosen for that region. This result will be used later.

8.3 Two Killing Vectors?

Now we consider a situation where a spacetime permits two Killing vectors, with special emphasis on the case where one of the coordinates is cyclic, such as the ϕ coordinate in axial symmetry (where $\phi = \phi_0$ and $\phi = \phi_0 + 2\pi$ are identified with each other). Let a spacetime permit two Killing vectors for which a set of coordinates adapted to both vectors exist. We will call the vectors ξ^α and ω^α and require that in our adapted coordinates, $\xi^\alpha = \delta^\alpha_0$ and $\omega^\alpha = \delta^\alpha_1$. Thus the metric is independent of x^0 and x^1. We will use capital Roman letters to represent the other $D-2$ indices. We will continue to use lower case Roman indices to cover indices from $1, ..., D-1$.

Assume that we have a region $\mathcal{R}$ for which we can define a Killing volume with respect to ξ^α. As usual, we can define $\mathcal{R}$ as $\mathcal{R} = \{x^\alpha | x^i \in \Sigma\}$ for some Σ. If we define a new Killing vector $\psi^\alpha = \xi^\alpha + H\omega^\alpha$ (where H is a non-zero constant), for what regions $\mathcal{R}$ can we calculate a Killing volume for $\mathcal{R}$ with respect to ψ^α? From Section 7.1.1, if we let n_α be the normal to the boundary $\partial\mathcal{R}$ of $\mathcal{R}$, then $n_\alpha \xi^\alpha = 0$. Similarly, the requirement that $\mathcal{R}$ be a valid region for which to calculate the Killing volume of ψ^α is that $n_\alpha \psi^\alpha = 0$, which implies

$$0 = n_\alpha \psi^\alpha = n_\alpha(\xi^\alpha + H\omega^\alpha) = Hn_\alpha \omega^\alpha, \tag{8.3.1}$$

or

$$n_\alpha \omega^\alpha = 0. \tag{8.3.2}$$

Also from Section 7.1.1, if $n_\alpha \omega^\alpha = 0$, then, if x^1 is the coordinate for which $\delta^\alpha_1 = \omega^\alpha$, any value of x^1 is permissible in $\mathcal{R}$. In other words, Σ should be:

$$\Sigma = \{x^i | x^A \in \Psi\} \tag{8.3.3}$$

where Ψ is some $(D-2)$-dimensional region which does not depend on either x^0 or x^1. This would imply, from (7.2.7), that the Killing volume ${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}$ for ξ^α would be

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = \int_\Sigma \sqrt{|g_D(x^A)|}\mathrm{d}^{D-1}x = \int \mathrm{d}x^1 \int_\Psi \sqrt{|g_D(x^A)|}\mathrm{d}^{D-2}x, \tag{8.3.4}$$

where $\mathrm{d}^{D-2}x$ is the product of the differentials for the $\mathrm{d}x^A$. We note first of all that the integral over Ψ is fully x^1-independent. The integral over $\mathrm{d}x^1$ is over all permissible values of x^1. If the coordinate x^1 can vary from, say, $-\infty$ to ∞ over the manifold, then this would imply an infinite volume. (Consider the case where x^1 is time t, or any of the spatial coordinates (x, y, z) in Minkowski space.) Clearly if we desire a finite volume, we require that $\int \mathrm{d}x^1$ integrated over all possible values of x^1 in the spacetime be finite.

One case for which the range of values of x^1 in the spacetime is finite is the case wherein x^1 is a cyclic coordinate with period P, such that the points $x^\alpha = (x^0, x^1, x^A)$ and $x^A = (x^0, x^1 + mP, x^A)$ are equivalent for integer m and some period P. In this case, $\int \mathrm{d}x^1 = P$, and so

$$ {}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = P \int_\Psi \sqrt{|g_D(x^A)|}\mathrm{d}^{D-2}x. \tag{8.3.5}$$

Now: what is the value of ${}^{D-1}\mathcal{V}^*_{\psi,\mathcal{R}}$? Let us write new coordinates $\tilde{x}^\alpha$ that are adapted to ψ^α and ω^α, demanding $\tilde{\psi}^\alpha = \tilde{\delta}^\alpha_0$, $\tilde{\omega}^\alpha = \tilde{\delta}^\alpha_1$. This implies

$$\tilde{\delta}^\alpha_1 = \tilde{\omega}^\alpha = \omega^\beta \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \delta^\beta_1 \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^1}. \tag{8.3.6}$$

and

$$\tilde{\delta}^\alpha_0 = \tilde{\psi}^\alpha = (\xi^\beta + H\omega^\beta)\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = (\delta^\beta_0 + H\delta^\beta_1)\frac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^0} + H\frac{\partial \tilde{x}^\alpha}{\partial x^1} \tag{8.3.7}$$

Combining the two equations, we conclude

$$\frac{\partial \tilde{x}^\alpha}{\partial x^0} = \tilde{\delta}^\alpha_0 - H\tilde{\delta}^\alpha_1, \quad \frac{\partial \tilde{x}^\alpha}{\partial x^1} = \tilde{\delta}^\alpha_1. \tag{8.3.8}$$

We can choose a new set of coordinates. If we choose $\tilde{x}^A = x^A$ and let $\tilde{x}^0$ and $\tilde{x}^1$ be independent of x^A, then we are left with:

$$\frac{\partial \tilde{x}^0}{\partial x^0} = 1, \frac{\partial \tilde{x}^0}{\partial x^1} = 0, \frac{\partial \tilde{x}^0}{\partial x^A} = 0, \tag{8.3.9}$$

which yields $\tilde{x}^0 = x^0 + \mathcal{C}_0$. We can choose to set the constant $\mathcal{C}_0$ to zero. Additionally, we have

$$\frac{\partial \tilde{x}^1}{\partial x^0} = -H, \frac{\partial \tilde{x}^1}{\partial x^1} = 1, \frac{\partial \tilde{x}^1}{\partial x^A} = 0, \tag{8.3.10}$$

which yields $\tilde{x}^1 = x^1 - Hx^0 + \mathcal{C}_1$. We can choose to set the constant $\mathcal{C}_1$ to zero. In summary, our transformation is $\tilde{x}^0 = x^0, \tilde{x}^1 = x^1 - Hx^0, \tilde{x}^A = x^A$. Of most interest is that $\tilde{x}^0 = x^0$ in these coordinates. If we define $\mathcal{R}(\bar{x}^0)$ using (7.2.4), we can see immediately that $\mathcal{R}(\bar{x}^0)$ is equal to both

$$\mathcal{R}(\bar{x}^0) = \mathcal{R} \cap \{x^\alpha | x^0 \in (a, \bar{x}^0)\} = \mathcal{R} \cap \{\tilde{x}^\alpha | \tilde{x}^0 \in (a, \bar{x}^0)\}. \tag{8.3.11}$$

This is to say, $\mathcal{R}(\bar{x}^0)$ plays the same role in coordinates adapted to either ξ^α or ψ^α. As a result, from (7.2.6), we see that the quantity $\mathrm{d}\mathcal{V}_{\mathcal{R}(\bar{x}^0)}/\mathrm{d}\bar{x}^0$ is equal to the Killing volume of $\mathcal{R}$ for *both* ξ^α and $\psi^\alpha = \xi^\alpha + H\omega^\alpha$, so that ${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = {}^{D-1}\mathcal{V}^*_{\psi,\mathcal{R}}$. In other words, if ω^α is a Killing vector which permits a cyclic coordinate, for certain regions $\mathcal{R}$, one can add any constant times ω^α to the Killing vector ξ^α, and not change the Killing volume. This remarkable result can be summarized in a theorem.

Theorem. Assume we have a spacetime or space for which adapted coordinates exist such two Killing vectors exist, $\delta^\alpha_0 \equiv \xi^\alpha$ and $\delta^\alpha_1 \equiv \omega^\alpha$, and such that x^1 is a cyclic coordinate with period P. Then it is possible to write the Killing volume for any region region $\mathcal{R}$ for which the normal n_α to the boundary hypersurface $\partial\mathcal{R}$ has $n_\alpha \xi^\alpha = n_\alpha \omega^\alpha = 0$. Further, the Killing volume of $\mathcal{R}$ with respect to ξ^α and the Killing volume with respect to $\xi^\alpha + H\omega^\alpha$ for any constant H are identical:

$${}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}} = {}^{D-1}\mathcal{V}^*_{\xi+H\omega,\mathcal{R}}. \tag{8.3.12}$$

8.4 Examples

The most obvious example is the spherically symmetric metric in four dimensions. We will show the invariance established in the previous subsection by calculating Killing volumes explicitly. We write the spherically symmetric metric in Eddington-Finkelstein-like form as in equation (5.1.2). As a reminder, this metric is:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}V^2 + 2\mathrm{d}r\mathrm{d}V + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{8.4.1}$$

We note the Killing vectors $\xi^\alpha = \delta^\alpha_V$ and $\omega^\alpha = \delta^\alpha_\phi$. These coordinates are then adapted. Obviously the period of ϕ is 2π. The metric determinant is $-g = r^4\sin^2\theta$.

We take the horizon to be at $r = r_+$ and wish to find the volume of the region below the horizon, i.e. set $\mathcal{R}$ to be $(r,\theta,\phi) \in \Sigma$, where Σ is $0 \le \phi < 2\pi$, $0 \le r \le r_+$, $0 \le \theta \le \pi$. Then the Killing volume for region $\mathcal{R}$ and Killing vector ξ^α is

$${}^3\mathcal{V}^*_{\xi,\mathcal{R}} = \int_0^{r_+}\mathrm{d}r\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi(r^2\sin\theta) = \frac{4\pi}{3}r_+^3. \tag{8.4.2}$$

Now set $\psi^\alpha = \xi^\alpha + H\omega^\alpha$ for some constant H. New adapted coordinates will be $(V, r, \theta, \tilde{\phi})$ where $\tilde{\phi} = \phi - HV$. This implies that we can write $\mathrm{d}\phi = \mathrm{d}\tilde{\phi} + H\mathrm{d}V$. The metric becomes

$$\mathrm{d}s^2 = (-f(r) + H^2r^2\sin^2\theta)\mathrm{d}V^2 + 2\mathrm{d}V\mathrm{d}r + Hr^2\sin^2\theta\mathrm{d}V\mathrm{d}\tilde{\phi} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\tilde{\phi}^2. \tag{8.4.3}$$

Here $\mathcal{R}$ is represented by the usual limits $0 \le r \le r_+$ and $0 \le \theta \le \pi$, as well as $0 \le \tilde{\phi} < 2\pi$. The determinant of the metric is once again $-r^4\sin^2\theta$. As a result, the Killing volume for vector ψ^α, region $\mathcal{R}$ is

$${}^3\mathcal{V}^*_{\psi,\mathcal{R}} = \int_0^{r_+}\mathrm{d}r\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\tilde{\phi}(r^2\sin\theta) = \frac{4\pi}{3}r_+^3. \tag{8.4.4}$$

Another important example is the Kerr-Newman metric. In Kerr coordinates, we write the metric as

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)\mathrm{d}V^2 + 2\mathrm{d}r\mathrm{d}V + \frac{\Upsilon}{\rho^2}\sin^2\theta\mathrm{d}\psi^2 - 2a\sin^2\theta\mathrm{d}\psi\mathrm{d}r\\ &- \frac{2a(2Mr - Q^2)\sin^2\theta}{\rho^2}\mathrm{d}\psi\mathrm{d}V + \rho^2\mathrm{d}\theta^2\end{aligned} \tag{8.4.5}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + Q^2, \quad \Upsilon = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \tag{8.4.6}$$

(Normally $(r^2 + a^2)^2 - a^2\Delta\sin^2\theta$ is written as Σ, but we choose Υ to avoid confusion with the 3-dimensional region Σ.) Clearly $\delta^\alpha_V \equiv \xi^\alpha$ and $\delta^\alpha_\psi \equiv \psi^\alpha$ are Killing vectors. The metric determinant is $g = -\rho^4\sin^2\theta$. The black hole event horizon is defined by $r = r_+$ where $r_+ = M + \sqrt{M^2 - a^2 - Q^2}$. Letting the black hole region be $\mathcal{R}$, we see that it is defined by setting $0 \le r \le r_+$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$. Calculating the Parikh volume for region $\mathcal{R}$ and Killing vector ξ^α, we find:

$${}^3\mathcal{V}^*_{\xi,\mathcal{R}} = \int_0^{r_+} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\psi(\rho^2\sin\theta) = \frac{4\pi}{3} r_+(r_+^2 + a^2). \tag{8.4.7}$$

Now, if we define a Killing vector $\zeta^\alpha = \xi^\alpha + \Omega\phi^\alpha$, where Ω is a constant, we can use the usual coordinate transformations to define a new set of coordinates $(V, r, \theta, \tilde{\psi})$ for which $\tilde{\psi} = \psi - \Omega V$. Then noting that $\mathrm{d}\phi = \mathrm{d}\tilde{\phi} + \Omega\mathrm{d}V$, the metric becomes:

$$\begin{aligned} \mathrm{d}s^2 = &\left(1 - \frac{(2Mr - Q^2)(1 + 2a\Omega\sin^2\theta) + \Omega^2\Upsilon\sin^2\theta}{\rho^2}\right)\mathrm{d}V^2 + 2(1 - \Omega a\sin^2\theta)\mathrm{d}r\mathrm{d}V \\ &+ \frac{\Upsilon}{\rho^2}\sin^2\theta\mathrm{d}\tilde{\psi}^2 - 2a\sin^2\theta\mathrm{d}\tilde{\psi}\mathrm{d}r - \frac{2(a(2Mr - Q^2) - \Omega\Upsilon)\sin^2\theta}{\rho^2}\mathrm{d}\tilde{\psi}\mathrm{d}V + \rho^2\mathrm{d}\theta^2 \end{aligned} \tag{8.4.8}$$

which also has determinant $-g = \rho^4\sin^2\theta$. Thus the volume using the black hole region $\mathcal{R}$ and the Killing vector ζ^α is

$${}^3\mathcal{V}^*_{\zeta,\mathcal{R}} = \int_0^{r_+} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\tilde{\psi}(\rho^2\sin\theta) = \frac{4\pi}{3} r_+(r_+^2 + a^2) = {}^3\mathcal{V}^*_{\xi,\mathcal{R}}. \tag{8.4.9}$$

Of particular interest is the case in which $\Omega = \Omega_H = a/(r_+^2 + a^2)$, in which ζ^α is tangent to the null generators of the horizon. This will be used in Section 9.1.

8.5 An Apparent Contradiction

The previous two sections present an interesting problem. How do we square the result that the Killing volume of a particular region calculated using $\xi^\alpha + H\omega^\alpha$ (where ω^α permits an adapted coordinate which is cyclic) is independent of H with the result from (8.2.4) that the Killing volume of a region calculated using $K\xi^\alpha$ is proportional to the value of K? The result (8.2.4) did not require us to specify whether or not the Killing vector ξ^α permitted a cyclic or non-cyclic adapted coordinate. Are these statements inconsistent, and should we be surprised by the fact that the constant multiplying the cyclic Killing vector ω^α has no impact on the final volume?

The answer to these final two questions is no. One of the essential features is that (8.3.12) requires that the coordinate x^1, the coordinate for which $\omega^\alpha = \delta^\alpha_1$, lies within the subspace formed by the x^i. This means that it is impossible to calculate a Parikh volume for this $\mathcal{R}$ using the Killing vector ω^α, since that would require both $\tilde{\omega}^\alpha = \tilde{\delta}^\alpha_0$ and $\tilde{\omega}^\alpha = \tilde{\delta}^\alpha_1$, clearly a contradiction if x^0 and x^1 are to be independent coordinates. Since the Killing volume ${}^{D-1}\mathcal{V}^*_{\omega,\mathcal{R}}$ is not defined for this $\mathcal{R}$, it is perhaps not surprising that the addition of $H\omega^\alpha$ to ξ^α does not affect the Killing volume for $\mathcal{R}$.

8.6 Combined Results

We can combine the results from (8.2.4) and (8.3.12) in a straightforward manner. If there exists a metric that can be written in adapted coordinates so that $\delta^\alpha_0 \equiv \xi^\alpha$ and $\delta^\alpha_1 \equiv \omega^\alpha$ are Killing vectors, and x^1 is a cyclic coordinate with period P, then we can write the Killing volume for any region $\mathcal{R}$ for which the normal n_α to the boundary hypersurface $\partial\mathcal{R}$ has $n_\alpha\xi^\alpha = n_\alpha\omega^\alpha = 0$. The relationship between the Killing volumes of $\mathcal{R}$ with respect to ξ^α and $K\xi^\alpha + H\omega^\alpha$ is:

$$ {}^{D-1}\mathcal{V}^*_{K\xi+H\omega,\mathcal{R}} = |K|\left({}^{D-1}\mathcal{V}^*_{\xi,\mathcal{R}}\right). \tag{8.6.1} $$

provided $K \neq 0$.

8.7 The Choice of t

Now that we have established (8.6.1), there is a new problem for the Parikh volume of black holes. If we have a metric which permits an axial (cyclic) Killing vector, as well as a non-cyclic one, which Killing vector should we use to define the volume for black holes? Obviously the region $\mathcal{R}$ should be the black hole region, but there may be infinitely many choices for the Killing vector, since we can choose any linear combination of the two Killing vectors. Since, as established by Hawking in 1972 [25], all stationary black holes are either static or axially symmetric, it is frequently the case that there are two Killing vectors. In spacetimes, it is assumed that the primary Killing vector will be the vector that preserves stationarity. But how do we choose it when we might not know *a priori* which linear combination of Killing vectors will represent the true, *most meaningful* measure of the volume?

Parikh's choice was the Killing vector which most closely represents the Killing vector $\delta_t^\alpha \equiv t^\alpha$ in Minkowski space. Minkowski space is axially symmetric (totally spherically symmetric) and so permits an axial Killing vector $\delta_\phi^\alpha \equiv \phi^\alpha$ which corresponds to the cyclic coordinate ϕ which has period 2π. Since we can write Minkowski space in spherically symmetric coordinates as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{8.7.1}$$

we note immediately that

$$t^\alpha t_\alpha = g_{tt} = -1 \tag{8.7.2}$$

and

$$t^\alpha \phi_\alpha = g_{t\phi} = 0. \tag{8.7.3}$$

This means that a desired Killing vector ζ^α will have, asymptotically, $\zeta^\alpha\zeta_\alpha \to -1, \zeta^\alpha\phi_\alpha \to 0$ where ϕ^α is the Killing vector for axial symmetry. In coordinates adapted to ζ^α where $\zeta^\alpha = \delta_0^\alpha$ and where $\phi^\alpha = \delta_\phi^\alpha$, this is equivalent to demanding that, asymptotically, $g_{00} \to -1$, $g_{0\phi} \to 0$.

Say there is some stationary and axially symmetric black hole spacetime. Let us call its Killing vectors ξ^α (stationarity) and ϕ^α (spherical symmetry) where trajectories of ϕ^α are closed, and in coordinates adapted so that $\phi^\alpha = \delta^\alpha_\phi$, ϕ is a cyclic coordinate with period 2π, as in Minkowski space. Further, let us demand that $\xi^\alpha \equiv \delta^\alpha_0$, so that it is adapted to coordinate x^0. At present we do not know if ξ^α is the "preferred" Killing vector for calculating the volume or not. Assume that this preferred Killing vector exists, and write it as (throughout using adapted coordinates)

$$\zeta^\alpha \equiv K\xi^\alpha + H\phi^\alpha. \tag{8.7.4}$$

Parikh suggests fixing the asymptotic form of the metric, i.e. choosing coordinate x^0 (or, rather, T in his case) such that $g_{00} = -1$. We shall do so as well, and additionally demand that, asymptotically, the Killing vector is "rotation free"—i.e. set (asymptotically) $\zeta^\alpha \phi_\alpha = 0$. So then we solve:

$$\zeta^\alpha \zeta_\alpha \to -1 \implies (K\xi^\alpha + H\phi^\alpha)(K\xi_\alpha + H\phi_\alpha) = K^2 g_{00} + 2KHg_{0\phi} + H^2 g_{\phi\phi} \to -1 \tag{8.7.5}$$

$$\zeta^\alpha \phi_\alpha \to 0 \implies (K\xi^\alpha + H\phi^\alpha)\phi_\alpha = Kg_{0\phi} + Hg_{\phi\phi} \to 0. \tag{8.7.6}$$

Note that, in general, $g_{\phi\phi}$ will not be finite in the asymptotic limit, since in the flat-space limit, $g_{\phi\phi} \to r^2 \sin^2\theta \neq 0$. At any rate, if the above admits a solution in (K, H) then it is possible to select a Killing vector which is asymptotically timelike and asymptotically has the same normalization as the time Killing vector in Minkowski space. That said, we will print out the general solution to this set of equations, assuming that $K > 0$:

$$K = \lim_{r\to\infty} \left(\frac{g_{\phi\phi}}{g_{0\phi}^2 - g_{00}g_{\phi\phi}} \right)^{\frac{1}{2}} \tag{8.7.7}$$

$$H = \lim_{r\to\infty} -g_{0\phi} \left(g_{\phi\phi} \left(g_{0\phi}^2 - g_{00}g_{\phi\phi} \right) \right)^{-\frac{1}{2}} \tag{8.7.8}$$

In case the reader is curious why there is an emphasis on finding the correct value for both K *and* H when H (or the component of the final Killing vector in the direction of the axial Killing vector ϕ^α) has no affect on the Killing volume, as has been verified previously, it is because of the condition (8.7.5) which depends on both K and H. If we choose a Killing vector which is asymptotically rotating, the condition on K to normalize g_{00} will be incorrect.

An alternate way to write the equation for ζ^α is, setting $|m|^2 \equiv m^\alpha m_\alpha$:

$$\zeta^\alpha = \lim_{r\to\infty} \frac{\phi_\gamma(\phi^\gamma \xi^\alpha - \xi^\gamma \phi^\alpha)}{\sqrt{|\phi|^2((\xi^\beta \phi_\beta)^2 - |\xi|^2|\phi|^2)}} \tag{8.7.9}$$

All this is a little ungainly. But we hope to demonstrate that merely possessing two independent Killing vectors, one of which is the vector for axial symmetry, is not sufficient to be able to calculate the Parikh volume which has the fixed asymptotic form; we must *find* that asymptotic form.

There are some spacetimes, however, in which it is not possible to find an asymptotically timelike Killing vector. Most obvious is the de Sitter metric which admits a cosmological horizon. But if we work with the spherically symmetric metric in general four-dimensional spacetime, we note

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_2^2 \tag{8.7.10}$$

which permits obvious Killing vectors $t^\alpha = \delta^\alpha_t$, $\phi^\alpha = \delta^\alpha_\phi$. Asymptotically, we have

$$t^\alpha t_\alpha = g_{tt} = -f(r) \tag{8.7.11}$$

$$\phi^\alpha \phi_\alpha = g_{\phi\phi} = r^2 \sin^2\theta. \tag{8.7.12}$$

If $f(r) \to C$ where C is some constant, as $r \to \infty$, then it is possible to find a "normalized" Killing vector which is a multiple of t; frequently we find that $f(r) \to 1$ (as in the Reissner-Nordström-Schwarzschild class of spacetimes). However, if $f(r) \to \Lambda r^2/3$ as in de Sitter, then there is no possible way to find a normalization that will lead to an asymptotically constant $t^\alpha t_\alpha$.

This was troubling for some time. Is there a consistent explanation for what the meaning of t was in metrics such as de Sitter, where the asymptotic form does not exist? Why is t used instead of some $\tilde{t} = 5t$? There is nothing wrong about t as a coordinate and t^α as a Killing vector, but there is little that seems uniquely *right* about it. The usual definition of the surface gravity of the horizon of a black hole makes reference to an observer at infinity in order to justify its normalization of κ, but this only makes any sense if the time coordinate t is such that $g_{tt} = -1$ at infinity. An argument for the use of t in the spherically symmetric metric is put forth by Jacobson [21], who suggests that there is something special about the condition

$$g_{tt}g_{rr} = -1 \tag{8.7.13}$$

where r is the aerial radius, in metrics with $g_{t\alpha} = 0$ for $\alpha \neq t$ and $g_{r\beta} = 0$ for $\beta \neq r$. This is still somewhat unsatisfying. This becomes particularly problematic when we move away from spherical symmetry. For example, the metric for a Kerr-Newman black hole in an anti-de Sitter spacetime can be written as (see, e.g. [27])

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\left(\mathrm{d}t - \frac{a\sin\theta}{\Xi}\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\phi\right)^2 \tag{8.7.14}$$

where $\Delta = \left(1 + \frac{r^2}{L^2}\right)(r^2 + a^2) - 2mr + Q^2$, $\Delta_\theta = 1 - \frac{a^2}{L^2}\cos^2\theta$, $\Xi = 1 - \frac{a^2}{L^2}$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $\Lambda = -\frac{3}{L^2}$. This metric has

$$g_{tt} = \frac{-\Delta + \Delta_\theta \sin^2\theta}{\rho^2} \tag{8.7.15}$$

so that we note

$$\lim_{r\to\infty} g_{rr}g_{tt} = \lim_{r\to\infty}\frac{-\Delta + \Delta_\theta\sin^2\theta}{\Delta} = -1. \tag{8.7.16}$$

As a result, in this case the Killing vector of choice, t^α, can only be justified by appealing to the limit $g_{rr}g_{tt}$ at spatial infinity in coordinates where $g_{r\alpha} = 0$ for $\alpha \neq r$. In these cases, we have to modify (8.7.5) so that $\zeta^\alpha\zeta_\alpha \to 1/g_{rr}$ asymptotically, which eventually changes (8.7.9) to

$$\zeta^\alpha = \lim_{r\to\infty}\frac{\phi_\gamma(\phi^\gamma\xi^\alpha - \xi^\gamma\phi^\alpha)}{\sqrt{g_{rr}|\phi|^2((\xi^\beta\phi_\beta)^2 - |\xi|^2|\phi|^2)}}. \tag{8.7.17}$$

Note, however, that we don't have an easy interpretation for r, which is not the aerial radius. Some discussion of the meaning of the coordinate r in Kerr-Newman spacetimes (and Kerr-Newman-anti-de Sitter) is found in Chapter 15. We revisit equation (8.7.17) in Section 9.2. At the moment we turn to using the ideas of this chapter and the last and apply them to our volume.

Chapter 9

Our Volume, Revisited

The introduction of the Killing volume raises the question: is it possible to use the Killing volume to define our volume in an even more direct fashion? The answer is yes. The definitions given earlier for our volume rely on ingoing null geodesics. The definition for the Killing volume is defined based on Killing trajectories. In this section we will show that it is possible to define our volume as the Killing volume where the Killing vector is normalized not according to asymptotic properties but according to the horizon itself.

Throughout this section, it is assumed that the region over which Killing volumes are calculated is the black hole region, so there is no need for the subscript $\mathcal{R}$ at any point.

9.1 Our Volume as a Killing Volume

Assume that we have a black hole spacetime which is stationary and axially symmetric. All such spacetimes permit a Killing vector ξ^α which is tangent to the null generators of the horizon, for which (again, on the horizon only)

$$\xi^\alpha_{;\beta}\xi^\beta = \kappa\xi^\alpha \tag{9.1.1}$$

where κ is the surface gravity of the horizon. Generally speaking, ξ^α is written as

$$t^\alpha + \Omega_H \phi^\alpha \tag{9.1.2}$$

where t^α is the Killing vector which has the usual asymptotic properties ($g_{tt} \to -1$ or $-1/g_{rr}$, $g_{t\phi} \to 0$) and ϕ^α is the Killing vector for axial symmetry where ϕ varies by 2π. If the vector which is tangent to the null generators of the horizon is known but t^α is not, t^α can be found using (8.7.17).

It should be obvious that any non-zero constant K times ξ^α will yield

$$(K\xi^\alpha)_{;\beta}(K\xi^\beta) = K^2\kappa\xi^\alpha = K\kappa(K\xi^\alpha) \tag{9.1.3}$$

which means that for the new vector $K\xi^\alpha$, we effectively have a new value for the constant of proportionality which was once held by κ. If we define $K = 1/\kappa$ and set $k^\alpha \equiv \xi^\alpha/\kappa$, then we have:

$$k^\alpha_{;\beta} k^\beta = k^\alpha, \tag{9.1.4}$$

where

$$k^\alpha = \frac{t^\alpha + \Omega_H \phi^\alpha}{\kappa}. \tag{9.1.5}$$

k^α is the unique Killing vector which is proportional to the null generators on the horizon and which satisfies (9.1.4) on the horizon. (If the horizon is degenerate, of course, no solution to (9.1.4) exists since the Killing vectors tangent to the horizon have $\xi^\alpha_{;\beta} = 0$.) We then note the following. Assume we have a coordinate system of adapted coordinates (k, x^i) where $k^\alpha = \delta^\alpha_k$. Since k^α is tangent to the null generators on the horizon, obviously on the horizon (which we will call $\mathcal{H}$), we have $x^i = const., k = k(\lambda)$, where λ is an affine parameter on the horizon. Then from Poisson [6] 1.3, an affine parameter λ can be found by noting that $k = \lambda^*$ where λ^* is the non-affine parameter of the curve as written, and then calculating

$$\frac{\mathrm{d}\lambda}{\mathrm{d}k} = \exp\left[\int^k \mathrm{d}k'\right] = \exp k, \tag{9.1.6}$$

which, rearranging, implies (up to a linear transformation of the affine parameter) that,

$$k = \ln \lambda. \tag{9.1.7}$$

We know from our work with the Killing volume that the Killing volume of a region calculated using the vector k^α will be expressible in terms of its adapted coordinate k as

$${}^{D-1}\mathcal{V}_k^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}k}, \tag{9.1.8}$$

which implies

$${}^{D-1}\mathcal{V}_k^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda} \tag{9.1.9}$$

which is part of our definition for $\mathcal{V}^*$! We need only to verify that λ can be attached to ingoing radial (or principal) null geodesics. We can do so by appealing to our work in Chapters 3-5 to show this is true at least in spherical symmetry (and Kerr-Newman). Having established this, we can use the slicing-invariance under slicing of the Killing volume to dispense with the necessity of λ being attached to a coordinate v which is constant on sets of ingoing radial null geodesics, noting only that it is necessary for such a coordinate to exist which is equivalent to λ on the horizon. So we can define our volume as the Parikh/Killing volume of the black hole region for Killing vector k^α:

$$\mathcal{V}^* = {}^{D-1}\mathcal{V}_k^* = \int_\Sigma \sqrt{|g_D(x^i)|}\mathrm{d}^{D-1}x \tag{9.1.10}$$

where k^α is the unique Killing vector for which $k^\alpha_{;\beta}k^\beta = k^\alpha$ on the horizon. For the second equality we require only that the coordinates x^α are adapted to k^α with $k^\alpha = \delta^\alpha_0$. Further, we demand that there exists a set of adapted coordinates for which ingoing null geodesics lie on $x^0 = const$. We can of course rewrite equation (6.2.8) now as the ratio of two Killing volumes:

$$|\kappa| \equiv \frac{{}^3\mathcal{V}_t^*}{{}^3\mathcal{V}_k^*} \tag{9.1.11}$$

where k^α is the Killing vector for which (9.1.4) holds on the horizon and t^α is the Killing vector which has the asymptotic properties as specified in Section 8.7.

Note the cleanness of this. It requires no knowledge of the global information about the spacetime. We do not have to refer to the asymptotic form of the metric or the asymptotic normalization of any Killing vectors. In practice, it may well be that we will never encounter a metric for which a Killing vector on the horizon is well known but its asymptotic normalization is too complicated to deal with. Nevertheless, our volume stands out starkly as using properties that depend on the spacetime structure of the horizon and below. This is not a surprise, of course, since the reference to the properties of the horizon alone was a key feature of our volume when we used our equivalent, original definition using the log of the affine parameter.

We now have a clear reason why the full four-volume increases with the log of the affine parameter. It is because the four-volume will always increase linearly with a Killing parameter (provided the hypersurface which is transported along the Killing trajectories remains constant), and the parameter which is affine on the horizon along which we are increasing the four-volume is logarithmically related to the Killing parameter.

We also note that we can also interpret why the derivative of the volume with respect to the log of the affine parameter was a constant if we chose radial shells ($r_1 < r < r_+$) for the boundaries of our volume, but not if we chose shells bounded by sets of outgoing null geodesics (as in Section 4.5). In the former case, the Killing vector k^α obviously lies within the boundary $r = r_1$, so that the (constant) Killing volume is well-defined for the region $r_1 < r < r_+$. Since k^α does not lie in the boundary $u = const.$, the Killing volume is not well-defined for the region $u_1 < u < 0$, so that $\mathrm{d}\mathcal{V}/\mathrm{d}\ln\lambda$ is not expected to be a constant.

9.2 Our Volume and Parikh's

Now we discuss the relationship between this and Parikh's definition of his volume—which references the asymptotic form of the metric.

If we know k^α and we know ϕ^α, the Killing field of azimuthal symmetry, we can find the Killing vector for stationarity which, asymptotically, appears as the usual t in Minkowski space (in Kerr-

Newman) or at least has $g_{tt}g_{rr} \to -1$ asymptotically from equation (8.7.17). We write

$$t^\alpha = \lim_{r\to\infty} \frac{\phi_\gamma(\phi^\gamma k^\alpha - k^\gamma \phi^\alpha)}{\sqrt{g_{rr}|\phi|^2((k^\beta\phi_\beta)^2 - |k|^2|\phi|^2)}}. \tag{9.2.1}$$

We know from (8.6.1) that ${}^{D-1}\mathcal{V}^*_{K\xi+H\phi} = K({}^{D-1}\mathcal{V}^*_\xi)$. As a result, since we see immediately that the black hole region for both the Parikh volume (based on the vector $t^\alpha = Kk^\alpha + H\phi^\alpha$) and for our region is defined by $0 < r < r_+$, we can write that the Parikh volume ${}^{D-1}\mathcal{V}^*_t$ can be written as our volume ${}^{D-1}\mathcal{V}^*_k$ times the constant

$$\lim_{r\to\infty} \left(g_{rr} \left(\frac{(k^\alpha\phi_\alpha)^2}{\phi^\beta\phi_\beta} - k^\alpha k_\alpha \right)\right)^{-\frac{1}{2}}. \tag{9.2.2}$$

Since we know that this constant of proportionality is in fact the surface gravity, this allows us to write another expression of the surface gravity in terms of the Killing vector on the horizon which has $k^\alpha_{;\beta}k^\beta = k^\alpha$ and the usual azimuthal Killing vector ϕ^α:

$$\kappa = \lim_{r\to\infty} \left(g_{rr} \left(\frac{(k^\alpha\phi_\alpha)^2}{\phi^\beta\phi_\beta} - k^\alpha k_\alpha \right)\right)^{-\frac{1}{2}}. \tag{9.2.3}$$

For Kerr-Newman, we can substitute $k^\alpha = (t^\alpha + \Omega_H\phi^\alpha)/\kappa$, $g_{rr} \to -1$ to verify that this equation holds.

Chapter 10

Surface Gravity Discussion

Now we revisit the result from (6.2.8). The usual physical meaning given to the surface gravity is, as explained by Poisson [6], "the force required of an observer at infinity to hold a particle (of unit mass) stationary at the horizon." The interpretation given here, that the surface gravity is the ratio of the Parikh volume to the rate of change with respect to the logarithm of the affine parameter of the invariant four-volume for a shell of arbitrary (but non-vanishing) thickness bounded by the horizon, is a local interpretation that would appear to be new. There is also the fully equivalent statement (9.1.11).

This interpretation does not remove one of the fundamental problems with the old definition of the surface gravity: finding the Killing vector with which to define the Parikh volume still requires appealing to the asymptotic form of the metric (or the asymptotic normalization of the magnitude of the Killing vector).

An important question is, can we use this to gain further insights into black hole mechanics? The most obvious conclusion regards the third law of black hole mechanics, $\kappa \nrightarrow 0$. Since ${}^3\mathcal{V}_s^* > 0$ even in the degenerate case (consider, for example, the static spherically symmetric case) we see that the third law demands that the rate of growth $\mathcal{V}_s^*$ must remain finite. In order to violate the third law we need $\mathcal{V}_s^* \to \infty$ and since $\mathrm{d}\mathcal{V}_s/\mathrm{d}\lambda$ is finite, this requires $\lambda \to \infty$ in agreement with the formulation of Israel [26]. That is, in a sequence of quasi-static steps, the reduction of κ to zero would take infinite advanced time.

Chapter 11

Comparison Between Surface Area and Volume of Black Holes

The surface area A of a black hole horizon is in general defined by making a cut Γ of $r = r_+$, t constant. For the Kerr class of spacetimes in Kerr coordinates, this cut leads to a two-space metric

$$\mathrm{d}s_\Gamma^2 = (r_\pm^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \frac{(r_\pm^2 + a^2)^2}{r_\pm^2 + a^2\cos^2\theta}\sin^2\theta\mathrm{d}\phi^2. \tag{11.0.1}$$

This metric has determinant $g = -(r_\pm^2 + a^2)^2\sin^2\theta$, so that integrating $\sqrt{|g|}$ over θ, ϕ yields

$$A = 4\pi(r_\pm^2 + a^2). \tag{11.0.2}$$

Similarly the two-metric for constant r, t for all spherically symmetric metrics such as (4.0.1), taking the horizon as $r = r_+$, is

$$\mathrm{d}s_\Gamma^2 = r_+^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{11.0.3}$$

which has determinant $-r_+^4 \sin^2\theta$. Integrating $\sqrt{|g|}$ over θ, ϕ gives

$$A = 4\pi r_+^2 \tag{11.0.4}$$

which is the surface area of the two-sphere of radius r_+.

We note a significant connection between these values of area and the calculated volume rates from previous sections. Our volume in spherical symmetry and in Kerr-Newman is given by (4.3.4) and (5.3.16) respectively, and the Parikh volume is related to our volume by (6.2.8). If we express our volume with r_+ as the outer horizon and $r_1 = 0$, we get $\mathcal{V}^* = 4\pi r_+^3/3\kappa$ (spherical symmetry) and $\mathcal{V}^* = 4\pi r_+(r_+^2 + a^2)/3\kappa$ (Kerr-Newman). This implies that, at least in spherical symmetry and in Kerr-Newman, we have:

$$\mathcal{V}^* = \frac{Ar_+}{3\kappa} \tag{11.0.5}$$

and

$${}^3\mathcal{V}^* = \frac{Ar_+}{3}. \tag{11.0.6}$$

Having established the method for calculation of the area as well as some simple relations between the volume and area, we now discuss formulas of black hole mechanics.

Chapter 12

Review of Black Hole Mechanics with Area

Black hole mechanics, a field which links the properties of black holes by analogy to thermodynamic properties of a system, is generally formulated in terms of the area A of a black hole. In this chapter we will review the major concepts of BH mechanics in terms of the area before writing new, reformulated rules in terms of these volume rates.

The laws of black hole mechanics were written by Bardeen, Carter and Hawking in 1973 [7], and bear a resemblance to the four laws of classical thermodynamics. Their formulation, which is the formulation most frequently used in the relativity community, has the area of the event horizon of a black hole as a central player. The laws concern stationary black holes.

The zeroth law of BH mechanics is that for a stationary black hole, the surface gravity κ is constant everywhere on the horizon.

A black hole in vacuum (with $\Lambda = 0$) can be described by three parameters: mass M, angular momentum J and charge Q in the Kerr-Newman metric (5.3.1). It was discovered by Larry Smarr [28] that the area A of a stationary, axially symmetric black hole (with $\Lambda = 0$) can be written as a function of M, J and Q as

$$\frac{\kappa A}{4\pi} = M - 2\Omega_H J - \Phi Q \tag{12.0.1}$$

where Ω_H is the angular speed of the black hole (and of a stationary observer just outside the BH), and Φ is the electric potential of the black hole. Since Hawking showed [25] that all stationary black holes are either static or axially symmetric, this covers all black holes that are stationary and non-static. In Kerr-Newman, these quantities are

$$\Omega_H \equiv \frac{a}{r_+^2 + a^2} \tag{12.0.2}$$

and

$$\Phi \equiv \frac{Qr_+}{r_+^2 + a^2}. \tag{12.0.3}$$

where $a = J/M$ and $r_+ = M + \sqrt{M^2 - Q^2 - J^2/M^2}$. Since $A = 4\pi(r_+^2 + a^2)$, we can also write A explicitly as

$$A = 4\pi(r_+^2 + a^2) = 4\pi\left(2M^2 - Q^2 + 2\sqrt{M^4 - Q^2M^2 - J^2}\right). \tag{12.0.4}$$

It is possible to write the change in area A of a black hole for a quasi-static process in terms of the change in these three parameters. We can do this by writing A as a function of (M, J, Q) and then considering variations of A with respect to variations, represented by δ, in the parameters:

$$\delta A = \left.\frac{\partial A}{\partial M}\right|_{J,Q} + \left.\frac{\partial A}{\partial J}\right|_{M,Q} + \left.\frac{\partial A}{\partial Q}\right|_{M,J}. \tag{12.0.5}$$

Following Poisson [6], we can begin by writing A as a function of $r_+ = M + \sqrt{M^2 - J^2/M^2 - Q^2}$ and $a = J/M$. Since $A = 4\pi(r_+^2 + a^2)$, we can write

$$\frac{\delta A}{8\pi} = r_+\delta r_+ + a\delta a. \tag{12.0.6}$$

From the defining equation of r_+, $r_+^2 - 2Mr_+ + a^2 + Q^2 = 0$, we can write

$$2r_+\delta r_+ - 2M\delta r_+ - 2M\delta r_+ + 2a\delta a + 2Q\delta Q = 0 \tag{12.0.7}$$

$$\therefore \delta r_+(r_+ - M) = r_+\delta M - a\delta a + Q\delta Q. \tag{12.0.8}$$

From $a = J/M$ we can write δa in terms of M, J as

$$\delta a = \frac{\delta J}{M} - a\frac{\delta M}{M}. \tag{12.0.9}$$

Combining equations (12.0.6), (12.0.8) and (12.0.9), we arrive at

$$\frac{\delta A}{8\pi} = \frac{r_+}{r_+ - M}\left(r_+\delta M - a\delta a + Q\delta Q\right) + a\delta a \tag{12.0.10}$$

$$\therefore \frac{\delta A}{8\pi} = (r_+ - M)\left[r_+^2\delta M + r_+ Q\delta Q + a\delta a\left(-r_+ + r_+ - M\right)\right] \tag{12.0.11}$$

$$\therefore \frac{\kappa}{8\pi}\delta A = \delta M + \frac{a}{r_+^2 + a^2}\delta J + \frac{r_+ Q}{r_+^2 + a^2}\delta Q \tag{12.0.12}$$

We can combine (12.0.12) with the definitions (12.0.2) and (12.0.3) to yield the following, which is known as the *first law of black hole mechanics*:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi\delta Q. \tag{12.0.13}$$

The second law of BH mechanics is that the area A of a BH cannot decrease through classical processes. (It turns out that it can decrease through quantum processes, such as through Hawking radiation.) This led to the identification of the black hole's area with the entropy S of a black hole, since thermodynamically the entropy of a closed system cannot decrease. Hawking's aforementioned discovery that black holes emit radiation with a thermal spectrum equivalent to a system with temperature

$$T = \frac{\hbar}{2\pi}\kappa \tag{12.0.14}$$

helped fix the constant of proportionality between the area and the presumed entropy S_{BH}, as we will see below.

The third law states that it is not possible to reduce the surface gravity of a black hole to zero in finite advanced time. Since the surface gravity is thought to be analogous to the temperature, this is similar to the first law of thermodynamics—which is that the temperature of a system cannot be brought to zero in finite time.

Now we revisit the first law. As is well known, the first law of thermodynamics can be written that the change in internal energy of some object (for which the number of particles is constant) is equal to the amount of heat added to the object minus the amount of work done by the object:

$$\mathrm{d}E = \delta\mathcal{Q} - \delta W \tag{12.0.15}$$

where $\delta\mathcal{Q} = T\delta S$ is the change in heat $\mathcal{Q}$, with temperature T and entropy S, and W is the mechanical work done by the object.

The mass M of a black hole can be likened to its total energy E. We have already proposed the identification of A with S (up to a multiplicative constant) and T with $\kappa\hbar/2\pi$. In classical processes, the amount of work required to produce an incremental change in angular momentum is the angular speed times the change in angular momentum, and the amount of work required to produce an incremental change in charge is the the electrostatic potential times the change in charge. As a result, we further interpret the quantity $\Omega_H\delta J + \Phi\delta Q$ as $-\delta W$, the amount of work done on the black hole in an incremental change. These identifications allow us to interpret (12.0.13) as

$$\delta M = \frac{\kappa}{8\pi}\delta A - \delta W, \tag{12.0.16}$$

which is analogous to (12.0.15). This along with (12.0.14) is consistent with an identification of

$$S_{BH} = \frac{A}{4\hbar} \tag{12.0.17}$$

for the black hole's entropy. One of the mysterious properties is that the entropy of the black hole should be equal (up to a multiplicative constant) to the area of the black hole horizon, rather than relating to the volume within the black hole.

Before continuing on, we will make another observation about the implications of the first law.

The surface area, revisiting (12.0.13), implies

$$\left.\frac{\partial A}{\partial M}\right|_{J,Q} = \frac{8\pi}{\kappa}. \tag{12.0.18}$$

Since κ is equal to $(r_+ - M)/(r_+^2 + a^2)$, κ is always finite provided that we do not have $r_+ = 0, a = 0$, which occurs only when $M = J = Q = 0$. As a result, we conclude that there are no black holes (with at least one non-zero parameter) for which $\partial A/\partial M_{J,Q} = 0$. There are no values of M, J, Q for which the area is at an extremum at a given value of M. Additionally, we find

$$\left.\frac{\partial A}{\partial J}\right|_{Q,M} = -\frac{8\pi\Omega_H}{\kappa}, \tag{12.0.19}$$

which is zero only when $\Omega_H = 0$ and so $J = 0$. Similarly,

$$\left.\frac{\partial A}{\partial Q}\right|_{J,M} = -\frac{8\pi\Phi}{\kappa} \tag{12.0.20}$$

which is zero only when $\Phi = 0$ and thus $Q = 0$. In general we can conclude that, for positive κ, A always increases with respect to M and decreases with respect to $|J|$ and $|Q|$.

Chapter 13

Black Hole Mechanics with Volume

We have described two invariantly defined quantities relating to the 4-volume. One is our volume, $\mathcal{V}^* \equiv \mathrm{d}\mathcal{V}/\mathrm{d}\ln|\lambda|$. The other is the Parikh volume ${}^3\mathcal{V}^* \equiv \mathrm{d}\mathcal{V}/\mathrm{d}t$. The surface gravity can be defined as the ratio of the two. As discussed in Chapter 10, we can use this definition to develop a simple demonstration of the third law of BH mechanics.

We now turn our attention to the first law. In this chapter we will write out our volume and the Parikh volume for a Kerr-Newman black hole in terms of M, J and Q, and then write a new variation law for each of these volumes in turn. Then we attempt to develop expressions for black hole entropy expressed in terms of the volume(s) rather than in terms of the area.

We assume here that $M^2 - J^2/M^2 - Q^2 \geq 0$, which implies that $r_+ = M + \sqrt{M^2 - J^2/M^2 - Q^2}$ is well-defined, real, and is greater than or equal to M.

In this chapter, we use the volume rates based on the region from $r = 0$ to $r = r_+$; this is perhaps misleading, particularly in the case where there is a non-zero charge, and thus there are Cauchy instabilities below the inner horizon. We assume that any impact from this is negligible in these initial calculations.

13.1 Our Volume Rate $\mathcal{V}^*$

From (5.3.16), we can express $\mathcal{V}^*$ in terms of r_+ and κ as $\mathcal{V}^* = 4\pi r_+(r_+^2 + a^2)/3\kappa$. Since we have

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}, \tag{13.1.1}$$

$$a = \frac{J}{M} \tag{13.1.2}$$

and

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2}, \tag{13.1.3}$$

$\mathcal{V}^*$ becomes

$$\mathcal{V}^* = \frac{4\pi}{3}\left[\frac{16M^6 - 20M^4Q^2 - 12M^2J^2 + 5M^2Q^4 + 4Q^2J^2}{\sqrt{M^4 - M^2Q^2 - J^2}} + 16M^4 + Q^4 - 12Q^2M^2 - 4J^2\right] \tag{13.1.4}$$

A simpler formulation (which we can find by combining (12.0.13) and (11.0.5)) is

$$\frac{3\kappa^2\mathcal{V}^*}{4\pi r_+} = M - 2\Omega_H J - \Phi Q \tag{13.1.5}$$

and the differential relationship, which is a new formulation of the first law of black hole mechanics for Kerr-Newman black holes, can be written as:

$$(6\kappa r_+ - 1)\delta M = \frac{3\kappa^3}{4\pi}\delta\mathcal{V}^* + (4\kappa r_+ - 1)\Omega_H\delta J + \left(4\kappa r_+ - \frac{M}{r_+}\right)\Phi\delta Q \tag{13.1.6}$$

As with the area A, we note that $\partial V^*/\partial J|_{M,Q} = 0$ when $\Omega_H = J = 0$ and $\partial V^*/\partial Q|_{M,J} = 0$ when $\Phi = Q = 0$. However, there are other values of (M, J, Q) in which the derivative of $\mathcal{V}^*$ with respect one of those parameters is zero. There are special values $\kappa r_+ = \frac{1}{4}$, $\frac{1}{6}$ and $\frac{M}{4r_+}$, at which points, respectively, $\partial\mathcal{V}^*/\partial M|_{J,Q}$, $\partial\mathcal{V}^*/\partial J|_{M,Q}$, and $\partial\mathcal{V}^*/\partial Q|_{M,J}$ become identically zero. Additionally, $\partial M/\partial\mathcal{V}^*_{Q,J} = 0$ when $\kappa = 0$, in the degenerate case. All these values for κr_+ permit physically acceptable solutions, which tabulated in Table 13.4. This means that there are particular values for which our volume achieves an extremum with respect to one of the three parameters (M, Q, J), after which it either increases or decreases.

From testing out special values explicitly in Maple, in the case in which $\kappa > 0$, it turns out that $\kappa r_+ = \frac{1}{4}$ is a minimum for $\mathcal{V}^*$ w.r.t. M, $\kappa r_+ = \frac{1}{6}$ is a local minimum for $\mathcal{V}^*$ with respect to $|J|$ (and $J = 0$ is a local maximum), and $\kappa = \frac{M}{4r_+}$ is a local minimum for $\mathcal{V}^*$ with respect to $|Q|$ (with $Q = 0$ as a local maximum). In all three cases, this is where the other two parameters are kept constant.

13.2 The Parikh Volume ${}^3\mathcal{V}^*$

From (6.1.7) we can express ${}^3\mathcal{V}^*$ in terms of r_+ and κ as ${}^3\mathcal{V}^* = 4\pi r_+(r_+^2 + a^2)/3$. This means that we can write ${}^3\mathcal{V}^*$ as:

$$ {}^3\mathcal{V}^* = \frac{4\pi}{3M}\left(4M^4 - 3M^2Q^2 - 2J^2 + (4M^2 - Q^2)\sqrt{M^4 - Q^2M^2 - J^2}\right). \tag{13.2.1} $$

We can also write this as (using (11.0.6) and (12.0.13))

$$ \frac{3\kappa {}^3\mathcal{V}^*}{4\pi r_+} = M - 2\Omega_H J - \Phi Q. \tag{13.2.2} $$

The differential relationship—another formulation of the first law—can be written as:

$$ \frac{\kappa}{4\pi r_+}\delta({}^3\mathcal{V}^*) = \left(1 + \frac{a^2}{3Mr_+}\right)\delta M - \left(1 + \frac{a^2}{3r_+^2}\right)\Phi\delta Q - \left(\frac{4}{3} - \frac{Q^2}{3Mr_+}\right)\Omega_H\delta J. \tag{13.2.3} $$

Unlike in Section 13.1, we do not discover any new special values. For example,

$$ \left.\frac{\partial({}^3\mathcal{V})}{\partial M}\right|_{J,Q} = \frac{4\pi r_+}{\kappa}\left(1 + \frac{a^2}{3Mr_+}\right). \tag{13.2.4} $$

Since for every black hole (which has $M > 0$), κ is finite and r_+ is non-zero, the only possible case in which this derivative is zero is $3Mr_+ + a^2 = 0$, which, since $M > 0, r_+ > 0$, has no real solutions. Similarly, excepting the trivial case $Q = 0$, $\partial({}^3\mathcal{V})/\partial Q = 0$ has no real solutions, since it requires

$3r_+^3 + a^2 = 0$. Besides the trivial case $J = 0$, $\partial(^3\mathcal{V})/\partial J = 0$ occurs only when $4Mr_+ - Q^2 = 0$, which looks a little more promising. This, however, turns out to have no real solutions either. This implies that there are no situations where the derivatives of the volume with respect to any of the three parameters become zero, except where $\Omega_H = 0$ (for the J-derivative) and $\Phi = 0$ (for the Q-derivative); and so they never change sign. This means we conclude

$$\frac{\partial(^3\mathcal{V}^*)}{\partial M} > 0, \frac{1}{\Omega_H}\frac{\partial(^3\mathcal{V}^*)}{\partial J} < 0, \frac{1}{\Phi}\frac{\partial(^3\mathcal{V}^*)}{\partial Q} < 0 \tag{13.2.5}$$

which have the same signs as the rate of change of A w.r.t. M, J, Q do.

13.3 Comparison

See Tables 13.1 through 13.3 for a comparison of different formulations of the first law written in terms of A, $\mathcal{V}^*$ and $^3\mathcal{V}^*$. We set $\hbar = 1$ throughout and also define $x \equiv r_+ - M = \sqrt{M^2 - Q^2 - J^2/M^2}$ so that we can write equations compactly in a form that makes the dependencies on M, Q and J as clear as possible. We have written the equations in many forms. Table 13.1 focuses on equivalents to the Smarr formula. Table 13.2 focuses on differential relationships. Table 13.3 begins with the assumption that the mass M does represent the total energy of the black hole, that the differential work δW is equivalent to $-\Omega_H \delta J - \Phi \delta Q$, that the temperature T is equivalent to $\kappa/2\pi$, and that the first law (12.0.15) does hold. These assumptions naturally imply that $S = A/4$ as usual, since these assumptions and the first law of classical thermodynamics are what is necessary to define $S = A/4$; even if S is a linear function of area, however, it is worth examining how this appears as a function of volume.

In Table 13.4, we print all the special values for which the volume derivative with respect to one of the parameters (M, J, Q) are identically zero. In addition we include the case in which the change in heat $\delta\mathcal{Q}$ with respect to the parameters is zero, under the assumption that $\delta M = dE$, $\delta W = -\Omega_H \delta J - \Phi \delta Q$ and $dE = \delta\mathcal{Q} - \delta W$. In this table, we have used α and q to represent $\alpha = a/M = J/M^2$ and $q = Q/M$, as well as $x = r_+ - M = \sqrt{M^2 - Q^2 - a^2}$.

One of the problems with writing out these formulas is that it is difficult to to decide on the

Area Relations	Vol. derivative $\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda}$	Killing Vol. derivative ${}^3\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t}$
$A = 4\pi\left(r_+^2 + a^2\right)$	$\mathcal{V}^* = \frac{4\pi r_+\left(r_+^2 + a^2\right)^2}{3(r_+ - M)}$	${}^3\mathcal{V}^* = \frac{4\pi r_+\left(r_+^2 + a^2\right)}{3}$
$= 4\pi\left(2Mr_+ - Q^2\right)$	$= \frac{4\pi r_+\left(r_+^2 + a^2\right)}{3\kappa}$	$= \frac{4\pi r_+(2Mr_+ - Q^2)}{3}$
$= 4\pi\left(2M^2 + 2Mx - Q^2\right)$	$= \frac{4\pi}{3x}\left\{\begin{matrix} 16M^5 - 20M^3Q^2 + 5Q^4M - 12J^2M \\ +4Q^2J^2/M + 16M^4x - 12M^2Q^2x \\ +Q^4x - 4J^2x \end{matrix}\right\}$	$= \frac{4\pi(4M^3 + 4M^2x - 3MQ^2 - 2J^2/M - Q^2x)}{3}$
$= \frac{4\pi}{\kappa}\left(M - 2\Omega_H J - \Phi Q\right)$	$= \frac{4\pi r_+}{3\kappa^2}\left(M - 2\Omega_H J - \Phi Q\right)$	$= \frac{4\pi}{3\kappa r_+}\left(M - 2\Omega_H J - \Phi Q\right)$
$= \frac{3\kappa\mathcal{V}^*}{r_+}$	$= \frac{Ar_+}{3\kappa}$	$= \frac{Ar_+}{3}$

Table 13.1. Comparison of area, our volume and Parikh volume.

Area Relations	Vol. derivative $\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda}$	Killing Vol. derivative ${}^3\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t}$
$\delta A = \frac{8\pi}{\kappa}\delta M - \frac{8\pi}{\kappa}\Omega_H\delta J - \frac{8\pi}{\kappa}\Phi\delta Q$	$\delta\mathcal{V}^* = \frac{4\pi}{3\kappa^3}\left(6\kappa r_+ - 1\right)\delta M - \frac{4\pi a\left(4\kappa r_+ - 1\right)}{3\kappa^3\left(r_+^2 + a^2\right)}\delta J + \frac{4\pi Q\left(4\kappa r_+^2 - M\right)}{3\kappa^3\left(r_+^2 + a^2\right)}\delta Q$	$\delta({}^3\mathcal{V}^*) = \frac{4\pi\left(3Mr_+ + a^2\right)}{3M\kappa}\delta M - \frac{4\pi a\left(4Mr_+ - Q^2\right)}{3M\kappa(r_+^2 + a^2)}\delta J - \frac{4\pi Q\left(3r_+^2 + a^2\right)}{3\kappa(r_+^2 + a^2)}\delta Q$
$= \frac{8\pi}{\kappa}\left(\delta M - \Omega_H\delta J - \Phi\delta Q\right)$	$= \frac{4\pi}{3\kappa^3}\Big[\left(6\kappa r_+ - 1\right)\delta M - \left(4\kappa r_+ - 1\right)\Omega_H\delta J - \left(4\kappa r_+ - \frac{M}{r_+}\right)\Phi\delta Q\Big]$	$= \frac{4\pi r_+}{\kappa}\Big[\left(1 + \frac{a^2}{3Mr_+}\right)\delta M - \left(\frac{4}{3} - \frac{Q^2}{3Mr_+}\right)\Omega_H\delta J - \left(1 + \frac{a^2}{3r_+^2}\right)\Phi\delta Q\Big]$
$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H\delta J - \Phi\delta Q$	$\frac{3\kappa^3}{4\pi}\delta\mathcal{V}^* = \left(6\kappa r_+ - 1\right)\delta M - \left(4\kappa r_+ - 1\right)\Omega_H\delta J - \left(4\kappa r_+ - \frac{M}{r_+}\right)\Phi\delta Q$	$\frac{\kappa}{4\pi r_+}\delta({}^3\mathcal{V}^*) = \left(1 + \frac{a^2}{3Mr_+}\right)\delta M - \left(\frac{4}{3} - \frac{Q^2}{3Mr_+}\right)\Omega_H\delta J - \left(1 + \frac{a^2}{3r_+^2}\right)\Phi\delta Q$

Table 13.2. Comparison of the differential changes of area, our volume and Parikh volume in quasi-static processes.

Area Relations	Vol. derivative $\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\ln\lambda}$	Killing Vol. derivative ${}^3\mathcal{V}^* = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t}$
$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi\delta Q$	$\delta M = \frac{3\kappa^3}{4\pi(6\kappa r_+ - 1)}\delta V + \left(1 - \frac{2\kappa r_+}{6\kappa r_+ - 1}\right)\Omega_H \delta J + \left(1 - \frac{2\kappa r_+ + \frac{M}{r_+} - 1}{6\kappa r_+ - 1}\right)\Phi\delta Q$	$\delta M = \left(3Mr_+ + a^2\right)^{-1}\left[\frac{3M\kappa}{4\pi}\delta({}^3\mathcal{V}^*) + \left(4Mr_+ - Q^2\right)\Omega_H \delta J + \left(3Mr_+ + \frac{Ma^2}{r_+}\right)\Phi\delta Q\right]$
$dE = \delta q - \delta W(\text{therm.})$ if $E = M, T = \frac{\kappa}{2\pi}, -\delta W = \Omega_H \delta J + \Phi\delta Q,$ then $\delta \mathcal{Q} = \frac{\kappa}{2\pi}\delta A$	$dE = \delta q - \delta W(\text{therm.})$ if $E = M, T = \frac{\kappa}{2\pi}, -\delta W = \Omega_H \delta J + \Phi\delta Q,$ then $\delta \mathcal{Q} = \frac{3\kappa^3}{4\pi(6\kappa r_+ - 1)}\delta\mathcal{V}^* - \frac{2\kappa r_+}{(6\kappa r_+ - 1)}\Omega_H\delta J - \frac{2\kappa r_+ + \frac{M}{r_+} - 1}{6\kappa r_+ - 1}\Phi\delta Q$	$dE = \delta q - \delta W(\text{therm.})$ if $E = M, T = \frac{\kappa}{2\pi}, -\delta W = \Omega_H \delta J + \Phi\delta Q,$ then $\delta \mathcal{Q} = \left(3Mr_+ + a^2\right)^{-1}\left[\frac{3M\kappa}{4\pi}\delta({}^3\mathcal{V}^*) + \left(\frac{M}{r_+} - 1\right)(r_+^2\Omega_H\delta J + a^2\Phi\delta Q)\right]$
and $S = \frac{A}{4}$	and $dS = (6kr_+ - 1)^{-1}\left[\frac{3\kappa^2}{2(6\kappa r_+ - 1)}\delta\mathcal{V}^* - \frac{4\pi r_+}{(6\kappa r_+ - 1)}\Omega_H\delta J - \frac{4\pi r_+ + \frac{2\pi M}{\kappa r_+} - \frac{2\pi}{\kappa}}{6\kappa r_+ - 1}\Phi\delta Q\right]$	and $dS = \left(3Mr_+ + a^2\right)^{-1}\left[\frac{3}{2}M\delta({}^3\mathcal{V}^*) + \frac{2\pi}{\kappa}\left(\frac{M}{r_+} - 1\right)(r_+^2\Omega_H\delta J + a^2\Phi\delta Q)\right]$
	$S = \frac{A}{4} = \frac{3\kappa\mathcal{V}^*}{4r_+}$	$S = \frac{A}{4} = \frac{3({}^3\mathcal{V}^*)}{4r_+}$

Table 13.3. Comparison of formulations about energy, heat, work and entropy in area, our volume and Parikh volume.

Special value	Consequence	Defining Equations	General Solutions	Special
$6\kappa r_+ = 1$	$\frac{\partial \mathcal{V}^*}{\partial M}\Big\|_{J,Q} = 0$	$36\alpha^4 - 32\alpha^2 + 60\alpha^2 q^2 - 24q^2 + 25q^4 = 0$	$\alpha = \pm\frac{1}{6}\sqrt{16 - 30q^2 + 4\sqrt{16 - 6q^2}}$ $q = \pm\frac{1}{5}\sqrt{12 - 30\alpha^2 + 4\sqrt{9 + 5\alpha^2}}$	$\alpha = \pm\frac{2\sqrt{2}}{3}$ (Kerr) $q = \pm\frac{2\sqrt{6}}{5}$ (R.-N.)
$4\kappa r_+ = 1$	$\frac{\partial \mathcal{V}^*}{\partial J}\Big\|_{M,Q} = 0$	$16\alpha^4 - 12\alpha^2 + 24\alpha^2 q^2 - 8q^2 + 9q^4 = 0$	$\alpha = \pm\frac{1}{4}\sqrt{6 - 12q^2 + \sqrt{9 - 4q^2}}$ $q = \pm\frac{1}{3}\sqrt{4 - 12\alpha^2 + 2\sqrt{4 + 3\alpha^2}}$	$\alpha = \pm\frac{\sqrt{3}}{2}$ (Kerr) $q = \pm\frac{2\sqrt{2}}{3}$ (R.-N.)
$4\kappa r_+^2 = M$	$\frac{\partial \mathcal{V}^*}{\partial Q}\Big\|_{M,J} = 0$	$4q^6 + 12\alpha^2 q^4 + 12\alpha^4 q^2 + 4\alpha^6 + 29q^4$ $+72\alpha^2 q^2 + 44\alpha^4 - 52q^2 - 64\alpha^2 + 20 = 0$	solutions to cubic in α^2, q^2 too complicated to print	$\alpha = \pm\sqrt{\frac{\sqrt{3}}{2}}$ (Kerr) $q = \pm\frac{\sqrt{15}}{4}$ (R.-N.)
$2\kappa r_+^2 + M = r_+$	$\frac{\partial \mathcal{Q}}{\partial Q}\Big\|_{\mathcal{V}^*, J} = 0$	$r_+^2 - a^2 = 0$	no physical solutions	no physical solutions
$3Mr_+ + a^2 = 0$	$\frac{\partial (^3\mathcal{V}^*)}{\partial M}\Big\|_{J,Q} = 0$	note: $M > 0, r_+^2 > 0, a^2 > 0$, so impossible		
$4Mr_+ = Q^2$	$\frac{\partial (^3\mathcal{V}^*)}{\partial J}\Big\|_{M,Q} = 0$	$q^4 + 8q^2 + 16\alpha^2 = 0$	no physical solutions	
$3r_+^2 + a^2 = 0$	$\frac{\partial^3 (\mathcal{V}^*)}{\partial Q}\Big\|_{M,J} = 0$	note: $r_+^2 > 0, a^2 > 0$, so impossible	no physical solutions	

Table 13.4. Special values for black hole parameters.

best form to show the equations. We tried to get the shortest form at all times. But essentially there are three independent quantities M, J, Q, which lead to three quantities κ, Ω_H, Φ (which are independent of each other) which have a complicated relationship with the first three, and there are also many other parameters like r_+, a, and the area and the two types of volumes. It is possible that there is some simplification that we have not found that renders the relationship between the change in volume and the change in these parameters in an easier-to-read and easier-to-interpret form.

We note immediately that the first law expressed in terms of area yields more aesthetically pleasing results. This is not necessarily important, but it may suggest that area is a more sensible parameter with which to express the laws of BH mechanics than volume.

In a comparison between the two volumes (the Parikh volume and our volume), we see that in the case of the Parikh volume, there are *no* points at which the derivatives of the volume with respect to any of the three parameters become zero, except where $\Omega_H = 0$ (for the J-derivative) and $\Phi = 0$ (for the Q-derivative); and so they never change sign. In our volume, on the other hand, there are several special values at which point $\mathcal{V}^*$ changes sign.

13.4 Notes on Entropy

What happens if the entropy actually is correctly defined as

$$S_{BH} = \frac{A}{4\hbar}? \tag{13.4.1}$$

This is the case if we assume $T = \hbar\kappa/2\pi$, $E = M$, $\delta W = -\Omega_H \delta J - \Phi \delta W$, and $dE = \delta\mathcal{Q} - \delta W$. Does this mean that entropy is no longer related to volume? Since we have a simple relationship between area and volume, we can write S_{BH} explicitly as a function of volume as shown below.

$$S_{BH} = \frac{A}{4\hbar} = \frac{3\kappa\mathcal{V}^*}{4r_+\hbar} = \frac{3({}^3\mathcal{V}^*)}{4r_+\hbar} \tag{13.4.2}$$

This means that the "average" entropy per volume element, or entropy density, say s for our volume and s' for the Parikh volume, becomes

$$s = \frac{3}{4}\frac{\kappa}{\hbar r_+} \tag{13.4.3}$$

and

$$s' = \frac{3}{4\hbar r_+} \tag{13.4.4}$$

This implies that the entropy density is proportional to the surface gravity divided by the black hole radius (our volume) or simply inversely proportional to the radius (Parikh volume). Either way, we find that the entropy density, as so defined, grows as the black hole radius shrinks.

Chapter 14

Connection to Other Works

Here we revisit the work of Parikh and introduce the work of Cvetič and of Hayward, which suggest interesting avenues to explore in terms of the connection between volume and black hole mechanics.

14.1 Parikh on His Volume and Thermodynamics: Free Energy

Parikh makes a claim which we quote here (changing some notation slightly in the equation):

> Actually, [the Parikh volume] is also the notion of volume that appears in thermodynamics. To see this, write the partition function as
>
> $$Z = \exp(-F\beta) = \exp\left(-\int \mathrm{d}^D x\sqrt{-g_D}L\right). \tag{14.1.1}$$
>
> Now, in the region where thermodynamics applies, the inverse temperature, β, is the period of a complexified time coordinate, τ. Notice: there is no $\sqrt{-g_{\tau\tau}}$ factor in β. Suppose the field is constant in τ. Then the free energy is $F = \int \mathrm{d}^{D-1}x\sqrt{-g_D}L$. For an extensive system, the free energy is proportional to the volume. We see that this is not inconsistent with regarding $\int \mathrm{d}^{D-1}x\sqrt{-g_D}$ as the volume.

Here, Z is the partition function and L is the Lagrangian for the system. This is an interesting idea worth exploring in the future. We note right away that if we interpret β not as the period of a complexified time coordinate but directly as $1/T_{BH}$, where the temperature T_{BH} of the black hole is usually expressed as $\kappa/2\pi$ (setting $\hbar = 1$), the inverse temperature β could be interpreted as $2\pi/\kappa$, in which case the free energy would be written as $F = \kappa \int \mathrm{d}^D x \sqrt{-g_D} L$. This is more difficult to interpret. It makes it worth investigating further whether there is any meat to Parikh's claim, and is relegated to future work.

14.2 Connection to Cvetič et al.

In a recent paper [2], M. Cvetič, G.W. Gibbons, D. Kubiznak and C.N. Pope consider black hole thermodynamics as well. There are several interesting points raised in the paper, but of particular interest for our work is their work with the first law in the case where $\Lambda \neq 0$. They write (in an equation that is valid in higher dimensions)

$$\mathrm{d}E = T\mathrm{d}S + \Omega_i \mathrm{d}J_i + \Phi_\alpha \mathrm{d}Q_\alpha + \Theta \mathrm{d}\Lambda \tag{14.2.1}$$

as the modified first law of black hole mechanics with E, still closely related to the mass M of the black hole, as the analogue to the *enthalpy*, Ω_i and J_i as the angular speeds and momenta (in various dimensions), Φ_α and Q_α as potentials and charges (permitting multiple kinds of charges), Λ as the cosmological constant and Θ as a conjugate to Λ. As usual, they identify $T\mathrm{d}S$ with $\kappa A/8\pi$, and then by expressing A in terms of M, J, Q, Λ they find Θ.

This formulation is based closely on the formula for the enthalpy,

$$\mathrm{d}E = \delta\mathcal{Q} - \delta W' + V\mathrm{d}P, \tag{14.2.2}$$

where E is enthalpy, $\mathcal{Q}$ is heat, W' is work done by the system *not including* through expansion, V is the volume and P is the pressure. Cvetič then identifies M with the enthalpy, $\Omega_i \mathrm{d}J_i + \Phi_\alpha \mathrm{d}Q_\alpha$ as $-\delta W'$ (the non-expansion work done by the black hole, in spinning up or spinning down or in charging), and $V\mathrm{d}P$ is interpreted as being equal to $\Theta\mathrm{d}\Lambda$. This is due to a common identification of Λ with the pressure of spacetime. In particular, they write:

$$P = -\frac{(D-2)\Lambda}{16\pi} \tag{14.2.3}$$

and, by treating E as an analogue to enthalpy and so writing $\Theta\mathrm{d}\Lambda \equiv V\mathrm{d}P$ for a volume V, they find

$$\Theta = -\frac{(D-2)}{16\pi}V. \tag{14.2.4}$$

They then write out the first law of thermodynamics (using the usual connection between entropy and surface area in the literature) and solve for Θ to develop a "thermodynamic volume" V.

Along the way, they define a "naive" geometrical volume V' which is calculated by taking

$$V' = \int_{r_0}^{r_+} \mathrm{d}r \int \mathrm{d}\Omega\sqrt{-g} \tag{14.2.5}$$

where r_0 is a complex radius which happens to be zero for even-dimensional spacetimes, $\mathrm{d}\Omega$ is the element of the unit sphere (in $D-2$ dimensions). It should be clear that this is the exact integration that (in four dimensions at least) is performed to calculate the Parikh volume. So they have in effect developed the Parikh volume independently. They note immediately the relationship

$$V' = \frac{r_+ A}{D-1}. \tag{14.2.6}$$

We recognize this as being akin to the Parikh volume, with (11.0.6) generalized to higher dimensions.

Are Cvetič's thermodynamic and geometric volumes equal? It turns out that the answer is yes if there is no rotation, and no if there is rotation. The formulas for the value of V in the case where there is rotation are somewhat complicated to print. In spherical symmetry, the volumes are equal. It is remarkable then that Cvetič accidentally rediscovered the Parikh volume and found that, in spherical symmetry, it plays a role very much like volume, w.r.t. the change in mass and entropy due to expansion.

Part of what is interesting about this track is that it relates the volume itself to the concept of the volume in thermodynamics. This might seem like an obvious analogy to make, but since the volume only enters into energy arguments if there is a pressure present and we have worked without Λ, it was not possible to relate the black hole volume to PdV terms. The result from this paper is an argument for the Parikh volume to have some sort of thermodynamic meaning, *in spherical symmetry*. If the Parikh volume has a more general meaning thermodynamically, then the surface gravity could be defined as the ratio of the Parikh/thermodynamic volume to our volume, the growth rate of the four-volume with respect to the log of the affine parameter.

Much of Cvetič's paper is spent examining the isoperimetric inequality for the volume and area of the black holes. This might be worth examining in future work as part of a continuing exploration of the relationship between the volume and surface area of black holes.

14.3 Connection to Hayward

In a series of papers (for example [3, 12, 29], among others), S. Hayward and others presented an argument for a generalized first law of black hole mechanics in dynamic spherical symmetry in four dimensions. He defines a volume in terms of the Kodama vector, an analogue to the Killing vector in dynamic spherical symmetry, which we will show is analogous to the Parikh volume. Hayward does a similar development for cylindrical symmetry in [30], but we will focus on the spherical symmetry case. Hayward assumes that there is a dynamic black hole, and develops black hole mechanics laws with reference to the black hole trapping horizon (rather than the event horizon as is used for stationary black holes).

The line element for four-dimensional dynamic spherically symmetric spacetime can be written in the form

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + r^2\mathrm{d}\Omega_2^2 \tag{14.3.1}$$

where there are two coordinates x^A in addition to the coordinates (θ, ϕ) within the 2-sphere metric $\mathrm{d}\Omega_2^2$. Here, r is the aerial radius; the spheres of symmetry at constant r have area $A = 4\pi r^2$. An invariant energy E well-defined here is the Misner-Sharp energy contained within a radius r, which

Hayward writes explicitly as

$$E = \frac{1}{2}r(1 - \nabla_\alpha r \nabla^\alpha r). \tag{14.3.2}$$

E is used by Hayward to represent “active gravitational energy.” (This reduces to M in the case of the Schwarzschild metric.) $\nabla_\alpha r$ is null on the trapping horizon, so that the trapping horizon can be defined by $E = \frac{1}{2}r$.

Hayward then defines two invariants of the energy tensor $T^{\alpha\beta}$. The first is a scalar representing energy density (“work density”), which is based on a two dimensional trace:

$$w = -\frac{1}{2}T^{AB}g_{AB}. \tag{14.3.3}$$

The other is a vector ψ^α which represents energy flux (“localized Bondi flux”),

$$\psi^\alpha = T^{\alpha\beta}\nabla_\beta r + w\nabla^\alpha r. \tag{14.3.4}$$

These will become useful shortly.

The analogue to the Killing vector in dynamic spherical symmetry is the Kodama vector, first defined in [31], which we will label by K^α. An important property of the Kodama vector is that it becomes null on and only on the trapping horizon of a dynamic black hole, a property that is analogous to the Killing vector becoming null on and only on the Killing horizon of a stationary black hole. The Kodama vector K^α is defined as the curl of the aerial radius,

$$K^\alpha = \epsilon^{\alpha\beta}\nabla_\beta r, \tag{14.3.5}$$

where $\epsilon^{\alpha\beta}$ is the volume form associated with the 2-metric g_{AB} from (14.3.1), or

$$\epsilon^{\alpha\beta} = \epsilon^{AB}\delta^\alpha_A\delta^\beta_B \tag{14.3.6}$$

where ϵ^{AB} is the Levi-Civita tensor for the two dimensions x^A (see (7.2.16)). The Kodama vector agrees with the standard Killing vector (the t^α Killing vector of Section 8.7) in stationary, spherically symmetric spacetimes if K^α commutes with $\nabla^\alpha r$; in these cases the line element can be written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2E(r)}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2E(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{14.3.7}$$

where the Killing and Kodama vectors are $t^\alpha = \delta^\alpha_t$.

The Kodama vector (in spherical symmetry only) has the property

$$\nabla_\alpha K^\alpha = 0, \tag{14.3.8}$$

which, along with the Gauss theorem (see, for example, [6]), implies a conserved quantity

$$V = \left|\int_\Sigma K^\alpha \mathrm{d}\Sigma_\alpha\right| \tag{14.3.9}$$

where Σ is a spacelike hypersurface (a hypersurface with a timelike normal) and $\mathrm{d}\Sigma_\alpha$ is the volume element of the surface times a future directed normal. The absolute value signs are to avoid having to deal with the sign of the result. The boundary is a surface of constant x^A, which implies a constant radius r. This V is interpreted as a volume (henceforth, we will call it a "Kodama volume"), equal to the Euclidean volume of a sphere with radius r, $4\pi r^3/3$. The form of (14.3.9) is appealing, and the analogy between the Kodama and Killing vectors led us to realize that the expression

$$\int_\Sigma \xi^\alpha \mathrm{d}\Sigma_\alpha \tag{14.3.10}$$

is the Killing volume for Killing vector ξ^α of a region $\mathcal{R}$ for which the hypersurface Σ is a possible slicing of $\mathcal{R}$. We showed this as equation (7.2.20). It is a particularly elegant formulation for the Killing volume, inspired by Hayward's Kodama volume.

By analogy with the relationship for the Killing vector t^α in stationary metrics

$$t^\beta \nabla_{[\beta} t_{\alpha]} = \kappa t_\alpha, \tag{14.3.11}$$

a dynamic surface gravity (also called κ) can be defined on the trapping horizon by

$$K^{\beta}\nabla_{[\beta}K_{\alpha]} = \kappa K_{\alpha}. \tag{14.3.12}$$

In both cases the square brackets represent antisymmetrization.

Hayward further defines a flux $j^{\alpha} = T^{\alpha\beta}K_{\beta}$, and finds that

$$\nabla_{\alpha}j^{\alpha} = 0, \tag{14.3.13}$$

from which another conserved current, $\int_{\Sigma} j^{\alpha}\mathrm{d}\Sigma_{\alpha}$, is found (where the hypersurface Σ is the same as for the Kodama volume integral). He then finds that this is E:

$$E = \left|\int_{\Sigma} j^{\alpha}\mathrm{d}\Sigma_{\alpha}\right|. \tag{14.3.14}$$

This equivalence is established in [32]. This relationship holds only if the hypersurface has a regular centre, which means that (14.3.14) doesn't hold in spaces like Schwarzschild, Reissner-Nordström with a central singularity. However, the more general relationship

$$E_{out} - E_{in} = \left|\int_{\Sigma} j^{\alpha}\mathrm{d}\Sigma_{\alpha}\right|, \tag{14.3.15}$$

where Σ has both an inner and outer boundary (and thus does not enclose the singularity), still holds. If Σ is regular at the centre then we can write E as a function of the coordinates x^A, where the choice of the boundary of Σ determines x^A. Hayward then calculates $\nabla_{\alpha}E$ from the equations of motion (the Einstein equations), and finds

$$\nabla_{\alpha}E = A\psi_{\alpha} + w\nabla_{\alpha}V \tag{14.3.16}$$

where all qualities are evaluated at x^A, and $A = 4\pi r^2$ is the surface area of a sphere. This equation holds everywhere, but it has particular significance on the trapping horizon. Defining z^{α} as a vector tangent to the trapping horizon (normalization irrelevant) and letting a prime represent intrinsic differentiation along the trapping horizon (e.g. $f' \equiv z^\alpha \nabla_\alpha f$), then a "first law of black-hole dynamics" can be written on the trapping horizon from (14.3.16) by multiplying both sides by z^α. This is

$$E' = \frac{\kappa A'}{8\pi} + wV'. \tag{14.3.17}$$

Note that in vacuum, $w = 0$, so that the volume dependence vanishes. This results because of a property (from the equations of motion) that

$$A z^\alpha \psi_\alpha = \frac{\kappa(z^\alpha \nabla_\alpha A)}{8\pi}. \tag{14.3.18}$$

In a more involved argument which can be found in [3], Hayward also establishes a relativistic thermodynamic relationship resulting from a material current ρu^α (with velocity vector u^α) based on (14.3.16). The result is summarized here. For a specific definition of heat supply Q, radial pressure p, and redshift factor γ along the path of the (spherically symmetric) matter flow, Hayward determines the equation

$$\dot{E} = \gamma \dot{Q} - p\dot{V} \tag{14.3.19}$$

where the dot represents the differentiation along the matter trajectory, $\dot{f} = u^\alpha \nabla_\alpha f$. This shows that (14.3.16) is a relationship that implies both a dynamic black hole mechanics law and a relativistic thermodynamics law, and both feature a definition of volume which is analogous to Parikh's and our definition of volume (in dynamic spherical symmetry).

The two most important results from Hayward's paper to our work are the following. First, as with Cvetič, a volume is defined which happens to have the same form as the Parikh volume, which becomes an important part of the energy variation law. Second, the actual form $\int K^\alpha \mathrm{d}\Sigma_\alpha$ for the Kodama volume seems elegant and instructive. The latter point in particular is the inspiration for the section which follows.

14.4 Killing Volume Element and Its Connections to Other Authors

The Kodama volume's formulation suggests a definition for a Kodama "volume element" on a hypersurface, which we will call $\mathrm{d}V$ or now, as

$$\mathrm{d}V = |K^{\alpha}\mathrm{d}\Sigma_{\alpha}|\,. \tag{14.4.1}$$

This volume element allows us to rewrite the definition (14.3.9) for the Kodama volume V as

$$V = \int_{\Sigma} \mathrm{d}V. \tag{14.4.2}$$

Similarly, we can define a Killing volume element on a hypersurface Σ, which we will call $\mathrm{d}\mathcal{V}^{*}_{\xi}$, for a specific Killing vector ξ^{α}, as

$$\mathrm{d}\mathcal{V}^{*}_{\xi} = \xi^{\alpha}\mathrm{d}\Sigma_{\alpha} \tag{14.4.3}$$

so that, following (7.2.20), we can write on a hypersurface Σ

$${}^{D-1}\mathcal{V}^{*}_{\xi} = \int_{\Sigma} \mathrm{d}\mathcal{V}^{*}_{\xi}. \tag{14.4.4}$$

The conception of $\xi^{\alpha}\mathrm{d}\Sigma_{\alpha}$ as a "Killing volume element" might be a potentially interesting avenue to explore. As an example, consider the Komar integrals for mass and angular momentum of a black hole. Using the definitions in [6], the Komar mass M_K and Komar angular momentum J_K in a stationary, axially symmetric spacetime with stationarity Killing vector t^{α} (see Section 8.7) and axial symmetry Killing vector ϕ^{α} enclosed by a two-surface S on the black hole horizon can be written as

$$M_K = -\frac{1}{8\pi}\oint_S \nabla^{\alpha}t^{\beta}\mathrm{d}S_{\alpha\beta} \tag{14.4.5}$$

$$J_K = \frac{1}{16\pi}\oint_S \nabla^{\alpha}\phi^{\beta}\mathrm{d}S_{\alpha\beta} \tag{14.4.6}$$

where $\mathrm{d}S_{\alpha\beta}$ is the surface element. We can now note Stokes' theorem,

$$\oint_S B^{\alpha\beta}\mathrm{d}S_{\alpha\beta} = 2\int_\Sigma B^{\alpha\beta}_{;\beta}\mathrm{d}\Sigma_\alpha, \tag{14.4.7}$$

where Σ is any closed hypersurface with boundary S and $B^{\alpha\beta}$ is an antisymmetric tensor. Additionally, we note the property that for any Killing vector ξ^α, $\nabla^\alpha\xi^\beta$ is antisymmetric and has

$$(\nabla^\alpha\xi^\beta)_{;\beta} = -(\nabla^\beta\nabla_\beta)\xi^\alpha = R^\alpha_\beta\xi^\alpha, \tag{14.4.8}$$

where R^α_β is the Ricci tensor, the last equality being a property satisfied by all Killing vectors. These allow us to rewrite the Komar formulas in the following way. Let S' be an arbitrarily small two-surface which encloses the singularity, and Σ be a hypersurface whose boundaries are S on the outside and S' on the inside. Then it is possible to write for any Killing vector ξ^α on this hypersurface, using (14.4.7) and (14.4.8),

$$2\int_\Sigma R^\alpha_\beta\xi^\beta\mathrm{d}\Sigma_\alpha = \oint_S \nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta} - \oint_{S'}\nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta}. \tag{14.4.9}$$

Then using the Einstein equations [14] with Λ in the form

$$R^\alpha_\beta = 8\pi\left(T^\alpha_\beta - \frac{1}{2}T\delta^\alpha_\beta\right) + \Lambda\delta^\alpha_\beta, \tag{14.4.10}$$

where $T = T^\alpha_\alpha$, and rearranging, we can write (14.4.9) as

$$\oint_S \nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta} = \oint_{S'}\nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta} + 2\int_\Sigma\left[8\pi\left(T^\alpha_\beta - \frac{1}{2}T\delta^\alpha_\beta\right) + \Lambda\delta^\alpha_\beta\right]\xi^\beta\mathrm{d}\Sigma_\alpha. \tag{14.4.11}$$

If the hypersurface Σ is regular at its centre, there is no need for the separate integral over S', but this is not the usual situation with black holes. We can rewrite (14.4.11) using our definition of the Killing volume element and the Killing volume proper as

$$\oint_S \nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta} = \oint_{S'}\nabla^\alpha\xi^\beta\mathrm{d}S_{\alpha\beta} + 16\pi\int_\Sigma T^\alpha_\beta\xi^\beta\mathrm{d}\Sigma_\alpha - 8\pi\int_\Sigma T\mathrm{d}\mathcal{V}^*_\xi + 2\Lambda\left({}^{D-1}\mathcal{V}^*_{\xi,\Sigma}\right). \tag{14.4.12}$$

where of course $\mathcal{V}^*_{\xi,\Sigma}$ is the Killing volume for Killing vector ξ^α and the portion of the black hole which is sliced by the hypersurface Σ.

Define quantities $M_{S'}$ and $J_{S'}$ as (respectively) the Komar integrals of mass and angular momentum over the inner surface S'; we can think of these as representing the "inner" mass and angular momentum:

$$M_{S'} = -\frac{1}{8\pi}\oint_{S'} \nabla^\alpha t^\beta \mathrm{d}S_{\alpha\beta} \qquad J_{S'} = \frac{1}{16\pi}\oint_{S'} \nabla^\alpha \phi^\beta \mathrm{d}S_{\alpha\beta}. \tag{14.4.13}$$

Then we can write the black hole mass as, using (14.4.12)

$$M_K = M_{S'} - 2\int_\Sigma T^\alpha_\beta t^\beta \mathrm{d}\Sigma_\alpha + \int_\Sigma T\mathrm{d}\mathcal{V}^*_t - \frac{{}^{D-1}\mathcal{V}^*_{t,\Sigma}}{4\pi}\Lambda. \tag{14.4.14}$$

As we make S' an arbitrarily small hypersurface, $\mathcal{V}^*_{t,\Sigma}$ approaches the full Parikh volume for the black hole. We note that the coefficient of Λ in our expansion of the Komar mass of the black hole now is the Parikh volume divided by 4π. This connects back to Cvetič's discovery that the coefficient of $\delta\Lambda$ in the variation law was (in some cases) equal to the Parikh volume (though they hadn't identified it as such). The coefficient there was the negative of the volume divided by 8π in four-dimensions, rather than 4π as we have here, but the similarity of the results is remarkable and should be further studied. We note that because the volume of the black hole itself is expected to change when Λ is varied, it is not necessarily the case that the Komar mass is strictly linear with the cosmological constant. The term $\int_\Sigma T\mathrm{d}\mathcal{V}^*_t$ can be interpreted as an energy density integrated over the volume of the black hole.

Chapter 15

The Kerr-Newman metric and r_+

The outer horizon in the Kerr-Newman metric is at $r = r_+ = M + \sqrt{M^2 - Q^2 - J^2/M^2}$. This value turns out to be 3 times the ratio of the Parikh volume to the area (in four dimensions), and, if Cvetič's generalization is correct, $(D-1)$ times the ratio of the Parikh volume to the area. We seek in this section some kind of greater understanding of why this is the case. It is not intuitively clear why the horizon should be at constant r, or even what the coordinate r (in Boyer-Lindquist, Kerr, and other coordinate systems) means, particularly as we are away from spherical symmetry. The following observations are our attempts to understand the significant role of r_+ (and r_-, the location of the inner horizon) and of the r coordinate generally. Is there a way we can better express the fact that (in four dimensions) $\mathcal{V}^* = Ar_+/3\kappa$ and ${}^3\mathcal{V}^* = Ar_+/3$?

The event horizons occur when $\Delta = r^2 - 2Mr + Q^2 + a^2 = 0$, which has two solutions

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{15.0.1}$$

In spherical symmetry, we can write that the horizon radius is, if nothing else, the aerial radius at the horizon. But this is not permitted in the Kerr-Newman metric, in which the area of the horizon is not $4\pi r_+^2$ but $4\pi(r_+^2 + a^2)$. We can of course specify that

$$r_+ = \sqrt{\frac{A}{4\pi} - a^2} \tag{15.0.2}$$

where A is the area of the horizon, but this does not feel very helpful. Here we examine some properties of the quantity r_+.

15.1 Affine Parameter

From [13], we note that in Kerr coordinates (5.3.1), the ingoing principal null geodesics are parameterized by $V, \theta, \psi = const$. Further, for a certain affine parameter λ, the ingoing principal null geodesics can be written as

$$m^\alpha = \left.\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right|_{\mathcal{C}} = -E\delta_r^\alpha \tag{15.1.1}$$

where we use $\mathcal{C}$ to denote the trajectory associated with a ingoing principal null geodesic and E is some multiplicative constant. Further, we note that since $g_{Vr} = 1$, if we let $t^\alpha = \delta_V^\alpha$ (the "good," properly normalized time-Killing vector, as detailed in Section 8.7—for which $t^\alpha t_\alpha \to -1$ asymptotically and $t^\alpha \phi_\alpha = 0$, where ϕ^α corresponds to azimuthal symmetry), then we have

$$t^\alpha m_\alpha = g_{\alpha\beta} t^\alpha m^\beta = g_{\alpha\beta}\delta_V^\alpha(-E\delta_r^\beta) = -Eg_{Vr} = -E. \tag{15.1.2}$$

We can now, if we wish, define m^α such that $t^\alpha m_\alpha = -1$, so that m^α is well normalized. With $m^\alpha = -\delta_r^\alpha$, then, obviously we can write

$$r_+ = \Delta\lambda, \tag{15.1.3}$$

where $\Delta\lambda$ is the difference in affine parameter along the path with tangent vector m^α between the horizon and $r = 0$. Alternatively, we can write

$$-1 = t_\alpha m^\alpha = t_\alpha \left.\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right|_{\mathcal{C}}, \tag{15.1.4}$$

so that we have

$$t_\alpha \mathrm{d}x^\alpha|_{\mathcal{C}} = -\mathrm{d}\lambda, \tag{15.1.5}$$

and finally

$$r_+ = \lambda|_{r=0} - \lambda|_{r=r_+} \equiv -\int_{\mathcal{B}} t_\alpha \mathrm{d}x^\alpha, \tag{15.1.6}$$

where $\mathcal{B}$ is the path from the outer horizon $r = r_+$ to $r = 0$ along an ingoing principal null geodesic. The right hand side, we see, is independent of our choice of parametrization of m^α and depends only on the vector t^α and the path $\mathcal{B}$. The advantage of (15.1.3) is its clarity in emphasizing that r_+ is an affine distance; the advantage of (15.1.6) is its clear invariance. As a result, we can write the Parikh volume ${}^3\mathcal{V}^*$ as:

$${}^3\mathcal{V}^* = \frac{r_+ A}{3} = \frac{\Delta\lambda A}{3} = -\frac{A}{3}\int_{\mathcal{B}} t_\alpha \mathrm{d}x^\alpha \tag{15.1.7}$$

where $\Delta\lambda$, $\mathcal{B}$ and t^α have their meanings as defined previously in this section.

It is tempting to try to apply the same arguments for $\mathcal{V}^*$, with $k^\alpha = (t^\alpha + \Omega_H \phi^\alpha)/\kappa$ replacing t^α (since, from Section 9, $\mathcal{V}^*$ is a Killing volume with Killing vector k^α). (Here $\phi^\alpha = \delta^\alpha_\psi$ is the Killing vector for axial symmetry.) We shall try writing $\int_{\mathcal{B}} k_\alpha \mathrm{d}x^\alpha = \int_{\mathcal{B}} k_\alpha m^\alpha \mathrm{d}\lambda$. First we note:

$$k_\alpha m^\alpha = g_{\alpha\beta} k^\alpha m^\beta = -g_{\alpha\beta}\left(\frac{\delta^\alpha_V + \Omega_H \delta^\alpha_\psi}{\kappa}\right)\delta^\beta_r = -\left(\frac{g_{rV} + \Omega_H g_{r\psi}}{\kappa}\right) \tag{15.1.8}$$

so that

$$k_\alpha m^\alpha = -\left(\frac{1 - \Omega_H a \sin^2\theta}{\kappa}\right). \tag{15.1.9}$$

Therefore,

$$-\int_{\mathcal{B}} k_\alpha \mathrm{d}x^\alpha = (\lambda|_{r=0} - \lambda_{r=r_+})\left(\frac{1 - \Omega_H a \sin^2\theta}{\kappa}\right) = \frac{r_+}{\kappa}\left(1 - \Omega_H a \sin^2\theta\right). \tag{15.1.10}$$

Unlike $-\int_{\mathcal{B}} t_\alpha \mathrm{d}x^\alpha$, this is not constant and has some angular dependence. However, if define $\mathcal{B}'$ to be one of the ingoing principal null trajectories *on the axis of axial symmetry*, i.e. on either $\theta = 0$

or π, then we find that $\int_{\mathcal{B}'} k_\alpha \mathrm{d}x^\alpha = r_+/\kappa$, so that we have

$$\mathcal{V}^* = \frac{r_+ A}{3\kappa} = -\frac{A}{3}\int_{\mathcal{B}'} k_\alpha \mathrm{d}x^\alpha. \tag{15.1.11}$$

Additionally, if we *define* $m'^\alpha \equiv \mathrm{d}x^\alpha/\mathrm{d}\lambda'|_{\mathcal{C}'}$ as the (affine) parametrization of the ingoing principal trajectory on the axis ($\theta = 0$ or π), say $\mathcal{C}'$, for which $m'^\alpha k_\alpha = -1$, then

$$m'^\alpha = \left.\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda'}\right|_{\mathcal{C}'} = -\kappa\delta^\alpha_r. \tag{15.1.12}$$

As a result, the change in parameter λ' between the horizon $r = r_+, \theta = 0$ or π and $r = 0, \theta = 0$ or π is $\Delta\lambda' = r_+/\kappa$, so that we can also write

$$\mathcal{V}^* = \frac{A}{3}\Delta\lambda' \tag{15.1.13}$$

for the particular (affine) parameter λ'. Obviously the formulation for the Parikh volume has a significant advantage over ours, which is that it does not require us to choose a particular ingoing null geodesic (on the axis). The other thing we could do is define another Killing vector $k'^\alpha = t^\alpha/\kappa = k^\alpha - \Omega_H\phi^\alpha/\kappa$, and then define our volume as the Killing volume of the black hole w.r.t. k'^α instead of k^α; in this case we could lift the restriction that $\mathcal{B}'$ is a path along the axis of rotational symmetry only. However, k'^α would not satisfy $k'^\beta k'^\alpha_{;\beta} = k'^\alpha$, and so its construction would be rather *ad hoc*.

15.2 Circumference

In spherical symmetry, the metric on the horizon can be written as the metric of the 2-sphere multiplied by r_+^2 (where r_+ is obviously the value of the aerial radius on the horizon). Then the circumference can be calculated by integrating along closed great circles around the two-sphere.

Obviously any of the closed curves will end up having a circumference of

$$C = 2\pi r_+. \tag{15.2.1}$$

So far, so good. Is it possible to define r_+ as a circumference divided by 2π, in Kerr-Newman?

In Kerr-Newman, we are not so fortunate, as the metric on the horizon is instead written as

$$\mathrm{d}s_\Gamma^2 = (r_\pm^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \frac{(r_\pm^2 + a^2)^2}{r_\pm^2 + a^2\cos^2\theta}\sin^2\theta \mathrm{d}\phi^2. \tag{15.2.2}$$

In this case the shape is something more akin to an ellipsoid. The most sensible measures then of the circumference are the minimal and maximal circumferences—C_ϕ, corresponding to a loop around ϕ at the equator $\theta = \pi/2$, and C_θ, corresponding to a loop around θ for any value of ϕ.

$$C_\phi = 2\pi\sqrt{r_+^2 + a^2} \tag{15.2.3}$$

$$C_\theta = 4\sqrt{r_+^2 + a^2}E\left(\frac{a}{\sqrt{r_+^2 + a^2}}\right) \tag{15.2.4}$$

where $E(x)$ is the complete elliptic integral of the second kind evaluated at x. The latter is the circumference for an ellipse with semi-major axis lengths $2r_+$ and $2\sqrt{r_+^2 + a^2}$. (This is not, however, the metric for an ellipsoid with these semi-major and semi-minor axes. It would be if the coefficient for $\mathrm{d}\phi^2$ were $(r_+^2 + a^2)\sin^2\theta$.)

We find then that the circumferences of the horizon do not particularly help in defining r_+. There is the curious fact that C_ϕ, defined on the equatorial plane, implies a "radius" $\sqrt{r_+^2 + a^2}$ which is the radius of a sphere that would produce the horizon's actual area, $4\pi(r_+^2 + a^2)$. That we can extract r_+ from these values is reassuring, but it doesn't really give us much of an intuitive understanding.

15.3 Area

In the literature, the surface area of a black hole is found by making a cut on the horizon ($r = r_\pm$) and at constant time (or V in Kerr coordinates). The surface in question (say Γ) then has the metric as written in (15.2.2).

$$\mathrm{d}s_\Gamma^2 = (r_\pm^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \frac{(r_\pm^2 + a^2)^2}{r_\pm^2 + a^2\cos^2\theta}\sin^2\theta\mathrm{d}\phi^2. \tag{15.3.1}$$

This metric has determinant $g = -(r_\pm^2 + a^2)^2\sin^2\theta$, so that integrating $\sqrt{|g|}$ over θ, ϕ yields

$$A = 4\pi(r_\pm^2 + a^2). \tag{15.3.2}$$

Now consider: what happens if one takes the area of slices of constant r, t (or r, V) that are not on the horizons? Then the metric of these surfaces is

$$\mathrm{d}s_\Gamma^2 = (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \frac{(r^2 + a^2)^2 - \Delta(r)a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\sin^2\theta\mathrm{d}\phi^2. \tag{15.3.3}$$

with $g = -((r^2 + a^2)^2 - \Delta(r)a^2\sin^2\theta)\sin^2\theta$. Integrating $\sqrt{|g|}$ over θ, ϕ gives a complicated result when $\Delta \neq 0$. For simplicity define

$$B \equiv \frac{\Delta a^2}{(R^2 + a^2)^2}, \tag{15.3.4}$$

and then the area can be written as

$$A = \begin{cases} 2\pi(R^2 + a^2)\left(1 + \left(|B|^{-\frac{1}{2}} + |B|^{\frac{1}{2}}\right)\arcsin\sqrt{\frac{B}{B-1}}\right) & \text{if } B < 0 \\ 4\pi(R^2 + a^2) & \text{if } B = 0 \\ 2\pi(R^2 + a^2)\left(1 + \left(B^{-\frac{1}{2}} - B^{\frac{1}{2}}\right)\operatorname{arcsinh}\sqrt{\frac{B}{1-B}}\right) & \text{if } 0 < B \leq 1 \\ 2\pi(R^2 + a^2)\left(1 + \left(B^{\frac{1}{2}} - B^{-\frac{1}{2}}\right)\left(\frac{\pi}{2} - \ln\left(1 - \frac{1}{\sqrt{B}}\right)\right)\right) & \text{if } B > 1 \end{cases}$$

which shows that the simple form $A = 4\pi(R^2 + a^2)$ is a property of the horizon radius alone.

15.4 Volume

Again consider the Kerr-Newman metric. Write the Killing volume w.r.t. Killing coordinate t (Killing vector is δ^α_t, which has asymptotic form, we note, $g_{tt} \to -1$) of the region $0 \leq r \leq R$ integrated over the 2-sphere. This is exactly equivalent to the usual Parikh volume of the black hole when $R = r_+$. Writing this out in Kerr coordinates, we find that the volume $\mathrm{d}\mathcal{V}/\mathrm{d}t$ is:

$$ {}^3\mathcal{V}^*_{t,0\leq r\leq R} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \sqrt{|g|}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \tag{15.4.1} $$

From the metric g in the Kerr section the integral for any value of R gives

$$ {}^3\mathcal{V}^*_{t,0\leq r\leq R} = \frac{4\pi R(R^2 + a^2)}{3}. \tag{15.4.2} $$

This has a remarkably simple form. It is clear that there is no trivial way to relate this quantity to the more complicated formula for area, which indicates that the result that the Parikh volume relationship to the surface area ${}^3\mathcal{V}^* = Ar/3$ is only valid on the horizon(s).

Chapter 16

Kerr-Newman with Λ

A Kerr-Newman black hole embedded in anti-de Sitter spacetime can be written using the metric [27]

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\left(\mathrm{d}t - \frac{a\sin\theta}{\Xi}\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\phi\right)^2 \tag{16.0.1}$$

where $\Delta = \left(1+\frac{r^2}{L^2}\right)(r^2+a^2) - 2mr + Q^2$, $\Delta_\theta = 1 - \frac{a^2}{L^2}\cos^2\theta$, $\Xi = 1 - \frac{a^2}{L^2}$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $L^2 = -3/\Lambda^2$, with cosmological constant Λ. Calculating $g = \det(g_{\alpha\beta})$, we find

$$-g = \frac{\sin^2\theta(r^2+a^2\cos^2\theta)^2}{\Xi^2} \tag{16.0.2}$$

so we write

$$^3\mathcal{V}^* = \int \sqrt{-g}\mathrm{d}^3x = \frac{4\pi(r_+^2+a^2)r_+}{3\left(1+\frac{\Lambda a^2}{3}\right)} \tag{16.0.3}$$

where r_+ is the outer horizon, which is the largest solution in r to $\Delta = 0$.

To calculate the surface area of the horizon, we write $r = r_+$, $t = const$. Applying this cut and taking the determinant of the two-metric σ_{AB}, we find

$$\sigma = \frac{(r^2+a^2)\sin^2\theta}{\Xi^2} \tag{16.0.4}$$

so that, integrating over θ, ϕ we find that the surface area is

$$A = \int \sqrt{\sigma} \mathrm{d}\theta \mathrm{d}\phi = \frac{4\pi(r_+^2 + a^2)}{1 + \frac{\Lambda a^2}{3}} \tag{16.0.5}$$

so that the standard relation

$${}^3\mathcal{V}^* = \frac{r_+ A}{3} \tag{16.0.6}$$

is preserved.

It should be possible to write out expanded thermodynamic relations as in the previous chapters (which were done for Kerr-Newman), but this is left for future work.

Chapter 17

Non-stationary metrics: The Vaidya metric

Non-stationary metrics are more difficult to deal with as they do not permit a (timelike) Killing vector. (If they possess rotational symmetry they may permit an azimuthal Killing vector.) We could attempt to deal with them, as Hayward did, using a Kodama vector (see Section 14.3). Instead, we will focus on calculating the volume in a manner similar to our work in Chapter 5. Let us examine the Vaidya spacetime briefly.

This has

$$\mathrm{d}s^2 = -f(r, V)\mathrm{d}V^2 + 2\mathrm{d}V\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 \tag{17.0.1}$$

where $\mathrm{d}\Omega_2$ is the two-sphere element. This has the form of Eddington-Finkelstein coordinates, but here, f depends not only on the radius function but on the advanced time coordinate V. "Ingoing" null geodesics—those which, for all values of r, decrease in r toward the future—have trajectories $k^\alpha = -\delta^\alpha_r$. So we find then that the set of ingoing null geodesics have $V = const$. If f happens to have the form

$$f = 1 - \frac{2m(V)}{r} \tag{17.0.2}$$

then the apparent horizon is located at $r = 2m(V)$. (From [6], p. 174.) The event horizon is more complicated and depends on the global structure of the spacetime—which means that we would have to define a specific function f and follow the calculations through. If we calculated the volume of the region bounded by the *apparent* horizon it is relatively simple. Let V vary from δ to V_{max}, $0 \leq r \leq 2m(V)$, and integrate over the two-sphere. We note $-g = \frac{1}{4}r^4 \sin^2\theta$, so we find

$$\mathcal{V} = \int_{\delta}^{V_{max}} \mathrm{d}V \int_0^{2m(V)} \mathrm{d}r \int \mathrm{d}\Omega_2 \sqrt{|g|} \tag{17.0.3}$$

$$\therefore \mathcal{V} = \int_{\delta}^{V_{max}} \mathrm{d}V \frac{4\pi}{3} \left(2m(V)\right)^3 \tag{17.0.4}$$

We can take a derivative respect to V. Unfortunately, V does not have a clear meaning here. Note also that the apparent horizon, which lies on $r = 2m(V)$, is in general non-geodesic. It is not difficult then to come up with a 4-volume for this space if we set sensible boundary conditions—but it is difficult to extract a rate, as we did initially for the stationary spacetimes we examined, that has any clear meaning. Note too that we have focused on the apparent rather than event horizon. In general, if spherical symmetry holds, the event horizon would occur at $r = R(V)$, so we could write (by analogy from above) the volume below the event horizon with $\delta < V < V_{max}$ as

$$\mathcal{V} = \int_{\delta}^{V_{max}} \mathrm{d}V \frac{4\pi}{3} R(V)^3 \tag{17.0.5}$$

One could work with this metric, as well as many other non-stationary metrics, at greater length than we have done here.

Chapter 18

Future work: Avenues for Exploration

There are many possible paths forward from this preliminary work. In particular, these questions are worth asking and, hopefully, answering:

- What quantities involved in these black holes will be measurable to outside observers directly? For example, stationary black holes in vacuum would be uniquely described by M, J, Q (and possibly Λ). So we need three (four, with Λ) distinct quantities to fix the black hole's properties. Are $\kappa, \Omega_H, \Phi, \Theta$ measurable? Are M, J, Q, Λ? If so, is it possible to write A, $\mathcal{V}^*$ and ${}^3\mathcal{V}^*$ out explicitly in terms of the parameters we would expect an outside observer to measure? Similarly, do we expect an external observer to be able to measure the area or volume(s)?

- Can we supply a better interpretation of our volume or Parikh's volume? Which comes closer to the way we usually think of volume in the classical (non-relativistic) sense? In what situations can these volumes be applied? Parikh [1] notes that his volume is the volume as measured by a stationary observer at infinity. In what way would this observer measure the volume?

- Are there other situations in which the generalized Parikh volume is useful, besides in black holes? Are there situations, even, in which it would be useful to use that quantity for spacelike (or null) Killing vectors, rather than asymptotically timelike ones? How would we interpret the "volume" in this sense?

- Which of our formulas derived with respect to four-dimensional spacetime still hold in arbitrary dimensions?

- Is it possible to find a deeper, richer understanding of the radius r and time coordinate t, which goes beyond the work in Section 8.7 and Chapter 15? What is at the root of the

$$\mathcal{V}^* = \frac{r_+ A}{3\kappa} \text{ and } {}^{D-1}\mathcal{V}^* = \frac{r_+ A}{3} \tag{18.0.1}$$

relationships, and do they generalize for D-dimensional spacetime to

$${}^{D-1}\mathcal{V}^* = \frac{r_+ A}{\kappa(D-1)} \text{ or } {}^{3}\mathcal{V}^* = \frac{r_+ A}{D-1} \tag{18.0.2}$$

as suggested by Cvetič's work?

- Can we prove the newly formulated laws of black hole mechanics not just in the specific case of the Kerr-Newman metric, but in some more general fashion? One possibility is to use the Komar formulae for black hole mass M_H (14.4.5) and angular momentum J_H (14.4.6). We should also try to verify the interpretation of δM as the incremental change in the black hole's energy, $\Omega_H \delta J$ as the work done by incrementally spinning up a black hole, and so on.

- A full literature review of the arguments for the equivalence between the black hole entropy S_{BH} and the surface area A should be undertaken. Are there arguments for our volume, or Parikh's volume, to play a more central role in defining the entropy? What are the best arguments for the entropy S_{BH} to be proportional to the area? If these arguments hold, is it possible to argue that entropy is still a function of volume, by using the relationship between volume and area?

- In what situations do people use "volumes" incorrectly in the literature, and what can be gained in those situations by applying our volume as an alternative?

- What are other stationary black hole metrics for which we could calculate the volume? Is it possible to do similar volume calculations for regions that are not stationary to develop (for example) ways of looking at, for example, apparent or trapping horizons (for example in the Vaidya metric)?

- Can we expand on Parikh's argument (Section 14.1) about Free Energy, Cvetič's argument about enthalpy (Section 14.2), or Hayward's discussion of dynamic black holes (Section 14.3)? Could we consider systems in which we treat the volume (perhaps the Parikh volume, perhaps "ours") as a volume in the classical thermodynamic sense with Λ as a pressure, and develop equations that deal with implications for constant-volume (isochoric) processes?

- Can we account for the instabilities below the Cauchy horizon in some way, for example by repeating the calculations from Chapter 13 for the region above the Cauchy horizon?

- Kerr-Newman permits a "Killing tensor" according to Walker and Penrose [33] which is closely related to the Boyer-Lindquist coordinates r and θ. Could Killing tensors be used to define another type of Killing volume or be used to gain some volume information in a similar manner to the Killing vectors? (Admittedly, as this will not be asymptotically timelike, it seems unlikely that this will give much physical insight.)

- Are there any physically significant metrics that have the form

$$\mathrm{d}s^2 = -h(r)(r-a)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{k(r)(r-a)} + r^2\mathrm{d}\Omega_2^2 \tag{18.0.3}$$

  where $h \neq k$, even up to a multiplicative constant? If they exist, these BHs will have a Parikh volume which is unequal to the Euclidean volume, despite possessing spherical symmetry.

Chapter 19

Summary and Conclusions

We have developed a new volume rate which represents the growth of the four-volume below the horizon of a black hole with respect to the log of the affine generator on the horizon. We have shown both how we developed our volume, and how it can be expressed very simply. We have offered several interpretations of it, and noted the difference in form between the logarithmic growth of the volume with respect to the affine parameter when the surface gravity $\kappa \neq 0$ and the linear growth with respect to the affine parameters when $\kappa = 0$.

We have examined the Parikh volume and have reformulated it in a robust fashion that can be used in any space or spacetime with a Killing vector. We reformulated this in several ways, including in a coordinate-independent manner. We verified directly Parikh's claim that the choice of slicing is irrelevant. Additionally, we examined what class of Killing vectors yield the same Killing volume result for a given region. We examined the link between the Parikh volume and our volume, and used this to develop an alternate definition/interpretation of our volume. We used the link between the two volumes to give a new definition for the surface gravity and then showed that this can be used to reestablish intuitively the third law of black hole mechanics.

We reformulated the laws of black hole thermodynamics by showing how both our and the Parikh volumes change under differential increase in mass, angular momentum and charge, and found expressions for the increment of heat and of entropy in terms of these volumes, assuming that the usual formulation in terms of the area is correct. We explored the connection between the volumes we have developed, the area, and the horizon radius r_+. We reviewed the work of Parikh [1], Cvetič et al. [2] and Hawyard [3] as it pertains to our work.

A particularly significant advantage of our volume over the Parikh volume is that it does not require specification of the normalization of the Killing vector. Both the Parikh/Killing volume and even the surface gravity κ require some kind of assumption about the magnitude of t (or whatever coordinate corresponding to stationarity is used), generally taking $g_{tt}g_{rr} = -1$ in spherical symmetry or $g_{tt}g_{rr} \to -1$ at large radius away from spherical symmetry. On the other hand, because our volume grows with respect to the log of the affine parameter, our volume is completely independent of the aymptotic value of $\xi^\alpha \xi_\alpha$ upon which κ and the Parikh volume depend so heavily. In that sense, our volume rate is striking in its independence of appeal to normalization.

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