Three Essays on Networks and Public Economics

by

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Abstract

This thesis is a collection of three essays. The first two study how ideas spread through a network of individuals, and how it an advertiser can exploit it. In the model I develop, users choose their sources of information based on the perceived usefulness of their sources of information. This contrasts with previous literature where there is no choice made by network users and thus, the information flow is fixed. I provide a complete theoretical characterization of the solution and define a natural measure of influence based on choices of users. I also present an algorithm to solve the model in polynomial time on any network, regardless of the scale or the topology. I also discuss the properties of a network technology from a public economic standpoint. In essence, a network allows the reproduction of ideas for free for the advertiser. If there is any free-riding problem, I show that coalitions of users on the network can solve such problem. I also discuss the social value of networks, a value that cannot be captured for profit. The third essay is completely distinct from the network paradigm and instead studies funding rules for public universities. I show that a funding rule that depends solely on enrolment leads to "competition by franchise" and that such behavior is sometimes inefficient. I suggest instead an alternate funding rule that allows government to increase welfare without increasing spending in universities.

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Chapter 1

Introduction

1.1 Spins, Networks and Universities

This thesis is a collection of three articles. Chapters two and three study diffusion problems on networks, while chapter four examines the incentives given to public universities through governmental funding rules. While the first two chapters relate to each other, the third one is standalone.

The articles about diffusion through networks stem from my interest in how flows of information influence economic decisions. I build an environment where individuals on a network exchange information about a product or an idea. They choose their source and share the information with their neighbours. Each individual has biases regarding sources of information and thus prefers some sources over others.

In this environment, a diffusion problem naturally arises: what is the best way to give initial information to individuals, what I call a “seed” in the paper, to maximize the spread of the message? This problem has various considerations. First, these individuals have biases towards some of their friends and are thus more likely to be selected. Second, some individuals have more friends and thus have more potential listeners. Third, the seed must account for the impact on the choice individuals will make.

The most obvious application of this question is viral marketing. For example, a company may be interested in broadcasting a message to some network of potential customers. Another interesting application is “herding maximization,” such as when a stockbroker tries to influence his colleagues by spreading a rumour about a stock. The paper reveals which individuals become critical to foster diffusion and, therefore, how herding maximization could be prevented or enhanced. If the phenomenon is persistent through time, it could provide a basis for the study of business cycle theory (which I do not explore in this thesis).

The first article focuses on finding the optimal solution to this diffusion problem. There is thus a theoretical characterization of the solution as well as a tractable way to solve it “fast enough” through an algorithm. The contribution of the essay lies in the fact that this characterization and the algorithm are in a model rich enough to embed a dynamic decision making process of individuals on any network topology. Hence, the solution can be found on any observed network, regardless of its scale. It is thus appealing for practical purposes.

I first show that the total value of the optimal seed does not depend on the structure of the network, but on the number of individuals in the network. This is intuitive, as what matters to the company is exposure of individuals to the information. The question is thus how to distribute this initial information to each agent. The solution is best understood as a Nash equilibrium that concentrates the delivery of information to a few selected individuals. One the one hand, the initial information induces a pattern of source selection by agents in the network. On the other hand, the pattern of selection by agents must be accounted for in order to maximize profits. Hence, the solution is a fixed point between the company and the network of individuals, or a Nash equilibrium.

There are many candidate equilibria for a given network structure. I show that the one that gives the most information on the most influential agent is then the most profitable equilibrium. The most influential agent is the one who can sustain the greatest number of direct or indirect listeners, given he has the largest amount of information. This maximizes the differences in initial information each user has and thus, the variance on the seeds given to individuals.

Diffusion problems are often cursed with dimensionality. Imagine a simple diffusion problem where a consumer can only become informed of the quality of a product from his or her neighbours. Imagine further that consumers only buy the product if their neighbour says it is good. A firm that has a good product must then choose who it should give a free product to in order to generate the best word-of-mouth advertising. Such a problem, as I illustrate in the first article, has an optimal solution which is a binary vector indicating who receives the product first. This solution can however only be found by enumerating all candidate solutions (all binary vectors). As there are 2^n of them, the optimum cannot be computed when n is large.

In order to have a theory of diffusion that can be applied in practice, one requires a framework and an algorithm that can compute the optimal solution "fast enough," or as computer scientists put it, in polynomial time. I provide an algorithm in the first chapter of the thesis, which can be used to answer practical questions. By doing so, I lay the groundwork for further articles and provide a tool for analysts who seek insights on large-scale networks. The key contribution is an algorithm that provides the optimal solution on *any* network topology, given the dynamic response of individuals to information.

In the chapter three, I use the same framework, but focus on the economic nature of word-of-mouth advertising. This analysis has never been done before. I find that, when using a network to spread a message, a company exploits the non-rival nature of ideas, or the fact that they can be reproduced free of charge. So in the context of the model, the variance maximizing solution is also the one that maximizes the free reproduction of information.

If individuals on the network share information without bearing any cost, this is simply a better technology for anyone who seeks to diffuse a message. However, if individuals on the network must incur a cost, either because maintaining links is costly (such as the efforts required to maintain a friendship) or if the action of sharing is costly, free-riding occurs from the seeder. I then show that coalitions of individuals can be used to avoid this free-riding. The maximum price a coalition can charge to an advertiser is determined by the "outside value" of the message sender — in other words, the value of the word-of-mouth advertising that the coalition provides to the advertiser, minus the profits the sender could make without this word-of-mouth advertising. If the coalition is a complete network, or if there is a sequence of coalitions that cover the whole network, then the free-riding problem is completely negated.

I also show that for any given word-of-mouth campaign, there is a social value that cannot be captured by the social network. The intuition is that individuals will have a conversation about a product regardless of word-of-mouth advertising campaigns. Since this conversation happens without the intervention of the advertiser, it cannot be captured. I show that the most influential individuals on the network play a key role in forming such social value. Recall that individuals have biases towards sources. These influential individuals have two characteristics: they have the highest biases and they lie within a cycle. These agents are thus placed in a cycle of self-reinforcement, where all other individuals connect to them, either directly or indirectly.

These two articles contain new scientific knowledge, but there are still opportunities for further research. The model is simple enough to be tractable analytically and computationally, while having individuals who make discrete choices. It could be further developed by connecting these choices to economic decisions about an asset or a stock, in order to study the impact of diffusion on asset pricing. This is briefly outlined in chapter two of this thesis.

The third essay in chapter four draws its origins from my observations of the higher education sector in the province of Québec, Canada. Circa 2000, the government of Québec modified the way it funds universities from lump-sum transfers to an enrolment-based scheme. With this reform, a university with more students would receive more funding than a smaller university. Although having more students is costlier, the funding rule neglects the incentives given to universities to "compete by franchise." With an enrolment-based rule, universities can open new facilities and campuses to increase their revenues. This is problematic because universities start to compete for the same students. If the competition is too fierce, there might be a misuse of public funds and there would be grounds for merging competing universities, therefore avoiding the costs of competition. Since the change in the funding rule, the province of Québec has seen many facilities open in different regions, sometimes with multiple universities competing in the same one.

In the essay, I explore the decision of two universities to open a new facility. I do so while using a generic funding rule that encompasses most enrolment-based funding policies. First, I characterize the decision of universities to open facilities under competition. I compare the results of the decentralized decision with a first-best outcome, where the government would direct university admission policies so as to maximize social welfare. I show that a funding rule based solely on enrolment can lead to an inefficient allocation of resources. This inefficiency goes both ways: it could be more efficient to foster access and have two universities compete in the same region, even though the universities might not find it profitable. It could also be socially efficient to have only one facility where the decentralized outcome leads to two universities competing. Unless the objectives of universities are aligned with the social objectives, a decentralized funding rule based solely on direct enrolment is inefficient.

I show that a rule which depends both on a university's enrolment and on competing universities' enrolments achieves social efficiency. In some cases, factoring in competing universities' enrolments acts solely as a penalty for a university to engage in competition. The penalty is high enough to deter a university from entering a market that is adequately served. In other cases, this second component acts as a reward to foster competition in the region. These two components of the funding rule can be used to drive universities towards the socially efficient outcome. In particular, this means that for a fixed amount of funding given to universities, governments can increase welfare by modifying how they distribute funds to universities.

1.2 Organization of the Thesis

The remaining chapters of this thesis are organized in the same way as the introduction. In chapter two, I present the essay “Getting the Right Spin: A Theory of Value of Social Networks.” This paper explains the model in detail, as well as the algorithm and findings. In chapter three, I present the essay “Externalities, Social Value and Word of Mouth: Notions of Public Economics on Networks,” which examines the economic concepts at play with word-of-mouth advertising. Chapter four contains the paper “University Funding Policies: Buildings or Citizens?”, which discusses’ the impact of funding rules on universities’ behaviour. Chapters two and four have their own appendix, where the proofs of mathematical statements are delegated. As chapter three has little mathematics, the proofs are directly in the chapter. The appendix to chapter two also contains a brief introduction to tropical algebras and a python implementation of the algorithm described. The thesis concludes in chapter five, where I discuss opportunities for future work based on these essays.

Chapter 2

A Theory of Value of Word-of-Mouth

2.1 Introduction

When Facebook announced plans for its initial public offering, many analysts wondered what the market price for its shares would be and to what extent the company could monetize the information it has about its users (see Nelson [33], Reid [38] or Blodget [4], for example). Aside from standard targeted publicity, critics argued that there was very little use for the information Facebook holds about its users. What could Facebook sell that would generate income?

This chapter suggests an answer to this question. In a nutshell, Facebook can craft, monitor and engineer "spins", word-of-mouth campaigns that maximize viral exposure of a given product through the sharing feature of the website. Hence, this essay focuses on the optimal "seeding" strategy of advertising given that users choose to share posts with their friends. In the following pages, I answer two related questions: first, what is the best word-of-mouth campaign that an advertiser can do on a social network and second, what is the market value of such a campaign? To answer these questions, one must consider the strategies of two types of agents: a company such as Facebook, which seeks the profit-maximizing exposure of some given information on the social network; and users, nodes in a network, who seek to reproduce the information they receive in some given fashion. These social network users share an opinion that can then be freely reproduced or modulated by users' friends. This process of replication and modulation of the information is what the company seeks to exploit.

Although the example I use throughout the chapter is Facebook, there are many situations to which the model developed in the following pages may apply. For instance, one can think of a broker seeking to spread the best rumor about a stock, or a President who seeks the Senators he should lobby in order to induce the adoption of a bill. Broadly speaking, this chapter yields a theory of the "optimal spin", that is, the best use of a social network of users who rebroadcast a message to generate some value.

The novel aspects are threefold. First, there is the explicit use of any given network structure to pin down profits. Second, there is a company that understands the structure of the social network and users' behavior and exploits it in a word-of-mouth campaign to make profits. Third, the optimal solution can be computed in polynomial time for any network topology and looks at any possible combination of seeds to users.

In short, the value of a network can be broken down into three components: the value of advertising purchased on the network, the net value of signal modifications induced by the choice of users, and finally the social value of the network, which cannot be captured by the company. I show that amongst a class of feasible solutions, the optimal solution maximizes the concentration of the seed in a small set of users. For any cost structure, the company has incentives to seed a higher advertising signal to users who have the ears of others, whether their friends listen to the message directly, or rebroadcast it to their friends (and so on). Such concentration allows for the replication of a more valuable signal. This is this process of replication that the company extracts from the network to generate higher sales. The model reproduces patterns similar to those identified in the Dynamics of Viral Marketing (Leskovec, Adamic and Huberman [29]).

The rest of this chapter is organized as follows. In section 2.2, I present a brief literature review on the diffusion of signals over a network and on the value of social networks. In section 2.3, I present the model itself, its main implications and the solution. In section 2.4, I present an algorithm to compute the optimal solution. In section 2.5, I use the model to derive a market value for Facebook and show some other applications. In section 2.6, I discuss some alternative specifications. A brief conclusion follows.

2.2 Literature Review

Existing research of signal diffusion over social networks can be roughly divided into four categories. The first category pertains to signal diffusion on networks (see, for example, Bala & Goyal [3], and chapters 7 & 8 of Jackson [24]). This literature focuses on whether social network users agree on the value of a signal in the long run. The main insight is that agents' actions converge in the long run in connected networks[1]. Although Bala & Goyal [3] and Jackson [24] examine signal diffusion, they do not focus on the value these signals can have to a social network owner, or how the differences in actions that occur before convergence can be used to generate value.

The second category of research relates to cascades or herding — the idea that when an agent makes a decision using information inferred from other agents' actions, this can cause all agents to "herd" toward a common outcome. A prominent paper in this area is Banerjee [5]. The author shows that users' past actions can generate "cascades" of decisions where people choose the same asset as previous agents, even when an agent's private information indicates this asset is not the best. From an asset seller's perspective, there is an incentive to pay buyers to generate such a cascade. Banerjee, however, fails to answer an important question: how is the sequence of agents formed? The paper assumes that the order in which agents receive information is random. It seems however that information does travel on paths allowed by a network of friends. Hence, the paper does not provide insights as to how one could exploit a pre-existing structure of acquaintances.

A third category of literature addresses diffusion in the context of network formation. Galeotti, Ghiglino & Squintani [17] have studied the problem of choosing a unique user who maximizes broadcasting of a message over a social network in a given finite time. Information is passed on through cheap talk between neighbors and the formation of links emerge from users who see truthful communication as a good strategy. This condition, from the user's perspective, can be interpreted as a trade-off between a personal perception about the state of the world and additional information received through truthful communication. They find that a company

[1] A network is connected if one can travel from any user to another one one by the edges of the network. This is defined formally in section 2.3.

should choose the user who can reach the highest number of nodes given the equilibrium condition imposed by truthful communication. This segment of their paper is similar to what I try to answer, but it does not address what happens if an advertiser seeks a seed is given to more than one user to maximize the exposure of a message. Having "more than one" node increases the space of possible solutions exponentially, which makes the purest version of the problem cursed with dimensionality. To solve the general case, one needs a proper model that keeps the dimensionality of the search for a solution low, regardless of the size of the network.

The computer science literature examines this problem of dimensionality. The purest version of viral marketing pertains to the set covering problem (see Cormen [9], for an overview): if there are n nodes (i.e. users) and k sets that cover some of these nodes (perhaps with some redundancy), one might ask what is the best collection of $m \leq k$ sets to maximize the number of nodes covered. This problem has been showed to be NP-hard, or in other words cursed with dimensionality.[2]

Thus, rather than seeking the optimal solution, this category of literature usually focuses on "fast" algorithms that can be shown to be within a given range of the optimal solution. For instance, Arthur, Motwani, Shama and Xu [1] present a framework for an optimal pricing strategy over a social network. They seek a set of initial users to whom they should send product information and offers of a discount to users who recommend the product to their friends. Their main contribution is algorithmic and they show a solution that is guaranteed to be within a constant factor of the optimal

[2]To illustrate, consider the following problem. There are n users on a network and they receive a binary signal, 0 or 1. The users reproduce the given signal only if they receive it from a neighbor. If a company has to choose the initial vectors of zeros and ones at a linear cost that maximizes the discounted sum of ones through time, there are 2^n possible candidate solutions. Although n is finite, on a network of the size of Facebook, there are $2^{300\times10^6}$ solutions to check. If it takes 10^{-9} seconds to compute one solution, the calculation would require a processing time of roughly $10^{90\times10^6}$ seconds. In comparison, the age of the universe is estimated to an order of 10^{17} seconds.

solution.

The fourth category of papers looks at profit maximization over a network where information is shared between users. Schiraldi and Liu [40] explore whether a firm should launch a product into different markets sequentially or simultaneously. Their main result is that the firm can increase its profit by manipulating the order of the launch sequence in different markets. Products that are highly anticipated should be launched simultaneously, as users already know enough about the product. However, products with an average level of anticipation should be launched sequentially to use momentum from reports from early markets. In this case, the order in which users receive information is not random, but rather selected by a company in order to maximize profits. However, the network structure remains unexploited. At each period, only one user is exposed to the product and past decisions, as in Banerjee, without any possibility that multiple users might get the same information at the same time.

Galeotti & Goyal [18] look at a profit-maximizing word-of-mouth strategy where agents randomly choose their sources of information given an in-degree distribution, a statistical distribution of connections between nodes. They find that the optimal fraction of selected nodes increases with degree if the profit function by degree increases and the opposite if the profit function decreases. Galeotti & Goyal [18] calculate the profit a social network owner can make from a given distribution if the network is large. They also exploit some information on the structure of a network, but their main assumption remains that users connect randomly, so their solution is optimal in this context but does not exploit the properties of a known network structure (aside from the network's distribution).

2.2.1 How This Chapter Relates to the Existing Literature

This chapter bridges these four categories of literature described in the previous paragraphs. It provides an optimal solution that exploits any known structure of social network, given that users choose to rebroadcast the message they like best. It accounts for what happens in the long-run as well as in the short-run and measures the trade-offs between the two. It also provides an optimal solution without restrictions on the number of nodes to be seeded and it does so while avoiding the curse of dimensionality. It is thus applicable on any large network. It also explains "cascades of recommendations" as they are observed empirically and explain how they are formed from a given network structure. The solution is a Nash equilibrium, in that neither the users nor the social network owner could be better off by changing their choices.

The model emphasizes the dynamic nature of the word-of-mouth process and accounts for the value of messages transmitted to "friends of friends." It can also answer questions such as "What is the value of a user in the network?" or "What happens if users' perceptions about a product change?" These questions cannot be answered by a statistical modeling of users.

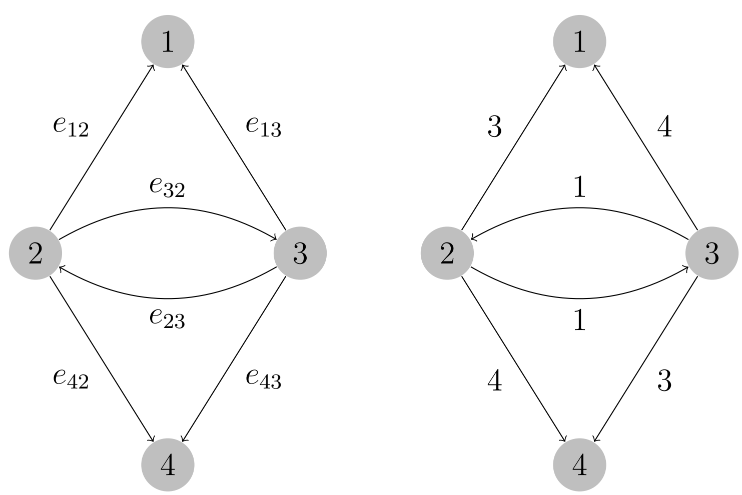
2.3 The Model

2.3.1 The Network and Users

I use a directed graph to model the relationship between users. In a graph, each node is an user and arrows between nodes are possible paths of information. Throughout this paper, I will use Figure 2.2 as a simple example of a possible network.

Definition (Network)**.** *A network* $G(V, E)$ *is a directed graph consisting of a set* V

Figure 2.1: An Example of A Facebook Post



(a) **Notation For Vertices And Edges**

(b) **Some Perceptions $\mathbf{b_{ij}}$ Along Edges**

Figure 2.2: The Example Used Throughout This Chapter

of vertices (nodes, users) and a set E of edges (links, sources) between vertices. A particular user of V will be denoted by an index $i \in V$ while a source going to vertex $i \in V$ from vertex $j \in V$ will be denoted by $e_{ij} \in E$.

In Figure 2.2(a), there are four users (labelled by the index i ranging from one to four). Each directed edge represents a possible flow of information as an arrow starting from node j to node i meaning that j is a possible source of information for i. For instance, user $i = 1$ has two sources of information, namely users $i = 2$ and $i = 3$. So information can flow from user 2 to user 1, but not from user 1 to user 2.

A useful definition is the set of sources of information for each user in the network. It specifies a unique network topology and it is also the choice set of any user.

Definition (Inbound Neighborhood)**.** *The inbound neighborhood of node i (η_i) is the set of users from whom i can listen to:*

$$\eta_i \equiv \{j \in V : e_{ij} \in E\}\,.$$

In Figure 2.2, the inbound neighborhood of consumer 4 is the set of users 2 and 3 ($\eta_4 = \{2, 3\}$). Each of these sets are assumed to be fixed. This constrains users to work with the set of friends each user already has. It is as if the advertising campaign is run in the short-run, where the social capital — the structure of the network — is fixed. Hence, users can draw information only from sources they have already developed.

In this essay, what users share over the network is a signal, a real number. It is a simple device that measures the value of information shared. It can either be of negative value, meaning that some users do not like information, or a positive value meaning the opposite. Since there is only one signal, it means that there is information about only one product. This is done for simplicity as the case with n products is a trivial extension.

Definition (Signal). *A signal s_{it} conveys the level of utility about information shared by user i at time t. I will denote $s_t \in \mathbb{R}^{|V|}$ the vector of all signals at time t.*

I define users by who they know, how they perceive who they know and how they choose signals.

Definition (Users). *An user $i \in V$ is defined by three elements $\left\{\eta_i, \{b_{ij}\}_{j\in\eta_i}, f\right\}$:*

1. *A set of sources of information η_i;*

2. *A set of perceptions $b_{ij} \in \mathbb{R}$ for each $j \in \eta_i$;*

3. *A choice function $f : \{s_j : j \in \eta_i\} \to \mathbb{R}$, or a reaction function, that governs the selection of signals.*

The set of sources of information has already been discussed and represents the choice set of each user. The set of perceptions b_{ij} encodes the correction user i gives to the signal from user j. It represents the attitude of user i towards user j. The higher the value of b_{ij}, the more highly agent i thinks of agent j. One way to think of this is personal preferences over sources. The model could embed changes in perceptions that vary with the signal by some given rule without loosing the ability to solve the model numerically. However, to clearly expose their role in a closed formed solution, I let them fixed. In section 2.5, I discuss a case where perceptions vary with the signal to talk about signal fluctuations and how this could apply to business cycles.

In this chapter, I specialize the choice function f to the max operator. Since the signal conveys the degree of informativeness about a product, this means that users pick the most informative signal.

Definition (Choice Function, Signal Law of Motion)**.** *A user i chooses to share a source of information $j^* \in \eta_i$ if it has the highest quality, given the perceptions.*

$$\begin{aligned} s_{i,t+1} &= s_{j^*t} + b_{ij^*}, \\ &= \max_{j \in \eta_i} \left(s_{jt} + b_{ij} \right). \end{aligned}$$

Denote $s_{t+1} = G(s_t)$ the law of motion generated by such reaction function.

There are several contexts in which this reaction function applies:

1. A product can be modelled as a vector in $\mathbb{R}^n$, where each dimension reveals something about the product (quality, color, probability of breaking, etc.). One piece of information on the product then represents a sigma-field with a measure on such vector, describing the likelihood of a particular realization of the object. If agents are endowed with strict preferences u over these sigma-fields, the signal represents the level of utility given by such a sigma-field. The choice is thus governed by how useful the information is to the user.

2. The signals are simply the econometric equivalent of the previous example, that is $s_{it} = X\beta + \epsilon_{it}$, where X is a vector of observed characteristics on the product, β is the (linear) weight given to each characteristic and ϵ_{it} is a "utility shock" following a random process.

3. The signal can also represent the evolution of demand, or the informativeness of a signal. Imagine that the firm has a good-quality product, but such quality is unknown to the users. Their priors are that there is a 50% chance that the product is of good quality and a 50% chance that it is not. The price of the product is set higher than 0.5 and the value of the product is one if it is of good quality and zero otherwise. Users can learn the quality about the product

if they buy it, or if they see one of their neighbours buy it. Users will want to buy a product only if it is of good quality, thus they will do so only if they see at least one of their neighbours buy it. Hence, the signal chosen can be represented as the highest signal in the neighbourhood.

4. Signals are simply a measure of the "quantity of information" the company provides.

For clarity, I keep the last case as the interpretation of the reaction function. This modeling choice implies that all users agree that information is useful. However, they disagree on the degree of usefulness, through perceptions. This implies that choices of sources remains an ordinal concept, meaning that what matters for selection is the relative difference between choices, given the perceptions. Users will only share a signal if it conveys the highest utility.

Since users only care about sending the signal they prefer, they have no consideration for possible loops in the network. They ignore the possibility that they could repeatedly receive a signal that they have received before. This also means that they do not predict the impact of their own behavior on the whole system. This assumption can be rooted in the classical idea that users think their own decisions have only a marginal effect on the market (or, in this case, the social network), and that they know little of the structure of the network outside their neighborhood. All in all, users do not know (or care[3]) that their actions can lead to a viral spread of a signal.

[3]It can actually be showed that if they cared about the fact that information came back to them, it would not change their decisions.

As a concrete example, let the users in Figure 2.2(a) behave according to :

$$s_{1,t+1} = \max(s_{2,t} + 3, s_{3,t} + 4), \tag{2.1}$$

$$s_{2,t+1} = \max(s_{3,t} + 1), \tag{2.2}$$

$$s_{3,t+1} = \max(s_{2,t} + 1), \tag{2.3}$$

$$s_{4,t+1} = \max(s_{2,t} + 4, s_{3,t} + 3). \tag{2.4}$$

This law of motion simply puts numbers on perceptions. This can be seen in Figure 2.2(b).

In such example, perceptions vary from one user to another. For instance, user $i = 1$ has a perception of 3 about user $j = 2$ and a perception of 4 about user $j = 3$.

If in time t, the vector of signals s_t is given by $[0, 0, 0, 0]$, the law of motion will yield the vector $[4, 1, 1, 4]$ in time $t + 1$. For instance, user one interprets the signals as $(0 + 3, 0 + 4)$. He thus picks the most informative signal, so $s_{1,t+1} = s_{3,t} + 4 = 4$. When all users have made their choice, the vector of signals s_{t+1} becomes the next set of signals and users come back in $t + 1$ to perform another selection.

With standard mathematics, this law of motion is non-linear. However, I will make the case that this law of motion is linear in a different algebra. I will use this algebra to find properties of the model that are easily expressed in terms of linear first difference equations. I will describe this in details when solving the model.

To sum up, I model that users log in to the social network, look at all the signals they have received, adjust signals based on their perceptions, rebroadcast the signal they like best, and repeat the process the next day. In the next section, I answer the following: how to optimize profits out of this reaction function?

2.3.2 The Company's Objective

The company maximizes profits for its shareholders. It does so by selling viral exposure to a third party at price p per unit. The demand for such exposure is rooted in the idea that advertising increases sales (see [41] for a review) and the fact that online social networks already sell standard advertising products. Word-of-mouth advertising is desirable because messages may have more influence when they come from friends than when they come from advertisers.

For example, Bond et al. [6] show that social transmission of positive messages about voting through Facebook has a significant effect on friends' decisions to vote and on political engagement:

> "The effect of social transmission on real-world voting was greater than the direct effect of the messages themselves, and nearly all the transmission occurred between 'close friends' who were more likely to have a face-to-face relationship. These results suggest that strong ties are instrumental for spreading both online and real-world behavior in human social networks."

Other researchers have found that word-of-mouth advertising can influence consumer behavior. With respect to the sharing of video content, Yoganarashimhan [49] finds that the size and structure of a social network is a significant driver of video propagation. Trusoc, Bucklin and Pauwels [43] find that word-of-mouth referrals have a greater and longer-lasting effect than standard advertising. In terms of price fluctuations, Yang [48] shows that the network structure of sources of information matters in forecasting prices and, thus, supply and demand based on those forecasts.

All in all, users are more likely to act on information if it comes from their friends.

This suggests that advertising messages transmitted by word-of-mouth can be valuable to advertisers. I formalize these ideas below.

First, I define the object of value to the third party company. That company wants the highest possible levels of utility reaching each user. For a unit of signal reaching a user at time t, that third party is willing to pay a unit price of $p\beta^t$. This means that the company values more current exposition than future exposition as it discounts time at the rate $\beta \in [0, 1)$. This defines an income function:

Definition (Company's Income Function)**.** *Let $t \in \mathbb{N}$, $p \in \mathbb{R}, \beta \in [0, 1)$ and define the set of all possible signals by :*

$$S \equiv \left\{ \{s_t\}_{t=0}^{\infty} : s_t \in \mathbb{R}^{|V|} \right\}.$$

For any $s \in S$, the company generates income with $s \in S$ by :

$$I(s, p) \equiv p \sum_{t=0}^{\infty} \beta^t \sum_{i \in V} s_{it}$$

This income function assumes that the campaign runs forever as t is allowed to go to infinity. As we shall see, performing the analysis in finite time would not change much of the insights as long as the number of periods is larger than a threshold t^* that is smaller than the largest directed diameter of the graph.

Facebook wants to let the network carry over the signals to profit from the strong ties between users on the network, so it does not interfere in the dynamics of the signal selection. It thus has only one instrument at its disposal, that is the initial vector of signals, s_0, reflecting the first information each consumer gets about the product.

A particular signal s_{i0} costs $\frac{s_{i0}^2}{2}$, so the total cost of the initial signal is $\sum_{i \in V} \frac{s_{i0}^2}{2}$. This assumption of quadratic costs reflects the idea that it becomes much harder to convince a user to share an initial signal if it has an high intensity. This discussion leads to the following principal problem:

Definition (Company's problem). *Let* $s_{t+1} = G(s_t)$ *denote the law of motion generated by the user's behavior in the network. The company then seeks the policy satisfying:*

$$\arg\max_{s_0} \Pi(s,p) \equiv I(s,p) - I(0,p) - c(s_0)$$
$$s.t.$$
$$s_{t+1} = G(s_t) \ \forall t$$
$$c(s_0) = \sum_{i \in V} \frac{s_{i0}^2}{2}$$

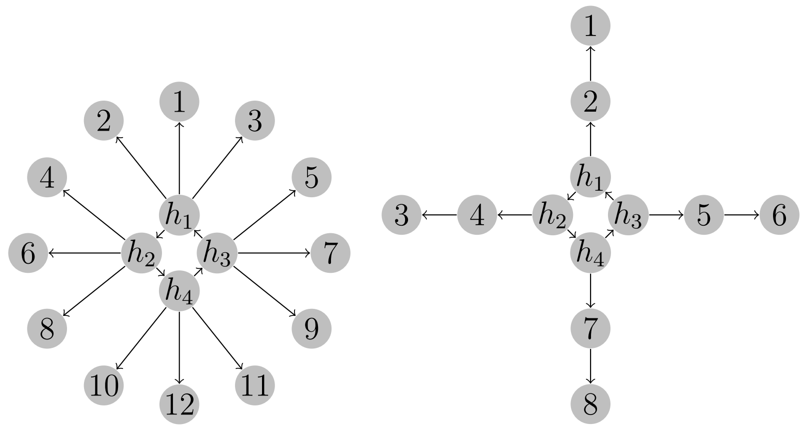
Denote s^* *the solution (if any) to this problem.*

Profits are calculated as the difference between net income generated by the chosen signal and the costs to build that signal. The net income is the difference between the value of the chosen campaign and the value of a campaign where the company does not seed any signal. As the value of this campaign will happen if the company performs no action, it cannot charge or capture the value associated with it. The component $I(0,p)$ can be thought as the social value of the network.

2.3.3 Solving the Model

The solution to the model can be understood in three key steps.

First, since perceptions are additive, any signal $s_{i,t}$ is the sum of some original signal $s_{j,0}$ plus a component $c_{j,t}$. Such number $c_{j,t}$ is the sum of perceptions over the path of selections that lead signal $s_{j,0}$ to user i in the course of time t.



(a) **A Star Network** (b) **A Constrained Star Network**

Figure 2.3: The Star and Constrained Star Networks.

Second, any optimal solution $s^*_{j,0}$ can be written as a discounted sum of signal selections over time. If we denote k_{jt} as the number of times the signal $s_{j,0}$ is selected in time t, the optimal solution solves:

$$s^*_{j,0} = p \sum_{t=0}^{\infty} k^*_{jt} \beta^t.$$

What generates value is exposure. Hence, the number of time a signal is carried over in the next period, as counted by k_{jt}, matters to measure exposure in the next period. Hence the discounted sum of the number of selections represents the marginal exposure of a signal $s_{j,0}$. Setting it equal to the marginal cost yields an optimal solution.

The third step requires the understanding of a network structure that emerges naturally in the dynamics of signal selection, namely a *constrained star network*. Loosely speaking, a star network has all its nodes attached to a central group of nodes (see Figure 2.3(a)), the center of the star. In a constrained star, some nodes are not directly connected to the center because the original network topology prevents them from doing so. They are however connected to the center through a chain of nodes (see Figure 2.3(b)). The notion of constrained star is defined formally below:

Definition (Path, Constrained Star Network)**.** *A path from i to j is a repetitionless sequence of directed edges $e_{ik_1}, e_{k_1k_2}, \ldots, e_{k_nj}$ that connects i to j.*

A constrained star network is a graph $G(V, E)$ with the following properties:

1. *An hub H: a subset of V such that for any two nodes $i, j \in H$, there exists at least one directed path from i to j.*

2. *For any other node $i \in V \setminus H$, there exists one directed path starting at some node $j \in H$ and going to i.*

If a network $G(V, E)$ admits at least one constrained star structure, the average value of the seed at each node is equal to the discounted value of the selling price $\frac{p}{1-\beta}$. So what matters to find the optimal solution is the spread of the value of initial signals around this average. This spread is maximized when all the seed is concentrated amongst a small group of users. When costs are quadratic, the maximization of the spread is equivalent to maximizing the variance amongst all Nash equilibria.

These key steps are described in the following sections. In section 2.3.3, I show that an optimal solution must be a Nash equilibrium between the network and the company. I use the example to show that there are many Nash equilibria. In section 2.3.3 I characterize the law of motion and use it to discuss profits and income functions. I show that profits depend on the spread of the initial signal and the value of contagion of signals. Since both terms increase when variance increases, the first component is a sufficient metric to optimize profits.

An Optimal Solution Must Be A Nash equilibrium

The following lemma shows how contagion of signals from one user to another can be tracked in the model.

Lemma 1 (Path Dependance)**.** *Let $s_{t+1} = G(s_t)$ be the law of motion of the users. Then, for all i, $s_{i,t+1} = s_{j,0} + c_{j,t}$ for some $j \in \{0, 1, 2, \ldots, |V|\}$ and where $c_{j,t} \in \mathbb{R}$ is a path dependent number.*

Proof. See appendix A.1.1 □

This lemma says that the signal chosen by user j at time t is nothing but the original signal of some other user i plus the sum of perceptions that were added over time through the path of diffusion from i to j. Hence, at each period, the signal s_{jt} is a function of one original signal. The following definition uses the result of this lemma to link the choices made by users in time t to a signal in time $t = 0$. This allows a discussion of contagion in terms of marginal income.

Definition (Indicator function, Number of choices)**.** *Let $\mathbb{1}_{ijt}$ be the indicator function. It equals one if user i chose $s_{j,0}$ at time t and zero otherwise. Furhermore define k_{jt} by*

$$k_{jt} \equiv \sum_{i=1}^{|V|} \mathbb{1}_{ijt},$$

that is the number of users who choose signal $s_{j,0}$ at time t.

If the signal $s_{j,0}$ has been chosen k_{jt} times in period t, the company can generate an income of $p\beta^t \left(k_{jt}s_{j,0} + \sum_{i\in\{i:\mathbb{1}_{ijt}=1\}} c_{i,t}\right)$. So it is important to count the number of times it has been chosen at each period to establish an optimal signal value.

The next proposition summarizes that any possible solution to the company's problem must be a Nash equilibrium.

Proposition 1 (Candidate Solutions). *Let* $\tilde{S} = \left\{ \{\tilde{s}_{jt}\}_{t=0}^{\infty}, \left\{\tilde{k}_{jt}\right\}_{t=0}^{\infty} \right\}_{j=1}^{|V|}$ *be the set of candidate solutions. Then any* $\tilde{s} \in \tilde{S}$ *must satisfy:*

1. $\tilde{s}_{t+1} = G(\tilde{s}_t)$;

2. $\tilde{k}_{j,0} = 1 \ \forall j$;

3. *The seed* $\{\tilde{s}_{j,0}\}_{j=1}^{|V|}$ *induces selections by users such that for all periods and users, the number of choices for a particular signal is given by* $\tilde{k}_{jt}$;

4. *Given these* $\tilde{k}_{jt}$, $\tilde{s}_{j,0}$ *solves:*
$$\tilde{s}_{j,0} = p \sum_{t=0}^{\infty} \beta^t \tilde{k}_{jt} \ \forall j;$$

5. *If the network admits at least one constrained star topology, the average value of the initial signals it then given by:*
$$\frac{1}{|V|} \sum_{j \in V} \tilde{s}_{j,0} = \frac{p}{1-\beta}.$$

Proof. See appendix A.1.2 □

The Nash equilibrium is summarized in points 3 and 4 of the proposition. Given that $\tilde{s}_{j,0}$ is the strategy of the company for user j, it must be that users over the network chooses that signal k_{jt} times at time t. Now given that users have chosen the signal j $\tilde{k}_{jt}$ times at time t, $\tilde{s}_{j,0}$ must account for the marginal contribution $p\beta^t\tilde{k}_{jt}$ at time t. Summing over all periods yields the marginal income and thus, a candidate solution.

The fourth item of this proposition makes quite clear what matters in seeding a node. It is a measure of the discounted social influence of each node. Any solution has to account for the discounted number of users who will select the information. The more a source is selected the more it is influent.

We can see this clearly in a node-time diagram. In Figure 2.4, I depict one possible case of the dynamics[4] of the example I previously described (Figure 2.2(b) and equations 2.1 to 2.4). Each column of nodes represents a user in time while arrows reflects the contagion of signals through time. At the center of each user is the initial signal that has been selected by this user. In the picture, the signal $s_{3,0}$ is such that it is selected three times by users at each future period while $s_{2,0}$ is selected only once every period. This means that $k_{3t} = 3$ for all $t > 0$ and $k_{2t} = 1$ for all periods. Signals $s_{1,0}$ and $s_{4,0}$ are never selected, so $k_{1t} = k_{4t} = 0$ for all $t > 0$. This means that given these k_{jt}, the company must set signals according to:

$$s_{1,0} = p, \tag{2.5}$$

$$s_{2,0} = p(1 + \beta + \beta^2 + \beta^3 + \dots), \tag{2.6}$$

$$s_{3,0} = p(1 + \beta 3 + \beta^2 3 + \beta^3 3 + \dots), \tag{2.7}$$

$$s_{4,0} = p. \tag{2.8}$$

The last statement of proposition 1 says that the average value of all signals must equal the discounted value of the marginal income. This means that all candidate solutions on a network with at least one constrained star in it have the same average value in $t = 0$. Thus, what discriminates the elements in $\tilde{S}$ is the degree of variation amongst signals given to users.

In the example of Figure 2.2(b), there is more than one solution satisfying the conditions of proposition 1. The signal $s_{3,0}$ is chosen at time $t = 1$ by user 4 only if $s_{3,0} + 3 > s_{2,0} + 4$. There are thus two cases. The first case leads to the previous optimal solution. The second case is when $s_{3,0}+3 < s_{2,0}+4$. In this case, $k_{21} = k_{31} = 2$ and since the network is symmetric, it is true for every subsequent period. For any

[4] Recall that an arrow represents the selection of a source.

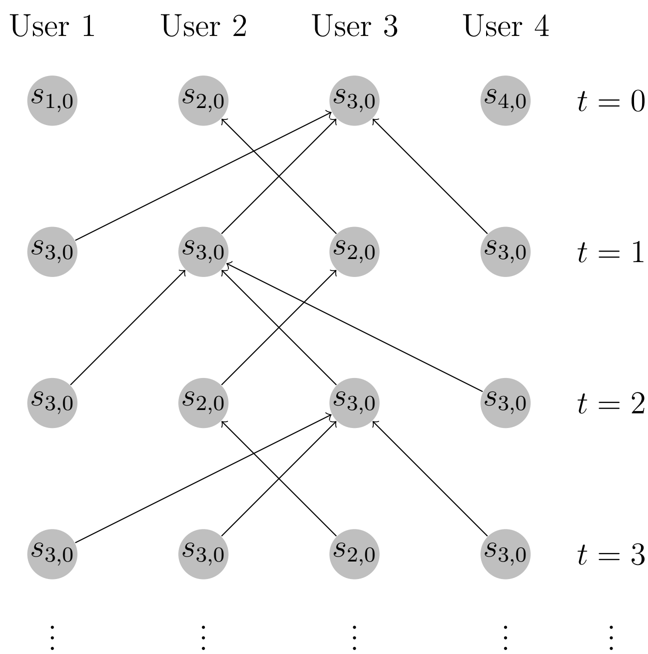


Figure 2.4: Node-Time Diagram of the Example in Figure 2.2(b)

of these two solutions, users one and four have no influence over other users because they have no friends who listen to them. This means that their optimal seeding value remains $\tilde{s}^*_{1,0} = \tilde{s}^*_{4,0} = p$ as it is the only solution to these equations. Thus, the two[5]

[5] There are actually three solutions on the network. But since it is symmetric, two solutions are identical, up to a permutation in seeds. By a permutation of the seed at $\tilde{s}_{3,0}$ with the seed at $\tilde{s}_{2,0}$ in the high variance solution, one gets another solution.

solutions are:

$$\left.\begin{aligned}
\tilde{s}_{1,0} &= p \\
\tilde{s}_{2,0} &= p + 2p\frac{\beta}{1-\beta} \\
\tilde{s}_{3,0} &= p + 2p\frac{\beta}{1-\beta} \\
\tilde{s}_{4,0} &= p
\end{aligned}\right\} \text{Smallest Variance Equilibrium (example of Figure 2.2(b))}$$

$$\left.\begin{aligned}
\tilde{s}_{1,0} &= p \\
\tilde{s}_{2,0} &= p + p\frac{\beta}{1-\beta} \\
\tilde{s}_{3,0} &= p + 3p\frac{\beta}{1-\beta} \\
\tilde{s}_{4,0} &= p
\end{aligned}\right\} \text{Highest Variance Equilibrium (example of Figure 2.2(b))}$$

I will refer to these solutions as the "highest" and "lowest" variance equilibrium. Notice that $\tilde{s}_{3,0} + 3 > \tilde{s}_{2,0} + 3$ holds only if $(2p+1)\beta > 1$, so this is a candidate equilibrium that belongs to $\tilde{S}$ if this is true. Otherwise, the smallest variance equilibrium is unique. In any case, all these candidate solutions are such that $\sum \tilde{s}_{i0} = \frac{p|V|}{1-\beta}$ since the network admits at least one constrained star.

If $(2p + 1)\beta < 1$, there is only one equilibrium, so it is the optimal solution. However, if $(2p+1)\beta > 1$, which of these equilibria should be picked? Intuitively, the one with the highest variance yields the highest profits because more value is kept through the process of replication. But in this equilibrium, there is a loss of value since information travels on edges of low perceptions (e_{12} rather than e_{13}). So there might be a trade-off between the replication process and the gain in perceptions. I will show later that this is not the case. In order to do so, I first characterize the conditions of emergence of candidate solutions. Each candidate solution is linked to a possible constrained star network. Each constrained star network is in turn linked to the eigenvalues of the law of motion.

On the Dynamics of Signal Selection

So far, I have written very little about the optimal profits. I have also written very little about the law of motion $s_{t+1} = G(s_t)$. The law of motion and profits/income are related as one must first characterize $I(0, p)$, the social value of the network, before calculating income and profits. I first introduce the main results and then show in details how to get them.

If the network admits at least one constrained star network, I show that it enters a periodic regime of period T after a finite transient time t^*. What this means is that after t^*, the law of motion G can be described by:

$$s_{i,t+T} = s_{i,t} + \lambda T \ \forall t > t^*.$$

The element λ is defined as the highest average perception over all possible loops of the network (I describe this in more depth later on). The value T is pinned down by the greatest common divisor of all cycles with average value λ. This result implies in turn that users select the same neighbors after a while as they remain the most valuable.

This pattern of source selection generates a constrained star network. After t^*, the loop with the highest average fuels all the other users who connects to them. The pattern of connections for those users, the "branches", are constrained by the sources available. Hence, after t^*, highly perceived agents are selected over and over directly or indirectly.

These results in turn gives a characterization of the profits formulation and the social value of the network. That value can be interpreted as the value the advertiser gets for free, without any campaign design from the company. It is the simple action of users talking about the product.

To prove these results, a bit of investment in mathematics is required. The law of motion is based on $s_{i,t+1} = \max_{j \in \eta_i}(s_{jt} + b_{ij})$ which is non-linear in standard algebra. However, it is linear in what is called a *tropical algebra* or a max-plus semiring. If one defines the addition operator over numbers a and b as the maximum operator ($a \oplus b \equiv \max(a, b)$) and the multiplication operator over numbers a and b as standard addition ($a \otimes b \equiv a + b$), one then gets the basic operations of a tropical algebra. An expression like:

$$s_{i,t+1} = \max_{j \in \eta_i}(s_{jt} + b_{ij}) + b_i, \qquad \text{(standard algebra)}$$

becomes

$$s_{i,t+1} = \bigoplus_{j \in \eta_i} b_{ij} \otimes s_{ij}, \qquad \text{(inner product in a tropical algebra)}$$

which is nothing but an inner product in this algebra. Hence, if we define a transition matrix B by the elements $[B]_{ij} = b_{ij}$, the system can be written as $s_{t+1} = B \cdot s_t$, a first difference linear equation. Thus, one can recover the essential tools of first difference equations like eigenvector/eigenvalue decomposition.

I introduce with greater details the main structure of linear systems on a tropical algebra in appendix A.2. Readers looking for a complete and formal treatment should read the book by Bacelli and coll. [2]. In the body of this article, I present only the important definitions and theorems to get the results. To begin with, I start by defining "highest average loops", that is the notion of critical cycle. Critical cycles pins down the increase of λ and the length of the periodic regime.

Definition (Critical Cycles)**.** *Let C be the set of all cycles on $G(V, E)$. For any cycle $c \in C$ of length $|c|$, let*

$$\rho(c) \equiv \sum_{e_{ij} \in c} \frac{b_{ij}}{|c|}$$

be the average value of perceptions and $C(G)$ be the set of all cycles in a network $G(V, E)$. A cycle c_c is critical if

$$c_c \in \arg \max_{c \in C(G)} \rho(c).$$

Denote $\lambda = \max_{c \in C(G)} \rho(c)$.

This definition assigns a number to all cycles in the network, namely the average increase of the signal through perceptions over each cycle. The critical cycles are the loops with the highest average. Notice that by construction, any cycle is an hub.

In Figure 2.2(b), there is only one cycle, that is $\{e_{23}, e_{32}\}$ with an average value of $\frac{1+1}{2} = 1$. Since it is the only cycle, it has the highest average and is thus a critical cycle.

Critical cycles are important because they contain users who create the highest consistent increase of the signal through perceptions. Since users choose signals they like best, they will eventually be influenced by the users in critical cycles, those that have the highest combination of perceptions.

Now, I will use results on tropical algebras to show the existence of the periodic regime. I start with the definition of eigenvectors and eigenvalues in the context of this model and then state the main mathematical result.

Definition (Iterated Law of Motion, Eigenvalues and Eigenvectors)**.** *Let $s_{t+1} = G(s_t)$ be the law of motion associated with a network $G(V, E)$. Define the iterated law of* *motion on $G(V, E)$ by the composition function:*

$$G^T(s) \equiv \underbrace{G \circ G \dots G \circ G(s)}_{T \text{ times}}.$$

Let furthermore $B^T \cdot s$ be its representation on a tropical algebra. An eigenvalue-eigenvector pair (λ, v) is a solution to the equation:

$$B^T \cdot v = \lambda^T \otimes v,$$

or in standard algebra:

$$G^T(v) = v + T\lambda.$$

With these definitions in hand, one can show the following:

Theorem 1. *For any network $G(V, E)$ which admits at least one constrained star, then:*

1. *There exists a transient time $t^* < \infty$ after which the network enters a periodic regime of frequency T. T is given by the least common multiple of the length of all critical cycles while t^* is no longer than the longest directed path between any two points;*

2. *There exists $v_1, v_2, \dots v_T$ eigenvectors such that $G^T(v_i) = v_i + T\lambda$. Each eigenvector corresponds to one period;*

3. *These vectors are formed by the columns of the matrix $[B^*]_{ij} \equiv \max([B^T - T\lambda]_{ij}, [B^{2T} - 2T\lambda]_{ij}, [B^{3T} - 3T\lambda]_{ij}, \dots)$.*

4. *In particular, define $[v^*]_i \equiv \max(v_{1i}, v_{2i}, \dots v_{Ti})$, then v^* is the unique eigenvector on $G(V, E)$:*

$$G(v^*) = v^* + \lambda;$$

Proof. See Bacelli, Cohen, J. Olsder and Quadrat [2]. In particular, section 3.2.4 pp 111-116 for the notion of eigenvectors and section 3.7 pp. 143-151 for the cyclicality and the finiteness of the transient time. □

The most stringent requirement is that there exists at least one cycle in the network. This means that there is a group of users who settle to listen to each other. If there is no cycle, then any word-of-mouth campaign will end in finite time t^*. In this case, all $k_{jt} = 0$ after t^* and the conditions for proposition 1 remains.

But if there is at least one cycle, the system will enter a periodic regime. The nodes that have no path connecting them to the cycle will have no long run value, as if there is no cycle. Hence, they change very little in the analysis. Given the clustering features of most real networks, having at least one cycle in a network seems like a weak condition.

In the example of Figure 2.2(b), the only critical cycle has an average value of 1, so $\lambda = 1$. There are only two nodes on this cycle, so $T = 2$. The eigenvectors are given by $v_1 = [3, 0, -\infty, 2]'$ and $v_2 = [2, -\infty, 0, 3]'$. This means that $v^* = [3, 0, 0, 3]'$. In particular, one can check that:

$$G(v_i) = v_j + 1 \ \forall i \neq j, \qquad G^2(v_i) = v_i + 2 \ \forall i,$$
$$G(v^*) = v^* + 1.$$

The eigenvalue measures the overall increase on the network due to the critical cycles, while the eigenvectors measures the increases net of λ due to choices made by users. The interpretation for the components of $v^* = v_1 \oplus v_2$ is similar: it gives the maximal increase that will occur over all periods given the selections made by users.

The last proposition allows to find the following results for the optimal solution:

Proposition 2. *Let $G(V, E)$ admit at least one constrained star topology and consider $\tilde{s} \in \tilde{S}$. Then:*

1. *Any $\tilde{s}$ can be written as:*
$$\tilde{s}_{j,0} = p \sum_{t=0}^{t^*} \beta^t \tilde{k}_{jt} + p \frac{\beta^{t^*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u \tilde{k}_{j,u},$$
where $\tilde{k}_{j,u}$ is the number of times signal $\tilde{s}_{j,0}$ is chosen in the u-th period of the periodic regime;

2. *The income generated by the network is given by:*
$$I(\tilde{s}, p) = \frac{1}{2} \sum_{i=0} (\tilde{s}_{i0})^2 + p\lambda \frac{|V|\beta}{(1-\beta)^2} + p \frac{\beta^{t^*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u \sum_{i \in V} \left[G^{t^*+u}(\tilde{s}_0) - \tilde{s}_0\right]_i$$
$$\cdots + p \sum_{t=0}^{t^*} \beta^t \sum_{i \in V} \left[G^t(\tilde{s}_0) - \tilde{s}_0\right]_i$$

3. *Denote $\mathbf{0}$ the zero vector. Then, profits are given by:*
$$\Pi(\tilde{s}, p) = \frac{1}{2} \sum_{i=0} (\tilde{s}_{i0})^2 + p \frac{\beta^{t^*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u \sum_{i \in V} \left[G^{t^*+u}(\tilde{s}_0) - \tilde{s}_0 - v^*\right]_i \cdots$$
$$\cdots + p \sum_{t=0}^{t^*} \beta^t \sum_{i \in V} \left[G^t(\tilde{s}_0) - \tilde{s}_0 - G^t(\mathbf{0})\right]_i.$$

4. *If $\tilde{s}' \in \tilde{S}$ is another candidate solution, then the difference in profits is given by:*
$$\Delta\Pi(\tilde{s}, \tilde{s}', p) \equiv \Pi(\tilde{s}, p) - \Pi(\tilde{s}', p),$$
$$= \frac{1}{2} \sum_{i=0} \left[(\tilde{s}_{i0})^2 - (\tilde{s}'_{i0})^2\right] + p \frac{\beta^{t^*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u \sum_{i \in V} \left[G^{t^*+u}(\tilde{s}_0) - G^{t^*+u}(\tilde{s}'_0)\right]_i \cdots$$
$$\cdots + p \sum_{t=0}^{t^*} \beta^t \sum_{i \in V} \left[G^t(\tilde{s}_0) - G^t(s'_0)\right]_i \quad (2.9)$$

Proof. See Appendix A.1.3. ☐

The first statement is a simple corollary of the periodic nature of the law of motion.

Since after t^* the choices becomes periodic, the optimal solution can be written as a discounted sum of every period.

The second statement characterizes the total value of the network. It states that the optimal income is the sum of four components. The first term is the income that stems out of the value of $\tilde{s}^*_{j,0}$. The second term is the value generated by users on critical cycles. As these users are those who increase the signal the most repeatedly, they are those who lead the increase of the signal. Because users look for higher signals, all of them eventually connect to them through the means (neighborhoods) they have.

Notice that this income is generated regardless of the chosen solution. This can be thought as the social value created from the network as users finds out about the product by themselves. If the highest average cycle λ is positive, this has a positive value, but nothing prevents from λ to be negative. This depends obviously on spins and perceptions on the network.

The third term is the value of contagion that is implied by the chosen campaign. It measures the increase of value from choosing highest valued sources in the periodic regime. It is thus independent of λ. To see why, imagine that all spins and perceptions $b_{ij} + b_i$ are subtracted by λ, the highest average cycle is thus zero (and so is the social value of the network). However, the marginal changes due to perceptions from one source to another remains the same. Hence, users still make the same choice of source over time.

The last term is the value of contagion generated by transitionary dynamics from ($t = 0$ to t^*) and has the same interpretation as the third term. It measures the increase of value from choosing highest valued sources in the transitionary dynamics.

The third statement describes the profits that the company can capture. It has almost the same interpretation as income. First, the two terms that depends on the value of perceptions implied by the campaign are now net of the social value ($I(\mathbf{0}, p)$). In the long run, the effortless campaign generates the vector v^* at every period. Second, the component of income implying λ disappears as the advertiser gets it for "free" if the company performs no campaign.

The formula of the last statement is a simple generalization of the third one: it compares the value of two equilibria in $\tilde{S}$.

In the example of Figure 2.2(b), if $(2p+1)\beta > 1$, the highest variance Nash equilibrium is given by $\tilde{s}_0 = \left[p, p + p3\frac{\beta}{1-\beta}, p + p\frac{\beta}{1-\beta}, p\right]'$. Since $\lambda = 1$, the social value is given by $p\frac{4\beta}{(1-\beta)^2}$. Given the solution $\tilde{s}_0$, the network enters the periodic regime with the state associated with the first eigenvector $v_1 = [3, 0, -\infty, 2]$ and $v_2 = [2, -\infty, 0, 3]'$ in the second period, so the transient time $t^* = 0$. Since nodes 2 and 3 are in the critical cycle, $v = \left[p + 3\frac{\beta}{1-\beta}, p + \frac{\beta}{1-\beta}, p + 3p\frac{\beta}{1-\beta}, p + 3p\frac{\beta}{1-\beta}\right]'$. If the vector $\mathbf{0}$ is seeded to the network, the system enters the eigenstate $v^* = [3, 0, 0, 3]$ after one period and remains in it forever. One can then calculate profits for the highest variance solution:

$$\Pi(s^*, p) = \underbrace{p^2 + \frac{1}{2}\left(p + \frac{3p\beta}{1-\beta}\right)^2 + \frac{1}{2}\left(p + \frac{p}{(1-\beta)}\right)^2}_{\text{Value of the N. eq.}} + \underbrace{\frac{p\beta}{1-\beta}\left[2\left(p + \frac{3p\beta}{1-\beta}\right) - 1\right]}_{\text{Value of Contagion}}.$$

(Highest Variance Profits)

With similar calculations, one can find the profits for the low variance equilibrium, which gives the following difference in profits:

$$\Delta\Pi(\tilde{s}, \tilde{s}', p) = \frac{2p^2\beta^2}{(1-\beta)^2} + \frac{p^2\beta^2}{(1-\beta)^2} - \frac{p\beta}{1-\beta}.$$

The first term measures the quadratic increase du to an higher dispersion of the seed. The second term measures the gain of value through contagion. However, this gain is lowered by the fact that agent one chose a source with a lower perception, which is measured by the third term.

Notice that the whole sum is positive if $(3p+1)\beta > 1$. Recall that the condition for the two equilibria to be sustainable is that $(2p+1)\beta > 1$, so the highest variance equilibria turns out to be s^*. As the next proposition state, this is not fortuitous.

Proposition 3. *Assume $G(V,E)$ admits at least one constrained star and define the variance on $\tilde{s} \in \tilde{S}$ by :*

$$var(\tilde{s}) \equiv var(\tilde{s}_0) = \frac{1}{|V|} \sum_i \left(\tilde{s}_{0,i} - \frac{p}{1-\beta} \right)^2 .$$

Then, s^* maximizes the variance on $\tilde{S}$.

Proof. See appendix A.1.4 □

This proposition establishes a complete order on all elements in $\tilde{S}$. If one candidate solution has a smaller variance than another, then it cannot be an optimal solution. In other words, the company has an incentive to steer the network clear of any path where users share the same signal.

The fact that variance can be used as a metric of optimality pertains to the specificity of quadratic costs. However, the idea that the company has incentives to concentrate the value of the seed amongst users pertains to the use of the max operator from individuals. So this result might change in its form if another cost function is chosen, but is robust to the choice of the cost function in nature.

The intuition for this last result can be illustrated. Higher signals increase chances of replication. To illustrate it, assume a complete network where users do not modify the signals ($b_{ij} = 0 \; \forall i, j$). In this framework, everybody listens to everybody and there is no change to signals due to perceptions. In this context, if the company sends an initial signal of value $\frac{p}{1-\beta} - \epsilon$ to all but one user who has $\frac{p}{1-\beta} + (|V| - 1)\epsilon$, it is certain that all users will choose this higher signal in the next period: perceptions are all identical and the signal is higher. So in only one iteration, a value of $(|V| - 1)\left(\frac{p}{1-\beta} - \epsilon\right)$ is lost no matter how small ϵ is. Given these choices, the social network owner has an interest in focusing on the selected user to increase the value of the signal, which in turns decrease the value of other signals. This increases the dispersion of signals across users[6].

2.3.4 Network Efficiency

This section explores what happens to the social network structure of information once the network has entered a periodic regime. In short, it becomes a constrained star network. The network is constrained because not all users can connect to the hub, as imposed initially by the neighborhood η of each user. So the periphery consists of nodes that are to the hub through a chain of users (see Figure 2.5). These paths are in turn characterized by the sequence of perceptions $(b_{ij} + b_i)$ that maximizes the average increase over the length of the path.

The star network structure is an efficient configuration under a vast number of circumstances (see Jackson ch. 6.3 for a description). Intuitively, this is the most efficient configuration of the network to maximize the spin from the center to the edges. Since node all nodes have a connection to the center of the star, the network can only achieve a constrained version of it.

Proposition 4. *Let $G(V, E)$ admit at least one constrained star topology. Then after the transient time t^*, users on the network generate at least one constrained star*

[6] In this simple argument, one can also see the roots of an algorithm to find the optimal solution.

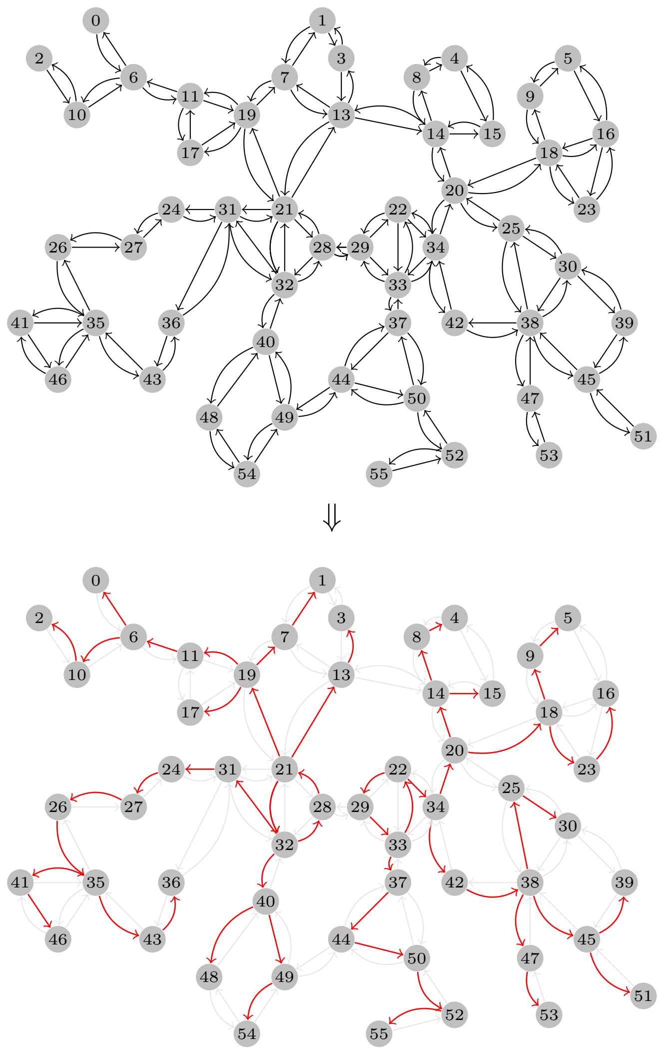


Figure 2.5: Networks Splits into Constrained Star Networks

network structure where:

1. *All the hubs are a critical cycle;*

2. *All users that are not in the hub are connected through a path that maximizes the increase of the signal between the hub and the user.*

Proof. See Appendix A.1.5 □

This means that even if users are not aware of the whole structure of the network, a structure of spin efficiency emerges. There is a cycle of users that share the most informative signal and other users attach themselves to this cycle in the way that maximizes the diffuse such most informative signal. This structure reproduces the patterns found in the paper by Leskovec, Adamic and Huberman [29]. It finds that cascades of recommendations are led by a small group of active users (the hub) to a chain of users (the constrained periphery).

This efficiency result must be interpreted with caution. It means that the networks organizes itself in a way that it maximizes the diffusion of the most informative signal. Nothing says, from a normative point of view, whether such signal with respect to a given product is a good thing. This could, however, serve as a basis of explanation for market bubble behaviors. I push this idea further in section 2.6.2.

2.4 An Algorithm to Find the Solution

In this section, I present the key elements of an algorithm that computes the solution in polynomial time.That means that for a given size $|V|$ of the network, the time to compute the optimal solution is a function of a polynomial in $|V|$.

The algorithm is a modified version of the greedy algorithm and can be described in a few steps. Each iteration seeks to seed as much of the total value $\left(\frac{p|V|}{1-\beta}\right)$ to each agent without modifying the selections of users implied by previous iterations. Hence, the algorithm needs to know what share of the total value remains at each iteration. It also needs to know an upper bound, the maximal value that untreated users can have without changing the pattern of selections induced by treated nodes.

1. Initialize the current budget to $p\frac{|V|}{1-\beta}$, the upper bound to ∞ and the set of treated nodes to the empty set.

2. While the set of treated nodes has less than $|V|$ elements:

   (a) Inititalize the seed in the following fashion. For all treated nodes, set their value found in the previous iterations. For untreated nodes, pick one node and set the seed for this particular node to the value of the current budget and set to zero to all other untreated nodes. Compute the closest sustainable Nash equilibrium.

   (b) Repeat for all untreated nodes and keep the resulting solution that has the highest value. Call the node $v^{(i)}$ at the i-th iteration leading to such solution.

   (c) Update the current budget by subtracting the value allocated to $v^{(i)}$. Find the upper bound such that $v^{(1)}, v^{(2)}, \ldots, v^{(i)}$ will remain unchanged. Set the current budget to the minimum between this value and the updated current budget.

   (d) Add $v^{(i)}$ to the set of treated nodes.

A formal description of the algorithm can be found in appendix A.4. This algorithm exploits two properties of the model. First, concentration is good. So it seeks to put as much value as possible in a node at every possible step given that the average value must be $\frac{p}{1-\beta}$. Second, it exploits the fact that the solution must be Nash equilibrium. Brouwer's fixed point theorem allows to compute a Nash equilibrium through the following formula:

$$h(s_0) \equiv \begin{bmatrix} \sum_{t=0}^{t^*} \beta^t k(1,t,s_0) + \frac{\beta^{t*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u k(1,u,s_0) \\ \vdots \\ \sum_{t=0}^{t^*} \beta^t k(i,t,s_0) + \frac{\beta^{t*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u k(i,u,s_0) \\ \vdots \\ \sum_{t=0}^{t^*} \beta^t k(|V|,t,s_0) + \frac{\beta^{t*+1}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u k(|V|,u,s_0) \end{bmatrix}, \tag{2.10}$$

where $k(i,t,s_0)$ returns the number of times a signal i is selected at time t (the vector of k_{jt}s).

I show below that if there exists a mean preserving spread of an existing Nash equilibrium, it must be that total value of the seed is concentrated further is some users. This thus establishes a complete lattice on users and one can separate the problem in finding the most influential user $v^{(1)}$ independently of other solutions. Then, one can find the second most influential user $v^{(2)}$ given that the value found for $v^{(1)}$ is part of the solution and so on. The search is thus greatly reduced as there is a lexicographic value of Nash equilibria.

Lemma 2. *Let $\tilde{s}, \tilde{s}'$ be two Nash equilibria on $G(V, E)$ and assume that $\tilde{s}'_0$ is a mean-preserving spread of $\tilde{s}_0$ so that :*

$$\tilde{s}'_{i0} = \tilde{s}_{i0} + \Delta s_{i0},$$
$$\textit{with } 0 = \sum_{i \in V} \Delta s_{i0}.$$

Then, the covariance between the solutions is positive:

$$\sum_{i \in V} \Delta \tilde{s}_{i0} \tilde{s}_{i0} > 0.$$

Proof. See Appendix A.1.6. □

Hence if a solution concentrates the value in some users, any better solution will concentrate more value amongst those users.

In order to prove that the steps described above is indeed an algorithm, I need to define the two key functions that I described informally before. I start by defining the function that builds a Nash equilibrium.

Definition (Choice Function and Choice Value)**.** *Let $G(V, E)$ admit at least one constrained star with a transient time t^* and a period T. Define the convex and compact set*

$$C = \left\{ s_0 : s_0 \in \mathbb{R}^{|V|}, 1 \leq s_{i0} \leq \frac{|V|}{1-\beta} \ \forall i \right\},$$

the choice function $k : C \times \mathbb{N} \to \mathbb{N}^{|V|}$ of G by $k(i, t, s_0) \equiv k_{it}$ and the Choice Value function $h : C \to C$ by equation 2.10.

This allows to show that 2.10 has a fixed point:

Proposition 5 (Fixed Point of Equation 2.10)**.** *Under the same conditions as in the last definition, the function h:*

1. *Admits a fixed point* $\tilde{s}_0$*;*

2. *Is a contraction.*

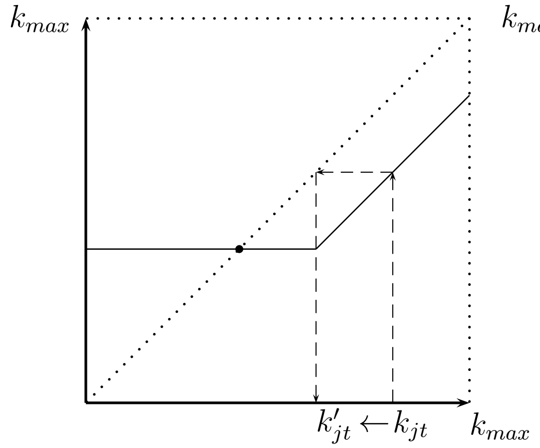
Proof. See Appendix A.1.7 □

The intuition that h is a contraction can be seen in Figure 2.6. That Figure depicts the decision of a user at some given time. This decision is visualized in the numbers k_{it}. On the horizontal axis, there is the current guess made by the company about the number of selections k_{it}. These guess forms the actual signals s_{i0} and given this signal, the user must decide which signal to choose. The solid line represents his decision rule $(\max(s_{it} + c_{mi}, s_{jt} + c_{mj}))$ and the vertical axis represents the actual choice k'_{it} of the user given the current signals.

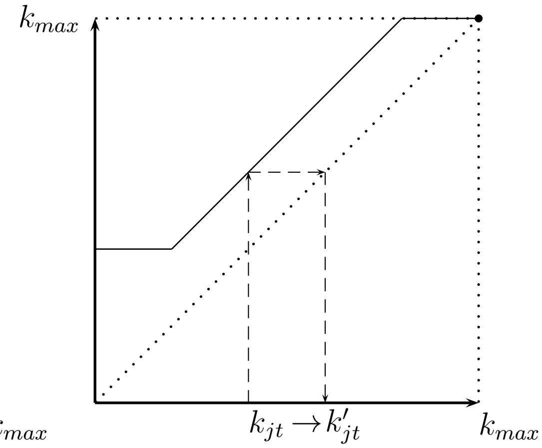
If the current k_{it} is too low, other signals are higher and it does not affect the decision, that is the flatline portion of the solid line. If however k_{it} is high enough, the signal is selected and leads to an increase of k_{it}. Thus, the kink of the solid line represents the threshold point at which the user change his mind.

In Panel 2.6(a), the current guess k_{it} is too high since the it does not induce the user to deliver the selection guessed in the seed. Hence, the signal must be adjusted downward to satisfy first order conditions. This is repeated until k'_{jt} equals the selection, the point where the 45 degree line meets the solid line.

Conversely, Panel 2.6(a) shows what happens when the signal is too low. In this case, the user i is more influential than what the current seed assumes and thus the signal must be increased. If the signal is high enough the user making the decision becomes the unique source and all his friends listen to him. In this case the value k_{it} equals k_{max} in the picture and the fixed point is in the top corner of the picture. Since the function h is a contraction, I can use it to find Nash Equilibra.



(a) **The Signal Is Too High**



(b) **The Signal Is Too Low**

Figure 2.6: The Function h is A Contraction

In the next proposition, I show that the code stops in finite time. I also show that the execution time is of an order of $|V|^5(t^* + T)$.

Proposition 6 (Execution Time)**.** *Consider the algorithm described in the previous pages (formally described in appendix A.4) on a graph $G(V, E)$. Let t^*, T be respectively the transient time and the period. Then:*

1. *Its execution time is finite;*

2. *Its execution time is at most of $O(|V|^5(t^* + T))$ for an exact solution.*

Proof. See Appendix A.1.8. □

2.5 Applications

In this section, I use the model to answer four practical questions. I start with a market valuation for Facebook. Second, I discuss how the model relates to current rules of thumb in network valuation. Third, I show how β influences the company's optimal strategy. Finally, I discuss the valuation of a single user.

2.5.1 On the Market Value of Facebook

The previous sections have defined what a social network sells and how it can be exploited to derive some value. So what remains is to use a model of market equilibrium.

I do this by assuming that Facebook is a monopoly and that its sole source of revenue is selling word-of-mouth advertising. As for the sole source of income, other sources could be included in the description below by simple addition of new terms. This is thus done for simplicity.

I assume there is a demand function $D(p)$ that represents the demand of interested buyers of a word-of-mouth marketing campaign. The demand function is differentiable and negatively sloped in p. The price of the campaign would thus solve:

$$p^* = \arg\max_p D(p)\Pi(s^*(p), p).$$

This function admits an optimum[7] p^*, yielding profits that can then be divided amongst shareholders.

2.5.2 Broad or Targeted Campaigns?

Should any company aiming for word-of-mouth target each user equally or should it focus on one user and give it all the initial information? This paper says that users should receive more importance by the company when they tend to be selected more by other users. So a user's influence is measured as the discounted value of direct and indirect listeners they have.

This means that the way time is discounted has an impact on the optimal solution. If β is close to zero, advertisers prefer fast, direct advertising and do not want to wait for messages to diffuse through a social network. So one can see that when β=0, the optimal solution is the average solution for all users and the variance is zero. The average solution is the only Nash equilibrium, so it is the variance maximizing solution. Everybody is targeted equally with the signal $s_i = p$, regardless of the long-run influence. This is, in some sense, standard advertising.

However, when β approaches one, it is the complete opposite. Notice that in this case, there is at least one original signal that remains in the network after some given time t^*. For the sake of discussion, assume it is the only signal that remains. In this

[7] To see this, notice that profits are increasing in variance and variance is increasing with the price. Hence, profits are increasing in price. One can easily show that profits are bounded and there thus exists at least one p^* in $[0, p_{max}]$ where p_{max} is defined as $D(p_{max}) = 0$.

case, the first-order condition for that signal commands that the signal has the form:

$$s_{j,0} = p \sum_{t=0}^{t^*-1} \beta^t k_{jt} + p \frac{\beta^{t^*}}{1-\beta} |V|.$$

As β goes to one, the value of the last term gets closer and closer to infinity, while the value of all other signals remains constant. So the relative importance of this signal becomes infinite. Relatively speaking, all the initial signaling resources are focused on one user and the social network lets users build the spin.

To sum up, a concentrated signal is a good strategy for a patient company, while it is a poor strategy for a company in a rush. Impatience comes at a price of less exposure over the network and thus, less income and profits.

2.5.3 On Rules of Thumb For Network Valuation

There are two current rules of thumb used by market analysts to evaluate networks, namely Metcalfe's Law and Zipf's law (see [46] for a discussion). Metcalfe's law assumes that the value of a network is based on the potential number of links in the network. So for n users, its value should be of an order of n^2. Zipf's law however assumes that the value of a network should be of $n \log n$.

In the model above, the profits inherit the properties of the cost structure. This is so because the optimal value of signals is bounded by a linear function of users. As such, the profits are bounded by a function that depends on the cost structure. If costs are quadratic, as it is the case in this paper, there is grounds for Metcalfe's law. However, if costs were exponential, Zipf's law could be recovered as an approximation.

Thus, the structure of the cost function seems important to find the correct rule of thumb when it comes to word-of-mouth networks. As income is bounded by a linear function of the users, it seems hard to reconcile theses rules without a look at costs.

2.5.4 What is The Value of A Node?

Imagine the following situation. Before the beginning of an advertising campaign, a user leaves the social network or willingly decides to stay out of the social network during the campaign. Although this can be computed using the profits difference equation in proposition 3, some broad intuitions can be discussed.

There are three possible types of losses that depends on the importance of the user in the network.

First, the loss due to the disconnection of some users to the hub of the constrained star. As these users become disconnected from the main source of exchanges, their long run value vanishes. So if there are n users of that sort, the loss of profits is at least $\frac{n}{2}\frac{\beta^{t^*}p^2}{(1-\beta)^2}$. As there is at least one user leaving, this loss is never zero.

Second, there are some additional losses if the user leaving is highly regarded by others. Since the user leaves, these users must select another source. This changes the income through the term $\sum_{t=1}^{\infty}\beta^t\sum_{j\in V}k_{jt}(b_{ij}+b_i+c_{j,t-1})$ and thus decreases profits. Conversely, if an agent with low perceptions from others leaves, these losses are minimal since users do not listen to him.

Third, if the user leaving is on a critical cycle, it can disconnect a lot of users from an hub as it vanishes. The magnitude of the losses depends on how users will reorganize themselves around another hub (if any). If there is no hub at all, then all the long run value of the network that comes from the periodic regime is lost.

2.6 Some Alternative Specifications of Users' Behaviors

In this section, I explore a different class of behavior of agents, namely what happens if users mix signals to form a new one. I discuss briefly what the first-order conditions look like in that context. I also discuss briefly what happens when perceptions fluctuate over time. I show how this can be used to model business cycles.

2.6.1 Signal Mixing

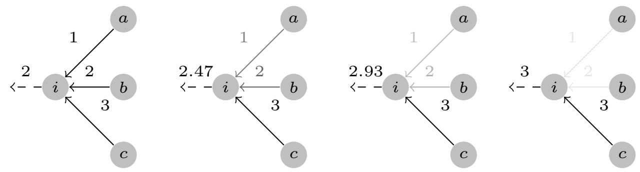
Here, I explore what happens when users mix the signals they receive. Instead of picking the signal they like best, I assume that they adopt a more prudent behavior and use some form of averaging across signals.

Definition (Users (modified))**.** *A user* $\left\{\eta_i, \{b_{ij}\}_{j\in\eta_i}, f\right\}$ *forms a new signal according to their perceptions of neighbors and the signal they get according to the function:*

$$s_{i,t+1} = \left(\frac{1}{|\eta_i|}\sum_{j\in\eta_j} b_{ij} s_{jt}^r\right)^{1/r},$$

where r *is a measure of importance to the highest signal.*

Notice that the modification of the signal according to perceptions depends now on the signal, as perceptions are multiplicative (through b_{ij}). The greatest advantage of this change is that perceptions can now change the sign of the signal. If a_{ij} is greater than one, it amplifies the signal while it is the opposite if it is smaller but positive. If this number is negative, it can also mean the two neighbors disagree on the informativeness of the signal. For instance, one might think a political party has a positive value while another one thinks it has a negative value.



(a) $r = 1$ (b) $r = 5$ (c) $r = 50$ (d) $r \rightarrow \infty$ (max behavior)

Figure 2.7: Weight of Sources In Signal Formation For user i Given Values of r.

The parameter r is a measure of excitement for the highest valued signal. Notice that if r goes to infinity and $b_{ij} = 1, \forall i, j$, the behavior boils down to $s_{i,t+1} = \max_{j \in \eta_i} (s_{ijt})$, which is the essence of the previous model. Users pick the highest signal. If $r = 1$, we have a standard linear model (as in the DeGroot Model) and users average. In Figure 2.7, I illustrate how r influences the selection of the signal mix using shades of gray. The higher is r, the more the highest signal comes in the mix.

Although there might be some grounds to model agents with this behavior, there are two dissatisfying things about it. First, users do not choose anything dynamically: they always average. Second, as all sources remain always employed, the constrained star structure is lost and the model thus loses its empirical match.

Nonetheless, I discuss below what first-order conditions looks like and provide one interesting example based on the fact that users can now disagree on the direction the signal should go.

The following definition introduces the adjacency matrix, which allows to use the standard inner product and linear algebra.

Definition (Adjacency Matrix, Vector of Spins)**.** *Let the adjacency matrix* $B \in \mathbb{R}^{|V|^2}$

be:

$$[B_{ij}] \equiv \begin{cases} b_{ij} & \text{if } j \in \eta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that a square matrix B can be decomposed in an unit-eigenvector/eigenvalue of the form $A = V\Lambda V^{-1}$ matrix form and thus, $B^t = V\Lambda^t V^{-1}$.

That decomposition separates the dispersion effect of the signal (through V) from the amplification effect of the signal (through Λ). The norm of a unit-eigenvector equals one so the interpretation of one of its component is the share of dispersion of the signal to a given user. For instance, if the element v_{ij} is equal to $\sqrt{0.5}$, this can be interpreted as if 50% of the signal in j is shared to i. The value λ_i, the ith element on the diagonal of the matrix of eigenvalues, has also a clear meaning: it tells how much of the signal passing through i is amplified by perceptions, at each period of the model. As the company is interested only in amplification, this is the element of interest. I present below the characteristics of the optimal solution.

Proposition 7. *Given the current behavior, the optimal solution solves:*

$$\begin{aligned} s^*_{i0} &= p\sum_{t=0}^{\infty}\beta^t \sum_{j=1}^{|V|}[B^t]_{ji}\left(\frac{s^*_{i,0}}{s^*_{j,t+1}}\right)^{r-1}, \\ &= p\sum_{t=0}^{\infty}\beta^t \sum_{j=1}^{|V|}\left[V_i\Lambda^t V_j^{-1}\right]\left(\frac{s^*_{i,0}}{s^*_{j,t+1}}\right)^{r-1}. \end{aligned}$$

In particular, if $r = 1$, the solution solves:

$$s^*_0 = pVDV^{-1}\imath$$

where $\imath$ is a vector of ones and $D = \sum_{t=0}^{\infty}\beta^t\Lambda^t$

Proof. See Appendix A.1.9. ∎

The structure of the first-order condition captures the amplification value of users in the network at each period in time. This amplification is weighted by the share of s_{i0} that goes in j at time $t+1$. This weight is also modified by $r-1$, to account for the tastes for the highest-valued signal.

When $r = 1$, the solution has a compact form. The total amplification is given by the adjacency matrix $V'DV^{-1}$ where D is a diagonal matrix with elements $\frac{1}{1-\beta\lambda_i}$, that is the total discounted value of amplification for a given user i.

An Example: Two Opposing Parties

In this section, I use the fact that spin can change the sign of the signal to show that reverse-psychology can be an effective word-of-mouth strategy. Consider the reduced form of members of two political parties. One party seeks the best message given that other members will always say the opposite. The network structure can then be characterized by the adjacency matrix:

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

with $b = 0$. I show in Appendix A.3, that if $r-1$ is odd, the optimal solution solves:

$$(s^*_{2,0})^{r-1} = \frac{p}{1-\beta^5}\left[1 - \beta + \beta^2 + \beta^3 + \beta^4\right](s^*_{1,0})^{r-2},$$
$$(s^*_{1,0})^{r-1} = \frac{p}{1-\beta^5}\left[-1 - \beta - \beta^2 - \beta^3 + \beta^4\right](s^*_{2,0})^{r-2},$$

and notice that if β is small, this can lead to a negative value for $s^*_{1,0}$. This means that reverse-psychology, initial values saying the opposite of what the party believes in, can be part of an effective word-of-mouth strategy if there are users who share the opposite of the message they receive.

2.6.2 A Network Foundation of Business Cycles

In this section, I illustrate that the company has incentives to deviate from a path where users agree on the signal over time. I do this by modifying the model to allow users to modify their net perceptions b_{ij} based on the past values of the signal they received. I do this in a simple fashion by assuming users react to a simple quadratic formulation of their preferences for information they receive and information they prefer (measured by ϕ):

$$\begin{aligned}
s_{i,t+1} &= \max_{j \in \eta_i}(s_{jt} + b_{ij}(s)), \\
b_{ij}(s) &= \arg\min_{a_{ij}} -(1-\theta)\frac{(s_{jt} + b_{ij}(s) - \phi)^2}{2} - \theta\frac{(s_{jt} + b_{ij}(s) - s_{j,t-1})^2}{2}, \\
&= (1-\theta)\phi + \theta s_{j,t-1} - s_{jt} \\
\Rightarrow s_{i,t+1} &= (1-\theta)\phi + \theta \max_{j \in \eta_i}(s_{j,t-1})
\end{aligned}$$

This modification of perceptions is simple to interpret. The user tries to balance the perceptions about each source between what he thinks is the right value for the signal and by how much the signal has changed from this source over time. Here, there is the assumption that if the signal changes over time, the source j is trying to oversell and thus, the perceptions will decrease the signal.

As all users have the same intrinsic preference for information, ϕ, it should be intuitive that the long-run value of the signal would converge to it. I characterize this equilibrium below:

Definition (Perfect Neighborhood Foresight)**.** *A Perfect Neighbor Foresight behavior is when* $a_{ij}(s_t) = 0 \ \forall i, j$.

In words, a perfect neighbor foresight is when all users receive the same signal and they think this value is correct. They thus form neutral perceptions about each user.

Proposition 8 (Perfect Neighborhood Foresight Regime)**.** *If users have a perfect neighborhood foresight, then:*

1. *The social network value is 0 ($\lambda = 0$);*

2. *Users are indifferent about their sources of information;*

3. *The signal shared over the network is ϕ for all users.*

Proof. See Appendix A.1.10 □

In a perfect neighborhood foresight regime, the structure of the network has no value as everybody agrees and shares the same signal over and over. Hence, any source is as good as others.

It should be however intuitive that, this is suboptimal from the company's perspective. If one user has a signal higher than others, some neighbors will choose it and this will increase the marketable value of the signal. In the long run, however, the signal will come back to ϕ, in a perfect neighborhood foresight regime. When there is no perfect neighbor foresight, there is a viral spin in the network and the principal can exploit it to derive some value. I formalize this in the following proposition.

Proposition 9 (Perfect Neighbor Foresight Regime Suboptimality)**.** *If $\theta \neq 1$, a Perfect Neighbor Foresight Equilibrium is suboptimal from the company's perspective.*

Proof. It is sufficient to notice that if $\theta \neq 1$, there exists a mean preserving spread of $[\phi, \ldots, \phi]'$ that can increase the company's profits. □

In other words, if there is some periodic regime where users agree on the signal to be shared over the network, the company has incentives to change the value of the signal to actually generate some cycle that will slowly converge back to the original equilibrium.

In this example, the change in perceptions is deliberately made simple to show that the variance maximizing component applies to changes in perceptions. However, some additions would be needed to make this interesting in terms of business cycles. So instead of generic signals, users could share valuations about the future through a strictly convex valuation function $V_{it}(\tilde{k})$. They would thus seek the best valuation given the ones they received from their neighbors trough the process:

$$V_{i,t+1} = \max_{u(k)+V_{j,t}(k)\in\eta_i} u(\tilde{k}(j)^*) + \beta V_{j,t}(\tilde{k}(j)^*)$$

$$\tilde{k}_j^* \equiv \arg\max_{\tilde{k}} u(\tilde{k}) + \beta V_{jt}(\tilde{k}).$$

Hence, what users would send are valuation functions (what they think is the best way to value an asset). In this case, the simple fact that valuations take time to travel from one agent to another could generate substantial persistence over the aggregate economy with simple technological shocks.

2.7 Conclusion

In this chapter, I characterize the optimal solution to a word-of-mouth campaign over a social network. I do so in an environment where a company can seed information to users who choose afterwards which information they share with their neighbors. The profit maximizing solution is a unique Nash equilibrium. I show that amongst all sustainable Nash equilibria, the optimal one maximizes concentration amongst a small set of users. Hence, concentration of the signal to a small group of users is good as the network replicates the information for free.

I also show that after a transition period, the word-of-mouth dynamics enters a periodic regime. Users on the network listen to the same sources over and over. Their uncoordinated actions are such that the social network becomes the most efficient structure to carry information given the possible sources available to users, namely a constrained star structure. The perceptions amongst users is a key element in determining who is at the center of this star of influence.

I present an algorithm that can compute the optimal solution in polynomial time on any network topology. Hence the model provides a tractable solution on large scale networks.

In particular, I show how this model can be used to price Facebook. If analysts measure the value of a social network company like Facebook solely on the value of direct advertising, they underestimate the value of the information the company has as it simply neglects the value of word-of-mouth.

Chapter 3

Some Notions of Public Economics on Networks

3.1 Introduction

Word-of-mouth communication is a topic widely discussed when it comes the economics of social networks. Some papers cover the endogenous formation of communications networks (Galeotti, Ghiglino & Squintani 2009), others are interested in harnessing the process from a marketing or sales perspective (Godes & Mayzlin, 2001), while others examine the impact of word-of-mouth on the behavior of agents on the network (Schiraldi & Liu, 2010). However, very little is said on contagion from the standpoint of public economics.

In this chapter, I use the same model of word-of-mouth communication I developed in the previous chapter to discuss network spillovers and the notion of social value of a network. I show how these concepts come in play in word-of-mouth communication and sometimes fuel the demand for such type of advertising. The key driver is that ideas and discussions are non-rivalrous. If an idea is disseminated to a neighborhood of friends, nothing prevents these friends to reproduce it at will. Since this effort of reproduction is in essence the action of the network agents, it comes for free for an advertiser. As such, a network is a technology that allows for the reproduction of ideas for free.

Anyone with a Twitter or Facebook account can see this process of information reproduction at work. Topics range from the latest Justin Bieber song to information about earthquakes (Sakaki, Okazaki & Matsuo, 2010). For instance, roughly 6000 Justin Bieber fans showed-up on a rumor of a concert spread on twitter (see Ryland 2012). In short, seeding one idea on the network might go a long way through a process of replication: that is, in essence, the contagion spillover.

The natural idea, from a sales point of view, is to try to exploit this phenomenon to increase sales. For instance, a company might want to target a group of users and let them evaluate a product to generate some word-of-mouth advertising. By selecting the right group, the company can use the network to generate a cascade of positive recommendations that can increase sales. This cascade of adoption might be caused by herding (as in Banerjee, 1992), by an informative signal about the quality of the product (Cambell 2005, Vettas 1997) or simply be the result of truthful communication (Galeotti and Goyal, 2009). The consistent feature of these models is that regardless of the *motives* of the agents on the network, or their behavior (Bayesian updating, randomly talking, or being fooled), they do *communicate* with each other. If what drives this communication is known to the company, it can be exploited to increase profits. The company, however, profits from such communication for free, and this is where the free rider problem occurs. I show below under which conditions this problem is important and how it might be solved.

If conversations between agents influence how consumers evaluate or perceive a product of the company, the company will try to exploit these perceptions as well. In this case, the process is not only about disseminating an idea, but also about ensuring the right conversations occur to generate a positive value. However, agents will discuss ideas even if the company performs no action. The conversation might be different in nature, but it can be taken as granted that some conversation happens. Such communication has some value and is, in essence, the social value generated by the network. Because this exchange of information does not depend on the actions of the company, it cannot be captured in any fashion. If the discussion is about the quality of the company's product, this conversation has certainly some value.

In section 3.2, I discuss the existing literature on network externalities. In section 3.3, I present again the model of word-of-mouth communication over a social network. In sections 3.4 and 3.5, I exploit the model to discuss its implications from a standpoint of externalities and social value. A brief conclusion follows.

3.2 Literature

To my knowledge, there are no academic articles on the social value of word-of-mouth communication, nor on the externalities it induces. Some of the academic articles dealing with network externalities refer to the adoption of a technology (see Katz & Shapiro, 1985 or Farrell & Saloner for an exposition). This area of literature focuses on the adoption of a product, where the value of the product depends on the number of people using it. Such articles are however of little help in the context of word of mouth because the externality in word-of-mouth is a consequence of discussion rather than the value of the product. The latter can have a value independent of the number of adopters but still lead to word-of-mouth communication. In the context of communication, the contagion effect stems from non-rivalry[1] of ideas rather than the adoption of the idea itself.

However, there are parallels to be made between word-of-mouth communication and the congestion literature (see Kelly, 2008 for a good introduction). In discussing about a product, agents on a network let the information produced circulate for free. As such, companies profit from this information flow although they bear no cost in the process of link formation. Such problem of flow has been studied in the design of information network or roads. In such cases, the externality is negative: an increase in the flow (traffic) leads to longer waiting times. The standard approach to address this externality is to introduce a toll, which influences road users to take the socially optimal route.

In the context of word-of-mouth communication, the externality is positive if the product being discussed is valuable: a firm can profit from the process of link creation between agents on a network to generate a contagion of ideas, or spillovers. Hence, the externality is different in nature. The idea of tolls can however still be applied to show how agents can address the free rider problem. This is discussed further below.

[1] A product is non-rival if its consumption does not prevent somebody else from consuming it as well. An idea has such a property.

3.3 Model

The Network

This section contains the same model description as in chapter two. Readers can easily skip this section if they read the previous chapter.

There is a directed network $G(V, E)$ with $|V|$ agents linked together by a set of $|E|$ directed edges. A particular agent is denoted by an index $i \in V$, while a link between agents $i, j \in V$ is denoted by e_{ij}. Since edges are directed, e_{ij} represents the direction of information traveling from $j \rightarrow i$. An example is provided in Figure 3.1. The set $V = \{0, 1, 2, 3, 4, 5, 6\}$ contains seven agents linked by the set of edges $E = \{e_{01}, e_{20}, e_{50}, e_{30}, e_{13}, e_{41}, e_{52}, e_{53}, e_{36}, e_{46}, e_{65}\}$.

This network topology is assumed to be fixed and the outcome of a network formation process. Various economic models can explain link formation. Some assume the formation is a consequence of random draws of a given distribution (see the Erdös-Renyi framework in Jackson, 2009) or a random rewiring of edges of a known network structure (see Watts and Strogratz, 1998). There are other processes of link formation based on the allocation of scarce resources in building costly links between agents. These allocations might be based on a trade-off between personal biases on the state of the world and the willingness to learn the truth (Galeotti, Ghiglino & Squintani 2009), or *homophily*, the tendency for people with similar preferences to be linked together (Boucher, 2012). Whatever the reason for the form of the network, it is taken as granted. For the discussion about the externality, it will be useful to assume that link formation is costly. This can be thought of as the cost to maintain a relationship or the additional time required to search through a larger set of information.

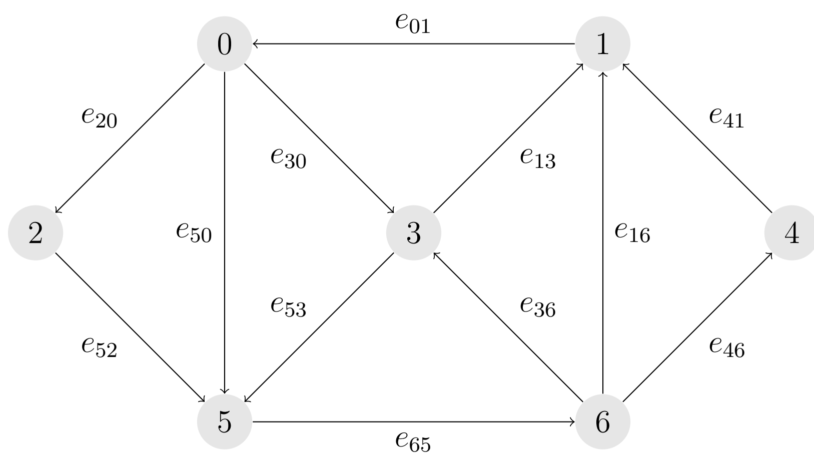


Figure 3.1: An Example of A Network

As in the previous chapter, a convenient definition is the notion of the neighborhood of sources of an agent i: η_i. This represents the set of agents from which agent i can choose to listen to. For example, in Figure 3.1, $\eta_1 = \{3, 4, 6\}$ since agent 1 has agents 3, 4 and 6 as sources of information. Agents exchange a signal $s_{it} \in \mathbb{R}$ on the network. This signal can take on various meanings: it can represent the quality of a product, the level of (dis)utility of a message or simply a truthful message about the state of the world. In all cases, the signal has some value to the agents and to the company. I give some examples below. This signal might change in time and might be different from one agent to another. I will denote $s_t \in \mathbb{R}^{|V|}$ the vector of signals distributed by each agent at time t.

3.3.1 The Agents

The next step is to define the behavior of agents regarding the information they have. Agents are mathematically modeled as three objects: a neighborhood η_i, a set of personal perceptions $b_{ji} \in \mathbb{R}$, or biases, and a choice function $f : \{s_j : j \in \eta_i\} \to s_j^*$. The notion of neighborhood has already been discussed: these are the possible sources of information for a given agent. When these are specified for every agent, the network structure is fully known. The notion of personal perception pertains to how agent i values information received from agent j. The higher the value of b_{ji}, the more highly agent i thinks of agent j. One way to think of this is personal preferences over sources. These numbers could change over time, as with a dynamic reputation function, but I hold them fixed for simplicity. The choice function represents how an agent will choose a source of information given its perceptions about other agents and the signals it receives. In this chapter, I will use the following model of agents :

$$s_{it+1} = \max_{j \in \eta_i} (s_{jt} + b_{ij}) \tag{3.1}$$

The motivation for this choice stems from numerous examples:

1. Agents learn about the quality of a product only from their neighbors (as in Campbell, 2012). Such quality is uncertain and can be high or low. They have a unit demand for it when the quality is high and no demand for it when it is low. Neighbors can observe the quality of the product when someone in the neighborhood has bought it. Hence, the vector of signals can be thought of as a binary vector where each element s_{it} is either one or zero. When the value is one, the signal means "this Neighbor conveys information: the product is of high quality," while when it is zero it means "this Neighbor does not convey information: it has not bought the product." In this context, the signalling process $s_{it+1} = \max_{j \in \eta_i}(s_{jt} + 0)$ represents the evolution of the beliefs (and the demand) about the product over time.

2. The information can be thought as a vector v drawn from a vector space of concepts. These concepts might be “liberty” and “equality” (a two-dimensional vector space) and a particular vector represents a level of support for those concepts. Agents have some given preferences $\tilde{v}$ over those concepts (e.g. what they believes the right level of liberty and equality) and they weight each concept according to a diagonal matrix W, each element of the diagonal being a weight for a concept. The distance function $(v - \tilde{v})'W(v - \tilde{v})$ represents a measure of homophily, or the distance between the message and what the agent prefers (see Currarini, Jackson & Pin for further discussion on homophily). Hence, an agent chooses its messages based on $s_{it+1} = \max_{j \in \eta_i}(-(v_j - \tilde{v})'W(v_j - \tilde{v}) + b_{ij})$. That is, they choose the message that is the closest to their own preferences, given the bias they have for individuals (b_{ij}).

3. The information can simply be the perceived quality of information. Hence, s_{jt} represents the degree of information communicated by agent j and b_{ij} represents the correction agent i applies to agent j's signal. Once the signal is corrected for perceptions, agent i sends his own most informative signal: $s_{it+1} = \max_{j \in \eta_i}(s_{jt} + b_{ij})$.

This behavior is sufficient to discuss the economic nature of word-of-mouth communication over a network. There is also some practical reasons for choosing this functional form.

First, it is tractable on large-scale networks. Social networks are, by design, large. Facebook has 900 million users, Twitter has 500 million users, and both are growing. The number of links between these agents is much larger. Although companies or scientists might be interested in subsets of these networks, they *still* represent large objects. Thus, the models used to represent them must be tractable to provide meaningful answers to practitioners. The model above can be described as a linear system, very similar to the linear system of equations $Bs_t = s_{t+1}$ in what is called a *tropical algebra* (or max-plus semirings). The solution can thus be found while avoiding the curse of dimensionality (as detailed in the previous chapter).

Second, the functional form $\max_{i \in \eta_j}(s_{it} + b_{ji})$ can be ported directly in an econometric setup. The numbers b_{ij} can be thought as a reduced form of a sum of characteristics ($X_i\beta$) and a type I extreme error term. This representation then leads to logistic regressions. The model thus has a probabilistic counterpart that can be estimated.

All of the examples above share the same constraint: information only flows through the network structure. In other words, agents are myopic; an agent cannot get more information than what his sources tell him. In the discussion on externalities, I will show that this assumption is critical for the firm to "free-ride" on the network structure. If agents are able to understand how critical they are to the spillover effect, they can solve the free rider problem by forming coalitions. This would, however, require a knowledge of the network structure.

3.3.2 The Company

There is one company that seeks to have information distributed about its product. It derives value from the fact that consumers are informed by word of mouth. I assume the company discounts time at the rate $\beta \in [0, 1)$, meaning that messages received earlier are better. Finally, I assume that the number of periods T is large enough to let the information reach all agents on the network. For simplicity, I assume that the number of periods is infinite, although this assumption does not affect the discussion. Hence, the company values :

$$I(s_0) = p \sum_{t=0}^{\infty} \beta^t \sum_{i \in V} s_{it}, \tag{3.2}$$

where p is some exogenous price representing the market value for word-of-mouth exposure. This formula captures, in essence, the value derived from the word-of-mouth discussion about the company's product. For instance, the company could value $h(s_{it})$ where h is some probability distribution of buying, given the signal. However, the main idea is embodied in I: more information is better, and sooner is better. This assumes that the information reveals something positive about the product. As noted by Goyal and Galeotti (2009), if agents perceive the product to be of poor quality, communication is harmful from the firm's standpoint. So it is assumed throughout the chapter that signals convey positive information about a product.

The firm prefers word-of-mouth advertising to standard advertising because information that comes from acquaintances has more impact than direct advertising (Bond et al., 2012). Hence, it seeks to "seed" its advertising message only once, to a particular group of users on the network, and allow word-of-mouth communication to disseminate its message further. The firm can choose the seed s_0 from a compact strategy space S. For each $s_0 \in S$, the company must pay a cost $c(s_0)$. This cost could represent what has to be paid to convince the seeded user to disseminate the information, or simply the cost of producing the signal. Thus, the firm seeks to maximize:

$$s_0^* = \arg\max_{s_0 \in S} I(s_0) - c(s_0)$$
$$\text{s.t.} s_{it+1} = \max_{j \in \eta_i}(s_{jt} + b_{ij}) \ \forall i \in V \tag{3.3}$$

Hence, the firm seeks to maximize exposure given the reaction function of agents. To stick with the examples above, this problem might represent:

1. A firm that sells a product of good quality that wants to get the word out through word of mouth. It must use a "money burning" signal to seed some initial agents on the network while other agents learn quality through their neighbors. The strategy space is the set of binary vectors with only $k \leq |V|$ seeds in it. The cost of any vector is the sum of its components.

2. A political party that seeks to design a popular message with the name of its candidate in it. It thus seeks the optimal vector v^* that will maximize diffusion given the homophilic preferences of agents. The cost of producing this vector could be linear in the distance to zero: $c(s_0) = v'v$ and the strategy space is defined implicitly as any v^* such that profits are non-negative.

3. A firm wants to exploit the perceptions of agents on the network to *spin* the perceived quality of a product. It might also seek herding to increase sales. The cost of producing this signal is quadratic in quality $\left(c(s_0) = \sum_{i \in V} \frac{s_{i0}^2}{2}\right)$ and the strategy space is defined implicitly by the set of vectors with positive profits.

The following statement can then be proven:

Theorem 2. *Given a graph $G(V, E)$ and the behavior of agents described in (3.1), then (3.3) admits at least one optimal solution $s_0^*(G)$.*

Proof. It is sufficient to notice that $I(s_0)$ is continuous in s_0 and that S is a compact set. It therefore has at least one maximum. □

In the next sections, I will work with $s_0^*(G)$ and $I(s_0^*(G))$ to discuss the relevant concepts.

In this chapter, I leave aside all the technical considerations to *find* this solution. For instance, the first example is known to be the set-covering problem (see Cormen, 2009), which is cursed with dimensionality (NP-hard). For a generic value of k, it is intractable on large-scale networks[2].

3.4 The Contagion Spillovers

3.4.1 The Baseline Case

In this section, I assume that the personal perceptions about other agents (b_{ij}) are equal to zero. As I am more interested in explaining the spillover effect and the externalities rather than a particular form of solution, I proceed without these perceptions to simplify the exposition. Much of the analysis remains the same when this is not the case.

As a baseline, I first use the case where $E = \emptyset$. In this case, there is no word-of-mouth discussion at all on the network, as agents are isolated. In this case, the firm's decision is quite simple: it will invest in seeding a node if it can derive some profit out of this note (if $s_i > c(s_i)$ for some $s_i \in S$). If agents are identical, this condition will be the only one. This leads to the following lemma:

[2] It is, however, tractable when $k = 1$, which is the optimal solution to the problem if β is "close enough" to one.

Lemma 3. *If $E = \emptyset$, the company seeds the network only if $s_i > c(s_i)$ for any agent.*

As there is no transmission of the signal over the network, this can be thought of as the company informing a consumer directly about the product. As the company pays all costs of seeding, the firm bears the whole cost of advertising. There is little to discuss about this case, but I use it as a benchmark in the following section.

3.4.2 Word-of-Mouth Externalities and Spillovers

When $E \neq \emptyset$, the company can exploit the network structure to increase profits. If producing links is costless to agents, the increase in profits is done at no expense. Hence, all the costs of producing the signal is captured by the firm and the additional profits over the benchmark case are a result of a better technology (a spillover). However, if producing links is costly to agents, the firm profits from these links at the expense of those who paid for it. There is a free rider problem.

To discuss this formally, I start by defining how the solution can be expressed in terms of choices of agents:

Definition (Choices)**.** *Let $k_{jt} \in \mathbb{N}$ denote the number of times the signal $s^*_{j,0}$ is chosen as the optimal signal by agents at time t, given their reaction function f. Then, for any signal $s^*_{j,0}$, the value generated is given by:*

$$I(s^*_{j0}) = p \sum_{t=0}^{\infty} \beta^t k_{jt} s^*_{j,0} \;\forall j.$$

The optimal solution thus accounts for the discounted number of times each seed j is being selected. In the baseline case, the signal is selected only once in time $t = 0$ and never again, since there is no network. But when the network has some edges, it takes different values in time $t > 0$. What this shows is that there are now spillovers associated with the same seed and there is thus some latitude in increasing profits. I formalize this idea in the proposition below:

Proposition 10 (Profits non-decreasing in links)**.** *Let $G(V,E)$ be a graph with its associated profits $\Pi(s_0^*(G))$ and consider $G'(V',E')$ such that $V'=V, E\subset E'$. Then, $\Pi(s_0^*(G'))\geq\Pi(s_0^*(G))$.*

Proof. It is sufficient to notice that $\Pi(s_0^*(G))$ is still available under G' and thus, adding links can only have a non-decreasing effect on the optimum. □

This proposition simply states that the company can increase its reach when there are more links for word-of-mouth discussion. The seed of one agent has a spillover effect from the agent to some neighbors who choose his signal and thus, increases exposure.

There is actually more that can be said if one additional assumption is made on the network topology. If one assumes that there is at least one cycle[3] on the network, then some signals never vanishes and over time all the branches connected to this cycle share the signals on the cycle. I call that structure a *constrained star network*. Hence, if a network can be partitioned in a sub-network containing constrained stars, the value of the signal never vanishes. Since the max operator ensures that signals are strictly increasing at any period, one can rewrite the Income in the following fashion:

$$I(s_0^*) = \underbrace{p\sum_{i\in V} s_{i0}^*}_{\text{Value of the seed}} + \underbrace{p\frac{\beta}{1-\beta}\sum_{i\in V} s_{i0}^*}_{\text{Long-run value of the seed}} + \underbrace{p\sum_{t=0}^{\infty}\beta^t\sum_{j\in V}(k_{jt}-1)s_{j,0}^*}_{\text{Value of non-rivalry}} \tag{3.4}$$

The first term is simply the value of the seed. The second term measures the value of the long-run value of the seed. The interpretation is that the firm would get this

[3] A cycle is a repetitionless path of edges whose starting point is also its endpoint. For instance, in Figure 3.1, the path e_{30}, e_{13}, e_{01} is a cycle.

value if agents repeated the signal they have to themselves. As any agent is connected to a cycle which repeats the signal periodically, such value is always there. So this second term measures the pure time effect of the network. The last term measures the non-rival effect. That is, the fact that agent can freely choose the information they have at their disposal. Since they pick the best available information through the max operator, this value is always positive. Hence, this last term measures the value of the non-rival aspect of information in time.

For instance, when the production costs of the firm are quadratic, the optimal solution s^*_{i0} satisfies:

$$s^*_{i0} = p \sum_{t=0}^{\infty} \beta^t k_{it}$$

and notice that since all agents make at least one choice at each period, we have that:

$$\sum_{i \in V} s^*_{i0} = \frac{p}{1-\beta}|V|$$

which is independent of the network structure. Hence, equation 3.4 becomes:

$$I(s^*_0) = \frac{p|V|}{(1-\beta)^2} + \underbrace{p \sum_{t=0}^{\infty} \beta^t \sum_{j \in V} (k_{jt} - 1) s^*_{j0}}_{\text{Value of non-rivalry}}$$

In other words, when production costs are quadratic, the only thing that distinguishes two networks is their ability to increase the signal through the non-rival effect.

When the cost of links are supported by agents, the company profits from a positive externality. The solution to this problem is for agents to introduce a toll on a given link. The following two propositions show that this solves the free rider problem.

Lemma 4. *Let $G(V, E)$ be a graph and C be a coalition of agents able to remove a set of links $e(C) \subseteq E$, generating a graph $G'(V, E \setminus e(C))$. Then, the maximum toll* *the coalition can charge for the links in $e(C)$ is given by:*

$$T(C) = \Pi(s_0^*(G)) - \Pi(s_0^*(G')) \geq 0$$

Proof. The proof is straightforward: with this toll, the company is indifferent between the links being removed on the network or paying the toll, as it leads to the same payoff. If the payoff is bigger, the company will prefer to avoid paying the toll. If it is lower, the agents make less profit. □

The next proposition summarizes the result:

Proposition 11. *Let $C_1, C_2 \dots C_i, \dots C_k$ be a sequence of coalitions covering E. Consider the sequence of graphs generated by the sequence:*

$$G(V, E), G(V, E \setminus e(C_1)), G(V, E \setminus e(C_1) \cup e(C_2)) \dots G\left(V, E \setminus \bigcup_{j=1}^{i} e(C_i)\right) \dots G(V, \emptyset).$$

Then, any sequence of coalitions solves the free rider problem.

Proof. For simplicity, let G_i the graph $G\left(V, E \setminus \bigcup_{j=1}^{i} e(C_i)\right)$. Then, by the previous lemma, the toll charged by C_i is given by:

$$T(C_i) = \Pi(s_0^*(G_i)) - \Pi(s_0^*(G_{i-1})).$$

In particular, for $i = k$, the toll is given by:

$$T(C_k) = \Pi(s_0^*(G_i)) - \Pi(s_0^*(G(V, \emptyset)),$$

that is, all the additional value above the baseline case. Summing over all tolls yield:

$$\begin{aligned}\sum_{i=1}^{k} T(C_i) &= \sum_{i=1}^{k} \Pi(s_0^*(G_i)) - \Pi(s_0^*(G_{i-1})) \\ &= \Pi(s_0^*(G(V, E)) - \Pi(s_0^*(G(V, \emptyset)),\end{aligned}$$

which is the whole spillover effect. □

If agents are able to form coalitions and charge a toll, they can sequentially break-up the spillover effect induced by the word-of-mouth on the network. They are aware of their word-of-mouth value and threaten the company not to share anything if they do not get paid. When this occurs, the company becomes indifferent between using the network and opting for the baseline case as the efforts of coalitions capture all the additional profits. As agents charge a number arbitrarily close to the maximal toll, they induce the company to use the network and capture the effect. In this case, the free rider problem no longer occurs. What this also means is that word-of-mouth advertising is a profitable strategy in an environment where agents have no means to form coalitions or if the efforts required to form a coalition are too high.

For instance, professional news organizations might have the means to cooperate and thus internalize some of the free-riding (convergence), but blogs might find it more profitable to let the traffic flow through their websites rather than spend efforts to organize.

I do not address how these coalitions could be formed, if at all, and what their motives could be. In particular, I have said nothing on how the structure (or the order) of these coalitions can influence the distribution of the profits among agents. These are potential areas for future research.

3.5 The Social Value of Word-of-Mouth Advertising

A company might have some interest in shaping messages about its product on a social network. In doing so, the firm hopes to use the network to its own advantage not only to increase diffusion, but also to change the perceived quality of the product. This can be done only if the network performs some active modifications of the "seed" the initial message the advertiser sends to certain agents on the network. If the seed is shared from one user to another without modifications, the signals remain identical to at least one seed, so the considerations are about diffusion and the "survival of the biggest".

Thus, in order to have a meaningful discussion, I assume that the perceptions about other users are no longer zero (i.e. $b_{ij} \neq 0$). This means that although one user j might send a signal of zero value about the product, a source i thinks j's valuation of the product is incorrect, so i corrects the signal to what i believes to be the true value, $0 + b_{ij}$. These perceptions can be positive or negative. The main assumption behind these numbers is that how we see our sources of information influences our perceptions of the products. Again, these numbers need not be fixed, but are kept so in the context of this thesis for tractability.

This introduces an additional dimension in the firm's optimization problem: there is now a trade-off between dispersion and credibility. One agent might have a high number of sources but his sources might think poorly of him. So by introducing heterogenous b_{ij}s in the model, the firm changes his optimal solution s_0^* to exploit these perceptions optimally.

In the context of this chapter, the interesting part is what happens when the company does not seed any message to the network (or if they seed a zero vector). This solution might not be optimal, but shows what happens in terms of valuation. After one period, every agent i chooses the message from the most highly perceived source in his neighborhood and shares it with his neighbors:

$$s_{i,t+1} = \max_{j \in \eta_i}(0 + b_{ij}).$$

In the next period, the signals, or the valuations about the product, will be changed by the perceived value between agents. Summing these numbers over time according to the income function I will yield the "natural" value generated by the network. I define this as the social value of the product:

Definition (Social Value)**.** *Let $G(V, E)$ be a network of agents responding to signals according to (3.1) and let $\vec{0} \in \mathbb{R}^{|V|}$ be the zero vector. Then, the social value of the product on the network is given by $I(\vec{0})$.*

By construction, this value is independent of the actions of the company. It therefore cannot be captured or internalized. To characterize the social value, a bit of investment in the mathematics of difference equations is required. I will state in the theorem below and refer to a treaty on tropical algebras for the proof.

Theorem 3. *Let $G(V, E)$ be a graph of agents and let $G(s_t) = s_{t+1}$ be the law of motion induced by the behavior of agents summarized in (3.1). Assume that this graph as at least one cycle. Then:*

1. *There exists a time $t^* < \infty$, a vector $\tilde{s}$ and constant $\lambda \in \mathbb{R}$ such that $G(\tilde{s}) = \tilde{s} + \lambda \imath$ for all $t > t^*$, where $\imath$ is the unit vector.*

2. *Denote the composition function by:*

$$G^n(s) = \underbrace{G \circ G \circ \cdots \circ G(s)}_{n \text{ times}},$$

then $\tilde{s}$ is given by $G^{t^*}(\vec{0}) - \lambda \imath t^*$.

3. *Consider the average value of perceptions b_{ij} over a given cycle on the network*[4]. *Then λ is the highest average amongst all cycles in the graph.*

Proof. See Bacelli, Cohen, Olsder, Geert and Quadrat, 1992 . □

This theorem explains the dynamics of signal selection over the network when the zero vector is seeded to agents. After a given finite time t^*, the network enters a periodic regime where agents choose the same sources of information over and over. As such, the vector of signals remains identical, up to an increase of λ. This value λ is given by the highest average perception of agents that lie on a cycle.

What this means is that after t^*, highly perceived agents are responsible for the dynamics of the network. Agents on the cycle listen to each other, generating at every step an increase in the perceived value of λ. As they are the most regarded, all other agents listen to them either directly by choosing them as a source, or indirectly by choosing a source that listens to them. After t^*, they generate the whole stream of value. This theorem allows for the following statement:

Proposition 12 (Social value)**.** *Let $G(V,E)$ be graph that admits at least one cycle of agents and let agents behave as in (3.1). Then, the social value can be decomposed in the following three terms:*

$$I(\vec{0}) = \underbrace{p\frac{\beta^{t^*+1}}{1-\beta}\sum_{i\in V}\tilde{s}_i}_{\text{Long-run value}} + \underbrace{p\lambda\frac{\beta|V|}{(1-\beta)^2}}_{\text{Opinion leaders' value}} + \underbrace{p\sum_{t=0}^{t^*}\beta^t\left(\sum_{i\in V}\left[G^t(\vec{0})\right]_i - \lambda t\right)}_{\text{Short-run value}} \tag{3.5}$$

Proof. After t^*, the vector is given by:

$$p\sum_{t=t^*+1}^{\infty}\beta^t\sum_{i\in V}(\tilde{s}_i+\lambda t) = p\frac{\beta^{t^*+1}}{1-\beta}\sum_{i\in V}\tilde{s}_i + p\lambda|V|\sum_{t=t^*+1}^{\infty}\beta^t t.$$

[4]Using the example in the last footnote, the average value if the cycle $\{e_{30}, e_{13}, e_{10}\}$ is given by $(b_{30}+b_{13}+b_{10})/3$.

Before t^*, it is sufficient to substract λt from the summation to obtain:

$$p\sum_{t=0}^{t^*}\beta^t\sum_{i\in V}\left[G^t(\vec{0})\right]_i = p\lambda|V|\sum_{t=0}^{t^*}\beta^t t + p\sum_{t=0}^{t^*}\beta^t\left(\sum_{i\in V}\left[G^t(\vec{0})\right]_i - \lambda t\right)$$

Combining the two elements leads to the desired answer. □

The last proposition shows that the social value of word-of-mouth advertising on the network can be decomposed into three components. The first is the long-run value of the choices induced by perceptions. As explained in the previous paragraphs, these choices reflect direct or indirect connections to the cycle of highly perceived agents. It measures the value of this choice, net of the increase λ at each period.

The second term is the value of the influence of highly perceived agents. As they generate an increase of λ at every step of the word-of-mouth process, it is measured as the discounted value of this increase.

The last term pertains to the transitionary dynamics from $t = 0$ to t^* and is explicitly dependent on the structure of the network. This is the short-run social value of the work-of-mouth advertising. This also illustrates that time plays a role in determining the value. If the number of periods to allow word-of-mouth communication is smaller than t^*, then only the last term and the relevant share of the second term matters in the social value.

It also shows that depending on how the company values time (through β), some components might be more valuable than others. In particular, if β is close to one, the value of influential persons is much higher, through the factor $(1-\beta)^{-2}$. Intuitively, all agents connect to these agents in the long run, so if the company is patient, they are the relevant persons to influence the discussion about the product. There are however two exceptions that I discuss in the following proposition.

Corrolary 1. *Let $G(V, E)$ be a graph with agents behaving as in (3.1). Then:*

1. *If the highest average on all cycles is zero $\lambda = 0$, the value of opinion leaders is zero.*

2. *If there is no cycle on the graph, then there is no long-run value of word-of-mouth.*

Proof. Both points are direct consequences of the previous theorem. □

One can substitute $\lambda = 0$ in the previous formula to find the result. As for the second point, the intuition is that if there is no way information can get back to its original starting point, there is no way a fixed point of the form $G(s) = s + \lambda$ can occur.

Hence, letting agents discuss for a long enough period is critical in the creation of social value. If there is no cycle, the flow of information on the network stops and the long-run value is lost. This is relevant in situations where agents do not repeat information they have already shared.

3.6 Conclusion

I have discussed how the notion of externalities and social value apply in the context of word-of-mouth communication. I have shown that a connected network can be used to increase sales. In a context where information is valuable, the more interconnected a network is, the more companies have flexibility to increases their profits through word-of-mouth advertising. If the network technology is costless, this can be seen as a simple improvement of the technology firms have at their disposition. However, if users on a network bear a cost to creating links, a free rider problem occurs as the firm benefits freely from the word-of-mouth advertising while it is costly for agents on the network. If agents can form a coalition and charge a "toll" for sharing information about a firm's product, then any sequence of coalitions that covers the whole set of agents will capture the value of the externality. The distributional aspects of this sequence of coalitions, or how each agent could capture some of the externality, remains a question open to further research.

If agents perform some active modifications of the information disseminated on the network, they generate a social value of information. In the context of a firm seeking to spread information about its product, this social value can be thought as the valuation that agents form about a product by talking to each other, without any intervention. This valuation can be positive, or negative, depending on how information is modified. Since this process occurs regardless of the action of the company, the firm has no way to capture this valuation.The opinion leaders, or highly regarded agents on the network, play a key role in the formation of this social value. As these agents are highly perceived, all other agents link directly or indirectly to these agents and they become central players in the modification of information.

Chapter 4

On University Funding Policies

Public universities are peculiar creatures from an economic perspective. They are largely subsidized through taxes, making them somewhat of a public service. On the other hand, it is generally recognized that universities should remain free to choose how they allocate their funds because they are better positioned to direct research and create educational programs than the government. This leaves very little room for governments to have their say on the way universities spend the money they receive. So governments usually rely on a particular policy to determine universities' levels of funding.

A popular scheme is based on enrolment. This rule allocates an amount of money per student in order to cover the costs associated with a student. Hence, an enrolment-based funding policy naively maintains a constant level of funding per student.

That policy however overlooks the incentives the policy bestows on universities. In particular, this funding policy encourages establishments to increase efforts in recruitment to raise revenues. If universities compete for the same students, this generates social losses. It therefore follows that this policy can lead to an inefficient allocation of resources by universities. Hence a change in the funding rule can increase funds available for teaching or research by modification of the incentives given to universities.

In this chapter, I use a simple theoretical model to look at the effect of an enrolment based policy and derive an optimal funding policy. A university's funding rule should depend on its enrolment as well as the enrolment in competing universities. If the government prefers an aggressive enrolment strategy, the optimal funding policy will put less weight on the number of registered students in other establishments. If the cost of competing for the same students is however too high, the government will prefer to put more weight on students in a competing establishment. By using these two channels of the funding rule, the government can achieve an efficient allocation of resources.

The remainder of this chapter is organized as follows. In section 4.1, I show why this work is important and how it can affect various jurisdictions across the world. In section 4.1.1, I show how this chapter relates to the existing literature. Then, I present the model in section 4.2 and the main propositions. A brief conclusion follows.

4.1 Why Is This Important ?

When it comes to university funding, the debate often boils down to questions about the level of tuition fees or public spending on universities. One the one hand, university presidents seek to increase funding to reach their institutional goals, while on the other hand, governments seek to strike a balance between public funding of universities, student pressure to keep tuition low, and other priorities. In this debate, very little is said about the incentives given to universities and what can be done to ensure they efficiently spend the money already raised.

In a public hearing on the quality of universities, the principal of the Université Laval, a publicly funded university in Canada's province of Québec, said that the prevailing funding policy in Québec causes universities to compete against each other:

> And what struck me, among other things, is that in order to have a balanced budget, ... we must necessarily have an increase in enrolment.... We are thus subjected to a system such that if our recruitment office is not extremely efficient, we will face a deficit next year. And even if I understand the logic behind the actual funding scheme — and that it seems reasonable that universities should be funded on an headcount basis — ... the Université Laval and perhaps inevitably, in the future, other institutions [in other cities of the province] will be led to adopt actions that do not appear to be of the nature of a university. (Translated from [37])

In the following years, some Quebec universities have built off-campus facilities to attract new students (see Crespo & Al. [10]). Regional universities have built facilities close to metropolitan areas, where other universities offer similar programs. Crespo also shows that some metropolitan universities built new facilities in areas with increasing shares of young adults, competing for the students in the region. In some cases, universities threatened by new campuses responded by building facilities close to the newly built one.

In 2011, the province of Ontario, Canada, decided to change its funding mechanism to reduce their endless pursuit for increases in the student body:

> Ontario is overhauling the way it finances universities and colleges, replacing some per-student funding with performance-based support intended to discourage an attitude of "growth at all costs" that has been acknowledged to have harmed quality. (Bradshaw, 2011 [8])

In Table 4.1, I provide the structure of the funding schemes in various jurisdictions in Canada and Europe. They fall into four broad categories: funding based on enrolment, lump-sum funding, output-based funding and funding based on achieving contractual targets. Contractual targets are written contracts between the university and the funding body stating what the university has to achieve in order to obtain funding.

In Canada, five provinces rely on the student body as the main indicator to determine transfers. The other five provinces rely mostly on unconditional transfers.

In Europe, there is a wide diversity of funding schemes. England sets teaching funds on the basis of a targeted number of students set nationally and research funds based on competition between universities. In Denmark, most of the funding for teaching is based on the success of students at passing exams, while research funding is partly lump sum and partly driven by a competition between universities. Some German länders set contractual targets in exchange for funding. Contracts are for a period of three to five years and ensure universities receive stable funding, provided they achieve their objectives.

Table 4.1 shows that different jurisdictions provide different incentives to universities. Some prefer to restrain academic liberty through "performance contracts", while others establish some rules and let institutions make their own decisions given those rules. It also shows that a significant number of jurisdictions choose a combination of lump-sum transfers and enrolment.

Table 4.2 shows the spending in immobilization as a percentage of total university spending from 1982 to 2004 in two provinces. The first one, British Columbia, has had a funding rule that fixed enrolment in the period ranging from 1982 to 2004. The second one, the province of Québec had a policy shift in 2000. Before 2000, transfers to university were based on past funding (or lump sum). Afterwards, the province moved to a enrolment based funding policy. The current data suggests that the difference in spending could be due to the funding rule, as there is a 5% shift in immobilization spending in Québec that does not follow the trend in British Columbia. The dataset provided is however limited and this evidence should be interpreted with caution.

To sum up, the way rules to transfer funds to universities are devised changes their behaviors and these rules are different from one jurisdiction to another. These rules might harm quality, enrolment or inter-institutional coordination (Darling & coll. [11]), depending on their composition. So the question is: what should be a good rule?

4.1.1 How This Chapter Relates to Other Contributions

The focus of this chapter is on a single decision of a university to create a new facility (new campus). It looks at the impact of the university's behavior on other universities when they compete in a region. In particular, it shows that it may not be socially efficient to have two facilities in a region competing for the same students. It also compares a set of funding policy schemes and determines which one leads to a social optimum.

The model draws some inspiration from the tax competition and fiscal federalism

Table 4.1: Funding Schemes for Public Universities In Given Jurisdictions

Jurisdiction	Share of Funding By Source	Reference
British Columbia, Canada	90% lump sum, 10% strategic.	See [36]
Alberta, Canada	85% FTE 10% strategic funding, 5% performance	
Saskatchewan, Canada	94% lump sum, 6% unspecified	
Manitoba, Canada	94% lump sum, 6% unspecified	
Ontario, Canada	74% FTE, 3.5% strategic, 22.5% unspecified	
Québec, Canada	80.5% FTE, 10% output, 9.5% strategic	See [30]
New Brunswick, Canada	75% lump sum, 25% enrolment	See [36]
Nova Scotia, Canada	84% FTE, 10% strategic, 6% unspecified	
Newfoundland & Labrador, Canada	95% lump sum, 5% facilities	
Prince Edward Island, Canada	100% lump sum	
Norway	60% lump sum and strategic, 15% research output, 25% graduate output	See [16]
Sweden	55% lump sum, 45% enrolment	See [42]
Finland	89.4% enrolment & research contract, 6.2% strategic, 4.3% ouput, 0.1% unspecified	See [31]
England	60% enrolment contract, 29% output in research, 11% strategic	See [23]
Denmark	22% lump-sum, 13% performance, 65% graduation output	See [26]
Baden-Württemberg, Germany	80% lump-sum, 20% output	See [22]
Bayern, Germany	100% contractual target	
Nordrhein-Westfalen, Germany	80% contractual target, 20% output	
France	40% contractual target, 60% FTE	See [25]
Austria	80% contractual target, 20% output	
Valencia, Spain	87% FTE, 10% output, 3% unspecified	
Italy	Part funding rule, part lump sum (% unspecified)	
Portugal	67% FTE, 24% output (prior to 2007)	See [13]

FTE: full-time enrolment, strategic refers to additional funding for specific goals set by the government, output refers to what is created by universities (graduates, research, etc).

Table 4.2: Difference in Average Immobilization Spending In Terms of Total Spending

Province	1982-1999 (std. error)	2000-2004 (average) (std. error)	Difference (std. error)
British Columbia	0.10	0.08	−0.02
	(0.013)	(0.014)	(0.016)
Québec	0.09	0.12	0.03
	(0.010)	(0.014)	(0.025)
Difference	-0.01	0.04	0.05
	(0.010)	(0.014)	(0.024)

Source : Statistics Canada, CANSIM 478-008 and estimation.

literatures. If one substitutes universities, funding policies and enrolment for competing jurisdictions, taxes and economic activity, the model below would be close to a canonical model (see [47] for a review). The solution presented, however, is quite novel and draws from the fact that universities are mostly funded under one jurisdiction.

The results of the model bears also resemblance with merger analysis: if two firms compete too much for the same market share, investors in those firms have incentives in merging the firms and gain value from the loss of spending for market shares. In the case of universities, the government has incentives to reduce competition between the two universities to generate some efficiencies.

The impact of funding schemes has been explored by del-Rey [12] and Gary-Bobo & Trannoy [19]. Del-Rey explores the relationship between university goals, competition and the impact of funding policies. Her focus is on the allocation of funds between research and teaching, and under what circumstances the government can influence a university's decisions. She finds that aside from extreme behaviors (corner solutions), a decentralized funding policy can lead to the optimal allocation of funds between teaching and research.

Gary-Bobo and Trannoy explore the impact of a funding scheme solely based on enrolment. They however include moral hazard from universities and show that the scheme induces universities to adopt a "funnel" behavior where more students are admitted than will graduate. They suggest adding a graduation component to the funding scheme to correct this behavior.

I make two contributions to the existing literature. First, I show that enrolment-based funding policies gives incentives to universities to attract more students. If universities compete for the same students, an optimal funding scheme depends either on the number of students at competing universities. Second, I show that the optimal policy increases quality or enrolment, holding the level of overall university funding constant, as compared to a funding rule that depends solely on the enrolment in a university.

4.2 The Model

The model can be broadly described in the following terms:

1. Prospective students in a given region decide if they apply or not to a university. That decision depends on quality of teaching (t_i), on the program available (p) and on the ability of the student. That decision forms the demand for programs in the region.

2. There are two identical universities i and j who must decide to open or not a facility in a region. If so, they maximize profits[1] from this facility to fund a

[1] A broader generic objective function is discussed later.

given institutional goal (like research on the main campus). If they decide to open the facility, they must choose the teaching quality, the size of the facility and a single program taught. This forms the supply in the region.

3. There is a common resource problem. If i opens a new facility while j is already opened, it creates a competitive alternative and some students that would go to j will enroll in i. Hence, university i's actions influence j's decision because they compete in part for the same students. The degree of competition is measured by $\rho \in [0, 1/2]$. The higher is ρ, the more universities compete in the same market.

4. The equilibrium concept studied is a Nash equilibrium between universities. The decision of a university to open a program depends on student demand, but also the funding rule of the government. Prevalent rules give an amount based on the number of registered students in a university. I explore a rule which also depends on students enrolled in competing universities.

In section 4.2.1, I model how students decide to go to university to generate demand for programs. I then present the funding rule made available by the government. Finally, I model in section 4.2.3 how universities exploit that environment when making the decision. Then, some analysis on funding rules is performed in sections 4.2.4 and 4.2.5.

4.2.1 Student Demand

For the sake of the discussion, I assume that there is one university in the region. In this region, there is a density of students $F(\tau)$, where $\tau \in [0, 1]$ is a measure of talent of individuals. An individual student with talent τ values the life term benefits of a program from a strictly increasing function $V_p(t_i\tau)$, where t_i is a choice variable that represents the quality of teaching. This benefit accounts for future earnings, but also for accrued social benefits of having a university degree.

If a student chooses to go to university, she bears a cost C_p that depends on tuition, foregone income during studies and other characteristics. Students can observe the quality of teaching and therefore weights if the benefits are higher than the costs C_p. This generates demand $D_p(t_i)$ for university programs.

If there is only one university in the region, all students who apply go to such university. However, if there are two universities, a fraction of students apply to both universities and choose where they decide to go. For simplicity, this fraction is fixed to ρ. This means that there are $\rho D_p(t_i)$ students that will go to facility j if i opens a new campus. This fraction is assumed to be independent of programs for simplicity[2]. This fraction is a measure of the degree of competition between universities, or the market shares at stake by competition. A depiction is given in Figure 4.1.

This leads to the following proposition:

Proposition 13. *Assume the previous behavior for students and assume a facility is opened in one region with a program p and quality t_i. Then:*

1. *All students above some given talent $\tau_p(t_i)$ apply to university;*

2. *The higher is the teaching quality t_i, the more the new campus will attract*

[2]One could imagine a similar fraction $\rho(p, p')$. By assuming that the fraction is constant, I avoid all the discussion on asymmetric equilibria, but keep the main insights, that is that some students move from one campus to another.

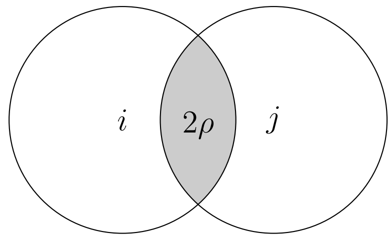


Figure 4.1: Student demand for i and j overlap by a factor of 2ρ.

students:

$$\frac{\partial \tau_p(t_i)}{\partial t_i} < 0.$$

Conversely, the more the university attracts students, the less an additional student is talented.

3. *Total demand for program p is given by $D_p(t_i) \equiv F(1) - F(\tau_p(t_i))$, with $D_p(0) = 0$.*

Proof. See appendix B.1.1. □

The first statement says that above some talent threshold, potential students choose to go to university because they weight the whole benefits higher than the costs. That talent has to be greater than $\tau_p(t_i)$, which is implicitly defined by $V_p(t_i \tau_p(t_i)) = C_p$. In this context, an increase in teaching quality will make it marginally profitable for students with less talent. Therefore, the number of students applying increases with teaching quality and each additional student is less talented.

4.2.2 The Funding Rule

The government announces a funding rule and commits to it. Such rule transfers funds to universities through three components (see equation (4.1)). The first component is a lump-sum transfer for each facility. This component encompasses general and particular transfer a university might receive for a given facility in the university. For instance, administration costs of the facility. I denote this transfer T_i for university i.

The second component depends on the number of students enrolled. It gives a certain amount of money per student enrolled in a given program at a given level (bachelor, masters, PhD). So one can think of an index p covering the set P of all possible programs at all levels. The government gives an amount α_p for each student enrolled in a program in a given university. So if there are $D_p(t_i)$ students registered in program p at university i, the amount given to the university through this component is $\alpha_p D_p(t_i)$. I will refer to such component later as the direct component of the funding rule.

The third component is the novel aspect of this chapter. As for the second one, it fixes an amount per student in each program, which I will denote β_p. However, it depends on the number of enrolled students in the competing facility (j). So I will refer to that component as the indirect component.

So with this third component, the total subsidy S_i to university i is given by:

$$S_i \equiv T_i + \sum_{p \in P} \alpha_p D_{p,}(t_i) + \sum_{p \in \text{competing facility}} \beta_p D_p(t_j). \tag{4.1}$$

This funding rule generalizes the existing scheme by introducing a component that depends on the other university. If β_p equals zero for all programs, the rule boils down to most existing rules. As an example, that restricted scheme prevailed[3] in the province of Ontario, Canada, from 1967 to 2011 (see [8] and [32]). It is also the current policy in the province of Québec, Canada (see [30]), although it is currently being

[3] These types of funding rules are actually based on moving averages of enrolled students in the past years to avoid steep variations. As I am not interested in the effects of time, I assume it depends solely on enrolment in past periods.

reviewed. In those provinces, the rule gives relative weights to different programs p and each of these weight is multiplied by a base amount in dollars. In Ontario, these weights ranged from 1.0 to 7.5 while in Québec, they ranged up to 10.7, depending on the program. This rule shows that if all universities increase enrolment and if the base amount does not change, the government increases funding.

4.2.3 Universities i and j

Universities conduct research and teaching. I assume throughout that their primary goal when opening a facility is to maximize funds available for research. In the context of this model, this means that they seek to maximize profits coming from the new facility. In section 4.3, I discuss that a broader objective for the facility leads to a similar qualitative analysis.

If a university opens a new facility, it must choose the programs offered in it. For the purpose of clarity in this chapter, I assume a single program is offered. Programs are indexed by p in the set of all possible programs $P \equiv \{p_1, p_2, \ldots, p_k\}$ subsidized by the government.

The university must also choose the size f_i of the facility and the teaching quality t_i in the program. Teaching quality is measured in terms of dollars per student.

Each university charges the same fees whether its on the main campus or in the new facility. This can be so either because fees are regulated by the jurisdiction, or because the university wants to charge consistent prices with its main campus. So fees can be thought as exogenous when opening the new program. From the standpoint of this chapter, they are implicitly accounted in the coefficients α_p.

In order to build a new facility, the university has to pay a fixed cost B and a linear component that increases with the size f_i of the facility. So the total cost is $f_i + B$.

For a given facility size f_i and a given student demand $D_p(t_i)$, the university i has a number of enrolled students given by the minimum between the facility size and the student demand:

$$\min\left(f_i, D_p(t_i)\right). \tag{4.2}$$

This means there cannot be more students enrolled than what the facility can hold or, if smaller, the number of students interested by the program.

Now, recall that if two campuses are open in the same region, they compete to some degree for the same students. This degree of competition is measured by $\rho \in [0, 1/2]$. If ρ is equal to zero, there is no competition. Conversely, if $\rho = 1/2$, there is full competition between the two institutions. Hence, when two universities are in the region, the demand is given by:

$$\min\left(f_i, D_p(t_i) - \rho D_{p'}(t_j)\right). \tag{4.3}$$

There will thus be two types of equilibria. In the first one, only one university is opened in the region and the second one does not find profitable to open a facility. In that case, enrolment in the open university (assuming it is i) is given by (4.2). In the second case, there is a symmetric equilibrium where both universities open a facility and demand is given by (4.3). Since $D_{p'}(0) = 0$ when the campus is not opened, (4.3) encompasses (4.2).

When university i decides to open a facility with a program $p \in P$ and a given facility size, it can generate additional profits given by:

$$\begin{aligned}\Pi_i(f_i, t_i, p, f_j, t_j, p') \equiv & \underbrace{\alpha_p \left[\min(f_i, D_p(t_i) - \rho D_{p'}(t_j))\right]}_{\text{Income from enrolment in } i} \dots \\ & \dots + \underbrace{\beta_{p'} \left[\min(f_j, D_{p'}(t_j) - \rho D_p(t_i))\right]}_{\text{Income from enrolment in } j} \dots \\ & \dots - \underbrace{t_i \left[\min(f_i, D_p(t_i) - \rho D_{p'}(t_j))\right]}_{\text{Cost of teaching quality in } i} \dots \\ & \dots + T_i - \underbrace{(f_i + B)}_{\text{Cost of building facility } i}\end{aligned} \tag{4.4}$$

where p' is the program chosen in the competing facility j.

The first component measures the additional income generated by enrolment in facility i. It is the amount given per student α_p multiplied by the number of students who go in this program. The second component measures the income based on the facility built by the competing university j. The third term measures the cost of teaching quality. Since the university commits to a quality level t_i in the new program, the university covers the costs of t_i for every student. The last component measures the additional income from lump-sum transfers and the costs of building the facility of size f_i.

By opening a new facility, the university seeks to maximize Π_i. The following proposition characterizes the two equilibria.

Proposition 14. *Let universities maximize the profits (4.4) and assume a funding rule as in (4.1). Then:*

1. *A symmetric Nash equilibrium where i and j both open a facility happens if for some $\alpha_{p^{2*}}, \beta_{p^{2*}}$, profits are positive for both universities:*

$$(\alpha_{p^{2*}} + \beta_{p^{2*}} - t_i^{2*} - 1)(1 - \rho) D_{p^{2*}}(t_i^{2*}) + T_i > B,$$

where $f_i^{2*}(\alpha_{p^{2*}}, \beta_{p^{2*}}), t_i^{2*}(\alpha_{p^{2*}}, \beta_{p^{2*}}), p^{2*}$ *are the optimal choices of each university. In that case:*

(a) *The facility size equals demand* ($f_i^{2*} = (1-\rho)D_{p^{2*}}(t_i^{2*})$);

(b) *The quality of teaching increases with the direct component of the funding rule and decreases with the indirect component. Likewise, the size of the facility increases in* $\alpha_{p^{2*}}$ *and decreases in* $\beta_{p^{2*}}$.

2. *There exists two asymmetric Nash equilibria where a single facility is opened with solution* $f_i^*(\alpha_{p^*}, \beta_{p^*}), t_i^*(\alpha_{p^*}, \beta_{p^*}), p^*$ *if profits are positive for one university:*

$$(\alpha_{p^*} - t_i^* - 1)D_{p^*}(t_i^*) + T_i > B, \tag{4.5}$$

but not for two:

$$(\alpha_{p^{2*}} + \beta_{p^{2*}} - t_i^{2*} - 1)(1-\rho)D_{p^{2*}}(t_i^{2*}) + T_i < B. \tag{4.6}$$

In that case:

(a) *The facility size equals demand* ($f_i^* = D_{p^*}(t_i^*)$);

(b) *The quality of teaching and the size of the facility increase with the direct component of the funding rule while the indirect component has no influence.*

Proof. See Appendix B.1.2 □

These results show the nature of the two possible types of equilibria. In the first one, both universities open a facility and this happens if it generates some additional funds for both. In particular, this is always the case if the marginal income is greater than the marginal cost ($\alpha_p + \beta_p > 1$) and if there are no fixed costs to this construction

($B = 0$) attached. Universities will select the program that yields the highest profits, as measured by the term $(\alpha_p + \beta_p - t_i^{2*} - 1)f_i^{2*}$.

In such equilibrium, both channels of the funding rule (direct and indirect) have an impact on the decision of universities. Increasing α_p increases both the quality of teaching and thus, demand. Increasing β_p has the opposite effect. One can thus see how the government can play with those channels to influence the behaviour of universities. Intuitively, increasing β_p increases the penalty that the university incurs by stealing students from the other university while increasing α_p increases the reward of recruitment, whether there is competition or not.

In this equilibrium, there can be a social loss. Because two universities enter the market, it becomes harder to recruit students for each university. Each must thus spend more to recruit. If this spending is too high, it would be efficient to have only one facility. This is explored in details in the next sections.

In the second equilibrium, only one university opens a facility. This is so because the funding rule or the region cannot support positive profits if two facilities are opened. In that case, one of the two universities does not enter the market. Again, this might not be socially efficient. Two universities might allow to enroll more students, even if it is not profitable to compete.

When there is only one university, only the first component of the funding rule has an impact on the behaviour of the university since the indirect component is multiplied by zero. There is no competition. The expense in the facility and the quality of teaching increases with the direct funding component (α_p).

Notice also that the size of the facility is a measure of quality. In laymen's terms, "the bigger, the better". This means that an increase in the direct funding component

(α_p) will increase both the quality of teaching in the new facility and the size of the facility.

The following proposition describes what happens to these optimal solutions when the degree of competition (ρ) increases:

Proposition 15. *Let the university problem be as decribed by (4.4) and assume a symmetric equilibrium. Then:*

1. *If per-student profits generated solely from the direct component of enrolment in the facility are greater than the per-student income generated by the indirect component, the quality in the new facility increases with the degree of competition:*

$$\alpha_{p^{2*}} - t_i^{2*} - 1 > \beta_{p^{2*}} \Rightarrow \frac{\partial t_i^{2*}}{\partial \rho} > 0$$

   Conversely, if the per-student income from students in the competing facility is greater than net profits, the degree of competition reduces teaching quality.

2. *These results also apply to the size of the facility as there is a one to one relationship between size and teaching quality.*

3. *In particular, under a standard funding scheme ($\beta_p = 0\ \forall p$), the quality always increases with the degree of competition.*

Proof. See Appendix B.1.3. □

This proposition shows how the indirect component affects quality and the size of the facility. If per-student profits from the facility are greater than the loss of income from the competing facility, the university will still compete by increasing teaching quality. In that case, it is worthwhile to steal students from the other facility. When the income per-student of the other facility is greater, the university finds it worthwhile to attract less students in its own new facility.

Notice that this result does not depend on the value of $\alpha_{p^{2*}}$ and $\beta_{p^{2*}}$ by themselves, but rather their relative difference.

One can see right away that if $\beta_p = 0\ \forall p$, as in a standard funding rule, the competition necessarily increases quality. Since the losses incurred in the other institution is not taken into account, this might lead to an inefficiency. In the next section, I show when this is the case.

4.2.4 An Optimal Centralized Recruitment Policy

In this section, I abandon the decentralized framework and assume the government can implement the decision to build facilities or not. I thus derive the government's preferred strategy if it had the power to make universities' decisions. The government seeks to maximize the welfare of individuals going to university in the region. It has a fixed amount G available to achieve this goal. If it decides to open two facilities in the region, it seeks to solve:

$$\begin{aligned}
&\max_{t_i,p} 2(1-\rho)\int_{\tau_p(t_i)}^{1} (V_p(t_i\tilde{\tau}) - C_p)dF(\tilde{\tau}) \\
&\text{s.t. } G \geq 2\left(t_i(1-\rho)\left[F(1) - F(\tau_p(t_i))\right] + B + f_i\right) \qquad (4.7)\\
&f_i = (1-\rho)\left[F(1) - F(\tau_p(t_i))\right]
\end{aligned}$$

If there is only one facility, the program is however:

$$\max_{t_i,p} \int_{\tau_p(t_i)}^{1} (V_p(t_i\tilde{\tau}) - C_p) dF(\tilde{\tau})$$
$$\text{s.t. } G \geq t_i \left[F(1) - F(\tau_p(t_i))\right] + B + f_i \tag{4.8}$$
$$f_i = F(1) - F(\tau_p(t_i))$$

The two problems are almost similar, saved for the budget constraint, which depends on the number of facilities built. The following proposition characterizes under which circumstances two competing facilities is better than one.

Proposition 16. *Consider the government problem of building one or two facilities as in (4.7) and (4.8). Then, it will build only one facility if the share of the fixed costs for building a facility is greater than the ratio of students unexposed to competition to those exposed to competition:*

$$\frac{B}{G} \geq \frac{1-2\rho}{2\rho}.$$

Proof. It is sufficient to notice that the two budgets constraints, taking into account the technology constraint, can be re-written in the following fashion:

$$\frac{G}{2(1-\rho)} - \frac{B}{1-\rho} \geq (t_i + 1)\left[F(1) - F(\tau_p(t_i))\right] \qquad \text{(two facilities)}$$
$$G - B \geq (t_i + 1)\left[F(1) - F(\tau_p(t_i))\right] \qquad \text{(one facility)}$$

Building one facility thus yields more income per student if

$$G - B > \frac{G}{2(1-\rho)} - \frac{B}{1-\rho},$$

which leads to the desired result. □

The ratio $\frac{1-2\rho}{2\rho}$ can be easily understood with figure 4.1. It is a measure of (the inverse of) "fixed costs" of competition. If competition is too costly (e.g. if the right hand side is too low), then the government prefers to open only one facility to save the extra fixed cost of a facility. In that case, competition is inefficient as it costs too much of public funds.

Let $\left(f_1^{opt}, t_1^{opt}, p_1^{opt}\right), \left(f_2^{opt}, t_2^{opt}, p_2^{opt}\right)$ be the solution to the problem of the government where there is respectively one or two facilities required. This solution is what the government is trying to implement given a decentralized policy.

4.2.5 An Optimal Decentralized Policy

In this section, I return to the original problem, where universities have control over how to allocate funding between programs, and show that there is an optimal policy that can achieve the first-best outcome. I still assume that the government wishes to spend G on universities, but instead lets universities make their own decisions given the government's choice of $T_i, T_j, \alpha_p, \beta_p$.

The next proposition is a simple corollary of the equilibrium analysis in proposition 14:

Proposition 17. *A funding rule composed only of transfers and a direct component (e.g.: $\beta_p = 0 \ \forall p$) cannot always achieve the social optimum.*

A trivial case is when there is full competition between the two universities ($\rho = 1/2$). In that case, it is better to let one university build a facility. However, if fixed costs are small enough, both universities will engage in competition.

Another trivial case is when the two universities do not compete at all ($\rho = 0$). The symmetric equilibrium is then the social optimal, but if the fixed costs are too high, both universities might not receive the right incentives solely through the direct component.

This shows that a funding rule that solely depends on direct enrolment and transfers cannot refrain universities to compete even if that would not be socially optimal from a social point of view. In both cases, the government has only two instruments to satisfy three constraints. It must thus let go one of its objectives: either budget balance, the optimal facility size, or let go incentives for universities to compete.

The next proposition shows that a funding rule with an indirect component can achieve the first best.

Proposition 18. *A funding rule with transfers, a direct and an indirect component reaches the first best:*

1. *If the social optimum is to build one facility, one efficient rule solves:*

$$\begin{aligned} f_i^*(\alpha_p^{opt}, 0) &= f_1^{opt} \textit{ for } p = p_1^{opt}, \\ \alpha_p^{opt} &= 0 \ \forall p \neq p_1^{opt}, \\ \alpha_p^{opt} + \beta_p^{opt} &= 0 \ \forall p \\ T_i^{opt} &= \alpha_p^{opt} f_1^{opt} - G \textit{ for } p = p_1^{opt}. \end{aligned}$$

2. *If the social optimum is to build two facilities, one efficient rule solves:*

$$\begin{aligned} f_i^*(\alpha_p^{opt}, \beta_p^{opt}) &= f_2^{opt} \textit{ for } p = p_2^{opt} \\ \alpha_p^{opt} + \beta_p^{opt} &> \frac{B}{f_2^{opt}} + 1 + t_2^{opt} \textit{ for } p = p_2^{opt}, \\ \alpha_p^{opt} + \beta_p^{opt} &= 0 \ \forall p \neq p^{opt} \\ T_i^{opt} &= (\alpha_p^{opt} + \beta_p^{opt}) f_2^{opt} - \frac{G}{2} \textit{ for } p = p_2^{opt}. \end{aligned}$$

Proof. See appendix B.1.6 □

This show that there exists at least one solution that reaches the first best outcome, whatever that first best is. Compared to the current funding scheme, the third channel β_p acts either as a penalty if the optimal choice is to open only one facility, or as an additional incentive to engage in competition if required.

This proposition and corollary (17) show that in general, a policy with solely lump-sum transfers and a direct component can lead to an inefficient allocation of quality and facility size. In particular, if two facilities are opened while there should be one, there is overspending. This means that for a given level of government spending, quality can be increased by introducing a component on β_{p^*} to correct for the decision to open a second facility while increasing α_p to increase quality. This mechanism works since income from the third channel is viewed as exogenous from the perspective of the university. Hence, with a new parameter in the funding policy, the government is able to increase quality and correct the inefficiency (if any).

4.3 Quality and Income

In the previous discussion, it is assumed that the goal of a university in establishing an off-campus facility is to maximize profits from the facility to fund some other activities (like research). In this section, I assume instead that they seek to strike a balance between quality of teaching and profits generated through a strictly concave utility function U. By doing so, the university still wants to extract some profits out of the facility, but also fosters quality of teaching as a goal in itself (rather than seeing it as a tool to extract profits). The goal of the university is thus:

$$\max_{t_i,p} \quad U\left(t_i, \Pi(f_i, t_i, p, f_j, t_j, p')\right), \tag{4.9}$$

$$\text{s.t.} f_i = \begin{cases} (1-\rho)\left[F(1) - F(\tau_p(t_i))\right], & \text{If two facilities,} \\ \left[F(1) - F(\tau_p(t_i))\right], & \text{If one facility.} \end{cases}$$

One can then show the following proposition:

Proposition 19. *Assume universities behave according to (4.9). Then there exists an optimal decentralized solution $T_i^{opt}, \alpha_p^{opt}, \beta_p^{opt}$ that matches the first best outcome.*

Proof. See appendix B.1.7 □

This proposition summarizes the idea that regardless the university's objective, the government can reach budget balance and the optimal outcome through a decentralized scheme as long as there are enough parameters in the funding rule.

4.4 Conclusion

The previous analysis provides a few insights about university funding policies. It studies the decision of a university to open a new campus given an enrolment based funding policy. It shows that such policy gives incentive for universities to do so, even if this decision might not be efficient.

A government that cares about the quality of teaching and research cannot simply provide funding based on a linear function of the students enrolled in each establishment. If universities compete for the same students, this policy increases spending in recruitment while it may not be socially efficient. This is so when the "cost of competition", namely the additional cost to recruit students in the presence of a competitor, is too high. To avoid this effect, the government can introduce a funding channel that depends on the number of students in a competing university. By doing so, the government can increase the efficiency of universities while spending the same amount.

This new component of the funding rule acts as a penalty, or a tax, when two universities compete for the same students. An efficient policy thus implement that penalty when competition costs too much in taxpayers dollars. The policy can then be used to keep the desirable effects of competition (fostering quality) by increasing the component that depends on enrolled students, but by penalizing universities who enter the market when it is not efficient. It can thus increase funds available for academic activities.

Chapter 5

Conclusion

5.1 Summary and Abstract

This thesis is a collection of three essays. The first two study diffusion problems on networks while the third one explores the impact of funding rules for public universities.

In the first essay, I explore how an external agent can find the best way to seed a network of individuals who talk to each other. In the first essay, I focus on the model, how to find its solution, and how can it can be used to find practical answers. The solution can be best understood as a Nash Equilibrium where the selection of sources made by agents is in accordance with the seed of the company and the seed of the company accounts for the selection of sources. As there are many Nash Equilibria that satisfy such criterion, the company selects the one that maximizes the spread of the signal. In the context of the model, this is equivalent to maximizing the variance of the seeds amongst users.

Such result is used to derive an algorithm. Networks models are often cursed with dimensionality, meaning that as the network size grows, the time required to compute the solution on the network becomes intractable. In this paper, I exploit the economic constraints and results on the model to present an algorithm to find such optimal solution. This algorithm is based on the known "greedy algorithm" which can be shown, with some minor modifications, to lead to the optimal solution.

The essay shows that the value of a network can be decomposed into three components: the value of the "seed", the value of the choices made by users given this campaign and finally, the social value of the network. The model suggests that advertising investments in users should be proportional to the discounted influence they have, where the influence of a user is measured as the number of users who choose to listen to him.

The second essay uses the same model but focuses on the interplay between diffusion and public economics concepts. Discussion over a social network generates spillover effects for firms when consumers can use the social network to inform each other about products. When the company can exploit a social network's structure, it can increase its sales. However, when the network formation process is costly, firms free-ride on such costs at the expense of agents on the network. If agents can form coalitions, I show that they can recoup the value of this externality by charging a toll.

When users actively modify the information, generating word-of-mouth advertising about a product provides a "social value." This social value stems from the discussions that agents have about the product, without any intervention. Since this process occurs regardless of the firm's actions, the firm cannot capture such valuation. The opinion leaders, or highly regarded agents on the network, play a key role in the formation of this social value.

The third essay leaves completely the network paradigm and studies funding rules for public universities. I show that if there is a common resource problem, namely if universities compete for the same students, a funding rule solely based on enrolment leads to an inefficient allocation of resources. I suggest instead a funding rule based both enrolment in the institution and enrolment in competing institutions. Such rule leads to an efficient allocation of resources.

5.2 Future Work

In the network paradigm, I might port the model to a practical test. I have gathered, in the course of the last year, all the tweets pertaining to the provincial election of Québec. I have also gathered the network structure between tweeters. If we extend the biases b_{ij} not only to individuals, but also to messages, the individual's decisions to retweet can then be framed as $\max_{j \in \eta_i}(b_{ij} + x\beta + \epsilon)$ where x is a vector of characteristics in the message, β are the relative weights given to each characteristics and ϵ is an error term. Given assumptions on the error term, such framework leads to the well known logit estimation technique. The meat of the work then consists of decomposing a million of tweets (words!) in a set of characteristics using data reduction technique such as principal component analysis. The estimation can then be used to infer optimal seeds as well as grounds for community detection.

Regarding the university paper, I still am looking in the CAUT database on university financial reports for a good identification strategy to put the theory to the test. As the policy in Québec has been implemented in 2000, I should be able to detect the impact of the funding rule by a difference in difference estimation strategy with a province that had steady funding over the period of analysis.

Both of these extensions are in my research agenda in the next year.

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Appendix A

Appendix For the Chapter Two

A.1 Various Proofs

A.1.1 Proof of Lemma 1

Proof. This is a simple proof by recursion. At time $t = 0$, the signal $s_{i,0}$ is the signal itself $(i = j)$ and $c_{i,0} = 0$. At time $t = 1$, the signal $s_{i,1}$ has a value given by some $s_{j^*,0} + b_{ij^*} = \max_{k \in \eta_i}(s_{k,0} + b_{jk})$ so $c_{i,1} = b_{ij^*}$. Now, for any given time, $s_{i,t}$ is equal to $s_{j^*,t-1} + b_{ij^*} = \arg\max_{k \in \eta_j}(s_{k,t} + b_{jk})$ which, by recursion, can be written as $s_{j^*,0} + b_{ij^*} + c_{j,t-1}$ for some i. The conclusion follows. □

A.1.2 Proof of Proposition 1

Proof. The first statement is simply a constraint of the company's problem. Now, for any $s^*_{j,0}$ there are some $k_{j,t}$ at each period that counts the number of times it has been selected by users. Hence, the total value of such signal is given by:

$$p \sum_{t=0}^{\infty} k_{j,t} \beta^t (s_{j,0}^* + c_{j,t}).$$

First-order conditions then commands that :

$$\begin{aligned} 0 &= \frac{\partial \Pi(s,p)}{\partial s_{j,0}} \\ \Rightarrow s_{j,0} &= p \sum_{t=0}^{\infty} \beta^t k_{j,t} \end{aligned}$$

Hence, an optimal solution must satisfy these two conditions together. The last item stems from the fact that $\sum_{j=1}^{|V|} k_{jt} = |V|$ at any period t where users make choices. If there is at least one constrained star network, users can make at least one choice at any period. Hence, by the first order condition:

$$\begin{aligned} \sum_{j \in V} s_{j,0} &= p \sum_{t=0}^{\infty} \beta^t \sum_{j \in V} k_{jt}, \\ &= |V| \frac{p}{1-\beta}, \end{aligned}$$

from which the conclusion follows. □

A.1.3 Proof of Proposition 2

Proof.

1. After t^*, the network enters a periodic regime of period T. Hence the choices made over $s_{j,0}^*$ repeat themselves every T iterations. Hence, the optimal solution can be written as:

$$\begin{aligned} s_{j,0}^* &= p \sum_{t=0}^{t^*} \beta^t k_{jt} + p\beta^{t^*} \sum_{t=1}^{\infty} \left[\beta^{tT+1} k_{j1} + \beta^{tT+2} k_{j2} + \cdots + \beta^{(t+1)T-1} \right], \\ &= p \sum_{t=0}^{t^*} \beta^t k_{jt} + p \frac{\beta^{t^*}}{1-\beta^T} \sum_{u=0}^{T-1} \beta^u k_{ju}, \end{aligned}$$

where k_{ju} is the number of times $s_{j,0}^*$ is selected in period $u \in [0, 1, \ldots, T-1]$.

2. Consider the law of motion G' :

$$s_{t+1} = G(s_t) - \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix}$$

As the relative value of each signal has not changed, s^* is also a solution to G'. The only difference is that each step, a value of λ is substracted. So for a particular user j, the law of motion G generates:

$$s^*_{j0}, s^*_{j1} + \lambda, s^*_{j2} + 2\lambda, \ldots, s^*_{jn} + n\lambda, \ldots$$

Hence, for an user j, the discounted sum of all terms involving λ can be written as:

$$\begin{aligned} I(\lambda) &= p\beta\lambda \sum_{t=0}^{\infty} \beta^t t + p\beta\lambda \sum_{t=0}^{\infty} \beta^t \\ &= p\beta\lambda \left(\frac{\beta}{(1-\beta)^2} + \frac{1}{1-\beta} \right) \\ &= p\beta\lambda \frac{1}{(1-\beta)^2} \end{aligned}$$

So summing over all nodes yields a value of :

$$p\beta\lambda \frac{|V|}{(1-\beta)^2},$$

which is the second term of the expression. The other segments of the expression simply the decompose the value of G' after and before t^*. For a given user j, the value of the solution looks like:

$$\begin{aligned}
p\sum_{t=0}^{\infty}\beta^t s^*_{j,t} &= p\sum_{t=0}^{\infty}\beta^t\left(s^*_{k^*,0}+b_{jk^*}+c_{t-1,k^*}-t\lambda\right)\\
&= p\sum_{t=0}^{\infty}\beta^t s^*_{k^*,0}+p\sum_{t=0}^{\infty}\beta^t\left(b_{jk^*}-+c_{t-1,k^*}-t\lambda\right)\\
&= p\sum_{t=0}^{\infty}\beta^t s^*_{k^*,0}+p\sum_{t=0}^{t^*}\beta^t\left(b_{jk^*}+c_{t-1,k^*}-t\lambda\right)+\dots\\
&\quad \dots p\frac{\beta^{t^*}}{1-\beta^T}\sum_{u=0}^{T-1}\beta^u\left(b_{jk^*}+c_{u-1,k^*}-(t^*+u)\lambda\right)
\end{aligned}$$

for some k^* being the source selection for user j at time t. By construction, the term $(b_{jk^*}+c_{u-1,k^*}-(t^*+u)\lambda)$ corresponds to the element $[v_{k^*}]_j$. The other term being a finite sum, it boils down to a term of value of an order of pM for some number $M \in \mathbb{R}$. Summing over all users yields the relevant results.

3. The last item of the proof is a simple rewriting of the difference of profits. The only key step to realize is that $\sum_i \tilde{s}_{i,0} - \tilde{s}'_{i,0} = 0$ since both solutions sums to $\frac{|V|p}{1-\beta}$.

□

A.1.4 Proof of proposition 3

Proof. Let $\tilde{s}, \tilde{s}' \in \tilde{S}$ be two candidate solutions. Since both solutions at time t_0 have a mean of $\frac{p}{(1-\beta)}$, one is a mean preserving spread of the other. So assume without loss of generality that :

$$\tilde{s}'_0 = \tilde{s}_0 + \Delta\tilde{s}_0,$$
$$\text{with } 0 = \sum_i \Delta\tilde{s}_{0,i}.$$

Consider now an agent m who chooses source j at time t under $\tilde{s}'$. This means that:

$$\Delta\tilde{s}_{j,0} + \tilde{s}_{j,0} + c_{j,t} > \tilde{s}_{i,0} + \Delta\tilde{s}_{i,0} + c_{i,t} \ \forall i$$
$$\Leftrightarrow (\Delta\tilde{s}_{j,0} - \Delta\tilde{s}_{i,0}) + (\tilde{s}_{j,0} - \tilde{s}_{i,0}) + (c_{j,t} - c_{i,t}) > 0 \ \forall i. \tag{A.1}$$

Consider also the difference in profits due to the variance:

$$\frac{1}{2}\sum_{i\in V}\left(\left[\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0} - \frac{p}{1-\beta}\right]^2 - \left[\tilde{s}_{i,0} - \frac{p}{1-\beta}\right]^2\right) = \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(2\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0}\right) > 0$$

This implies that :

$$\begin{aligned} 0 &< \sum_{i\in V}(\Delta\tilde{s}_{i,0})^2 + \sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}_{i,0}, \\ &= \sum_{i\in V}\Delta\tilde{s}_{i,0}(\Delta\tilde{s}_{i,0} + \tilde{s}_{i,0}), \\ &= \sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}'_{i,0}. \end{aligned}$$

Now, consider the total change in profits:

$$\begin{aligned} \Delta\Pi(\tilde{s},\tilde{s}',p) &= \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(2\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0}\right) + \sum_{t=0}^{\infty}\beta^t\sum_{m\in V}\sum_{i\in V}\mathbb{1}_{mit}\left[\tilde{s}_{j,0} + c_{j,t} + \Delta\tilde{s}_{j,0} - \tilde{s}_{i,0} - c_{i,t}\right], \\ &= \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(2\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0}\right) + \sum_{t=0}^{\infty}\beta^t\sum_{i\in V}\tilde{k}_{it}\left[\tilde{s}_{j,0} + c_{j,t} + \Delta\tilde{s}_{j,0} - \tilde{s}_{i,0} - c_{i,t}\right], \\ &= \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(2\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0}\right) + \sum_{t=0}^{\infty}\beta^t\sum_{i\in V}\tilde{k}_{it}\left[\tilde{s}_{j,0} + c_{j,t} + \Delta\tilde{s}_{j,0} - \tilde{s}_{i,0} - c_{i,t}\right]\dots \\ &\quad \dots + \sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}_{i,0} - \sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}_{i,0}. \end{aligned}$$

By construction, we have :

$$\begin{aligned}\sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}_{i,0} &= \sum_{i\in V}\Delta\tilde{s}_{i,0}\left(\sum_{t=0}^{\infty}\beta^t\tilde{k}_{i,t}\right),\\ &= \sum_{t=0}^{\infty}\beta^t\sum_{i\in V}\tilde{k}_{i,t}\Delta\tilde{s}_{i,0}.\end{aligned}$$

Hence, we can combine the fourth term into the second term and the third term into the first term:

$$\Delta\Pi(\tilde{s},\tilde{s}',p) = \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(3\tilde{s}_{i,0}+\Delta\tilde{s}_{i,0}\right) + \sum_{t=0}^{\infty}\beta^t\sum_{i\in V}\tilde{k}_{it}\underbrace{\left[\tilde{s}_{j,0}+c_{j,t}+\Delta\tilde{s}_{j,0}-\Delta\tilde{s}_{i,0}-\tilde{s}_{i,0}-c_{i,t}\right]}_{>0}.$$

By equation A.1, the last term is positive. Now, for the first term, consider that:

$$\begin{aligned}\frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(3\tilde{s}_{i,0}+\Delta\tilde{s}_{i,0}\right) &= \frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\left(3\tilde{s}_{i,0}+\Delta\tilde{s}_{i,0}\right)+\frac{1}{2}\sum_{i\in V}\left(\frac{3}{2}\tilde{s}_{i,0}\right)^2-\frac{1}{2}\sum_{i\in V}\left(\frac{3}{2}\tilde{s}_{i,0}\right)^2,\\ &= \frac{1}{2}\sum_{i\in V}\left(\frac{3}{2}\tilde{s}_{i,0}+\Delta\tilde{s}_{i,0}-\frac{3}{2}\frac{p}{1-\beta}\right)^2-\frac{1}{2}\sum_{i\in V}\left(\frac{3}{2}\tilde{s}_{i,0}-\frac{3}{2}\frac{p}{1-\beta}\right)^2,\end{aligned}$$

which is positive.

Since all terms in the difference in profits are increasing with a mean preserving spread, the variance is a sufficient measure for maximizing profits. □

A.1.5 Proof of Proposition 4

Proof. 1. Assume that a user that is not on a critical cycle is in the hub. By definition, this means that there exits a directed cycle starting at i. But since i is not in a critical cycle, this means that the increase λ' over the cycle is smaller than the increase on a critical cycle. Thus, for any s_k^* in the periodic regime, $s_k^*+T\lambda' < s_k^*+T\lambda$. Thus, the agent does not select this cycle as a source. By the same token, it does not select any other possible cycle that is not a critical cycle. Hence, the user cannot be in the hub. Hence, only users in critical cycles can be in the hub.

2. Assume that a path does not maximize the increase of the signal. This means that there exists another path from the hub with an higher increase of the signal. So this means that for the same signal, one has chosen $s^*_{k,0} + c_{kt} < s^*_{k,0} + c'_{kt}$, which contradicts that users choose the highest utility signal.

□

A.1.6 Proof of Lemma 2

Proof. This is a simple corollary of the increase of variance of proposition 3. Since we have:

$$\frac{1}{2}\sum_{i\in V}\left(\tilde{s}_{i,0} + \Delta\tilde{s}_{i,0}\right)^2 - \frac{1}{2}\sum_{i\in V}\left(\tilde{s}_{i,0}\right)^2 > 0,$$
$$\Rightarrow \underbrace{\frac{1}{2}\sum_{i\in V}\Delta\tilde{s}_{i,0}\tilde{s}_{i,0}}_{\text{Covariance}} + \underbrace{\frac{1}{2}\sum_{i\in V}\Delta\tilde{s}^2_{i,0}}_{>0}.$$

Hence, for two identical increases $\Delta s_{i,0}$, the one allocated to an already high solution will increase profits more. □

A.1.7 Proof of Proposition 5

Proof. For the first element of the proposition, it is sufficient to notice that h is a mapping a convex and compact set to itself. By Brouwer's Fixed Point Theorem, h admits a fixed point.

For the second element of the proposition, denote s'_0 the result of $h(s_0)$ for some s_0 and consider a change $\Delta k'_{it}$ from user m at time t. Assume this user changes from source j to source i. For $s'_{i,0}$ to increase, it must be that:

$$s_{i,0} + \beta^t \Delta k'_{i,t} + c_{i,t-1} + b_{mi} > s_{j,0} - \beta^t \Delta k'_{i,t} + c_{j,t-1} + b_{mj} > 0,$$
$$\Rightarrow 2\beta^t \Delta k'_{i,t} > s_{j,0} - s_{i,0} + c_{j,t-1} - c_{i,t-1} + b_{mj} - b_{mi} > 0,$$

for some user m. At the same time, we must have that $\eta_m - k_{it} \geq 2\beta^t \Delta k'_{i,t}$ for all values of t. So this leads to the following constraint:

$$\eta_m - k_{it} > 2\beta^t \Delta k'_{i,t} > s_{j,0} - s_{i,0} + c_{j,t-1} - c_{i,t-1} + b_{mj} - b_{mi} > 0.$$

Hence, if $2\beta^t \Delta k_{i,t} \geq \eta_m - k_{it}$, we have that Δk_{it} must be reduced. So any increase in k_{it} is bounded by the neighborhood of m and previous choices.

Now consider the decision of user m' who faces the change $\Delta k_{j,t}$ but does not change his decision from source j to source k, the second highest source. For this to hold, it must be that:

$$s_{j,0} - s_{k,0} + c_{j,t-1} - c_{k,t-1} + b_{mj} - b_{mk} > -\beta^t \Delta k_{j,t},$$
$$\Rightarrow s_{j,0} - s_{k,0} + c_{j,t-1} - c_{k,t-1} + b_{mj} - b_{mk} > \beta^t \Delta k_{i,t}$$

Hence, if $s_{j,0}$ is decreasing without any change, its decrease is bounded by the value of the second highest source.

By Lemma 2, if a signal increases (decreases), it will keep increasing (decreasing). But the changes in ks are bounded for any t. Thus, they converge to a point in $\mathbb{N}$, which means that h converges as well. □

A.1.8 Proof of Proposition 6

Proof. 1. Each time the function is called, the number of nodes treated increases by at least one. Hence, the stopping condition is reached eventually. Now since h converges on $\mathbb{N}$, the number of steps required to reach convergence is finite.

Hence, the algorithm stops in finite time.

2. The findValue function is called at most $|V|^2$ times in the general algorithm. Such function converges in the worst case in $|V|(t^* + T)$ and each step requires $|V|^2$ computations. So the computational speed of findValue is of $O(|V|^3(t^* + T))$.

   The function findUpperBounds is called $|V|$ times in the general algorithm and requires $|V|^2(t^*+T)$ steps to compute G^t, $|V|^2(t^*+T)$ steps to compute all values and it calls the function sort $|V|$ times. A sort function takes at most $|V|^2$ steps to sort. Hence, the function algorithm requires an order of $O(|V|^5(t^*+T))$ steps to use findUpperBounds. The conclusion follows.

□

A.1.9 Proof of Proposition 7

Proof. The total income generated by the users is given by :

$$I(s,p) = p\sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{|V|}\left(\sum_{j=1}^{|V|}[B^t]_{ij}s_{j,0}^r\right)^{1/r}.$$

Thus, the first order condition for any given $s^*_{k,0}$ is given by:

$$\begin{aligned} s^*_{k,0} &= p\sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{|V|}\left(\sum_{j=1}^{|V|}[B^t]_{ij}(s^*_{j,0})^r\right)^{(1-r)/r}[B^t]_{ik}(s^*_{k,0})^{r-1}, \\ &= p\sum_{t=0}^{\infty}\beta^t\sum_{i=1}^{|V|}\frac{[B^t]_{ik}(s^*_{k,0})^{r-1}}{(s^*_{i,t+1})^{r-1}}. \end{aligned}$$

which is the expression in the proposition. If $p = 1$, then this summation can be expressed as :

$$s_0^* = pV\frac{\Lambda}{1-\beta}V^{-1}\iota,$$

which corresponds to the expression stated. □

A.1.10 Proof of Proposition 8

Proof.

1. Since the matrix B contains only zeros, the highest average cycle in the matrix is 0. Thus $\lambda = 0$ and the network effect is null.

2. Users face the same signal for any neighborhood. They thus pick any source in their inbound neighborhood.

□

A.2 A Primer On Tropical Algebras

In this appendix, I introduce various definitions of the mathematical algebra I use in the last segment of the paper, namely a tropical algebra. This algebra is useful as it allows to describe the problem in a linear space, which reveals a tractable analytic solution.

Intuitively, this algebra is almost identical to the classical algebra teached in high-school except for one property: there is no unique inverse under addition. Hence $a \oplus x = b$ has no unique solution for x. Otherwise, properties are similar.

Definition (Tropical Algebra)**.** *A tropical algebra is a semiring over a set* $\mathbb{R}_{\max} \equiv \mathbb{R} \cup \{\infty\}$ *with an addition operator and a multiplication operator defined by:*

$$a \oplus b \equiv \max\{a, b\} \;\; \forall a, b \in \mathbb{R}_{\max}$$

$$a \otimes b \equiv a + b \; \forall a, b \in \mathbb{R}_{\max}$$

With such an algebra, the multiplicative and additive identities are given by $e = 0$ and $\epsilon = -\infty$. One can easily check that some usual properties of standard algebra (and some others) are met:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \qquad x \otimes (y \otimes z) = (x \otimes y) \otimes z \qquad \text{(associativity)}$$

$$x \oplus y = y \oplus x \qquad x \otimes y, = y \otimes x \qquad \text{(commutativity)}$$

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) \qquad \text{(distributivity)}$$

$$x \oplus \epsilon = \epsilon \oplus x = x \qquad \text{(zero element)}$$

$$x \otimes e = e \otimes x = x \qquad \text{(unit element)}$$

$$x \otimes \epsilon = \epsilon \otimes x = \epsilon \qquad \text{(absorbing element)}$$

$$\forall x \in \mathbb{R}, \exists! y : x \otimes y = e \qquad \text{(unique multiplicative inverse)}$$

$$x \oplus x = x \qquad \text{(idempotency of addition)}$$

Unacoutumed readers might want to check that the following equations are true:

$$10^2 \equiv 10 \otimes 10 = 20 \qquad (-0.5)^2 = -1$$

$$6 = 2 \otimes 4 \oplus 5 \qquad \sqrt{-1} = -0.5$$

$$7 = 3 \otimes 2^2 \oplus 1 \otimes 2 \oplus 1 \qquad 4^5 = 5^4 = 20$$

Matrix algebra can also be defined. I here describe its relationship with a Network:

Definition (Adjacency Matrix)**.** *Define the weighted adjacency matrix* $B \in \mathbb{R}_{\max}^{|V| \times |V|}$

by :

$$[B_{ij}] = \begin{cases} b_{ij} & \textit{if } j \in \eta_i, \\ -\infty & \textit{if } j \notin \eta_i. \end{cases}$$

The Adjacency matrix contains the entries of perceptions that must be added from each signal sent from the neighborhood. users that are not in the neighborhood of user i are weighted with the neutral element $(-\infty)$ under $\oplus$ and are thus never chosen (not accessible). For instance, the matrix of the network presented in Figure 2.2(b) is given by :

$$B = \begin{bmatrix} -\infty & 3 & 4 & -\infty \\ -\infty & -\infty & 1 & -\infty \\ -\infty & 1 & -\infty & -\infty \\ -\infty & 4 & 3 & -\infty \end{bmatrix}.$$

Definition (Irreducible Matrix)**.** *Let $B \in \mathbb{R}_{\max}^{|V|\times|V|}$, the matrix B is irreducible if it spans the full $|V|$-dimensional vector space or equivalently, if its determinant $|B|$ is not equal to ϵ.*

Definition (Inner Product)**.** *Let $a, b \in \mathbb{R}_{\max}^{|V|}$ be two vectors in a tropical space. Their inner product is defined as*

$$a \cdot b = \bigoplus_{i \in V} a_i \otimes b_i.$$

With such definition, the global behavior of the network can be modeled as an homogenous, first-difference equation:

$$s_{t+1} = B \cdot s_t$$

where B depicts the perceptions between each users.

Such matrix system then allows to define them in terms of eigenvalues and eigenvectors.

Definition (Eigenvalues and Eigenvectors)**.** *Let $B \in \mathbb{R}_{\max}^{|V|\times|V|}$. An eigenvalue λ of such matrix is a solution to the equation*

$$B \cdot v = \lambda \otimes v.$$

A particular vector v satisfying this equation for a given λ is called an eigenvector.

The (unique) eigenvalue of a network has an interpretation that depends on the notion of cycles and critical cycles. Every possible cycle in the network has an average net-spin increase over its edges. The eigenvalue is the highest average net spin.

Definition (Cycles)**.** *A simple cycle of length n in a graph $G(V, E)$ is a set of edges $\{a_{12}, a_{23}, \ldots, a_{n-1n}\} \subseteq E$ such that nodes $1 = n$, that no nodes is repeated and that $b_{ij} \neq \epsilon \forall b_{ij}$.*

Definition (Maximal Average Weight)**.** *Let $G(V, E)$ be a graph defined on a tropical algebra and denote C_G the set of all cycles . The maximal weight of a cycle of length is given by :*

$$\lambda = \max_{c \in C_G} \frac{\bigotimes_1^{n-1} c_{i,i+1}}{|c|} = \max_{c \in C_G} \frac{\sum_{i=1}^{n-1} c_{i,i+1}}{|c|}$$

For instance, the set of edges $\{e_{12}, e_{21}\}$ is a cycle (a loop) in Figure 2.2(b). It is a critical cycle since the average net spin over this loop $\left(\frac{1+1}{2}\right)$ is the highest amongst all possible cycles.

The following theorem sews all these definitions together:

Theorem 4 (Mathematical Results)**.** *Let $G(V, E)$ be an irreducible graph and let B be its adjacency matrix. Then:*

1. *B has a unique eigenvalue* $\lambda = \rho(c_c), c_c \in C_c(G)$*;*

2. *there exists eigenvectors* $B_1^*, \ldots A_p^*$ *that spans part of the graph space. Such vectors are the different permutations of the cycles belonging to* $C_c(G)$*;*

3. *Define* $A^* \equiv \bigoplus_{i=0}^{t^*}(B \otimes \lambda^{-1})^{pi}$ *where* t^* *is the transient time of the graph, then the different modes of the unique eigenvector are represented by the columns in* A^* *for which nodes are in at least one* $C_c(G)$*.*

4. *For any* s_0*, the system will enter a steady state after* t^* *periods. The steady state is characterized by* $s_{t+p} = s_t - \lambda p$*, and* $s_t = B^* \cdot s_0$ *when* $t > t^*$ *.*

5. *The transient time is finite.*

Proof. See Bacelli, Cohen, J. Olsder and Quadrat [2]. □

Each of the columns in B^* represent a different mode of the periodic regime after $t >^*$, so each of them have a neat interpretation. The components v_i of the eigenvector j represents the net spin increase of the signal $s_{j,0}$ to user i in the given mode.

For instance, the matrix B^* of the example in Figure 2.2(b) is given by the matrix $B \otimes -1$:

$$B^* = \begin{bmatrix} -\infty & 3 & 2 & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & 0 & -\infty & -\infty \\ -\infty & 2 & 3 & -\infty \end{bmatrix}.$$

One can thus verify that the vector $v_1 = [3, 0, -\infty, 2]'$ and $v_2 = [2, -\infty, 0, 3]'$ are eigenvectors that solve:

$$B^2 \cdot v_i = 1^2 \otimes v_j \ \forall i \neq j,$$

$$B \cdot (v_1 \oplus v_2) = 1 \otimes (v_1 \oplus v_2), i \neq j,$$

$$B^* \cdot v_i = v_i \ \forall i.$$

A.3 Details of the Example In Section 2.6.1

The optimal solution solves:

$$\frac{1}{(s^*_{1,0})^{r-2}} = p \sum_{t=0}^{\infty} \beta^t \left[[B^t]_{11} \left(\frac{1}{s^*_{1,t+1}} \right)^{r-1} + [B^t]_{21} \left(-\frac{1}{s^*_{1,t}} \right)^{r-1} \right],$$

$$\frac{1}{(s^*_{2,0})^{r-2}} = p \sum_{t=0}^{\infty} \beta^t \left[[B^t]_{12} \left(\frac{1}{s^*_{2,t}} \right)^{r-1} + [B^t]_{22} \left(\frac{1}{s^*_{2,t+1}} \right)^{r-1} \right].$$

Exploiting the fact that $A^5 = A$ and that signals only alternate in signs.

$$\frac{1}{(s^*_{1,0})^{r-2}} = \frac{p}{1-\beta^5} \left[\left(\frac{1}{s^*_{2,0}} \right)^{p-1} - \beta \left(-\frac{1}{s^*_{2,0}} \right)^{r-1} - \beta^2 \left(\frac{-1}{s^*_{2,0}} \right)^{r-1} + \beta^3 \left(\frac{1}{s^*_{2,0}} \right)^{r-1} + \beta^4 \left(\frac{1}{s^*_{2,0}} \right)^{r-1} \right],$$

$$\frac{1}{(s^*_{2,0})^{r-2}} = \frac{p}{1-\beta^5} \left[\left(\frac{-1}{s^*_{1,0}} \right)^{r-1} + \beta \left(\frac{-1}{s^*_{1,0}} \right)^{p-1} - \beta^2 \left(\frac{1}{s^*_{1,0}} \right)^{r-1} - \beta^3 \left(\frac{1}{s^*_{1,0}} \right)^{r-1} + \beta^4 \left(\frac{-1}{s^*_{1,0}} \right)^{r-1} \right].$$

So if $r-1$ is even, one can find:

$$(s^*_{2,0})^{r-1} = \frac{p}{1-\beta^5} \left[1 - \beta - \beta^2 + \beta^3 + \beta^4 \right] (s^*_{1,0})^{r-2},$$

$$(s^*_{1,0})^{r-1} = \frac{p}{1-\beta^5} \left[1 + \beta - \beta^2 - \beta^3 + \beta^4 \right] (s^*_{2,0})^{r-2},$$

which leads to :

$$s^*_{1,0} = \frac{p}{1-\beta^5} \left[\left[1 + \beta - \beta^2 - \beta^3 + \beta^4 \right]^{(r-1)} \left[1 - \beta - \beta^2 + \beta^3 + \beta^4 \right]^{(r-2)} \right]^{\frac{1}{(r-1)+(r-2)}},$$

$$s^*_{2,0} = \frac{p}{1-\beta^5} \left[\left[1 + \beta - \beta^2 - \beta^3 + \beta^4 \right]^{(r-2)} \left[1 - \beta - \beta^2 + \beta^3 + \beta^4 \right]^{(r-1)} \right]^{\frac{1}{(r-1)+(r-2)}}.$$

If $r-1$ is odd, one can find:

$$(s_{2,0}^*)^{r-1} = \frac{p}{1-\beta^5}\left[1-\beta+\beta^2+\beta^3+\beta^4\right](s_{1,0}^*)^{r-2},$$
$$(s_{1,0}^*)^{r-1} = \frac{p}{1-\beta^5}\left[-1-\beta-\beta^2-\beta^3+\beta^4\right](s_{2,0}^*)^{r-2},$$

which is the expression used in the paper.

A.4 A Formal Description of the Algorithm

A.4.1 Pseudo-Code

Data: $G(V, E)$, $\beta \in [0, 1)$ and $p \in \mathbb{R}^+$.
Result: $s_0^* \in \tilde{S}$.
Find λ;
Find t^* and T;
Initialize the set of nodes to treat to the entire set of nodes $v_t \leftarrow V$;
Initialize some seed for the solution: $s_0^* \leftarrow [0, 0, \ldots, 0]'$;
Initialize $maxBudget_i \leftarrow p\frac{|V|}{1-\beta}\ \forall i \in v_t$;
Initialize $upperBounds_i \leftarrow \infty\ \forall i \in v_t$;
while $v_t \neq \emptyset$ **do**
  $\tilde{s}_{\text{candidates}} \leftarrow \emptyset$;
  for $i \in v_t$ **do**
    for $j \in v_t$ **do**
      $s_{j,0}^* \leftarrow 0$;
    end
    $s_{i,0}^* \leftarrow maxBudget_i$;
    $s_{\text{candidates}} \leftarrow s_{\text{candidates}} \cup \{$`findValue`
    $(G, v_t, s_0^*, upperBounds, \beta, p, t^*, T, maxBudget))\ \}$;
  end
  $(i) \leftarrow i : i = \arg\max_{\tilde{s}_i \in s_{\text{candidates}}} I(\tilde{s}_i) - c(\tilde{s}_i)$;
  $s_0^* \leftarrow \tilde{s}_{(i)}$;
  $v_t \leftarrow v_t \cap (i)$;
  $upperBounds \leftarrow$ `findUpperBounds`(G, v_t, s_0^*, t^*, T);
  for $i \in v_t$ **do**
    $maxBudget_i \leftarrow \min(maxBudget_i - s_{(i),0}^*, upperBounds_i)$;
  end
end

Algorithm 1: A Simple Algorithm to Find the Optimal Solution

A.4.2 Python Code

```
"""
```

Input: $G(V,E), v_t \subseteq V, s_0^* \in \mathbb{R}^{|V|}, upperBounds \in \mathbb{R}^{|V|}, \beta \in [0,1), p \in \mathbb{R}^+, t^* \in \mathbb{N}, T \in \mathbb{N}$ and $maxBudget \in \mathbb{R}$.
Result: $s_0^* \in \tilde{S}$.
Set $\tilde{s}'_0 \leftarrow \tilde{s}^*_0$;
Set $[\tilde{s}'_0]_i \leftarrow [\tilde{s}'_0]_i + 1$;
while $\|\tilde{s}_0 - \tilde{s}'_0\| \geq 0$ **do** Find the closest Nash Equilibrium that supports such increase:
    $\tilde{s}_0 \leftarrow \tilde{s}'_0$;
    $\tilde{s}'_0 = h(\tilde{s}_0)$;
end
return $\tilde{s}'_0$;

Algorithm 2: The Function findValue Approaches s^*

Input: $G(V,E), v_t \subseteq V, s_0^* \in \mathbb{R}^{|V|}, t^* \in \mathbb{N}, T \in \mathbb{N}$.
Result: $upperBounds \in \mathbb{R}^{|V|}$.
$upperBounds_i \leftarrow \infty \ \forall i$;
for $t \in [0, \ldots, t^* + T]$ **do**
    $A \leftarrow A^{\otimes t}$;
    for $i \in V$ **do**
        $values = \emptyset$;
        for $j \in V$ **do**
            $values \leftarrow values \cup (value = A_{ij} + s^*_{j,0}, index = j)$;
        end
        $values \leftarrow \texttt{sort}(values, \text{sort on } value, \text{descending})$;
        $chosenSource = value_0$;
        if $chosenSource.index \notin v_t$ **then**
            for $j \in v_t$ **do**
                $currentItem \leftarrow values.find(index = j)$;
                $currentValue \leftarrow currentItem.value$;
                $upperBounds_j \leftarrow$
                $\min\left(upperBounds_i, chosenSource.value - currentValue + s^*_{j,0}\right)$;
            end
        end
    end
end
return $upperBounds$;

Algorithm 3: The Function findUpperBounds Ensures Stability on Past Solutions

```
#!/usr/bin/env python2.3
#
# contagion.py : A package to deal with the right spin.
#
#
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#
# The goal of the maxplus class is to have seemless integration of the maxplus
# algebra on standard operators.
# Hence:
# - a + b will is redefined as max(a, b) for any two vectors/matrices/scalar with
#   the same dimensions
# - a * b will yield:
# i) a + b for any two scalars
# ii) a_ij + b if a is a matrix and b is a scalar
# iii) A matrix/vector C with c_ik = max_j(a_ij + b_jk) if a, b are
# matrices/vectors with suitable dimensions
# - A**i will define the matrix A*A*...(i times)...A with the inner product
#    defined above
#
# The object maxplus is essentially the adjacency matrix of a graph and its
# components can be accessed through standard brackets :
```

```
# A[i][j] will yield the element a_ij of the matrix.
#
# It also has some other objects within:
# - eig (float) : the eigenvalue of the matrix. It must be computed before it
#    is acceeessed.
# - dim [int, int] : the dimension [i, j] of the matrix.
# - infinity (float): an approximation of infinity (10^18). This is so because
# a-infinity = a is the neutral operator in a maxplus algebra.
# - transient_time: the time required to reach a steady state. Must be
# computed before it is accessed.
# - period (int): the period of the network in the periodic regime. Must be
#  computed before it is accessed.
# - cycling_vectors ([maxplus]): a list of eigenvectors on the cyclical
# regime.
# E.g.: (A ** p)*v = (eig**p)*v for v an eigenvector on the list and
# p the period.
# - maxiter (int): the highest number of allowed iterations to find
# eig, p, etc. (should never be atteined)
# - epsilon (float): a tolerance factor for numerical comparisons (unused)
# - debug (bool): if true, standard output will display some things for
# debugging purpose while some function are running.
"""
class maxplus:
def __init__(self, size_i, size_j):
```

```
self.dim = [size_i, size_j]
self.infinity = -1 * pow(10, 18)
self.transient_time = pow(10, 18)
self.adj = [[self.infinity for j in range(size_j)] for i in range(size_i)]
self.epsilon = 10**(-4)
self.eig = self.infinity
self.cycling_vectors = []
self.maxiter = size_i * size_j + max(size_i, size_j)
self.period = self.infinity
self.debug = False

#Overloading the addition operator so that a + b = max(a,b) pointwise
def __add__(self, otherMaxPlusObject):
toto = maxplus(self.dim[0], self.dim[1])
if otherMaxPlusObject.dim[0] == self.dim[0] and otherMaxPlusObject.dim[1] \
== self.dim[1]:
for i in range(self.dim[0]):
for j in range(self.dim[1]):
toto.adj[i][j] = max(self[i][j], otherMaxPlusObject[i][j])
else:
print "No. Dimension mismatch!"
return toto

#Overloading the multiplication operator so that A \otimes b = A + b pointwise.
```

```
def __mul__(self, someObject):
if type(someObject) in {float, int}:
value = maxplus(self.dim[0], self.dim[1])
value.adj = [[self.adj[i][j] + someObject for j in range(self.dim[1])] for \
i in range(self.dim[0])]
elif isinstance(someObject, maxplus):
if self.dim[1] == someObject.dim[0]:
value = maxplus(self.dim[0], someObject.dim[1])
for i in range(self.dim[0]):
for j in range(someObject.dim[1]):
current = self.infinity
for k in range(self.dim[1]):
if self.adj[i][k] != self.infinity:
current = max(current, self.adj[i][k] + someObject.adj[k][j])
else:
current = max(current, self.adj[i][k])
value.adj[i][j] = current
else:
value = maxplus(1,1)
print "No. Dimension Mismatch"
else:
print "No. Types unsupported"
value = maxplus(1,1)
return value
```

```
#Right multiplication for integers: (for instance 3*A)
#Otherwise 3*A would not be supported
def __rmul__(self, someObject):
if type(someObject) in {float, int}:
value = self.__mul__(someObject)
else:
print "Right multiplication not supported for "+str(type(someObject))
value = False
return value

#Overloading the exponent operator A ** 2 == A*A
#(with * being the overloaded operator)
def __pow__(self, power):
value = maxplus(self.dim[0], self.dim[1])
if power > 0 :
value.adj = self.adj
for i in range(1, power):
value = self*value
elif power == 0:
for i in range(self.dim[0]):
for j in range(self.dim[1]):
if i == j:
value[i][j] = 0
```

```
else:
value[i][j] = self.infinity
else:
print "Impossible. The inverse does not exists."
return value

#Overloading of the bracket operator so that A[i][j] == A.adj[i][j]
def __getitem__(self, index):
return self.adj[index]

#Overloading of the bracket operator so that
#A[i][j] = number == A.adj[i][j] = number
def __setitem__(self, index, element):
self.adj[index] = element

#Returns a maxplus object of [size_i, size_j] with zeros everywhere
def unit(self, size_i, size_j):
unit = maxplus(size_i, size_j)
for i in range(size_i):
for j in range(size_j):
unit[i][j] = 0
return unit

#Returns the set of sources of node_index
```

```
def n_i(self, node_index):
neighbor_set = []
for i in range(self.dim[1]):
if self[node_index][i] != self.infinity:
neighbor_set.append(i)
return neighbor_set

#Returns the potential listeners of node_index
def audience(self, node_index):
audience_set = []
for i in range(self.dim[0]):
if self[i][node_index] != self.infinity:
audience_set.append(i)
return audience_set

#A quick display function that shows the adjacency matrix on standard output
#(da = "Display adjacency")
def da(self):
for i in range(len(self.adj)):
output = ""
for j in range(len(self.adj[i])):
if self.adj[i][j] == self.infinity:
output += "-\t"
else:
```

```
output += str(self.adj[i][j])+"\t"
if j == len(self.adj[i])-1:
print output+"\n"

"""
# Function grn: generates a random network.
# Input :
# - nodeSize: the number of vertex on the network.
# - min_perception : the smallest allowed value for b_ij (an integer)
# - max_perception: the highest allowed value for b_ij (an integer)
# Ouput: A maxplus object with the following properties:
# - All elements b_ij of the adjacency matrix are integers
# - There is at least one constrained star on the network.
# - The hub of the constrained star is a critical cycle.
# (this avoids network splits)
#   - All the other links are built at random.
"""
def grn(self, nodeSize, min_perception, max_perception):
the_matrix = maxplus(nodeSize, nodeSize)
from random import Random
test = Random()
possible_links = []
for i in range(self.dim[0]):
for j in range(self.dim[1]):
```

```
if i != j :
possible_links.append([i,j])
#Building at least one cycle
cycle_size = test.randint(2, nodeSize)
for i in range(0, cycle_size):
if i != cycle_size-1:
the_matrix.adj[i][i+1] = test.randint(max_perception-1, max_perception)
possible_links.remove([i,i+1])
else:
the_matrix.adj[i][0] = test.randint(max_perception-1, max_perception)
possible_links.remove([i,0])
#Introduce a constrained star structure in the network
connection_set = range(0, cycle_size)
to_be_connected = range(cycle_size, nodeSize)
while len(to_be_connected) > 0:
inside_node = to_be_connected.pop()
outside_node = connection_set[test.randint(0,len(connection_set)-1)]
the_matrix.adj[inside_node][outside_node] = \
test.randint(min_perception, max_perception - 1)
possible_links.remove([inside_node,outside_node])
connection_set.append(inside_node)
#Constrained star built and currently nodeSize+1 links
additional_links = test.randint(0, len(possible_links) - 1)
#Add other links everywhere but on the diagonal
```

```
for i in range(additional_links):
  link_id = test.randint(0, len(possible_links) - 1)
  link = possible_links[link_id]
  the_matrix.adj[link[0]][link[1]] = \
test.randint(min_perception, max_perception - 1)
  possible_links.remove(link)
return the_matrix

"""
#find_properties: A function that finds eigenvalues,
#  cycling eigenvectors (A^t x = lambda^t x),
# the transient time and the period of the network.
#Input: nothing
#Output: nothing, but some parts of the object are instanciated:
# - self.eig is set
# - self.cycling_vectors is set
# - self.transient_time is set
# - self.period is set.
#Algorithm: The algorithm is an adapation of the power method as described in
# Linear Algebra and its Applications 302-303 (1999) pp. 17-32, Elsevier.
# Retrieved online at: yaroslavvb.com/papers/elsner-on.pdf
#
# The logic of Algoritmh 3.4 (p. 28) is used here, with
# adaptation to cyclical matrices.
```

```
# A cyclical matrix is periodically irreducible.
#
# The algorithm finds:
#  - the eigenvalue
#  - the cycling eigenvectors
#  - the transit time
#  - the period
#
#Key requirement: The adjacency matrix must be filled with integers
#  (as with self.gnr)
# (Comparison between numbers is strict. Numerical discrepancies will
#   lead to errors.
# This can be easily be fixed by changing the condition:
# if q_current.adj[k-1][0] != q_current[k][0]:
# By:
# if abs(q_current.adj[k-1][0] -q_current[k][0]) > self.epsilon:
# )
"""
def find_properties(self):
i = 0
A_current = self**0
A_star = maxplus(self.dim[0], self.dim[1])
A_star = A_star + A_current
filled = False
```

```
while i < self.maxiter and filled == False:
filled = True
for i in range(self.dim[0]):
for j in range(i+1):
if A_star[i][j] == self.infinity:
filled = False

A_current = A_current * self
A_star = A_star + A_current
i += 1

self.transient_time = i - 1
current_vector = self.unit(self.dim[0], 1)
current_vector[0][0] = self.dim[0]**2 #One large value for state memory.
#Otherwise, the network cannot always distinguish nodes
#in a periodic regime.
vector_sequence = [current_vector]
while i < self.maxiter:
next_vector = self * current_vector
vector_sequence.append(next_vector)
j = 0
if self.debug == True:
print "Current vector at hand: ", next_vector.adj
while j < len(vector_sequence) - 1:
```

```
if self.debug == True:
print "Vector in the sequence and its index j: ", vector_sequence[j].adj, j
q_current = self.q_i(next_vector, vector_sequence[j], \
len(vector_sequence) - j - 1)
if self.debug == True:
print "Current difference : ", q_current.adj
print "Length of vec sequence and index j :", len(vector_sequence), j
raw_input("Press a key")
all_equal = True
for k in range(1, q_current.dim[0]):
if q_current.adj[k-1][0] != q_current[k][0]:
all_equal = False
if self.debug == True:
print "All equal == ", all_equal
if all_equal == True:
self.eig = q_current.adj[0][0]
self.period = len(vector_sequence) - j - 1
self.cycling_vectors = []
for k in range(j, len(vector_sequence)-1):
self.cycling_vectors.append(vector_sequence[k] * (-k * self.eig))
i = self.maxiter
j = i + 1
j += 1
```

```
current_vector = next_vector
i = i + 1

if len(self.cycling_vectors) == 0:
self.cycling_vectors.append(maxplus(self.dim[0], 1))
else:
#Normalize to unit vectors
for i in range(len(self.cycling_vectors)):
max_el = -1 * self.infinity
for j in range(self.dim[0]):
if max_el > self.cycling_vectors[i][j][0]:
max_el = self.cycling_vectors[i][j][0]
self.cycling_vectors[i] = -max_el * self.cycling_vectors[i]

#See Lemma 3.1 in the paper for the definition of q_i
#In short:  q_i ==  ((A^t*x)_i - x_i) / t whith a max-plus inner product on Ax
def q_i(self, vec_ATx, vec_x, t):
q_i = maxplus(vec_ATx.dim[0], vec_ATx.dim[1])
for i in range(len(vec_ATx.adj)):
q_i.adj[i][0] = float(vec_ATx.adj[i][0] - vec_x.adj[i][0]) / t
return q_i

#Normalizes the matrix so that the eigenvalue equals zero.
#The eigenvalue must be known.
```

```
def normalize(self):
if self.eig != 0:
for i in range(self.dim[0]):
for j in range(self.dim[1]):
if self.adj[i][j] != self.infinity:
self.adj[i][j] -= self.eig
self.eig = 0


"""
# argmmaxA: returns the index of each node selected at each period
#
# Input: v0 (maxplus): the solution to check.
# Output: sol_return (s_tilde): all the choices made.
#
# Order of O((t+p)2|V|^2)
"""
def argmaxA(self, v0):
sol_return = s_tilde(self.dim[0], self.transient_time,  self.period)
At_current = self ** 1
for t in range(1, self.transient_time + self.period):
for i in range(self.dim[0]):
current_value = self.infinity
current_choice = -1000
```

```
for j in range(self.dim[1]):
if At_current[i][j] + v0[j][0] >= current_value:
current_choice = j
current_value = At_current[i][j] + v0[j][0]
sol_return.k_it_form[current_choice][t] += 1
At_current = At_current * self
return sol_return

"""
# A Class that manages a nash equilibrium in its exact form:
# - k_it_form, an array of arrays containing the choices k_it
# for t in range of 0 up to the transient time + period
# - A function that collapses the k_it_form into a max_plus vector,
# containing an inexact (float) version of the solution.
# This is handy as the exact solution can be used to perform exact
# tests without worries about numerical errors.
"""
class s_tilde:
#Initialize the instance
def __init__(self, dim_s, transient_time, period):
self.k_it_form = []
for i in range(dim_s):
some_k_i = [1]
for t in range(1, transient_time + period):
```

```
some_k_i.append(0)
self.k_it_form.append(some_k_i)
self.transient_time = transient_time
self.period = period
#Operator overloading of the equality operator:
#Checks if the arrays containing the exact form are equal
def __eq__(self, other_tilde):
return self.k_it_form == other_tilde.k_it_form

#Operator overloading of the inequality operator:
#Checks if the arrays containing the exact form are not equal
#(I don't know why the two are not tied together)
def __ne__(self, other_tilde):
return self.k_it_form != other_tilde.k_it_form

"""
# Function that collapses the k_it array into a maxplus vector
# For instance, if the solution is:
# b^0 b^1  b^2
# n1  1    3  3
#   n2 1   1  1
#   n3  1  0 0
# n4  1  0 0,
#   the function will collapse this k_it_form into (if t* = 1 and p = 2)
```

```
# n1  1 + 3 * b / (1-b)
#   n2 1 + b / (1-b)
#   n3  1
# n4  1.
"""
def find_init_form(self, beta):
init_form = maxplus(len(self.k_it_form), 1)
init_form = init_form.unit(len(self.k_it_form), 1)
for i in range(len(self.k_it_form)):
for t in range(len(self.k_it_form[i])):
if t < self.transient_time:
init_form[i][0] += (beta ** t) * self.k_it_form[i][t]
else:
init_form[i][0] += (beta ** t) / (1 - beta ** self.period) * self.k_it_form[i][t]
return init_form

"""
# This class basically handles the previous objects. It executes the algorithm
# and finds the optimal solution.
"""
class wom:
def __init__(self, size, min_perception, max_perception, beta):
self.A = maxplus(size, size)
self.A = self.A.grn(size, min_perception, max_perception)
```

```
self.A.find_properties()
self.beta = beta
self.debug = False

"""
# Computes the value \sum_t beta^t \sum_i s_jt - \sum_i 1/2(s_i0)^2
# for a given initial seed v0
"""
def value(self, v0):
A_norm = maxplus(self.A.dim[0], self.A.dim[1])
for i in range(self.A.dim[0]):
for j in range(self.A.dim[1]):
A_norm[i][j] = self.A[i][j]

if len(A_norm.cycling_vectors) == 0:
A_norm.find_properties()
A_norm.normalize()
if self.debug == True:
print "After normalization: "
A_norm.da()
v0_max = maxplus(self.A.dim[0], 1)
for i in range(self.A.dim[0]):
v0_max[i][0] = v0[i][0]
vector_set = [v0_max]
```

```
for i in range(1,self.A.transient_time + self.A.period + 1):
vector_set.append((self.A**i)*v0_max)
value = 0
beta = self.beta
if self.debug == True:
print "Value : ", value
for i in range(0, self.A.transient_time):
current_vector = 0
vi = vector_set[i]
for j in range(self.A.dim[0]):
current_vector += vi[j][0]
current_vector *= (beta**i)
value+=current_vector
if self.debug == True:
print "Value of transient time: ", value
raw_input("press a key...")
p = self.A.period
for i in range(0, p):
current_vector = 0
if self.debug == True:
vector_set[self.A.transient_time + i].da()
for j in range(self.A.dim[0]):
current_vector += vector_set[self.A.transient_time + i][j][0]
current_vector = current_vector * (beta ** \
```

```
(self.A.transient_time + i)) / (1 - (beta ** p))
value += current_vector
if self.debug == True:
print "Value of transient time + period: ", value
raw_input("press a key...")
for i in range(self.A.dim[0]):
value += -1 * float((v0[i][0] ** 2)) / 2
value += float(self.A.eig * self.A.dim[0] * self.beta) / (1 - self.beta)
if self.debug == True:
print "Total Value - costs: ", value
raw_input("press a key...")
return value


"""
# For a given initial vector v0, the function below
# finds the closest vectors s_tilde (nash eq)
# that can be sustained.
#
# Typical use: overload one node to a very large value
# to see which nash eq. can be sustained.
#
# Input: a |V|x1 maxplus object
# Output: an s_tilde object
```

```
# Order of at most |V|*(t+p)*O(ArgmaxA)
"""
def find_nash_eq(self, v0):
s_current = self.A.argmaxA(v0)
s_current_next = self.A.argmaxA(s_current.find_init_form(self.beta))
while s_current != s_current_next:
s_current = s_current_next
s_current_next = self.A.argmaxA(s_current_next.find_init_form(self.beta))
return s_current

"""
# compute_upper_bound: Computes what is the maximal value a node can reach
# so that it does not change the value of nodes in the set "treated_nodes".
#
# Typical use: once a set of nodes have been found to be part of the variance
# maximizing solution (the treated nodes), one wants to find the highest value
# that other nodes can support without changing the values in treated nodes.
# In order to do this, one must know the largest value each node can take
# without changing treated nodes.
#
# Input:
# - v0 (maxplus): a Nash equilibria with some treated nodes.
# - treated_nodes (list): a list of nodes that must remain unchanged in v0
# - node_index (int): the node for which one computes the upper bound.
```

```
#
# Output: upper_bound (float): the highest value that node_index can take.
#
"""
def find_upper_bounds(self, v0, treated_nodes):
import operator
A_current = self.A
upper_bound = maxplus(self.A.dim[0], 1)
for i in range(self.A.dim[0]):
upper_bound[i][0] = -1 * self.A.infinity
for t in range(1, self.A.transient_time + self.A.period):  #for each period
for i in range(self.A.dim[0]): #for each row
current_value = self.A.infinity
current_choice = -1
for j in range(self.A.dim[0]): #for each column
if A_current[i][j] + v0[j][0] > current_value:
current_choice = j
current_value = A_current[i][j] + v0[j][0]
if current_choice in treated_nodes:
for j in range(self.A.dim[0]):
if j != current_choice:
upper_value = current_value - A_current[i][j]
if upper_value < upper_bound[j][0]:
upper_bound[j][0] = upper_value
```

```
return upper_bound

"""
# find_value: finds if there is any nash equilibria where some node has to be
# within some upper and lower bounds.
#
# Typical use: find what is the largest nth seed given that the nth-1 seeds have
# been found.
#
# Input:
# - v_current (maxplus): the current nash equilibrium.
# - lower_bound (float): the lowest value one node can have.
# - higher_bound (float): the highest value one node can have.
# - node_index (int): the node investigated
# - max_budget (float): the maximal amount of seed the node can have.
#
# Output: return_value, a nash eq. (s_tilde object) wich has a better or equal
# value than v_current.
"""
def find_value(self, v_current, treated_nodes, untreated_nodes, \
upper_bounds, max_budget):
upper_bound_check = False
solution_check = maxplus(self.A.dim[0], 1)
return_value = self.find_nash_eq(v_current)
```

```
upper_bound = 0
max_index = untreated_nodes[0]
#print "Max budget: ", max_budget
#print "Set of treated nodes: ", treated_nodes
#print "Set of untreated nodes: ", untreated_nodes
#print "Current optimal solution: ", return_value.k_it_form
for j in treated_nodes:
solution_check[j][0] = v_current[j][0]

solution_value = self.value(return_value.find_init_form(self.beta))

for i in untreated_nodes:
if max_budget <= upper_bounds[i][0]:
upper_bound = max_budget
else:
upper_bound = upper_bounds[i][0] - pow(10, -10)
for j in untreated_nodes:
solution_check[j][0] = 0
solution_check[i][0] = upper_bound
solution_check_tilde = self.find_nash_eq(solution_check)
solution_check_tilde_value = self.value(solution_check_tilde.\
find_init_form(self.beta))

if self.debug == True:
```

```
print "Current solution value: ", solution_value
print "Newly calculated solution: ", solution_check_tilde_value
print "Newly calculeted equilibrium: ", solution_check_tilde.k_it_form
if solution_value < solution_check_tilde_value:
return_value = solution_check_tilde
solution_value = solution_check_tilde_value
max_index = i
#print "Value of max_index in if: ", max_index
return [max_index, return_value]

"""
# optimize: Finds s_star over the given network.
#
# Input: void
#
# Output: optimal (s_tilde object), the optimal solution.
"""
def optimize(self):
optimal = s_tilde(self.A.dim[0], self.A.transient_time, self.A.period)
nodes_to_treat = range(0, self.A.dim[0])
treated_nodes = []
while len(nodes_to_treat) > 1:
optimal_init_form = optimal.find_init_form(self.beta)
max_budget = self.A.dim[0] / (1-self.beta)
```

```
for i in treated_nodes:
max_budget -= optimal_init_form[i][0]
node_max = -1
upper_bounds = self.find_upper_bounds(optimal_init_form, treated_nodes)
[node_max, optimal] = self.find_value(optimal.find_init_form(self.beta), \
treated_nodes, nodes_to_treat, upper_bounds, max_budget)
if self.debug == True:
print "Optimal solution updated with: ", optimal.k_it_form
nodes_to_treat.remove(node_max)
treated_nodes.append(node_max)
return optimal
```

Appendix B

Appendix For the Third Essay

B.1 Proof of various propositions

In the proofs below, I will denote F_p^2, F_p' as respectively shorthands for the second and first derivatives of F with respect to τ. Likewise, I will denote $\tau^2(p, t_i, \tau'(p, t_i)$ for the derivatives with respect to t_i.

B.1.1 Proposition 13

Proof. 1. A student wit talent τ will go to university if $V_p(\tau t_i) > C_p$ and since V is strictly increasing, this is equivalent to :

$$\tau > \frac{V_p^{-1}(C_p)}{t_i}$$

hence, there exists a value of $\tau(p, t_i) = \frac{V_p^{-1}(C_p)}{t_i}$ that is decreasing with quality. Such number represents the marginal student indifferent between going to university or not.

2. The quantity $F_p(1) - F_p(\tau(p, t_i))$ represents the total number of persons willing

to go to universities. Since F_p is decreasing in τ and τ is decreasing in t_i, it follows at once that $F_p(1) - F(\tau)$ is increasing in t_i.

□

B.1.2 Proposition 14

Proof. For any p, the min operator implies that

$$f_i = F_p(1) - F(\tau(p, t_i)). \tag{B.1}$$

Since $\tau(p, t_i)$ is strictly decreasing in t_i and that F_p is strictly decreasing in τ, there is a unique solution to such equation. Given this equality and symmetry, the first order condition satisfies:

$$[F_p(1) - F_p(\tau(p, t_i))] = (\alpha_p - t_i - 1 - \beta_p \rho) \underbrace{-F_p' \tau'(p, t_i)}_{>0}$$

which defines a unique t_i^*. Since this is a maximum, the second order condition is negative at the optimum:

$$0 > 2F_p'(\tau(p, t_i))\tau'(p, t_i) - (\alpha_p - t_i - 1 - \beta_p \rho) \left[F_p^2(\tau(p, t_i))^2 + F_p' \tau^2(p, t_i)\right]$$

The derivative of B.2 with respect to α_p is given by the implicit function:

$$-F_p' \tau'(p, t_i^*) \frac{\partial t_i^*}{\partial \alpha_p} = \left(1 + \frac{\partial t_i^*}{\partial \alpha_p}\right) F_p' \tau'(p, t_i^*) - (\alpha_p - t_i - 1 - \beta_p \rho) \left[F_p^2(\tau(p, t_i))^2 + F_p' \tau^2(p, t_i)\right] \frac{\partial t_i^*}{\partial \alpha_p}$$

$$\Rightarrow \frac{\partial t_i^*}{\partial \alpha_p} = \frac{F_p' \tau'(p, t_i^*)}{2F_p'(\tau(p, t_i))\tau'(p, t_i) - (\alpha_p - t_i - 1 - \beta_p \rho) \left[F_p^2(\tau(p, t_i))^2 + F_p' \tau^2(p, t_i)\right]} \tag{B.2}$$

$$> 0$$

With similar calculations, the derivatives with respect to β_p can be found to be:

$$\frac{\partial t_i^*}{\partial \beta_p} = -\rho \frac{F_p' \tau'(p, t_i^*)}{2F_p'(\tau(p,t_i))\tau'(p,t_i) - (\alpha_p - t_i - 1 - \beta_p \rho)\left[F_p^2(\tau(p,t_i))^2 + F_p' \tau^2(p,t_i)\right]} \tag{B.3}$$

By (B.1), this implies that:

$$\frac{\partial f_i^*}{\partial \alpha_p} = \underbrace{-F_p' \tau'(p, t_i^*)\frac{\partial t_i^*}{\partial \alpha_p}}_{>0} \tag{B.4}$$

$$\frac{\partial f_i^*}{\partial \beta_p} = \underbrace{-F_p' \tau'(p, t_i^*)\frac{\partial t_i^*}{\partial \beta_p}}_{<0} \tag{B.5}$$

Now the selection of the program p^* will be the one that maximizes income. Since the set of programs is finite, this p^* exists and the university will open a new facility if the generated income is positive. □

B.1.3 Proposition 15

Proof. From equation B.2, the implicit function yields the following derivative with respect to ρ:

$$\frac{\partial t_i^*}{\partial \rho} = -\beta_p \frac{F_p' \tau'(p, t_i^*)}{2F_p'(\tau(p,t_i))\tau'(p,t_i) - (\alpha_p - t_i - 1 - \beta_p \rho)\left[F_p^2(\tau(p,t_i))^2 + F_p' \tau^2(p,t_i)\right]}.$$

From equation B.1, one can find that :

$$\frac{\partial f_i^*}{\partial \rho} = -F_p' \tau'(p, t_i^*)\frac{\partial t_i^*}{\partial \rho}$$

□

B.1.4 Proposition 16

Proof. Let U_χ, U_S denote the first and second derivatives of the function U. The first order conditions are given by:

$$0 = -U_\chi 2\frac{\partial f_i}{\partial t_i}\frac{1}{S} - U_\chi \frac{G-2(f_i+F)}{S^2}2\frac{\partial f_i}{\partial t_i} + U_S 2\frac{\partial f_i}{\partial t_i}$$

Since $\frac{\partial f_i}{\partial t_i} \neq 0$, we get:

$$U_\chi \left[\frac{1}{S} + \frac{\chi}{S}\right] = U_S$$

The term between brackets is positive and there is thus an optimal solution S^{opt}. Such S^{opt} can then be used to recover f_i^{opt}, t_i^{opt} through the equation:

$$\begin{aligned} S^{opt} &= 2f_i \\ &= 2(1-\rho)\left[F_p(1) - F_p(\tau(p, t_i^{opt}))\right]. \end{aligned}$$

□

B.1.5 Proposition 17

Proof. If $f_i^{opt} = t_i^{opt} = 0$, the optimal solution is to have no facility. Hence, setting $\alpha_p = 0\ \forall p$ sets the income generated by the facility to zero, which is below the fixed cost. Finding such solution for any program leads to

If the optimal solution is to open the facility, there exists a solution to $f_i^*(\alpha_p, 0) = f_i^{opt}$ since f_i^* is strictly increasing in α_p. Hence, it crosses f_i^{opt} at one point. The rest of the income is simply transferred through T_i:

$$T_i^{opt} = \frac{G - \alpha_p^{opt} f_i^{opt}}{2}.$$

□

B.1.6 Proposition 18

Proof. The proof is quite similar to the last one. If $t_i^{opt} = f_i^{opt}$, the government sets $\beta_p = \frac{\bar{\alpha}_p}{\rho}$ for all programs so that the university has no additional income from the facility.

So if $t_i^{opt} \neq 0$, the equation $f_i^*(\bar{\alpha}_p, \beta_p) = f_i^{opt}$ has a solution since f_i is strictly decreasing in β_p and therefore crosses f_i^{opt} only once. Now, we find T_i^{opt} to obtain budget balance:

$$T_i^{opt} = \frac{G - 2(\bar{\alpha}_p + \beta_p^{opt})f_i^{opt}}{2},$$

which concludes the proof. □

B.1.7 Proposition 19

Proof. For a given program p and a given behaviour of the competing university, the first order condition of the university is given by:

$$U_{t_i} + U_\Pi \frac{\partial \Pi}{\partial t_i} = 0,$$
$$U_\pi \frac{\partial \Pi}{\partial f_i} = 0.$$

Because the function U is strictly increasing in both of its argument, there is a unique solution $t_i(\alpha_p, \beta_p), f_i(\alpha_p, \beta_p)$. Likewise, because U is concave, the derivatives of these quantities with respect to α_p, β_p are of the same sign as in proposition 19. Thus, the government can still use these channels to reach the first best. □