STIFFNESS ANALYSIS OF CABLE-DRIVEN PARALLEL ROBOTS

by

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Abstract

The aim of this thesis is the stiffness analysis of cable-driven parallel robots. Cable-driven parallel robots have drawn considerable attention because of their unique abilities and advantages such as the large workspace, light weight of cable actuators, easy disassembly and transportation of the robot. The mobile platform of a cable-driven parallel robot is attached to the base with multiple cables.

One of the parameters that should be studied to make sure a robot is able to execute a task accurately is stiffness of the robot. In order to investigate the stiffness behaviour of a robot, the stiffness matrix can be calculated as the first step. Because cables act in tension, keeping the positive tension in cables becomes a challenge. In order to have a fully controllable robot, an actuation redundancy is needed. These complexities are addressed in the thesis and simulations.

In this thesis, the complete form of the stiffness matrix is considered without neglecting any terms in calculation of the stiffness. Some stiffness indices such as single-dimensional stiffness based on stiffness ellipse, directional stiffness and condition number of the stiffness matrix are introduced and calculated and stiffness maps of the robot are developed. In addition, the issue of unit inconsistency in calculating the stiffness index is addressed.

One of the areas which is also addressed in this thesis is failure analysis based on the stiffness of robot. The effect of the failure in one or more cables or motors is modelled and stiffness maps are developed for the failure situation. It is shown that by changing the
anchor position and mobile platform orientation, the lost stiffness after failure of a cable or motor can be retrieved partially. Optimum anchor position and mobile platform orientation are identified to maximize the area of the stiffness map.

Condition number of the stiffness matrix while robot is following a trajectory is optimized. In addition, when one cable fails during the path planning, the recovery of the robot is studied. Finally, these analyses on stiffness and failure provide the designer with the necessary and valuable information about the anchor positions and actuator torques.
I would like to express my sincere gratitude and appreciation toward my supervisor, Professor Ron Anderson for his kind and never-failing support. I could not have imagined completing my thesis without his continued guidance, endless encouragement and unique teaching methods. It has been a privilege to work with him which cannot be described in words.

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# Table of Contents

Abstract i  
Acknowledgements iii  
Table of Contents iv  
List of Tables viii  
List of Figures ix  
Notation xiv  

## Chapter 1: Introduction

1.1 Background and Motivation 2  
1.2 Proposed Work 6  
1.3 Contributions and Thesis Outline 8  

## Chapter 2: Literature Review

2.1 Terminologies and Theories in Robotics 12  
2.1.1 Position Analysis of Robots 12  
2.1.2 Jacobian Analysis of Robots 13  
2.1.3 Pseudo-inverse of Jacobian Matrix 13  
2.1.4 Statics and Stiffness Analysis of Robots 15  
2.1.5 Dynamics of Robots 15
Chapter 5: Failure Analysis ........................................ 61
  5.1 Failure Formulation ........................................ 61
    5.1.1 Failure of a Cable .................................... 61
    5.1.2 Failure of a Motor .................................... 62
  5.2 Effect of Failure on the Stiffness Maps .................... 63
    5.2.1 Stiffness Maps of 3DOF Planar Robot after Failure .... 63
    5.2.2 Stiffness Maps of 2DOF Translational Robot after Failure 66
  5.3 Retrieving Lost Stiffness of Robot after Failure ............ 67
  5.4 Optimum Layouts of Robots after Failure ................... 70
    5.4.1 Optimizing Anchor Position ............................ 70
    5.4.2 Optimizing Mobile Platform Orientation ................ 72
    5.4.3 Optimizing Anchor Position and Mobile Platform Orientation 77
  5.5 Conclusions ................................................ 77

Chapter 6: Trajectory Planning ..................................... 79
  6.1 Optimum configuration .................................... 79
  6.2 Optimum Trajectory Planning ............................... 82
    6.2.1 Straight Line Path .................................... 82
    6.2.2 Circular Path ......................................... 85
  6.3 Effect of Redundancy on Optimum Trajectory Planning ...... 89
  6.4 Failure Recovery in Trajectory Planning ..................... 91
    6.4.1 Straight Line Path .................................... 91
    6.4.2 Circular Path ......................................... 94
  6.5 Conclusions ................................................ 100
# Chapter 7: Conclusions

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1 Summary</td>
<td>101</td>
</tr>
<tr>
<td>7.2 Future Work</td>
<td>103</td>
</tr>
</tbody>
</table>

# Bibliography

| Bibliography    | 106  |
List of Tables

4.1 GA function parameters ........................................... 54
6.1 GA function parameters ........................................... 81
# List of Figures

1.1 A KUKA Pick and Place serial robot in a factory \[\text{48}\] ........................................ 3
1.2 A parallel robot formed by six linear actuators (hexapod) controls the movement for the Airbus A3200 Full Flight Simulator (FFS) at Baltic Aviation Academy \[\text{9}\] ........................................ 3
1.3 Terracotta Ancient Greek dolls, exhibited in the National Archaeological Museum in Athens, room 56, 5th/4th century BC \[\text{73}\] ........................................ 4
1.4 NIST RoboCrane striping paint off a U.S. Air Force C-130 \[\text{3}\] ................. 5
1.5 Skycam HD at an ESPN on ABC-broadcast University of California, Berkeley football game \[\text{74}\] ........................................ 5

3.1 Coordinates and variables for the 3DOF planar cable-driven parallel robot. 26
3.2 Coordinates and variables for the 2DOF translational cable-driven parallel robot. ........................................ 28
3.3 Comparing the distribution of the single dimensional stiffness index and directional stiffness for an example stiffness matrix. ................................. 39

4.1 Cross-section of $7 \times 7$ wire rope \[\text{80}\] ........................................ 43
4.2 Example 3DOF planar four-cable-driven parallel robots. ......................... 44
4.3 Stiffness maps of the shown in Figure 4.2(a) robot without gravity. 45
4.4 Stiffness maps of the robot shown in Figure 4.2(a) with gravity. 46
4.5 Stiffness maps along X direction of the robot shown in Figure 4.2(a) without gravity. 46
4.6 Stiffness maps of the robot shown in Figure 4.2(b) with gravity. .......... 47
4.7 Stiffness maps about Z direction of the robot shown in Figure 4.2(b) without gravity at (ϕ = 5°). ................................................................. 48
4.8 Stiffness maps about Z direction of the robot shown in Figure 4.2(b) without gravity at (ϕ = 0°). ................................................................. 48
4.9 Stiffness maps of the robot shown in Figure 4.2(c) with gravity. .......... 49
4.10 2DOF translational three-cable-driven parallel robot. ....................... 50
4.11 Stiffness maps of the robot shown in Figure 4.10 without gravity. ....... 51
4.12 Stiffness maps of the robot without gravity for the required minimum stiffness. 51
4.13 Stiffness maps of the robot shown in Figure 4.10 with gravity. .......... 52
4.14 Stiffness maps of the robot with gravity for the required minimum stiffness. 53
4.15 Stiffness maps of the robot with the optimum layout. .................... 54
4.16 Stiffness maps of the robot for the required minimum stiffness with the optimum layout. ................................................................. 55
4.17 Elastic potential energy maps of the robot in Figure 4.2(a) with gravity at (ϕ = 0°). ................................................................. 55
4.18 Deflection maps of the robot where the unit external force in X direction is applied. ................................................................. 57
4.19 Deflection maps of the robot where the unit external force in Y direction is applied. ................................................................. 57
4.20 Deflection maps of the robot where the unit external moment about Z direction is applied. ................................................................. 57
4.21 Condition number maps of the robot shown in Figure 4.10. .............. 58
4.22 Condition number maps of the robot shown in Figure 4.2(a). ............ 59
5.1 Example 3DOF planar four-cable-driven parallel robots. ................. 63
5.2 Stiffness maps of the robot shown in Figure 5.1(a) with gravity at (ϕ = 5°). 63
5.20 Stiffness maps of the robot after failure of cable 1 with the optimum mobile platform orientation and anchor position for cable 4.

6.1 Coordinate and variables of cable $i$ of 2DOF planar cable-driven parallel robot.

6.2 Optimum configurations where cable 1 to 4 are respectively shown by lines (---, ......, ----, ). Mobile platform and anchor positions are respectively shown by markers (⊙, ○).

6.3 Following a straight line while keeping the condition number close to 1. Cable 1 to 3 are respectively shown by lines (---, ......, ----). Mobile platform and anchor positions are respectively shown by markers (⊙, ○).

6.4 Condition number of stiffness matrix while following the straight line shown in Fig. 6.3.

6.5 Area of stiffness ellipse while following the straight line shown in Fig. 6.3.

6.6 Anchor position of cables while following the straight line shown in Fig. 6.3.

6.7 Following a circular path (counterclockwise) while keeping the condition number close to 1. Cable 1 to 3 are respectively shown by lines (---, ......, ----). Mobile platform and anchor positions are respectively shown by markers (⊙, ○).

6.8 Condition number of stiffness matrix while following the circular path shown in Fig. 6.7.

6.9 Area of stiffness ellipse while following the circular path shown in Fig. 6.7.

6.10 Anchor position of cables while following the circular path shown in Fig. 6.7.

xii
6.11 Condition number of stiffness matrix while following the circular path by 4 cable-driven robot. ................................................................. 89
6.12 Area of stiffness ellipse while following the circular path by 4 cable-driven robot. ................................................................. 90
6.13 Anchor position of cables while following the circular path by 4 cable-driven robot. Anchor position of cable 1 to 4 are respectively shown by lines (—-—- , —-—- , —-—- , —-—- ). ................................................................. 90
6.14 Configuration of cables when failure occurs at $t_f = 5$ s. Cable 1 to 4 are respectively shown by lines (—-—- , —-—- , —-—- , —-—- ). ......................... 91
6.15 Condition number while following the straight line path and failure occurs at instant $t_f = 5$ s. ................................................................. 92
6.16 Area of stiffness ellipse following the straight line path and failure occurs at instant $t_f = 5$ s. ................................................................. 93
6.17 Anchor position of cables following the straight line path and failure occurs at instant $t_f = 5$ s. Anchor position of cable 1 to 4 are respectively shown by lines (—-—- , —-—- , —-—- , —-—- ). ................................................................. 95
6.18 Configuration of cables when failure occurs at $t_f = 10$ s. Cable 1 to 4 are respectively shown by lines (—-—- , —-—- , —-—- , —-—- ). ......................... 96
6.19 Condition number while following the circular path and failure occurs at instant $t_f = 10$ s. ................................................................. 97
6.20 Area of stiffness ellipse following the circular path and failure occurs at instant $t_f = 10$ s. ................................................................. 98
6.21 Anchor position of cables following the circular path and failure occurs at instant $t_f = 10$ s. Anchor position of cable 1 to 4 are respectively shown by lines (—-—- , —-—- , —-—- , —-—- ). ................................................................. 99
Notation

\(a_i = [a_{ix}, a_{iy}]^T\)  The position vector of anchor \(A_i\)  \\
\(A_c = \pi D^2 / 4\)  The cross-sectional area of the cable  \\
\(A_i\)  Anchor point of cable \(i\)  \\
\(b_i\)  The length of the line segment \(PB_i\)  \\
\(B_i\)  Attachment point on the mobile platform  \\
\(C\)  Compliance matrix  \\
\(D\)  The nominal (outer) diameter of the cable which can be measured by caliper  \\
\(e_i = [\cos(\phi + \theta_i), \sin(\phi + \theta_i)]^T\)  The unit vector in the direction from point \(P\) to the attachment point on the mobile platform \(B_i\)  \\
\(E\)  The equivalent Young’s modulus of elasticity of the cable, which can be identified by tension test  \\
\(f\)  Magnitude of the external force applied on the mobile platform  \\
\(f\)  Vector of force  \\
\(\mathbf{F} = [F_x, F_y]^T\)  Cartesian forces applied on the mobile platform, which is also called the wrench acting on the mobile platform  \\
\(f_c\)  Centrifugal force applied to the mobile platform
\( f_s \)  
Part of force vector resulting from the translational part of the stiffness matrix

\( f_\phi \)  
Part of force vector resulting from the rotational part of the stiffness matrix

\( G_f \)  
Unit-homogenized matrix corresponding to the translational part of the stiffness matrix

\( G_M \)  
Unit-homogenized matrix corresponding to the rotational part of the stiffness matrix

\( h \)  
An arbitrary scalar

\( h \)  
An arbitrary vector of size \((n - M) \times 1\)

\( h_{\text{max}} \)  
Maximum scalar which keeps the cable tensions in the allowable bound

\( h_{\text{min}} \)  
Minimum scalar which keeps the cable tensions in the allowable bound

\( H_s \)  
Orthogonal matrix whose columns are the eigenvectors of matrix \( K_{12} K_{12}^T \)

\( H_\phi \)  
Orthogonal matrix whose columns are the eigenvectors of matrix \( K_{22} L_{22} \)

\( J \)  
The Jacobian matrix of the robot

\( J^\# \)  
Pseudo-inverse of the Jacobian matrix of the robot

\( J_f \)  
Jacobian matrix after failure of a motor or cable

\( J_i^T \)  
The \( i \)th column of matrix \( J^T \)

\( J_\alpha \)  
The Jacobian matrix relating the differential of cable orientations and the twist vector

\( k_e \)  
Directional value of stiffness obtained from the stiffness ellipse
Stiffness of $i$th cable actuator

Element $(i,j)$ of stiffness matrix

Maximum stiffness at a given position of the mobile platform

Minimum stiffness at a given position of the mobile platform

Stiffness matrix

Diagonal matrix of cable stiffness

Part of the stiffness matrix related to the cable tensions

Vector of cable lengths

The constant length of cable between the motor spool and the pulley

The variable length of cable between the pulley and the attachment point on the mobile platform

Mass of mobile platform

Degree of freedom of the task space of the robot which is 2 in this paper

Vector of moment

Part of moment vector resulting from the translational part of the stiffness matrix

Part of moment vector resulting from the rotational part of the stiffness matrix

Number of cables in the robot

Matrix of size $n \times (n - M)$ whose columns correspond to the orthonormal basis for the null space of the transposed Jacobian matrix

The position vector of the mobile platform

Vector of joint variables (generalized coordinates)
\( q_i \)  
\( Q_i \)  
\( r \)  
\( r_c \)  
\( s \)  
\( S_s \)  
\( S_\phi \)  
\( t \)  
\( t_f \)  
\( u \)  
\( u_i = [\cos \alpha_i, \sin \alpha_i]^T \)  
\( v \)  
\( \alpha = [\alpha_1, ..., \alpha_n]^T \)  
\( \alpha_i \)  
\( \beta \)  
\( \Gamma(X', Y') \)  
\( \delta r = [\delta p_x, \delta p_y]^T \)  
\( \Delta r \)  
\( \eta_i \)  
\( \theta_i \)  

\( q_i \)  
\( Q_i \)  
\( r \)  
\( r_c \)  
\( s \)  
\( S_s \)  
\( S_\phi \)  
\( t \)  
\( t_f \)  
\( u \)  
\( u_i = [\cos \alpha_i, \sin \alpha_i]^T \)  
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\( \alpha = [\alpha_1, ..., \alpha_n]^T \)  
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\( \Gamma(X', Y') \)  
\( \delta r = [\delta p_x, \delta p_y]^T \)  
\( \Delta r \)  
\( \eta_i \)  
\( \theta_i \)  

\( q_i \)  
\( Q_i \)  
\( r \)  
\( r_c \)  
\( s \)  
\( S_s \)  
\( S_\phi \)  
\( t \)  
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\( u \)  
\( u_i = [\cos \alpha_i, \sin \alpha_i]^T \)  
\( v \)  
\( \alpha = [\alpha_1, ..., \alpha_n]^T \)  
\( \alpha_i \)  
\( \beta \)  
\( \Gamma(X', Y') \)  
\( \delta r = [\delta p_x, \delta p_y]^T \)  
\( \Delta r \)  
\( \eta_i \)  
\( \theta_i \)  

\( q_i \)  
\( Q_i \)  
\( r \)  
\( r_c \)  
\( s \)  
\( S_s \)  
\( S_\phi \)  
\( t \)  
\( t_f \)  
\( u \)  
\( u_i = [\cos \alpha_i, \sin \alpha_i]^T \)  
\( v \)  
\( \alpha = [\alpha_1, ..., \alpha_n]^T \)  
\( \alpha_i \)  
\( \beta \)  
\( \Gamma(X', Y') \)  
\( \delta r = [\delta p_x, \delta p_y]^T \)  
\( \Delta r \)  
\( \eta_i \)  
\( \theta_i \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa(K)$</td>
<td>Condition number of the stiffness matrix</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>$i$th eigenvalue of the stiffness matrix</td>
</tr>
<tr>
<td>$\lambda_{\text{max}}$</td>
<td>Maximum eigenvalue of the stiffness matrix</td>
</tr>
<tr>
<td>$\lambda_{\text{min}}$</td>
<td>Minimum eigenvalue of the stiffness matrix</td>
</tr>
<tr>
<td>$\nu_i$</td>
<td>$i$th eigenvector of the stiffness matrix</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Dimensionless space</td>
</tr>
<tr>
<td>$\tau = [\tau_1, ..., \tau_n]^T$</td>
<td>Vector of cable tensions</td>
</tr>
<tr>
<td>$\tau_{\text{max}}$</td>
<td>Maximum allowable tension</td>
</tr>
<tr>
<td>$\tau_{\text{min}}$</td>
<td>Minimum allowable tension</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Vector of infinitesimal rotation of the mobile platform about the unit vector $\phi/|\phi|$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>The angle at which the mobile platform is oriented with respect to the base frame</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Potential energy of robot</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Dimensionless space</td>
</tr>
<tr>
<td>$\Psi(X,Y)$</td>
<td>The reference frame attached to the base at point 0</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

In the design of a robot many criteria, such as the robot workspace, dexterity, and stiffness have to be considered. The stiffness matrix can be used to study the stiffness of robots in Cartesian axes or to investigate the stiffness bounds by using the eigenvalues of the stiffness matrix [27, 39].

Stiffness modelling and symbolic formulation of the stiffness matrix in terms of the design parameters, such as robot geometry, facilitate a systematic analysis of the influence of these parameters, and identification of the critical parameters. Some of these parameters affect the acceptable region of workspace with minimum/maximum required stiffness.

Knowledge of the stiffness or compliance of a robot reflected at its end effector is important when accomplishment of the contact and non contact tasks is studied. In fact, the stiffness of robots can be used as an index of the accuracy at the position and force level.

The background and motivation behind the proposed research is given in Section 1.1. Section 1.2 gives an introduction to the proposed research. The proposed research contributions and thesis outline come in Section 1.3, respectively.
1.1 Background and Motivation

Robot Institute of America defines robot as a programmable multi-function manipulator designed to move material, parts, or specialized devices through variable programmed motions for the performance of a variety of tasks. Since the industrial revolution, there has been an ever-increasing demand to improve product quality, assure accuracy, reduce manufacturing cost and increase the safety at work. It explains the increasing attention to the robots in repetitive operations, mass production, high precision and in hazardous environments [24, 88].

A robot consists of bodies/links which are connected with joints which allow relative motion of links and it leads to the relative motion of the end effector with respect to the ground. “The number of degrees of freedom that a robot possesses is the number of independent variables which would have to be specified in order to locate all parts of the mechanism” [24].

According to the architecture of the robots, they can be classified in two categories. Serial robots consist of an open-loop kinematic chain of links and parallel robot consists of closed-loop kinematic chains. An example of a serial robot is shown in Figure 1.1 and an example of a parallel robot is shown in Figure 1.2. Parallel robots possess advantages such as high stiffness and accuracy, high payload capacity and improved dynamics. However, they have relatively small workspace [24, 88].

A robot whose links move in a plane or parallel planes is called planar. In this research cable-driven parallel robots are considered. May be one of the first cable-driven mechanisms was marionette. Marionette is a kind of puppet which is controlled by the puppeteer from above using cables and strings. Thus, one of the oldest known cable-driven systems is the ancient Greek terracotta puppet doll shown in Figure 1.3. Cable-driven robots are parallel robots which use the actuated cables for the branches. Mobile platform of the cable-driven robot is attached to the base with multiple cables and by using long cables; larger workspace
Figure 1.1: A KUKA Pick and Place serial robot in a factory [48]

Figure 1.2: A parallel robot formed by six linear actuators (hexapod) controls the movement for the Airbus A3200 Full Flight Simulator (FFS) at Baltic Aviation Academy [9]
for these kinds of robots is possible. In addition, the light weight of cable actuators, easy

disassembly and transportation of the robot are the other advantages of cable-driven robots.

Because of the light weight of robot, cable-driven robots can perform high speed tasks
[42, 43]. One of the first cable-driven robots built was used in early 1980s in shipping ports
which was called RoboCrane. This robot is shown in Figure 1.4 and it was developed by

National Institute of Standards and Technology (NIST) [3]. Another application in which

cable-driven robots have been used widely is for camera systems to maneuver through three
dimensions in the open space over a playing area of a stadium or arena. Figure 1.5 shows an

example of these cable-driven camera systems called Skycam. There are several examples of
designed and developed cable-driven systems for different applications, e.g., CHARLOTTE
[19] was developed for International Space Station, WireMan was designed as a portable
Figure 1.4: NIST RoboCrane striping paint off a U.S. Air Force C-130

Figure 1.5: Skycam HD at an ESPN on ABC-broadcast University of California, Berkeley football game
haptic device driven by cables [14], and Kawamura in [42] introduced an ultrahigh speed robot named FALCON (FAst Load CONveyance) based on a cable-driven system.

On the other hand, by using cable actuators, some limits are forced to the design and abilities of the robot. Cables can be used only when they are in tension, so keeping the positive tension in cables is a challenge. Because of this characteristic of cables, to design a fully controllable robot in absence of gravity, at least $n+1$ cables are needed to have a robot with $n$ degrees of freedom (DOF) [22, 82].

Because of using the cables instead of solid links and actuators, the stiffness of cable-driven parallel robots is relatively low; hence stiffness analysis is necessary to avoid the failure in the system. Failure can be defined as not meeting the required stiffness for a robot.

1.2 Proposed Work

Cable-driven parallel robots have drawn considerable attention because of their unique abilities and advantages which were discussed earlier in Section 1.1. One of the parameters which should be studied to make sure a robot is able to execute a task accurately is stiffness of the robot. In order to investigate the stiffness behavior of a robot, stiffness matrix can be calculated as the first step. The aim of the proposed research is the stiffness analysis of the cable-driven parallel robots. Because the cables can only be in tension, keeping the positive tension in cables is a challenge.

On the other hand to resolve this challenge, as it was mentioned earlier in Section 1.1, redundancy in actuation is used to keep the cable tensions positive. This redundancy results in the infinite solutions of the cable tensions for the given external wrench on the mobile platform. Having infinite solutions makes choosing the cable tensions a challenge. In addition, choosing the cable tensions among the possible infinite solutions can be used in a way to increase or decrease the stiffness. Therefore, the redundancy adds more complexity.
to the stiffness analysis of the cable-driven parallel robots. These issues will be addressed in the proposed research and simulations.

In the literature review section, most of the published researches were on the stiffness matrix formulation of the serial and parallel robots and not on cable-driven robots. As discussed in Section 2.2 some research use the simplest form of the stiffness matrix (conventional form) and neglect some terms in calculation of the stiffness. In this research, the complete form of the stiffness matrix is considered without neglecting any terms in calculation of the stiffness. Different methods and definitions for calculation of the stiffness matrix have been introduced. In this research, stiffness matrix is formulated in the moving frame which results in a symmetric matrix. Earlier this frame was introduced in Section 2.2.

To compare the different robot architectures from the stiffness point of view, and investigate the required stiffness for the robot over the workspace, some stiffness indices should be reviewed and calculated for the robot. The proposed research will introduce and use the stiffness indices based on the stiffness matrix. Some issues like unit inconsistency in calculation of the stiffness index will be addressed in the this work.

Applying and modifying the existing stiffness matrix calculations used to serial and parallel robots for cable-driven robots, stiffness matrix of the cable-driven robots is formulated. In the formulations all effective terms in calculation of the stiffness matrix are considered and no term is neglected.

Stiffness indices maps are the guides for the designers. Thus, stiffness maps for the cable-driven robots are developed which can be used in design of the robots. One of the areas which is studied in the research is failure analysis based on the stiffness. By defining a required stiffness for doing a task, the robot might not meet the required stiffness in some points and regions in its workspace. Thus, stiffness failure in cable-driven robots will be investigated in this research and effects of different factors such as the robot configuration (layout), position and orientation of the mobile platform and tension in cables will be studied. In addition, the failure of one or more cables will affect the mobility and stiffness
map of the robot. By having the general formulation for the stiffness matrix of a cable-driven robot, the effect of the failure in one or more cables can be modelled and stiffness maps will be developed for the failure situation.

After stiffness and failure analysis, another study which will be done is failure recovery. Different methods for retrieving the lost stiffness after failure of a motor or cable in a the robot are introduced. These methods are applied on example cases of failure in cable-driven robots and their effectiveness is measured and optimized.

The final goal of this research is performing a task within a desired/required stiffness behavior without failure or recovering in case of failure. Thus, a trajectory is defined and the robot will follow the trajectory while a stiffness index is optimized. It is also studied how the stiffness index is affected by failure of a cable and how the robot can recover to its optimum configuration and continue the trajectory.

1.3 Contributions and Thesis Outline

The aim of the proposed research is the stiffness analysis of cable-driven robots considering the fact that keeping the positive tensions in the cables and choosing the desired cable tensions from the stiffness point of view are challenging.

The proposed work develops the complete form of the stiffness matrix of planar n-cable-driven parallel robots. The differential form of the static force and moment equations is used to formulate the stiffness matrix for a given pose of the robot. Variation of the cable stiffness with the cable length is formulated and implemented in the model. One of the contributions is considering the variable stiffness of cables and all the effective terms in overall stiffness of cable-driven robots based on complete models used for solid link parallel robots.

In order to evaluate and compare the different layouts of the robots or the robots in the different poses in terms of stiffness, stiffness indices based on the stiffness matrix are
introduced and formulated. Another contribution is on comparing the indices and their different physical meanings for cable-driven robots.

Stiffness indices are mapped over the workspace of robot. e.g., directional stiffness about and along different directions are developed and the effect of cable or motor failure on the stiffness maps is investigated. Specifically, the stiffness matrix of the robot before and after failure is formulated and the maps based on the stiffness matrix are developed and compared. Defining the modes of failure and modelling them are also contributions of this research.

Some strategies for retrieving the lost stiffness after the motor or cable failure are discussed. It is demonstrated that by changing the anchor position and mobile platform orientation, the lost stiffness due to failure of a cable or motor can be retrieved partially. The effects of the introduced strategies on stiffness maps over the workspace of example planar cable-actuated robots are presented. The idea of retrieving the lost stiffness using the introduced strategies is another contribution of this work.

Optimum layouts of the robots are also identified in this research, i.e., the genetic algorithm is used to maximize the area of the stiffness maps. Also, before and after failure of a cable or motor, variation of the stiffness map area versus anchor position and/or mobile platform orientation of an example cable-driven parallel robot is plotted. Through examination of these plots, optimum anchor position and mobile platform orientation can be identified to maximize the area of the stiffness map. One of the contributions of this research is the idea of changing the anchor positions to optimize different qualities of cable-driven robots.

Condition number of the stiffness matrix while robot is following a trajectory is optimized. In addition, when one cable fails during the path planning, the recovery of the robot is studied. Finally, optimizing the cable tensions based on stiffness and failure analyses provide the designer with the necessary information about the anchor positions and actuator torques which can be used for choosing the control scheme and designing the controller.
This research has been laid out as follows. A literature review of work relevant to this research is given in Chapter 2. In Chapter 3, planar \( n \)-cable-driven parallel robots are studied. In Section 3.1, the modelling starts with the kinematics analysis and the Jacobian matrix is formulated. Based on Jacobian matrix, force analysis of the robot is conducted in Section 3.1.2. The differential form of the static force and moment equations is used to formulate the stiffness matrix of the robot in Sections 3.1.3.1 and 3.1.3.2. By considering all the terms in the differential form of equations, the complete form of stiffness matrix of planar cable-driven parallel robots is developed which is symmetric. Based on the developed stiffness matrix, different stiffness indices are introduced and formulated in Section 3.2. Single dimensional stiffness based on stiffness ellipse and directional stiffness are studied in Sections 3.2.1 and 3.2.2 and the results for a given stiffness matrix are compared in Section 3.2.2 for all the directions in the plane of mobile platform. For the 3DOF planar case which has unit inconsistent degrees of freedom, i.e., both rotational and translational degrees of freedom exist, issue of unit inconsistency in calculation of the stiffness indices is addressed in Section 3.2.1.1 and meaningful indices are introduced and formulated.

In Chapter 4, directional stiffness index is mapped over the workspace of the example cable-driven robots. In Section 4.1, stiffness maps are shown for directional stiffness in X and Y directions (translational stiffness) and for the 3DOF planar robot also about Z direction (rotational stiffness). Robots are considered to move on the horizontal plane (without gravity) or on the vertical plane (with gravity). In Section 4.1.1, for different cases of 3DOF planar robots with different cable configurations such as two cable attachments on mobile platform, symmetrical four cable attachments on mobile platform, and crossed four cable configuration, stiffness maps are developed and compared. Those maps are also developed for the 2DOF translational robots in Section 4.1.2. It is also discussed that in some cases a minimum required stiffness might be defined for the robot, thus the corresponding maps considering a minimum required stiffness are also developed and shown. In addition in Section 4.2, an optimization problem is defined to maximize the area of the
maps by changing the layout of the robots, the optimum results are identified and the optimum layouts are shown. Also in Sections 4.3, 4.4, and 4.5, other maps are also introduced and developed such as, potential energy maps, deflection maps and condition number maps.

In Chapter 5, failure of cable-driven parallel robots is studied. Failure modes are introduced and formulated in Section 5.1. In Section 5.2, effect of failures on the stiffness maps of the robots is investigated. Failure recovery methods are introduced and investigated in Section 5.3. Finally in Section 5.4, optimum layouts of robots after failure are identified.

In Chapter 6, trajectory planning of cable-driven parallel robots is studied. The methodology to find the optimum configuration of 3 or 4 cable-driven robots to maximize the directional stiffness or condition number is introduced in Section 6.1. In Section 6.2, the same methodology is used to follow a straight line or circular trajectory while keeping the optimum configuration. The results of optimum trajectory planning for straight line and circular path are presented in Sections 6.2.1 and 6.2.2 respectively. Effect of actuation redundancy on optimum trajectory planning is studied in Section 6.3 by comparing the results for 3 and 4 cable-driven robots following the same trajectory. Failure recovery during the optimum trajectory planning is studied in Section 6.4.
Chapter 2

Literature Review

In Section 2.1 some terminologies and basic theories used in the study of robots are explained. A literature review of work relevant to this research is given in Section 2.2.

2.1 Terminologies and Theories in Robotics

2.1.1 Position Analysis of Robots

When operation of a task by a robot end effector is studied, knowledge of the location of the end effector relative to the base (ground) is necessary. This information can be obtained through the position analysis of the robot. There are two different approaches in position analysis of the robots: direct position or direct kinematics and inverse position or inverse kinematics. In direct kinematics, given the joint variables, e.g., joint angle in revolute joints, position and orientation of the end effector is calculated. However, in inverse kinematics, position and orientation of the end effector is known and the goal is calculation of the joint variables. Generally direct kinematics is a straight forward analysis for serial robots unlike parallel robots which have a more straight forward inverse kinematics. Knowledge of kinematics is one of the primary information needed by the designers of robots which will
provide valuable information regarding position, velocity, acceleration and even higher-order derivatives of the end effector position and orientation or joint variables [88].

2.1.2 Jacobian Analysis of Robots

In the previous section kinematics was introduced which would define a relationship between the joint variables and position and orientation of the end effector. The space that the joint variables are defined in is called joint space and the space spanned by end effector is called end effector space. Based on kinematics, the transformation matrix between rate of joints in joint space and the vector of linear and angular velocity of end effector in end effector space is defined and it is called Jacobian matrix. Knowing the Jacobian matrix of a robot, it can be determined in which configuration the matrix loses its full rank or in other words becomes singular. The physical meaning of this condition for serial robot is where the robot loses one or more degrees of freedom. However, in parallel robots the same condition leads in the robot gaining one or more degrees of freedom. This is one example of Jacobian analysis but knowing the Jacobian matrix of the robot is also necessary in trajectory and path planning of the end effector [88].

2.1.3 Pseudo-inverse of Jacobian Matrix

In the previous section the transformation between the end effector rate space and joint rate space was defined by Jacobian matrix. In order to inverse this transformation, inverse of the Jacobian matrix needs to be calculated. When the task assigned to the robot has th dimension of $M$ and the robot joint space has the dimension of $n$ and $M \neq n$, the Jacobian matrix $J_{(M \times n)}$ is not square and consequently not invertible [13, 26, 95]. When a task is operated by a robot with higher degrees of freedom than required, i.e., $M < n$, the robot is redundant and the inverse velocity problem will have infinite number of solutions. On the other hand, if the assigned task is operated by a robot with less degrees of freedom as required, i.e., $M > n$, there will be no exact solution to the inverse velocity problem.
In these cases where the Jacobian matrix is non-square, instead of inverse, pseudo-inverse (generalized inverse or Moore-Penrose inverse) is used which is calculated as [95]

\[ \mathbf{J}^\# = \mathbf{J}^T (\mathbf{J} \mathbf{J}^T)^{-1} \quad M < n \]  

(2.1)

\[ \mathbf{J}^\# = (\mathbf{J}^T \mathbf{J})^{-1} \mathbf{J}^T \quad M > n \]  

(2.2)

For the case that \( M < n \) the inverse velocity analysis using the pseudo-inverse of Jacobian matrix introduced in Equation (2.1) gives a solution for joint velocities which have the minimum norm and corresponding the desired end effector velocity. On the other hand, for the case that \( M > n \) the inverse velocity analysis using the pseudo-inverse of Jacobian matrix introduced in Equation (2.2) gives a solution for joint velocities which have the minimum error norm from the desired end effector velocity. Pseudo inverse of a matrix also satisfies the following Penrose conditions [95]:

\[ \mathbf{J} \mathbf{J}^\# \mathbf{J} = \mathbf{J} \]  

(2.3)

\[ \mathbf{J}^\# \mathbf{J} \mathbf{J}^\# = \mathbf{J}^\# \]  

(2.4)

\[ (\mathbf{J} \mathbf{J}^\#)^T = \mathbf{J} \mathbf{J}^\# \]  

(2.5)

\[ (\mathbf{J}^\# \mathbf{J})^T = \mathbf{J}^\# \mathbf{J} \]  

(2.6)

In [26], it is also studied where the joint variables or end effector variables are not unit consistent and when it is necessary to use weighted pseudo-inverse to end up having a meaningful minimized norm.
2.1.4 Statics and Stiffness Analysis of Robots

Performing different tasks by robots will make the end effector to apply/resist different forces and moments to/from the environment. These forces and moments should be supplied by actuators. It can be shown that the forces and moments in joint space are related to the forces and moments of end effector by the transpose of the Jacobian matrix of the robot. This transformation can be inversed using the inverse of Jacobian transpose. As it was discussed in Section 2.1.3, if a matrix in non-square instead of the inverse, pseudo-inverse of the matrix is calculated, e.g., in redundant robots, Jacobian transpose is non-square so for the statics analysis and inversing the transformation between forces and moments between joint space and end effector, pseudo-inverse of the Jacobian transpose is used. The knowledge of statics of robots will provide the designer with the required information to size the links, actuators, bearings and etc. Also knowing the applied forces and moments to the end effector of robot, deflection of the end effector will be determined which is the basis of stiffness analysis. In the following chapters this relationship between the forces and moments and the deflections is identified by a matrix called stiffness matrix. Compliance control of robots is also based on the results of stiffness analysis [88].

2.1.5 Dynamics of Robots

When studying the operation of high speed tasks using robots, the dynamic force and moments become significantly important and cannot be neglected. In dynamics of robots, the inertia forces and moments are added to the static forces and moments discussed in Section 2.1.4. Development of the dynamic model of the robot can provide the controller designer with valuable information to design an optimum controller. Similar to other analyses in robotics, we have direct and inverse dynamics. Direct dynamics will study the response of the robot due to applied torques and forces by the actuator where in inverse
dynamics the desired trajectory is known and the required forces and torques of the actuators to produce that response is questioned. There are different methods for dynamic analysis such as Newton-Euler equations, Lagrangian equations of motion, and the principle of virtual work [88].

2.2 Literature Review

As it was discussed earlier in Section 1.1, cables can only apply force in one direction, i.e., cables can only pull on objects. This phenomenon is very similar and opposite of grasping. Thus, the studies on grasping can be applied to the cable-driven systems as well. Some early studies on grasping have been conducted in 1980s. The number of required contacts to constrain an object in a plane was studied in [68] which is similar to the later study of number of required cables to design a fully controllable robot in [22, 82].

Later Nguyen in [62] investigated different kinds of contacts in grasping to withstand the external force and moment on the grasped object which is the base theory to obtain the force and moment closure workspace of cable-driven robots [4, 10, 15, 16, 17, 28, 29, 30, 34, 35, 47, 55, 57, 66, 70, 71, 72, 78, 91, 92, 93].

Roberts et al. in [78] conducted one of the earliest studies on kinematics and statics of cable-driven robots. They also developed the workspace of 3DOF planar robots by identifying the fully constrained configurations using the null space of the Jacobian matrix. In their study the developed workspaces correspond to the minimum cable tension of zero and maximum of infinity. Later Maeda et al. in [55] built a redundant cable-driven parallel robot called WARP for high speed assembly of lightweight objects such as semiconductors. They also simulated the workspace of the robot to generate arbitrary amounts of force and moment and also avoid the cable contacts. Another study by Won Jeong et al. investigated a 6DOF cable-driven robot for measuring the 6DOF of an industrial robot [93]. They also developed and presented the robot workspace in different planes. This study took the cable
sag due to gravity in to the account in the kinematics analysis.

In [10], Barrette and Gosselin defined the dynamic workspace as set of all configurations that mobile platform with a given linear and angular acceleration in a specific direction, can reach while keeping positive tension in all the cables. They also proposed an analytical method for the determination of the workspace boundaries of 2DOF translational cable-driven robot. Bosscher et al. were another research group who focused on deriving the complete analytical expressions for the workspace boundaries of a planar cable-driven robot and a spatial point-mass cable-driven robot [15, 16, 17]. They also showed how to extend their workspace-generation approach to determine other workspaces. Their analytical approach results were compared and verified by numerical workspace-generation method.

Agrawal research group developed the workspace of cable-driven robots as well and investigated the design variables such as attachment points of the cables, size and shape of the mobile platform. In their work the workspace area and global condition index were used as the objective functions to optimize the design parameters. They demonstrated the effectiveness of the optimization on their three-cable-driven planar experimental robot [29, 66]. Hay and Snyman in [37] looked in to the optimal configurations of planar cable-driven parallel robots to identify a robot with a large dexterous workspace. Pham and his research group studied the workspace of cable-driven parallel robots based on convex hull theory [72]. They also focused on generating a workspace with optimized performance [70, 71].

Another important factor in designing a robot is the force/moment capability of the robot. The force/moment capability of parallel robots was analyzed in [32, 65, 64] by Nokelby and his research group. They investigated analytical and optimization-based methodologies to determine the force/moment capabilities of non-redundantly and redundantly-actuated parallel robots. They also applied these developed methodologies for generating the force/moment capabilities of redundantly-actuated parallel robots with different parallel robot architectures such as 3-RRR, 3-RPR, 3-PRR layouts. Similar to parallel robots,
force/moment capability study has been conducted on cable-driven robots as well. Bosscher et al. in [15, 16] investigated the force/moment that a cable-driven robot can apply to its surroundings without violating tension limits in the cables. Their research resulted in necessary application information such as payload specification of the robot. In [47], force performance indices adapted to cable-driven parallel robots were studied and the authors adapted the maximum operational isotropic force to characterize cable-driven parallel robots force behavior. Bouchard et al. presented a geometry-based method to determine if a cable-driven robot can generate a given set of force/moment in a given pose, considering allowable minimum and maximum tensions in the cables [18].

For a robot to perform a task successfully with a given precision, specific range of stiffness is required. Thus, stiffness analysis of robotic systems is necessary and inevitable. One of the earliest studies on the stiffness was conducted by Dimentberg in 1965 [25], which developed the stiffness matrix about an equilibrium position based on screw theory. Some other researches in 1980s also used the stiffness modelling for various applications [54, 63, 89], e.g. Salisbury in [81] studied the active stiffness control of a robot using a basic stiffness formulation. Gosselin in [33] applied the existing formulation for serial robots for the stiffness matrix formulations for parallel robots. Developed stiffness matrix is the simplest form of stiffness matrix. Ciblak and Lipkin derived the stiffness matrix for a robot subjected to external load. Method and reference frame used by [23] lead to an asymmetric stiffness matrix. The method is based on calculation of the matrix which relates small linear and angular displacements (deflections) from an initial configuration of the robot to small changes in the applied force and torque. In [98], it is discussed that the stiffness matrix is dependent on the choice of an affine connection on the Lie group. In the literature [23] by assuming an asymmetric connection, an asymmetric stiffness matrix has been resulted in a general loaded configuration. In [98] it is proven that by choosing a symmetric connection, a symmetric Cartesian stiffness matrix is obtained which is formulated in this thesis as well.
The definition used by Pigoski and Griffis which is valid for serial, parallel and cable-driven robots, describes the stiffness as a matrix which transforms a differential displacement (deflection) of the end effector of a serial robot or mobile platform (output link, end effector) of a parallel or cable-driven robot (twist) into the corresponding incremental change in the applied force and moment to the end effector of the robot (wrench) [36, 75].

Cartesian stiffness matrix relates the forces and moment applied at the end effector of the robot in Cartesian space to the corresponding linear and angular displacements (deflections) of the end effector of the robot. In [20] the differential form of the static force and moment balance equations has been used to derive the complete form of the stiffness matrix.

On the other hand, Zefran and Kumar in [97] and Kovecses and Angeles in [45] defined the stiffness matrix of a robot as the Hessian of the potential energy of the robot with respect to the generalized coordinates. For instance, if the connection between the end effector of the robot is modelled as a conservative coupling of springs representing links, joints and actuation which connect the end effector to the ground, the potential energy of the coupling will be the summation of the potential energy of the spring elements [97]. For the case that potential energy of the robot, \( \Phi \), is a function of the generalized coordinates, \( q_i \), and the robot is subjected to external forces and moments, the \( i \)th generalized force, \( Q_i \), is given by

\[
Q_i = \frac{\partial \Phi}{\partial q_i}
\]  

(2.7)

The stiffness matrix of the robot describes the changes in the generalized forces with changes in the generalized coordinates [38]. Thus, the stiffness matrix elements which are the elements of the Hessian of the potential energy can be obtained by differentiating Equation 2.7 as

\[
k_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j}
\]

(2.8)
The same moving frame which gives a symmetric stiffness matrix was discussed in \([38, 75]\). In another word, this special frame translates with the mobile platform, but it does not rotate with the mobile platform. This special moving frame is called "symmetric body of reference" in \([75]\).

Another issue addressed by Kovecses and Angeles in \([45]\), which may cause the asymmetry, is the matter of the incompatibility between the twist and the wrench increment. The fixed point on the mobile platform that undergoes the displacement increment and the point of application of the external wrench should be coincident.

In \([50]\), fundamental properties of stiffness matrix as applied to analysis of grasping and dexterous manipulation was studied. As discussed earlier grasp theory analysis is very similar but opposite to that of cable-driven robots. Kao et al. extended their study of stiffness to human grasping. Grasp stiffness was demonstrated to be useful for modeling and controlling the robots as well \([41]\). In \([44]\), adjustability of the output compliance matrix of a planar 3DOF parallel robot was studied, by employing redundancy on either joint compliances or on actuators. This purpose was achieved through redundant passive springs or decoupled feedback stiffness gains or through antagonistic actuation of the system actuators.

Huang and Schimmels in \([39]\) used the screw theory method for investigating the compliant behavior of the robots by modelling them as simple springs. In their analysis, the connection between the end effector and base were considered as source of compliance/stiffness and modelled as translational and rotational simple springs. This method resulted in a symmetric stiffness matrix which is identical to the conventional form of the stiffness matrix given by \([33]\) earlier. In \([53]\), the conditioning and stiffness indices for 3DOF spherical parallel robots were obtained to optimize the link lengths of the robot. For the stiffness index, condition number of the stiffness matrix was used. Sivinin and his research group worked on stiffness analysis of Gough-Stewart platform and used the derived stiffness matrix to investigate the stability of the system \([83]\). Later in \([84]\), they extended the stiffness
analysis on the redundant robots and used it in stiffness control of those robots. Yoon et al. in [94] designed a modified Delta mechanism, with a well-balanced stiffness, based on their method of stiffness analysis including elastic deformations of both parts and bearings. Their method is based on standard concepts such as static elastic deformations.

In [56], Majou et al. presented a parametric stiffness analysis of the Orthoglide. They studied the influence of the geometric design parameters on overall stiffness and identified the stiffest areas of the workspace for a specific machining task. Pashkevich et al. in [69] investigated the stiffness of over constrained parallel robots with flexible links and compliant actuating joints. They implemented a FEA-based link stiffness evaluation in computation of the stiffness matrix. They used their method for stiffness analysis of two translational parallel robots of 3-PUU and 3-PRPaR architectures.

Quennouelle and Gosselin worked on the stiffness matrix of parallel manipulators. The formulated stiffness matrix took into account the stiffness of the passive joints even with large displacements [77]. They claimed the developed stiffness matrix to be conservative, i.e., symmetric stiffness matrix, and positive definite, semipositive definite or non-positive definite. They applied their kinematostatic model on a 3-PRRR mechanism, the Tripteron in [76]. Li and Gosselin also formulated the stiffness matrix for 3-RPR planar parallel robot. They developed the stiffness maps for the robot with and without external forces. They used the diagonal entries of the stiffness matrix as the mapped stiffness index [51]. Li and Meng formulated the stiffness matrix for 6-SPS parallel robot and showed the stiffness is dependent on the configuration of robot, actuating forces, joint stiffness and direction and magnitude of the external forces acting on the robot [52].

Behzadipour and Khajepour in [12] studied the stiffness of cable-driven robots. They considered the effect of cable tensions and antagonistic forces on the stiffness matrix. They used the resultant stiffness matrix in stability analysis and concluded that the root of instability is a rotational stiffness caused by the internal cable forces. They derived a set
of sufficient conditions to ensure the robot never becomes unstable by increasing the antagonistic forces. Generally the total stiffness matrix is dominated by the part of stiffness corresponding to the stiffness of each cable (elastic stiffness) not the part corresponding to the cable tensions (antagonistic stiffness). But Behzadipour and Azadi continued the study on antagonistic stiffness of cable-driven robots and showed that the geometry of the robot can be devised such that the antagonistic stiffness becomes fully dominant [11]. Later in [7, 8], they presented the concept of variable stiffness elements based on antagonistic forces in cable-driven robots. They designed an adaptive engine mount using variable antagonistic stiffness and a translational or rotational spring based on the concept of tensegrity structures.

The word tensegrity came from the combination of the words tension and integrity, introduced by Fuller in [31]. Tensegrity mechanisms are the assemblies of axially loaded components where each component stays in tensile or compressive loading during the operation. This constant role of tensile loading for some components in tensegrity mechanisms will enable them to be replaced by cables. Arsenault in [5] analyzed the stiffness of a 2DOF planar tensegrity mechanism. He used the directional stiffness index based on stiffness matrix to develop the stiffness maps for the tensegrity mechanism over its workspace. Later in [6], he did the stiffness analysis for a 2DOF cable-driven system with long cables, without neglecting the mass of cables.

Failure analysis is necessary when a robot is performing a task. Knowing the nature of robot and its actuators, different forms of failure are predictable. Ting et al. investigated the internal shock phenomena due to the failure of joint actuation. They also presented a recovery algorithm for both serial and parallel robots and a control structure for fault-tolerant operation of robots [86]. In [21], Chen et al. worked on robots which can achieve secondary goals without compromising primary performance. For example the robot would follow a velocity and static force mapping precisely where accomplishing and optimizing some secondary goals such as reliability enhancement, obstacle and singularity avoidance,
fault tolerance, or joint limit avoidance. Abdi and Nahavandi tried to achieve a robotic task of moving on a specified trajectory for a fault tolerant operation where the robot continues the trajectory with a minimum velocity jump when a fault occurs within a joint. They proposed a way to tolerate the fault by optimally redistributing the joint velocities for the remained healthy joints of the robots [1]. One of the first studies looking into the failure of cable-driven robots was done by Roberts and his research group in [78]. They investigated the fault tolerance of cable-driven robots that are redundantly actuated using the null space of Jacobian matrix. They studied how removing a cable affects the ability of robot to achieve static equilibrium.

The performance of cable-driven robots is strictly related to the cable configuration. In this thesis and [60, 61], cable-driven robots are introduced which are able to re-configure by changing the anchor positions and this ability is used to optimize different performances of the robots or recover from a failure. Later in [79], similar design has been used to introduce "adaptive cable-driven systems".

To have an operational robot, it is necessary to be able to follow a trajectory and have a control system. One of the first studies on trajectory control of cable-driven robots was conducted by Kawamura and his research group in [42]. They developed an ultrahigh speed cable-driven robot named FALCON which could achieve peak acceleration of up to 43 g and maximum velocities of 1.7 m/s with considerably small D.C. motors of 60 W. Using cable actuators arises the problem of vibration but later in [43], they claimed that internal forces from redundant cable actuators could effectively reduce the vibration in high-speed point to point position control. Williams II in [90] studied a haptic interface based on cable-driven systems and later Williams II and his research group worked on dynamic modelling and control of a planar translational cable-driven robot in [91, 92].

Kamishima et al. in [40] designed and built a prototype hybrid cable-driven parallel arm and developed its motion control system. Alp and Agrawal worked on feedback controllers for cable-driven robots [4]. They compared their simulation and experimental results on a
6DOF cable-driven robot. Oh and Agrawal also worked on approaches to design positive tension controllers for redundant cable-driven robots in [66]. Later in [67], they worked on a novel 6DOF two-stage cable-driven robot proposed by NIST for skin-to-skin transfer of cargo. They considered a planar version of this robot and the disturbance motion from the sea was considered in dynamic modelling. They designed a robust controller for the robot in the presence of unknown disturbances while maintaining positive tensions in the cables. Fang et al. presented the motion control of a 6DOF tendon-based parallel robot for high speed applications using seven cables. For trajectory control of the robot, nonlinear feed forward control laws were used. Having a redundant robot, the optimal tension distribution was considered to the advantage of the control laws [28].

Yu investigated the simultaneous trajectory planning and stiffness control of cable-driven parallel robots in [96]. In [49], a collision free path planning developed for serial robots was adapted to cable-driven robots. A trajectory planning method ensuring positive and bounded cable tensions for the planar cable-driven robots was studied in [87]. Agahi in her PhD Thesis investigated the redundancy resolution of cable-driven robots and developed the dynamic model of the robot. She used the redundancy to achieve impact reduction in a trajectory planning task [2].
Chapter 3

Stiffness Formulation

The following stiffness formulations are valid for both serial and parallel robots, but the Jacobian matrix of the parallel (solid-link and cable-driven) robots corresponds to the inverse of the Jacobian matrix for a serial robot. In this chapter, planar $n$-cable-driven parallel robots are studied. In Section 3.1, the modelling starts with the kinematics analysis and the Jacobian matrix is formulated. Based on Jacobian matrix, force analysis of the robot is conducted in Section 3.1.2. The differential form of the static force and moment equations is used to formulate the stiffness matrix of the robot in Sections 3.1.3.1 and 3.1.3.2. By considering all the terms in the differential form of equations, the complete form of stiffness matrix of planar cable-driven parallel robots is developed which is symmetric. This complete model has also been used in other stiffness analysis work such as [77] on compliant parallel mechanisms. Based on the developed stiffness matrix, different stiffness indices are introduced and formulated in Section 3.2. Single dimensional stiffness based on stiffness ellipse and directional stiffness are studied in Sections 3.2.1 and 3.2.2 and the results for a given stiffness matrix are compared in Section 3.2.2 for all the directions in the plane of mobile platform. For the 3DOF planar case which has unit inconsistent degrees of freedom, i.e., both rotational and translational degrees of freedom exist, issue of unit inconsistency
3.1 Modelling

3.1.1 Kinematics

For the parallel robots, the relationship between the twist vector $\delta \mathbf{r}$ and the vector of differential changes in the joint variables $\delta \mathbf{q}$ can be defined using the Jacobian matrix $\mathbf{J}$ as

$$\delta \mathbf{q} = \mathbf{J} \delta \mathbf{r}$$

(3.1)

In a cable-driven robot, vector $\delta \mathbf{l} = [\delta l_1, ..., \delta l_n]^T$ is the vector of differential changes in the cable lengths which is equivalent to vector $\delta \mathbf{q}$.

3.1.1.1 Jacobian Matrix of 3DOF Planar Robot

The robot studied in this section is planar and it has two translations along the X and Y directions and one rotation about the Z direction. It should be noted that the same
methodology, i.e., vector loop closure method, can be applied to obtain the kinematics for 3D robots with 6DOF task space.

The parameters and reference frames of the planar cable-driven parallel robot are depicted in Figure 3.1. The coordinate frame $\Psi(X, Y)$ is attached to the base at point 0, and the moving frame $\Gamma(X', Y')$ is attached to the mobile platform at point $P$ with position vector of $\mathbf{p} = [p_x, p_y]^T$ with respect to the base frame $\Psi(X, Y)$. In the following formulations, all the vectors will be defined with respect to the base frame unless otherwise stated. The vector loop closure for cable $i$, $i = 1, \cdots, n$, can be written as

$$\mathbf{p} + b_i \mathbf{e}_i - l_i \mathbf{u}_i = \mathbf{a}_i$$

(3.2)

where $\mathbf{u}_i = [\cos \alpha_i, \sin \alpha_i]^T$ is the unit vector along cable $i$, $i = 1, \cdots, n$, from its attachment point to the base $A_i$ to the attachment point on the mobile platform $B_i$. Vector $\mathbf{e}_i = [\cos(\varphi + \theta_i), \sin(\varphi + \theta_i)]^T$ is the unit vector in the direction from point $P$ to the attachment point on the mobile platform $B_i$. The angle at which the mobile platform is oriented with respect to the base frame $\Psi(X, Y)$ is measured by $\varphi$ and the orientation of lines $PB_i$ with respect to the mobile platform frame $\Gamma(X', Y')$ are given by angles $\theta_i$. Vector $\mathbf{a}_i = [a_{ix}, a_{iy}]^T$ is the position vector of point $A_i$, $b_i$ is the length of the line segment $PB_i$, and $l_i$ is the length of cable $i$ which is calculated as

$$l_i = \sqrt{(\mathbf{p} + b_i \mathbf{e}_i - \mathbf{a}_i)^T(\mathbf{p} + b_i \mathbf{e}_i - \mathbf{a}_i)}$$

(3.3)

By taking the derivative of Equation 3.2 and rearranging it in the form of Equation 3.1, the Jacobian matrix of the cable-driven parallel robot can be calculated as

$$\mathbf{J} = \begin{pmatrix} \cos \alpha_1 & \sin \alpha_1 & b_1 \sin(\alpha_1 - \varphi - \theta_1) \\ \vdots & \vdots & \vdots \\ \cos \alpha_n & \sin \alpha_n & b_n \sin(\alpha_n - \varphi - \theta_n) \end{pmatrix}$$

(3.4)
3.1.1.2 Jacobian Matrix of 2DOF Translational Robot

The robot studied in this section is planar but unlike the one studied in Section 3.1.1.1, it does not have the rotational degree of freedom as the mobile platform is a point mass and it only has two translations along the X and Y directions. Figure 3.2 shows the parameters for cable $i$ and the reference frame of the planar translational cable-driven parallel robot. Coordinate system $\Psi(X,Y)$ is attached to the base at point 0. The cable is released from a spool attached to an electric motor. A pulley is placed at anchor points $A_i$ between the spool of cable $i$ and its attachment point on the mobile platform. The anchor points $A_i$ can move on a circular rail shown by dashed-line in Figure 3.2. From the inverse kinematic analysis of the robot, $l_i$ and $\alpha_i$ are obtained as

$$l_i = \sqrt{(p_x - a_{ix})^2 + (p_y - a_{iy})^2}$$  \hspace{1cm} (3.5)

$$\alpha_i = \text{atan2}((p_y - a_{iy}), (p_x - a_{ix}))$$  \hspace{1cm} (3.6)
The Jacobian matrix of the translational robot with $n$ cables in terms of the cable orientations is formulated as

$$
J = \begin{pmatrix}
\cos \alpha_1 & \sin \alpha_1 \\
\vdots & \vdots \\
\cos \alpha_n & \sin \alpha_n 
\end{pmatrix}
$$

(3.7)

where

$$
\sin \alpha_i = \frac{p_y - a_{iy}}{\sqrt{(p_x - a_{ix})^2 + (p_y - a_{iy})^2}}
$$

(3.8)

$$
\cos \alpha_i = \frac{p_x - a_{ix}}{\sqrt{(p_x - a_{ix})^2 + (p_y - a_{iy})^2}}
$$

(3.9)

### 3.1.2 Force Analysis

The static force balance for the robot can be written as

$$
F = J^T \tau
$$

(3.10)

For the robot introduced in Section 3.1.1.1 with the degrees of freedom of $M = 3$, vector of external wrench $F = [F_x, F_y, M_z]^T$ consists of external forces along X and Y directions and external moment about Z direction. For the robot introduced in Section 3.1.1.2 with the degrees of freedom of $M = 2$, vector $F = [F_x, F_y]^T$ only consists of external forces along X and Y directions. The robots consist of $n$ cable actuators; thus, the degree of actuation redundancy is $n - M$. According to the static force balance equation, vector of cable tensions $\tau = [\tau_1, \ldots, \tau_n]^T$ is formulated using the pseudo-inverse of the Jacobian transpose as

$$
\tau = J^{\#} F + Nh
$$

(3.11)

$N$ is a matrix of size $n \times (n - M)$ whose columns correspond to the orthonormal basis for the null space of the transposed Jacobian matrix and $h$ is an arbitrary vector of size
\((n - M) \times 1\). For the case with 1 degree of actuation redundancy, i.e., three-cable-driven 2DOF robot or four-cable-driven 3DOF robot, the arbitrary vector of \(\mathbf{h}\) is reduced to an arbitrary scalar and matrix \(\mathbf{N}\) is reduced to the vector of size \(n \times 1\).

### 3.1.3 Stiffness Matrix Formulation

To calculate the stiffness matrix, Equation 3.10 is differentiated as

\[
\delta \mathbf{F} = \mathbf{J}^T \delta \mathbf{\tau} + \delta \mathbf{J}^T \mathbf{\tau}
\]

(3.12)

Stiffness matrix \(\mathbf{K}\) can be formulated by re-arranging Equation 3.12 as

\[
\delta \mathbf{F} = \mathbf{K} \delta \mathbf{r}
\]

(3.13)

where \(\delta \mathbf{r}\) is the twist vector. For cable-driven parallel robots, the stiffness of each cable is modelled as a simple spring. Thus, the changes in cable forces is written as

\[
\delta \mathbf{\tau} = \mathbf{K}_q \delta \mathbf{l}
\]

(3.14)

where

\[
\mathbf{K}_q = diag[k_1, ..., k_n]
\]

(3.15)

The Jacobian matrix of the cable-driven robots gives the relationship between the vector of differential change in the cable lengths \(\delta \mathbf{l}\) and the twist vector as

\[
\delta \mathbf{l} = \mathbf{J} \delta \mathbf{r}
\]

(3.16)

Upon substituting Equation 3.16 in Equation 3.14 the relationship between \(\delta \mathbf{\tau}\) and the twist vector is obtained as
\[ \delta \tau = K_q J \delta r \] (3.17)

Thus, the first term on the right-hand side of Equation 3.12 can be written as

\[ J^T \delta \tau = J^T K_q J \delta r \] (3.18)

The second term on the right-hand side of Equation 3.12 can be expanded as

\[ \delta J^T \tau = \sum_{i=1}^{n} \delta J_i^T \tau_i = K_\tau \delta r \] (3.19)

where \( J_i^T \) is the \( i \)th column of matrix \( J^T \). Matrix \( K_\tau \) is the part of the stiffness matrix corresponding to the second term on the right-hand side of Equation 3.12. Thus, the overall stiffness matrix is the summation of the matrices corresponding to first and second term on the right-hand side of Equation 3.12.

\[ K = J^T K_q J + K_\tau \] (3.20)

3.1.3.1 Stiffness Matrix of 3DOF Planar Robot

For the 3DOF planar robot, the transpose of the Jacobian matrix is in terms of the cable orientation \( \alpha_i \) and the mobile platform orientation \( \phi \). Thus, \( \delta J_i^T \) can be written as

\[ \delta J_i^T = \frac{\partial J_i}{\partial \alpha_i} \delta \alpha_i + \frac{\partial J_i}{\partial \phi} \delta \phi \] (3.21)

The mobile platform pose in terms of the parameters of cable \( i \) can be written as

\[
\begin{pmatrix}
p_x \\
p_y \\
\phi
\end{pmatrix} =
\begin{pmatrix}
a_{ix} + l_i \cos \alpha_i - b_1 \cos(\phi + \theta_i) \\
a_{iy} + l_i \sin \alpha_i - b_1 \sin(\phi + \theta_i) \\
\phi
\end{pmatrix},
i = 1, \ldots, n
\] (3.22)
A relationship between the differential of cable orientations $\delta \alpha$ and the twist vector $\delta r$ could also be derived by taking the derivative of Equation 3.22 as

$$\delta \alpha = J_\alpha \begin{pmatrix} \delta p_x \\ \delta p_y \\ \delta \varphi \end{pmatrix}$$  \hspace{1cm} (3.23)$$

where

$$J_\alpha = \begin{pmatrix} \frac{-\sin \alpha_1}{l_1} & \frac{\cos \alpha_1}{l_1} & \frac{b_1}{l_1} \cos(\alpha_1 - \varphi - \theta_1) \\ \vdots & \vdots & \vdots \\ \frac{-\sin \alpha_n}{l_n} & \frac{\cos \alpha_n}{l_n} & \frac{b_n}{l_n} \cos(\alpha_n - \varphi - \theta_n) \end{pmatrix}$$  \hspace{1cm} (3.24)$$

$J_\alpha$ is the Jacobian matrix relating the differential of cable orientations $\delta \alpha$ and the twist vector $\delta r$. Considering the Jacobian matrix given in Equation 3.24, Equation 3.21 can be written in terms of the vector of differential of cable orientations $\delta \alpha = [\alpha_1, ..., \alpha_n]^T$ and the twist vector $\delta r = [\delta p_x, \delta p_y, \delta \varphi]^T$ as

$$\delta J_i^T = \begin{pmatrix} 0 & \cdots & -\sin \alpha_i & \cdots & 0 \\ 0 & \cdots & \cos \alpha_i & \cdots & 0 \\ 0 & \cdots & b_1 \cos(\alpha_i - \varphi - \theta_1) & \cdots & 0 \end{pmatrix} \begin{pmatrix} \delta \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b_1 \cos(\alpha_i - \varphi - \theta_1) \end{pmatrix} \begin{pmatrix} \delta \alpha \\ 0 \\ \delta r \end{pmatrix}$$  \hspace{1cm} (3.25)$$

Complete form of the stiffness matrix for the 3DOF planar robot considering both the terms of changes in cable tensions and changes in Jacobian matrix is formulated as
\[ K = J^T K_q J + \sum_{i=1}^{n} \tau_i \begin{pmatrix} 0 & \cdots & -\sin \alpha_i & \cdots & 0 \\ 0 & \cdots & \cos \alpha_i & \cdots & 0 \\ 0 & \cdots & b_1 \cos(\alpha_i - \varphi - \theta_1) & \cdots & 0 \end{pmatrix} J_{\alpha} \]

\[ + \sum_{i=1}^{n} \tau_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b_1 \cos(\alpha_i - \varphi - \theta_1) \end{pmatrix} \]

(3.26)

### 3.1.3.2 Stiffness Matrix of 2DOF Translational Robot

For the 2DOF translational robot, the transpose of the Jacobian matrix is only in terms of the cable orientation \( \alpha_i \). Thus, \( \delta J_i^T \) can be written as

\[ \delta J_i^T = \frac{\partial J_i}{\partial \alpha_i} \delta \alpha_i \] (3.27)

The mobile platform pose in terms of the parameters of cable \( i \) can be written as

\[ \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a_{ix} + l_i \cos \alpha_i \\ a_{iy} + l_i \sin \alpha_i \end{pmatrix}, i = 1, \ldots, n \] (3.28)

A relationship between the differential of cable orientations \( \delta \alpha \) and the twist vector \( \delta r = [\delta p_x, \delta p_y]^T \) could also be derived by taking the derivative of Equation (3.28) as

\[ \delta \alpha = J_{\alpha} \begin{pmatrix} \delta p_x \\ \delta p_y \end{pmatrix} \] (3.29)

where

\[ J_{\alpha} = \begin{pmatrix} -\sin \alpha_1 t_1 & \cos \alpha_1 t_1 \\ \vdots & \vdots \\ -\sin \alpha_n t_n & \cos \alpha_n t_n \end{pmatrix} \] (3.30)
Considering the Jacobian matrix given in Equation 3.7, \( \delta J_i^T \) can be written in terms of twist vector as

\[
\delta J_i^T = \begin{pmatrix}
0 & \cdots & -\sin \alpha_i & \cdots & 0 \\
0 & \cdots & \cos \alpha_i & \cdots & 0 \\
\end{pmatrix}_{2 \times n}
\]

where

\[
J_\alpha = \begin{pmatrix}
-\sin \alpha_1 & \cos \alpha_1 \\
\vdots & \vdots \\
-\sin \alpha_n & \cos \alpha_n \\
\end{pmatrix}_{2 \times n}
\]

(3.32)

Complete form of the stiffness matrix for the 2DOF translational robot considering both the terms of changes in cable tensions and changes in Jacobian matrix is formulated as

\[
K = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix} = J^T K_q J + \sum_{i=1}^{n} \tau_i \begin{pmatrix}
0 & \cdots & -\sin \alpha_i & \cdots & 0 \\
0 & \cdots & \cos \alpha_i & \cdots & 0 \\
\end{pmatrix}_{2 \times n}
\]

(3.33)

\[
k_{11} = \sum_{i=1}^{n} \left( \frac{E_i A_c i \cos^2 \alpha_i}{l_i + l_c} + \frac{\tau_i \sin^2 \alpha_i}{l_i} \right)
\]

(3.34)

\[
k_{12} = k_{21} = \sum_{i=1}^{n} \left( \frac{E_i A_c i \cos \alpha_i \sin \alpha_i}{l_i + l_c} - \frac{\tau_i \sin \alpha_i \cos \alpha_i}{l_i} \right)
\]

(3.35)

\[
k_{22} = \sum_{i=1}^{n} \left( \frac{E_i A_c i \sin^2 \alpha_i}{l_i + l_c} + \frac{\tau_i \cos^2 \alpha_i}{l_i} \right)
\]

(3.36)

### 3.2 Stiffness Indices

In order to evaluate and compare the different layouts of the robots or the robots in the different poses in terms of stiffness, stiffness indices based on the stiffness matrix developed
in Section 3.1 are presented in this section.

3.2.1 Single Dimensional Stiffness Based on Stiffness Ellipse

For the unit consistent stiffness matrices, single dimensional stiffness index can be defined as follows. The $2 \times 2$ stiffness matrix of the 2DOF translational robot for a given pose, where both degrees of freedom are translational is unit consistent and the stiffness ellipse has a physical meaning. Stiffness ellipse can be formed as the major and minor axes coincide with the eigenvectors of the stiffness matrix, $\nu$, and the corresponding eigenvalues, $\lambda$, indicate the value of stiffness in the direction of the eigenvectors. It should be noted for the unit consistent $3 \times 3$ stiffness matrix three eigenvectors and eigenvalues exist which will form a stiffness ellipsoid. Thus for the $2 \times 2$ stiffness matrix, value of stiffness when an external force is applied in the direction of vector $u$, is calculated as

$$k_e = \frac{\lambda_1 \lambda_2}{\sqrt{\lambda_1^2 (u \cdot \nu_2)^2 + \lambda_2^2 (u \cdot \nu_1)^2}}$$

The stiffness index calculated in Equation (3.37) can be physically interpreted in a relationship between changes in the magnitude of external force and the total displacement of the mobile platform such as $\Delta f = k_e \Delta r$. Thus, this stiffness index can also be formulated as

$$k_e = \frac{\|F\|}{\|K^{-1}F\|} = \frac{1}{\|K^{-1}u\|} \text{ where } F = fu$$

It should be noted that the displacement of mobile platform is not collinear with the direction of the external force applied on the mobile platform unless the external force is applied in the direction of the eigenvectors of the stiffness matrix.
3.2.1.1 Unit Inconsistent Stiffness Matrices

For the case that the stiffness matrix is not unit consistent, the matrix can be partitioned and unit-homogenized using the methods introduced in [46, 85]. Consequently, for each unit-homogenized translational or rotational part of the stiffness matrix a point (scalar value), stiffness ellipse or ellipsoid can be formulated using the eigenvalue and eigenvector method studied in Section 3.2.1.

When the task space degrees of freedom include both translational and rotational, the stiffness matrix of the robot is unit inconsistent. Unit inconsistent stiffness matrix is formed by four block matrices whose components have different units. Thus, stiffness matrix can be partitioned and Equation 3.13 can be re-written as

\[
\begin{pmatrix}
  f \\
  M
\end{pmatrix}
= 
\begin{pmatrix}
  K_{11}(N/m) & K_{12}(N/rad) \\
  K_{12}^T(N/rad) & K_{22}(N.m/rad)
\end{pmatrix}
\begin{pmatrix}
  s \\
  \phi
\end{pmatrix}
= 
\begin{pmatrix}
  f_s \\
  f_{\phi}
\end{pmatrix}
\begin{pmatrix}
  s \\
  \phi
\end{pmatrix}
(3.39)
\]

where \( f \) and \( M \) are force and moment vectors respectively. Vector \( s \) is the vector of infinitesimal translation of the mobile platform and vector \( \phi \) is the infinitesimal rotation of the mobile platform about the unit vector \( \phi/\|\phi\| \). According to Equation 3.39, force and moment equations can be written as

\[
f = K_{11}s + K_{12}\phi = f_s + f_{\phi}
\]

\[
M = K_{12}^Ts + K_{22}\phi = M_s + M_{\phi}
\]

Each independent part of Equations 3.40 and 3.41 can be associated with a physically meaningful quadratic form [46, 85] which defines a point, ellipse or ellipsoid in the space of \( \beta \) such as \( \|f_\beta\|^2 = \beta^T K_\beta^T K_\beta \beta \). Here \( \beta \) can be the translational space, i.e., \( \beta = s \), or rotational space, i.e., \( \beta = \phi \). The eigenvectors of the matrix \( K_\beta^T K_\beta \) are used to define the
transformation from the space $\beta$ to dimensionless spaces $\psi$ and $\rho$ as

$$
\begin{pmatrix}
  s \\
  \phi
\end{pmatrix} =
\begin{pmatrix}
  S_s & 0 \\
  0 & S_\phi
\end{pmatrix}
\begin{pmatrix}
  \psi_s \\
  \psi_\phi
\end{pmatrix} \quad (3.42)
$$

$$
\begin{pmatrix}
  s \\
  \phi
\end{pmatrix} =
\begin{pmatrix}
  H_s & 0 \\
  0 & H_\phi
\end{pmatrix}
\begin{pmatrix}
  \rho_s \\
  \rho_\phi
\end{pmatrix} \quad (3.43)
$$

where $S_s$, $S_\phi$, $H_s$, and $H_\phi$ are orthogonal matrices whose columns are the eigenvectors of matrices $K_{11}^T K_{11}$, $K_{12}^T K_{12}$, $K_{12}^T K_{12}^T$ and $K_{22}^T K_{22}$ respectively. By substituting the Equations 3.42 and 3.43 into the Equations 3.40 and 3.41, force and moment vectors are related to the dimensionless vectors of $\psi$ and $\rho$ as

$$
f = G_f \psi \quad (3.44)
$$

$$
M = G_M \rho \quad (3.45)
$$

In which $G_f$ and $G_M$ are unit-homogenized matrices which are calculated as

$$
G_f = [K_{11} S_s \ K_{12} S_\phi] \quad (3.46)
$$

$$
G_M = [K_{12}^T H_s \ K_{22} H_\phi] \quad (3.47)
$$

Using the eigenvectors and square root of the eigenvalues of the unit-homogenized matrices $G_f G_f^T$ and $G_M G_M^T$, single dimensional stiffness index for the translational and rotational part of the unit inconsistent stiffness matrix are defined respectively similar to Section 3.2.1.
It should be noted that the stiffness matrix of the 3DOF planar robot studied in Section 3.1.3.1 is unit inconsistent. Using the method studied in this section, the stiffness matrix of the 3DOF planar robot can be partitioned and unit-homogenized such that \( G_f \) is a \( 2 \times 2 \) matrix which forms an ellipse for the 2DOF translational part of the stiffness and because there is only 1 degree of rotation, matrix \( G_M \) is reduced to a scalar which represents the rotational part of the stiffness.

### 3.2.2 Directional Stiffness

In engineering applications, it might be more interesting to consider the displacement of the mobile platform only in the direction of the applied external force or moment and calculate the directional stiffness as an index. In other words, in the calculation of the directional stiffness, the projected part of the displacement in the direction of the external force or moment is taken into account. Thus, stiffness in the direction of vector \( \mathbf{u} \) is formulated as

\[
k_{ds} = \frac{\| \mathbf{F} \|}{\| (\mathbf{K}^{-1} \mathbf{F})^T \mathbf{u} \|} = \frac{1}{\| \mathbf{u}^T \mathbf{K}^{-1} \mathbf{u} \|} \quad \text{where} \quad \mathbf{F} = f \mathbf{u}
\]

(3.48)

In [27] the directional stiffness has also been formulated in terms of the eigenvalues and eigenvectors of the stiffness matrix as

\[
k_{ds} = \frac{\sum_{i=1}^{2} \eta_i \lambda_i \nu_i \nu_i^T \sum_{j=1}^{2} \eta_j \lambda_j \nu_j}{\sum_{i=1}^{2} \eta_i \nu_i \nu_i^T \sum_{j=1}^{2} \eta_j \lambda_j \nu_j} = \frac{\sum_{i=1}^{2} \eta_i^2 \lambda_i^2}{\sum_{i=1}^{2} \eta_i^2 \lambda_i^2} \quad \text{(3.49)}
\]

where

\[
\eta_i = \Delta r \cdot \nu_i = (\mathbf{K}^{-1} f \mathbf{u}) \cdot \nu_i = f \mathbf{u}^T \mathbf{K}^{-T} \nu_i \quad \text{(3.50)}
\]

Figure 3.3 shows the different distribution of the single dimensional stiffness index studied in Section 3.2.1 and directional stiffness for the example stiffness matrix of

\[
\mathbf{K} = \begin{pmatrix}
1.5 \times 10^5 & 1.1 \times 10^5 \\
1.1 \times 10^5 & 1.5 \times 10^5
\end{pmatrix} \quad (N/m)
\]

38
Figure 3.3: Comparing the distribution of the single dimensional stiffness index and directional stiffness for an example stiffness matrix.

Where the stiffness is maximum/minimum, the external force is applied in the direction of the eigenvectors of the stiffness matrix and the displacement of mobile platform is collinear with the direction of the external force. Thus, it can be seen that two definitions, the single dimensional stiffness index and directional stiffness coincide.

The same method can be applied to unit inconsistent stiffness matrices. In order to calculate the translational stiffness in the direction of unit vector $\mathbf{u}_t$, vector $\mathbf{u} = [\mathbf{u}_t^T \mathbf{0}]^T$ is used and to calculate the rotational stiffness about the unit vector $\mathbf{u}_r$, vector $\mathbf{u} = [\mathbf{0} \mathbf{u}_r^T]^T$ is used in the calculation of Equation 3.48.

For the example unit inconsistent stiffness matrix of 3DOF planar robot studied in Section 3.1.3.1, to calculate the rotational stiffness about Z axis, vector $\mathbf{u} = [0 0 1]^T$ is used. In order to calculate the translational stiffness in the direction of vector $\mathbf{u}_t$, vector $\mathbf{u} = [\mathbf{u}_t^T \mathbf{0}]^T$ is used.
3.2.3 Condition Number of Stiffness Matrix

According to Figure 3.3, value of the stiffness in different directions are different, but in some applications it might be desirable to have a uniform distribution for the stiffness. Thus, condition number of the stiffness matrix can be taken as an index which is the ratio of the minimum value of stiffness over the maximum stiffness and it is formulated as

$$\kappa(K) = \frac{k_{\text{min}}}{k_{\text{max}}} = \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}}$$ (3.51)

where $0 \leq \kappa \leq 1$, and when its value is close to one the distribution of stiffness is uniform and it forms a circle shape and both definitions, the single dimensional stiffness index and directional stiffness are identical in this case. It should be noted that condition number in traditional matrix algebra is the ratio of maximum eigenvalue over the minimum eigenvalue, i.e., inverse of the condition number used in this research.

For unit inconsistent stiffness matrices, condition number can be defined for the translational and rotational parts separately. Thus, the square root of eigenvalues of matrices $G_f G_f^T$ and $G_M G_M^T$ are used in Equation 3.51 to calculate $\kappa$ for translational and rotational part of the stiffness respectively.

3.3 Conclusion

In this chapter, planar $n$-cable-driven parallel robots were studied. The modelling started with the kinematics analysis and the Jacobian matrix was formulated.

Based on Jacobian matrix, force analysis of the robot was conducted. The differential form of the static force and moment equations was used to formulate the stiffness matrix of the robot.

By considering all the terms in the differential form of equations, the complete form of stiffness matrix of planar cable-driven parallel robots was developed which is symmetric.
Based on the developed stiffness matrix, different stiffness indices were introduced and formulated. For the 3DOF planar case which has unit inconsistent degrees of freedom, i.e., both rotational and translational degrees of freedom exist, issue of unit inconsistency in calculation of the stiffness indices was addressed and meaningful indices were introduced and formulated.

Single dimensional stiffness based on stiffness ellipse and directional stiffness were studied and the results for a given stiffness matrix were compared for all the directions in the plane of mobile platform.
Chapter 4

Stiffness and Other Quality Maps\(^1\)

In this chapter, directional stiffness index is mapped over the workspace of the example cable-driven robots. In Section 4.1 stiffness maps are shown for directional stiffness in X and Y directions (translational stiffness) and for the 3DOF planar robot also about Z direction (rotational stiffness). Robots are considered to move on the horizontal plane (without gravity) or on the vertical plane (with gravity). In Section 4.1.1 for different cases of 3DOF planar robots with different cable configurations such as two cable attachments on mobile platform, symmetrical four cable attachments on mobile platform, and crossed four cable configuration, stiffness maps are developed and compared. Those maps are also developed for the 2DOF translational robots in Section 4.1.2. It is also discussed that in some cases a minimum required stiffness might be defined for the robot, thus the corresponding maps considering a minimum required stiffness are also developed and shown.

In addition in Section 4.2 an optimization problem is defined to maximize the area of the maps by changing the layout of the robots, the optimum results are identified and the optimum layouts are shown. Also in Sections 4.3, 4.4, and 4.5 other maps are also introduced and developed such as, potential energy maps, deflection maps and condition number maps.

\(^1\)Parts of the work presented in this chapter have been published in [58, 59, 60]
4.1 Stiffness Maps

In this section, the directional stiffness index introduced in Section 3.2.2 is mapped over the workspace of the example cable-driven robots. It should be noted that the anchor positions are shown by marker ◦ in the stiffness map figures. Robots are considered to move on the horizontal plane (without gravity) or on the vertical plane (with gravity). For the case with gravity, the mobile platform mass of \( m = 2 \) kg is considered.

In this thesis, for the cable actuators, \( 7 \times 7 \) wire rope is considered. The cross-section of the cable \( A_c \) is shown in Figure 4.1. The cable diameter is 0.0015 m and the cable is manufactured from AISI 316 stainless steel grade 1.4401. The equivalent modulus of elasticity of \( E = 57.3 \) GPa has been reported for the cable in [80]. The constant length \( l_c = 0.3 \) m is considered for the cable actuators. Stretch of the cable \( \Delta L \) under the effect of an axial load \( F \) is calculated as [80]

\[
\Delta L = \frac{FL}{EA_c}
\]

The length of cable \( i \) for a given position of the mobile platform is \( l_i + l_c \). The stiffness of cable \( i \), \( k_i \), which varies with the cable length, is formulated as

\[
k_i = \frac{E_i A_{c_i}}{l_i + l_c}
\]

For the cable, minimum and maximum allowable tensions of \( \tau_{min} = 5 \) N and \( \tau_{max} = 500 \) N are considered.

As it was shown in Equation 3.11, there are infinite possible solutions for the cable
tensions of redundantly-actuated robots. For the example robots in this chapter with 1 degree of actuation redundancy a feasible region for arbitrary scalar of $h$ can be found such as $h_{\min} < h < h_{\max}$ and in order to have the maximum stiffness at each point, $h_{\max}$ is chosen.

To find the feasible region of $h$, each cable tension is once set to $\tau_{\min}$, minimum allowable tension, and then to $\tau_{\max}$, maximum allowable tension, and the minimum and maximum values of $h$ for each cable are calculated. The intersection of the $n$ feasible regions of $h$ for $n$ cables will be the feasible region of $h$ for the robot. Inside the workspace of the robot, this feasible region of $h$ exists, i.e., $h_{\min} < h_{\max}$. Outside the workspace of the robot, the feasible region of $h$ does not exist i.e., there is no $h$ to keep the tension of the $n$ cables in the allowable tension limits in other words, $h_{\min} > h_{\max}$. On the boundaries of the workspace, the feasible region of $h$ reduces to one point, i.e., $h_{\min} = h_{\max}$.

### 4.1.1 Stiffness Maps of 3DOF Planar Robots

The stiffness characteristics of the example 3DOF planar cable-driven robots shown in Figure 4.2 are investigated in this section. Specifically, the stiffness matrix of each of these robots and the corresponding directional stiffness is formulated over the robot workspace and the stiffness maps are developed.
4.1.1.1 Two Cable Attachments on Mobile Platform

For the robot shown in Figure 4.2(a), the anchor positions of \{(-1, -0.75), (1, -0.75), (1, 0.75), (-1, 0.75)\} are used. Stiffness maps are plotted over the robot workspace for the following coordinates \{(-0.125, 0), (0.125, 0), (0.125, 0), (-0.125, 0)\} as the cable attachment points on the mobile platform in the moving frame $\Gamma(X', Y')$. These points are calculated by using a radius of 0.125 m for the mobile platform with the angles \{180°, 0°, 0°, 180°\} for the four attachments. The external wrench is considered to be zero and the mobile platform orientation defined in Figure 3.1 is constant at $\varphi = 20°$. It should be noted that all the length values in this thesis are reported in meters and adopted from [2] on redundancy resolution of cable-driven robots. Comparing Figure 4.3 and Figure 4.4, it is seen that the gravity extends the stiffness maps in the negative Y direction. Comparing the maximum values of directional stiffness along X and Y directions for the cases with and without gravity shows a slight increase for the cases with gravity.

In the plot of Figure 4.5(a), the stiffness map along X direction for the robot in Figure 4.2(a) has been developed. The external wrench is considered to be zero and the mobile platform orientation defined in Figure 3.1 is constant at $\varphi = 0°$. In addition, the robot is considered to move on the horizontal plane (without gravity). In the plots of Figure 4.5(b) and 4.5(c) the effect of external force on the stiffness map is investigated.
It can be seen from Figure 4.5(b) that applying the external force, $F_x$, extends the stiffness map in X direction and Figure 4.5(c) shows that the stiffness map is extended in Y direction after applying the external force $F_y$. By applying the external force $F_y$, the rectangular form of the stiffness map is slightly tapered as the value of Y increases.

Comparing Figures 4.5(b) and 4.5(c), the effect of applying external forces in X and Y directions on the stiffness maps of the example robot of Figure 4.2(a) are not identical. This is because the arrangement of cables and cable attachment points affect the stiffness map and behavior of the robot under effect of external force in different directions.
4.1.1.2 Symmetrical Four Cable Attachments on Mobile Platform

Stiffness maps are plotted over the robot workspace for the following coordinates \{(-0.088, -0.088), (0.088, -0.088), (0.088, 0.088), (-0.088, 0.088)\} as the cable attachment points on the mobile platform in the moving frame, for the robot shown in Figure 4.2(b). These points are calculated by using a radius of 0.125 m for the mobile platform with the angles \{225°, 315°, 45°, 135°\} for the four attachments.

For this robot, the anchor positions are the same as the robot of Figure 4.2(a). In the following plots, the external wrench is zero, the orientation of mobile platform defined in Figure 3.1 is constant at \(\phi = 5°\), and the mobile platform is moving on the vertical plane (with gravity).

It should be noted that the stiffness maps of the example robot of Figure 4.2(b) at the orientation of mobile platform of \(\phi = 20°\) would shrink to zero area and it can be seen that even at the orientation of mobile platform of \(\phi = 5°\) the area of the maps shown in Figure 4.6 are smaller than the ones for the robot of Figure 4.2(a) shown in Figure 4.4. In other words, changing the configuration of cable attachments from "two cable attachments" to "symmetrical four cable attachments" makes the stiffness maps shrink. Figure 4.6(c) shows a significant decrease in the average value of rotational stiffness about Z direction of the stiffness map for the robot of Figure 4.2(b) compared to the robot of Figure 4.2(a).
Figure 4.7: Stiffness maps about Z direction of the robot shown in Figure 4.2(b) without gravity at ($\varphi = 5^\circ$).

(a) $F_x = F_y = M_z = 0$

(b) $F_y = F_x = 0$, $M_z = 5$ Nm

(c) $F_y = F_x = 0$, $M_z = -5$ Nm

Figure 4.8: Stiffness maps about Z direction of the robot shown in Figure 4.2(b) without gravity at ($\varphi = 0^\circ$).

(a) $F_x = F_y = M_z = 0$

(b) $F_y = F_x = 0$, $M_z = 5$ Nm

(c) $F_y = F_x = 0$, $M_z = -5$ Nm

Figure 4.7(a) shows the rotational stiffness map of the robot without gravity. In the plots of Figure 4.7(b) and 4.7(c), the effect of external moment on the stiffness map is investigated.

Figure 4.7(b) shows the effect of the positive external moment of 5 Nm. It can be seen that the stiffness map is extended significantly and the maximum stiffness increases from 332 Nm/rad for the case with zero external wrench to 383 Nm/rad. Figure 4.7(c) shows the effect of the external moment of 5 Nm. It can be seen that the stiffness map shrinks and the maximum stiffness decreases from 332 Nm/rad for the case with zero external wrench to 311 Nm/rad.
4.1.1.3 Crossed Four Cable Configuration

For the robot shown in Figure 4.2(c), the stiffness maps are plotted over the workspace of robot for the following coordinates \{(-0.088, -0.088), (0.088, -0.088), (0.088, 0.088), (-0.088, 0.088)\} as the cable attachment points on the mobile platform in the moving frame $\Gamma(X', Y')$. These points are calculated by using a radius of 0.125 m for the mobile platform with the angles \{135°, 45°, 315°, 225°\} for the four attachments. For the considered robot, the anchor positions are the same as the previous cases.

In the plots of Figure 4.9 the external wrench is zero, the orientation of mobile platform defined in Figure 3.1 is constant at $\varphi = 0^\circ$, and mobile platform is in the vertical plane (with gravity). It can be seen the crossed cable configuration results in bigger work space and stiffness map. In addition, the maximum value of the rotational stiffness about Z direction...
for the robot of Figure 4.2(c) is 4,466 Nm/rad which is higher than the maximum value of rotational stiffness about Z direction for the robots of Figures 4.2(a) and 4.2(b). However, the crossing cables are not desirable from the design point of view.

### 4.1.2 Stiffness Maps of 2DOF Translational Robots

The stiffness matrix of the robot shown in Figure 4.10 is derived, the corresponding directional stiffness is formulated over the robot workspace and the stiffness maps are developed. The anchor points \( A_i \) can move on a circular rail shown by dashed-line in Figure 4.10 with a radius of 1 m and centered at the origin of the coordinate system \( \Psi(X,Y) \) at point 0. For the stiffness maps developed in this section, the coordinates of anchors in terms of their angular positions for a radius of 1 meter are \( \{45^\circ, 165^\circ, 285^\circ\} \), i.e., the anchor positions of \( \{(0.7071, 0.7071), (-0.9659, 0.2588), (0.2588, -0.9659)\} \) are used. However, these anchor points can be modified, i.e., they can move on the circular rail, to identify the optimum layout of the robot in Section 4.2.

Plots of Figure 4.11 show the stiffness maps along the X and Y directions, when the force applied on the mobile platform is zero and the robot is moving on the horizontal...
Figure 4.11: Stiffness maps of the robot shown in Figure 4.10 without gravity.

Figure 4.12: Stiffness maps of the robot without gravity for the required minimum stiffness.
plane (without gravity). Assuming a minimum stiffness of 150 kN/m, the corresponding deflection for the payload of 150 N is 0.001 m. Accuracy in positioning in the order of 0.001 m is reasonable for an assembly task using a robot of the size studied in this section. Stiffness maps of the robot for the required minimum stiffness of 150 kN/m are shown in Figure 4.12.

Plots of Figure 4.13 show the stiffness maps along the X and Y directions, when the force applied on the mobile platform is zero and the robot is under the effect of gravity. Stiffness maps of the robot under effect of gravity for the required minimum stiffness of 150 kN/m are shown in Figure 4.14.

4.2 Optimum Layout of the Robot to Maximize the Area of Stiffness Maps

In this section, anchor positions of the robot are optimized to maximize the stiffness map area. In other words, the optimum layout of the robot is identified to maximize the area of the stiffness maps considering the constraint that the anchor positions should be on the circular rail shown in Figure 4.10.
The stiffness maps for the optimum layouts of the robot are developed with and without the required minimum stiffness. The cable tension constraints are checked for each potential position of mobile platform to identify the workspace. For all the positions of the mobile platform inside the workspace, the stiffness matrix is derived following the procedure discussed in Section 3.1.3.2.

After deriving the stiffness map, the number of poses inside the map corresponds to the area of the stiffness map. The expressions for the boundaries of the stiffness maps in terms of the anchor positions are quite lengthy. To identify the optimum anchor positions analytically, the symbolic expression for the area of the stiffness map needs to be formulated which is not easily obtainable. Thus, for each case, the optimization is carried out in MATLAB using the genetic algorithm (GA) function to identify the optimum anchor positions that maximize the stiffness map area. It should be noted that the area of the stiffness map is calculated numerically for each case. After deriving the stiffness map, the number of poses inside the map corresponds to the area of the stiffness map. This integer number can be converted to the area of the stiffness map knowing that the increment of 0.02 m in generating the poses in X and Y directions is considered. Parameters used in the GA function are listed in Table 4.1.
Table 4.1: GA function parameters

<table>
<thead>
<tr>
<th>GA function variable</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population size</td>
<td>20</td>
</tr>
<tr>
<td>Generations</td>
<td>100</td>
</tr>
<tr>
<td>Migration fraction</td>
<td>0.2</td>
</tr>
<tr>
<td>Crossover fraction</td>
<td>0.8</td>
</tr>
<tr>
<td>Initial penalty</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Penalty factor</td>
<td>$10^4$</td>
</tr>
<tr>
<td>TolFun</td>
<td>$10^{-6}$</td>
</tr>
</tbody>
</table>

(a) along X direction

(b) along Y direction

Figure 4.15: Stiffness maps of the robot with the optimum layout.

The optimum anchor positions for the robot shown in Figure 4.10 when the mobile platform is under the effect of gravity have been identified as $\{(0.9977, -0.0775), (0.0000, 1.0000), (-0.9977, -0.0775)\}$ and the value of the optimum stiffness map area is $3.3605 \text{ m}^2$. The optimum layout of the robot and its stiffness maps are shown in Figure 4.15.

The optimum anchor positions for the robot shown in Figure 4.10 when the mobile platform is under the effect of gravity with the required minimum stiffness of 150 kN/m have been identified as $\{(0.939, -0.343), (0.770, 0.638), (-0.999, 0.015)\}$ with area of $1.1729 \text{ m}^2$ for the stiffness along X direction and $\{(0.4815, -0.8764), (0.0000, 1.0000), (-0.5184, -0.8551)\}$ with area of $1.0195 \text{ m}^2$ for the stiffness along Y direction. The stiffness maps for
Figure 4.16: Stiffness maps of the robot for the required minimum stiffness with the optimum layout.

Figure 4.17: Elastic potential energy maps of the robot in Figure 4.2(a) with gravity at $\phi = 0^\circ$.

these optimum layouts are shown in Figure 4.16

4.3 Potential Energy Maps

Potential energy of a robot for a given pose of the mobile platform is formulated as

$$V_e = \frac{1}{2} \mathbf{r}^T \mathbf{Kr}$$  \hspace{1cm} (4.3)
where vector $\mathbf{r}$ is the displacement/rotation (deflection) of the mobile platform. For instance, for a unit displacement (deflection) along X direction $\mathbf{r} = [1, 0, 0]^T$. For the robot shown in Figure 4.2(a) the elastic potential energy maps are developed and shown in Figure 4.17.

According to Equation 4.3, the elastic potential energy for a unit displacement (deflection) along X and Y directions and unit rotation (deflection) about Z direction are $\frac{1}{2}k_{11}$, $\frac{1}{2}k_{22}$ and $\frac{1}{2}k_{33}$ respectively. Entries $k_{11}$, $k_{22}$ and $k_{33}$ are the diagonal entries of the stiffness matrix of the robot. It shows unlike the directional stiffness index, the potential energy map is not reflecting the effect of coupling terms of the stiffness matrix. However, the diagonal entries have widely been used in developing the stiffness maps in the literatures such as [51].

### 4.4 Deflection Maps

As it was discussed in Section 3.2.1, applying a unit force in X or Y direction or unit moment about Z direction does not necessarily result in deflection in the direction of applied force or moment. In general case, applying the unit force or moment will cause deflection in all directions. Vector of deflection of the mobile platform $\delta \mathbf{r}$ under effect of an external wrench $\mathbf{F}$ is calculated as

$$
\delta \mathbf{r} = \mathbf{C} \mathbf{F}
$$

where $\mathbf{C}$ is the compliance matrix which is the inverse of the stiffness matrix. Figures 4.18, 4.19 and 4.20 show the deflection maps of the robot shown in Figure 4.2(a) where the unit external wrench of $\mathbf{F} = [1, 0, 0]^T$ N, $\mathbf{F} = [0, 1, 0]^T$ N and $\mathbf{F} = [0, 0, 1]^T$ Nm are applied on the mobile platform respectively. The robot is under effect of gravity and the orientation of mobile platform defined in Figure 3.1 is constant at $\varphi = 0^\circ$.

Figures 4.18(a), 4.19(b) and 4.20(c) show when the positive unit external wrench is applied, the deflection in the direction of the applied wrench is positive all over the map.
Figure 4.18: Deflection maps of the robot where the unit external force in X direction is applied.

Figure 4.19: Deflection maps of the robot where the unit external force in Y direction is applied.

Figure 4.20: Deflection maps of the robot where the unit external moment about Z direction is applied.
which is not the case for the deflection in the other directions. From Equation 4.4, it can be seen that the deflection maps under effect of the unit wrenches are basically the map of entries of the compliance matrix $C$ over the workspace of the robot. Stiffness and compliance matrices are symmetric matrices, that is why the maps shown in Figures 4.19(c) and 4.20(b) are similar although the units are different. The same observation can be made for the maps in Figures 4.18(c) and 4.20(a) and also for the maps in Figures 4.18(b) and 4.19(a).

4.5 Condition Number Maps

In this section, condition number of the robots defined in Section 3.2.3 is mapped over their workspaces and condition number maps are developed. Figure 4.21 shows the condition number maps for the robot shown in Figure 4.10 where the coordinates of anchors in terms of their angular positions for a radius of 1 meter are \{45°, 165°, 285°\}, i.e., the anchor positions of \{(0.7071, 0.7071), (-0.9659, 0.2588), (0.2588, -0.9659)\} are used.

As it was discussed in Section 3.2.3, condition number can also be defined for unit
inconsistent stiffness matrices after unit-homogenizing them. For the robot shown in Figure 4.2(a) at the constant mobile platform orientation of $\varphi = 0^\circ$, condition number of the translational part of the stiffness matrix is calculated and mapped over the workspace in Figure 4.22.

4.6 Conclusion

In this chapter, directional stiffness index was mapped over the workspace of the example cable-driven robots. Stiffness maps were shown for directional stiffness in X and Y directions (translational stiffness) and for the 3DOF planar robot also about Z direction (rotational stiffness).

Robots were considered to move on the horizontal plane (without gravity) or on the vertical plane (with gravity). For different cases of 3DOF planar robots with different cable configurations such as two cable attachments on mobile platform, symmetrical four cable attachments on mobile platform, and crossed four cable configuration, stiffness maps were developed and compared. Those maps were also developed for the 2DOF translational robots.

It was also discussed that in some cases a minimum required stiffness might be defined.
for the robot, thus the corresponding maps considering a minimum required stiffness were also developed and shown.

In addition, an optimization problem was defined to maximize the area of the maps by changing the layout of the robots, the optimum results were identified and the optimum layouts were shown.

Other maps were also introduced and developed such as, potential energy maps, deflection maps and condition number maps.
Chapter 5

Failure Analysis\textsuperscript{1}

Failure of cable-driven parallel robots is studied in this chapter. Failure modes are introduced and formulated in Section 5.1. In Section 5.2, effect of failures on the stiffness maps of the robots is investigated. Failure recovery methods are introduced and investigated in Section 5.3. Finally in Section 5.4, optimum layouts of robots after failure are identified.

5.1 Failure Formulation

In this section two types of cable and motor failures are considered. These failure modes are modelled and the stiffness matrix of the robot after failure is derived.

5.1.1 Failure of a Cable

One type of failure is when cable $i$ is disconnected or slack. In this type of failure the $i$th row of the Jacobian matrix derived in equation 3.4 or 3.7 and matrix $J_\alpha$ in equation 3.24 or 3.33 are eliminated.

For the 3DOF four-cable-driven parallel robots shown in Figure 5.1 for instance when cable 1 fails, the Jacobian matrix $\mathbf{J}$ and matrix $\mathbf{J}_\alpha$ reduce to

\textsuperscript{1}Parts of the work presented in this chapter have been published in [58, 60, 61]
The stiffness matrix for the failure case can be derived using the Jacobian matrices after failure $\mathbf{J}_f$ and $\mathbf{J}_{\alpha f}$ and the same method presented in Sections 3.1.3.1 and 3.1.3.2. It should be noted that after failure of a cable, the example robots are no longer redundant and the tension of the remaining cables is calculated as

$$\mathbf{\tau} = \mathbf{J}_f^{-T} \mathbf{F}$$

The tension constraints are checked to be satisfied for all the poses of the mobile platform inside the stiffness map after failure.

### 5.1.2 Failure of a Motor

Another type of failure in the robot is when one of the motors fails. In this mode of failure, the motor shaft jams and the length of the cable cannot be larger than the length of released cable at failure $l_f$. In other words, the robot will not be able to reach the poses that need a length of cable larger than $l_f$. In this failure mode, for the poses that the length of cable should be smaller than $l_f$, the cable is slack and the formulation is similar to the case that the cable fails. Thus, the $i$th row of the Jacobian matrix $\mathbf{J}$ and matrix $\mathbf{J}_{\alpha}$ are eliminated.
Figure 5.1: Example 3DOF planar four-cable-driven parallel robots.

Figure 5.2: Stiffness maps of the robot shown in Figure 5.1(a) with gravity at ($\phi = 5^\circ$).

5.2 Effect of Failure on the Stiffness Maps

5.2.1 Stiffness Maps of 3DOF Planar Robot after Failure

For the robot shown in Figure 5.1(a), the anchor positions of \{(-1, -0.75), (1, -0.75), (1, 0.75), (-1, 0.75)\} are used. Stiffness maps are plotted over the robot workspace for the following coordinates \{(-0.125, 0), (0.125, 0), (0.125, 0), (-0.125, 0)\} as the cable attachment points on the mobile platform in the moving frame $\Gamma(X', Y')$. The external wrench is considered to be zero and the mobile platform orientation defined in Figure 3.1 is constant at $\varphi = 5^\circ$ under effect of gravity. For this case the stiffness maps are developed and shown in Figure 5.2. For the same case after failure of cable 1 the stiffness maps are developed and shown in Figure 5.3.
When cable 1 of the robot shown in Figure 5.1(a) fails, stiffness of the robot decreases. Comparing the stiffness maps of the robot before failure in Figure 5.2 and after failure in Figure 5.3, it can be seen that the stiffness maps shrink significantly after cable 1 fails. Figure 5.4 shows the stiffness maps for the case that motor 1 fails and $l_f$ is 1.75 m.

Stiffness maps are plotted over the robot workspace for the following coordinates $\{(-0.177, -0.177), (0.177, -0.177), (0.177, 0.177), (-0.177, 0.177)\}$ as the cable attachment points on the mobile platform in the moving frame, for the robot shown in Figure 5.1(b).

For this robot, the anchor positions are the same as the robot of Figure 5.1(a). In the stiffness maps of Figure 5.5, the external wrench is zero, the orientation of mobile platform defined in Figure 3.1 is constant at $\varphi = 0^\circ$, and the mobile platform is moving on the
Figure 5.5: Stiffness maps of the robot shown in Figure 5.1(b) with gravity at $\varphi = 0^\circ$.

Figure 5.6: Stiffness maps of the robot shown in Figure 5.1(b) after failure of cable 1.
vertical plane (with gravity). For the same case after failure of cable 1 the stiffness maps are developed and shown in Figure 5.6. When cable 1 fails, the stiffness of robot decreases and the stiffness maps shrink.

For the robot shown in Figure 5.1(c) the following coordinates {(-0.088, -0.088), (0.088, -0.088), (0.088, 0.088), (-0.088, 0.088)} as the cable attachment points on the mobile platform in the moving frame $\Gamma'(X', Y')$ are considered. These points are calculated by using a radius of 0.125 m for the mobile platform with the angles \{135°, 45°, 315°, 225°\} for the four attachments. For the considered robot, the anchor positions are the same as the previous cases. The external wrench is zero, the orientation of mobile platform defined in Figure 3.1 is constant at $\varphi = 0°$, and the mobile platform is in the vertical plane (with gravity). Stiffness maps for this case were shown Figure 4.9. For the same case when cable 1 fails, the stiffness maps are developed and shown in Figure 5.7.

5.2.2 Stiffness Maps of 2DOF Translational Robot after Failure

The stiffness maps of the robot shown in Figure 4.10 with the anchor positions of \{(0.7071, 0.7071), (-0.9659, 0.2588), (0.2588, -0.9659)\} were developed and shown in Figure 4.13. The external wrench is zero and the mobile platform is under effect of gravity. For the same case after failure of cable 1, the stiffness maps are developed and shown in Figure 5.8.
5.3 Retrieving Lost Stiffness of Robot after Failure

As presented in Section 5.2, after failure of a cable, the stiffness maps of robots shrink drastically. In this section, some strategies for retrieving the lost stiffness are presented. New stiffness maps are developed and the results are compared with the failure results before recovery. These strategies will increase the overall area of stiffness maps but it should be mentioned that besides gaining the area the shape of maps will also change which might result in losing some parts of the original stiffness map after failure. This fact should be considered in case that those specific regions of task space are needed for executing a given task.

One of the plans for retrieving the lost stiffness after failure is to apply an appropriate external wrench on the mobile platform. To illustrate the effect of applying an external wrench, the stiffness maps shown in Figure 5.3 are regenerated in Figure 5.9 after applying a positive external moment of 2.5 Nm about Z direction.

Comparing the stiffness maps of Figures 5.9 and 5.3, it can be seen that applying an external moment in the same direction of the orientation of mobile platform (positive Z direction here because of $\varphi = 5^\circ$) increases the area of stiffness map. If the moment was
Figure 5.9: Regenerated stiffness maps of the case study in Figure 5.3 after applying an external moment of 2.5 Nm.

Figure 5.10: Regenerated stiffness maps of the case study in Figure 5.6 when orientation of the mobile platform is $\varphi = 5^\circ$.

applied in the opposite direction, the stiffness map would have shrunk.

Stiffness maps in Figure 5.6 are for the case study that the orientation of the mobile platform is constant at $\varphi = 0^\circ$. To investigate the effect of non-zero orientation of the mobile platform, stiffness maps are regenerated in Figure 5.10 when orientation of mobile platform is $\varphi = 5^\circ$ and failure occurs.

Comparing the stiffness maps of Figures 5.10 and 5.6, it can be seen that the stiffness maps after failure, when the orientation of the mobile platform is $\varphi = 5^\circ$, does not shrink as much as the case that the orientation of the mobile platform was $\varphi = 0^\circ$ (Figure 5.6).
Thus, orientation of the mobile platform is an effective parameter in the stiffness loss after failure.

Another strategy which is implemented on an example robot in this section is modification of the anchor positions. As it has been shown in Figure 5.1, the anchor positions for the four-cable-driven robots are on a rectangle of \((-1, -0.75), (1, -0.75), (1, 0.75), (-1, 0.75)\). If anchor positions can move, e.g., on rails shown by dashed line in Figure 5.1, there is a possibility to retrieve some of the lost stiffness after failure.

For example, when cable 1 fails (with anchor position at \((-1, -0.75)\)), the anchor position of cable 4 can be moved down on the rail from point \((-1, 0.75)\) to point \((-1, 0)\). Point \((-1, 0)\) is the midpoint of the line segment \(A_1A_4\) shown in Figure 5.1. By moving the anchor position of cable 4 to this point, cable 4 could partially compensate for the lost cable 1.

Anchor position modification is applied to the case study whose stiffness maps were shown in Figure 5.7. Figure 5.11 shows the stiffness maps after relocating the anchor position of cable 4 to point \((-1, 0)\). Comparing the stiffness maps of Figures 5.11 and 5.7, it can be seen that relocating the anchor position of cable 4 has modified and increased the area that the robot has non-zero stiffness to some extent. This modification can be applied to the cases that retrieving the failed cable is not possible. Thus, changing the anchor position of other cables may be helpful.
5.4 Optimum Layouts of Robots after Failure

As presented in Section 5.2 after the failure, the stiffness maps of robots shrink drastically. In Section 5.3 some strategies for retrieving the lost stiffness after failure were applied to the example robots.

In this section optimum anchor position and mobile platform orientation are identified in order to retrieve the lost stiffness as much as possible. In these optimizations, the goal is to achieve the stiffness map with the largest area after failure. In the simulation, the cable tension constraints are checked for different poses of mobile platform to identify the workspace after failure according to Equation 5.3. In addition, for all the poses inside the workspace, the stiffness matrix is derived using the Jacobian matrices after failure introduced in Equations 5.1 and 5.2. After developing the stiffness map, the number of poses inside the map corresponds to the area of the stiffness map. This integer number can be converted to the area of the stiffness map knowing that the increment of 0.0125 m in generating the poses in X and Y directions is considered.

For each anchor position and mobile platform orientation, the stiffness map area is calculated and the graph of the variation of the stiffness map area versus anchor position and mobile platform orientation is developed. From this graph the optimum parameters which maximize the stiffness map area are identified. It should be noted that in this algorithm the variation of the stiffness map area is investigated for the increment of 0.015 m in anchor position and the increment of 1° in mobile platform orientation for an off-line implementation.

5.4.1 Optimizing Anchor Position

One of the strategies for retrieving the lost stiffness after failure is the modification of the anchor positions. The anchor positions for the four-cable-driven robot shown in Figure 5.1(a) are on the vertices of a rectangle at \{(-1, -0.75), (1, -0.75), (1, 0.75), (-1, 0.75)\}. If anchors
can move on the rails on the sides of the rectangle shown by dashed line in Figure 5.1(a), there is a possibility to retrieve some of the lost stiffness after failure. For the case that cable 1 fails, the position of anchor 4, $a_4 = [a_{4x}, a_{4y}]^T$, is changed and the optimum anchor position for cable 4 is identified to maximize the area of the stiffness map after failure.

Stiffness maps of the robot shown in Figure 5.1(a) after failure of cable 1 were shown in Figure 5.3. Figure 5.12(a) shows how the stiffness map area varies when the Y component of position of anchor 4, i.e., $a_{4y}$ changes and the stiffness map area is maximized for $a_{4y} = 0.24$ m. Comparing the stiffness maps of Figures 5.3 and 5.13, it can be seen that some of the lost stiffness after failure of cable 1 is retrieved by moving the anchor position of cable 4.
Figure 5.14: Stiffness maps of the robot after failure of motor 1 with the optimum anchor position for cable 4.

From point (-1, 0.75) to point (-1, 0.24), which is the optimum position to maximize the area of the stiffness map after failure.

For the case that motor 1 fails, the stiffness maps were shown in Figure 5.4. The optimum anchor position for cable 4 is identified using the plot of Figure 5.12(b) which shows how the stiffness map area varies when the Y component of position of anchor 4, i.e., \( a_{4y} \), changes and that the stiffness map area is maximized where \( a_{4y} = -0.03 \text{ m} \). Figure 5.14 shows the stiffness maps after the optimization.

Stiffness maps of the robot shown in Figure 4.10 after failure of cable 1 were shown in Figure 5.8. Using the GA optimization in MATLAB similar to Section 4.2, the optimum anchor positions for the robot after failure of a cable have been identified as \{(-0.9769, 0.2135), (0.9819, 0.1892)\} and the value of the optimum stiffness map area is 3.2013 m². It should be noted that the parameters used in the GA function are listed in Table 4.1. The stiffness maps for this optimum layout are shown in Figure 5.15.

5.4.2 Optimizing Mobile Platform Orientation

Another strategy for retrieving the lost stiffness after failure is changing the mobile platform orientation. This strategy is applicable when the orientation of the mobile platform is not a concern, e.g., when the robot is handling a symmetric object.
Figure 5.15: Stiffness maps of the robot shown in Figure 4.10 after failure of a cable with the optimum layout.

Stiffness maps of the robot shown in Figure 5.1(a) for $\varphi = 5^\circ$ after failure of cable 1 are developed and shown in Figure 5.16. It can be seen that the stiffness maps shrink drastically. Figure 5.17 shows how the stiffness map area varies when the orientation of mobile platform changes in case a cable fails. The optimum orientation of the mobile platform which maximizes the area of the stiffness maps after failure is identified and the optimum stiffness maps are developed and shown in Figure 5.18(a)-(d) respectively for the failure of cables 1 through 4.

In Figures 5.18(a) and 5.18(b), and also 5.18(c) and 5.18(d), the magnitude of the optimum orientation of mobile platform is the same; $-9^\circ$ for the case of Figure 5.18(a) and $9^\circ$ for Figure 5.18(b), while orientation is $-29^\circ$ for the case of Figure 5.18(c) and $29^\circ$ for Figure 5.18(d). Because of the symmetric geometry of the robot, the resultant maps shown in Figure 5.18(a) and Figure 5.18(b) are mirror images of each other and the magnitude of the optimum orientation of mobile platform is the same. The same observation is made for the maps shown in Figure 5.18(c) and Figure 5.18(d) and their corresponding optimum values of the mobile platform orientation.
Figure 5.16: Stiffness maps about Z direction of the robot shown in Figure 5.1(a) after failure of one cable with gravity at ($\phi = 5^\circ$).
Figure 5.17: Variation of stiffness map area with changes in orientation of mobile platform after failure of a cable.
Figure 5.18: Stiffness maps of the robot about Z direction after failure with optimum orientation of mobile platform.
5.4.3 Optimizing Anchor Position and Mobile Platform Orientation

The two strategies of retrieving the lost stiffness when cable 1 fails are combined and the optimum orientation for the mobile platform and anchor position for cable 4, to maximize the area of the stiffness maps after failure, are identified. Figure 5.19 shows how the stiffness map area varies when the orientation of mobile platform and position of anchor 4 change after failure of cable 1. It is identified that the stiffness map area is maximized where $\varphi = 0^\circ$ and $a_{4y} = 0.39$ m. Figure 5.20 shows the stiffness map after optimizing the orientation of mobile platform and position of anchor 4.

5.5 Conclusions

In this chapter, stiffness of planar cable-driven parallel robots after the loss of a cable or motor was investigated.

The stiffness matrix was formulated and stiffness maps for the example robots after failure of a cable or motor were developed.

Strategies for retrieving the lost stiffness after failure, namely applying proper external
wrench, relocating the anchor position of cables and changing the orientation of the mobile platform, as well as the combination of the last two, were implemented on the example robot.

It was discussed that applying an external wrench on the mobile platform in a specific direction can increase the area of the stiffness maps. It was also shown that changing the mobile platform orientation is applicable when the orientation of the mobile platform is not a concern, e.g., when the robot is handling a symmetric object. For the case that replacing the failed cable is not easily feasible, rearranging the anchor position of other cables may be helpful to partially retrieve the lost stiffness.

In addition, the optimum values for the mobile platform orientation and anchor position to maximize the area of the stiffness maps after failure were found and the optimum stiffness maps after failure were developed and presented.

The optimum layouts for the robot can be used when designing the robot. This knowledge can be used in control of the robot to retrieve the lost stiffness after failure partially and to achieve a better fault tolerant robot.

Figure 5.20: Stiffness maps of the robot after failure of cable 1 with the optimum mobile platform orientation and anchor position for cable 4.
Chapter 6

Trajectory Planning

Trajectory planning of cable-driven parallel robots is studied in this chapter. The methodology to find the optimum configuration of three-cable-driven or four-cable-driven robots to maximize the directional stiffness or condition number is introduced in Section 6.1. In Section 6.2, the same methodology is used to follow a straight line or circular trajectory while keeping the optimum configuration. The results of optimum trajectory planning for straight line and circular path are presented in Sections 6.2.1 and 6.2.2 respectively. Effect of actuation redundancy on optimum trajectory planning is studied in Section 6.3 by comparing the results for three-cable-driven or four-cable-driven robots following the same trajectory. Failure recovery during the optimum trajectory planning is studied in Section 6.4.

6.1 Optimum configuration

As it was discussed in section 3.1.2 for the 2DOF translational robot with 3 cables, i.e., one degree of actuation redundancy, an arbitrary scalar $h$ should be chosen to keep all the entries of the vector of cable tensions formulated in Eqn. 3.7 in the allowable tension bound of $\tau_{min} < \tau < \tau_{max}$. Thus, for a given position of the mobile platform, a feasible region of $h_{min} < h < h_{max}$ is found and in order to have the maximum stiffness at each point, $h_{max}$
For the case with 4 cables, i.e., 2 degrees of actuation redundancy, an arbitrary vector $\mathbf{h}$ of size $2 \times 1$ should be chosen while keeping the tension in the allowable bound of $\tau_{\text{min}} < \tau < \tau_{\text{max}}$ for all the cables. Thus, an optimization is carried out using the $fmincon$ function in MATLAB to find the entries of the vector $\mathbf{h}$ to maximize the area of the stiffness ellipse at the given point which represents the overall value of the stiffness at that point.

For a given position of the mobile platform, configuration of the anchor positions on the circular rail shown in Fig. 6.1 can be found to optimize any of the stiffness indices by defining and solving an optimization problem without violating the given bound for the cable tensions.

An optimization using GA\(^1\) in MATLAB is carried out to find the optimum position of anchors on a circular rail with the radius of 1 m and centered at point 0 shown in Fig. 6.1 to achieve the goal which is having the condition number of 1. Parameters used in the GA function are listed in Table 6.1. Fig. 6.2(a) shows the optimum configuration for the case 1

\(^1\)Genetic Algorithm
Table 6.1: GA function parameters

<table>
<thead>
<tr>
<th>GA function variable</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population size</td>
<td>20</td>
</tr>
<tr>
<td>Generations</td>
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</tr>
<tr>
<td>Migration fraction</td>
<td>0.2</td>
</tr>
<tr>
<td>Crossover fraction</td>
<td>0.8</td>
</tr>
<tr>
<td>Initial penalty</td>
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</tr>
<tr>
<td>Penalty factor</td>
<td>100</td>
</tr>
<tr>
<td>TolFun</td>
<td>$10^{-6}$</td>
</tr>
</tbody>
</table>

Figure 6.2: Optimum configurations where cable 1 to 4 are respectively shown by lines (--- , ..., --- , ---). Mobile platform and anchor positions are respectively shown by markers (○, ◦).

with 3 cables where Mobile platform is at point (-0.5, -0.5) and no external force is applied to it. Fig. 6.2(b) shows the optimum configuration for the case 2 with 4 cables where Mobile platform is at point (0, -0.5) and external force of $[F_x, F_y]^T = [1, 1]$ N is applied to it. As it was formulated in section 3.2.2, stiffness can be defined in a given direction. Thus, it might be desirable to maximize the stiffness in a given direction for a special task. For a similar situation in case 1, directional stiffness is maximized at $45^\circ$ and the result is shown as case 3 in 6.2(c). It can be seen that all cables are almost oriented at $45^\circ$ to make the highest stiffness at that direction while keeping the cable tensions in the allowable bound.
However, the configuration is very close to infeasible case and not stable for an optimum trajectory planning.

6.2 Optimum Trajectory Planning

In this section, mobile platform of the robot follows a given path. While following the path, an optimization using GA in MATLAB is carried out to change the anchor position of the cables on the circular rail to keep the condition number close to 1. As it was discussed in Section 3.2.3 and formulated in Equation 3.51, condition number of the stiffness matrix is the ratio of the maximum value of stiffness over the minimum stiffness. Parameters used in the GA function are listed in Table 6.1. It should also be noted that this is a constrained optimization problem as the anchor position of each cable cannot move more than 3° on the circular rail every second in order to have a continuous and smooth graph for the anchor positions, i.e., in an application mobile platform of the robot would move in a continuous and smooth manner. It has also been studied that if the constraints are tighter, i.e., anchors can only move less than 1° every second, the goal which is condition number of close to 1 cannot be achieved. On the other hand if the constraints do not exist or let the anchors move big angles at every second, then the anchor positions can do big jumps over the rail which is not physically achievable by any driving system. It should be mentioned that the driving systems are designed in a way to avoid any cable interference, if not those cases should be considered in the analysis as well.

6.2.1 Straight Line Path

A straight line starting at point (-0.5, -0.5) and ending at point (0.5, 0.5) is assumed to be followed in 50 seconds with a constant velocity. Thus, no dynamic force is applied to the mobile platform because of the motion. Anchor positions are found by GA to optimize the condition number, i.e., to keep the condition number close to 1.
Figure 6.3: Following a straight line while keeping the condition number close to 1. Cable 1 to 3 are respectively shown by lines (— — , ——— , ——— ). Mobile platform and anchor positions are respectively shown by markers (○, ◦).

Fig. 6.3 shows the cables configurations and anchor positions during the trajectory planning.

During the trajectory planning the condition number has been able to be kept close to 1 successfully which can be seen in Fig. 6.4. Fig. 6.5 also shows the changes in the area of stiffness ellipse which represents an overall value for stiffness at each point while following the straight line path. As i was discussed in Section 3.2.1 stiffness ellipse can be formed as the major and minor axes coincide with the eigenvectors of the stiffness matrix, \( \nu \), and the corresponding eigenvalues, \( \lambda \), indicate the value of stiffness in the direction of the eigenvectors. It can be seen that the stiffness is higher at the beginning and end of the path and it is lower in the middle of the line, because where the cable length is shorter, the stiffness of the cable is higher according to Eqn. 4.2.

Fig. 6.6 shows the anchor position of each cable on the circular rail during the trajectory planning. The robot starts the path from an optimum configuration for the anchor positions and while the position of the mobile platform changes along the path, the anchor positions change accordingly to keep the condition number of the stiffness matrix close to 1. It should
Figure 6.4: Condition number of stiffness matrix while following the straight line shown in Fig. 6.3.

Figure 6.5: Area of stiffness ellipse while following the straight line shown in Fig. 6.3.
be noted that the anchor positions are the angular positions of the anchors on the circular rail in radian.

6.2.2 Circular Path

Similar to section 6.2.1, an optimization is carried out to follow a circular path this time. One of the differences, however, is that the path is followed with a constant velocity, but because of the circular path, the centrifugal force is applied to the mobile platform which is a dynamic force caused by the motion and it is calculated in terms of the mass of mobile platform, velocity of the motion and radius of the circular path (curvature) as

\[ f_c = \frac{mv^2}{r_c} \]  

(6.1)

Mass of the mobile platform \( m \) is assumed to be 2kg and radius of the circular path \( r_c \) is 0.5 m. Fig. 6.7 shows the cable configurations and anchor positions during the trajectory planning. Mobile platform starts the motion at point (0, -0.5) and completes the circular path in 180 seconds, counterclockwise. For a better visualization, the first half of the motion
Figure 6.7: Following a circular path (counterclockwise) while keeping the condition number close to 1. Cable 1 to 3 are respectively shown by lines (— — —, —— —, ———). Mobile platform and anchor positions are respectively shown by markers (○, ○).
is shown in Fig. 6.7(a) and the continuation of the motion is shown in Fig. 6.7(b). Because if the whole path was shown in one figure, it would be so hard to see the cable traces.

Fig. 6.8 shows the condition number during the trajectory planning. In most part of the motion, optimization code has been able to keep the condition number close to 1. For $20s < t < 90s$, it can be seen in Fig. 6.7 that the anchor position of cable 1 is stationary and the configuration of the cables is transforming from one optimum configuration at the lower part of the circle at the beginning of the path to another optimum configuration at upper part of the circle which basically is the mirror of the configuration at lower part of the path about the X axis. Thus, during this transition, the condition number is lower than 1. Because, as it was said earlier in section 6.2, the optimization is constrained to have a continuous graph for the anchor positions. Thus, the anchor positions can not jump on the circular rail to achieve the optimum configuration.

Fig. 6.9 shows the area of the stiffness ellipse and Fig. 6.10 shows the anchor position of each cable on the circular rail during the trajectory planning.
Figure 6.9: Area of stiffness ellipse while following the circular path shown in Fig. 6.7.

Figure 6.10: Anchor position of cables while following the circular path shown in Fig. 6.7.
Anchor position of cable 1 to 3 are respectively shown by lines (—— , ——— , --- ).
6.3 Effect of Redundancy on Optimum Trajectory Planning

It is seen in Fig. 6.8 that in some parts of the motion the robot with 3 cables is not able to follow the circular path and keep the condition number close to 1. Thus, in this section it is tried to follow the same path by 4 cable-driven robot, i.e., one degree of actuation redundancy is added to the robot. Fig. 6.11 shows the condition number during the trajectory planning for 4 cable-driven robot and by comparing that with Fig. 6.8 it can be seen that the more redundant robot is more successful than 3 cable-driven robot in keeping the symmetry and having the condition number of close to 1. By comparing Fig. 6.12 and Fig. 6.9 as it is expected the area of the stiffness ellipse, i.e., overall value of stiffness for the 4 cable-driven robot is higher than the 3 cable-driven robot.
Figure 6.12: Area of stiffness ellipse while following the circular path by 4 cable-driven robot.

Figure 6.13: Anchor position of cables while following the circular path by 4 cable-driven robot. Anchor position of cable 1 to 4 are respectively shown by lines (—, - - - - , - - - , - - - ).
6.4 Failure Recovery in Trajectory Planning

6.4.1 Straight Line Path

In this section, the same straight line path followed by the 3 cable-driven robot in section 6.2.1 is followed by the 4 cable-driven robot and at instant $t_f = 5$ s failure occurs in one of the cables. Thus, the rest of the path is followed by 3 cables and the optimization code tries to recover the failure and continue the trajectory while keeping the condition number close to 1. Figure 6.14 shows the configuration of cables when failure occurs at $t_f = 5$ s.

Fig. 6.15 shows the condition number during the trajectory planning and also before and after the failure. It should be noted that in some cases when the failure occurs, according to the position of mobile platform and configuration of the cables at that instant, keeping the position of the mobile platform on the line by 3 remaining cables is not feasible. In other words, the position of mobile platform at $t_f$ is out of workspace of the 3 cable-driven robot with its given configuration at that instant. In order to recover from this
Figure 6.15: Condition number while following the straight line path and failure occurs at instant $t_f = 5$ s.
situation, an emergency stop system will keep the mobile platform in position till the 3 remaining cables reconfigure to the closest configuration that includes the given position of mobile platform at that instant in its workspace. During this transition, the condition number is noted as zero as seen in Fig. 6.15(a) and Fig. 6.15(b). After this recovery, the optimization will try to bring the anchor position of the remaining cables to the closest optimum configuration with condition number of 1.

Fig. 6.16 shows the area of stiffness ellipse while doing the trajectory planning. In
Fig. 6.16(a) and Fig. 6.16(b) where the area is zero is the transition period where the mobile platform is out of the workspace and the emergency stop is in effect till the cables reconfigure. It is also seen that the value of area which represents the overall value of stiffness drops after failure because of losing one of the cables.

Fig. 6.17 shows the anchor position of cables while following the straight line path. It can be seen that after failure in the cable, the corresponding anchor position stays stationary. In Fig. 6.17(b), it can be seen that anchors 1 and 2 will cross over at the end of the path so this should be considered in the design. It would be possible to design multiple rails for the anchors to be able to do cross over if necessary.

### 6.4.2 Circular Path

In this section, the same circular path followed by the 3 cable-driven robot in section 6.2.2 is followed by the 4 cable-driven robot and at instant $t_f = 10$ s failure occurs in one of the cables. Thus, the rest of the path is followed by 3 cables and the optimization code tries to recover the failure and continue the trajectory while keeping the condition number close to 1. Figure 6.18 shows the configuration of cables when failure occurs at $t_f = 10$ s.

Fig. 6.19 shows the condition number during the trajectory planning and also before and after the failure. It should be noted that in some cases when the failure occurs, according to the position of mobile platform and configuration of the cables at that instant, keeping the position of the mobile platform on the line by 3 remaining cables is not feasible. In other words, the position of mobile platform at $t_f$ is out of workspace of the 3 cable-driven robot with its given configuration at that instant. In order to recover from this situation, an emergency stop system will keep the mobile platform in position till the 3 remaining cables reconfigure to the closest configuration that includes the the given position of mobile platform at that instant in its workspace. During this transition, the condition number is noted as zero as seen in Fig. 6.19(a) and Fig. 6.19(b). After this recovery, the optimization will try to bring the anchor position of the remaining cables to the closest
Figure 6.17: Anchor position of cables following the straight line path and failure occurs at instant $t_f = 5$ s. Anchor position of cable 1 to 4 are respectively shown by lines (—, ..., ——, ---).
optimum configuration with condition number of 1.

Fig. 6.20 shows the area of stiffness ellipse while doing the trajectory planning. In Fig. 6.20(a) and Fig. 6.20(b) where the area is zero is the transition period where the mobile platform is out of the workspace and the emergency stop is in effect till the cables reconfigure. It is also seen that the value of area which represents the overall value of stiffness drops after failure because of losing one of the cables.

Fig. 6.21 shows the anchor position of cables while following the straight line path. It can be seen that after failure in the cable, the corresponding anchor position stays stationary. It can be seen that some anchors will cross over at some points on the path so this should be considered in the design. It would be possible to design multiple rails for the anchors to be able to do cross over if necessary.
Figure 6.19: Condition number while following the circular path and failure occurs at instant $t_f = 10$ s.
Figure 6.20: Area of stiffness ellipse following the circular path and failure occurs at instant $t_f = 10$ s.
Figure 6.21: Anchor position of cables following the circular path and failure occurs at instant $t_f = 10$ s. Anchor position of cable 1 to 4 are respectively shown by lines (— , ..., -- , --- ).
6.5 Conclusions

Two example trajectories were tried to be followed while optimizing the stiffness indices. The results were developed and compared for different cases and it was shown that in an example case of following a circular path adding the actuation redundancy improved the condition number during the trajectory planning. Failure recovery during the optimum trajectory planning was also studied and the results were shown and discussed for the cases of failure in different cables.
Chapter 7

Conclusions

7.1 Summary

In this thesis, stiffness of cable-driven parallel robots was studied. Considering the challenges that using cable actuators add to the stiffness analysis of these robots, complete form of the stiffness matrix was formulated and based on that stiffness analysis was developed. Some methods for improving the stiffness of the robot were introduced and optimized such as changing the layout of the robot to maximize the area of the stiffness maps, and optimizing the condition number and directional stiffness of the robot.

In Chapter 3, planar n-cable-driven parallel robots were studied. The modelling started with the kinematics analysis and the Jacobian matrix was formulated. Based on Jacobian matrix, force analysis of the robot was conducted. The differential form of the static force and moment equations was used to formulate the stiffness matrix of the robot. By considering all the terms in the differential form of equations, the complete form of stiffness matrix of planar cable-driven parallel robots was developed which is symmetric. Based on the developed stiffness matrix, different stiffness indices were introduced and formulated. For the 3DOF planar case which has unit inconsistent degrees of freedom, i.e., both rotational and translational degrees of freedom exist, issue of unit inconsistency in calculation of
the stiffness indices was addressed and meaningful indices were introduced and formulated. Single dimensional stiffness based on stiffness ellipse and directional stiffness were studied and the results for a given stiffness matrix were compared for all the directions in the plane of mobile platform.

In Chapter 4, directional stiffness index was mapped over the workspace of the example cable-driven robots. Stiffness maps were shown for directional stiffness in X and Y directions (translational stiffness) and for the 3DOF planar robot also about Z direction (rotational stiffness). Robots were considered to move on the horizontal plane (without gravity) or on the vertical plane (with gravity). For different cases of 3DOF planar robots with different cable configurations such as two cable attachments on mobile platform, symmetrical four cable attachments on mobile platform, and crossed four cable configuration, stiffness maps were developed and compared. Those maps were also developed for the 2DOF translational robots. It was also discussed that in some cases a minimum required stiffness might be defined for the robot, thus the corresponding maps considering a minimum required stiffness were also developed and shown. In addition, an optimization problem was defined to maximize the area of the maps by changing the layout of the robots, the optimum results were identified and the optimum layouts were shown. Other maps were also introduced and developed such as, potential energy maps, deflection maps and condition number maps.

In Chapter 5, stiffness of planar cable-driven parallel robots after the loss of a cable or motor was investigated. The stiffness matrix was formulated and stiffness maps for the example robots after failure of a cable or motor were developed. Strategies for retrieving the lost stiffness after failure, namely applying proper external wrench, relocating the anchor position of cables and changing the orientation of the mobile platform, as well as the combination of the last two, were implemented on the example robot. It was discussed that applying an external wrench on the mobile platform in a specific direction can increase the area of the stiffness maps. It was also shown that changing the mobile platform orientation is applicable when the orientation of the mobile platform is not a concern, e.g., when the
robot is handling a symmetric object. For the case that replacing the failed cable is not easily feasible, rearranging the anchor position of other cables may be helpful to partially retrieve the lost stiffness. In addition, the optimum values for the mobile platform orientation and anchor position to maximize the area of the stiffness maps after failure were found and the optimum stiffness maps after failure were developed and presented. The optimum layouts for the robot can be used when designing the robot. This knowledge can be used in control of the robot to retrieve the lost stiffness after failure partially and to achieve a better fault tolerant robot.

In Chapter 6, two example trajectories were tried to be followed while optimizing the stiffness indices. The results were developed and compared for different cases and it was shown that in an example case of following a circular path adding the actuation redundancy improved the condition number during the trajectory planning. Failure recovery during the optimum trajectory planning was also studied and the results were shown and discussed for the cases of failure in different cables.

7.2 Future Work

To continue the research on the stiffness of cable-driven robots, different criteria should be looked at, such as:

1) Using finite element analysis to validate the modelling.
2) Fabricating an experimental cable-driven robot to validate the modelling results using the experiment as well.
3) Improving the modelling by considering more sophisticated model for each element of the robot involved in the stiffness.
4) Designing a controller for the robot using the result of stiffness and failure analysis.

In the modelling section, stiffness of the mobile platform also affects the overall stiffness of the robot. So modelling the stiffness of this element could result in a more realistic
analysis. In addition, specially in high speed operations, weight of the cables will play a role in the force analysis which will influence the modelling. Where long cables are used in the robot, cable sag error is another parameter to be considered in the calculations.

This work mostly focuses on the stiffness of the cables and their effect on the overall stiffness, where a simple spring model for the cable has been used. However, more sophisticated models for the cables could be implemented and also the driving system of the cable actuators could be modelled in details.

For having a running prototype robot to perform the introduced failure recovery methods and trajectory planning, a controller is needed which should be designed based on the information provided in this study. The combination of the introduced models, optimum layouts, simulated trajectories for the robot and a sophisticated control scheme can provide a better understanding of the robot behavior. To provide enough feedback from the system, sensors should also be implemented in the robot which might affect the analysis.
Bibliography


