Abstract

This dissertation studies volatility measurement and modeling issues when asset prices are subject to price limits based on Bayesian approaches. Two types of estimators are developed to consistently estimate integrated volatility in the presence of price limits. One is a realized volatility type estimator, but using both realized asset prices and simulated asset prices. The other is a discrete sample analogue of integrated volatility using posterior samples of the latent volatility states. These two types of estimators are first constructed based on the simple log-stochastic volatility model in Chapter 2. The simple log-stochastic volatility framework is extended in Chapter 3 to incorporate correlated innovations and further extended in Chapter 4 to accommodate jumps and fat-tailed innovations. For each framework, a MCMC algorithm is designed to simulate the unobserved asset prices, model parameters and latent states. Performances of both type estimators are also examined using simulations under each framework. Applications to Chinese stock markets are also provided.
Dedication

To my family for their unconditional love, support and understanding throughout my life.
Acknowledgments

I am grateful to Morten Nielsen for his encouragement and invaluable guidance regarding my research. His profound knowledge and great passion for research have always been and will continue to be an inspiration.

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I take full responsibility for any remaining errors.
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Chapter 1

Introduction

Volatility measurement, modeling and forecasting have been among the most active research areas in financial econometrics due to their central role in asset pricing, portfolio choice and risk management. Since volatilities are latent, many models and inference procedures have been developed to study the dynamic behavior of latent volatilities including ARCH (Engle 1982) and GARCH (Bollerslev 1986) classes of models, stochastic volatility (SV) models (Taylor 1982, Heston 1993, Harvey, Ruiz, and Shephard 1994, Duffie, Pan, and Singleton 2000), ARCH filters and smoothers (Nelson 1992, Nelson 1996), and realized volatility (Merton 1980, Andersen and Bollerslev 1998). These models and procedures can be classified into two categories, parametric volatility models and nonparametric volatility measurement, depending on whether they assume certain functional forms for the underlying volatility process. Another distinction between these two categories is that parametric volatility models usually model expected volatilities while nonparametric measurements usually focus on notional volatilities. By its essence, integrated volatility is a notional volatility. It
measures the aggregated volatility level over a certain time interval. Realized volatility (RV), as a nonparametric ex-post volatility measurement, can estimate integrated volatility (IV) very well with finely-sampled financial data in a frictionless market. With the availability of increasing amounts of high frequency financial data, realized volatility has become one of the most popular volatility measurements, see e.g. Andersen, Bollerslev, and Diebold (2009) and the references therein for a review.

Although microstructure noise has been widely discussed in the realized volatility literature (Aït-Sahalia, Mykland, and Zhang 2005, Zhang, Mykland, and Aït-Sahalia 2005, Bandi and Russell 2006, Bandi and Russell 2008, Hansen and Lunde 2006), there are other frictions, such as price limits, that have not been investigated. Many countries, both developing and developed, impose daily fluctuation limits on prices of different financial assets, such as stocks, foreign exchange rates, options, etc. Price limits pre-specify the maximum range, usually both upward and downward, within which asset prices are allowed to move within a single day. Not all countries (e.g. the US) impose price limits on financial markets. Countries that do not impose such price limits employ similar regulatory tools (e.g. market wide circuit-breakers and individual stock circuit-breakers). Although such government intervention provides a cooling period for investors to evaluate all of the relevant information and can help to deter market manipulation, it prevents financial asset prices from fully revealing all the information. Such information loss could bias investors’ evaluation of underlying risks of financial assets.

My thesis focuses on the estimation of integrated volatility when daily asset price fluctuations are restricted by price limits. In the presence of price limits, transactions out of the limit are forbidden and thus do not occur. Properties of realized volatility
based on frictionless market assumptions can not be guaranteed.

It is important to correctly measure integrated volatility when price limits are present. Hull and White (1987) show that integrated volatility is a key parameter in determining the options price on assets with stochastic volatilities when asset prices are allowed to fluctuate freely. Even with price limits, it is still the integrated volatility over the life of options that determines the value of the options because the installation of price limits only prevents equilibrium asset prices from revealing themselves when they fall outside of the pre-specified range. Once investors adjust their expectations to incorporate price limits, the underlying asset process does not depend on whether or not price limits take effect. Therefore, results in Hull and White (1987) remain correct as long as we assume stochastic volatilities for the underlying asset process.

In Chapter 2 I suggest a Bayesian approach to simulate the unobserved transactions and latent volatilities based on a simple log-stochastic volatility model. Two new estimators, which I call quasi-realized volatility and simulated integrated volatility, respectively, are suggested. Quasi-realized volatility is constructed in a way that is analogous to realized volatility which itself is a discrete analogue of the quadratic variation of the underlying asset process, by using both realized and simulated asset prices, while simulated integrated volatility is constructed as a discrete analogue of integrated volatility.

In Chapter 3 I include the leverage effect in the log-stochastic volatility model to generalize the methods developed in Chapter 2. In Chapter 4 I incorporate jumps and fat-tailed innovations in returns in the log-stochastic volatility model with the leverage effect to further generalize the application of quasi-realized volatility and
simulated integrated volatility. In Chapter 5 I conclude.
Chapter 2

Bayesian Estimation of Integrated Volatility with Price Limits: Simple Log-Stochastic Volatility Model

2.1 Introduction

In modern dynamic asset pricing theory, asset prices and latent volatility states are described by differential equations for reasons of analytical tractability. Different from the GARCH family models, stochastic volatility models allow for separate error terms for conditional mean and conditional variance processes. Among all stochastic volatility models, the log-stochastic volatility model (Taylor 1982) and the square-root volatility model (Heston 1993) are among the most popular. The log-stochastic
volatility model assumes that the log of conditional variance follows an AR(1) process. This guarantees the positivity of conditional variances. However, this class of models assumes constant volatility for the conditional variance process and does not fall into the class of affine processes, which implies additional computational cost to calculate options prices and optimal portfolio weights. The square-root volatility model assumes that the diffusive volatility of the conditional variance process is a function of the square root of conditional variance itself. This class of models allows for time-varying volatilities for the conditional variance process and also falls into the affine process class, which leads to analytical solutions to option pricing and optimal portfolio weights. However, the positivity of conditional variances can not be guaranteed in this class of models. Since this dissertation focuses on the estimation of integrated volatility, and not option pricing, I use the log-stochastic volatility model for its simplicity.

I make the following contributions in this chapter. First, I identify a volatility measurement issue caused by price limits using a simulation experiment. Second, I incorporate price limits in the simple log-stochastic volatility model. Third, I design a MCMC algorithm to simulate the unobserved asset prices, model parameters and latent states. Forth, I develop two types of estimators for integrated volatility when price limits are present. I also apply both newly developed estimators to the Chinese stock markets and provide some empirical results.

The rest of the chapter is organized as follows. In Section 2.2 I specify the log-stochastic volatility model and introduce price limits into the framework. I also use a simulation experiment to illustrate how the introduction of price limits affects the estimation of integrated volatility. In Section 2.3 I briefly review the estimation
methods developed for log-stochastic volatility models and design a MCMC sampling
scheme to incorporate price limits. I also discuss the convergence property of the
designed MCMC algorithm and introduce quasi-realized volatility and simulated-
integrated volatility as estimators of integrated volatility when price limits are present.
In Section 2.4 I examine the performance of the newly developed estimators through
simulation studies. In Section 2.5 I apply my method to the Chinese stock markets
using high frequency data from both the Shanghai Stock Exchange and the Shenzhen
Stock Exchange and provide some empirical results. I conclude the chapter in Section
2.6.

2.2 Price Limits and Log-Stochastic Volatility Model

In the log-stochastic volatility model, the log asset price $p_t$ and its log diffusive volatil-
ity $h_t$ solve the following two differential equations

\begin{align*}
    dp_t &= \mu_t dt + \exp(h_t/2) dW^s_t, \quad (2.1) \\
    dh_t &= \kappa(\beta - h_t) dt + \sigma dW^\sigma_t, \quad (2.2)
\end{align*}

where $\mu_t$ is the equity risk premium, $\sigma$ is the diffusive volatility of conditional vari-
ances, $\kappa$ and $\beta$ are two parameters of the conditional variance process, and $W^s_t$, $W^\sigma_t$
are two independent Brownian motions.

Then the integrated volatility over period $[t_0, t_m]$ is defined as

\begin{equation}
    IV_{[t_0,t_m]} = \int_{t_0}^{t_m} \exp(h_s) ds.
\end{equation}
To focus on volatilities, I assume $\mu_t$ is equal to zero. An Euler time discretization of the model in equations 2.1 and 2.2 implies that

$$y_{t\Delta} \equiv p_{t\Delta} - p_{(t-1)\Delta} = \exp\left(\frac{h_{t\Delta}}{2}\right) \varepsilon_{t\Delta},$$  \hspace{1cm} (2.3)

$$h_{(t+1)\Delta} = \beta + \phi (h_{t\Delta} - \beta) + \eta_{t\Delta},$$  \hspace{1cm} (2.4)

with $\phi \equiv (1 - \kappa \Delta)$ where $\Delta$ is the distance between two observations. Here, $\varepsilon_{t\Delta}$ and $\eta_{t\Delta}$ are normally distributed with zero mean, and covariance matrix $\Sigma = \Delta \begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 \end{bmatrix}$.

In this chapter I assume there are $m$ equidistant observations in each trading day. This assumption can be relaxed easily and would not change the results as long as the distances between adjacent observations are very small ($\rightarrow 0$). Therefore $\Delta = \frac{1}{m}$.

When a symmetric price limit $l > 0$ (in percentage) is imposed, that is the price of a stock in a trading day can not increase or decrease by more than $l$ percent of its previous closing price, the following observation rule takes effect:

$$p_{ti} = \begin{cases} p_{ti} & \text{if } \ln(1 - l) \leq p_{ti} - p_{t0} \leq \ln(1 + l), \\ \text{Unobservable} & \text{otherwise}, \end{cases}$$  \hspace{1cm} (2.5)

for $i = 1, 2, \cdots, m$, where $p_{t0}$ denotes the closing price of the previous trading day.

As shown in Figure 2.1, the introduction of price limits undoubtedly affects the properties of realized volatility. If the length of the truncated period is not negligible, volatility within that period would not be picked up by using realized volatility. As a result, realized volatility tends to underestimate the volatility for periods when price limits do take effect.
From Figure 2.1, it also can be observed that the downward bias would not disappear even when higher frequency samples are employed. Moreover, for a given volatility level, a more restrictive price limit (a smaller $l$) is more likely to take effect. In this situation, the total length of the truncated sub-periods within each trading day is likely to be longer. As a result the downward bias tends to be larger. These relationships are clearly shown by the following simulation experiment.

In order to illustrate the convergence properties of realized volatility at different sampling frequencies, I choose four sampling frequencies that correspond to 10, 100,
1000 and 10,000 intra-daily observations. At each frequency, 10,000 random samples are generated from the model in equations 2.3 and 2.4 with \((\beta, \kappa, \sigma) = (-7.3618, 0.05, 0.26)\). These parameter values are used in Jacquier, Polson, and Rossi (1994). For the sake of simplicity, the closing price of the previous trading day, \(p_{t_0}\), is normalized to zero. For each random sample, observation rules corresponding to 100 symmetric price limits, 0.001, 0.002, \cdots, 0.1, are applied to compute the realized volatilities. At each price limit level, bias and root mean squared errors of realized volatility at the four sampling frequencies are shown in Figure 2.2. This figure shows that when price limits are not restrictive, realized volatility appears consistent. But when price limits are very restrictive, realized volatility underestimates integrated volatility and the bias does not converge to zero as we increase the sampling frequency.

The downward bias of realized volatility is clearly a result of information loss due to price limits. Therefore, one would assume that if we could recover the lost information based on realized asset prices and the implied relationship from the stochastic volatility model, then we should be able to measure the volatility level for the truncated sub-periods. As a state space model, the stochastic volatility model has many advantages in regards to recovering lost information and is even able to reveal latent information.
2.3 Price Limits and Bayesian Estimation of Log-Stochastic Volatility Model

To recover the missing information resulting from price limit truncation, we need to resort to estimation procedures for stochastic volatility models. Due to the popularity of stochastic volatility models, many estimation methods have been developed.

depend on approximate linear filtering methods whose accuracy relies on the properties of the underlying data generating processes. Moment-based methods, including method of moments (MM) (Taylor 1982, Melino and Turnbull 1990, Vetzal 1997) and implied-state GMM (Pan 2002), are usually thought to be inefficient relative to likelihood-based methods. Bayesian approaches seem to be well suited for estimation and inference on stochastic volatility models. Markov-Chain Monte Carlo (MCMC) algorithms have been designed for log-stochastic models (Carter and Kohn 1994, Geweke 1994, Jacquier, Polson, and Rossi 1994, Jacquier, Polson, and Rossi 2004, Kim, Shephard, and Chib 1998, Mahieu and Schotman 1998, Omori, Chib, Shephard, and Nakajima 2007), square-root stochastic volatility models (Eraker, Johannes, and Polson 2003) and jump-diffusion stochastic volatility models (Eraker, Johannes, and Polson 2003). Compared to other competing methods, the Bayesian approach has the following advantages. First, it provides estimates of both parameters and latent state variables. Second, estimation and model risk can be quantified using this approach. Third, it has been shown that the Bayesian estimators outperform the competing methods in both parameter estimation and filtering (Jacquier, Polson, and Rossi 1994, Andersen, Chung, and Sørensen 1999). Fourth, the Bayesian approach is based on conditional simulation and is computationally efficient.

Many MCMC algorithms have been suggested even for log-stochastic volatility models. These algorithms differ from each other mainly in regards to sampling latent volatility states. Since the conditional distribution of the latent volatility state is not standard, a Metropolis-Hastings algorithm needs to be used to generate random samples. Jacquier, Polson, and Rossi (1994) use an accept/reject independence

Contrary to the previous stochastic volatility models, asset prices may not be observable due to the price limits. Therefore, besides parameters and latent volatility states, we also need to estimate truncated asset prices.

2.3.1 Bayesian Perspective

To separate the observed asset prices from the unobserved prices, let us define

$$p_{t\Delta} = \begin{cases} S_{t\Delta} & \text{if observed,} \\ Z_{t\Delta} & \text{otherwise.} \end{cases}$$

Also let $\theta = (\beta, \phi, \sigma)$, $h = (h_{2\Delta}, h_{3\Delta}, \cdots, h_{(T+1)\Delta})'$, $p = (p_{\Delta}, p_{2\Delta}, \cdots, p_{T\Delta})'$. Let $S$ denote a column vector of all observed prices and $Z$ denote a column vector of all unobserved prices. Since at each time the asset price can be either observed or unobserved, the combined dimension of $S$ and $Z$ must equal the dimension of $p$. 
Suppose $h_\Delta$ is given. From a Bayesian’s perspective we are interested in

$$P(Z, \theta, h|S),$$

that is, the posterior distribution of unobserved asset prices, parameters and latent volatility states given the observed asset prices. According to the Clifford-Hammersley theorem, this posterior distribution is uniquely determined by the two following conditional distributions: $P(Z|\theta, h, S)$ and $P(\theta, h|p)$.

The idea of the MCMC algorithm is to iteratively draw from these two conditional distributions, which form a Markov chain. Tierney (1994) shows that if the chain has a proper invariant distribution $\pi$ and it is irreducible and aperiodic, then this invariant distribution is unique and the unique invariant distribution $\pi$ is also the equilibrium distribution of the chain.

I will first design a MCMC algorithm for the objective posterior distribution. Then I will discuss the convergence properties of the designed MCMC algorithm.

### 2.3.2 MCMC Algorithm

Jacquier, Polson, and Rossi (1994) have developed an efficient algorithm for $P(\theta, h|p)$. If we were able to sample from $P(Z|\theta, h, S)$ too, we achieve our goal. So, before summarizing the method in Jacquier, Polson, and Rossi (1994), I will first derive the conditional distribution for unobserved asset prices.
Conditional Distribution of Unobserved Asset Prices

Using the definition of conditional distributions, we have

\[ P(Z|\theta, h, S) = \frac{P(p, h|\theta)}{P(h, S|\theta)} \propto P(p, h|\theta). \]

This indicates that the conditional distribution of unobserved asset prices is proportional to the likelihood of the model.

Conditionally on \( h_\Delta \), the likelihood function is given by

\[
P(Z, S, h|\theta) \propto \prod_{t=1}^{T} \exp\left( -h_{(t+1)\Delta} \right) P\left( \frac{y_{t\Delta}}{\exp\left( \frac{h_{t\Delta}}{2} \right)}, h_{(t+1)\Delta}|\theta \right)
\]

\[ = \exp\left( -\frac{1}{2} \text{tr}\left( \Sigma^{-1} A \right) \right) \prod_{t=1}^{T} \exp\left( -h_{(t+1)\Delta} \right) |\Sigma|^{-\frac{1}{2}}, \]

where \( A = \sum_t r_{t\Delta} r_{t\Delta}' \) and \( r_{t\Delta} = (\varepsilon_{t\Delta}, \eta_{t\Delta})' \).

We can use a Gibbs sampler to sample each unobserved asset price individually. By the Markov property of asset prices, we have

\[
P(Z_{t\Delta}|p_{-t\Delta}, \theta, h) \propto P(Z_{t\Delta}|p_{(t-1)\Delta}, p_{(t+1)\Delta}, \theta, h)
\]

\[ \propto \exp\left( -\frac{1}{2} \left( \text{tr}(\Sigma^{-1} r_{t\Delta} r_{t\Delta}') + \text{tr}(\Sigma^{-1} r_{(t+1)\Delta} r_{(t+1)\Delta}') \right) \right). \]

It can be shown that the above density corresponds to the following normal distribution (see Appendix A for details):

\[
Z_{t\Delta}|\theta, h, p_{-t\Delta}
\]

\[ \sim N\left( \frac{\exp(h_{(t+1)\Delta})p_{(t-1)\Delta}}{\exp(h_{t\Delta}) + \exp(h_{(t+1)\Delta})} + \frac{\exp(h_{t\Delta})p_{(t+1)\Delta}}{\exp(h_{t\Delta}) + \exp(h_{(t+1)\Delta})}, \frac{\exp(h_{t\Delta}) \exp(h_{(t+1)\Delta})}{\exp(h_{t\Delta}) + \exp(h_{(t+1)\Delta})} \right). \]
for $t = 1, 2, \cdots, T$.

From the observation rule \[2.5\] we know that $Z_{t\Delta}$ is truncated either because

$$Z_{t\Delta} > p_0 + \ln (1 + l)$$

or

$$Z_{t\Delta} < p_0 + \ln (1 - l),$$

where $p_0$ is the corresponding closing price of the previous trading day. Also, in reality we usually have this information. Therefore, instead of sampling unobserved asset prices from the above normal distribution, we will generate random samples from the corresponding truncated normal distributions.

**Conditional Distribution of Parameters and Volatility States**

Once we finish sampling unobserved asset prices, we return to the standard estimation problem of the SV model in \[2.3\] and \[2.4\]. Jacquier, Polson, and Rossi (1994) develop an efficient MCMC algorithm to sample parameters and latent volatility states. These procedures are briefly summarized below.

Since parameters only enter the conditional variance equation, according to Bayes rule

$$P(\beta, \phi, \sigma^2|h) \propto P(h|\beta, \phi, \sigma^2)P(\beta, \phi, \sigma^2).$$

Conditional on $h_{\Delta}$, the first factor on the right hand side is the likelihood function of latent volatility states, and the second factor is the prior distribution of parameters.
The likelihood function of latent volatility states is given by

\[
P(h|\theta) = \prod_{t=1}^{T} P(h_{t+1}\Delta|h_{t}\Delta, \beta, \phi, \sigma^2)
= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma\Delta}} \exp \left\{ -\frac{1}{2\sigma^2\Delta} (h_{t+1}\Delta - \beta - \phi(h_{t}\Delta - \beta))^2 \right\}.
\]

So we have

\[
P(\beta, \phi, \sigma^2|h) \propto \prod_{t=1}^{T} P(h_{t+1}\Delta|h_t, \beta, \phi, \sigma^2)P(\beta, \phi, \sigma^2).
\]

This is a standard Bayesian regression with normal innovations. Many priors could be used depending on the purpose of researchers. I use the non-informative prior given by equation (2.6) to minimize the effect of the prior.

\[
P(\beta, \phi, \sigma^2) \propto \frac{1}{\sigma^2}
\]

(2.6)

To simplify notation, let us define \( \alpha = \beta (1 - \phi) \), \( B = \begin{bmatrix} \alpha \\ \phi \end{bmatrix} \), \( X_{T\times2} = \begin{bmatrix} 1 & h_\Delta \\ 1 & h_{2\Delta} \\ \vdots & \vdots \\ 1 & h_{T\Delta} \end{bmatrix} \),
and \( Y_{T \times 1} = \begin{bmatrix} h_2 \Delta \\ h_3 \Delta \\ \vdots \\ h_{(T+1)\Delta} \end{bmatrix} \), then we have

\[
P(\alpha, \phi|\sigma^2, h) \propto \prod_{t=1}^{T} P(h_{(t+1)\Delta}|h_t \Delta, \beta, \phi, \sigma^2) \\
\qquad \propto \exp\left\{-\frac{1}{2\sigma^2\Delta} (XB - Y)'(XB - Y)\right\},
\]

which implies that

\[
B|\sigma^2, h \sim N \left( (X'X)^{-1} X'Y, \sigma^2\Delta (X'X)^{-1} \right).
\]

The conditional distribution of latent volatility states is

\[
P(\sigma^2|\beta, \phi, h) \propto \prod_{t=1}^{T} P(h_{(t+1)\Delta}|h_t \Delta, \beta, \phi, \sigma^2) P(\beta, \phi, \sigma^2) \\
\qquad \propto (\sigma^2)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma^2\Delta} (Y - XB)'(Y - XB)\right\},
\]

which implies that

\[
\sigma^2|\beta, \phi, h \sim IG \left( \frac{T}{2}, \frac{(Y - XB)'(Y - XB)}{2\Delta} \right),
\]

where IG denotes an inverse gamma distribution. The latent volatility states enter both the asset price equation and the state evolutionary equation. According to the
Markov property of latent volatility states and Bayes rule,

\[
P(h_{t \Delta} | \theta, p, h_{-t \Delta}) = P(h_{t \Delta} | \theta, p, h_{(t-1) \Delta}, h_{(t+1) \Delta})
\]

\[
\propto P(y_{t \Delta} | h_{t \Delta}) P(h_{t \Delta} | \theta, h_{(t-1) \Delta}) P(h_{(t+1) \Delta} | \theta, h_{t \Delta})
\]

\[
\propto (\exp(h_{t \Delta}))^{-\frac{3}{2}} \exp \left( -\frac{y^2_{t \Delta}}{2 \exp(h_{t \Delta}) \Delta} \right)
\]

\[
\exp \left( -\frac{(h_{t \Delta} - m_{t \Delta})^2}{2\sigma^2_{t \Delta}} \right), \quad (2.7)
\]

where \( m_{t \Delta} = \frac{\alpha(1-\phi) + \phi(h_{t+1 \Delta} + h_{t \Delta})}{1+\phi^2} \) and \( \sigma^2_{t \Delta} = \frac{\sigma^2}{1+\phi^2} \).

Since the conditional distribution of latent volatility state is not standard, a Metropolis-Hastings algorithm will be used to generate random samples. Many strategies have been proposed to sample from \( P(h_{t \Delta} | \theta, p, h_{-t \Delta}) \). Jacquier, Polson, and Rossi (1994) use an accept/reject independence Metropolis-Hastings algorithm to update the states individually. The suggested proposal density is

\[
\exp(h_{t \Delta}) \sim IG \left( \frac{2 \exp(\sigma^2_{t \Delta}) - 1}{\exp(\sigma^2_{t \Delta}) - 1} + \frac{1}{2} \frac{\exp(m_{t \Delta} + \frac{3\sigma^2_{t \Delta}}{2})}{\exp(\sigma^2_{t \Delta}) - 1} + \frac{y^2_{t \Delta}}{2\Delta} \right).
\]

An inverse gamma distribution is used to approximate the log-normal distribution term in equation [2.7] by matching the first two moments.

**Convergence**

The above sampling scheme clearly shows that \( \theta \in R \), where

\[
R = \left\{ (\beta, \phi, \sigma^2)^\prime : \beta \in \mathbb{R}, |\phi| < 1, \sigma^2 > 0 \right\}.
\]
And $h \in \mathbb{R}^T$,

$$Z_{t\Delta} \in \begin{cases} (p_0 + \ln (1 + l), \infty), & \text{if truncated above,} \\ (-\infty, p_0 + \ln (1 - l)), & \text{if truncated below.} \end{cases}$$

In each iteration, it is possible for $\theta$, $h$ and $Z$ to take any values in the corresponding space. That is, the constructed Markov chain is irreducible. It is also clear that no portions of the state spaces of $\theta$, $h$ or $Z$ can only be visited at certain regularly spaced times. That is, the constructed Markov chain is also aperiodic. From Tierney (1994), we know that if this Markov chain has an invariant distribution $\pi$ this invariant distribution is unique and it is also the equilibrium distribution, which is the posterior distribution $P(Z, \theta, h|S)$ in which we are interested.

Using this property, as long as we observe the convergence of the suggested sampling scheme, we just need to continue the iterations for a long enough period so that we could make inferences based on samples from the convergent portion.

Instead of the convergence of the Markov chain itself, what we are usually interested in is the convergence of sample averages of functionals along the chain. The following two propositions from Johannes and Polson (2010) provide powerful tools.

**Proposition 1.** Suppose $\Theta^{(g)}$ is an ergodic chain with stationary distribution $\pi$ and suppose $f$ is a real-valued function with $\int |f| d\pi < \infty$. Then for all $\Theta^{(g)}$ for any initial starting value $\Theta^{(0)}$

$$\lim_{G \to \infty} \frac{1}{G} \sum_{g=1}^{G} f(\Theta^{(g)}) = \int f(\Theta) \pi(\Theta) d\Theta$$

almost surely.
**Proposition 2.** Suppose $\Theta^{(g)}$ is an ergodic chain with stationary distribution $\pi$ and suppose $f$ is real-valued and $\int |f| \, d\pi < \infty$. Then there exists a real number $\sigma(f)$ such that
\[
\sqrt{G} \left( \frac{1}{G} \sum_{g=1}^{G} f(\Theta^{(g)}) - \int f(\Theta) \, d\pi \right)
\]
converges in distribution to a mean zero normal distribution with variance $\sigma^2(f)$ for any starting value.

### 2.3.3 Quasi-Realized Volatility and Simulated-Integrated Volatility

As the objective of this chapter is to provide estimators for integrated volatility within periods when some parts of the asset price process are truncated due to price limits, I suggest the following two estimators by exploiting the generated posterior samples and existing estimators for integrated volatility. The first estimator is constructed utilizing the posterior sample of truncated asset prices. By treating all simulated asset prices as realizations of an unobserved asset price process, we can use the idea of realized volatility to construct the following estimator,
\[
\text{QRV}^{(g)}_{[t_0, t_m]} = \sum_{i=1}^{m} \left( p_t^{(g)} - p_{t-1}^{(g)} \right)^2,
\]
for $g = 1, 2, \cdots, G$. Here, $p_t^{(g)}$ is defined as
\[
p_t^{(g)} = \begin{cases} 
S_t & \text{if observed}, \\
Z_{t_i}^{(g)} & \text{otherwise},
\end{cases}
\]
where \( g \) denotes that the corresponding value is from iteration \( g \) after the burn-in period and \( G \) is the total number of iterations after the burn-in period. Since this estimator utilizes both realized asset prices and simulated asset prices, I call it quasi-realized volatility (QRV).

According to the results in Barndorff-Nielsen and Shephard (2002), Meddahi (2002) and Andersen, Bollerslev, Diebold, and Labys (2003), realized volatility is a consistent estimator of integrated volatility under some regularity conditions. In the posterior sample, each iteration can be thought of as a realization of integrated volatility over the period of interest. And in each iteration this realization of integrated volatility can be consistently estimated by QRV. Therefore, the derived posterior distribution of QRV converges to the posterior distribution of integrated volatility. These results are summarized in Theorem 1.

**Theorem 1.** Quasi-realized volatility provides a consistent measure of integrated volatility. That is,

\[
\lim_{m \to \infty} QRV_{[t_0, t_m]}^{(g)} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv IV_{[t_0, t_m]}^{(g)}
\]

for \( g = 1, 2, \cdots, G \) and

\[
\lim_{m \to \infty} P \left( QRV_{[t_0, t_m]} | S \right) = P \left( IV_{[t_0, t_m]} | S \right).
\]

Proof: By the convergence property of MCMC, we have

\[
p^{(g)}, \theta^{(g)}, h^{(g)} \sim P (p, \theta, h | S).
\]
Since \( p \) and \( h \) are from the stochastic volatility model in equations 2.1 and 2.2, the asset price process belongs to the class of special semi-martingales as detailed by Back (1991). According to Propositions 1 and 2 from Andersen, Bollerslev, Diebold, and Labys (2003), we have

\[
\text{plim}_{m \to \infty} \sum_{i=1}^{m} \left( p_{t_i}^{(g)} - p_{t_{i-1}}^{(g)} \right)^2 = \left[ r^{(g)}, r^{(g)} \right]_{t_m} - \left[ r^{(g)}, r^{(g)} \right]_{t_0} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv \text{IV}_{[t_0, t_m]}^{(g)},
\]

where \( r_t^{(g)} = p_t^{(g)} - p_0^{(g)} \) is the cumulative log return at time \( t \) and \([r, r]\) denotes the quadratic variation process of the return process. That is, for each iteration QRV is a consistent measure of integrated volatility. Therefore, the posterior distribution of QRV converges to the posterior distribution of IV.

The second estimator is even more natural in the context of stochastic volatility models. Since the MCMC algorithm not only provides us with the posterior sample of the truncated asset prices but also the posterior sample of the latent volatility states, we can directly derive the posterior distribution of integrated volatility using the discrete sample analogue by

\[
\text{SIV}_{[t_0, t_m]}^{(g)} = \frac{1}{m} \sum_{i=1}^{m} \exp \left( h_{t_i}^{(g)} \right)
\]

for \( g = 1, 2, \cdots, G \). I call this estimator the simulated integrated volatility (SIV).

**Theorem 2.** Simulated integrated volatility provides a consistent measure of integrated volatility. That is,

\[
\text{plim}_{m \to \infty} \text{SIV}_{[t_0, t_m]}^{(g)} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv \text{IV}_{[t_0, t_m]}^{(g)}
\]
for \( g = 1, 2, \cdots, G \) and

\[
\text{plim}_{n \to \infty} P \left( SIV_{[t_0, t_n]} \big| S \right) = P \left( IV_{[t_0, t_n]} \big| S \right).
\]

Proof: These are direct results from the convergence of numerical integration. That is, for each iteration, SIV is a consistent measure of integrated volatility. Thus the posterior distribution of SIV converges to the posterior distribution of IV. ■

2.4 Simulation

2.4.1 Illustrative Example

To show how the suggested procedure works more intuitively, I provide the following example. I simulate a random sample from the stochastic volatility model in equations 2.3 and 2.4. Parameter values are given in Table 2.1, which are used in the simulation study of Jacquier, Polson, and Rossi (1994).

<table>
<thead>
<tr>
<th>Table 2.1: Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
</tr>
<tr>
<td>-7.3618</td>
</tr>
</tbody>
</table>

A symmetric price limit of 0.02 is used, which is close to the most restrictive price limit level in reality.\(^1\) My simulation study shows that the lower the price limit is, the more difficult it is to estimate the integrated volatility due to an increase

\(^1\)The Amman Stock Exchange (ASE) set the daily price limit to 2% during the Gulf War in 1991. The Taiwan Stock Exchange (TSE) installed a 2.5% symmetric price limit from December 19, 1978 to January 4, 1987.
in information loss. I choose $m = 240$ for the sampling frequency\footnote{This corresponds to a sampling frequency of 1 minute for the Chinese stock markets because there are only 4 trading hours in both the Shanghai Stock Exchange (SSE) and the Shenzhen Stock Exchange (SZSE).} As in other Bayesian analyses of stochastic volatility models, I use a reasonably large sample size in this example and the simulation studies. A large sample not only brings more information, but also mitigates the effect of priors. Therefore, in this example, along with the trading day of interest, I also include 15 days before and 14 days after. The simulated data are shown in Figure 2.3.

For the trading day of interest, 97 price observations are truncated after applying the price limit, which corresponds to 40.42\% of the total observations in a trading day. The realized volatility on the day of interest is 4.7593 ($\times 10^{-4}$) while the true

Figure 2.3: Simulated returns, log volatilities and log prices
underlying integrated volatility is 7.9190 ($\times 10^{-4}$). That is, the realized volatility underestimates the integrated volatility by about 40%. Figure 2.4 zooms in on the simulated asset prices of day 16.

![Simulated log price (\(p_t\))](image)

Figure 2.4: Simulated log prices and price limits

The two horizontal lines correspond to the upper and lower limits of the trading day. According to the observation rule, only asset prices between the two price limits are observable.

The MCMC algorithm developed in Section 3 is applied to the simulated sample for 20,000 iterations. The first 5,000 iterations are discarded as the burn-in period. Posterior distributions are estimated from the remaining 15,000 iterations and results are shown in Figure 2.5 and Table 2.2.
Table 2.2: Estimation results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>Mean</th>
<th>St. dev.</th>
<th>95% interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>-7.3618</td>
<td>-7.3689</td>
<td>0.1959</td>
<td>[-7.7309,-7.1305]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.9998</td>
<td>0.9957</td>
<td>0.0015</td>
<td>[0.9925,0.9986]</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.2600</td>
<td>0.2971</td>
<td>0.0302</td>
<td>[0.2436,0.3626]</td>
</tr>
<tr>
<td>QRV($\times10^{-4}$)</td>
<td>7.9190</td>
<td>8.3293</td>
<td>0.8048</td>
<td>[6.9922,10.1470]</td>
</tr>
<tr>
<td>SIV($\times10^{-4}$)</td>
<td>7.9190</td>
<td>8.1463</td>
<td>0.9990</td>
<td>[6.4080,10.3590]</td>
</tr>
</tbody>
</table>

In Figure 2.5, the columns show the trace plots, the autocorrelograms, and the kernel densities of the parameters and two suggested estimators. As shown in all of the autocorrelograms, the autocorrelations of all posterior samples of the parameters decrease to zero very quickly, which indicates that the sampling scheme achieves very high computational efficiency as suggested by Jacquier, Polson, and Rossi (1994). Also, trace plots of all parameters and two estimators do not show any trends or patterns, which provides very strong evidence of convergence. The horizontal line in each trace plot shows the true value of the corresponding parameter or estimator. The observation that each horizontal line is very close to the center of the corresponding trace plot clearly suggests MCMC algorithm provides very good estimates for all parameters and for the integrated volatility. The trace plots, the autocorrelograms and the kernel densities of QRV and SIVE are very similar, which indicates that both estimators provide very similar estimates of integrated volatility. Due to the way SIV is defined, whether SIV provides a good estimate of IV depends closely on how well we can estimate the latent volatility states. Figure 2.6 shows the posterior mean of the latent log volatility states and the true simulated log volatility states.

We see that the posterior means of latent volatility states are very close to the true latent volatility states. Therefore, the posterior mean of SIV suggests a good
point estimator candidate for integrated volatility. Similarly, due to the similarity of QRV and SIV as suggested above we can also use the posterior mean of QRV as another point estimator for IV.

2.4.2 Simulation Study

The convergence properties of QRV and SIV guarantee that if the sampling frequency is very high, the posterior distributions of QRV and SIV converge to the posterior distribution of integrated volatility. The above example also shows that the modes of the posterior distributions of QRV and SIV are both very close to the true integrated volatility level. In this section, I study the performance of QRV and SIV based on 1,000 simulated samples. I use the same parameter values to generate the random samples. Due to the computational burden, I only run 10,000 iterations for each sample and keep the last 5,000 iterations for the posterior distribution. I use the mean of QRV\(^{(g)}\) and the mean of SIV\(^{(g)}\), which both converge to the posterior mean of integrated volatility according to Proposition 1, as point estimates for integrated volatility. For reasons of comparison, I also include the following three estimators

\[
RV^*_\{t_0, t_m\} = \sum_{i=1}^{m} (p_{t_i} - p_{t_{i-1}})^2,
\]

\[
RV_{\{t_0, t_m\}} = \sum_{t_0 < t_i \leq t_m} (S_{t_i} - S_{t_{i-1}})^2,
\]

\[
RV_{adj\{t_0, t_m\}} = \frac{T}{T - M} RV_{\{t_0, t_m\}}.
\]

Here, \(RV^*_\{t_0, t_m\}\) is the realized volatility over period \([t_0, t_m]\) assuming we know all of the asset prices. Therefore it is not a feasible estimator if price limits are present. \(RV_{\{t_0, t_m\}}\)
is the traditional realized volatility based only on realized asset prices, and $M$ is the total number of truncated asset prices. Therefore $\text{RV}_{adj}^{[t_0, t_m]}$ is an adjusted measurement assuming volatilities are constant over time, which contradicts the empirical finding of time varying volatilities.

Because the realized integrated volatility level is different in each simulated sample, I compare the relative errors of all estimators as in Nielsen and Frederiksen (2008). Figure 2.7 shows the line graph of relative errors.

From this graph we can see that QRV and SIV are almost the same. In addition, they both behave very similarly to the infeasible estimator $\text{RV}^*$. From the realized volatility literature, we know that realized volatility can estimate integrated volatility very well if we can observe all asset prices. Therefore both QRV and SIV provide very reliable estimates for integrated volatility if the dynamics of the financial asset can be described by a simple log-stochastic volatility model. The relative error plot of RV again clearly shows the downward bias. Although the relative error plot of $\text{RV}_{adj}^{adj}$ does not clearly show evidence of bias, the variation of the relative errors (note the different scale on the y-axes) is much higher than QRV and SIV. Table 2.3 presents summary statistics of the relative errors.

<table>
<thead>
<tr>
<th></th>
<th>QRV</th>
<th>SIV</th>
<th>RV$^*$</th>
<th>RV</th>
<th>RV$^{adj}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0029</td>
<td>0.0109</td>
<td>-0.0021</td>
<td>-0.2647</td>
<td>-0.0354</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.1052</td>
<td>0.0949</td>
<td>0.0926</td>
<td>0.3881</td>
<td>0.1368</td>
</tr>
<tr>
<td>s.d.</td>
<td>0.1051</td>
<td>0.0943</td>
<td>0.0926</td>
<td>0.2838</td>
<td>0.1321</td>
</tr>
</tbody>
</table>
2.5 Empirical Application

In this section, I apply the method developed in this chapter to Chinese stock markets, where a symmetric 10% price limit has been imposed on regularly traded stocks listed on both the Shanghai Stock Exchange (SSE) and the Shenzhen Stock Exchange (SZSE) since December 16, 1996. Although price limits do not take effect for most of the stocks in each trading day, it is not rare to observe price limits taking effect. For normally traded stocks, price limits take effect in 2% of the trading days on average. During the sampling period of this empirical application, price limits take effect for 14 stocks each day on average in the SSE out of approximately 900 regularly traded stocks. The maximum number of stocks hitting price limits in a trading day within the sampling period is 24 in the SSE, and the minimum is 1. During the sampling period of this empirical application, price limits take effect for 17.4 stocks each day on average in the SZSE out of approximately 900 regularly traded stocks. The maximum number of stocks hitting price limits in a trading day within the sampling period is 29 in the SSE, and the minimum is 5.

I use the one minute previous tick price exported from Great Wisdom, a financial information provider whose products are widely used by many Chinese investors. I select two normally traded stocks that triggered the price limits on November 8, 2010. For the same reason as stated in the simulation section, besides the data from the trading day of interest, I also include the data for the previous 15 trading days and the following 14 trading days. The first stock is issued by Zhejiang Haiyue Co.,

\footnote{For a detailed description of the price limit regulation, please visit the official websites of SSE (www.sse.com.cn) and SZSE (www.szse.cn).}

\footnote{The official website is www.gw.com.cn}

\footnote{In both the SSE and the SZSE, the price limits for normally traded stocks are \( \pm 10\% \), while the price limits for specially traded stocks are \( \pm 5\% \).}

\footnote{These two stocks are selected such that no prices were truncated besides the day of interest in}
Lt. (code: HYGF) and the second is issued by Unisplendour Guhan Group Co., Ltd. (code: ZGGH). The realized volatility on the day of interest for HYGF is $3.5706 \times 10^{-3}$ and 229 observations are truncated due to the price limits. For ZGGH the realized volatility on the day of interest is $1.7582 \times 10^{-3}$ and 25 observations are truncated due to the price limits. For both stocks, price limits only take effect on the day of interest. That is, for the rest of days within the sample period, stock prices fluctuate within the pre-specified ranges. Also both stocks exhibit high liquidity within the sample period and no price manipulation behavior is detected for both stocks. Figure 2.8 shows plots of log returns for these two stocks for the period 10/18/2010 to 11/26/2010. In order to have a detailed view for the day of interest, I plot the asset prices and price limits on November 8, 2010 for both stocks in Figure 2.9.

For each stock, the MCMC algorithm runs for 10,000 iterations. The first 5,000 are discarded as the burn-in period and the remaining 5,000 are used in analyzing the posterior distributions. Results are given in Figures 2.10 and 2.11 and Table 2.4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Stock: HYGF</th>
<th>Stock: ZGGH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>-7.2164</td>
<td>-6.8245</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.8842</td>
<td>0.8285</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>4.4801</td>
<td>5.3346</td>
</tr>
<tr>
<td>QRV($\times 10^{-3}$)</td>
<td>6.0802</td>
<td>2.3183</td>
</tr>
<tr>
<td>SIV($\times 10^{-3}$)</td>
<td>4.1145</td>
<td>2.3230</td>
</tr>
</tbody>
</table>

Table 2.4: Estimation results

I also apply the developed method to several other stocks, which hit price limits within the sampling period. Similar results are found.
Figures 2.10 and 2.11 clearly demonstrate convergence for both stocks. For both stocks the QRV and the SIV seem very similar except that QRV takes higher values than SIV for ZGGF. From Table 2.4 both stocks exhibit high conditional variances of latent volatility states, which is consistent with the fact that the Chinese stock markets are very volatile at that time. Latent volatility states also exhibit relatively low persistency for both stocks, which is also consistent with the fact that market wide and individual stock specific shocks are mainly temporary during the period of interest. For HYGF, the posterior mean of QRV is $6.0802 \times 10^{-3}$, while the posterior mean of SIV is $4.1145 \times 10^{-3}$. This can be explained by the observed return jumps on the day of interest and the fact that realized volatility tends to over estimate integrated volatility in the presence of return jumps. Therefore, the posterior mean of SIV provides more reliable estimate for integrated volatility under this circumstance. Compared with the realized volatility, the posterior mean of SIV increases by 15.23%. For ZGGH, the posterior mean of QRV and SIV are very close and increase by about 32% compared with the realized volatility.

To further examine the difference between the QRV and the SIV, I plot the daily realized volatilities and the SIVs for the sampling period in Figure 2.12. For the day of interest, which corresponds to Day 16 in Figure 2.8, I replace the RV with the posterior mean of the QRV.

We can see that RV and SIV are very close. Both estimates seem reasonable when comparing the posterior mean of both QRV and SIV with the neighboring days. As discussed earlier, on Day 16 the presence of return jumps explains the difference between QRV and SIV for HYGF. Similarly, we can explain the difference between RV ans SIV for ZGGH in the last two days in the sampling period.
2.6 Concluding Remarks

This chapter shows that when we assume log-stochastic volatilities for the underlying asset process, Bayesian methods can help us to recover the lost information due to the presence of price limits. The MCMC algorithm designed based on Jacquier, Polson, and Rossi (1994) clearly shows convergence and efficiency. Simulation results indicate that both QRV and SIV provide very good estimates for the underlying integrated volatility and both behave very similarly to the realized volatility assuming we know all of the asset prices. The application to the Chinese stock markets also shows that both QRV and SIV provide reasonable estimates for financial practitioners. Results from the empirical application also indicate that as realized volatility, QRV is not jump consistent. Adjustments have to be made to mitigate the effect of return jumps on the volatility measurement. This extension is shown in Chapter 4.
Figure 2.5: Trace plots, correlograms and kernel densities
Figure 2.6: True and posterior mean volatilities
Figure 2.7: Relative errors
Figure 2.8: Log returns for the period 10/18/2010 to 11/26/2010
Figure 2.9: Stock prices and price limits on November 8, 2010
Figure 2.10: Trace plots, correlograms and kernel densities for HYGF
Figure 2.11: Trace plots, correlograms and kernel densities for ZGGH
Figure 2.12: RV and SIV plots
Chapter 3

Bayesian Estimation of Integrated Volatility with Price Limits: Log-Stochastic Volatility Model with Correlated Innovations

3.1 Introduction

Developed in Black (1976) and Christie (1982), the leverage effect refers to a prevalent explanation of the negative correlation between an asset’s return and its changes of volatility. When asset prices decrease, the ratio between a company’s value of debt and the value of equity rises. As a result, the company’s stock is expected to be riskier, hence more volatile. Whereas, French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992) discuss another volatility asymmetry documented as
the volatility feedback effect, which argues that an anticipated increase in volatility would raise the required rate of return. In this chapter, I allow asset return innovations and volatility innovations to be correlated to capture both the leverage effect and the volatility feedback effect.

This chapter is an extension of Chapter 2 where asset return innovations and volatility innovations are assumed to be independent. By allowing for nonzero correlations between asset return innovations and volatility innovations, the methods developed in Chapter 2 can be used in a more general framework to capture the important empirical findings, the leverage effect and the volatility feedback effect. This chapter contributes to the literature in the following ways. First, I incorporate price limits in the log-stochastic volatility model with correlated innovations. Second, I design a MCMC algorithm to simulate the unobserved asset prices, model parameters and latent states allowing for correlated innovations. Third, I generalize the two types of estimators for integrated volatility in the presence of price limits to a log-stochastic volatility model with correlated errors. I also apply both estimators to the Chinese stock markets and provide some empirical results.

The rest of this chapter is organized as follows. In Section 3.2, I specify the log-stochastic volatility model with correlated errors and introduce price limits into the framework. In Section 3.3, I design a MCMC sampling scheme to incorporate price limits. I also discuss the convergence property of the designed MCMC algorithm. In Section 3.4, I examine the performance of quasi-realized volatility and simulated integrated volatility through simulation studies. In Section 3.5, I apply my method to the Chinese stock markets using high frequency data from both the Shanghai Stock Exchange and the Shenzen Stock Exchange and provide some empirical results. I
3.2 Price Limits and Log-Stochastic Volatility Model with Correlated Innovations

In the log-stochastic volatility model with correlated innovations, the log asset price $p_t$ and its log diffusive volatility $h_t$ solve the following two differential equations

\begin{align}
    dp_t &= \mu_t dt + \exp \left( \frac{h_t}{2} \right) dW_t^s, \\ 
    dh_t &= \kappa(\beta - h_t) dt + \sigma dW_t^\sigma,
\end{align}

where $\mu_t$ is the equity risk premium, $\sigma$ is the diffusive volatility of conditional variances, $\kappa$ and $\beta$ are two parameters of the conditional variance process, and $W_t^s, W_t^\sigma$ are two Brownian motions with correlation $\rho$, which is assumed to be zero in Chapter 2. As discussed in Section 3.1, a negative $\rho$ indicates the existence of the leverage effect, while a positive $\rho$ indicates the existence of the volatility feedback effect.

Then the integrated volatility over period $[t_0, t_m]$ is defined as

$$IV_{[t_0, t_m]} = \int_{t_0}^{t_m} \exp (h_s) ds.$$ 

To focus on volatilities, I assume $\mu_t$ is equal to zero. An Euler time discretization
of the model in equations 3.1 and 3.2 implies that

\[ y_{t\Delta} \equiv p_{t\Delta} - p_{(t-1)\Delta} = \exp \left( h_{t\Delta}/2 \right) \varepsilon_{t\Delta}, \quad (3.3) \]

\[ h_{(t+1)\Delta} = \beta + \phi(h_{t\Delta} - \beta) + \eta_{t\Delta}, \quad (3.4) \]

with \( \phi \equiv (1 - \kappa \Delta) \) where \( \Delta \) is the distance between two observations. Here, \( \varepsilon_{t\Delta} \) and \( \eta_{t\Delta} \) are normally distributed with zero mean and covariance matrix \( \Sigma = \Delta \begin{bmatrix} 1 & \rho \sigma \\ \rho \sigma & \sigma^2 \end{bmatrix} \).

In this chapter I assume there are \( m \) equidistant observations in each trading day. This assumption can be relaxed easily and would not change the results as long as the distances between adjacent observations are very small (\( \to 0 \)). Therefore \( \Delta = \frac{1}{m} \).

When a symmetric price limit \( l > 0 \) (in percentage) is imposed, that is the price of a stock in a trading day can not increase or decrease by more than \( l \) percent of its previous closing price, the following observation rule takes effect:

\[ p_{ti} = \begin{cases} 
  p_{ti} & \text{if } \ln(1 - l) \leq p_{ti} - p_{t0} \leq \ln(1 + l), \\
  \text{Unobservable} & \text{otherwise},
\end{cases} \quad (3.5) \]

for \( i = 1, 2, \cdots, m \), where \( p_{t0} \) denotes the closing price of the previous trading day.

In Chapter 2 I develop two types of estimators for integrated volatility by recovering the lost information due to price limits based on realized asset prices and the simple log-stochastic volatility model. In this chapter, I extend this idea to the log-stochastic volatility model with correlated errors to accommodate two important empirical findings, the leverage effect and the volatility feedback effect.
3.3 Price Limits and Bayesian Estimation of Log-
Stochastic Volatility Model with Correlated
Innovations

To accommodate the leverage effect and the volatility feedback effect, I use the
log-stochastic volatility model with correlated innovations instead of the simple log-
stochastic volatility model. Considering the computational efficiency advantage, I
extend the suggested method in Omori, Chib, Shephard, and Nakajima (2007) to
generate random samples for truncated asset prices, parameters and latent volatilities.

3.3.1 Bayesian Perspective

To separate the observed asset prices from the unobserved prices, let us define

\[ p_{t\Delta} = \begin{cases} S_{t\Delta} & \text{if observed,} \\ Z_{t\Delta} & \text{otherwise.} \end{cases} \]

Also let \( \theta = (\phi, \rho, \sigma) \), \( h = (h_{2\Delta}, h_{3\Delta}, \ldots, h_{(T+1)\Delta})' \), \( p = (p_{\Delta}, p_{2\Delta}, \ldots, p_{T\Delta})' \). Let
\( S \) denote a column vector of all observed prices and \( Z \) denote a column vector of
all unobserved prices. Since at each time the asset price can be either observed or
unobserved, the combined dimension of \( S \) and \( Z \) must equal the dimension of \( p \). From
a Bayesian’s perspective, we are interested in

\[ P(Z, \theta, \beta, h|S). \]
That is the posterior distribution of unobserved asset prices, parameters and latent volatility states given the observed asset prices. According to the Clifford-Hammersley theorem, this posterior distribution is uniquely determined by the two following conditional distributions: $P(Z|\theta, \beta, h, S)$ and $P(\theta, \beta, h|p)$.

The idea of the MCMC algorithm is to iteratively draw from these two conditional distributions, which form a Markov chain. Tierney (1994) shows that if the chain has a proper invariant distribution $\pi$ and it is irreducible and aperiodic then this invariant distribution is unique and the unique invariant distribution $\pi$ is also the equilibrium distribution of the chain.

I will first design a MCMC algorithm for the objective posterior distribution. Then I will discuss the convergence properties of the designed MCMC algorithm.

### 3.3.2 MCMC Algorithm

Omori, Chib, Shephard, and Nakajima (2007) have developed an efficient algorithm for $P(\theta, \beta, h|p)$. If we were able to sample from $P(Z|\theta, \beta, h, S)$ too, we achieve our goal. So, before summarizing the method in Omori, Chib, Shephard, and Nakajima (2007), I will first derive the conditional distribution for unobserved asset prices.

**Conditional Distribution of Unobserved Asset Prices**

Using the definition of conditional distributions, we have

$$P(Z|\theta, \beta, h, S) = \frac{P(p, h|\theta, \beta)}{P(h, S|\theta, \beta)} \propto P(p, h|\theta, \beta).$$

This indicates that the joint conditional marginal distribution of unobserved asset prices is proportional to the likelihood of the model.
Conditionally on $h_{\Delta}$, the likelihood function is given by

$$P(Z, S, h | \theta, \beta) \propto \prod_{t=1}^{T} \exp \left( -h_{(t+1)\Delta} \right) P \left( \frac{y_{t\Delta}}{\exp \left( \frac{h_{t\Delta}}{2} \right)}, h_{(t+1)\Delta} | \theta, \beta \right)$$

$$= \exp \left( -\frac{1}{2} \text{tr} \left( \Sigma^{-1} A \right) \right) \prod_{t=1}^{T} \exp \left( -h_{(t+1)\Delta} \right) |\Sigma|^{-\frac{1}{2}},$$

where $A = \sum_{t} r_{t\Delta} r_{t\Delta}'$ and $r_{t\Delta} = (\varepsilon_{t\Delta}, \eta_{t\Delta})'$.

We can use a Gibbs sampler to sample each unobserved asset price individually. By the Markov property of asset prices, we have

$$P(Z_{t\Delta} | \theta, \beta, h_{-t\Delta}) \propto P(Z_{t\Delta} | \theta, \beta, h, p_{(t-1)\Delta}, p_{(t+1)\Delta})$$

$$\propto \exp \left( -\frac{1}{2} \left( \text{tr} \left( \Sigma^{-1} r_{t\Delta} r_{t\Delta}' \right) + \text{tr} \left( \Sigma^{-1} r_{(t+1)\Delta} r_{(t+1)\Delta}' \right) \right) \right).$$

It can be shown that the above density corresponds to the following normal distribution (see Appendix B for details):

$$Z_{t\Delta} | \theta, \beta, h, p_{-t\Delta} \sim N \left( \frac{\sigma^2_{Z_{t\Delta}}}{(1 - \rho^2) \Delta} \left( \frac{p_{(t-1)\Delta}}{\exp (h_{t\Delta})} + \rho \frac{\eta_{t\Delta}}{\sigma \exp (h_{t\Delta}/2)} + \frac{p_{(t+1)\Delta}}{\exp (h_{(t+1)\Delta})} - \rho \frac{\eta_{(t+1)\Delta}}{\sigma \exp (h_{(t+1)\Delta}/2)} \right), \sigma^2_{Z_{t\Delta}} \right),$$

for $t = 1, 2, \ldots, T$, where

$$\sigma^2_{Z_{t\Delta}} = \frac{\exp (h_{t\Delta} + h_{(t+1)\Delta})}{\exp (h_{t\Delta}) + \exp (h_{(t+1)\Delta})} (1 - \rho^2) \Delta.$$
From the observation rule 3.5, we know that $Z_{t\Delta}$ is truncated either because

$$Z_{t\Delta} > p_0 + \ln(1 + l)$$

or

$$Z_{t\Delta} < p_0 + \ln(1 - l),$$

where $p_0$ is the corresponding closing price of the previous trading day. Also, in reality we usually have this information. Therefore, instead of sampling unobserved asset prices from the above normal distribution, we will generate random samples from the corresponding truncated normal distributions.

**Conditional Distribution of Parameters and Volatility States**

Once we finish sampling unobserved asset prices, we return to the standard estimation problem of the SV model in equations 3.3 and 3.4. Omori, Chib, Shephard, and Nakajima (2007) develop a very efficient MCMC algorithm to sample parameters and latent volatility states. These procedures are briefly summarized below.

It is equivalent to express $y_{t\Delta}$ as

$$y_{t\Delta} = d_{t\Delta} \sqrt{\Delta} \exp(y_{t\Delta}^*/2),$$

where $y_{t\Delta}^* = \log y_{t\Delta}^2 - \log \Delta = h_{t\Delta} + \varepsilon_{t\Delta}^*$ and $d_{t\Delta} = I(\varepsilon_{t\Delta} \geq 0) - I(\varepsilon_{t\Delta} < 0)$, and $\varepsilon_{t\Delta}^* = \log \varepsilon_{t\Delta}^2 - \log \Delta$ follows a log-$\chi^2_1$ distribution. Then the SV model can be formulated as

$$
\begin{pmatrix}
  y_{t\Delta}^* \\
  h_{(t+1)\Delta}
\end{pmatrix} =
\begin{pmatrix}
  h_{t\Delta} \\
  \beta + \phi(h_{t\Delta} - \beta)
\end{pmatrix} +
\begin{pmatrix}
  \varepsilon_{t\Delta}^* \\
  \eta_{t\Delta}
\end{pmatrix}.
$$
Kim, Shephard, and Chib (1998) use a matched mixture of 7 normal distributions to approximate the log $\chi^2_1$ distribution. Omori, Chib, Shephard, and Nakajima (2007) follow this approach but use a matched mixture of 10 normal distributions, which results in a better approximation. That is,

$$g(\varepsilon^*_\Delta) = \sum_{j=1}^{10} w_j N(\varepsilon^*_\Delta|m_j, v^2_j), \varepsilon^*_\Delta \in \mathbb{R},$$

where $N(\varepsilon^*_\Delta|m_j, v^2_j)$ denotes the density function of a normal distribution with mean $m_j$ and variance $v^2_j$.

Given $\varepsilon^*_\Delta \sim N(m_j, v^2_j)$, if we approximate $\exp(\varepsilon^*_\Delta/2)$ by

$$\exp(m_j/2) (a_j + b_j (\varepsilon^*_i - m_j)),$$

the joint density $f(\varepsilon^*_\Delta, \eta_{t\Delta}|d_{t\Delta}, \rho, \sigma)$ can be approximated by

$$g(\varepsilon^*_\Delta, \eta_{t\Delta}|d_{t\Delta}, \rho, \sigma) = \sum_{j=1}^{10} w_j N(\varepsilon^*_\Delta|m_j, v^2_j) N[\eta_{t\Delta}|d_{t\Delta}\sqrt{\Delta}\sigma \exp(m_j/2) \{a_j + b_j (\varepsilon^*_i - m_j)\}, \Delta\sigma^2 (1 - \rho^2)],$$

where the values of $a_j$ and $b_j$, for $j = 1, 2, \cdots, 10$, are determined by

$$(a_j, b_j) = \arg \min_{a,b} E \{\exp(\varepsilon^*_i/2) \exp(-m_j/2) - a - b (\varepsilon^*_i - m_j))^2, \varepsilon^*_i \sim N(m_j, v^2_j).$$

Component parameters are given in Table 3.1.
Table 3.1: Component parameters in the approximating mixture distribution

<table>
<thead>
<tr>
<th>$j$</th>
<th>$w_j$</th>
<th>$m_j$</th>
<th>$v_j^2$</th>
<th>$a_j$</th>
<th>$b_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00609</td>
<td>1.92677</td>
<td>0.11265</td>
<td>1.01418</td>
<td>0.50710</td>
</tr>
<tr>
<td>2</td>
<td>0.04775</td>
<td>1.34744</td>
<td>0.17788</td>
<td>1.02248</td>
<td>0.51124</td>
</tr>
<tr>
<td>3</td>
<td>0.13057</td>
<td>0.73504</td>
<td>0.26768</td>
<td>1.03403</td>
<td>0.51701</td>
</tr>
<tr>
<td>4</td>
<td>0.20647</td>
<td>0.02266</td>
<td>0.40611</td>
<td>1.05207</td>
<td>0.52604</td>
</tr>
<tr>
<td>5</td>
<td>0.22715</td>
<td>-0.85173</td>
<td>0.62699</td>
<td>1.08153</td>
<td>0.54076</td>
</tr>
<tr>
<td>6</td>
<td>0.18842</td>
<td>-1.97278</td>
<td>0.98583</td>
<td>1.13114</td>
<td>0.56557</td>
</tr>
<tr>
<td>7</td>
<td>0.12047</td>
<td>-3.46788</td>
<td>1.57469</td>
<td>1.21754</td>
<td>0.60877</td>
</tr>
<tr>
<td>8</td>
<td>0.05591</td>
<td>-5.55246</td>
<td>2.54498</td>
<td>1.37454</td>
<td>0.68728</td>
</tr>
<tr>
<td>9</td>
<td>0.01575</td>
<td>-8.68384</td>
<td>4.16591</td>
<td>1.68327</td>
<td>0.84163</td>
</tr>
<tr>
<td>10</td>
<td>0.00115</td>
<td>-14.65000</td>
<td>7.33342</td>
<td>2.50097</td>
<td>1.25049</td>
</tr>
</tbody>
</table>

Notes: This table is from Omori, Chib, Shephard, and Nakajima (2007).

Let $s_{t\Delta} = \{1, 2, \cdots, 10\}$. Conditional on the realization of the state $s_{t\Delta}$, the above approximation implies that

$$
\begin{align*}
\left( \begin{array}{c}
\varepsilon^*_{t\Delta} \\
\eta_{t\Delta}
\end{array} \right)_{d_{t\Delta}, s_{t\Delta} = j, \rho, \sigma} & \sim \mathcal{N} \left( \left( \begin{array}{c}
m_j + v_j z_{t\Delta} \\
\frac{m_j}{2} + \sqrt{\Delta} \sigma \sqrt{1 - \rho^2} z^*_{t\Delta}
\end{array} \right), \begin{pmatrix}
d_{t\Delta} \rho \sqrt{\Delta} \sigma (a_j + b_j v_j z_{t\Delta}) & \sqrt{\Delta} \sigma \sqrt{1 - \rho^2} z^*_{t\Delta} \\
\sqrt{\Delta} \sigma \sqrt{1 - \rho^2} z^*_{t\Delta} & \frac{d_{t\Delta} \rho \sqrt{\Delta} \sigma (a_j + b_j v_j z_{t\Delta})}{2}
\end{pmatrix} \right),
\end{align*}
$$

where $(z_{t\Delta}, z^*_{t\Delta})' \sim \mathcal{N}_2 (0, I)$.

Through such approximation, the SV model with correlated innovations can be formulated as the following linear Gaussian state space model (see e.g. West and Harrison (1997), Durbin and Koopman (2001), and Harvey (2001)).
for \( t = 1, 2, \cdots, T \), and

\[
\begin{pmatrix}
    h_\Delta \\
    \widetilde{\beta}_\Delta
\end{pmatrix}
\sim
N
\left(
\begin{pmatrix}
    \beta_0 \\
    \beta_0
\end{pmatrix},
\begin{pmatrix}
    \Delta \sigma^2 / (1 - \phi^2) + \sigma_0^2 & \sigma_0^2 \\
    \sigma_0^2 & \sigma_0^2
\end{pmatrix}
\right)
\]

Assuming \( h_\Delta | \beta, \theta \sim N (\beta, \Delta \sigma^2 / (1 - \phi^2)) \) and \( \widetilde{\beta}_\Delta \sim N (\beta_0, \sigma_0^2) \), where \( \widetilde{\beta}_\Delta = \beta_2 \Delta = \cdots = \beta_T = \beta \).

Then posterior sample from

\[
g (s, h, \beta, \theta| y^*, d),
\]

where \( s = (s_\Delta, s_2 \Delta, \cdots, s_T \Delta)' \), \( y^* = (y^*_\Delta, y^*_2 \Delta, \cdots, y^*_T \Delta)' \) and \( d = (d_1, d_2, \cdots, d_T)' \) can be generated by iteratively drawing \( s| y^*, d, \beta, \theta \) and \( h, \beta, \theta| y^*, d, s \).

Mixture states can be sampled from

\[
\pi (s_{t \Delta} = j| y^*, d, h, \beta, \theta)
\propto \Pr (s_{t \Delta} = j) \exp \left\{ -\frac{(\varepsilon^*_t \Delta - m_j)^2}{2v_j^2} - \frac{[\eta_{t \Delta} - d_{t \Delta} \rho \sqrt{\Delta \sigma} \exp (m_j / 2) \{a_j + b_j (\varepsilon^*_t \Delta - m_j)\}]^2}{\Delta \sigma^2 (1 - \rho^2)}\right\}.
\]

Sampling of \( h, \beta, \theta| y^*, d, s \) can be generated by iteratively drawing \( \theta| y^*, d, s \) and \( h, \beta| y^*, d, s, \theta \). According to Bayes rule,

\[
g (\theta| y^*, d, s) \propto g (y^*| d, s, \theta) \pi (\theta),
\]

where \( g (y^*| d, s, \theta) \) can be found by applying Kalman filter recursions to the formulated state space model and \( \pi (\theta) \) is the prior distribution of \( \theta \). Random samples
can be generated using a Metropolis-Hastings algorithm with truncated Gaussian proposal density,

\[ TN_R \left( \hat{\theta}, \Sigma_* \right), \]

where

\[ \hat{\theta} = \arg \max_{\theta} g \left( y^*|d, s, \theta \right) \pi \left( \theta \right), \]

\[ \Sigma_*^{-1} = - \frac{\partial^2 \log g \left( y^*|d, s, \theta \right) \pi \left( \theta \right)}{\partial \theta \partial \theta'} \bigg|_{\theta = \hat{\theta}}, \]

and

\[ R = \left\{ (\phi, \sigma, \rho)^{'} : |\phi| < 1, \sigma^2 > 0, |\rho| < 1 \right\}. \]

Sampling of \( h, \beta | y^*, d, s, \theta \) can be done using the Gaussian simulation smoother (Carter and Kohn 1994, Frühwirth-Schnatter 1994, de Jong and Shephard 1995, Durbin and Koopman 2002).

**Prior**

In Omori, Chib, Shephard, and Nakajima (2007), the following prior distributions are used in generating the posterior samples:

\[ \beta \sim N \left( 0, 1 \right), \]

\[ \frac{\phi + 1}{2} \sim \text{Beta} \left( 20, 1.5 \right), \]

\[ \sigma^{-2} \sim \text{Gamma} \left( \frac{5}{2}, \frac{0.05}{2} \right), \]

\[ \rho \sim U \left( -1, 1 \right). \]
A detailed discussion of these priors is also provided. The same prior distributions are used in this chapter.

**Convergence**

The above sampling scheme clearly shows that $\theta \in \mathbb{R}$, $\beta \in \mathbb{R}$, $h \in \mathbb{R}^T$ and $Z_t \Delta \in \begin{cases} (p_0 + \ln(1 + I), \infty), & \text{if truncated above,} \\ (-\infty, p_0 + \ln(1 - I)), & \text{if truncated below.} \end{cases}$

In each iteration, it is possible for $\theta$, $\beta$, $h$ and $Z$ to take any values in the corresponding space. That is, the constructed Markov chain is irreducible. It is also clear that no portions of the state spaces of $\theta$, $\beta$, $h$ or $Z$ can only be visited at certain regularly spaced times. That is, the constructed Markov chain is also aperiodic. From Tierney (1994), we know that if this Markov chain has an invariant distribution $\pi$ this invariant distribution is unique and it is also the equilibrium distribution, which is the posterior distribution $P(Z, \theta, \beta, h | S)$ in which we are interested.

Using this property, as long as we observe the convergence of the suggested sampling scheme, we just need to continue the iterations for a long enough period so that we could make inferences based on samples from the convergent portion.

### 3.3.3 Quasi-Realized Volatility and Simulated-Integrated Volatility

As the objective of this chapter is to provide estimators for integrated volatility within periods when some parts of the asset price process are truncated due to price limits, I suggest the following two estimators by exploiting the generated posterior samples
and existing estimators for integrated volatility. The first estimator is constructed utilizing the posterior sample of truncated asset prices. By treating all simulated asset prices as realizations of an unobserved asset price process, we can use the idea of realized volatility to construct the following estimator,

$$\text{QRV}^{(g)}_{[t_0, t_m]} = \sum_{i=1}^{m} \left( p^{(g)}_{t_i} - p^{(g)}_{t_{i-1}} \right)^2,$$

for $g = 1, 2, \cdots, G$. Here, $p^{(g)}_{t_i}$ is defined as

$$p^{(g)}_{t_i} = \begin{cases} S_{t_i} & \text{if observed,} \\ Z^{(g)}_{t_i} & \text{otherwise,} \end{cases}$$

where $g$ denotes that the corresponding value is from iteration $g$ after the burn-in period and $G$ is the total number of iterations after the burn-in period. Since this estimator utilizes both realized asset prices and simulated asset prices, I call it quasi-realized volatility (QRV).

According to the results in Barndorff-Nielsen and Shephard (2002), Meddahi (2002) and Andersen, Bollerslev, Diebold, and Labys (2003), realized volatility is a consistent estimator of integrated volatility under some regularity conditions. In the posterior sample, each iteration can be thought of as a realization of integrated volatility over the period of interest. And in each iteration this realization of integrated volatility can be consistently estimated by QRV. Therefore, the derived posterior distribution of QRV converges to the posterior distribution of integrated volatility. These results are summarized in Theorem 3.

**Theorem 3.** Quasi-realized volatility provides a consistent measure of integrated
volatility. That is,

$$plim_{m \to \infty} QRV_{[t_0, t_m]}^{(g)} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv IV_{[t_0, t_m]}^{(g)}$$

for $g = 1, 2, \cdots, G$ and

$$plim_{m \to \infty} P \left( QRV_{[t_0, t_m]} | S \right) = P \left( IV_{[t_0, t_m]} | S \right).$$

Proof: By the convergence property of MCMC, we have

$$p^{(g)}, \theta^{(g)}, \beta^{(g)}, h^{(g)} \sim P \left( p, \theta, \beta, h | S \right).$$

Since $p$ and $h$ are from the stochastic volatility model in equations 3.1 and 3.2, the asset price process belongs to the class of special semi-martingales as detailed by Back (1991). According to Propositions 1 and 2 from Andersen, Bollerslev, Diebold, and Labys (2003), we have

$$plim_{m \to \infty} \sum_{i=1}^{m} \left( p_{t_i}^{(g)} - p_{t_{i-1}}^{(g)} \right)^2 = [r^{(g)}, r^{(g)}]_{t_m} - [r^{(g)}, r^{(g)}]_{t_0} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv IV_{[t_0, t_m]}^{(g)},$$

where $r_t^{(g)} = p_t^{(g)} - p_0^{(g)}$ is the cumulative log return at time $t$ and $[r, r]$ denotes the quadratic variation process of the return process. That is, for each iteration QRV is a consistent measure of integrated volatility. Therefore, the posterior distribution of QRV converges to the posterior distribution of IV.

The second estimator is even more natural in the context of stochastic volatility models. Since the MCMC algorithm not only provides us with the posterior sample of the truncated asset prices but also the posterior sample of the latent volatility
states, we can directly derive the posterior distribution of integrated volatility using the discrete sample analogue by

\[
SIV^{(g)}_{[t_0,t_m]} = \frac{1}{m} \sum_{i=1}^{m} \exp \left( h^{(g)}_{t_i} \right)
\]

for \( g = 1, 2, \cdots, G \). I call this estimator the simulated integrated volatility (SIV).

**Theorem 4.** Simulated integrated volatility provides a consistent measure of integrated volatility. That is,

\[
\lim_{m \to \infty} SIV^{(g)}_{[t_0,t_m]} = \int_{t_0}^{t_m} \exp \left( h^{(g)}_s \right) ds \equiv IV^{(g)}_{[t_0,t_m]}
\]

for \( g = 1, 2, \cdots, G \) and

\[
\lim_{n \to \infty} P \left( SIV_{[t_0,t_m]} | S \right) = P \left( IV_{[t_0,t_m]} | S \right).
\]

Proof: These are direct results from the convergence of numerical integration. That is, for each iteration, SIV is a consistent measure of integrated volatility. Thus the posterior distribution of SIV converges to the posterior distribution of IV.

\[ \blacksquare \]

### 3.4 Simulation

#### 3.4.1 Illustrative Example

To show how the suggested procedure works more intuitively, I provide the following example. I simulate a random sample from the stochastic volatility model in equations (3.3) and (3.4). Parameter values are given in Table 3.2 which are used in the

Table 3.2: Parameter values

<table>
<thead>
<tr>
<th>β</th>
<th>κ</th>
<th>σ</th>
<th>ρ</th>
<th>m</th>
<th>l</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7.3618</td>
<td>0.05</td>
<td>0.26</td>
<td>-0.6</td>
<td>240</td>
<td>0.02</td>
</tr>
</tbody>
</table>

A symmetric price limit of 0.02 is used, which is close to the most restrictive price limit level in reality. My simulation study shows that the lower the price limit is, the more difficult it is to estimate the integrated volatility due to an increase in information loss. I choose $m = 240$ for the sampling frequency. As in other Bayesian analyses of stochastic volatility models, I use a reasonably large sample size in this example and the simulation studies. A large sample not only brings more information, but also mitigates the effect of priors. Therefore, in this example, along with the trading day of interest, I also include 15 days before and 14 days after. The simulated data are shown in Figure 3.1.

For the trading day of interest, 111 price observations are truncated after applying the price limit, which corresponds to 46.25% of the total observations in a trading day. Figure 3.2 zooms in on the simulated asset prices of day 16.

The two horizontal lines correspond to the upper and lower limits of the trading day. According to the observation rule, only asset prices between the two price limits are observable.

The MCMC algorithm developed in Section 3 is applied to the simulated sample for 20,000 iterations. The first 5,000 iterations are discarded as the burn-in period. Posterior distributions are estimated from the remaining 15,000 iterations and results are shown in Figure 3.3 and Table 3.3.
In Figure 3.3, the columns show the trace plots, the autocorrelograms, and the kernel densities of the parameters and two suggested estimators. As shown in all of the autocorrelograms, the sampling scheme achieves very high computational efficiency as suggested by Omori, Chib, Shephard, and Nakajima (2007). Also, trace plots provide very strong evidence of convergence. The horizontal line in each trace plot shows the true value of the corresponding parameter or estimator. Clearly the suggested MCMC algorithm provides very good estimates for all parameters and for the integrated volatility. This example also shows that QRV and SIV provide very similar estimates of integrated volatility. Due to the way SIV is defined, whether SIV provides a good estimate of IV depends closely on how well we can estimate the latent volatility states.
Figure 3.2: Simulated log prices and price limits

Figure 3.4 shows the posterior mean of the latent log volatility states and the true simulated log volatility states.

We see that the posterior means of latent volatility states are very close to the true latent volatility states. Therefore, the posterior mean of SIV suggests a good point estimator candidate for integrated volatility. Similarly, we can also use the posterior mean of QRV as another point estimator for IV.

### 3.4.2 Simulation Study

The convergence properties of QRV and SIV guarantee that if the sampling frequency is very high, the posterior distributions of QRV and SIV converge to the posterior
distribution of integrated volatility. The above example also shows that the modes of the posterior distributions of QRV and SIV are both very close to the true integrated volatility level. In this section, I study the performance of QRV and SIV based on 1,000 simulated samples. I use the same parameter values to generate the random samples. Due to the computational burden, I only run 10,000 iterations for each sample and keep the last 5,000 iterations for the posterior distribution. I use the mean of $\text{QRV}^{(g)}$ and the mean of $\text{SIV}^{(g)}$, which both converge to the posterior mean of integrated volatility according to Proposition 1, as point estimates for integrated volatility. For reasons of comparison, I also include the following three estimators

$$\text{RV}^{*}[t_0,t_m] = \sum_{i=1}^{m} (p_{t_i} - p_{t_{i-1}})^2,$$

$$\text{RV}[t_0,t_m] = \sum_{t_0 < t_i \leq t_m} (S_{t_i} - S_{t_{i-1}})^2,$$

$$\text{RV}^{adj}[t_0,t_m] = \frac{T}{T - M} \text{RV}[t_0,t_m].$$

Here, $\text{RV}^{*}[t_0,t_m]$ is the realized volatility over period $[t_0, t_m]$ assuming we know all of the asset prices. Therefore it is not a feasible estimator if price limits are present. $\text{RV}[t_0,t_m]$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True value</th>
<th>Mean</th>
<th>St. dev.</th>
<th>95% interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>-7.3618</td>
<td>-8.2035</td>
<td>0.9499</td>
<td>[-10.4150,-7.0413]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.9998</td>
<td>0.9992</td>
<td>0.0009</td>
<td>[0.9970,0.9999]</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.2600</td>
<td>0.2607</td>
<td>0.0243</td>
<td>[0.2176,0.3145]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.6000</td>
<td>-0.5657</td>
<td>0.1224</td>
<td>[-0.7618,-0.2946]</td>
</tr>
<tr>
<td>QRV($\times 10^{-3}$)</td>
<td>9.0456</td>
<td>9.0154</td>
<td>0.7665</td>
<td>[7.7546,10.7400]</td>
</tr>
<tr>
<td>SIV($\times 10^{-3}$)</td>
<td>9.0456</td>
<td>8.7814</td>
<td>0.9940</td>
<td>[7.0536,10.8710]</td>
</tr>
</tbody>
</table>
is the traditional realized volatility based only on realized asset prices, and \( M \) is the total number of truncated asset prices. Therefore, \( RV^{adj}_{[t_0, t_m]} \) is an adjusted measurement assuming volatilities are constant over time, which contradicts the empirical finding of time varying volatilities.

Because the realized integrated volatility level is different in each simulated sample, I compare the relative errors of all estimators as in Nielsen and Frederiksen (2008). Figure 3.5 shows the line graph of relative errors.

From this graph we can see that QRV and SIV are almost the same. In addition, they both behave very similarly to the infeasible estimator \( RV^* \). Therefore both QRV and SIV provide reliable estimates for integrated volatility. The relative error plot of RV again clearly shows the downward bias. Although the relative error plot of \( RV^{adj} \) does not clearly show evidence of bias, the variation of the relative errors (note the different scale on the y-axes) is much higher than QRV and SIV. Table 3.4 presents summary statistics of the relative errors.

<table>
<thead>
<tr>
<th></th>
<th>QRV</th>
<th>SIV</th>
<th>RV*</th>
<th>RV*</th>
<th>RV</th>
<th>RV^{adj}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0103</td>
<td>0.0127</td>
<td>0.0023</td>
<td>-0.3717</td>
<td>-0.0312</td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>0.1059</td>
<td>0.0948</td>
<td>0.0909</td>
<td>0.4586</td>
<td>0.1889</td>
<td></td>
</tr>
<tr>
<td>s.d.</td>
<td>0.1054</td>
<td>0.0940</td>
<td>0.0909</td>
<td>0.2686</td>
<td>0.1863</td>
<td></td>
</tr>
</tbody>
</table>

### 3.5 Empirical Application

In this section, I apply the method developed in this chapter to the same data that I use in Chapter 2. For each stock, the MCMC algorithm runs for 10,000 iterations.
The first 5,000 are discarded as the burn-in period and the remaining 5,000 are used in analyzing the posterior distributions. Results are given in Figures 3.6 and 3.7 and Table 3.5.

Table 3.5: Estimation results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Stock: HYGF</th>
<th>Stock: ZGGH</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>-7.2139 0.0640 [-7.3363,-7.0850]</td>
<td>-6.8302 0.0512 [-6.9298,-6.7304]</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.8826 0.0110 [0.8600,0.9031]</td>
<td>0.8239 0.0148 [0.7936,0.8518]</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>4.4957 0.2300 [4.0467,4.9527]</td>
<td>5.4044 0.2546 [4.9165,5.9370]</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.0807 0.0264 [0.0288,0.1337]</td>
<td>0.0668 0.0243 [0.0194,0.1147]</td>
</tr>
<tr>
<td>QRV((\times 10^{-3}))</td>
<td>6.0169 1.0194 [4.6584,8.5094]</td>
<td>2.2899 0.3937 [1.8823,3.2942]</td>
</tr>
<tr>
<td>SIV((\times 10^{-3}))</td>
<td>3.9408 1.1310 [2.2572,6.5505]</td>
<td>2.3128 0.4409 [1.6672,3.3970]</td>
</tr>
</tbody>
</table>

Figures 3.6 and 3.7 clearly demonstrate convergence for both stocks. For both stocks the QRV and the SIV seem very similar except that QRV takes higher values than SIV for ZGGF. From Table 3.5, both stocks exhibit high conditional variances of latent volatility states, which is consistent with the fact that the Chinese stock markets are very volatile at that time. Latent volatility states also exhibit relatively low persistency for both stocks, which is also consistent with the fact that market wide and individual stock specific shocks are mainly temporary during the period of interest. For HYGF, the posterior mean of QRV is \(6.0169 \times 10^{-3}\), while the posterior mean of SIV is \(3.9408 \times 10^{-3}\). This can be explained by the observed return jumps on the day of interest and the fact that realized volatility tends to over estimate integrated volatility in the presence of return jumps. Therefore, the posterior mean of SIV provides more reliable estimate for integrated volatility under this circumstance. Compared with the realized volatility, the posterior mean of SIV increases by 10.37%.
For ZGGH, the posterior mean of QRV and SIV are very close and increase by about 31% compared with the realized volatility. For both stocks, the 95% confidence intervals of $\rho$ contain only positive values. This indicates a favor of the volatility feedback effect (positive $\rho$) over the leverage effect (negative $\rho$) for both stocks, which is consistent with the empirical finding that the volatility feedback effect dominates the leverage effects in Bekaert and Wu (2000) and Wu (2001). In the literature, it has been found that the magnitude of the leverage effect is generally larger for market index returns than for individual stocks and the volatility feedback effect is generally larger for individual stocks.

To further examine the difference between the QRV and the SIV, I plot the daily realized volatilities and the SIVs for the sampling period in Figure 3.8. For the day of interest, I replace the RV with the posterior mean of the QRV.

As in Chapter 2, we can see that RV and SIV are very close. Both estimates seem reasonable when comparing the posterior mean of both QRV and SIV with the neighboring days. As discussed earlier, on Day 16 the presence of return jumps explains the difference between QRV and SIV for HYGF. Similarly, we can explain the difference between RV ans SIV for ZGGH in the last two days in the sampling period.

3.6 Concluding Remarks

This chapter extends the ideas developed in Chapter 2 to a log-stochastic volatility model allowing correlations between the innovations of asset returns and latent volatilities. Bayesian methods can still help us to recover the lost information due to the presence of price limits. The MCMC algorithm designed based on Omori, Chib,
Shephard, and Nakajima (2007) clearly shows convergence and efficiency. Simulation results indicate that both QRV and SIV provide very good estimates for the underlying integrated volatility and both behave very similarly to the realized volatility assuming we know all of the asset prices. The application to the Chinese stock markets also shows that both QRV and SIV provide reasonable estimates for financial practitioners.
Figure 3.3: Trace plots, correlograms and kernel densities
True and posterior mean of volatility ($h_t$)

Figure 3.4: True and posterior mean volatilities
Figure 3.5: Relative errors
Figure 3.6: Trace plots, correlograms and kernel densities for HYGF
Figure 3.7: Trace plots, correlograms and kernel densities for ZGGH
Figure 3.8: RV and SIV plots
Chapter 4

Bayesian Estimation of Integrated Volatility with Price Limits: Log-Stochastic Volatility Model with Correlated Innovations, Fat Tails and Jumps

4.1 Introduction

A great deal of research has shown that diffusive stochastic volatility models cannot fully capture the empirical features of equity returns or option prices (see, e.g. Bakshi, Cao, and Chen (1997), Bates (2000), and Pan (2002)). For diffusive stochastic...
volatility models, market extremes, such as the October 1987 crash, require an implausible high volatility level both prior to and after the crash. To capture the large market movements, fat-tailed errors and jump components have been included in the generalized stochastic volatility models in the recent literature (see, e.g., Gallant, Hsieh, and Tauchen (1997), Geweke (1994), Bates (2000) and Pan (2002)). Chan and Maheu (2002) and Maheu and McCurdy (2004) also incorporate the autoregressive conditional jump intensity parametrization in the GARCH framework.

In this chapter, I extend the analysis in Chapter 2 and Chapter 3 to include fat-tailed errors and discontinuous jump components in asset returns. This chapter contributes to the literature from the following aspects. First, I incorporate price limits in the log-stochastic volatility model with correlated innovations, fat tails and return jumps. Second, I design a MCMC algorithm to simulate the unobserved asset prices, model parameters and latent states allowing for correlated innovations, fat tails and return jumps. Third, I revise the two types of estimators for integrated volatility in the presence of price limits which are developed in Chapter 2 and Chapter 3 to adjust for fat tails and return jumps.

The rest of this chapter is organized as follows. In Section 4.2, I specify the log-stochastic volatility model with jumps and fat-tailed innovations in the return equation and introduce price limits into the framework. In Section 4.3, I design a MCMC sampling scheme to incorporate price limits. I also discuss the convergence property of the designed MCMC algorithm. In Section 4.4 I examine the performance of adjusted Quasi-realized volatility and simulated integrated volatility through simulation studies. I conclude this chapter in Section 4.5.
CHAPTER 4. SV MODEL WITH CORRELATED INNOVATIONS, FAT-TAILS AND JUMPS

4.2 Price limits and Log-Stochastic Volatility Model with Jumps and Fat-Tails

In the log-stochastic volatility model with correlated innovations, fat tails and jumps, the log asset price $p_t$ and its log diffusive volatility $h_t$ solve the following two differential equations

\begin{align}
    dp_t &= \mu_t dt + \exp\left(\frac{h_t}{2}\right) \sqrt{\lambda_t} dW_t^s + \xi^y dN^y, \quad (4.1) \\
    dh_t &= \kappa (\beta - h_t) dt + \sigma dW_t^\sigma, \quad (4.2)
\end{align}

where $\mu_t$ is the equity risk premium, $\sigma$ is the diffusive volatility of conditional variances, $\kappa$ and $\beta$ are two parameters of the conditional variance process, and $W_t^s$, $W_t^\sigma$ are two Brownian motions with correlation $\rho$. As in Nakajima and Omori (2009), I assume

$$
\lambda_t \sim \text{Gamma} \left( \frac{\nu}{2}, \frac{\nu}{2} \right)
$$

to obtain Student-t innovations for the asset return process, where $\nu$ denotes the degrees of freedom. $N^y$ is a Poisson process with constant intensity $\lambda_y$, and $\xi^y$ is the jump size in returns. For simplicity, I only include jumps in the asset return process. However, the techniques developed in this chapter could be extended to more general jump diffusion frameworks.

The integrated volatility over period $[t_0, t_m]$ is still defined as

$$
IV_{[t_0, t_m]} = \int_{t_0}^{t_m} \exp(h_s) \, ds.
$$
CHAPTER 4. SV MODEL WITH CORRELATED INNOVATIONS, FAT-TAILS AND JUMPS

To focus on volatilities, I assume $\mu_t$ is equal to zero. An Euler time discretization of the model in equations 4.1 and 4.2 implies that

$$
\begin{align*}
  y_{t\Delta} &= k_{t\Delta} \gamma_{t\Delta} + \sqrt{\lambda_{t\Delta}} \varepsilon_{t\Delta} \exp(h_{t\Delta}/2), \\
  h_{(t+1)\Delta} &= \beta + \phi (h_{t\Delta} - \beta) + \eta_{t\Delta},
\end{align*}
$$

with $\phi \equiv (1 - \kappa \Delta)$ where $\Delta$ is the distance between two observations. Here, $\varepsilon_{t\Delta}$ and $\eta_{t\Delta}$ are normally distributed with zero mean, and covariance matrix $\Sigma = \Delta \begin{bmatrix} 1 & \rho \sigma \\ \rho \sigma & \sigma^2 \end{bmatrix}$.

The jump flag $\gamma_{t\Delta}$ is defined as a Bernoulli random variable with

$$
\pi (\gamma_{t\Delta} = 1) = \omega, \quad \pi (\gamma_{t\Delta} = 0) = 1 - \omega, \quad 0 < \omega < 1,
$$

and the jump size $k_{t\Delta}$ is assumed to satisfy

$$
\psi_{t\Delta} \equiv \log (1 + k_{t\Delta}) \sim N (-0.5\delta^2, \delta^2)
$$

as in Andersen, Benzoni, and Lund (2002). In this chapter, I also assume there are $m$ equidistant observations in each trading day. This assumption can be relaxed easily and would not change the results as long as the distances between adjacent observations are very small ($\rightarrow 0$). Therefore $\Delta = \frac{1}{m}$.

When a symmetric price limit $l > 0$ (in percentage) is imposed, that is the price of a stock in a trading day can not increase or decrease by more than $l$ percent of its
previous closing price, the following observation rule takes effect:

\[
p_t = \begin{cases} 
  p_t & \text{if } \ln(1-l) \leq p_t - p_0 \leq \ln(1+l), \\
  \text{Unobservable} & \text{otherwise},
\end{cases}
\]

for \( i = 1, 2, \ldots, m \), where \( p_0 \) denotes the closing price of the previous trading day.

In Chapter 2, I show that realized volatility tends to underestimate integrated volatility in the presence of price limits. The downward bias of realized volatility is a result of information loss due to price limits. If we could recover the lost information based on realized asset prices and the implied relationship from the stochastic volatility model, then we should be able to measure the volatility level for the truncated sub-periods. In this chapter, I extend this idea to the framework of the log-stochastic volatility model with correlated innovations, fat tails and jumps.

### 4.3 Bayesian Estimation of Log-Stochastic Volatility Model with Correlated Innovations, Fat Tails and Jumps

To recover the missing information resulting from price limit truncation, we need to resort to estimation procedures for stochastic volatility models with correlated innovations, fat tails and jumps. I extend the suggested method in Nakajima and Omori (2009) to generate random samples for truncated asset prices, parameters and latent states.
4.3.1 Bayesian Perspective

To separate the observed asset prices from the unobserved prices, let us define

\[ p_{t\Delta} = \begin{cases} 
S_{t\Delta} & \text{if observed,} \\
Z_{t\Delta} & \text{otherwise.}
\end{cases} \]

Also let \( \theta = (\phi, \rho, \sigma) \), \( h = (h_{2\Delta}, h_{3\Delta}, \cdots, h_{(T+1)\Delta})' \), \( \gamma = (\gamma_{\Delta}, \gamma_{2\Delta}, \cdots, \gamma_{T\Delta})' \), \( k = (k_{\Delta}, k_{2\Delta}, \cdots, k_{T\Delta})' \), \( \lambda = (\lambda_{\Delta}, \lambda_{2\Delta}, \cdots, \lambda_{T\Delta})' \), \( p = (p_{\Delta}, p_{2\Delta}, \cdots, p_{T\Delta})' \). Let \( S \) denote a column vector of all observed prices and \( Z \) denote a column vector of all unobserved prices. Since at each time the asset price can be either observed or unobserved, the combined dimension of \( S \) and \( Z \) must equal the dimension of \( p \). From a Bayesian’s perspective, we are interested in

\[ P(Z, \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda | S). \]

That is the posterior distribution of unobserved asset prices, parameters and latent states given the observed asset prices. According to the Clifford-Hammersley theorem, this posterior distribution is uniquely determined by the two following conditional distributions: \( P(Z | \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, S) \) and \( P(\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda | p) \).

The idea of the MCMC algorithm is to iteratively draw from these two conditional distributions, which form a Markov chain. Tierney (1994) shows that if the chain has a proper invariant distribution \( \pi \) and it is irreducible and aperiodic then this invariant distribution is unique and the unique invariant distribution \( \pi \) is also the equilibrium distribution of the chain.

I will first design a MCMC algorithm for the objective posterior distribution.
Then I will discuss the convergence properties of the designed MCMC algorithm.

### 4.3.2 MCMC Algorithm

Nakajima and Omori (2009) have developed an efficient algorithm for $P(\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda \mid p)$. If we were able to sample from $P(Z \mid \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, S)$ too, we achieve our goal.

So, before summarizing the method in Nakajima and Omori (2009), I will first derive the conditional distribution for unobserved asset prices.

**Conditional Distribution of Unobserved Asset Prices**

Using the definition of conditional distributions, we have

$$P(Z \mid \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, S) = \frac{P(p, h \mid \theta, \beta, \omega, \delta, \nu, k, \gamma, \lambda)}{P(h, S \mid \theta, \beta, \omega, \delta, \nu, k, \gamma, \lambda)} \propto P(p, h \mid \theta, \beta, \omega, \delta, \nu, k, \gamma, \lambda).$$

This indicates that the conditional distribution of unobserved asset prices is proportional to the likelihood of the model.
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Conditionally on $h_{\Delta}$, the likelihood function is given by

$$
P(y, h, k, \gamma, \lambda|\theta, \beta, \omega, \delta, v)
\begin{align*}
&= \frac{1}{\sqrt{2\pi}} \frac{\sigma_{\Delta}}{\sqrt{1-\phi^2}} \exp \left( -\frac{1}{2} \frac{\sigma^2_{\Delta}}{1-\sigma^2} (h_{\Delta} - \beta)^2 \right) \\
&\quad \prod_{t=1}^{T} \left( \omega^{\gamma_{\Delta}} (1 - \omega)^{1-\gamma_{\Delta}} \right) \\
&\quad \prod_{t=1}^{T} \frac{1}{1 + k_{t\Delta}} \frac{1}{\sqrt{2\pi} \delta} \exp \left( -\frac{1}{2\delta^2} \left( \log (1 + k_{t\Delta}) + 0.5\delta^2 \right)^2 \right) \\
&\quad \prod_{t=1}^{T} \frac{\left( \frac{\nu_{t\Delta}}{2} \right)}{\Gamma \left( \frac{\nu_{t\Delta}}{2} \right)} (\lambda_{t\Delta})^{-\frac{\nu_{t\Delta}}{2} - 1} \exp \left( -\frac{\nu_{t\Delta}}{2\lambda_{t\Delta}} \right) \\
&\quad \prod_{t=1}^{T} \frac{1}{2\pi \sqrt{\lambda_{t\Delta}}} \exp \left( \frac{h_{t\Delta} \Delta}{\lambda_{t\Delta}} \exp(h_{t\Delta}/2) \right) \exp \left( -\frac{1}{2} \frac{1}{(1-\rho^2)} \left( \frac{y_{t\Delta} - k_{t\Delta} \gamma_{t\Delta}}{\sqrt{\lambda_{t\Delta}}} \exp(h_{t\Delta}/2) \right)^2 \right. \\
&\quad \left. + \left( \frac{h_{(t+1)\Delta} \Delta - \beta - \phi (h_{t\Delta} - \beta)}{\sigma \sqrt{\Delta}} \right)^2 \right) - 2\rho \left( \frac{y_{t\Delta} - k_{t\Delta} \gamma_{t\Delta}}{\sqrt{\lambda_{t\Delta}}} \exp(h_{t\Delta}/2) \right) \\
&\end{align*}
$$

We can use a Gibbs sampler to sample each unobserved asset price individually.

By the Markov property of asset prices, we have

$$
P(Z_t|\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, p_{-1}) \propto P(Z_t|\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, p_{-1}, p_{t+1}).
$$

It can be shown that the above density corresponds to the following normal distribution (see Appendix C for details):

$$
Z_{t\Delta}|\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda, p_{-t} \sim N \left( \mu_{Z_{t\Delta}}, \sigma_{Z_{t\Delta}}^2 \right),
$$
for \( t = 1, 2, \cdots, T \), where

\[
\mu_{Z_t\triangle} = \frac{\sigma_{Z_t\triangle}^2}{(1 - \rho^2) \Delta} \left( \frac{p(t-1)\triangle + k_t\triangle \gamma_t\triangle}{\lambda_t\triangle \exp(h_t\triangle)} + \frac{\rho \eta_{t\triangle}}{\sqrt{\lambda_t\triangle \exp(h_t\triangle/2)\sigma}} + \frac{p(t+1)\triangle - k(t+1)\triangle \gamma(t+1)\triangle}{\lambda(t+1)\triangle \exp(h(t+1)\triangle)} \right) - \frac{\rho \eta_{(t+1)\triangle}}{\sqrt{\lambda(t+1)\triangle \exp(h(t+1)\triangle/2)\sigma}},
\]

\[
\sigma_{Z_t\triangle}^2 = \frac{(1 - \rho^2) \Delta \lambda_t\triangle \exp(h_t\triangle) \lambda_{t+1}\triangle \exp(h_{t+1}\triangle)}{(\lambda_{t+1}\triangle \exp(h_{t+1}\triangle) + \lambda_t\triangle \exp(h_t\triangle))}.
\]

From the observation rule 4.5 we know that \( Z_{t\triangle} \) is truncated either because

\[
Z_{t\triangle} > p_0 + \ln (1 + l)
\]

or

\[
Z_{t\triangle} < p_0 + \ln (1 - l),
\]

where \( p_0 \) is the corresponding closing price of the previous trading day. Also, in reality we usually have this information. Therefore, instead of sampling unobserved asset prices from the above normal distribution, we will generate random samples from the corresponding truncated normal distributions.

**Conditional Distribution of Parameters and States**

Once we finish sampling unobserved asset prices, we return to the standard estimation problem of the SV model in equations 4.3 and 4.4. I follow the procedure suggested in Nakajima and Omori (2009) to generate random samples for parameters and latent states. Please refer to the Appendix of Nakajima and Omori (2009) for technique details.
Prior

The following prior distributions are used in generating the posterior samples:

\[
\beta \sim N(-10, 1),
\]

\[
1 - \frac{m(1 - \phi)}{2} \sim \text{Beta}(20, 1.5),
\]

\[
\sigma^{-2} \sim \text{Gamma}\left(\frac{5}{2}, \frac{0.05}{2}\right),
\]

\[
\rho \sim U(-1, 1),
\]

\[
\omega \sim \text{Beta}(0.2, 100),
\]

\[
\log(\delta) \sim N(-2.5, 0.15),
\]

\[
\nu \sim \text{Gamma}(16, 0.8).
\]

The prior distributions of \(\phi\) and \(\omega\) are adjusted considering the difference between the sampling frequency in this chapter and that in Nakajima and Omori (2009). For a detailed discussion of these priors, please refer to Nakajima and Omori (2009).

Convergence

In each iteration, it is possible for \(\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda\) and \(Z\) to take any values in the corresponding space. That is, the constructed Markov chain is irreducible. It is also clear that no portions of the state spaces of \(\theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda\) or \(Z\) can only be visited at certain regularly spaced times. That is, the constructed Markov chain is also aperiodic. From Tierney (1994), we know that if this Markov chain has
an invariant distribution $\pi$ this invariant distribution is unique and it is also the equi-
librium distribution, which is the posterior distribution $P(Z, \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda | S)$
in which we are interested.

Using this property, as long as we observe the convergence of the suggested sam-
pling scheme, we just need to continue the iterations for a long enough period so that
we could make inferences based on samples from the convergent portion.

### 4.3.3 Adjusted Quasi-Realized Volatility and Simulated-Integrated Volatility

As the objective of this chapter is to provide estimators for integrated volatility within
periods when some parts of the asset price process are truncated due to price limits,
I suggest the following two estimators by exploiting the generated posterior samples
and existing estimators for integrated volatility. The first estimator is constructed
utilizing the posterior sample of truncated asset prices. By treating all simulated
asset prices as realizations of an unobserved asset price process, we can use the idea
of realized volatility to construct the following estimator,

$$\text{Adj-QRV}_{[t_0, t_m]}^{(g)} = \sum_{i=1}^{m} \frac{\left(p_{t_i}^{(g)} - p_{t_{i-1}}^{(g)} - k_{t_i}^{(g)} \gamma_{t_i}^{(g)} \right)^2}{\lambda_{t_i}^{(g)}}$$

for $g = 1, 2, \cdots, G$. Here, $p_{t_i}^{(g)}$ is defined as

$$p_{t_i}^{(g)} = \begin{cases} S_{t_i} & \text{if observed}, \\ Z_{t_i}^{(g)} & \text{otherwise}, \end{cases}$$
where \( g \) denotes that the corresponding value is from iteration \( g \) after the burn-in period and \( G \) is the total number of iterations after the burn-in period. Due to the presence of fat-tailed innovations and jumps, the traditional realized volatility need to be adjusted accordingly. To differentiate this estimator from the quasi-realized volatility defined in the previous chapters, I call it adjusted quasi-realized volatility (Adj-QRV).

According to the results in Barndorff-Nielsen and Shephard (2002), Meddahi (2002) and Andersen, Bollerslev, Diebold, and Labys (2003), realized volatility is a consistent estimator of integrated volatility under some regularity conditions. In the posterior sample, each iteration can be thought of as a realization of integrated volatility over the period of interest. And in each iteration this realization of integrated volatility can be consistently estimated by Adj-QRV. Therefore, the derived posterior distribution of Adj-QRV converges to the posterior distribution of integrated volatility. These results are summarized in Theorem 5.

**Theorem 5.** Adjusted Quasi-realized volatility provides a consistent measure of integrated volatility. That is,

\[
\text{plim}_{m \to \infty} \text{Adj-QRV}^{(g)}_{[t_0,t_m]} = \int_{t_0}^{t_m} \exp(h_s^{(g)}) \, ds \equiv IV^{(g)}_{[t_0,t_m]}
\]

for \( g = 1, 2, \ldots, G \) and

\[
\text{plim}_{m \to \infty} P \left( \text{Adj-QRV}_{[t_0,t_m]}|S \right) = P \left( IV_{[t_0,t_m]}|S \right).
\]
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Proof: By the convergence property of MCMC, we have

\[ p^{(g)}, \theta^{(g)}, \beta^{(g)}, \omega^{(g)}, \delta^{(g)}, \nu^{(g)}, h^{(g)}, k^{(g)}, \gamma^{(g)}, \lambda^{(g)} \sim P(Z, \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda|S). \]

Since \( p \) and \( h \) are from the stochastic volatility model in equations 4.1 and 4.2, the asset price process belongs to the class of special semi-martingales as detailed by Back (1991) if adjusted for the fat-tails and jump components. According to Propositions 1 and 2 from Andersen, Bollerslev, Diebold, and Labys (2003), we have

\[
\text{plim}_{m \to \infty} \sum_{i=1}^{m} \left( \frac{p_{t_i}^{(g)} - p_{t_{i-1}}^{(g)} - k_{t_i}^{(g)} \lambda_{t_i}^{(g)}}{\lambda_{t_i}^{(g)}} \right)^2 = \left[ r^{(g)}, r^{(g)} \right]_{t_m} - \left[ r^{(g)}, r^{(g)} \right]_{t_0} = \int_{t_0}^{t_m} \exp \left( h^{(g)}_s \right) ds \equiv IV^{(g)}_{[t_0,t_m]},
\]

where \( r^{(g)}_{t_i} = \frac{p_{t_i}^{(g)} - p_{t_{i-1}}^{(g)} - k_{t_i}^{(g)} \lambda_{t_i}^{(g)}}{\sqrt{\lambda_{t_i}^{(g)}}} \) and \([r, r]\) denotes the quadratic variation process of the adjusted return process. That is, for each iteration Adj-QRV is a consistent measure of integrated volatility. Therefore, the posterior distribution of Adj-QRV converges to the posterior distribution of IV.

The second estimator is even more natural in the context of stochastic volatility models. Since the MCMC algorithm not only provides us with the posterior sample of the truncated asset prices but also the posterior sample of the latent volatility states, we can directly derive the posterior distribution of integrated volatility using the discrete sample analogue by

\[
\text{SIV}^{(g)}_{[t_0,t_m]} = \frac{1}{m} \sum_{i=1}^{m} \exp \left( h^{(g)}_{t_i} \right).
\]
for $g = 1, 2, \cdots, G$. I call this estimator the simulated integrated volatility (SIV).

**Theorem 6.** Simulated integrated volatility provides a consistent measure of integrated volatility. That is,

$$\text{plim}_{m \to \infty} SIV_{[t_0, t_m]}^{(g)} = \int_{t_0}^{t_m} \exp \left( h_s^{(g)} \right) ds \equiv IV_{[t_0, t_m]}^{(g)}$$

for $g = 1, 2, \cdots, G$ and

$$\text{plim}_{m \to \infty} P \left( SIV_{[t_0, t_m]} | S \right) = P \left( IV_{[t_0, t_m]} | S \right).$$

Proof: These are direct results from the convergence of numerical integration. That is, for each iteration, SIV is a consistent measure of integrated volatility. Thus the posterior distribution of SIV converges to the posterior distribution of IV.

### 4.4 Simulation

#### 4.4.1 Illustrative Example

To show how the suggested procedure works more intuitively, I provide the following example. I simulate a random sample from the stochastic volatility model in equations 4.3 and 4.4. Parameter values are given in Table 4.1 which are used in the simulation study of Nakajima and Omori (2009).

A symmetric price limit of 0.02 is used, which is close to the most restrictive price limit level in reality. My simulation study shows that the lower the price limit is, the more difficult it is to estimate the integrated volatility due to an increase in information loss. I choose $m = 240$ for the sampling frequency. Since the jump
Table 4.1: Parameter values

<table>
<thead>
<tr>
<th>β</th>
<th>κ</th>
<th>σ</th>
<th>ρ</th>
<th>ω</th>
<th>δ</th>
<th>ν</th>
<th>m</th>
<th>l</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7.3618</td>
<td>0.05</td>
<td>0.26</td>
<td>-0.3</td>
<td>0.0005</td>
<td>0.04</td>
<td>15</td>
<td>240</td>
<td>0.02</td>
</tr>
</tbody>
</table>

intensity is very small, estimates of the jump parameters tend to have large variances if the sample size is not large enough. Therefore, in this example, along with the trading day of interest, I also include 30 days before and 29 days after. That is, the sample size is twice as big as what I used in the previous two chapters. The simulated data are shown in Figure 4.1.

Figure 4.1: Simulated returns, log volatilities and log prices

For the trading day of interest, 54 price observations are truncated after applying the price limit, which corresponds to 22.5% of the total observations in a trading day.
Figure 4.2 zooms in on the simulated asset prices of day 31.

The two horizontal lines correspond to the upper and lower limits of the trading day. According to the observation rule, only asset prices between the two price limits are observable.

The MCMC algorithm developed in Section 4.3 is applied to the simulated sample for 20,000 iterations. The first 5,000 iterations are discarded as the burn-in period. Posterior distributions are estimated from the remaining 15,000 iterations and results are shown in Figure 4.3 and Table 4.2.

In Figure 4.3, the columns show the trace plots, the autocorrelograms, and the kernel densities of the parameters and two suggested estimators. As shown in all of the
autocorrelograms, the sampling scheme achieves very high computational efficiency as suggested by Nakajima and Omori (2009). Also, trace plots provide very strong evidence of convergence. The horizontal line in each trace plot shows the true value of the corresponding parameter or estimator. Clearly the suggested MCMC algorithm provides very good estimates for all parameters and for the integrated volatility. This example also shows that Adj-QRV and SIV provide very similar estimates of integrated volatility. Due to the way SIV is defined, whether SIV provides a good estimate of IV depends closely on how well we can estimate the latent volatility states. Figure 4.4 shows the posterior mean of the latent log volatility states and the true simulated log volatility states.

We see that the posterior means of latent volatility states are very close to the true latent volatility states. Therefore the posterior mean of SIV suggests a good point estimator candidate for integrated volatility. Similarly, we can also use the posterior mean of Adj-QRV as another point estimator for IV.
4.4.2 Simulation Study

The convergence properties of Adj-QRV and SIV guarantee that if the sampling frequency is very high, posterior distributions of Adj-QRV and SIV converge to the posterior distribution of integrated volatility. The above example also shows that the modes of the posterior distributions of Adj-QRV and SIV are both very close to the true integrated volatility level. In this section, I study the performance of Adj-QRV and SIV based on 1000 simulated samples. I use the same parameter values to generate the random samples. Due to the computational burden, I only run 10,000 iterations for each sample and keep the last 5,000 iterations for the posterior distribution. I use the mean of Adj-QRV\(_g\) and the mean of SIV\(_g\), which both converge to the posterior mean of integrated volatility according to Proposition 1, as point estimates for integrated volatility. For reasons of comparison, I also include the following four estimators

\[
RV^*[t_0,t_m] = \sum_{i=1}^{m} (p_{t_i} - p_{t_{i-1}})^2;
\]

\[
\text{Adj-RV}^*[t_0,t_m] = \sum_{i=1}^{m} \frac{(p_{t_i} - p_{t_{i-1}} - k_t \gamma_t)^2}{\lambda_t};
\]

\[
RV[t_0,t_m] = \sum_{t_0 < t_i \leq t_m} (S_{t_i} - S_{t_{i-1}})^2;
\]

\[
RV_{adj}^{*}[t_0,t_m] = \frac{T}{T-M}RV[t_0,t_m].
\]

Here, \(RV^*[t_0,t_m]\) is the realized volatility over period \([t_0, t_m]\) assuming we know all of the asset prices. Therefore it is not a feasible estimator if price limits are present. Due to the presence of fat-tailed innovations and jumps in the asset return process, \(RV^*[t_0,t_m]\)
CHAPTER 4. SV MODEL WITH CORRELATED INNOVATIONS, FAT-TAILS AND JUMPS

tends to overestimate the underlying integrated volatility. While \( \text{Adj-RV}^*_{[t_0,t_m]} \), which takes into account both fat-tails and jumps could provide a good estimate for the underlying integrated volatility. It assumes all of the asset prices, jumps and fat-tailed states are known. Therefore it is also not feasible in the presence of price limits. \( \text{RV}_{[t_0,t_m]} \) is the traditional realized volatility based only on realized asset prices. And \( M \) is the total number of truncated asset prices. Therefore \( \text{RV}^{adj}_{[t_0,t_m]} \) is an adjusted measurement assuming volatilities are constant over time, which contradicts the empirical finding of time varying volatilities.

Because the realized integrated volatility level is different in each simulated sample, I compare the relative errors of all estimators as in Nielsen and Frederiksen (2008). Figure 4.5 shows the line graph of relative errors.

From this graph we can see that \( \text{RV}, \text{RV}^{adj} \) and \( \text{RV}^* \) are not jump robust estimators for integrated volatility as expected. The errors of \( \text{RV}^* \) can be more than 10 times larger than the underlying integrated volatility level. And the errors of \( \text{RV} \) and \( \text{RV}^{adj} \) can be 4 times larger than the underlying integrated volatility level. This is because all three estimators include squared return jumps in their measurement of integrated volatility, which by its definition only measures the volatility of the continuous sample path. The relative errors of \( \text{Adj-QRV}, \text{SIV} \) and \( \text{Adj-RV}^* \) fluctuate around zero with very small variances, indicating that these three estimators provide very good estimates for integrated volatility. \( \text{Adj-RV}^* \), which requires full information, provides the best estimate for integrated volatility. SIV behaves very similarly to the infeasible estimator \( \text{Adj-RV}^* \). \( \text{Adj-QRV} \) is also similar to \( \text{SIV} \) and \( \text{Adj-RV}^* \), but shows a little bit higher variation. Overall, both \( \text{Adj-QRV} \) and \( \text{SIV} \) provide reliable estimates for integrated volatility due to their closeness to the infeasible \( \text{Adj-RV}^* \).
CHAPTER 4. SV MODEL WITH CORRELATED INNOVATIONS, FAT-TAILS AND JUMPS

whose properties have been approved in the realized volatility literature.

Table 4.3: Summary statistics for relative errors

<table>
<thead>
<tr>
<th></th>
<th>Adj-QRV</th>
<th>SIV</th>
<th>RV*</th>
<th>Adj-RV*</th>
<th>RV</th>
<th>RV_{adj}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0373</td>
<td>-0.0387</td>
<td>0.3047</td>
<td>0.0017</td>
<td>0.0166</td>
<td>0.1668</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.1901</td>
<td>0.1312</td>
<td>0.9670</td>
<td>0.0926</td>
<td>0.3509</td>
<td>0.3921</td>
</tr>
<tr>
<td>s.d.</td>
<td>0.1864</td>
<td>0.1253</td>
<td>0.9177</td>
<td>0.0926</td>
<td>0.3505</td>
<td>0.3549</td>
</tr>
</tbody>
</table>

Table 4.3 presents summary statistics of the relative errors. Relative errors of Adj-QRV, SIV and Adj-RV* all have means very close to zero. Relative errors of the infeasible Adj-RV* has the smallest RMSE, while the RMSE of Relative errors of SIV is slightly higher. The RMSE of relative errors of Adj-QRV is about twice as large as the RMSE of relative errors of Adj-RV*, but still less than 20%. These results show that all these three estimators provide very good estimates for integrated volatility. The relative errors of all other three estimators either have significant positive means or have very large RMSE. We notice that the relative errors of RV has a mean very close to zero. This is the result of the total effect of the upward bias from return jumps and the downward bias due to price limits. Therefore when I use RV_{adj} to naively compensate for the downward bias due to price limits, significant upward bias shows up.

4.5 Concluding Remarks

This chapter further extends the ideas developed in Chapter 2 to a log-stochastic volatility model allowing correlated innovations, fat-tailed innovations and jumps in returns. Bayesian methods can still help us to recover the lost information due to
the presence of price limits. The MCMC algorithm designed based on Nakajima and Omori (2009) clearly shows convergence and efficiency. Simulation results indicate that both Adj-QRV and SIV provide very good estimates for the underlying integrated volatility, and both behave very similarly to the realized volatility assuming we know all of the asset prices.
Figure 4.3: Trace plots, correlograms and kernel densities
True and posterior mean of volatility ($h_t$)

Figure 4.4: True and posterior mean volatilities
CHAPTER 4. SV MODEL WITH CORRELATED INNOVATIONS, FAT-TAILS AND JUMPS

Figure 4.5: Relative errors
Chapter 5

Conclusion

This dissertation focuses on the estimation of integrated volatility when asset prices are subject to price limits. The information contained in asset prices beyond price limits is unavailable in the presence of price limits. Thus realized volatility calculated only from realized asset prices would underestimate the integrated volatility. Also, this bias would not disappear even when sampling frequency is very high. Parametric stochastic volatility models allow us to make inferences about the unobserved asset processes based on realized prices and, in this way, can help us retrieve missing information. MCMC is particularly well suited for inference on stochastic volatility models due to its advantages of being able to deal with latent state variables, the curse of dimensionality, non-Gaussian distributions and nonlinearity. By using MCMC algorithms, I simulate the unobserved asset prices based on stochastic volatility models, and develop two types of estimators, which I call quasi-realized volatility (adjusted quasi-realized volatility when jumps and fat-tailed innovations in returns are considered), using both realized and simulated asset prices, and simulated integrated volatility, which is directly constructed as a discrete analogue of integrated
volatility using the posterior sample of latent volatility states. The convergence of the Markov chain guarantees the consistency of quasi-realized volatility (or adjusted quasi-realized volatility) and simulated-integrated volatility. The performances of both type estimators are justified using simulated high frequency data.

Although the method developed in this dissertation focuses on stock market price limits, it can be directly applied to other financial markets, such as the futures markets and the foreign exchange markets\footnote{For example, a 3\% symmetric daily price limit is installed on copper and aluminum contracts in the Shanghai Futures Exchange.} Minor revisions would enable us to analyze other types of price restrictions such as circuit breakers installed on US stock market.

Since financial volatility is key to option pricing and portfolio allocation, QRV (or Adj-QRV) and SIV can help financial practitioners correctly price financial instruments and locate the risk-return frontier, which is important when determining optimal portfolio weights. Since RV tends to underestimate integrated volatility, option prices based on RV are lower than their theoretical prices, which means financial institutions bear more risk than they should. Similarly, when price limits are present, RV can not correctly estimate the true risk of corresponding financial assets. As a result, the derived risk return frontier is not correct nor are the optimal portfolio weights determined from the risk return frontier. QRV (or Adj-QRV) and SIV can help us correct these issues. Therefore, in addition to their theoretical significance, QRV (or Adj-QRV) and SIV also provide valuable reference for financial practitioners.
Bibliography


BIBLIOGRAPHY


Appendix A

Conditional Distribution of Unobserved Asset Prices: Chapter 2

The conditional distribution of unobserved asset prices is

\[ P(Z_{t\Delta}|p_{-t\Delta}, \theta, h) \]
\[ = P(Z_{t\Delta}|p_{(t-1)\Delta}, p_{(t+1)\Delta}, \theta, h) \]
\[ \propto \exp \left( -\frac{1}{2} \left( \text{tr} \left( \Sigma^{-1} r_{t\Delta} r_{t\Delta}' \right) + \text{tr} \left( \Sigma^{-1} r_{(t+1)\Delta} r_{(t+1)\Delta}' \right) \right) \right) \]
\[ = \exp \left( -\frac{1}{2} \text{tr} \left( \Sigma^{-1} r_{t\Delta} r_{t\Delta}' \right) \right) \exp \left( -\frac{1}{2} \text{tr} \left( \Sigma^{-1} r_{(t+1)\Delta} r_{(t+1)\Delta}' \right) \right) \]
\[ = \exp \left( -\frac{1}{2} \left( \frac{\varepsilon_{t\Delta}^2}{\Delta} + \frac{\eta_{t\Delta}^2}{\sigma^2 \Delta} \right) \right) \exp \left( -\frac{1}{2} \left( \frac{\varepsilon_{(t+1)\Delta}^2}{\Delta} + \frac{\eta_{(t+1)\Delta}^2}{\sigma^2 \Delta} \right) \right) \]
\[
\begin{align*}
&\propto \exp \left( -\frac{1}{2\sigma^2\Delta} \left( \frac{y_{t\Delta}}{\exp \left( \frac{h_{t\Delta}}{2} \right)} \right)^2 + \frac{\eta^2_{t\Delta}}{2} \right) \\
&\propto \exp \left( -\frac{1}{2\sigma^2\Delta} \left( \frac{y_{(t+1)\Delta}}{\exp \left( \frac{h_{(t+1)\Delta}}{2} \right)} \right)^2 + \frac{\eta^2_{(t+1)\Delta}}{2} \right) \\
&\propto \exp \left( -\frac{1}{2\Delta} \left( \frac{Z_{t\Delta} - p_{(t-1)\Delta}}{\exp \left( \frac{h_{t\Delta}}{2} \right)} \right)^2 + \left( \frac{p_{(t+1)\Delta} - Z_{t\Delta}}{\exp \left( \frac{h_{(t+1)\Delta}}{2} \right)} \right) \right) \\
&\propto \exp \left( -\frac{1}{2\Delta} \left( \frac{Z_{t\Delta}^2 - 2p_{(t-1)\Delta}Z_{t\Delta}}{\exp \left( h_{t\Delta} \right)} \right) + \left( \frac{Z_{t\Delta}^2 - 2p_{(t+1)\Delta}Z_{t\Delta}}{\exp \left( h_{(t+1)\Delta} \right)} \right) \right) \\
&\propto \exp \left( -\frac{1}{2\Delta} \left( \frac{\exp \left( h_{t\Delta} \right) + \exp \left( h_{(t+1)\Delta} \right)}{\exp \left( h_{t\Delta} \right) \exp \left( h_{(t+1)\Delta} \right)} \right) Z_{t\Delta}^2 - 2\frac{\exp \left( h_{(t+1)\Delta} \right) p_{(t-1)\Delta} + \exp \left( h_{t\Delta} \right) p_{(t+1)\Delta}}{\exp \left( h_{t\Delta} \right) \exp \left( h_{(t+1)\Delta} \right)} \right) \\
&\propto \exp \left( -\frac{1}{2\exp(h_{t\Delta})\exp(h_{(t+1)\Delta})} \left( Z_{t\Delta}^2 - 2\frac{\exp \left( h_{(t+1)\Delta} \right) p_{(t-1)\Delta} + \exp \left( h_{t\Delta} \right) p_{(t+1)\Delta}}{\exp \left( h_{t\Delta} \right) + \exp \left( h_{(t+1)\Delta} \right)} \right) \right) \\
&\propto \exp \left( -\frac{1}{2\exp(h_{t\Delta})\exp(h_{(t+1)\Delta})} \left( Z_{t\Delta} - \frac{\exp \left( h_{(t+1)\Delta} \right) p_{(t-1)\Delta} + \exp \left( h_{t\Delta} \right) p_{(t+1)\Delta}}{\exp \left( h_{t\Delta} \right) + \exp \left( h_{(t+1)\Delta} \right)} \right)^2 \right)
\end{align*}
\]

for \( t = 1, 2, \ldots, T \).
Appendix B

Conditional Distribution of Unobserved Asset Prices: Chapter 3

The conditional distribution of unobserved asset prices is

\[
P(Z_t | p_{-t \Delta}, \theta, \beta, h) = P(Z_t | p_{(t-1) \Delta}, p_{(t+1) \Delta}, \theta, \beta, h) \\
\propto \exp \left( -\frac{1}{2} \left( \text{tr} \left( \Sigma^{-1} r_{t \Delta} r_{t \Delta}' \right) + \text{tr} \left( \Sigma^{-1} r_{(t+1) \Delta} r_{(t+1) \Delta}' \right) \right) \right) \\
= \exp \left( -\frac{1}{2} \text{tr}(\Sigma^{-1} r_{t \Delta} r_{t \Delta}') \right) \exp \left( -\frac{1}{2} \text{tr}(\Sigma^{-1} r_{(t+1) \Delta} r_{(t+1) \Delta}') \right)
\]
\[
\begin{align*}
= \exp \left( - \frac{1}{2(1 - \rho^2)} \left( \frac{\varepsilon_{t+1}^2}{\Delta \exp (h_{t+1})} + \frac{\eta_{t+1}^2}{\sigma^2 \Delta} - 2\rho \sigma \varepsilon_{t+1} \eta_{t+1} \exp \left( \frac{h_{t+1}\Delta}{2} \right) \sigma \Delta \right) \right) \\
\times \exp \left( - \frac{1}{2(1 - \rho^2)} \left( \frac{\varepsilon_{(t+1)\Delta}^2}{\Delta \exp (h_{(t+1)\Delta})} + \frac{\eta_{(t+1)\Delta}^2}{\sigma^2 \Delta} - 2\rho \sigma \varepsilon_{(t+1)\Delta} \eta_{(t+1)\Delta} \exp \left( \frac{h_{(t+1)\Delta}\Delta}{2} \right) \sigma \Delta \right) \right) \\
\times \exp \left( - \frac{1}{2\sigma^2(1 - \rho^2)\Delta} \left( \sigma^2 \left( \frac{y_{t\Delta}}{\exp \left( \frac{h_{t\Delta}\Delta}{2} \right)} \right)^2 + \frac{\eta_{t\Delta}^2}{\exp \left( \frac{h_{t\Delta}\Delta}{2} \right)} - 2\rho \sigma \frac{y_{t\Delta}}{\exp \left( \frac{h_{t\Delta}\Delta}{2} \right)} \eta_{t\Delta} \right) \right) \\
\times \exp \left( - \frac{1}{2\sigma^2(1 - \rho^2)\Delta} \left( \sigma^2 \left( \frac{y_{(t+1)\Delta}}{\exp \left( \frac{h_{(t+1)\Delta}\Delta}{2} \right)} \right)^2 + \frac{\eta_{(t+1)\Delta}^2}{\exp \left( \frac{h_{(t+1)\Delta}\Delta}{2} \right)} - 2\rho \sigma \frac{y_{(t+1)\Delta}}{\exp \left( \frac{h_{(t+1)\Delta}\Delta}{2} \right)} \eta_{(t+1)\Delta} \right) \right) \\
\times \exp \left( - \frac{1}{2(1 - \rho^2)\Delta} \left( \frac{\exp (h_{t\Delta}) + \exp (h_{(t+1)\Delta})}{\eta_{t\Delta} + \exp (h_{(t+1)\Delta})} Z_{\Delta}^2 - \frac{p_{(t+1)\Delta}}{\exp (h_{(t+1)\Delta})} \right) Z_{\Delta} - 2 \frac{p_{(t-1)\Delta}}{\exp (h_{t\Delta}) \exp (h_{(t+1)\Delta})} \frac{\eta_{t\Delta}}{\sigma \exp \left( \frac{h_{t\Delta}\Delta}{2} \right) \sigma \Delta} \frac{Z_{\Delta}}{Z_{(t+1)\Delta}} \right)
\end{align*}
\]
for $t = 1, 2, \cdots, T$. 
The conditional distribution of unobserved asset prices is

\[
P(Z_{t\Delta} | p_{-t\Delta}, \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda) \\
= P(Z_{t\Delta} | p_{(t-1)\Delta}, p_{(t+1)\Delta}, \theta, \beta, \omega, \delta, \nu, h, k, \gamma, \lambda) \\
\propto \exp \left( -\frac{1}{2 (1 - \rho^2)} \left( \frac{p_{t\Delta} - p_{(t-1)\Delta} - k_{t\Delta} \gamma_{t\Delta}}{\sqrt{\lambda_{t\Delta} \Delta \exp(h_{t\Delta}/2)}} \right)^2 \\n- 2\rho \left( p_{t\Delta} - p_{(t-1)\Delta} - k_{t\Delta} \gamma_{t\Delta} \right) \left( h_{(t+1)\Delta} - \beta - \phi \left( h_{t\Delta} - \beta \right) \right) \right) \\
\exp \left( -\frac{1}{2 (1 - \rho^2)} \left( \frac{p_{(t+1)\Delta} - p_{t\Delta} - k_{(t+1)\Delta} \gamma_{(t+1)\Delta}}{\sqrt{\lambda_{(t+1)\Delta} \Delta \exp(h_{(t+1)\Delta}/2)}} \right)^2 \\n- 2\rho \left( p_{(t+1)\Delta} - p_{t\Delta} - k_{(t+1)\Delta} \gamma_{(t+1)\Delta} \right) \left( h_{(t+2)\Delta} - \beta - \phi \left( h_{(t+1)\Delta} - \beta \right) \right) \right) \\
\right)
\]
\[
\begin{align*}
\propto \exp \left( -\frac{1}{2 (1 - \rho^2)} \left( \frac{p_t - p_{t-1}}{\sqrt{\lambda_t} \Delta \exp(h_t \Delta)} \right)^2 - \frac{2 \rho p_t \Delta}{\sqrt{\lambda_t} \Delta} \left( h_{t+1} \Delta - \beta - \phi (h_t \Delta - \beta) \right) \right) \\
\exp \left( -\frac{1}{2 (1 - \rho^2)} \left( \frac{p_t - p_{t-1} - k_t \Delta \gamma_t}{\sqrt{\lambda_t} \Delta \exp(h_t \Delta)} \right)^2 - \frac{2 \rho p_t \Delta}{\sqrt{\lambda_t} \Delta} \left( h_{t+1} \Delta - \beta - \phi (h_t \Delta - \beta) \right) \right) \\
\propto \exp \left( -\frac{1}{2 (1 - \rho^2)} \left( \frac{p_t - p_{t-1} - k_t \Delta \gamma_t}{\sqrt{\lambda_t} \Delta \exp(h_t \Delta)} \right)^2 - \frac{2 \rho p_t \Delta}{\sqrt{\lambda_t} \Delta} \left( h_{t+1} \Delta - \beta - \phi (h_t \Delta - \beta) \right) \right) \\
+ \left( \frac{p_t - p_{t-1} - k_t \Delta \gamma_t}{\sqrt{\lambda_t} \Delta \exp(h_t \Delta)} \right)^2 - \frac{2 \rho p_t \Delta}{\sqrt{\lambda_t} \Delta} \left( h_{t+1} \Delta - \beta - \phi (h_t \Delta - \beta) \right) \right) \\
\end{align*}
\]
\[
\exp \left( -\frac{1}{2(1-\rho^2)\Delta \lambda_t \Delta \exp(h_t \Delta) \lambda_t \Delta \exp(h_t \Delta)} \right) \left( p_t - \frac{\lambda_t \Delta \exp(h_t \Delta) \lambda_{(t+1)} \Delta \exp(h_{(t+1)} \Delta)}{(\lambda_{(t+1)} \Delta \exp(h_{(t+1)} \Delta) + \lambda_t \Delta \exp(h_t \Delta))} \right) \right.
\]
\[
\exp \left( \frac{p_t \Delta + k_t \Delta \gamma_t \Delta}{\lambda_t \Delta \exp(h_t \Delta)} + \frac{\rho \left( h_{(t+1)} \Delta - \beta - \phi (h_t \Delta - \beta) \right)}{\sqrt{\lambda_t \Delta \exp(h_t \Delta / 2) \sigma}} \right)
\]
\[
+ \frac{p_{(t+1)} \Delta - k_{(t+1)} \Delta \gamma_{(t+1)} \Delta}{\lambda_{(t+1)} \Delta \exp(h_{(t+1)} \Delta)} - \frac{\rho \left( h_{(t+2)} \Delta - \beta - \phi (h_{(t+1)} \Delta - \beta) \right)}{\sqrt{\lambda_{(t+1)} \Delta \exp(h_{(t+1)} \Delta / 2) \sigma}} \right) \right)^2 \right)
\]

for \( t = 1, 2, \cdots, T \).