ARITHMETIC AND INTERMEDIATE JACOBIANS
OF CALABI-YAU THREEFOLDS

by

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A thesis submitted to the
Graduate Program in Mathematics and Statistics
in conformity with the requirements for
the degree of Doctor of Philosophy

Queen’s University
Kingston, Ontario, Canada
September 2015

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Abstract

This thesis is centered around particular Calabi-Yau threefolds. Borcea [14] and Voisin [77] construct Calabi-Yau threefolds using elliptic curves and K3 surfaces with non-symplectic involutions. This family has an incredible property, that a general member has a mirror pair within this family. We start by investigating if this construction works only for Calabi-Yau threefolds with non-symplectic involutions or with non-symplectic automorphisms of higher order as well. Thereafter, we generalize this construction to Calabi-Yau fourfolds.

After this, we focus on the underlying construction that lead Borcea to the families above, using a product of three elliptic curves with non-symplectic involutions. These threefolds do not come in families, so we cannot ask about mirror symmetry, but if we have models defined over \( \mathbb{Q} \), we may ask arithmetic questions. Many arithmetic properties of the Calabi-Yau threefolds can be studied via the underlying elliptic curves. In particular, we are able to show (re-establish in the rigid case) that the Calabi-Yau threefolds are all modular by computing their \( L \)-functions. Then, guided by a conjecture of Yui, we investigate their (Griffiths) intermediate Jacobians and a relationship between their respective \( L \)-functions.
Acknowledgments

Over these last four years, I have come across many people that deserve special thanks and mention here. Not only is four years enough time for anyone to come across helpful colleagues, but when I first arrived at Queen’s University I was recovering from a severe concussion after a skiing accident, so the patience everyone showed me was particularly fruitful (and quite likely the only reason I survived these last 4 years!).

First and foremost, I would like to thank my supervisor, Professor Noriko Yui, who has introduced me to this fascinating field of Calabi-Yau varieties and mirror symmetry. Every time we talked, she treated me as if I was not concussed, but normal, and this was no doubt partly responsible for my surviving the very tough first year.

A thanks to everyone at the Calabi-Yau seminar, that somehow listened to my first two hour talk when I could not remember a complete sentence! To Dr. Hector Pasten, for his passion and helpful guidance, and especially his 2 pages of notes on modularity that I learn something new from every single time I read it. Without that, much of Chapter 4 would not have been possible. To Dr. Simon Rose, without whom I would probably still be blowing up singularities in an endless loop, wondering if it might resolve next time... And to Dr. Andrija Peruničić, while not always answering
my questions, always saying exactly what it took for me to realize the answer. Both their help as colleagues in the office, and friends outside the office, was invaluable.

The staff in the math department, most especially Jennifer, took the time to ensure my entrance (and my tenure) would be the least stressful as possible. Carla and Tyson at the KATC helped put me back together, physically, piece by piece.

I was given opportunities to give talks at the Fields Institute for Research in Mathematical Sciences in Toronto, the University of Copenhagen, Leibniz Universität Hannover and Louisiana State University, as well as a ‘few’ at Queen’s University. The first few talks in Kingston (and the associated adrenaline from standing in the front of the room) likely helped with the recovery of the concussion, as I noticed distinct improvements the morning after. The rest of the talks gave me an opportunity to interact with the many members of the seminar, and hone my skills, learning what makes the audience appear bored, and what does not.

With regard to the travels, many thanks to Dr. Ian Kiming and Dr. Matthias Schütt for warmly welcoming me to their departments, for spending time discussing my work with me, and for many insightful comments that are a part of the final draft of this thesis. I am also grateful to the Fields Institute for providing funding for my 6 month visit for the Fields Thematic Program on Calabi-Yau Varieties: Arithmetic, Geometry and Physics, during the period of July to December 2013.

Lastly, academically, I am indebted to the many agencies that provided me with funding. In particular, Queen’s University and the Department of Mathematics and Statistics, for always finding extra money every year, for my supervisor’s Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (NSERC), for the Ontario government and their Ontario Graduate Scholarship (OGS)
awards, and the people and institutions I visited, the trips all being possible only because of the travel expenses being covered.

On a personal note, I would be nowhere without the support of everyone in my life. It would take too many pages to list ‘everyone’, so to everyone not listed here, I am still grateful! A final thank you to Lily, who came out to Kingston when I most needed the support, and for everything after.
Statement of Originality

This thesis is original work, written by me with help from many colleagues. We use results due to other authors at times, but these are presented as such, following the current citation standards in mathematics.

Chapter 1 gives an introduction to the ideas investigated in this thesis, as well as a short outline of each of the chapters. Chapter 2 reviews preliminary material required for Chapters 3, 4 and 5.

Chapter 3 collects many known results, simplifies arguments, and is an original exposition towards solving an unsolved problem. However, while writing this chapter we learned of [17] and [28] studying the same problems, so we continued in a different direction in Chapter 4 and 5. These Chapters contain original research.
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Chapter 1

Introduction

1.1 Motivation

Calabi-Yau threefolds have gained a lot of attention in the last 20 to 30 years, mostly due to their possible applications to physics, in string theory. As such, mathematicians have also found great interest in them. Physicists have made many predictions about things that should be true for general Calabi-Yau threefolds, if string theory is correct, but cannot prove them in many cases. Mathematicians then spent years laying down foundations of new areas, presenting mathematically rigorous proofs, finding out these predictions – coming from the underlying intuition from the physics – seem to be correct.

For example, after Yau’s proof of the Calabi conjecture [81], string theorists found that if ‘strings’ were to exist, then they must come in pairs, a notion they called mirror symmetry. In some conjectured forms of string theory, the universe is not just 4-dimensional, coming from space and time, but 10-dimensional, with the extra dimensions coming from a 6-dimensional (or 3 complex dimensions) Calabi-Yau varieties. Consequently, the predicted mirror symmetry leads to an expectation that a
1.1. MOTIVATION

general Calabi-Yau threefold has a ‘mirror partner’. Mathematically, this translates to meaning (among other things) two families $F$ and $F'$ of Calabi-Yau threefolds are ‘mirrors’ if for any member $X$ of $F$ and any member $Y$ of $F'$, the Hodge numbers of both $X$ and $Y$ are ‘mirrored’, i.e., $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$. Mirror symmetry makes enumerative predictions in Gromov–Witten theory of Calabi-Yau threefolds, numerical evidence started piling up, and this symmetry seems very likely to exist for Calabi-Yau threefolds. Hence, regardless of whether string theory is correct or not, there is a very interesting mathematics to be studied here. Moreover, not just for Calabi-Yau threefolds, but the mathematics generalizes to Calabi-Yau varieties of any dimension. A mathematical framework was put in place to try to explain the symmetry rigorously, outlined in [21]. It is still unknown in general, but many results in this direction have been shown, e.g., [9] and [77], constructing families of Calabi-Yau threefolds that (provably) do have mirror partners. Moreover, many different mathematical theories to explain this phenomenon have been conjectured, for instance Berghlund-Hübsch-Krawitz mirror symmetry, [13] and [52], for finite quotients of hypersurfaces in weighted projective spaces, and a toric approach by Batyrev and Borisov, [8], [9] and [10] for toric Calabi-Yau varieties.

While the interest in string theory has been significant, this is not the only appeal to study Calabi-Yau varieties. The proof of Fermat’s Last Theorem given by Wiles, [80], came down to understanding arithmetic – in particular, the modularity – of Calabi-Yau 1-folds, elliptic curves, defined over $\mathbb{Q}$. Therefore, there is great interest in Calabi-Yau varieties for their complex geometry when defined over $\mathbb{C}$, as well as their arithmetic properties when defined over $\mathbb{Q}$ or a number field.

The Galois representation of an elliptic curve defined over $\mathbb{Q}$ is two-dimensional,
and Wiles showed that any such representation coming from a semistable elliptic curve defined over $\mathbb{Q}$ was modular, [80]. Continuing in this fashion, Breuil, Conrad, Diamond and Taylor were able to show that all elliptic curves defined over $\mathbb{Q}$ were modular [15], and while this was one of the greatest results of the 20th century, this was just a very small first step in understanding arithmetic of Calabi-Yau varieties.

While a higher dimensional variety typically has a Galois representation of dimension greater than 2, Serre conjectured [71] that any odd residual two-dimensional Galois representation should be modular. This conjecture was proved by Khare and Wintenberger [47], [48], and Kisin [50].

Building on the validity of the Serre conjecture, Gouvêa and Yui [41], and independently Dieulefait [25], have established the modularity of two-dimensional Galois representations associated to rigid Calabi-Yau threefolds defined over $\mathbb{Q}$.

## 1.2 Main results

In Chapter 4 we investigate Calabi-Yau threefolds using CM elliptic curves and their CM automorphisms. Let

$$E_6 : y^2 = x^3 - 1 \quad \text{with} \quad \iota_6(x, y) = (\zeta_3 x, -y)$$

for a primitive third root of unity $\zeta_3$. Then the group

$$G_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6 \times \text{id} \times \iota_6^5 \rangle \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

acts component-wise on the triple product $E_6^3$, fixing the holomorphic 3-form. Similarly, any subgroup of $G_6$ that fixes the holomorphic 3-form leads naturally to a
1.2. MAIN RESULTS

Calabi-Yau threefold by taking a crepant resolution of a quotient. We show that all of these Calabi-Yau threefolds are modular, by computing their $L$-series.

**Theorem 1.1.** Consider the following groups of automorphisms acting on $E^3_6$.

\[
G_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6 \times \text{id} \times \iota_6^5 \rangle, \quad H_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^2 \times \text{id} \times \iota_6^4 \rangle,
\]

\[
I_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6 \rangle, \quad J_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6^4 \times \iota_6^5 \times \iota_6^3 \rangle,
\]

\[
K_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^3 \times \text{id} \times \iota_6^3 \rangle, \quad L_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6 \rangle,
\]

\[
M_6 = \langle \iota_6^2 \times \iota_6^2 \times \iota_6^2 \rangle, \quad N_6 = \langle \iota_6 \times \iota_6^2 \times \iota_6^3 \rangle, \quad O_6 = \langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle.
\]

For each of these groups, the quotient with $E^3_6$ admits a Calabi-Yau resolution. These have models defined over $\mathbb{Q}$ and are all modular. The quotients with $G_6, H_6, L_6$ and $M_6$ give rise to rigid Calabi-Yau threefolds.

Noting that $J_6, K_6$ and $N_6$ only act by an involution or identity on some of the elliptic curves, we may construct the threefolds using any elliptic curve in those components, instead of simply the CM curve $E_6$. Doing this, we show these families of Calabi-Yau threefolds are automorphic.

A similar construction can be made with the CM elliptic curve

\[
E_4 : y^2 = x^3 - x \quad \text{with} \quad \iota_4(x, y) = (-x, iy)
\]

the group

\[
G_4 = \langle \iota_4 \times \iota_4^3 \times \text{id}, \iota_4 \times \text{id} \times \iota_4^3 \rangle \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}
\]

and some of its subgroups.
**Theorem 1.2.** Consider the groups of automorphisms

\[ G_4 = \langle \iota_4 \times \iota_4^3 \times \text{id}, \iota_4 \times \text{id} \times \iota_4^3 \rangle, \quad H_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^3 \times \text{id} \rangle, \]

\[ I_4 = \langle \iota_4^2 \times \iota_4 \times \text{id}, \iota_4^2 \times \text{id} \times \iota_4^2 \rangle, \quad J_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2 \rangle, \]

acting on the threefold \( E_4^3 \). Crepant resolutions of the respective quotients are modular Calabi-Yau threefolds defined over \( \mathbb{Q} \). The quotient with \( G_4 \) gives a rigid Calabi-Yau threefold.

Again, noting that \( H_4, I_4 \) and \( J_4 \) only act by an involution or identity on some of the underlying elliptic curves, we consider the families lying over the \( j \)-line and show they are automorphic.

One may twist the underlying elliptic curves in both cases, and the modularity still follows in the same manner of our computing the \( L \)-functions. In particular, we are able to find the \( L \)-functions of all of the twists of the Calabi-Yau threefolds above, and show they are modular (or automorphic).

With this in hand, we then shift our attention to a conjecture relating the \( L \)-functions of rigid Calabi-Yau threefolds of CM-type (meaning their Hodge groups are commutative) and the \( L \)-functions of particular models of their intermediate Jacobians.

**Conjecture 1.3** (Yui, [83]). Let \( X \) be a rigid Calabi-Yau threefold of CM-type defined over a number field \( F \). Then the intermediate Jacobian \( J(X) \) is an elliptic curve with CM by an imaginary quadratic field \( K \), and has a model defined over the number field \( F \).
If $\chi$ is a Hecke character associated to $J(X)$ and

$$L(J(X), s) = \begin{cases} L(\chi, s)L(\overline{\chi}, s) & \text{if } K \subset F, \\ L(\chi, s) & \text{otherwise}, \end{cases}$$

then

$$L(X, s) = \begin{cases} L(\chi^3, s)L(\overline{\chi^3}, s) & \text{if } K \subset F, \\ L(\chi^3, s) & \text{otherwise}. \end{cases}$$

With our rigid Calabi-Yau threefolds, we show the conjecture is true except when the CM by order 3 (or 6) cannot be recognized by the cube of the Hecke character. In particular, with $E_4$ the conjecture is true for all of our Calabi-Yau threefolds above. Denote by $E_4(D)$ the twist

$$E_4(D) : y^2 = x^3 - Dx.$$

**Theorem 1.4.** Let $Y(D_1, D_2, D_3)$ be a crepant resolution of

$$(E_4(D_1) \times E_4(D_2) \times E_4(D_3))/G_4$$

where $D_1, D_2$ and $D_3$ are non-zero integers. Then $Y(D_1, D_2, D_3)$ is a Calabi-Yau threefold defined over $\mathbb{Q}$. Furthermore, the intermediate Jacobian $J(Y(D_1, D_2, D_3))$ is isomorphic to $E_4(D_1D_2D_3)$ as a complex variety, and hence there is a model for the intermediate Jacobian satisfying

$$L(J(Y(D_1, D_2, D_3)), s) = L(\chi, s)$$
and
\[ L(Y(D_1, D_2, D_3), s) = L(\chi^3, s). \]

With the rigid Calabi-Yau threefolds coming from $E_6$ we have to be more careful, but are still able to show the conjecture is true in many cases. Denote by $E_6(D)$ the twist
\[ E_6(D) : y^2 = x^3 - D. \]

**Theorem 1.5.** Let $Z(D_1, D_2, D_3)$ be a crepant resolution of
\[ (E_6(D_1) \times E_6(D_2) \times E_6(D_3))/G \]

where $G$ is one of $G_6, H_6, L_6$ or $M_6$, and $D_1, D_2$ and $D_3$ are non-zero integers. Then $Z(D_1, D_2, D_3)$ is a Calabi-Yau threefold defined over $\mathbb{Q}$. Furthermore, the intermediate Jacobian $J(Z(D_1, D_2, D_3))$ is isomorphic to $E_6(D_1D_2D_3)$ as a complex variety, and hence there is a model for the intermediate Jacobian satisfying
\[ L(J(Y(D_1, D_2, D_3)), s) = L(\chi, s) \]

and
\[ L(Y(D_1, D_2, D_3), s) = L(\chi^3, s) \]

if and only if $D_1D_2D_3$ is the cube of an integer, or in other words $E_6(D_1D_2D_3)$ is a quadratic twist of $E_6$.

Using the computations involved in these two results, we also investigate higher dimensional rigid Calabi-Yau $n$-folds constructed from the CM elliptic curves and the natural generalization of this conjecture, as well as a possible generalization of this
1.3. ORGANIZATION OF THE THESIS

conjecture for the non-rigid Calabi-Yau threefolds.

1.3 Organization of the thesis

In Chapter 2 we present some of the preliminary material needed in the rest of thesis, as well as set up some standard notation used throughout.

In Chapter 3, we recall the Borcea-Voisin construction of Calabi-Yau threefolds, and Voisin’s mirror relationship within this family. Then we ask and investigate whether it generalizes to non-symplectic automorphisms of higher order. For this, we need a classification of the K3 surfaces used in the construction that gives a classification, up to isomorphism, of the corresponding threefolds. We present results due to Artebani and Sarti [2] as well as Artebani, Sarti and Taki [4] doing this, so that we may compute the Hodge numbers of the families of resulting Calabi-Yau threefolds. We search for mirror symmetry, and find it does not exist as with the original Borcea-Voisin construction. We then investigate similar constructions of fourfolds using involutions, similar to Dillies in [27] and [28], finding some families with a similar mirror relationship as Voisin found, and some families that do not, raising questions about what may cause it to exist at times, and not in others.

Chapter 4 then shifts the focus to the Borcea construction, that of a triple product of elliptic curves. Generalizing as before, we use any non-symplectic automorphisms, not only involutions, to construct Calabi-Yau threefolds. This construction uses a triple product of CM elliptic curves, as we need CM automorphisms acting on them. We construct all possible Calabi-Yau threefolds generalizing the Borcea construction with automorphisms from the underlying elliptic curves acting component-wise. We compute their Hodge numbers and show they are all modular, in that their respective
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Galois representations have semi-simplifications that decompose into two-dimensional modular representations.

After this, we work with our groups of automorphisms that only require an involution on one component of the threefold, thus providing a family of Calabi-Yau threefolds parametrized by the \(j\)-line in that component. Modularity here is beyond reach, but we are able to show the families are all automorphic due to results in the Langlands program.

Lastly, in Chapter 5 we shift our attention to the (Griffiths) intermediate Jacobians of the Calabi-Yau threefolds in Chapter 4. Having computed their \(L\)-functions, Conjecture 1.3 relating these \(L\)-functions to the \(L\)-functions of models of their intermediate Jacobians is our main interest. As the intermediate Jacobian is a complex torus by construction, we compute the torus structure of the respective intermediate Jacobians, and try to find a natural structure over \(\mathbb{Q}\) from this.

We find Conjecture 1.3 is true for the untwisted rigid Calabi-Yau threefolds as well as their quadratic and biquadratic twists, but not for cubic and sextic twists. We then observe this conjecture generalizes to Calabi-Yau \(n\)-folds, and investigate when it is true for our Calabi-Yau \(n\)-folds coming from the generalized Borcea construction. Lastly, the non-rigid Calabi-Yau threefolds in Chapter 4 do not have elliptic curves as the intermediate Jacobians, so we compute their intermediate Jacobians to see if a natural generalization of Conjecture 1.3 may exist.
Chapter 2

Background

2.1 Calabi-Yau varieties and complex geometry

The central objects in this work are Calabi-Yau varieties (or manifolds).

**Definition 2.1.** A *Calabi-Yau* variety $X$ of dimension $n$ (or $n$-fold) is a compact Kähler variety of dimension $n$ with

(i) trivial canonical bundle, i.e., $K_X = \wedge^n \Omega^1_X \simeq \mathcal{O}_X$, and

(ii) $h^{0,i}(X) := \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X) = 0$ for every $i$, $0 < i < n$.

There are many different definitions for Calabi-Yau varieties found throughout the literature, and our interest in these (common) defining conditions is the simplicity induced on the cohomology. The triviality of the canonical bundle implies

$$h^{n,0}(X) = \dim_{\mathbb{C}} H^0(X, \Omega^n_X)$$

$$= \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1,$$

so Serre duality (see for example [46]) gives $h^{0,0}(X) = 1$, and all of the holomorphic Hodge numbers are fixed for Calabi-Yau varieties.
A Kähler variety has a Hodge decomposition

\[ H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \]

for each \( k \), giving the Hodge diamond

\[
\begin{array}{ccccccc}
& & & & h^{0,0} & & \\
& & & h^{1,0} & & h^{0,1} & \\
& & h^{n,0} & & \cdots & & \cdots & h^{0,n} \\
& h^{n,n} & & \cdots & & \cdots \end{array}
\]

Complex conjugation, Serre duality, Poincare duality and the Calabi-Yau condition combine to simplify the Hodge diamond of a Calabi-Yau \( n \)-fold as

\[
\begin{array}{ccccccc}
& & & & 1 & & \\
& & & 0 & & 0 & \\
& & 0 & & \cdots & & \cdots & 0 \\
& 0 & & \cdots & & \cdots & 0 \\
& 1 & & \downarrow & & \downarrow & \downarrow & 1 \\
& 0 & & \cdots & & \cdots & 0 \\
& 0 & & 0 & & \cdots & \cdots & 0 \\
& & & & 1 & & \\
\end{array}
\]
where the arrows represent the symmetries from the dualities. The $k$-th *Betti number* of $X$ is $h^k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$. The (topological) *Euler characteristic* of $X$ is

$$\chi(X) := \sum_{k=0}^{2n} (-1)^k h^k(X),$$

and the *holomorphic Euler characteristic* of $X$ is

$$\chi(\mathcal{O}_X) := \sum_{k=0}^{n} (-1)^k h^{0,i}(X).$$

Note that if $X$ is a Calabi-Yau $n$-fold, then when $n$ is odd we have $\chi(\mathcal{O}_X) = 0$, and when $n$ is even, $\chi(\mathcal{O}_X) = 2$.

The most studied and important examples of Calabi-Yau varieties are the following.

The simplest example, integral in our later constructions, is the Calabi-Yau 1-fold, an elliptic curve $E$, with Hodge diamond

$$
\begin{array}{c}
1 \\
1 & 1 \\
1
\end{array}
$$

The Betti numbers are $h^0(E) = 1$, $h^1(E) = 2$ and $h^2(E) = 1$, and so both the topological and holomorphic Euler characteristic vanish.

A 2-dimensional Calabi-Yau variety is called a K3 surfaces, named so after Kähler, Kodaira and Kummer (and the mountain K2 in Kashmir) by Weil, [79]. Such a surface
$S$ has Hodge diamond

1

0 0

1 $h^{1,1}(S)$ 1

0 0

1

by definition. It is a priori unclear what values may occur for $h^{1,1}(S)$ as the Calabi-Yau condition only prescribes the holomorphic cohomology, but it turns out the Calabi-Yau condition characterizes this completely as well. Indeed, for surfaces one has Max Noether’s formula (see for example [12])

$$\chi(\mathcal{O}_S) = \frac{K_S^2 + \chi(S)}{12}$$

relating the holomorphic Euler characteristic of $S$, which is 2, and the topological Euler characteristic of $S$, which is $4 + h^{1,1}(S)$. The Calabi-Yau condition gives $K_S^2 = 0$ and so we find

$$\chi(S) = 24 = 4 + h^{1,1}(S)$$

so that any K3 surface must have $h^{1,1}(S) = 20$. Hence, the Betti numbers are always

$$h^0(S) = h^4(S) = 1, \quad h^1(S) = h^3(S) = 0 \quad \text{and} \quad h^2(S) = 22.$$ 

There are two important lattices associated to any K3 surface $S$. One is $H^2(S, \mathbb{Z})$, commonly called the K3 lattice. The second, the Picard lattice of $S$, is the Picard group of $S$, the group of divisor classes together with the intersection pairing. The Picard lattice is a sublattice of $H^2(S, \mathbb{Z})$, isomorphic to $H^1(S, \mathcal{O}_S^\times)$. Its rank is called
the *Picard number* of $S$. All K3 surfaces considered in this thesis will be algebraic, implying Pic($S$) has rank at least 1. A K3 surface defined over $\mathbb{C}$ is called *singular* or *extremal* if its Picard rank is the maximum possible, 20.

The first case where the cohomology is no longer fixed is in 3-dimensions, the Calabi-Yau threefold. For a Calabi-Yau threefold $X$, our Hodge diamond is

\[
\begin{array}{ccccccc}
1 & & & & & & \\
& 0 & 0 & & & & \\
& & h^{1,1}(X) & 0 & & & \\
& h^{2,1}(X) & h^{2,1}(X) & 1 & & & \\
& h^{1,1}(X) & 0 & & & & \\
& & & 0 & & & \\
& & & & 1 & & \\
\end{array}
\]

and there is no analogue of Noether’s formula for 3-dimensional varieties allowing any information about $h^{1,1}(X)$ of $h^{2,1}(X)$ to be discerned. As a Calabi-Yau variety is Kähler, we have $h^{1,1}(X) > 0$, and nothing else is known to hold for all Calabi-Yau threefolds in general. The Betti numbers are

\[h^0(X) = h^6(X) = 1, \quad h^1(X) = h^5(X) = 0,\]

and

\[h^2(X) = h^4(X) = h^{1,1}(X), \quad h^3(X) = 2 + 2h^{2,1}(X).\]
2.1. CALABI-YAU VARIETIES AND COMPLEX GEOMETRY

Hence the Euler characteristic is

\[ \chi(X) = 2(h^{1,1}(X) - h^{2,1}(X)). \]

Many examples with varying Hodge numbers have been constructed. Some conjectures in Physics [32] predict there is a finite bound on the number of Hodge pairs a Calabi-Yau threefold may have, while on the other hand, some mathematicians conjecture the Euler characteristic is unbounded [63]. Unfortunately, there is little evidence to support either.

We can introduce another duality that can be expressed as a symmetry in the Hodge diamond - one that is getting a lot of attention in recent years. Two Calabi-Yau threefolds \( X_1 \) and \( X_2 \) are called a (topological) mirror pair if the Hodge diamond of \( X_1 \) is “a reflection” of the Hodge diamond of \( X_2 \) along the main diagonal. More precisely, if

\[ h^{1,1}(X_1) = h^{2,1}(X_2) \quad \text{and} \quad h^{2,1}(X_1) = h^{1,1}(X_2). \]

The Hodge numbers being equal means the respective Hodge spaces are isomorphic, so interpreting \( H^{1,1}(X_i) \) as the Kähler moduli, and \( H^{2,1}(X_i) \) as the complex moduli, this gives an interesting isomorphism of the moduli spaces, motivated by the fact that physicists classify strings naturally in pairs, [57].

Since its inception by physicists, there have been refinements made and there are many different notions of mirror symmetry presented. The idea of ‘flipping’ the Hodge diamond gives a non-trivial duality for Calabi-Yau \( n \)-folds when \( n \geq 3 \) but the physical phenomenon string theorists observe is mathematically more subtle to generalize to elliptic curves and K3 surfaces. Without getting into too much detail,
to any (complex) elliptic curve $E$ we can identify $E$ by some complex number $\tau$ in the upper half plane $\mathfrak{h}$ and write

$$E = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}),$$

with $\tau = \tau_1 + i\tau_2$ and $\tau_2 > 0$. Similarly, $E$ has a Kähler class $[\omega] \in H^2(X, \mathbb{C})$ which can similarly be parametrized by $t \in \mathfrak{h}$ as

$$\omega = -\frac{\pi t}{\tau_2} dz \wedge d\bar{z}.$$

We can identify this elliptic curve by these two parameters, so may write $E_{\tau,t}$. A mirror symmetry of elliptic curves is then given by interchanging these two parameters,

$$E_{\tau,t} \leftrightarrow E_{t,\tau}.$$

While this may appear fairly simple, it is by no means trivial, see [26] for more details and the use of this symmetry.

For K3 surfaces, the situation is far more involved. There have been many approaches to develop a symmetry for K3 surfaces consistent with the predictions in string theory. The toric and Berglund-Hübsch-Krawitz approaches mentioned in Chapter 1 have been shown equivalent – when they both exist – to Dolgachev’s work ([1], [19] and [31]) showing for K3 surfaces, mirror symmetry lies within the Picard lattice: two K3 surfaces $S$ and $S'$ are mirrors if their Nikulin invariants (defined in Theorem 3.1) are

$$(r,a,\delta) \quad \text{and} \quad (20 - r,a,\delta),$$
respectively.

When a Calabi-Yau threefold has no complex deformations, i.e., \( h^{2,1}(X) = 0 \), we say the threefold is \textit{rigid}.

One may continue this investigation with Calabi-Yau 4-folds, 5-folds, etc... but even Calabi-Yau threefolds are very hard to study, so far less effort has been applied to the higher dimensional Calabi-Yau varieties. (We remark, however, that Calabi-Yau fourfolds have more recently become of interest to string theorists as well, [76].)

For our threefolds, we will use additional tools to help study our varieties. With our interest in these varieties for their special cohomology, we define an automorphism \( \sigma \) of a Calabi-Yau \( n \)-fold to be \textit{symplectic} if the induced action on the holomorphic \( n \)-form is trivial, i.e.,

\[
\sigma_{\ast}(\omega) = \omega.
\]

Otherwise, the automorphism is \textit{non-symplectic}.

In light of Hironaka’s result [44] that all singularities can be resolved by blowing up, and the following theorem giving a cohomological description of blow ups, these automorphisms allow us to construct many interesting examples of Calabi-Yau varieties from other Calabi-Yau varieties.

**Theorem 2.2** ([42], Section VII). Let \( X \) be a smooth projective irreducible variety over \( \mathbb{C} \) of dimension \( n \). Let \( Y_1, Y_2, \ldots, Y_r \subset X \) be mutually disjoint closed irreducible subvarieties of \( X \) of codimension \( d \geq 2 \), and let \( Y \) be their (disjoint) union.

We have

\[
H^k(\tilde{X}) \cong H^{k-2}(Y) \oplus \cdots \oplus H^{k-2-2(d-2)}(Y) \xi^{d-2} \oplus H^k(X)
\]
where $\xi$ be the image of the canonical line bundle $\mathcal{O}_{\tilde{X}}(\tilde{Y})$ in $H^2(\tilde{X}, \mathbb{Z})$.

This comes from considering the blowup of $Y$ (in $X$) as a vector bundle over $Y$ and using the Chern classes $\xi^k$ to generate the cohomology of the resolution. One can also give an integral version of this result, descending to an isomorphism of $\mathbb{Z}$-modules, so that

$$H^k(\tilde{X}, \mathbb{Z}) \simeq H^{k-2}(Y, \mathbb{Z}) \oplus \cdots \oplus H^{k-2-2(d-2)}(Y, \mathbb{Z})\xi^{d-2} \oplus H^k(X, \mathbb{Z})$$

Thus, for $X$ a Calabi-Yau variety with symplectic automorphism $\sigma$, we see that if a crepant resolution of $X/\sigma$ exists, it is a Calabi-Yau variety as well. When $\sigma$ is non-symplectic one can be creative by combining multiple Calabi-Yau varieties with non-symplectic automorphisms to get new Calabi-Yau varieties as follows.

Let $X_k$ be a Calabi-Yau variety of dimension $n_k$, with non-symplectic automorphisms $\sigma_k$, for $k = 1, 2$, such that $\sigma_1 \times \sigma_2$ is symplectic on $X_1 \times X_2$. Then, a crepant resolution of $$(X_1 \times X_2)/(\sigma_1 \times \sigma_2)$$ if one exists, is a Calabi-Yau $(n_1 + n_2)$-fold.

We will be constructing Calabi-Yau threefolds from smaller dimensional Calabi-Yau varieties. Automorphisms of elliptic curves are well understood, see Theorem 3.4 below. Similarly, effort has been put in to classifying automorphisms of K3 surfaces. Symplectic automorphisms of K3 surfaces have been completely classified by Mukai [59] and interestingly (though not yet known why) the group of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group $M_{23}$. No similar characterization of non-symplectic automorphisms of K3 surfaces exists. The possible non-symplectic automorphisms of prime order have been classified [4], but
when the order is composite no known characterization exists. See [3] and [29] for partial results.

This idea of constructing Calabi-Yau threefolds from smaller dimensional Calabi-Yau varieties was studied by Borcea [14] and Voisin [77] to construct threefolds amenable to study. Any elliptic curve $E$ over a field of characteristic 0 can be expressed by a Weierstrass equation

$$y^2 = x^3 + ax + b$$

which has the hyperelliptic involution given by $\iota((x,y)) = (x,-y)$. Note that

$$\iota_* \left( \frac{dx}{y} \right) = - \frac{dx}{y}$$

and so this action is non-symplectic. Moreover, for any K3 surface $S$ with non-symplectic involution $\sigma$ acting on $H^{2,0}(S)$ as multiplication by $-1$, the action by $\iota \times \sigma$ on $H^{3,0}(E \times S)$ is trivial, i.e., the action is symplectic. Hence, a crepant resolution of the quotient

$$(E \times S)/\langle \iota \times \sigma \rangle$$

is a Calabi-Yau threefold.

Voisin showed that this family of threefolds is canonically closed under mirror symmetry. If $X$ is a crepant resolution of

$$(E \times S)/\langle \iota \times \sigma \rangle$$
then its topological mirror $X^\vee$ is a crepant resolution of

$$(E \times S^\vee)/\langle \iota \times \sigma^\vee \rangle$$

where $S^\vee$ is the Dolgachev mirror of $S$. The construction uses the fact that $\text{Pic}(S)^\sigma$ is a 2-elementary lattice, which have been classified by Nikulin [60], and can be stated precisely as follows.

**Theorem 2.3 ([77]).** Let $E$ be an elliptic curve with non-symplectic involution $\iota$, and let $S$ be a K3 surface with non-symplectic involution $\sigma$. Then there is a Calabi-Yau threefold

$$X = (E \times S)/\langle \iota \times \sigma \rangle$$

with Hodge numbers

$$h^{1,1}(X) = 11 + 5N - N',$$

$$h^{2,1}(X) = 11 + 5N' - N,$$

where $N$ is the number of curves fixed by $\sigma$ on $S$, and $N'$ is the maximum of their genera.

With the possibility of mirror symmetry quite clear from this representation of the Hodge numbers, Voisin uses Nikulin’s classification to show that one has mirror symmetry for all but one of the examples with $N' \neq 0$.

A useful result when studying the fixed locus of actions on K3 surfaces will be the Hodge Index Theorem (for surfaces).

**Theorem 2.4 ([43] Theorem 6.1).** The wedge product is positive definite on $(H^{2,0} \oplus H^{0,2})_\mathbb{R}$ and has signature $(1, h^{1,1} - 1)$ on $H^{1,1}_\mathbb{R}$. 
2.1. CALABI-YAU VARIETIES AND COMPLEX GEOMETRY

One last tool we will use to study our resolutions will be fixed point formulas.

**Theorem 2.5** (Topological Lefschetz fixed point formula, [55]). Let $X$ be a variety, and consider a continuous map $f : X \to X$. If the alternating sum

$$
\sum_{k \geq 0} (-1)^k \text{tr}(f_*|H_k(X, Q))
$$

is not equal to 0, then the map $f$ has a fixed point.

Our particular interest will be applying this to the identity map to compute the Euler characteristic. The more involved holomorphic version we need only for surfaces.

**Theorem 2.6** ([5]). Let $X$ be a compact complex surface, and let $G$ be a finite group of automorphisms of $X$. For any $g \in G$, let $X^g$ denote the fixed point set of $g$, and let

$$
N^g = \sum N^g(\theta)
$$

denote the (complex) normal bundle of $X^g$ decomposed according to the eigenvalues $e^{i\theta}$ of $g$. Then we have

$$
\sum (-1)^p \text{tr}(g|H^p(X, O_X)) = \sum_j a(P_j) + \sum_k b(D_k)
$$

where the numbers $a(P)$ and $b(D)$ are given as follows: for any point $P$,

$$
a(P) = \frac{1}{\det(1 - g|T_P)},
$$
where $T_P$ is the tangent space at $P$, and for any curve $D$, 

$$b(D) = \frac{1 - gD}{1 - e^{-i\theta}} - \frac{e^{-i\theta}}{(1 - e^{-i\theta})^2} D^2.$$ 

### 2.2 Arithmetic on Calabi-Yau varieties

If we want to study arithmetic questions about varieties, we no longer wish to work over the complex numbers. In general, arithmetic questions may be asked about varieties defined over any number field, though given the examples we will be studying will be defined over $\mathbb{Q}$, we will content ourselves only discussing varieties defined over $\mathbb{Q}$. By this, we mean our varieties are defined by the vanishing of a finite number of polynomial equations defined over $\mathbb{Q}$.

Let $X$ be a Calabi-Yau variety of dimension $n$, defined over $\mathbb{Q}$. Then there exists a model defined over $\mathbb{Z}$, an integral model, so we may consider the reduction of the variety for any (integral) prime $p$. A prime $p$ is called a *good prime* (or simply *good*) if the reduction of $X$ at $p$ is again a Calabi-Yau variety. Otherwise, we say the prime is *bad*. For a fixed Calabi-Yau variety $X$, the set of bad primes is finite.

Let $p$ be a good prime, and denote the reduction of $X$ at $p$ by $X_p$. Similarly, consider $\overline{X}$ the variety base extended to an algebraic closure $\overline{\mathbb{Q}}$, and the reduction $X_p$ extended to $\overline{\mathbb{F}_p}$ by $\overline{X}_p$. The *geometric Frobenius* $F_p : \overline{X}_p \to \overline{X}_p$ acts by mapping coordinates to their $p$-th power. For any prime $\ell \neq p$, we may consider the induced action of the (geometric) Frobenius on the $\ell$-adic cohomology groups

$$F_p^* : H^k_\ell(\overline{X}) \to H^k_\ell(\overline{X}),$$

where $H^k_\ell(\overline{X}) = H^k_{\text{ét}}(\overline{X}, \overline{\mathbb{Q}}_\ell)$. By the Weil conjectures (now theorems) we know the
eigenvalues of $F_p^*$ acting on $H^k_\ell(X)$ are algebraic integers having absolute value $p^{k/2}$.

Moreover, for $0 \leq k \leq 2n$, the ‘characteristic polynomials’

$$P_{k,p}(t) = \det(1 - t \cdot F_p^*|H^k_\ell(X))$$

are all polynomials in $\mathbb{Z}[t]$, where $P_{k,p}(t)$ has degree $h^k(X)$. We define the $k$-th cohomological $L$-series of $X$ in terms of the $P_{k,p}(t)$ above, which are all independent of $\ell$. As an Euler product, we define the $k$-th cohomological $L$-series of $X$ as

$$L(H^k_\ell(X), s) = (*) \prod_{p \text{ good}} \frac{1}{P_{k,p}(p^{-s})},$$

where the product runs over the good primes of $X$, and $(*)$ denotes possible Euler factors for the bad primes. We then define the $k$-th $L$-series of $X$ as

$$L^k(X, s) := L(H^k_\ell(X), s).$$

When we discuss an $L$-series of $X$ without a degree, we are referring to $L^n(X, s)$, the $L$-series corresponding to the dimension of $X$.

The primes of bad reduction are usually a lot less understood, and we frequently denote equality up to finitely many bad primes, e.g.,

$$L^k(X, s) = \prod_{p \text{ good}} \frac{1}{P_{k,p}(p^{-s})}.$$
2.2. ARITHMETIC ON CALABI-YAU VARIETIES

Expanding the product, we can write the $L$-series as a sum

$$L^k(X, s) = \sum_{j=1}^{\infty} \frac{a_j(X)}{j^s}$$

where

- $a_1(X) = 1$,
- $a_p(X) = \text{tr}(F_p^*|H^k(X))$,
- $a_j(X)$ is determined by the $a_p(X)$, where $p$ is prime dividing $j$.

For example, when $E$ is an elliptic curve defined over $\mathbb{Q}$, we have

$$L(E, s) = \prod_p \frac{1}{1 - a_p(E)p^{-s} + \chi(p)p^{1-2s}}$$

where $\chi$ is the trivial Dirichlet character modulo $N_E$, the conductor of $E$. For a good prime $p$ we have

$$a_p(E) = p + 1 - \#E(\mathbb{F}_p)$$

and otherwise, if $p$ is bad the value of $a_p(E)$ is either 0, 1 or $-1$, and is determined by the reduction type. For non-prime coefficients we have

- $a_{mn}(E) = a_m(E)a_n(E)$ when $\gcd(m, n) = 1$, and
- $a_{p^r+1}(E) = a_p(E)a_{p^r+1}(E) - \chi(p)pa_{p^r}(E)$ for $r \geq 0$.

As $\chi$ depends on the conductor $N_E$, we see why the relationship between the coefficients for higher dimensional Calabi-Yau varieties is not easy to write down, as there is no known classification of bad reduction.
2.3. MODULAR FORMS

Remark. Consider the Galois representation

$$\rho_X : G_\mathbb{Q} \to \text{Aut}(H^n_\ell(X))$$

for a variety $X$ of dimension $n$, where $G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the absolute Galois group. The $L$-series of $X$ is also called the $L$-series of the Galois representation $\rho_X$.

2.3 Modular forms

A modular form is an analytic function having many symmetries in the complex upper half plane. Standard introductions include [24] or [51], with a focus on their use in the proof of Fermat’s Last Theorem, and other number theoretic problems.

While they are analytic by construction, our interest in arithmetic questions on varieties leads naturally to this contrasting setting. The symmetries are described by the action of the modular group

$$\Gamma(1) := \text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

or a finite index subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$.

The subgroups we will require come from the principal congruence subgroups of $\Gamma(1)$. For any positive integer $N$ we have the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \pmod{N}, \ a \equiv d \equiv 1 \pmod{N} \right\}.$$
The subgroups of interest are

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}, \]

and

\[ \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0, \ a \equiv d \equiv 1 \pmod{N} \right\}. \]

In general, a finite index subgroup \( \Gamma \subset \Gamma(1) \) is called a \emph{congruence subgroup} if \( \Gamma \) contains \( \Gamma(N) \) for some \( N \). We say the least such \( N \) is the \emph{level} of \( \Gamma \).

**Definition 2.7.** A function \( f : \mathfrak{h} \to \mathbb{C} \) is a \emph{modular form of weight} \( k \) for the congruence subgroup \( \Gamma \subset \Gamma(1) \) if it satisfies the following three conditions:

1. \( f \) is holomorphic on \( \mathfrak{h} \);
2. for any \( z \in \mathfrak{h} \) and any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), we have
   \[ f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z); \]
3. \( f \) is holomorphic at the cusps.

Modular forms are frequently described by their Fourier expansion at infinity,

\[ f(z) = \sum_{n \geq 0} a_n q^n \quad q = e^{2\pi i z}, \]

coming from the relation \( f(z + 1) = f(z) \).
If, instead, we have

\[ f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k f(z) \]

for some Dirichlet character mod \( N \), then \( f \) is said to have *Nebentypus* \( \chi \).

If a modular form \( f \) on \( \Gamma \) vanishes at each of the cusps, it is called a *cusp form*. The set of all modular forms of weight \( k \) on a subgroup \( \Gamma \) is denoted by \( M_k(\Gamma) \), and the set of all cusp forms of weight \( k \) on \( \Gamma \) is denoted by \( S_k(\Gamma) \). These are both vector spaces under the natural addition and scalar multiplication.

If \( f \) is a modular form on \( \Gamma_0(N) \), we say \( f \) is *of level \( N \)* or *has level \( N \)*. We now have what we need to define our main objects of interest, newforms.

Rewriting the modular transformation as

\[ f(z) = (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) \]

we may generalize to any matrix

\[ \gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}) \]

using the *slash operator*

\[(f|\gamma)(z) = \det(\gamma)^{k/2}(tz + u)^{-k} f(\gamma \cdot z).\]

(The superscript + above means we are only looking at the connected component of \( \text{GL}_2(\mathbb{R}) \) containing the identity.)
With this notation we see that if \( f(z) \in S_k(\Gamma_1(N)) \), then \( g(z) := f(\alpha z) \in S_k(\Gamma_1(\alpha N)) \) for any positive integer \( \alpha \). Indeed, consider

\[
\gamma_\alpha := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2^+(\mathbb{R}).
\]

Then as \( \Gamma_1(\alpha N) \subset \gamma_\alpha^{-1}\Gamma_1(N)\gamma_\alpha \), we have

\[
\alpha^{k/2}g(z) = \alpha^{k/2}f(\alpha z) = (f|\gamma_\alpha)(z)
\]
is a modular (cusp) form on \( \Gamma_1(\alpha N) \).

Such \( g \) are called old forms for \( \Gamma_0(N\alpha) \), and any cusp form that is orthogonal (under the Petersson inner product) to the space of old forms is a newform.

We always assume our newforms are normalized in the sense that if \( f(z) = \sum_{n=1}^{\infty} a_n q^n \) is a newform, then \( a_1 = 1 \).

We say a variety \( X \) defined over \( \mathbb{Q} \) of dimension \( d \) is modular if its \( L \)-series is the \( L \)-series of a newform \( f \) of weight \( d + 1 \) on \( \Gamma_0(N) \) for some \( N \). We say the newform \( f \) has CM by a quadratic number field \( K \) if \( a_n = \varphi_K(n)a_n \) for almost all \( n \), where \( \varphi_K \) is the non-trivial Dirichlet character associated to \( K \).

As mentioned in the introduction, one of the major results in the last 25 years was the proof that elliptic curves defined over \( \mathbb{Q} \) are modular. Similarly, every singular K3 surface defined over \( \mathbb{Q} \) was shown to be modular by Livné, \([56]\), in the sense that the Galois representation coming from the 2-dimensional transcendental lattice is modular.

All rigid Calabi-Yau threefolds defined over \( \mathbb{Q} \) are modular by \([25]\) or \([41]\), in
particular, all of the rigid Calabi-Yau threefolds in our construction. One can use Serre’s conjecture to determine the associated newforms (hence $L$-series) as the (now proven) conjecture gives a bound on the level of the newform - it is a small product of powers of the primes of bad reduction of the corresponding threefold. However, this method is not practical when there are large primes of bad reduction, as one must construct spaces of cusp forms of high level and search for the correct newform. We will be able to avoid this in Chapter 4 by computing the $L$-series without the a priori knowledge of modularity.

Remark. As is common in the literature, we will use $L$-series and $L$-function interchangeably.

2.4 Hecke characters

For any quadratic imaginary number field $K$ with ideal $m$ we say a Hecke character $\chi$ of modulus $m$ and infinity type $c$ is a homomorphism $\chi : I_m \to \mathbb{C}^\times$ on fractional ideals of $K$, relatively prime to $m$, given by setting

$$\chi(aO_K) = a^c$$

for all $a \in K^\times$ with $a \equiv 1 \pmod{m}$. We may then extend it by setting it equal to 0 for any fractional ideal not coprime to $m$. 
The $L$-series of $\chi$ is given by the product over all prime ideals $p$ of $K$

$$L(s, \chi) = \prod_p \left( 1 - \chi(p)N(p)^{-s} \right)^{-1}$$

where $N(p)$ is the norm of $p$. Hecke’s insight was to associate a newform to $\chi$: the inverse Mellin transform

$$f_\chi := \sum a \chi(a)q^{N(a)}$$

of the $L$-series is a newform of weight $c + 1$, level $\Delta_K N(m)$ where $\Delta_K$ is the absolute value of the discriminant of $K$, and Nebentypus character

$$\eta(n) := \frac{\chi(nO_K)}{n^c}.$$ 

Note that $\eta(n) = 0$ when $n$ is not coprime to the norm of the modulus, $N(m)$.

We will need one small lemma to describe the newform for the cube of a Hecke character and the newform of the Hecke character itself.

**Lemma 2.8.** Let $\psi$ be a Hecke character of infinity type $c$ of an imaginary quadratic field $K$ and suppose its associated newform $f_\psi$ has trivial Nebentypus. Suppose that we have Fourier $q$-expansions

$$f_\psi = \sum_{n=1}^{\infty} a_n q^n \quad f_{\psi^3} = \sum_{n=1}^{\infty} b_n q^n.$$  

Then

$$b_p = a_p^3 - 3p a_p$$

for any prime $p$ not ramified in $K$.  

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Proof. If $p = p_1 p_2$ splits in $K$, then $a_p = \psi(p) + \psi(p_2)$, so

$$b_p = \psi(p)^3 + \psi(p_2)^3 = a_p^3 - 3pa_p.$$ 

If $p$ is inert, then $f_{\psi^3}$ has CM by $K$, so $a_p = b_p = 0$. 

Remark. The requirement that $f_\psi$ have trivial Nebentypus is subtle. For example, note that in $\mathbb{Q}(\sqrt{-3})$, we can take a Hecke character $\psi$ such that $f_\psi$ has trivial Nebentypus, and a cubic twist $\psi \otimes \chi$ has associated newform $f_{\psi \otimes \chi}$ with non-trivial Nebentypus. If $f_\psi = \sum a_n q^n$ has trivial Nebentypus, then by Lemma 2.8 we have $b_p = a_p^3 - 3pa_p$. On the other hand, the cubic twist has $(\psi \otimes \chi_3)^3 = \psi^3$, however a coefficient $a_p$ twisted by a third root of unity $\zeta_3$ (for example, twisting by a primitive Dirichlet character modulo 9) satisfies

$$(\zeta_3 a_p)^3 - 3p(\zeta_3 a_p) = a_p^3 - 3p\zeta_3 a_p \neq b_p.$$ 

In general one has a decomposition of the space of weight $k$ cusp forms on $\Gamma_1(N)$ over the Nebentypus characters $\epsilon$ modulo $N$,

$$S_k(\Gamma_1(N)) = \bigoplus_\epsilon S_k(\Gamma_0(N), \epsilon).$$ 

In our cases of interest in Chapters 4 and 5, all our newforms have real coefficients and we will only be interested in weight 2 or 4 newforms with CM by a quadratic imaginary number field $K$, so we recall

**Theorem 2.9** (Ribet [65] Prop 4.4, Thm 4.5). A newform has CM by a quadratic field $K$ if and only if it comes from a Hecke character of $K$. In particular, the field
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\( K \) is imaginary and unique.

As our newforms have CM and even weight \( k \), any Nebentypus character \( \epsilon \) satisfies \( \epsilon(-1) = (-1)^k = 1 \), so all of our newforms have trivial Nebentypus, and Lemma 2.8 will always apply.

A standard reference for Hecke characters, and their application with CM newforms is Schuett’s thesis, [69].

2.5 Intermediate Jacobians

To any complex curve \( C \) one may associate the Jacobian variety of \( C \), \( J(C) \), which is defined as the quotient

\[
H^{0,1}(C)/H^1(C, \mathbb{Z}).
\]

It is known that this is isomorphic to the Picard variety of \( C \) as well as the Albanese variety of \( C \), and so these varieties are important tools in studying \( C \), both the complex geometry, and the arithmetic if we have a model defined over a number field.

In order to generalize this to an arbitrary variety \( X \) of dimension \( n \) we can define, for any \( 0 \leq k \leq n \), the \((k+1)\)-th (Griffiths) intermediate Jacobian

\[
J^{k+1} := H^{2k+1}(X, \mathbb{R})/H^{2k+1}(X, \mathbb{Z})
\approx \bigoplus H^{2k+1}(X, \mathbb{C})/(F^{k+1}H^{2k+1}(X) \oplus H^{2k+1}(X, \mathbb{Z})_{tor}
\]
2.5. INTERMEDIATE JACOBIANS

where the integral cohomology in the quotient is the torsion free part of $H^{2k+1}(X, \mathbb{Z})$, and $F^{k+1}H^{2k+1}(X)$ is the second half of the Hodge filtration

$$F^{k+1}H^{2k+1}(X, \mathbb{C}) = \bigoplus_{p+q=2k+1, \ q<k+1} H^{p,q}(X).$$

For any curve $C$ we have $J^1(C) = J(C)$, and in general the intermediate Jacobians $J^1(X)$ and $J^n(X)$ are isomorphic to the Picard and Albanese varieties of $X$, respectively.

These have been instrumental in algebraic geometry, as Griffiths and Clemens [18] used the intermediate Jacobians of a cubic threefold to show the threefold was not rational, while being unirational. This showed that threefolds are more difficult to study than (complex) surfaces and curves where unirationality implies rationality. An interest in the physics literature comes from the use of Calabi-Yau varieties in string theory, e.g., in [6], [23] and [58]. In particular, certain families of Calabi-Yau threefolds can be characterized by their intermediate Jacobians, and so studying their intermediate Jacobians gives information about the threefolds themselves.

For elliptic curves it is known that $J^1(E) \simeq E$, and so their (intermediate) Jacobians are well understood. For K3 surfaces, the intermediate Jacobians are trivial by the Calabi-Yau definition, and so our focus will be on Calabi-Yau threefolds. For a Calabi-Yau threefold $X$, $J^1(X) = J^3(X) = 0$ are trivial, and only the ‘middle’ intermediate Jacobian $J^2(X)$ is non-trivial, so we simplify notation and denote this by

$$J(X) := J^2(X).$$

Any Calabi-Yau threefold $X$ is Kähler by definition, so the intermediate Jacobian
$J(X)$ is a complex torus of dimension $1 + h^{2,1}(X)$. In particular, when $X$ is rigid the intermediate Jacobian is a complex 1-torus.
Chapter 3

Mirror Symmetry in Generalized Borcea-Voisin families

In the mid 90s, Claire Voisin and Ciprian Borcea independently constructed new families of Calabi-Yau threefolds with marvelous properties. Voisin showed that a very elementary construction of a family of threefolds was closed under mirror symmetry, while Borcea found a characterization for when these threefolds have complex multiplication.

Since then this construction has been generalized for many different applications. The construction, at its simplest, is to take the product of an elliptic curve $E$ and a K3 surface $S$, and look at crepant resolutions of the quotient of $E \times S$ by the product of non-symplectic involutions on the curve and surface. Voisin’s main interest in this family was showing a general member had a mirror variety in the family, coming from the underlying mirror of $S$. In this chapter we briefly recall the construction, then ask the question of mirror symmetry with a construction using higher order automorphisms on the elliptic curve and K3 surface, instead of involutions.
3.1 The Borcea-Voisin construction

It will be beneficial to see both Borcea’s and Voisin’s approach to the construction of this family of threefolds. We start with Borcea’s work [14].

The construction starts by considering a product of three elliptic curves, say $E_1, E_2$ and $E_3$ and taking the quotient with respect to some involutions. Let $\iota_j$ be the hyperelliptic involution on $E_j$, for $j = 1, 2, 3$, and consider the group

$$G = \langle \text{id} \times \iota_2 \times \iota_3, \iota_1 \times \text{id} \times \iota_3 \rangle$$

acting component-wise on the product $E_1 \times E_2 \times E_3$. Then $X := (E_1 \times E_2 \times E_3)/G$ has a Calabi-Yau resolution, say $\tilde{X}$. To see this, note that, if we let $\omega_i$ be the holomorphic 1-form on $E_i$, then $\omega_1 \wedge \omega_2 \wedge \omega_3$ gives a holomorphic 3-form on $X$, fixed by the elements of $G$. Similarly, the holomorphic 1-forms and the holomorphic 2-forms are not fixed by all of $G$, so $h^{1,0}(\tilde{X}) = h^{2,0}(\tilde{X}) = 0$. The resolution cannot introduce $(n, 0)$-classes for any $n \leq 2$, so a crepant resolution of $X$ will be Calabi-Yau threefold, as the canonical bundle of $E_1 \times E_2 \times E_3$ is trivial.

To see how the resolution affects the middle cohomology, we need to investigate the fixed locus of $G$. If we set $g = \text{id} \times \iota_2 \times \iota_3$ and $h = \iota_1 \times \text{id} \times \iota_2$, then $G = \{e, g, h, gh\}$ so we note that $(E_1 \times E_2 \times E_3)^g$ is 16 copies of $E_1 \times \{pt\} \times \{pt\}$, and the action by $h$ makes these rational curves in the quotient, so there are 16 curves of the form $C \times \{pt\} \times \{pt\}$ in the fixed locus of $G$. Similarly, $h$ gives rise to 16 fixed copies of $\{pt\} \times C \times \{pt\}$ and $gh$ gives 16 fixed copies of $\{pt\} \times \{pt\} \times C$, so the singular locus of the quotient is 48 rational curves with $4^3$ points of intersection.

Every $\omega \in H^0(C)$ for $C$ in the singular locus gives rise to a $(1, 1)$-class in $\tilde{X}$ and
every \((1,0)\)-form gives rise to a \((2,1)\)-form in \(\widetilde{X}\). Indeed, by Poincaré duality, a fixed rational curve \(C\) and the ruling in the resolution satisfies

\[
\begin{array}{c}
H^k(C \times \mathbb{P}^1) \\ \downarrow \\
H_{4-k}(C \times \mathbb{P}^1) \\ \downarrow \\
H_{4-k}(\widetilde{X}) \\ \downarrow \\
H^{2+k}(\widetilde{X})
\end{array}
\]

Each of the fixed curves was a transversal \(A_1\) singularity, so a single blowup is a crepant resolution. Hence, the resolution introduces 48 \((1,1)\)-classes and no \((2,1)\)-classes. Together with the Künneth formula, this implies

\[
h^{1,1}(\widetilde{X}) = 51,
\]

\[
h^{2,1}(\widetilde{X}) = 3,
\]

and

\[
\chi(\widetilde{X}) = 96.
\]

Noting the similarity with each automorphism in \(G\) and the Kummer construction, that of constructing a K3 surface as a resolution of

\[
(E \times E)/\langle \iota \times \iota \rangle,
\]
Borcea also considered products of the form \( E \times S \) where \( E \) is an elliptic curve and \( S \) is a K3 surface. Let \( \iota \) be the hyperelliptic involution on \( E \) and let \( \sigma \) be an automorphism of \( S \), acting by \(-1\) on the holomorphic 2-form of \( S \). Then, in the case that \( S \) is a resolution of 6 lines in \( \mathbb{P}^2 \) he found that the quotient of \( E \times S \) by \( \iota \times \sigma \) has a Calabi-Yau resolution with \( h^{1,1} = 41 \) and \( h^{2,1} = 5 \), so \( \chi = 72 \).

In [77], Voisin starts with the more general situation where \( S \) can be any K3 surface. These all have (crepant) Calabi-Yau resolutions, and using Nikulin’s classification of K3 surfaces with non-symplectic involutions, she finds that the Hodge numbers and Euler characteristic of the respective Calabi-Yau threefolds depend only on the isomorphism class \((S, \sigma)\) of the K3 surface and the non-symplectic automorphism.

**Theorem 3.1** (Nikulin, [60]). A pair \((S, \sigma)\), where \( S \) is a K3 surface and \( \sigma \) a non-symplectic involution on \( S \), is determined, up to deformation, by a triplet \((r, a, \delta)\), where \( r = \text{rank} \text{Pic}(S)^\sigma \), \( a \) is the rank of the 2-elementary lattice \((\text{Pic}(S)^\sigma)^* / \text{Pic}(S)^\sigma \) and \( \delta = 0 \) if \((x^*)^2 \in \mathbb{Z}\) for any \( x^* \in (\text{Pic}(S)^\sigma)^* \), and 1 otherwise.

The classification of such triples \((r, a, \delta)\) has a very nice, almost symmetric, graphical representation given in Figure 3.1. The \( g \) and \( k \) values in the figure comes from the geometric side of things, studying the fixed locus of the involution on the K3 surface, described in the following theorem.

**Theorem 3.2** ([4]). Let \((S, \sigma)\) be a K3 surface with a non-symplectic automorphism \( \sigma \) corresponding to the triple \((r, a, \delta)\).

(i) If \((r, a, \delta) = (10, 10, 0)\), then \( \sigma \) has no fixed points.

(ii) If \((r, a, \delta) = (10, 8, 0)\), then \( S^\sigma \) is the disjoint union of two elliptic curves.
(iii) For any other \((r, a, \delta)\) we have that the fixed locus of \(\sigma\) is a disjoint union

\[ S^\sigma = C \cup L_1 \cup L_2 \cup \cdots \cup L_k, \]

where \(C\) is a curve of genus \(g\) and \(L_1, \ldots, L_k\) are rational curves. Moreover, we have

\[ g = \frac{1}{2}(22 - r - a) \]

and

\[ k = \frac{1}{2}(r - a). \]

Relating the Nikulin invariants to the geometric invariants is a matter of applying fixed point theorems and other Hodge theory to the resolution. With this in hand, one may compute the Hodge numbers of Borcea-Voisin threefolds and look for mirror
symmetry.

**Theorem 3.3** (Voisin, [77]). Let $E$ be an elliptic curve with non-symplectic involution $\iota$, and let $S$ be a K3 surface with non-symplectic involution $\sigma$. Then there is a Calabi-Yau threefold

$$X = \widetilde{E \times S}/\iota \times \sigma$$

with Hodge numbers

$$h^{1,1}(X) = 11 + 5N - N' \quad \text{and} \quad h^{2,1}(X) = 11 + 5N' - N,$$

where $N$ is the number of fixed curves on $S$ under $\sigma$, and $N'$ is the sum of their genera.

In particular, when $N = 0$ the quotient is smooth, so we are only interested in the case when $N > 0$. We can now see the mirror symmetry hinted to us in Nikulin’s pyramid. There is a symmetry about the line $r = 10$, which does not involve the line at $g = 0$ (in Voisin’s notation, $N' = 0$) where we did not expect a mirror relationship. Otherwise, only the point $(14, 6, 0)$ is missing a mirror K3. The problem here, Voisin shows that a K3 surface with these Nikulin invariants is extremal (i.e., Picard rank 20) and there are only finitely many of these [72], so no mirror symmetry is expected once more.

### 3.2 Generalizing the Borcea-Voisin construction

There are many approaches to generalizing this construction - automorphisms instead of just involutions, $n$-folds instead of only threefolds, or products not of the forms seen above. For the latter, we note that in [35], Garbagnati and Penegini classify K3
surfaces that are constructed using (a resolution of) a quotient of the product of not necessarily elliptic curves, and so the Borcea-Voisin construction may generalize to Calabi-Yau threefolds with arbitrary curves and surfaces, and non-crepant resolutions. However, crepant resolutions seem more natural, so we do not address this here. See for example [64], and the references therein, for approaches in this direction.

We first start by altering the Borcea-Voisin construction by choosing different automorphisms. Symplectic automorphisms will not give Calabi-Yau threefolds, and so we must focus on different non-symplectic automorphisms. As involutions are simply automorphism of order 2, one can ask for the same construction with automorphisms of order larger than 2. One finds immediately that, in order for a crepant resolution of the quotient of the form

$$\left(E \times S\right)/\langle \iota \times \sigma \rangle$$

to be Calabi-Yau, one needs the automorphisms of $E$ and $S$ to have the same order. Because of this, the question reduces to asking for which integers $n > 2$ do elliptic curves and K3 surfaces have non-symplectic automorphisms of order $n$?

For elliptic curves the question is not too difficult.

**Theorem 3.4** (Silverman, [74]). Let $E/\mathbb{Q}$ be an elliptic curve. Then its automorphism group is a finite group of order dividing 6.

This is simply a matter of picking a Weierstrass model for $E$ and looking at what an automorphism does with the coefficients. So other than involutions, an elliptic curve may only have non-symplectic automorphisms of order $n = 3, 4$ or 6. Examples of each are well known to exist, e.g., the curves

$$E_3 : y^2 = x^3 - 1$$
with automorphism \( \iota_3(x, y) = (\zeta_3 x, y) \),

\[
E_4 : y^2 = x^3 - x
\]

with \( \iota_4(x, y) = (-x, iy) \) and \( E_6 = E_3 \) with automorphism \( \iota_6(x, y) \mapsto (\zeta_3 x, -y) \), where \( \zeta_3 \) is a primitive third root of unity. These curves will be central to most of Chapters 4 and 5.

For K3 surfaces, the result is predictably less simple. For this we use a simplified version of Nikulin’s result pioneering this area of study.

**Theorem 3.5** (Nikulin, [60]). *Let \( S \) be a complex K3 surface. If \( \sigma \) is a non-symplectic automorphism of \( S \) of finite order \( n \), then \( \varphi(n) \leq 21 \), where \( \varphi \) is the Euler \( \varphi \)-function. In particular, \( n \leq 66 \) and when \( \sigma \) has prime order, its order is at most 19.*

**Proof.** Recall the real extension of the transcendental lattice of \( S \) decomposes as

\[
T(S)_{\mathbb{R}} = (H^{2,0}(S) \oplus H^{0,2}(S))_{\mathbb{R}} \oplus P,
\]

where \( P \) is the orthogonal complement of \( (H^{2,0}(S) \oplus H^{0,2}(S))_{\mathbb{R}} \) with respect to the intersection pairing, in \( T(S)_{\mathbb{R}} \). The first part is positive definite, so if \( f \in \text{Aut}(T(S)_{\mathbb{R}}) \) preserves \( H^{2,0}(S) \), it must preserve this orthogonal decomposition. If \( f \) has finite order, the eigenvalue of the action on \( H^{2,0}(S) \) must be a root of unity.

If we now consider an automorphism \( g \) of \( S \), then \( g \) induces maps \( g^* : T(S) \to T(S) \) and (abusing notation) \( g^* : H^{2,0}(S) \to H^{2,0}(S) \). By the above, this last map is multiplication by a root of unity, so has minimal polynomial \( \Phi(m) \), the cyclotomic polynomial of order \( m \), for some \( m \) dividing \( n \). This polynomial must divide the characteristic polynomial of the map \( g^* : T(S) \to T(S) \), and \( T(S) \) has rank at
most 21, so $\Phi(m)$ has degree at most 21. The degree of $\Phi(n)$ is $\phi(n)$, so we must have $\phi(n) \leq 21$, and $n \leq 66$, as desired. In particular, if $g^*$ has prime order, it is multiplication by a root of unity $\zeta_p$ with $p \leq 19$.

As with the elliptic curve case, it has been shown that K3 surfaces with non-symplectic automorphisms with these orders do exist (see, for example, [4], [29], [70], and [75]). However, the elliptic curves will force us to be interested only in K3 surfaces with automorphisms of order 3, 4 or 6, so we only focus on these.

### 3.3 Automorphisms of order three

To start with, we begin the search for mirror symmetry with generalized Borcea-Voisin threefolds with quotients of the form

$$X := (E \times S)/(\iota \times \sigma^2)$$

with $E$ an elliptic curve with non-symplectic automorphism $\iota$ of order 3, and $S$ a K3 surface with non-symplectic automorphism $\sigma$ of order 3. We will use the same approach as Voisin.

*Remark.* We use $\iota \times \sigma^2$, instead of $\iota \times \sigma$, so we may assume both $\iota$ and $\sigma$ act on the respective holomorphic top forms of $E$ and $S$ as multiplication by the same primitive third root of unity. When only dealing with the K3 surface $S$ we will still work with $\sigma$ as the fixed locus is the same.

There are three main steps to Voisin’s process. Firstly, we must show this construction gives rise to (simply connected) Calabi-Yau threefolds, and with automorphisms of order 3 instead of 2, this is a little more work. We must then compute the Hodge
numbers using a classification of K3 surfaces with non-symplectic automorphisms of order 3, similar to that of Nikulin’s classification. This is available from [4]. Lastly, with the possible Hodge numbers we find, we simply look for a mirror symmetry relationship in this family.

Remark. We are not just interested in whether these Calabi-Yau threefolds have mirrors, we are interested in whether they have mirrors in the family of generalized Borcea-Voisin threefolds, as with Voisin’s original discovery.

Step 1 - Calabi-Yau resolution
To start, note that the (holomorphic) Hodge number $h^{1,0}$ is a birational invariant, and so the resolution of the quotient has $h^{1,0} = h^{2,0} = 0$ as the $(1,0)$-forms of $E \times S$ are not preserved by the group. Furthermore, the holomorphic threeform is fixed by $\iota \times \sigma^2$, and so descends to the quotient, and a crepant resolution will then be a Calabi-Yau threefold.

A crepant resolution exists by [16] and Lemma 3.7, though to compute the Hodge numbers we will actually need to construct this crepant resolution explicitly below.

Step 2 - Computing the Hodge numbers
To explicitly compute the resolution we will need a result similar to Theorem 3.1, but for non-symplectic automorphisms of order 3. The same ideas above apply, and we may classify the pairs $(S, \sigma)$ where $S$ is a K3 surface with a non-symplectic automorphism of order three using a classification of 3-elementary lattices.

Theorem 3.6 (Rudakov-Shafarevich, [68]). An even hyperbolic 3-elementary lattice $L$ of rank $r > 2$ is uniquely determined by its discriminant, $a := a(L)$. Moreover,
given $r$ and $a \leq r$, such a lattice exists if and only if the following conditions are satisfied: we must have $r \equiv 0 \mod 2$, and

for $a \equiv 0 \mod 2$ we have $r \equiv 2 \mod 4$

for $a \equiv 1 \mod 2$ we have $(-1)^{r/2-1} \equiv 3 \mod 4$

and when $r \not\equiv 2 \mod 8$ we must have $r > a > 0$.

For $r = 2$, the binary forms have similarly been classified in [20], so all 3-elementary lattices can be classified by the invariants $r$ and $a$. Hence, as with Nikulin’s original work, K3 surfaces with non-symplectic automorphisms of order 3 may be classified using their underlying 3-elementary lattices.

Remark. Given the classification of 3-elementary lattices in terms of their rank and discriminant, one may present all the information for each lattice in the following diagram, and note the similar symmetry about the line $r = 12$ as before in Nikulin’s pyramid.

To study the cohomology of our threefolds, we will work with the fixed locus of $E \times S$ under $\iota \times \sigma^2$. For this, we want to have a good understanding of the fixed loci
of $E$ under $\iota$ (three fixed points) and $S$ under $\sigma$. We will need several tools in order to compute the fixed locus of $S$, starting with the following.

**Lemma 3.7** (Taki, [75] Lemma 3.1). Let $S$ be an algebraic $K3$ surface, with $\sigma$ a non-symplectic automorphism of order 3 acting on the holomorphic 2-form of $S$ by multiplication

$$\sigma^* \omega_S = \zeta_3 \omega_S$$

where $\zeta_3$ is a primitive third root of unity. Then

(a) The restriction $\sigma_T := \sigma^*|T(S) \otimes \mathbb{C}$ can be diagonalized as

$$\begin{pmatrix} \zeta_3 I_r & 0 \\ 0 & \overline{\zeta_3} I_r \end{pmatrix}$$

where $I_r$ the $r \times r$ identity matrix.

(b) Let $P$ be an isolated fixed point of $\sigma$ on $S$. Then $\sigma^*$ the induced map on the tangent space about $P$, can be written as

$$\begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$$

in appropriate local coordinates about $P$.

(c) Let $C$ be a fixed irreducible curve and $Q$ a point on $C$. Then $\sigma^*$ the induced map around the tangent space at $Q$, can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$$
in appropriate local coordinates about \( Q \). In particular, fixed curves are non-singular.

**Proof.** Part (a) follows from considering the \( \zeta_3 \)-eigenspace, and the \( \zeta_3^2 \)-eigenspace, and noting that complex conjugation maps one to the other, so they must have the same dimension. To see that \( \sigma_T \) has no fixed points, hence no eigenvalues equal to 1, consider the map (abusing notation)

\[
\omega_S : T(S) \rightarrow \mathbb{C}
\]

given by \( \omega_S(x) = \omega_S \cdot x \), where \( \omega_S \) is a holomorphic 2-form on \( S \). This map does not send any \( x \in T(S) \) to 0, and we have

\[
(\sigma_T \omega_S)(\sigma_T x) = \omega_S(x).
\]

We know

\[
\sigma_T \omega_S = \lambda \omega_S
\]

for some \( 1 \neq \lambda \in \mathbb{C}^\times \), thus

\[
\frac{\omega_S(\sigma_T x)}{\omega_S(x)} = \lambda^{-1} \neq 1,
\]

and \( \sigma_T x \neq x \).

Parts (b) and (c) come from the fact that \( \sigma^* \) acts on \( H^{2,0}(S) \) as multiplication by some primitive third root of unity \( \zeta_3 \). We may pick a basis on the cotangent space of a fixed point so that the action on \( H^{2,0}(S) \) is

\[
dz_1 \wedge dz_2 \mapsto \zeta_3 dz_1 \wedge dz_2,
\]
coming from an action on the one-forms

\[ dz_1 \mapsto \zeta_3^k dz_1 \quad dz_2 \mapsto \zeta_3^\ell dz_2. \]

Wedging these actions, we must have \( k + \ell \equiv 1 \mod 3. \)

Let \( \mathcal{E} = \mathbb{Z}[\zeta_3] \) be the ring of Eisenstein integers, the ring of integers in \( \mathbb{Q}(\zeta_3). \) An \( \mathcal{E} \)-lattice is a pair \((L, \rho)\) where \( L \) is an even lattice and \( \rho \), a fixed point free isometry on \( L \) of order three. Any \( \mathcal{E} \)-lattice \((L, \rho)\) is a free module over \( \mathcal{E} \) with the action

\[ (a + \zeta_3 b) \cdot x = ax + b \rho^*(x). \]

Thus, the rank of an \( \mathcal{E} \)-lattice is \( 2m \) where \( m \) is its rank as an \( \mathcal{E} \)-module. In particular, every \( \mathcal{E} \)-lattice has even rank. Let \( N(\sigma) \) denote the invariant lattice

\[ \{x \in H^2(S, \mathbb{Z}) : \sigma^*(x) = x\}. \]

Part (a) of the previous Lemma can now be stated as: \((T(S), \sigma^*)\) is an \( \mathcal{E} \)-sublattice of \( N(\sigma)^\perp \), the orthogonal complement taken in \( H^2(S, \mathbb{Z}) \), and our transcendental lattices always have even rank. Note that any \( \mathcal{E} \)-lattice \((L, \rho)\) with trivial \( \rho \)-action on \( L^\vee/L \) is necessarily 3-elementary, as \( \rho \) is fixed point free on \( L \). Indeed \( \rho^2 + \rho + \text{id} \) is the zero map, so for any \( x \in L^\vee/L \) we have

\[ 0 = \rho^2(x) + \rho(x) + \text{id}(x) = 3x. \]
With all this in mind, we are now able to classify the K3 surfaces with non-symplectic automorphisms of order 3. Let $S$, our K3 surface with an automorphism $\sigma$ of order 3, have the 3-elementary lattice pair $(r, a)$, namely

$$r = \operatorname{rank} H^2(S, \mathbb{Z})^{\sigma},$$

and $a$ the discriminant.

**Theorem 3.8** (Artebani-Sarti, [2]). *The fixed locus of $\sigma$ is the disjoint union of

$$n = \frac{2r - 2}{2} \leq 9$$

points and

$$k = \frac{2 + r - 2a}{4}$$

smooth curves with

(a) one curve of genus $g \geq 0$ and $k - 1$ rational curves, or

(b) $k = 0$ and $n = 3$.

Moreover, letting $\operatorname{rank}(N(\sigma)^\perp) = 2m$, we have

$$n + m = 10 \quad \text{and} \quad g = 3 + k - n,$$

in case (a).

**Proof.** We start by using multiple Lefschetz fixed point formulas. The restricted map
\[ \sigma : S^\sigma \to S^\sigma \text{ is the identity, and so the topological Lefschetz formula gives} \]

\[ \chi(S^\sigma) = \sum_{k \geq 0} (-1)^k \text{Tr}(\sigma^*|H^k(S^\sigma, \mathbb{R})). \]

The action on the Néron-Severi lattice is not necessarily trivial, and we have

\[ \chi(S^\sigma) = 2 + \text{rank}(N(\sigma)) - m. \]

As \( \text{rank } N(\sigma) = 22 - 2m \), we have

\[ n + \sum_{i=1}^{k} \chi(C_i) = \chi(S^\sigma) = 2 - m + (22 - 2m) = 3(8 - m), \]

where the \( C_i \) are the fixed curves. The holomorphic Lefschetz trace formula gives

\[ 2n - \sum_{i=1}^{k} \chi(C_i) = 6, \quad (3.1) \]

and part (b) follows immediately from this. Furthermore, the trace formulas combine to give \( n + m = 10 \).

The Hodge Index Theorem (Theorem 2.4) tells us there can only be one curve of genus \( g > 1 \) on \( S \). Indeed, if there is such a curve \( C \), then any other fixed curve \( C' \), disjoint from \( C \), satisfies \( C \cdot C' = 0 \), while the Riemann-Roch Theorem gives

\[ C' \cdot C' = 2g(C') - 2, \]

so the genus of \( C' \) must be 0.

If we have an elliptic curve \( E \) on \( S \), fixed by \( \sigma \), the above shows there cannot be
a fixed curve of genus greater than 1. If we have some other elliptic curve $E'$ fixed on $S$, then $E$ and $E'$ are linearly equivalent and the linear system $|E|$ gives an elliptic fibration $S \to \mathbb{P}^1$.

To see this, recall the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0.$$ 

Tensoring with $\mathcal{O}_S(E)$ gives

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_S(E)|_E \longrightarrow 0$$

and the long exact sequence in cohomology starts as

$$0 \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(S, \mathcal{O}_S(E)) \longrightarrow H^0(E, \mathcal{O}_S(E)|_E) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow \cdots$$

where we know that $H^0(S, \mathcal{O}_S)$ is 1-dimensional and $H^1(S, \mathcal{O}_S)$ is 0-dimensional. The adjunction formula gives

$$H^0(E, \mathcal{O}_S(E)|_E) \simeq H^0(E, K_E)$$

which is 1-dimensional, and the map $H^0(S, \mathcal{O}(E)) \to H^0(E, \mathcal{O}_S(E)|_E)$ is surjective, so we have

$$h^0(S, \mathcal{O}(E)) \leq 2.$$ 

On the other hand, the Riemann-Roch formula for surfaces shows that $\chi(E) = 2$,
and so
\[ h^0(S, \mathcal{O}(E)) = 2 + h^1(S, \mathcal{O}(E)) \geq 2. \]

Hence \( h^0(S, \mathcal{O}(E)) = 2 \), and there are two linearly independent sections \( E \) and \( E' \).
Looking at their vanishing sets we must have \( E \sim E' \), so
\[ E \cdot E' = E \cdot E = 0, \]
and the curves are disjoint. Thus we have a fibration \( S \to \mathbb{P}^1 \) via the linear system \(|E|\). The induced action of \( \sigma \) on \( \mathbb{P}^1 \) cannot be trivial, as otherwise \( \sigma \) acts trivially on the tangent space of a point on \( E \). Hence, \( \sigma \) has exactly two fixed points on \( \mathbb{P}^1 \), and we know \( E \) and \( E' \) are fixed, so (3.1) becomes
\[ 4 - 0 = 2n - \sum_{i=1}^{k} \chi(C_i) = 6, \]
which is not possible.

Thus, there can only be one non-rational curve on \( S \), fixed by \( \sigma \). Set \( g := \max_{0 \leq i \leq k} (g(C_i)) \). Now (3.1) is
\[ 2n - (2 - 2g) - 2(k - 1) = 6, \]
and so \( g = 3 + k - n. \)

\[ \Box \]

**Note.** A subtle, but important point, is that our fixed locus is a *disjoint* union of curves and isolated points. The Hodge Index Theorem only applies to linearly inequivalent divisors, and if the fixed locus were not a disjoint union, these curves would not necessarily be linearly inequivalent. To see why, we can use an equivalent
version of the Hodge Index Theorem.

**Theorem 3.9.** Let $E, C$ be divisors of $S$ with $E^2 > 0$. Then

(i) $E^2C^2 < (E \cdot C)^2$, or

(ii) $E^2C^2 = (E \cdot C)^2$, and $C \equiv sE$.

To see why this is equivalent to the Hodge Index Theorem, recall that on $S$ the signature of $\text{Pic}(S)$ is $(1, \rho - 1)$, and the span of $E$ and $C$ in $\text{Pic}(S)$ is either 1- or 2-dimensional. In the 2-dimensional case, the signature of

$$
\begin{pmatrix}
E^2 & E \cdot C \\
E \cdot C & C^2
\end{pmatrix}
$$

is $(1, 1)$, so the determinant is negative. On the other hand, if 1-dimensional, then the determinant is 0 and $C \equiv sE$ for some $s$.

In this setting, the fixed curves being disjoint means $(E \cdot C)^2 = 0$ and so any two linearly equivalent curves $E$ and $C$ must have $g(E) = 1$ or $g(C) = 1$. Moreover, the elliptic fibration coming from $|E|$ gives us that any other curve in the fixed locus (which must be disjoint from $E$) must be one of the fibres, so $E \cdot C > 0$ whenever $g(C) > 0$.

Thus, the proof crucially relies on the fixed locus being disjoint, given by Lemma 3.7.

If we are specifically interested in automorphisms that are trivial on the Néron-Severi group of $S$, one can classify the possible fixed loci using the Picard number and the discriminant $a$. This is done in [75].
Determining the Hodge numbers of our generalized Borcea-Voisin threefolds is now straightforward from just knowing the fixed locus. A subtle point to note, however, is that the resolution in the case with involutions is simple because the action of the automorphism on the divisors arising from the resolution is trivial, since both $\iota$ and $\sigma$ act by $-1$ on the (co)tangent space. This is not the case now, as we saw above the action of $\iota \times \sigma^2$ is given by

$$
\begin{pmatrix}
\zeta_3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}
$$

so we must actually study the resolution a little more closely to know what the fixed locus will be.

**Theorem 3.10.** Let $E$ and $S$ be as above, and let $\widetilde{X}$ be a crepant resolution of $(E \times S)/(\iota \times \sigma^2)$. Then

$$h^{1,1}(\widetilde{X}) = r + 1 + 3n + 6k = 7 + 4r - 3a.$$

$$h^{2,1}(\widetilde{X}) = m - 1 + 6g = 43 - 2r - 3a.$$

**Proof.** Recall, if we denote the fixed locus by $D$, then Poincaré duality gives a simple description of the cohomology of the resolution as

$$H^n((E \times S)/(\iota \times \sigma^2)) = H^{n-2}(D) \oplus H^n((E \times S)/(\iota \times \sigma^2)) = H^{n-2}(D) \oplus H^n(E \times S)/\iota \times \sigma^2.$$

Since we are blowing up points and curves, the only non-trivial cases are when $n = 2$
and \( n = 3 \). The right hand side is easy to deal with via the Künneth formula. We have

\[
H^2(E \times S)^{\iota \times \sigma^2}
\]

\[
= (H^2(E) \otimes H^0(S))^{\iota \times \sigma^2} \oplus (H^1(E) \otimes H^1(S))^{\iota \times \sigma^2} \oplus (H^0(E) \otimes H^2(S))^{\iota \times \sigma^2}
\]

\[
= (H^2(E) \otimes H^0(S)) \oplus (H^0(E) \otimes H^2(S)^{\sigma^2}).
\]

The last term is simply the value of \( r \) in the classification of \( S \), so when \( n = 2 \) the space is \((r + 1)\) dimensional. Similarly,

\[
H^3(E \times S)^{\iota \times \sigma^2} = (H^1(E) \otimes H^2(S))^{\iota \times \sigma^2}
\]

\[
= (H^{1,0}(E) \otimes H^2(S)^{\zeta_3^2}) \oplus (H^{0,1}(E) \otimes H^2(S)^{\zeta_3}),
\]

where the notation \( H^2(S)^{\lambda} \) denotes the \( \lambda \)-eigenspace of \( H^2(S) \). As the transcendental lattice of \( S \) has no fixed elements, this is \((22 - r)\)-dimensional.

For the exceptional part, recall the simple application of Poincaré duality described above. Each element in the fixed locus gives a \((1,1)\)-class, while each fixed genus \( g \) curve also contribute \( g \) \((2,1)\)-classes to the cohomology of \( \tilde{X} \). Now we want a crepant resolution, so need to be careful in our resolution.

To see how many fixed components arise in the resolution, note that the action on the tangent space of an isolated fixed point is an action by

\[
\begin{pmatrix}
\zeta_3^2 & 0 & 0 \\
0 & \zeta_3^2 & 0 \\
0 & 0 & \zeta_3^2
\end{pmatrix}
\]
which is trivial, projectively, so after blowing up the fixed points, the exceptional \( \mathbb{P}^2 \)
associated to each is fixed and the singularities are resolved.

To see this resolution is crepant, note that locally around the singularity we have
\( \mathbb{C}^3 \) quotiented by the action above, which is a cone over the triple Veronese embedding
of \( \mathbb{P}^2 \) into \( \mathbb{P}^8 \) given by

\[
[x : y : z] \rightarrow [x^3 : y^3 : z^3 : x^2y : xy^2 : x^2z : xz^2 : y^2z : yz^2]
\]

Embedding a hyperplane \( H \) in general position in the cone \( Y \subset \mathbb{P}^9 \) over the triple
Veronese then gives that \( H|_H \) is linearly equivalent to a cubic.

Denote the blowup of this point by \( f \), giving a resolution \( Z \) of the cone \( Y \). We
have \( K_Z \sim f^*K_Y + dE \), where \( E \) is the exceptional divisor. The adjunction formula
gives

\[
K_E = (K_Z + E)|_E \\
= (f^*K_Y + (d+1)E)|_E \\
= (d+1)E|_E
\]

since \( f^*K_Y|_E = 0 \). Hence, as \( E \simeq \mathbb{P}^2 \), taking degrees gives \(-3 = (d+1)(-3)\) as \(-E|_E \)
is linearly equivalent to the restriction of the hyperplane class of the exceptional \( \mathbb{P}^8 \).
Thus \( d = 0 \), \( K_Z \simeq f^*K_Y \) and the resolution is crepant.

The fixed curve singularities are all transversal \( A_2 \) singularities. Blowing up the
curve once does not resolve the singularity, as the \( \mathbb{P}^1 \) fibers are not fixed by the
(extended) action of \( \iota \times \sigma^2 \). Each \( \mathbb{P}^1 \) fiber only has two fixed points, the origin, and
the point at infinity. Blowing up these two fixed curves now gives two \( \mathbb{P}^1 \) bundles
that are fixed, so the singularities are resolved.

If we denote the exceptional divisor as $D$, then as $D.D < 0$ (being exceptional), the adjunction formula on a crepant resolution gives

$$-4 = \deg K_E = E.E,$$

as (locally) a $\mathbb{P}^1$ bundle is of the form $\mathbb{P}^1 \times \mathbb{P}^1$. Now each exceptional $\mathbb{P}^1$ bundle has self intersection $-2$ by a similar argument, so a crepant resolution requires only two of the $\mathbb{P}^1$-bundles. Hence, we must contract the first exceptional set that was not fixed to get a Calabi-Yau threefold.

As $E'$ consists of three points, we get the first form of the Hodge numbers, i.e.,

$$h^{1,1}(\tilde{X}) = 1 + r + 3n + 6k,$$

$$h^{2,1}(\tilde{X}) = m - 1 + 6g.$$

To get the second form, simply apply the relations in Theorem 3.8.

With these values, it is easy to see there is no notion of mirror symmetry in this family. The possible values of $h^{1,1}$ and $h^{2,1}$ are given in the following table.
### 3.3. Automorphisms of Order Three

<table>
<thead>
<tr>
<th>$r$</th>
<th>$a$</th>
<th>$h_1^{1,1}$</th>
<th>$h_2^{2,1}$</th>
<th>$\chi$</th>
<th>镜像存在？</th>
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<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>15</td>
<td>39</td>
<td>-48</td>
<td>no</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>47</td>
<td>23</td>
<td>48</td>
<td>no</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>79</td>
<td>7</td>
<td>144</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>20</td>
<td>32</td>
<td>-24</td>
<td>no</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>36</td>
<td>24</td>
<td>24</td>
<td>no</td>
</tr>
<tr>
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<td>1</td>
<td>52</td>
<td>16</td>
<td>72</td>
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</tr>
<tr>
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<td>1</td>
<td>68</td>
<td>8</td>
<td>120</td>
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</tr>
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<td>0</td>
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</tr>
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<td>33</td>
<td>-48</td>
<td>no</td>
</tr>
<tr>
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<td>25</td>
<td>25</td>
<td>0</td>
<td>yes (self)</td>
</tr>
<tr>
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<td>17</td>
<td>48</td>
<td>no</td>
</tr>
<tr>
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<td>2</td>
<td>57</td>
<td>9</td>
<td>96</td>
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</tr>
<tr>
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<td>1</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>3</td>
<td>30</td>
<td>18</td>
<td>24</td>
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</tr>
<tr>
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<td>10</td>
<td>72</td>
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<td>2</td>
<td>120</td>
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</tr>
<tr>
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<td>4</td>
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<td>19</td>
<td>0</td>
<td>yes (self)</td>
</tr>
<tr>
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<td>35</td>
<td>11</td>
<td>48</td>
<td>no</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>51</td>
<td>3</td>
<td>96</td>
<td>no</td>
</tr>
<tr>
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<td>5</td>
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</tr>
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<td>4</td>
<td>72</td>
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<td>6</td>
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<td>48</td>
<td>no</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>18</td>
<td>6</td>
<td>24</td>
<td>no</td>
</tr>
</tbody>
</table>
The only cases where a mirror exists are the two self-mirror threefolds. We do not know of a reason for this. Under the symmetry about $r = 12$, one has a partner while the other does not. Their invariants $g$ and $k$ are not unique in the family, while $r$ is (and hence $n$ as well). This corresponds to the rank of $\text{Pic}(S)^\sigma$.

Despite the lack of mirror symmetry in the classifying pyramid, the Euler characteristic still acts as though mirror symmetry occurs here. If $X$ is the generalized Borcea-Voisin variety coming from a K3 surface with invariants $(r, a)$, we have

$$
\chi(\tilde{X}) = 72 - 12r,
$$

so about the $r = 12$ line in the order 3 pyramid, associating $(r, a)$ to $(12 - r, a)$ gives a the threefold with Euler characteristic

$$
12r - 72 = -\chi(\tilde{X}).
$$

On possible explanation for the lack of mirror symmetry here, is the elliptic curve with an automorphism of order 3 is unique up to isomorphism over $\mathbb{C}$, while mirror symmetry is a relation between families of threefolds.

### 3.4 Automorphisms of higher order

We can now ask for a possible mirror correspondence using a generalized Borcea-Voisin construction with automorphisms of (non-prime) order 4 and 6.

These cases have already been studied by Cattaneo and Garbagnati [17], and are more technical, as the automorphisms no longer have prime order. More care needs to be taken to study the type of automorphisms on the K3 surfaces, and there are many
different cases to consider, so we leave the details of the computation to their paper. They were not interested in mirror symmetry, but constructing elliptic fibrations on the threefolds. Nevertheless, they compute the Hodge numbers with respect to the geometric invariants of the singular locus, e.g., how many points and curves are fixed, their genera, etc...

Unfortunately no known Nikulin-style invariants exist, to classify K3 surfaces with automorphisms of order 4 or 6 up to isomorphism, so it is not possible to check mirror symmetry here. Much work towards a classification is done in [3] where K3 surfaces with non-symplectic automorphisms of order 4 whose square is symplectic, or whose fixed locus contains at least a curve and all the curves fixed by the square are rational.

As for non-symplectic automorphisms of order 6, one has work of Dillies [29], where a complete classification is given in terms of the fixed locus of the action, as well as some details about the 'stacky' complication involved in analyzing the moduli space of K3 surfaces with a non-symplectic automorphisms of order 6 using the squares of the respective automorphisms.

3.5 Higher dimensional construction

As the Borcea-Voisin construction is, in a sense, the natural extension of the Kummer construction for K3 surfaces, one can ask if we can go further and construction Calabi-Yau varieties of dimension $n$, for any $n > 3$.

One immediate complication: a crepant resolution is no longer guaranteed as the singularities may have codimension 4, [30].

Recall the classification of K3 surfaces with non-symplectic automorphisms of
3.5. HIGHER DIMENSIONAL CONSTRUCTION

finite order. The same ideas above show that if we consider a quotient

\[(S_1 \times S_2)/(\sigma_1 \times \sigma_2)\]

with \(S_j\) a K3 surface with non-symplectic automorphism \(\sigma_j\), and want it to have a (crepant) Calabi-Yau resolution, then \(\sigma_1\) and \(\sigma_2\) must have the same order. By Theorem 3.5, we can attempt this construction when both \(\sigma_i\) have order \(n\), for some \(2 \leq n \leq 19\). Again, while under consideration we discovered this work was also being consider by Dillies, [28]. He shows that there exist Calabi-Yau fourfolds of this form if and only if the \(\sigma_j\) are involutions, or have order 3 and both K3 surfaces have Nikulin invariant \(r_j = 2\). This comes down to results of Batyrev-Dais [11] and Reid [62] that the K3 surfaces have singular points of type \(\frac{1}{n}(2, n - 1)\), meaning the group locally acts as

\[\left\langle \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \epsilon^{n-1} \end{pmatrix} \right\rangle\]

where \(\epsilon = \exp(2\pi i/n)\), and this gives rise to a point on the fourfold that does not have a crepant resolution.

Thus, we cannot even ask if there is a mirror symmetry relationship when the automorphisms are not involutions.

On the other hand, involutions were the only situation where a mirror relationship was found, so it is still quite natural to ask if this occurs for the fourfolds. Indeed, Dillies shows this to be the case, though his use of orbifold cohomology obscures the symmetry, so we present our geometric approach as above.

**Theorem 3.11.** Let \(S_1\) and \(S_2\) be K3 surfaces with non-symplectic involutions \(\sigma_1\)
and \( \sigma_2 \), respectively. The quotient

\[ X := (S_1 \times S_2)/(\sigma_1 \times \sigma_2) \]

has a crepant resolution \( \tilde{X} \) with Hodge numbers

\[
\begin{align*}
    h^{1,1}(\tilde{X}) &= \frac{1}{4} (r_1 - a_1 + 2)(r_2 - a_2 + 2) + r_1 + r_2 \\
    h^{2,1}(\tilde{X}) &= \frac{1}{4} (22 - r_1 - a_1)(r_2 - a_2 + 2) + \frac{1}{4} (22 - r_2 - a_2)(r_1 - a_1 + 2) \\
    h^{2,2}(\tilde{X}) &= 4 + (20 - r_1)(20 - r_2) + r_1 r_2 \\
    &\quad + \frac{1}{2} (22 - r_1 - a_1)(22 - r_2 - a_2) + \frac{1}{2} (r_1 - a_1 + 2)(r_2 - a_2 + 2).
\end{align*}
\]

where \( (S_j, \sigma_j) \) has Nikulin invariants \((r_j, a_j, \delta_j)\).

As before, the mirror symmetry relation with the K3 surfaces with non-symplectic involutions was

\[
(r, a, \delta) \leftrightarrow (20 - r, a, \delta)
\]

and we see immediately this exchanges \( h^{1,1}(\tilde{X}) \) and \( h^{3,1}(\tilde{X}) \), and preserves \( h^{2,1}(\tilde{X}) \) and \( h^{2,2}(\tilde{X}) \), the desired mirror relationship.

Proof. That \( H^{1,0}(\tilde{X}) = H^{2,0}(\tilde{X}) = H^{3,0}(\tilde{X}) = 0 \), and the canonical bundle being trivial, follows immediately from the \( S_j \) being K3 surfaces. For the rest of the Hodge numbers, note that the fixed locus under the involutions is resolved after a single blowup, and so for \( 2 \leq j \leq 4 \) we have

\[
H^j(\tilde{X}) = H^j(X)^{\sigma_1 \times \sigma_2} \oplus H^{j-2}(S_1^{\sigma_1} \cup S_2^{\sigma_2}).
\]
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Suppose $\sigma_1$ fixes $k$-rational curves $C_1, C_2, \ldots, C_k$ and possibly one curve $C$ of genus $g = (22 - r_1 - a_1)/2$. Similarly, suppose $\sigma_2$ fixes $\ell$-rational curves $D_1, D_2, \ldots, D_\ell$ and possibly one curve $D$ of genus $h = (22 - r_2 - a_2)/2$.

Remark. In the two cases with Nikulin invariants $(10, 10, 0)$ and $(10, 8, 0)$ we have to be slightly careful, but the computation below still works.

With $(10, 10, 0)$ the fixed locus is empty, so we simply ignore all the terms coming from the resolution. When we have $(10, 8, 0)$ there are two fixed curves, both of genus 1, however the genus formula given earlier predicts the highest genus fixed curve has

$$g = \frac{22 - 10 - 8}{2} = 2,$$

and using this gives the same result as though two genus 1 fixed curves are being resolved.

We have

$$H^0(X^\sigma) = H^0(\bigcup_{i,j} C_i \times D_j) = \bigoplus_{i,j} H^0(C_i \times D_j)$$

which has rank $(k + 1)(\ell + 1)$.

$$H^1(X^\sigma) = H^1(\bigcup_{i,j} C_i \times D_j)$$

$$= \bigoplus_{i,j} [H^1(C_i) \otimes H^0(D_j)] \oplus [H^0(C_i) \otimes H^1(D_j)]$$

which has rank $2g(\ell + 1) + 2h(k + 1)$.

$$H^2(X^\sigma) = H^2(\bigcup_{i,j} C_i \times D_j)$$

$$= \bigoplus_{i,j} [H^2(C_i) \otimes H^0(D_j)] \oplus [H^1(C_i) \otimes H^1(D_j)] \oplus [H^0(C_i) \otimes H^2(D_j)]$$
which has rank $2(k + 1)(\ell + 1) + 2gh$. For the quotient, we have

$$H^0(X)^\sigma = H^0(S_1 \times S_2)^\sigma = [H^0(S_1) \otimes H^0(S_2)]^\sigma = H^0(S_1) \otimes H^0(S_2),$$

which has rank 1. We also have $H^1(X)^\sigma = H^3(X)^\sigma = 0$.

Now

$$H^2(X)^\sigma = H^2(S_1 \times S_2)^\sigma$$

$$= [H^{1,1}(S_1) \otimes H^0(S_2)]^\sigma \oplus [H^0(S_1) \otimes H^{1,1}(S_2)]^\sigma$$

which has rank $(20 - r_1) + (20 - r_2)$. For the middle cohomology, we have

$$H^4(X)^\sigma = H^4(S_1 \times S_2)^\sigma$$

$$= [H^4(S_1) \otimes H^0(S_2)] \oplus [H^2(S_1) \otimes H^2(S_2)]^\sigma \oplus [H^0(S_1) \otimes H^4(S_2)].$$

The interesting piece is the middle term, coming from the middle cohomology. We have

$$[H^2(S_1) \otimes H^2(S_2)]^\sigma$$

$$= [H^{2,0}(S_1) \otimes H^{2,0}(S_2)] \oplus [H^{2,0}(S_1) \otimes H^{1,1}(S_2)]^\sigma \oplus [H^{2,0}(S_1) \otimes H^{0,2}(S_2)]$$

$$\oplus [H^{1,1}(S_1) \otimes H^{2,0}(S_2)]^\sigma \oplus [H^{1,1}(S_1) \otimes H^{1,1}(S_2)]^\sigma \oplus [H^{1,1}(S_1) \otimes H^{0,2}(S_2)]^\sigma$$

$$\oplus [H^{0,2}(S_1) \otimes H^{2,0}(S_2)] \oplus [H^{0,2}(S_1) \otimes H^{1,1}(S_2)]^\sigma \oplus [H^{0,2}(S_1) \otimes H^{0,2}(S_2)].$$
which has rank $4 + 2(20 - r_1) + 2(20 - r_2) + (20 - r_1)(20 - r_2) + r_1r_2$, leaving

$$h^4(X)^\sigma = 6 + 2(20 - r_1) + 2(20 - r_2) + (20 - r_1)(20 - r_2) + r_1r_2.$$  

Note the $(2, 2)$-forms comprise of $4 + (20 - r_1)(20 - r_2) + r_1r_2$ of these, so that $h^{3, 1}(X)^\sigma = 40 - r_1 - r_2$.

As for the resolution, note that for a fixed divisor $D$ on $X$ we still have the cohomological pushforward on

$$\begin{array}{c}
\tilde{D} \\
\downarrow f
\end{array} \longrightarrow \begin{array}{c}
D \\
\tilde{S}_1 \times \tilde{S}_2
\end{array}$$

where $\tau$ is the blowup of $X/\sigma$ restricted to $D$ and $\tilde{D}$ and $f$ is an inclusion. Thus in cohomology, we have each element $\xi$ of $H^{1, 1}(D)$ giving an element $f_!(\tau^*(\xi))$ in $H^{2, 2}(\tilde{X})$, i.e.,

$$h^{2, 2}(\tilde{X}) = 4 + (20 - r_1)(20 - r_2) + r_1r_2 + gh$$

$$= 4 + (20 - r_1)(20 - r_2) + r_1r_2 + \frac{1}{2}(22 - r_1 - a_1)(22 - r_2 - a_2)$$

and

$$h^{3, 1}(\tilde{X}) = 40 - r_1 - r_2 + \frac{1}{4}(r_1 - a_1 + 2)(r_2 - a_2 + 2).$$
Putting all these together, we see our fourfold has Hodge numbers

\[ h^{1,1}(\tilde{X}) = \frac{1}{4}(r_1 - a_1 + 2)(r_2 - a_2 + 2) + r_1 + r_2 \]
\[ h^{2,1}(\tilde{X}) = \frac{1}{4}(22 - r_1 - a_1)(r_2 - a_2 + 2) + \frac{1}{4}(22 - r_2 - a_2)(r_1 - a_1 + 2) \]
\[ h^{3,1}(\tilde{X}) = 40 - r_1 - r_2 + \frac{1}{4}(22 - r_1 - a_1)(22 - r_2 - a_2) \]
\[ h^{2,2}(\tilde{X}) = 4 + (20 - r_1)(20 - r_2) + r_1 r_2 + \frac{1}{2}(22 - r_1 - a_1)(22 - r_2 - a_2) + \frac{1}{2}(r_1 - a_1 + 2)(r_2 - a_2 + 2). \]

as desired, with the mirror relationship clear.

Since we have had so much luck with involutions, it is interesting to note that attempting to search for mirror symmetry among fourfolds of the form

\[(E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle\]

fails, where \( E_1 \) and \( E_2 \) are elliptic curves with the respective hyperelliptic involutions, and \( S \) is a K3 surface with non-symplectic involution \( \sigma \). Hence, the remark about mirror symmetry with the threefolds using automorphisms of order 3 failing due to a lack of underlying family in the construction may not be the entire story.

Indeed, for fourfolds coming from the product \( E_1 \times E_2 \times S \), note that each elliptic curve \( E_j \) has 4 fixed points under \( \iota_j \), and so every fixed curve in \( S^\sigma \) gives rise to 16 singular curves on the threefold. With computations as above, one relates the Hodge numbers to the Nikulin invariants of \( (S, \sigma) \) as follows.
3.5. HIGHER DIMENSIONAL CONSTRUCTION

**Theorem 3.12.** Let $E_1$ and $E_2$ be elliptic curves, and let $\iota_j$ be their respective hyperelliptic involutions. Let $S$ be a $K3$ surface with non-symplectic involution $\sigma$ corresponding to the Nikulin triple $(r, a, \delta)$. Then the quotient

$$Y = (E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle$$

has a crepant resolution with Hodge numbers

$$h^{1,1} = 23 + \frac{3r - a}{2},$$

$$h^{2,1} = 53 - \frac{5r + a}{2},$$

$$h^{3,1} = 42 + k,$$

$$h^{2,2} = r,$$

where $S^\sigma$ consists of $k$ rational curves and one of genus $g$.

**Proof.** This is essentially no different from the original Borcea-Voisin construction with the quotient $(E \times S)/\langle \iota \times \sigma \rangle$. The only notable difference, other than there being 16 fixed points on $E_1 \times E_2$ instead of 4 on $E$ is that the 1-forms of $E_1$ and $E_2$ combine to give more forms on the fourfold.

Indeed, we have

$$h^{1,1}((E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle) = 23 + r,$$

$$h^{2,1}((E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle) = 2 + 2(20 - r),$$

$$h^{2,2}((E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle) = 2r + 2(20 - r) + 1,$$

$$h^{3,1}((E_1 \times E_2 \times S)/\langle \iota_1 \times \iota_2 \times \sigma \rangle) = r,$$
The only different part with the resolution is that the $(1, 1)$-classes on the singular curves give rise to $(2, 2)$-classes on the fourfold, which is not already accounted for by Poincaré duality as on threefolds. With $k$ fixed rational curves, and a genus

$$g = \frac{22 - r - a}{2}$$

curve in $S^\sigma$, we get the desired Hodge numbers.

We can try using the product of an elliptic curve with a Calabi-Yau threefold, however the lack of classification of these threefolds make it quite unlikely we would have any viable Nikulin-style invariants to use in the computation.
Chapter 4

A Generalized Borcea Construction

In this chapter we move from the Borcea-Voisin construction, to the construction of Calabi-Yau threefolds Borcea was initially studying. Recall, the original Borcea construction was to consider three elliptic curves $E_1, E_2$ and $E_3$ each with a non-symplectic involution $\iota$, and then resolve the quotient

$$(E_1 \times E_2 \times E_3)/\langle \iota \times \iota \times \text{id}, \iota \times \text{id} \times \iota \rangle.$$ 

We can generalize this combining different automorphisms, as in Chapter 3, except that unlike before we no longer require the order of all the automorphisms to be equal. Using complex multiplication (CM) automorphisms will enable us to construct numerous examples.

The Borcea family is a three parameter family of non-rigid Calabi-Yau threefolds. This is already a large space of complex deformations, but we will see that the CM automorphisms reduce this in a stratified way, to make the middle cohomology, and hence the threefolds, easier to study. This will allow us to study arithmetic properties of models defined over $\mathbb{Q}$. 
4.1 Construction of threefolds

Over \( \mathbb{C} \), there is only one elliptic curve with an automorphism of order 3 and one elliptic curve with an automorphism of order 4, up to isomorphism. Denote these by \( E_3 \) and \( E_4 \), with their respective CM automorphisms \( \iota_3 \) and \( \iota_4 \), and note that \( \iota_6 := -\iota_3 \) is an automorphism of order 6 on \( E_6 = E_3 \). (We will use both \( E_3 \) and \( E_6 \) to emphasize whether we are considering the curve as only having an automorphism of order 3, or the full automorphism of order 6.)

On the triple product \( E_j^3 := E_j \times E_j \times E_j \) we have an action of the group

\[
G_j := \langle \iota_j \times \iota_j^{-1} \times \text{id}, \iota_j \times \text{id} \times \iota_j^{-1} \rangle \cong \mathbb{Z}/j\mathbb{Z} \times \mathbb{Z}/j\mathbb{Z}
\]

for \( j = 3, 4 \) and 6. As \( G_j \) preserves the holomorphic three-form of \( E_j^3 \) we have \( h^{3,0}(E_j^3/G_j) = 1 \) and \( h^{1,0}(E_j^3/G_j) = 0 \), so a crepant resolution of \( E_j^3/G_j \) is a Calabi-Yau threefold. As \( E_j^3/G_j \) is a quotient of dimension 3, such a crepant resolution exists by [16]. The same is true for many subgroups of \( G_j \) but the geometry varies widely with the choice of subgroup.

**Theorem 4.1.** Consider the following groups of automorphisms acting on \( E_6^3 \):

\[
G_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6 \times \text{id} \times \iota_6^5 \rangle, \quad H_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^2 \times \text{id} \times \iota_6^4 \rangle,
\]

\[
I_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^4 \times \iota_6^5 \times \text{id} \rangle, \quad J_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6^5 \times \iota_6^5 \times \iota_6^3 \rangle,
\]

\[
K_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^3 \times \text{id} \times \iota_6^3 \rangle, \quad L_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6 \rangle,
\]

\[
M_6 = \langle \iota_6^2 \times \iota_6^2 \times \iota_6^2 \rangle, \quad N_6 = \langle \iota_6 \times \iota_6 \times \iota_6^3 \rangle, \quad O_6 = \langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle.
\]
4.1. CONSTRUCTION OF THREEFOLDS

Crepant resolutions of the respective quotients are Calabi-Yau threefolds with Hodge numbers

\begin{align*}
    h^{1,1}(\tilde{E}_6^3/G_6) &= 84, & h^{2,1}(\tilde{E}_6^3/G_6) &= 0, \\
    h^{1,1}(\tilde{E}_6^3/H_6) &= 84, & h^{2,1}(\tilde{E}_6^3/H_6) &= 0, \\
    h^{1,1}(\tilde{E}_6^3/I_6) &= 73, & h^{2,1}(\tilde{E}_6^3/I_6) &= 1, \\
    h^{1,1}(\tilde{E}_6^3/J_6) &= 51, & h^{2,1}(\tilde{E}_6^3/J_6) &= 3, \\
    h^{1,1}(\tilde{E}_6^3/K_6) &= 51, & h^{2,1}(\tilde{E}_6^3/K_6) &= 3, \\
    h^{1,1}(\tilde{E}_6^3/L_6) &= 36, & h^{2,1}(\tilde{E}_6^3/L_6) &= 0, \\
    h^{1,1}(\tilde{E}_6^3/M_6) &= 36, & h^{2,1}(\tilde{E}_6^3/M_6) &= 0, \\
    h^{1,1}(\tilde{E}_6^3/N_6) &= 35, & h^{2,1}(\tilde{E}_6^3/N_6) &= 11, \\
    h^{1,1}(\tilde{E}_6^3/O_6) &= 29, & h^{2,1}(\tilde{E}_6^3/O_6) &= 5.
\end{align*}

Remark. All of the Hodge numbers using $G_6, H_6, L_6$ and $M_6$ can be found in [33] as well as references therein. The pair $(h^{1,1}, h^{2,1}) = (73, 1)$ can be found in [34] and [53], and a large set of pairs from a toric construction including $(35, 11), (29, 5)$ can be found in [53]. Lastly, as mentioned above, the pair $(51, 3)$ is the original Borcea construction [14]. These exhaust all the Calabi-Yau threefolds one can obtain from this construction, up to isomorphism, noting that any subgroup of $G_6$ that does not act trivially on one coordinate is isomorphic to one of the above subgroups.

Remark. While the rigid examples cannot have Calabi-Yau mirror partners, all of the non-rigid examples have (topological) mirrors that have been constructed in the literature. All mirror pairs except $(1, 73)$ can be found in the toric construction of [53], while the last mirror can be found in [7] which constructs Calabi-Yau varieties.
and their mirrors via conifold transitions.

**Proof.** As all the examples are similar, we only look at the cyclic example

\[ O_6 = \langle \iota_4 \times \iota_6 \times \iota_6 \rangle \]

which contains all the geometry necessary for the resolution of each example.

We must investigate the fixed points under the action of each element of this group, so we break things up into steps.

By continuously extending the appropriate automorphisms we have a birational diagram

\[
\begin{array}{ccc}
E_6^3 & \rightarrow & E_6^3/O_6 \\
\downarrow & & \downarrow \\
E_6^3/M_6 & \leftarrow & E_\sim^3/M_6 \rightarrow \left( E_6^3/M_6 \right) / \langle \bar{O}_6 \rangle
\end{array}
\]

with which we can resolve our threefold in straightforward manner. Note that at each step we will be blowing up fixed points or fixed curves, so \( h^{1,0} \) is fixed. Moreover, we blowup precisely the ramification locus of the quotient and so our resolution is crepant, and the resulting threefold is Calabi-Yau.

**Step 1.**

The K"unneth formula gives

\[ h^{1,1}(E_6^3/M_6) = 9, \quad \text{and} \quad h^{2,1}(E_6^3/M_6) = 0, \]
and the resolution involves blowing up the 27 fixed points. Using Theorem 2.2 to compute the cohomology of a resolution, or (picking a model, and) looking at an affine patch explicitly and seeing the induced action on the exceptional $\mathbb{P}^2$ is trivial, we find

\[ h^{1,1}(\widetilde{E^3_6}/M_6) = 36, \quad \text{and} \quad h^{2,1}(\widetilde{E^3_6}/M_6) = 0. \]

Moreover, this is the triple Veronese resolution from chapter 3, so we again have this resolution is crepant.

**Step 2.**

We now quotient this resolution by (continuous extensions of) the remaining elements in $O_6$ to see what $(1,1)$-classes remain. Note that only 5 classes from the Künneth formula are preserved. Furthermore, the action of $id \times \iota_6^3 \times \iota_6^3$ identifies many of the 27 exceptional divisors from the previous blowup, so that

\[ h^{1,1}((\widetilde{E^3_6}/M_6)/O_6) = 20, \quad \text{and} \quad h^{2,1}((\widetilde{E^3_6}/M_6)/O_6) = 0. \]

**Step 3.**

The final resolution can now be done with two separate blowups. We start with the three codimension 3 subvarieties fixed under $\iota_6^4 \times \iota_6 \times \iota_6$, and then the six codimension 2 subvarieties from $id \times \iota_6^3 \times \iota_6^3$. If we denote by $O$ the identity of $E_6$ as a group, the point $(O, O, O)$ is fixed by each automorphism in $O_6$. After the blowup of the fixed locus of $\iota_6^2 \times \iota_6^2 \times \iota_6^2$ we have a $\mathbb{P}^2$ lying over these points, but these are not fixed by $\iota_6^4 \times \iota_6 \times \iota_6$ or $id \times \iota_6^3 \times \iota_6^3$. Instead, only the $\mathbb{P}^1$ corresponding to the latter two coordinates is fixed, so the fixed locus includes a $\mathbb{P}^1$ as well as the remaining fixed
points. There are four fixed points $O, P_1, P_2, P_3$ on $E_6$ under the involution $\iota_6^3$, but $\iota_6^3(P_1) = P_2$, $\iota_6^3(P_2) = P_3$ and $\iota_6^3(P_3) = P_1$. Hence, the fixed locus of (the continuous extension of) $\text{id} \times \iota_6^3 \times \iota_6^3$ on $(\overline{E_6^3/M_6})/O_6$ are the rational curves above, as well as the (genus 1) fixed curves

$$E_6 \times O \times P_1,$$

$$E_6 \times P_1 \times O,$$

$$E_6 \times P_1 \times P_1,$$

$$E_6 \times P_1 \times P_2,$$

$$E_6 \times P_1 \times P_3.$$

Each of these is a transversal $A_1$ singularity, so (crepant) resolve after a single blowup, and $\iota_6^4 \times \iota_6 \times \iota_6$ fixes all of the exceptional divisors, so a crepant resolution of $E_6^3/O_6$ has Hodge numbers

$$h^{1,1}(\overline{E_6^3/O_6}) = 29 \quad \text{and} \quad h^{2,1}(\overline{E_6^3/O_6}) = 5,$$

as desired.

Note that for each subgroup $H$ of $G_3$, the variety $E_3^3/H$ is isomorphic to $E_6^3/J$ for some subgroup $J$ of $G_6$, so the above covers all the Calabi-Yau threefolds that arise using $E_3$ and $G_3$ as well.

All of these Hodge pairs, except for $(36, 0)$, can be found using a generalization
of a construction studied by Borcea [14] and Voisin [77]. For example, with the pair
$(h^{1,1}, h^{2,1}) = (29, 5)$ we can look at the the action of

$$\langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle = \langle \iota_6^4 \times \iota_6 \times \iota_6, \text{id} \times \iota_6^3 \times \iota_6^3 \rangle$$

first as a “birational Kummer construction” taking the quotient of $E_3^3$ by $\text{id} \times \iota_6^3 \times \iota_6^3$, and then noting the induced action of $\iota_6^4 \times \iota_6 \times \iota_6$ is simply $\iota_3 \times \iota_3 \times \iota_3$ which is a generalized Borcea-Voisin threefold. See [34].

Similarly, one can find the Calabi-Yau threefolds using this construction with $E_3^3$.

**Theorem 4.2.** Consider the groups of automorphisms

$$G_4 = \langle \iota_4 \times \iota_4^3 \times \text{id}, \iota_4 \times \text{id} \times \iota_4^3 \rangle, \quad H_4 = \langle \iota_4 \times \iota_4 \times \iota_4^3, \iota_4 \times \iota_4^3 \times \text{id} \rangle,$$

$$I_4 = \langle \iota_4^2 \times \iota_4^3 \times \text{id}, \iota_4^2 \times \text{id} \times \iota_4^3 \rangle, \quad J_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2 \rangle,$$

acting on the threefold $E_3^3$. Crepant resolutions of the respective quotients are Calabi-Yau threefolds with Hodge numbers

$$h^{1,1}(\widetilde{E_3^3}/G_4) = 90, \quad h^{2,1}(\widetilde{E_3^3}/G_4) = 0,$$
$$h^{1,1}(\widetilde{E_3^3}/H_4) = 61, \quad h^{2,1}(\widetilde{E_3^3}/H_4) = 1,$$
$$h^{1,1}(\widetilde{E_3^3}/I_4) = 51, \quad h^{2,1}(\widetilde{E_3^3}/I_4) = 3,$$
$$h^{1,1}(\widetilde{E_3^3}/J_4) = 31, \quad h^{2,1}(\widetilde{E_3^3}/J_4) = 7.$$

**Remark.** Again, this exhausts all the possible Calabi-Yau threefolds arising from this generalized Borcea construction with $E_4$, up to isomorphism, as any subgroup of $G_4$ for which a crepant resolution of the quotient is Calabi-Yau is isomorphic to one of the above groups.
The example with $G_4$ is seen in [22], while $H_4$ is studied in [34] as a generalized Borcea-Voisin construction, and $I_4$ is the Borcea construction again, so we will only prove the result for the cyclic example with $J_4$.

**Proof.** The main difference between this and the previous construction with $E_6$ is, instead of combining the automorphisms of order 2 and 3, we now have the fixed points of $E_4$ satisfying

$$E_4^{i_4} \subset E_4^{i_2^2}. \quad (4.1)$$

In particular, we may denote the fixed points of the involution $i_4$ by $P_0, P_1, P_2$ and $P_3$, where $i_4(P_2) = P_3$ and $i_4(P_3) = P_2$, while $i_4$ fixes both $P_0$ and $P_1$.

To compute the Hodge numbers, we start by noting the Küneth formula gives

$$h^{1,1}(E_4^3/\langle i_4 \times i_4 \times i_4^2 \rangle) = 5 \quad \text{and} \quad h^{2,1}(E_4^3/\langle i_4 \times i_4 \times i_4^2 \rangle) = 1.$$

We first blowup the fixed curves, resolving the singularities under the action of $i_4^2 \times i_4^2 \times \text{id}$. This fixes the $\mathbb{P}^1$ fiber in the exceptional divisor corresponding to each point fixed under $i_4 \times i_4 \times i_4^2$ (and acting as an involution on the other fibers) as transversal $A_1$ singularities as well. Hence, once again, a single blowup gives a crepant resolution.

Each $P_j \times P_k \times E_4$ fixed under $i_4 \times i_4 \times \text{id}$, where $0 \leq j, k \leq 1$, corresponds to a rational curve in the quotient, while the other twelve $P_j \times P_k \times E_4$ are identified in pairs under the action of the $i_4$ on the first two coordinates, and remain genus 1 curves in the quotient. Thus, we find

$$h^{1,1}(\widetilde{E_4}/J_4) = 5 + 10 + 16 = 31 \quad \text{and} \quad h^{2,1}(\widetilde{E_4}/J_4) = 1 + 6 = 7,$$
4.1. CONSTRUCTION OF THREEFOLDS

the desired Hodge numbers.

Remark. The Hodge pairs \((90, 0)\) and \((61, 1)\) can again be found in [34] while \((61, 1)\) also arises from a toric construction in [53], as well as \((31, 7)\), and these both have topological mirrors in [7]. The Borcea example with \((51, 3)\) completes the list once more.

Complete proof of Theorem 4.1

For completeness, we calculate the Hodge numbers claimed in Theorem 4.1 here.

Recall the subgroups of interest

\[
G_6 = \langle \iota_6 \times \iota_6^2 \times \text{id}, \iota_6 \times \text{id} \times \iota_6^2 \rangle, \quad H_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^2 \times \text{id} \times \iota_6^4 \rangle,
\]

\[
I_6 = \langle \iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6 \rangle, \quad J_6 = \langle \iota_6 \times \iota_6^5 \times \text{id}, \iota_6^4 \times \iota_6^5 \times \iota_6^3 \rangle,
\]

\[
K_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^3 \times \text{id} \times \iota_6^3 \rangle, \quad L_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6 \rangle,
\]

\[
M_6 = \langle \iota_6^2 \times \iota_6^2 \times \iota_6^2 \rangle, \quad N_6 = \langle \iota_6 \times \iota_6^2 \times \iota_6^3 \rangle, \quad O_6 = \langle \iota_6^4 \times \iota_6 \times \iota_6 \rangle.
\]

For simplicity, we start with the cyclic examples \(M_6\) and \(N_6\) (with \(O_6\) in the proof of Theorem 4.1). In what follows, it is helpful to recall there is one fixed point of \(E_6\) under \(\iota_6\), say \(O\). Then under \(\iota_6^2\) there are three fixed points, \(Q_1, Q_2\) and the point \(O\) again. Lastly, under the involution \(\iota_6^3\) there are four fixed points, \(P_1, P_2, P_3\) and \(O\) again. Also, up to labeling we have

\[
\iota_6(P_1) = P_2 = \iota_6^2(P_3),
\]
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\[ \iota_6(P_2) = P_3 = \iota_6^2(P_1), \]
\[ \iota_6(P_3) = P_1 = \iota_6^2(P_2), \]
as well as
\[ \iota_6^3(Q_1) = Q_2 \quad \text{and} \quad \iota_6^3(Q_2) = Q_1. \]

The quotient by \( M_6 \) is perhaps the simplest (and, in fact, implicit in the proof of Theorem 4.1). Indeed, each of the \((1, 1)\)-forms is fixed by \( M_6 \), and there are 27 fixed points that all resolve after a single blowup. Hence

\[ h^{1,1}(E_6^3/M_6) = 9 + 27 = 36. \]

We now address the resolution with \( N_6 \). Here the action by \( \iota_6 \times \iota_6^2 \times \iota_6^3 \) has 12 fixed points that are all fixed by the other elements of \( N_6 \).

To get a crepant resolution of the quotient, we use the birational diagram

\[
\begin{array}{ccc}
E_6^3 & \rightarrow & E_6/N_6 \\
\downarrow & & \downarrow \\
E_6^3/\langle \iota_6^2 \times \iota_6^4 \times \text{id} \rangle & \leftarrow & E_6^3/\langle \iota_6^2 \times \iota_6^4 \times \text{id} \rangle/\langle \iota_6^3 \times \text{id} \times \iota_6^3 \rangle
\end{array}
\]

The only \((2, 1)\)-form fixed by \( N_6 \) is \( dz_1 \wedge dz_2 \wedge dz_3 \), and as for the \((1, 1)\)-forms, only those coming from a \((1, 1)\)-form on one of the underlying \( E_6 \) are fixed.

Now, the fixed locus of \( E_6^3 \) under \( \iota_6^2 \times \iota_6^4 \times \text{id} \) is 9 genus 1 curves. Blowing these up does not resolve the singularities, and gives rise to another genus 1 curve lying above each of the previous 9 curves. Blowing up the 18 fixed curves resolves the singularities, and we contract the first exceptional \( \mathbb{P}^1 \) so the resolution will be Calabi-Yau.
The fixed locus of $\iota_6 \times \iota_6^2 \times \iota_6^3$ is now, instead of 12 fixed points, 12 rational curves, and resolves with a single blowup.

Lastly, the quotient by $\iota_6^3 \times \text{id} \times \iota_6^3$ identifies $2 \cdot 6$ of the elliptic ruled surfaces above in pairs, to $2 \cdot 3$ distinct (elliptic ruled) surfaces, as well as converting the other $2 \cdot 3$ to rational ruled surfaces. The fixed locus of $\iota_6^3 \times \text{id} \times \iota_6^3$ is now 4 rational curves and 4 elliptic curves, after the identification by $\iota_6^2 \times \iota_6^4 \times \text{id}$, and all of these resolve after a single blowup.

Hence, blowing all of the singularities in $E_6^3/N_6$ in the order above gives

$$h^{1,1}(\widetilde{E_6^3/N_6}) = 3 + 2 \cdot 6 + 4 + 4 + 12 = 35,$$

$$h^{2,1}(\widetilde{E_6^3/N_6}) = 1 + 2 \cdot 3 + 4 = 11.$$

We now use these cyclic computations to our advantage to compute the Hodge numbers with the non-cyclic groups.

For $L_6$, we may use our work with $N_6$ above, as we have

$$
\begin{array}{ccccc}
E_6^3 & \rightarrow & E_6^3/L_6 & \leftarrow & \widetilde{E_6^3/L_6} \\
\downarrow & & \downarrow & & \downarrow \\
E_6^3/N_6 & \leftarrow & \widetilde{E_6^3/N_6} & \rightarrow & \widetilde{E_6^3/N_6/L_6}
\end{array}
$$

We note that the contribution from the K"unneth formula remains, as the three $(1,1)$-forms on each $E_6$ are still preserved by $L_6$.

The 15 fixed projective planes lying over the fixed points of $\iota_6^2 \times \iota_6^2 \times \iota_6^2$ are identified to 8 distinct classes, the 3 fixed planes lying over the fixed points of $\iota_6^4 \times \iota_6 \times \iota_6$ are identified to 2 distinct classes, and the 5 elliptic ruled surfaces from $\text{id} \times \iota_6^3 \times \iota_6^3$ become
rational ruled surfaces under the involution $\iota_6^3 \times \iota_6^3 \times \text{id}$.

As to the singularities coming from the new elements, note that $L_6$ contains all permutations of $\iota_6^4 \times \iota_6 \times \iota_6$, and $\text{id} \times \iota_6^3 \times \iota_6^3$, so we get three times as many $(1,1)$-classes coming from their contributions. Altogether, this means

$$h^{1,1}(\widetilde{E_6^3}/L_6) = 3 + 8 + 3 \cdot 2 + 3 \cdot 6 = 35,$$

and

$$h^{2,1}(\widetilde{E_6^3}/L_6) = 0.$$

The group $K_6$ is the original Borcea construction, dealt with in Chapter 3.

The group $J_6$ contains $N_6$ as a subgroup, so we can use this as a starting point. The singular points on $E_6^3$ fixed under $\iota_6^2 \times \iota_6 \times \iota_6^3$ become fixed rational curves on $\widetilde{E_6^3}/N_6$. These and the fixed curves under $\iota_6 \times \iota_6^5 \times \text{id}$ all resolve after a single blowup. Similarly, independent of this, the fixed curves under $\iota_6^3 \times \iota_6^3 \times \text{id}$ and $\text{id} \times \iota_6^3 \times \iota_6^3$ all resolve after a single blowup, and so it is only a matter of counting how many classes descend to the quotient with $J_6$ as well as how many ‘new’ singularities under $J_6$ have already been identified by $N_6$.

The $2 \cdot 3$ rational ruled surfaces under $\iota_6^2 \times \iota_6^4 \times \text{id}$ are identified into $2 \cdot 2$, and the $2 \cdot 3$ elliptic ruled surfaces are identified into $2 \cdot 2$ ruled surfaces, two still elliptic, two rational in the quotient.

The four elliptic ruled surfaces coming from $\iota_6^3 \times \text{id} \times \iota_6^3$ become rational ruled surfaces in the quotient, and with the other four rational ruled surfaces we still have 8 classes in the quotient.

The fixed planes from $\iota_6 \times \iota_6^2 \times \iota_6^3$ identify to 8 distinct classes, as well as the fixed $\mathbb{P}^2$ under $\iota_6^2 \times \iota_6 \times \iota_6^3$ are 8 distinct planes in $\widetilde{E_6^3}/N_6$.
The elliptic curves fixed by \( \iota_6 \times \iota_6^5 \times \text{id} \) become rational in the quotient (there are two curves because this uses the same charts when we blew up the fixed curves under \( \iota_6^2 \times \iota_6^4 \times \text{id} \) at which we had an extra curve ‘at infinity’ fixed), as well as all the elliptic curves fixed by \( \iota_6^3 \times \iota_6^8 \times \text{id} \) and \( \text{id} \times \iota_6^3 \times \iota_6^3 \). The curves under the former identify into 6 classes, while the latter are identified into 8 classes.

Putting everything together, we have

\[
h^{1,1}(\tilde{E}_6^3/J_6) = 3 + 2 \cdot 2 + 2 \cdot 2 + 8 + 8 + 8 + 2 \cdot 1 + 6 + 8 = 51, \quad \text{and} \]

\[
h^{2,1}(\tilde{E}_6^3/J_6) = 1 + 2 = 3.
\]

For \( I_6 \), no \((2,1)\)-classes of \( E_6^3 \) are fixed by \( I_6 \), and only \((1,1)\)-classes with trivial action are preserved. We can investigate the resolution with the diagram

\[
\begin{array}{ccc}
E_6^3 & \longrightarrow & E_6^3/I_6 \\
\downarrow & & \downarrow \\
E_6^3/O_6 & \longleftarrow & \tilde{E}_6^3/O_6 \\
\end{array}
\longrightarrow
\begin{array}{ccc}
\tilde{E}_6^3/\tilde{I}_6 & \longleftarrow & \tilde{E}_6^3/O_6/I_6
\end{array}
\]

In the quotient with \( I_6 \), the 27 points fixed under \( \iota_6^2 \times \iota_6^2 \times \iota_6^2 \) are identified to 15 points. The 3 points fixed under \( \iota_6^4 \times \iota_6 \times \iota_6 \) remain distinct, and the 16 genus 1 curves fixed under \( \text{id} \times \iota_6^3 \times \iota_6^3 \) are identified into 3 rational curves and 1 genus 1 curve (coming from \( E \times \{\text{pt}\} \times \{\text{pt}\} \)).

The 9 rational curves fixed under \( \iota_6^3 \times \text{id} \times \iota_6^4 \) as well as the 9 rational curves fixed under \( \iota_6^2 \times \iota_6^4 \times \text{id} \) are identified into 6 and 6 distinct rational curves respectively, while the 9 rational curves fixed under \( \text{id} \times \iota_6^2 \times \iota_6^4 \) are identified into 5 rational curves.

The 12 points fixed under \( \iota_6^2 \times \iota_6^3 \times \iota_6 \) and the 12 points fixed under \( \iota_6^2 \times \iota_6 \times \iota_6^3 \) are
identified into 6 and 6 points respectively, and lastly, there is only one rational curve fixed by \( \text{id} \times \iota_6 \times \iota_6^5 \).

The rational curves fixed by any permutation of \( \iota^2_6 \times \iota^4_6 \times \text{id} \) do not resolve after a single blowup, and give another fixed curve after the first blowup. Blowing up these two resulting curves resolves the singularities. Similarly, the rational curve fixed by \( \text{id} \times \iota_6 \times \iota_6^5 \) is two fixed curves after the blowup with \( \text{id} \times \iota^2_6 \times \iota^4_6 \), and one blowup resolves these. To make the resolution Calabi-Yau, we contract all of the rulings that are not fixed. The rest of the singularities now resolve after a single blowup, so altogether, we have

\[
h^{1,1}(\tilde{E}_6^3/I_6) = 3 + 15 + 3 + 4 + 2 \cdot 6 + 2 \cdot 6 + 2 \cdot 5 + 6 + 6 + 2 = 73,
\]

\[
h^{2,1}(\tilde{E}_6^3/I_6) = 1.
\]

With \( H_6 \), first note that each \((1,1)\)-forms in \( H^2(E_6^3) \) is either fixed by \( H_6 \) if it comes from a \((1,1)\)-form on one of the \( E_6 \), and otherwise not fixed by the action of the group.

The fixed locus contains 27 points fixed under the action of \( \iota^2_6 \times \iota^2_6 \times \iota^2_6 \) and \( \iota^4_6 \times \iota^4_6 \times \iota^4_6 \). These all resolve after a single blowup. The remaining elements in the fixed locus are 27 rational curves that blowup a single \( \mathbb{P}^1 \) that is not fixed, though the origin and point at infinity are, giving another fixed rational curve ‘at infinity’. Blowing up these two rational curves now resolved the singularities, and contracting the rulings that were not fixed, makes the resolution Calabi-Yau, giving

\[
h^{1,1}(\tilde{E}_6^3/H_6) = 3 + 27 + 2 \cdot 27 = 84.
\]
Lastly, for $G_6$, note that every permutation of every element is in the group, so all fixed curves are rational, and we need only count how many singularities are identified for a single element in each permutation class.

Using $J_6$ as a starting to point, to see what resolutions to do in what order (for each permutation) we only have the resolutions by $\iota_6^4 \times \iota_6 \times \iota_6$ and $\iota_6^2 \times \iota_6^2 \times \iota_6^2$ left. There are two fixed points under $\iota_6^4 \times \iota_6 \times \iota_6$ and 8 fixed points under $\iota_6^2 \times \iota_6^2 \times \iota_6^2$, all of which resolve after a single blowup, and so we find

$$h^{1,1}(\widetilde{E_6^3}/G_6) = \sum a_n q^n$$

$$h^{1,1}(\widetilde{E_6^3}/G_6) = 3 + 6 \cdot 4 + 6 \cdot 1 + 6 \cdot 4 + 3 \cdot 4 + 3 \cdot 2 + 1 \cdot 9 = 84,$$

$$h^{2,1}(\widetilde{E_6^3}/G_6) = 0.$$
automorphisms

\[ E_3 : y^2 = x^3 - 1 \quad \iota_3 : (x, y) \mapsto (\zeta_3 x, y), \]
\[ E_4 : y^2 = x^3 - x \quad \iota_4 : (x, y) \mapsto (-x, iy), \]

where \( \zeta_3 \) is a fixed primitive third root of unity.

Note that the orbits of each of the quotients in Theorems 4.1 and 4.2 are fixed under the action of the absolute Galois group \( G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), so each of the threefolds in this construction is defined over \( \mathbb{Q} \).

Let \( \mathbb{Q}_\ell \) denote the field of \( \ell \)-adic numbers. As our threefolds require the CM automorphisms that are not defined over every \( \mathbb{Q}_\ell \), we will always work with a base extension to \( \overline{\mathbb{Q}_\ell} \). We will write

\[ L(X, s) := L(H^3(X), s) \]

where \( H^k_{\ell}(X) = H^k_{\ell}(X \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}, \overline{\mathbb{Q}_\ell}) \).

**Theorem 4.3.** Let \( H \) be a subgroup of \( G_3 \) such that \( X_3 \), a crepant resolution of \( E^3_3/H \), is a rigid Calabi-Yau threefold defined over \( \mathbb{Q} \). We have

\[ L(X_3, s) = L(s, \chi^3_3) \]

where \( \chi_3 \) is the Hecke character of \( E_3 \), i.e., such that

\[ L(E_3, s) = L(s, \chi_3). \]

**Proof.** With the rigid cases, the resolution does not add any classes to the middle
4.2. MODULARITY OF RIGID THREEFOLDS OVER \( \mathbb{Q} \)

cohomology. Hence, by [67] we have

\[
H^3_\ell(X_3) \simeq H^3_\ell(E_3^3/H) \simeq H^3_\ell(E_3^3)^H.
\]

The Künneth formula gives

\[
H^3_\ell(E_3^3)^H = (H^1_\ell(E_3) \otimes H^1_\ell(E_3) \otimes H^1_\ell(E_3))^H,
\]

which is 2-dimensional. To be explicit, since

\[
H^1_\ell(E_3)^\vee \simeq V_\ell(E_3) := T_\ell(E_3) \otimes \overline{\mathbb{Q}_\ell}
\]
as Galois modules, we will work with the homology representation

\[
\text{Aut}_{\overline{\mathbb{Q}_\ell}}((V_\ell(E_3) \otimes V_\ell(E_3) \otimes V_\ell(E_3))^H)
\]

where

\[
V_\ell(E_3) = T_\ell(E_3) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \left( \lim_{\leftarrow} E_3[\ell^n] \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,
\]

the extended Tate module of \( E_3 \).

To understand the action of Frobenius under the Galois representation, we start by investigating the action on \( E_3 \). Denote the automorphism \( \iota_3 \) by \([\zeta_3]\) for notational convenience. Indeed, this induces a non-trivial action \([\zeta_3]^*_\) on the Tate module \( V_\ell(E_3) \) with characteristic polynomial \( T^2 + T + 1 \). The eigenvalues of this action are thus the distinct primitive third roots of unity, and the action of \( \iota_3^2 \) is then \([\zeta_3^2]\) and we may compute cleanly with the eigenvalues.
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For any $\sigma \in G_{\mathbb{Q}}$, and any $(x, y) \in E_3(\overline{\mathbb{Q}})$, we have

$$\sigma([\zeta_3](x, y)) = \sigma((\zeta_3 x, y)) = (\sigma(\zeta_3)\sigma(x), \sigma(y)) = [\sigma(\zeta_3)]\sigma((x, y)). \quad (4.2)$$

For any vector $v \in V_\ell(E_3)$ that is an $\zeta_3$-eigenvector of $[\zeta_3]_*$, we then have

$$\zeta_3\sigma(v) = \sigma(\zeta_3v)$$

$$= \sigma([\zeta_3]_*(v))$$

$$= (\sigma \circ [\zeta_3])_*(v)$$

$$= ([\sigma(\zeta_3)] \circ \sigma)_*(v) \quad \text{by (4.2)}$$

$$= [\sigma(\zeta_3)]_*\sigma(v).$$

Hence, taking complex conjugates of both sides if necessary, $\sigma(v)$ is in the $\sigma(\zeta_3)$-eigenspace of $[\zeta_3]_*$. In particular, if $c \in G_{\mathbb{Q}}$ is complex conjugation, then $w := c_* (v)$ is a $\zeta_3^2$-eigenvector for $[\zeta_3]_*$. 

Let $\chi : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}^\times$ be the non-trivial character of $\mathbb{Q}(\zeta_3)$. Fix a prime $p \neq 2, 3, \ell$, so that $E_3$ has good reduction at $p$. If $\chi(\text{Frob}_p) = 1$, then the above shows that $(\text{Frob}_p)_*(v)$ is a $\zeta_3$-eigenvector for $(\text{Frob}_p)_*$ and $(\text{Frob}_p)_*(w)$ gives another eigenvector so that the induced action of Frobenius on $V_\ell(E_3)$ with the basis $v, w$ is given by a matrix

$$\begin{pmatrix}
\alpha_p & 0 \\
0 & \beta_p
\end{pmatrix}$$

where $\alpha_p, \beta_p$ are the eigenvalues of $(\text{Frob}_p)_*$.

On the other hand, if $\chi(\text{Frob}_p) = -1$, then $(\text{Frob}_p)_*(v)$ is a $\zeta_3^2$-eigenvector of $[\zeta_3]_*$. 

and \((\text{Frob}_p)_*(w)\) is a \(\zeta_3\)-eigenvector of \([\zeta_3]_*\), so the action in the basis \(v, w\) is given by

\[
\begin{pmatrix}
0 & h_p \\
\alpha_p & 0
\end{pmatrix}
\]

for some \(h_p, k_p\) such that \(h_p k_p = -p\).

On \((V_{\ell}(E_3)^{\otimes 3})^H\), we know the pure tensors

\[v \otimes v \otimes v \quad \text{and} \quad w \otimes w \otimes w\]

are fixed by \(H\), and span the space, hence are a basis. If we denote the Galois representation by

\[\rho_3 : G_{\mathbb{Q}} \to \text{Aut}_{\mathbb{Q}_\ell}((V_{\ell}(E) \otimes V_{\ell}(E) \otimes V_{\ell}(E))^H),\]

then we have two possibilities for \(\rho_3(\text{Frob}_p)\). If \(\chi(\text{Frob}_p) = 1\), the action of \(\rho(\text{Frob}_p)\) is given by the matrix

\[
\begin{pmatrix}
0^3 & 0 \\
0 & \beta_p^3
\end{pmatrix}
\]

while if \(\chi(\text{Frob}_p) = -1\) the action is given by

\[
\begin{pmatrix}
0 & k_p^3 \\
k_p^3 & 0
\end{pmatrix}.
\]
Now, as $\alpha_p, \beta_p = \pm i \sqrt{p}$ when $\chi(\text{Frob}_p) = -1$, we simply have

$$\text{tr}(\rho(\text{Frob}_p)) = \alpha_p^3 + \beta_p^3 = (\alpha_p + \beta_p)^3 - 3p(\alpha_p + \beta_p).$$

Lemma 2.8 completes the proof.

The main interesting piece of arithmetic that comes from working over $\mathbb{Q}$ instead of $\mathbb{C}$ is that our elliptic curves are no longer unique up to isomorphism, and we can investigate what occurs if we pick another model. Using twists of the elliptic curves and proceeding with the construction, we get an appropriate twist of the $L$-series of the threefold. In this sense, this defines twists of our threefolds, as in [40]. Let $D$ be a non-zero integer, and denote by $E_3(D)$ the curve

$$E_3(D) : y^2 = x^3 - D.$$ 

Then $E_3 = E_3(1)$, and if $D$ is square-free, contains a square but cube-free, or contains a cube but fourth-power-free, the curve $E_3(D)$ is a sextic, cubic or quadratic twist of $E_3$ respectively. The action of Frobenius on $E_3(D)$ is the action of the Frobenius on $E_3$ twisted by $\psi_D$, a non-trivial sextic, cubic or quadratic Dirichlet character of $\mathbb{Q}(\sqrt{D})$ respectively, when $D \neq 1$.

**Remark.** If $1 \neq D = (-1)^k p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ is a prime factorization of $D$, then

$$\psi_D = \psi_{p_1}^{k_1} \psi_{p_2}^{k_2} \cdots \psi_{p_n}^{k_n},$$

so if $D$ is not fourth-power-free there are fourth-power-free integers $k_j' \leq k_j$ for $1 \leq j \leq n$ such that $D' = (-1)^k p_1^{k_1'} p_2^{k_2'} \cdots p_n^{k_n'}$ gives the same twist, i.e., $E_3(D') \simeq E_3(D)$.
over $\mathbb{Q}$. As such, we do not concern ourselves with always ensuring our twists are fourth-power-free.

On a crepant resolution of

$$(E_3(D) \times E_3(D) \times E_3(D))/H$$

we have

$$\text{tr}(\rho(\text{Frob}_p)) = \psi_{D^3}(\text{Frob}_p)(\alpha_p^3 + \beta_p^3).$$

This extends to the case where we do not twist all three curves by the same $D$. Suppose $D_1, D_2$ and $D_3$ are not necessarily equal. On a crepant resolution of

$$E_3^3(D_1, D_2, D_3) := (E_3(D_1) \times E_3(D_2) \times E_3(D_3))/H$$

we have

$$\text{tr}(\rho(\text{Frob}_p)) = \psi_{D_1D_2D_3}(\text{Frob}_p)(\alpha_p^3 + \beta_p^3).$$

With this notation set up, we have

**Theorem 4.4.** Let $H$ be a subgroup of $G_3$ such that a crepant resolution of

$$E_3^3(D_1, D_2, D_3)/H$$

is a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. If we denote this resolution by $Y_3$, then

$$L(Y_3, s) = L(s, \chi_3^3)$$
where \( \chi_3 \) is the Hecke character such that

\[
L(E_3(D_1D_2D_3), s) = L(s, \chi_3).
\]

In a similar fashion, we define the twists

\[ E_4(D) : y^2 = x^3 - Dx. \]

If \( D \) is square-free, or the cube of an integer, this is a biquadratic twist of \( E_4 \), and otherwise if \( D \) is a square this is a quadratic twist. The twisted threefolds are

\[
E_4^3(D_1, D_2, D_3) := (E_4(D_1) \times E_4(D_2) \times E_4(D_3))/G_4
\]

and we again have \( E_4(1) \) being our model of \( E_4 \) above.

**Theorem 4.5.** Let \( D_1, D_2 \) and \( D_3 \) be non-zero integers such that a crepant resolution \( Y_4 \) of \( E_4^3(D_1, D_2, D_3)/G_4 \) is a rigid Calabi-Yau threefold defined over \( \mathbb{Q} \). We have

\[
L(Y_4, s) = L(s, \chi_4^3)
\]

where \( \chi_4 \) is the Hecke character such that

\[
L(E_4(D_1D_2D_3), s) = L(s, \chi_4).
\]

**Proof.** As a (non-trivial) twist only multiplies the trace of the action of Frobenius by the respective quadratic or biquadratic character as above, we need only show the \( L \)-series are as described in the case without twisting the underlying elliptic curves.
The induced action on the (extended) Tate module \( V_\ell(E_4) \) is the only difference here. If we denote the action on \( E_4 \) by \([i]\), we have \([i]^2 = [-1]\), and the eigenvalues of the action are \( \pm i \). For any \( \sigma \in G_\mathbb{Q} \) and \((x, y) \in E_4(\overline{\mathbb{Q}})\) we have

\[
\sigma([i](x, y)) = [\sigma(i)]\sigma((x, y)).
\]

Hence, for an \( i \)-eigenvector \( v \) of \([i]_*\) we have

\[
i\sigma_* (v) = \sigma_* ([i]_* (v))
= (\sigma \circ [i])_* (v)
= (\sigma \circ [i] \circ \sigma)_* (v)
= \sigma([i])_* (\sigma_* (v))
= \chi(\sigma) [i]_* (\sigma_* (v))
\]

where \( \chi \) is the non-trivial character of \( \mathbb{Q}(i) \). We again find that if \( c \) denotes complex conjugation, then \( w = c_* (v) \) gives a \((-i)\)-eigenvector of \([i]_*\) under \( \sigma_* \) so that \( v, w \) gives a basis for \( V_\ell(E_4) \). The computation of the action of Frobenius again divides into the two cases where \( \chi(\text{Frob}_p) = \pm 1 \), and is otherwise as above with \( E_3 \).

With a little adjustment to the action on the Tate module of \( E_3 \) we can find the \( L \)-series associated to the rest of the constructions of rigid Calabi-Yau threefolds using the automorphism of order 6 on \( E_3 \). (Which, in a sense, reproves Theorem 4.4, but does a separate computation.)
Theorem 4.6. Let $J$ be a subgroup of $G_6$ such that a crepant resolution of 

$$E_6^3(D_1, D_2, D_3)/J$$

is a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. If we denote this resolution by $Y_6$, then 

$$L(Y_6, s) = L(s, \chi_6^3)$$

where $\chi_6$ is the Hecke character such that 

$$L(E_6(D_1D_2D_3), s) = L(s, \chi_6).$$

4.3 Modularity of non-rigid threefolds over $\mathbb{Q}$

More generally, one may call a Calabi-Yau threefold modular if the semisimplification of its Galois representation decomposes into 2-dimensional (modular) pieces.

We will separate our investigation into three cases. Our first step is to investigate the non-rigid threefolds in Theorems 4.1 and 4.2, where all the $(2,1)$-classes lie in the Künneth component. Then we will study the threefolds where all of the $(2,1)$-classes come from the resolution, and lastly, when there is a combination of both $(2,1)$-classes from the Künneth component as well as the resolution. We will show they are all modular non-rigid threefolds.

4.3.1 The Künneth construction with $E_4$

Starting with $E_4$, we have the quotient with $H_4$, where $h^{2,1}(E_4^3/H_4) = 1$, coming from the 3-form $dz_1 \wedge dz_2 \wedge d\bar{z}_3$. As with the rigid cases, we let $H_3^X = H_{\text{et}}^3(X, \mathbb{Q}_\ell)$ and
find
\[ H_3^3(E_3^3/H_4) \simeq H_3^3(E_4^3/H_4) \simeq \left( H_1^1(E_4)^{\otimes 3} \right)^{H_4}. \]

As \( H_1^1(E_4) \) is dual to the (extended) Tate module \( V_\ell(E_4) = T_\ell(E_4) \otimes \overline{\mathbb{Q}}_\ell \), we will study the Galois (homology) representation
\[
\rho_{H_4} : G_\mathbb{Q} \to \text{Aut}_{\overline{\mathbb{Q}}_\ell} \left( (V_\ell(E_4)^{\otimes 3})^{H_4} \right).
\]

If we denote by \( x, y \) a basis for \( V_\ell(E_4) \), the classes
\[
u_1 = x \otimes x \otimes x, \quad \nu_2 = x \otimes x \otimes y, \quad \nu_3 = y \otimes y \otimes x, \quad \nu_4 = y \otimes y \otimes y
\]
are linearly independent and fixed by \( H_4 \), hence form a basis for the representation.

Consider a (good) prime \( p \neq 2, \ell \), and let \( \chi : G_\mathbb{Q} \to \overline{\mathbb{Q}}_\ell^\times \) be the non-trivial character of \( \mathbb{Q}(i) \). If \( \chi(\text{Frob}_p) = 1 \), then action of \( \text{Frob}_p \) on \( V_\ell(E_4) \) is given by
\[
\begin{pmatrix}
    \alpha_p & 0 \\
    0 & \beta_p
\end{pmatrix}
\]
where \( \alpha_p, \beta_p \) are the eigenvalues of the action, and otherwise if \( \chi(\text{Frob}_p) = -1 \), the action of \( \text{Frob}_p \) is given by
\[
\begin{pmatrix}
    0 & h_p \\
    k_p & 0
\end{pmatrix}
\]
for some algebraic numbers \( h_p, k_p \) such that \( h_p k_p = -p \). Thus, in the basis \( u_1, u_2, u_3, u_4 \),
the action of $\text{Frob}_p$ is given by

$$
\begin{pmatrix}
\alpha_p^3 & 0 & 0 & 0 \\
0 & \alpha_p^2 \beta_p & 0 & 0 \\
0 & 0 & \alpha_p \beta_p^2 & 0 \\
0 & 0 & 0 & \beta_p^3
\end{pmatrix}
$$

if $\chi(\text{Frob}_p) = 1$, and

$$
\begin{pmatrix}
0 & 0 & 0 & h_p^3 \\
0 & 0 & h_p k_p^2 & 0 \\
0 & h_p k_p^2 & 0 & 0 \\
k_p^3 & 0 & 0 & 0
\end{pmatrix}
$$

if $\chi(\text{Frob}_p) = -1$. When $\chi(\text{Frob}_p) = -1$, we have $\alpha_p, \beta_p = \pm i \sqrt{p}$, and so we have

$$
\text{tr}(\rho_{H_4}(\text{Frob}_p)) = (\alpha_p^3 + \beta_p^3) + p(\alpha_p + \beta_p)
$$
as $\alpha_p \beta_p = p$. Hence,

$$
L(E_4^3/H_4, s) = L(s, \chi_4^3)L(s - 1, \chi_4),
$$

where $\chi_4$ is the Hecke character of $E_4$, i.e, $L(E_4, s) = L(s, \chi_4)$. Both $L(s, \chi_4^3)$ and $L(s - 1, \chi_4)$ are modular, and so our non-rigid Calabi-Yau threefold is modular as well. Hence, we have the following.

**Proposition 4.7.** Let $E_4$ be the elliptic curve with an automorphism $\iota_4$ of order 4 above, and let

$$
H_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^3 \times \text{id} \rangle.
$$
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There exists a crepant resolution of $E^3_4/H_4$, this is a modular non-rigid Calabi-Yau threefold, and its $L$-series is given by

$$L(s, \chi^3_4)L(s - 1, \chi_4),$$

where $\chi_4$ is the Hecke character of $E_4$.

Remark. Note that we may also write

$$\text{tr}(\rho_{H_4}(\text{Frob}_p)) = \alpha^3_p + \alpha^2_p\beta_p + \alpha_p\beta^2_p + \beta^3_p$$

and so

$$L(\widetilde{E^3_4}/H_4, s) = L(\text{Sym}^3 f_4, s)$$

where $f_4$ is the modular form associated to $E_4$, i.e., $L(E_4, s) = L(f_4, s)$. This $L$-function point of view has been studied in [36] and [37] where special values of the $L$-functions are shown to be multiples of powers of $\pi$, algebraic up to a term coming from the newforms.

Similarly, the Borcea construction with $I_4$ has

$$H_\ell(\widetilde{E^3_4}/I_4) \cong H_\ell(E_4)^{\otimes 3},$$

and so our Galois (homology) representation of interest will be

$$\rho_{I_4} : G_\mathbb{Q} \to \text{Aut}_{\overline{\mathbb{Q}_\ell}}(V_\ell(E_4)^{\otimes 3}).$$

Here, as every class is preserved, with basis $x, y$ for $V_\ell(E_4)$ as above, the set of all
elementary 3-tensors of $x$ and $y$ is a basis for the representation. For a prime $p \neq 2, \ell$, we can order the basis so that the action of $\text{Frob}_p$ is given by

$$
\begin{pmatrix}
\alpha_p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_p^2 \beta_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_p^2 \beta_p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_p^2 \beta_p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_p \beta_p^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_p \beta_p^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_p \beta_p^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_p^3 \\
\end{pmatrix}
$$

if $\chi(\text{Frob}_p) = 1$, and otherwise

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & k_p^3 \\
0 & 0 & 0 & 0 & 0 & 0 & h_p^2 k_p & 0 \\
0 & 0 & 0 & 0 & 0 & h_p^2 k_p & 0 & 0 \\
0 & 0 & 0 & 0 & h_p^2 k_p & 0 & 0 & 0 \\
0 & 0 & 0 & h_p k_p^2 & 0 & 0 & 0 & 0 \\
0 & 0 & h_p k_p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & h_p k_p^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
(k_p^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

As before, the CM conditions on the eigenvalues gives

$$
L(E_3^3/I_4, s) = L(s, \chi_3^3) L(s - 1, \chi_4)^3,
$$
and the threefold is again modular.

**Proposition 4.8.** Let $E_4$ be the elliptic curve above. The group

$$I_4 = \langle \iota_4^2 \times \iota_4^2 \times \text{id}, \iota_4^2 \times \text{id} \times \iota_4^2 \rangle$$

acts on $E_4^3$ and there is a crepant resolution of $E_4^3/I_4$. This is a non-rigid modular Calabi-Yau threefold with $L$-series

$$L(s, \chi_4^3)L(s-1, \chi_4)^3$$

where $L(E_4, s) = L(s, \chi_4)$.

### 4.3.2 The Künneth construction with $E_6$

The only non-rigid Calabi-Yau threefold in the construction with $E_6$, where the resolution does not produce $(2,1)$-classes, is the original Borcea construction. With the non-trivial character associated to $\mathbb{Q}(\sqrt{-3})$ this time, instead of $\mathbb{Q}(i)$, to separate the CM condition on (good) primes $p \neq 2, 3$, the work above holds again and we have

$$\widetilde{L(E_6^3/M_6, s)} = L(s, \chi_6^3)L(s-1, \chi_6)^3$$

where $L(E_6, s) = L(s, \chi_6)$. Hence, all the Calabi-Yau threefolds where all of the $(2,1)$-classes lie in the Künneth component are modular.

**Proposition 4.9.** Let $E_6$ be as above, and let

$$K_6 = \langle \iota_6^3 \times \iota_6^3 \times \text{id}, \iota_6^3 \times \text{id} \times \iota_6^3 \rangle,$$
act on $E_6^3$. There exists a crepant resolution of $E_6^3/K_6$, and such a resolution is a modular non-rigid Calabi-Yau threefold with $L$-series

$$L(s, \chi_{E_6}^3)L(s-1, \chi_6)^3$$

where $L(E_6, s) = L(s, \chi_6)$.

### 4.3.3 Elliptic curve singularities

When a quotient in our construction has a $(2,1)$-class corresponding to an elliptic curve in the fixed locus that is blown up in the resolution, we are no longer able to apply our previous approach to get the entire third $\ell$-adic cohomology from the product of the underlying Tate modules. Instead, we use an idea of Hulek and Verrill [45] using the fact that each of the elliptic curves singularities gives rise to an elliptic ruled surface in the resolved threefold.

Suppose we have a threefold $X = \widetilde{E_6^3}/G$, defined over $\mathbb{Q}$, whose only $(2,1)$ classes come from the resolution of $m$ distinct elliptic curves in the singular locus of $E_6^3/G$, each of which is also defined over $\mathbb{Q}$. Each of these corresponds to an elliptic ruled surface $Y_1, Y_2, \ldots, Y_m$ birational to $E_k \times \mathbb{P}^1$. As ruled surfaces have no $(3,0)$ or $(0,3)$ forms, we get an exact sequence

$$0 \longrightarrow H^{3,0}(X) \oplus H^{0,3}(X) \longrightarrow H^3(X, \mathbb{C}) \longrightarrow \bigoplus_{j=1}^m H^3(Y_j, \mathbb{C}) \longrightarrow 0$$

with the natural maps. As our varieties are all defined over $\mathbb{Q}$, this corresponds to
4.3. MODULARITY OF NON-RIGID THREEFOLDS OVER $\mathbb{Q}$

maps of Galois modules

$$0 \longrightarrow U \longrightarrow H^3_\ell(X) \longrightarrow \bigoplus_{j=1}^{m} H^3_\ell(Y_j) \longrightarrow 0$$

where, by an abuse of notation, we consider the $\ell$-adic cohomology groups their respective (dual) Galois representations. The Galois representation $U$ coming from the rigid piece of $X$ is simply the Galois representation we have already studied,

$$U = \text{Aut}_{\mathbb{Q}_\ell}((H_\ell(E^3_k) \otimes^3_{G_k})$$

for $k = 4$ or 6. Hence we have

$$H^3_\ell(X) = U \oplus \left( \bigoplus_{j=1}^{m} H^3_\ell(Y_j) \right)$$

where $U$ and each of the $H^3_\ell(Y_j)$ are modular 2-dimensional representations. Hence $X$ is modular. Moreover, as $L$-functions are preserved by birational maps, we have

$$L(X, s) = L(U, s) L(E_k, s - 1)^m.$$ 

Hence, we have

**Theorem 4.10.** Let $G$ be one of the groups of automorphisms $I_6$ or $O_6$ in Theorem 4.1. A crepant resolution of $E^3_6/G$ is a modular non-rigid Calabi-Yau threefold, and

$$L(\overline{E^3_6}/I_6, s) = L(s, \chi_6^3)L(s - 1, \chi_6),$$

$$L(\overline{E^3_6}/O_6, s) = L(s, \chi_6^3)L(s - 1, \chi_6)^5,$$
where $\chi_6$ is the Hecke character of $E_6$.

This leaves only the threefolds coming from $J_6, N_6$ and $J_4$, with both Künneth components and classes from the resolution to deal with.

Let $Y_k = \widetilde{E}_k^3/J_k$ for $k = 4, 6$, and $Z = \widetilde{E}_6^3/N_6$. We no longer have a 2-dimensional Galois representation $U$ coming from the rigid piece, which we already know to be modular. Instead, write

$$H^3(Y_k, \mathbb{C}) \simeq H^3(E_k^3/J_k, \mathbb{C}) \oplus H^3(E_k \times \mathbb{P}^1, \mathbb{C})^{m_k}$$

where $m_4 = 6$ and $m_6 = 2$, and

$$H^3(Z, \mathbb{C}) \simeq H^3(E_6^3/N_6, \mathbb{C}) \oplus H^3(E_6 \times \mathbb{P}^1, \mathbb{C})^{10}.$$ 

The decomposition of the middle cohomology into the Künneth component and the resolution then gives exact sequences

$$0 \longrightarrow H^3(E_k^3/J_k, \mathbb{C}) \longrightarrow H^3(Y_k, \mathbb{C}) \longrightarrow H^3(E_k \times \mathbb{P}^1, \mathbb{C})^{m_k} \longrightarrow 0$$

and

$$0 \longrightarrow H^3(E_6^3/N_6, \mathbb{C}) \longrightarrow H^3(Z, \mathbb{C}) \longrightarrow H^3(E_6 \times \mathbb{P}^1, \mathbb{C})^{10} \longrightarrow 0$$

respectively. This again corresponds to a sequence of Galois representations and so abusing notation once again, we have

$$0 \longrightarrow V_k \longrightarrow H^3(Y_k) \longrightarrow H^3(E_k \times \mathbb{P}^1)^{m_k} \longrightarrow 0$$
is exact under the natural maps, where the $V_k$ are 4-dimensional Galois representation, as well as

$$0 \rightarrow W \rightarrow H^3_{\ell}(Z) \rightarrow H^3_{\ell}(E_6 \times \mathbb{P}^1)^{10} \rightarrow 0$$

for a 4-dimensional Galois representations $W$. Conveniently having studied our three-folds in this order, we have already seen $V_4$ before. Up to isomorphism we have

$$V_4 \simeq \text{Aut}_{\mathbb{Q}}((H^1_{\ell}(E_4)^{\otimes 3})^H_4)$$

and hence

$$L(Y_4, s) = L(E_4^3/H_4, s)L(E_4, s - 1)^6$$

$$= L(E_4^3/G_4, s)L(E_4, s - 1)^7$$

$$= L(s, \chi_4^3)L(s - 1, \chi_4)^7.$$

While we have not seen the 4-dimensional Galois representations in the $J_6$ and $N_6$ quotients, they are similarly constructed. Indeed, both quotients fix the 3-form $dz_1 \wedge dz_2 \wedge d\bar{z}_3$ on $E_6^3$, and so the representations are isomorphic to

$$\text{Aut}_{\mathbb{Q}}((V_{\ell}(E_6)^{\otimes 3})^{J_6}), \quad \text{and} \quad \text{Aut}_{\mathbb{Q}}((V_{\ell}(E_6)^{\otimes 3})^{N_6})$$

generated by the four classes

$$u_1 = x \otimes x \otimes x, \quad u_2 = x \otimes x \otimes y, \quad u_3 = y \otimes y \otimes x, \quad u_4 = y \otimes y \otimes y$$

where $x, y$ are an appropriate basis of $V_{\ell}(E_6)$. Analyzing the action of $\text{Frob}_p$ for a
good prime then gives

\[ L(Y_6, s) = L(s, \chi_4^3)L(s - 1, \chi_4)^3, \]

\[ L(Z, s) = L(s, \chi_6^3)L(s - 1, \chi_6)^{11}. \]

Putting everything in Sections 4.2 and 4.3 together, we finally prove Theorems 1.1 and 1.2: the Calabi-Yau threefolds in Theorems 4.1 and 4.2 are all modular.

### 4.4 Automorphic constructions

Note that in some of the groups of automorphisms above, the only non-trivial action on one component of \( E_4^3 \) was by an involution. For example, each element of

\[ H_4 = \langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^3 \times \text{id} \rangle \]

acts on the third coordinate of \( E_4^3 \) by an involution, or trivially. Hence, there is no requirement for that component to be a CM elliptic curve, and we may instead consider the family of threefolds

\[ (E_4 \times E_4 \times E)/I_4, \]

abusing notation. As the resolutions do not depend on the elliptic curve \( E \) having CM, the generalization of Theorem 4.1 and 4.2 are true.

On the other hand, the underlying elliptic curves having CM was integral in our showing modularity of the threefolds, and our methods for understanding the Galois representations in this family will need to be changed. Fortunately, as a starting
point, we may still use the idea of using the Tate modules of the underlying elliptic curves.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$. We start with the simplest of such non-rigid families,

$$X_E = (\tilde{E_4 \times E_4})/H_4.$$ 

The decomposition of $H^3(X_E, \mathbb{C})$ still comes from the K"unneth component, but the Galois representation of interest is now

$$\text{Aut}_{\overline{\mathbb{Q}_\ell}}((V_\ell(E_4) \otimes V_\ell(E_4) \otimes V_\ell(E))^H_4).$$

Let $N$ be the conductor of $E$, so that any prime $p \nmid 2N$ is a good prime for $X_E$. Fix such a prime $p$. Let $x, y$ be an eigenbasis for the induced action of the non-symplectic involution $[i]^*$ of $V_\ell(E)$. Then with $v, w$ an eigenbasis for $[i]^*$ of $V_\ell(E_4)$, the following four tensors are fixed by $H_4$:

$$u_1 = v \otimes v \otimes x, \quad u_2 = v \otimes v \otimes y, \quad u_3 = w \otimes w \otimes x, \quad u_4 = w \otimes w \otimes y.$$ 

As they are all linearly independent, they are a basis for our representation. We have two cases to consider.

If $E$ does not have CM, then the action of $\text{Frob}_p$ on $V_\ell(E)$ is given by

$$\begin{pmatrix} \gamma_p & 0 \\ 0 & \delta_p \end{pmatrix},$$

where $\gamma_p, \delta_p$ are the eigenvalues of the action. If $\alpha_p, \beta_p$ are again the eigenvalues of
Frob\textsubscript{p} acting on \(V_\ell(E_4)\), then the action of Frob\textsubscript{p} on the basis \(u_1, u_2, u_3, u_4\) is given by

\[
\begin{pmatrix}
\alpha_p^2 \gamma_p & 0 & 0 & 0 \\
0 & \alpha_p^2 \delta_p & 0 & 0 \\
0 & 0 & \beta_p^2 \gamma_p & 0 \\
0 & 0 & 0 & \beta_p^2 \delta_p
\end{pmatrix}
\]

if \(\chi(\text{Frob}_p) = 1\), and otherwise if \(\chi(\text{Frob}_p) = -1\), by

\[
\begin{pmatrix}
0 & 0 & h_p^2 \gamma_p & 0 \\
0 & 0 & 0 & h_p^2 \delta_p \\
k_p^2 \gamma_p & 0 & 0 & 0 \\
0 & k_p^2 \delta_p & 0 & 0
\end{pmatrix}
\]

Hence,

\[
\text{tr}(\rho(\text{Frob}_p)) = \begin{cases} 
(\alpha_p^2 + \beta_p^2)(\gamma_p + \delta_p) & \text{if } \chi(\text{Frob}_p) = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

If \(E\) has CM by \(\mathbb{Q}(\sqrt{-D})\), then we have four cases to consider. Let \(\chi_D\) be the non-trivial character of \(\mathbb{Q}(\sqrt{-D})\). If \(\chi_D(\text{Frob}_p) = 1\), then the action of Frob\textsubscript{p} is the diagonal action by \(\gamma_p\) and \(\delta_p\) above, while if \(\chi_D(\text{Frob}_p) = -1\), the action is given by

\[
\begin{pmatrix}
0 & r_p \\
s_p & 0
\end{pmatrix}
\]

where \(r_p s_p = -p\). Hence, if \(\chi(\text{Frob}_p) = \chi_D(\text{Frob}_p) = 1\), the action of Frob\textsubscript{p} is given
by

\[
\begin{pmatrix}
\alpha_p^{2\gamma_p} & 0 & 0 & 0 \\
0 & \alpha_p^{2\delta_p} & 0 & 0 \\
0 & 0 & \beta_p^{2\gamma_p} & 0 \\
0 & 0 & 0 & \beta_p^{2\delta_p}
\end{pmatrix},
\]

if \( \chi(\text{Frob}_p) = 1 \) and \( \chi_D(\text{Frob}_p) = -1 \), the action is given by

\[
\begin{pmatrix}
0 & \alpha_p^{2r_p} & 0 & 0 \\
\alpha_p^{2s_p} & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_p^{2r_p} \\
0 & 0 & \beta_p^{2s_p} & 0
\end{pmatrix},
\]

if \( \chi(\text{Frob}_p) = -1 \) and \( \chi_D(\text{Frob}_p) = 1 \), the action is given by

\[
\begin{pmatrix}
0 & 0 & h_p^{2\gamma_p} & 0 \\
0 & 0 & 0 & h_p^{2\delta_p} \\
k_p^{2\gamma_p} & 0 & 0 & 0 \\
0 & k_p^{2\delta_p} & 0 & 0
\end{pmatrix},
\]

and lastly, if \( \chi(\text{Frob}_p) = \chi_D(\text{Frob}_p) = -1 \), the action is given by

\[
\begin{pmatrix}
0 & 0 & 0 & h_p^{2r_p} \\
0 & 0 & h_p^{2s_p} & 0 \\
0 & k_p^{2r_p} & 0 & 0 \\
k_p^{2s_p} & 0 & 0 & 0
\end{pmatrix}.
\]
Thus, we have

\[
\text{tr}(\rho(Frob_p)) = \begin{cases} 
(\alpha_p^2 + \beta_p^2)(\gamma_p + \delta_p) & \text{if } \chi(Frob_p) = \chi_D(Frob_p) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Now, as before, \(E_4\) has CM so we know that when \(\chi(Frob_p) = -1\), the eigenvalues are complex conjugates, \(\alpha_p, \beta_p = \pm i\sqrt{p}\). Similarly, if \(E\) has CM, then the eigenvalue of \(Frob_p\) are complex conjugates and \(\gamma_p = -\delta_p\). Hence, we may write

\[
\text{tr}(\rho(Frob_p)) = (\alpha_p^2 + \chi(Frob_p)\beta_p^2)(\gamma_p + \delta_p)
\]

\[
= (\alpha_p^2 + \alpha_p\beta_p + \beta_p^2 - \chi(Frob_p)p)(\gamma_p + \delta_p)
\]

as \(\alpha_p\beta_p = p\). Thus, for any elliptic curve \(E\) (defined over \(\mathbb{Q}\)), we find

\[
L(X_E, s) = L(\text{Sym}^2 f_4 \times f_E, s)L(\chi \otimes f_E, s - 1)^{-1},
\]

where \(f_4\) is the modular form associated to \(E_4\) and \(f_E\) is the modular form of \(E\).

**Theorem 4.11.** For any elliptic curve \(E\) defined over \(\mathbb{Q}\), the non-rigid Calabi-Yau threefold \(X_E\) described above is automorphic, with \(L\)-series

\[
L(\text{Sym}^2 f_4 \times f_E, s)L(\chi \otimes f_E, s - 1)^{-1}
\]

where \(f_E\) is the cusp form associated to \(E\), \(f_4\) is the cusp form associated to \(E_4\), and \(\chi\) is the non-trivial character of \(\mathbb{Q}(i)\).

**Proof.** It is well known that \(L(E, s)\) is modular for any elliptic curve defined over \(\mathbb{Q}\), so automorphic on \(\text{GL}_2\) as well. Gelbart and Jacquet [38] use the natural lift
of \( f_4 \) to \( \text{Sym}^2 f_4 \) to show that \( \text{Sym}^2 f_4 \) is automorphic on \( \text{GL}_3 \). Work of Kim and Shahidi [49] shows that since \( f_E \) is automorphic on \( \text{GL}_2 \) and \( \text{Sym}^2 f_4 \) is automorphic on \( \text{GL}_3 \), then \( f_E \times \text{Sym}^2 f_4 \) is automorphic on \( \text{GL}_6 \). Finally, since the product of automorphic L-functions is automorphic by the seminal work of Langlands [54], our family of threefolds \( X_E \) are automorphic.

We are grateful for Pasten [61] making this observation that these Calabi-Yau threefolds are designed perfectly to apply the known results in the Langlands program.

Similarly, one may compute the L-function of a crepant resolution of

\[(E_4 \times E_4 \times E)/J_4.\]

**Theorem 4.12.** For any elliptic curve \( E \) defined over \( \mathbb{Q} \), a crepant resolution of

\[(E_4 \times E_4 \times E)/J_4\]

is automorphic, with L-series

\[L(\text{Sym}^2 f_4 \otimes f_E, s)L(\chi \otimes f_E, s - 1)^{-1} L(f_E, s)^6\]

where \( f_E \) is the cusp form associated to \( E \), \( f_4 \) is the cusp form associated to \( E_4 \), and \( \chi \) is the non-trivial character of \( \mathbb{Q}(i) \).

We leave the last example, the original Borcea construction with \( I_4 \) for later.

The construction with \( E_6 \) similarly has some families of threefolds, with our previous threefolds simply special cases in this family: the quotients by \( J_6, K_6, \) and \( N_6 \). The computations above extend to show that replacing the third elliptic curve by any
elliptic curve $E$ defined over $\mathbb{Q}$, the quotient with $J_6$ has $L$-function

$$L(\text{Sym}^2 f_6 \times f_E, s)L(f_E \otimes \chi, s - 1)^{-1}L(f_E, s - 1)^2,$$

while the quotient with $N_6$ has $L$-function

$$L(\text{Sym}^2 f_6 \times f_E, s)L(f_E \otimes \chi, s - 1)^{-1}L(f_E, s - 1)^6L(f_6, s - 1)^4.$$

**Theorem 4.13.** Let $G$ be one of $J_6$ or $N_6$. For any elliptic curve $E$ defined over $\mathbb{Q}$, the Calabi-Yau threefold obtained as a crepant resolution of

$$(E_6 \times E_6 \times E)/G$$

is automorphic, and the respective $L$-series are

$$L(\text{Sym}^2 f_6 \times f_E, s)L(f_E \otimes \chi, s - 1)^{-1}L(f_E, s - 1)^2,$$

and

$$L(\text{Sym}^2 f_6 \times f_E, s)L(f_E \otimes \chi, s - 1)^{-1}L(f_E, s - 1)^6L(f_6, s - 1)^4,$$

where $f_E$ is the cusp form associated to $E$, $f_6$ is the cusp form associated to $E_6$, and $\chi$ is the non-trivial character of $\mathbb{Q}(\sqrt{-3})$.

All that remains are the original Borcea constructions. Here we may replace any of the elliptic curves in the respective threefolds by any elliptic curve $E$, as the only non-trivial action on them is an involution. Let $E_1$, $E_2$ and $E_3$ be elliptic curves
4.4. AUTOMORPHIC CONSTRUCTIONS

defined over \( \mathbb{Q} \). The Galois representation of interest this time is

\[ \rho : G_{\mathbb{Q}} \to \text{Aut}_{\overline{\mathbb{Q}}} \left( H^1_{\ell}(E_1) \otimes H^1_{\ell}(E_2) \otimes H^1_{\ell}(E_3) \right) \]

as the Borcea group, the group of symplectic automorphisms coming from the involutions, acts by \(-1\) on the respective 1-forms. Let \( p \) be a prime not dividing any of their conductors. For a good prime \( p \), studying the action of \( \text{Frob}_p \) on each of the (extended) Tate modules \( V_\ell(E_j) \) then gives the natural guess for what the \( L \)-function of the Calabi-Yau threefold should be,

\[ L(f_1 \times f_2 \times f_3, s) \]

where \( f_j \) is the modular form associated to \( E_j \).

Indeed, if \( E_1, E_2 \) and \( E_3 \) do not have CM, and have eigenvalues \( \alpha_p, \beta_p, \gamma_p, \delta_p \) and \( \epsilon_p, \eta_p \) respectively, for \( \text{Frob}_p \), then the action of the Frobenius on the threefold can be given by

\[
\begin{pmatrix}
\alpha_p \gamma_p \epsilon_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_p \gamma_p \eta_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_p \delta_p \epsilon_p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_p \delta_p \eta_p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_p \gamma_p \epsilon_p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_p \gamma_p \eta_p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \beta_p \delta_p \epsilon_p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta_p \delta_p \eta_p & 0
\end{pmatrix}
\]
and hence
\[ \text{tr}(\rho(\text{Frob}_p)) = (\alpha_p + \beta_p)(\gamma_p + \delta_p)(\epsilon_p + \eta_p). \]

Similarly, if any of the $E_j$ have CM, the expression above becomes 0, and the action is off diagonal in all the other cases, so this is the trace for the entire family.

**Theorem 4.14.** Let $E_1, E_2$ and $E_3$ be elliptic curves defined over $\mathbb{Q}$, with non-symplectic involutions $\iota_1, \iota_2$ and $\iota_3$ respectively. Further suppose

\[ L(E_1, s) = L(f_1, s), \quad L(E_2, s) = L(f_2, s), \quad \text{and} \quad L(E_3, s) = L(f_3, s) \]

for cusp forms $f_1, f_2$ and $f_3$. Then a crepant resolution of

\[ X := (E_1 \times E_2 \times E_3)/\langle \iota_1 \times \iota_2 \times \text{id}, \iota_1 \times \text{id} \times \iota_3 \rangle \]

has $L$-function

\[ L(X, s) = L(f_1 \times f_2 \times f_3, s), \]

which is known to be automorphic [36].

One can approach the problem of proving modularity/automorphy of Borcea-Voisin Calabi-Yau threefolds in a similar manner, when the elliptic curve $E$ and K3 surface $S$ are both defined over $\mathbb{Q}$, and known to both be modular/automorphic. This has been addressed in [39] using toric methods and K3 surfaces of CM-type from (Reid and) Yonemura’s classification of K3 surfaces in weighted projective space, [82], [84].
Chapter 5

Intermediate Jacobians

In this chapter, we compute the intermediate Jacobians of our Calabi-Yau threefolds from Chapter 4 when possible. As the intermediate Jacobians generalize the Picard varieties and Albanese varieties, it is natural to ask if we can study arithmetic questions of Calabi-Yau threefolds using their intermediate Jacobians. Recall the only non-zero intermediate Jacobian of a Calabi-Yau threefold $X$ is the ‘middle’ one, namely

$$J(X) = J^2(X) = H^{2k+1}(X, \mathbb{C})/(F^{k+1}H^{2k+1}(X) \oplus H^{2k+1}(X, \mathbb{Z})_{tor}).$$

Shafarevich conjectures that every variety of CM-type (meaning its Hodge group is abelian) has the $L$-series of a Grossencharacter, a Hecke $L$-series, and Borcea [14] shows that a rigid Calabi-Yau threefold with CM-type over an imaginary quadratic number field $K$ has an elliptic curve, also with CM by $K$, as its intermediate Jacobian. As our threefolds are defined over $\mathbb{Q}$ and have CM by $\mathbb{Q}(i)$ and $\mathbb{Q}(e^{2\pi i/3})$, their intermediate Jacobians are well known to have a Hecke $L$-series. Thus, if the conjecture of Shafarevich is true, it is a natural question to ask if there is any relation between
the associated Grossencharacters of a rigid Calabi-Yau threefold and its intermediate Jacobian. The motivation for this chapter is to study a precise conjecture of Yui to this effect, Conjecture 1.3. With all of our threefolds defined over $\mathbb{Q}$, we may simplify the statement we are after.

**Conjecture 5.1** (Yui, [83]). *Let $X$ be a rigid Calabi-Yau threefold of CM-type defined over $\mathbb{Q}$. Then the intermediate Jacobian $J(X)$ is an elliptic curve with CM by an imaginary quadratic field $K$, and has a model defined over $\mathbb{Q}$.

If $\chi$ is a Hecke character associated to $J(X)$ and

$$L(J(X), s) = L(\chi, s),$$

then

$$L(X, s) = L(\chi^3, s).$$

We will show that most of our rigid Calabi-Yau threefolds of CM-type satisfy this conjecture, but not all. In particular, we will show that if one of our examples, $X$, satisfies the conjecture, then all quadratic and quartic twists of $X$ satisfy the conjecture, while all cubic and sextic twists of $X$ do not. After this we generalize our construction to Calabi-Yau $n$-folds and show that for infinitely many $n$ satisfying a congruence with the order of the CM automorphisms we have a natural generalization of the conjecture is true. Similarly, for infinitely many $n$ the conjecture will not be true because of the CM twists on the varieties.

We start by computing the intermediate Jacobians of the rigid threefolds constructed in Chapter 4. The method is to reconstruct the torus structure from a
suitable quotient of cohomology groups, extending Roan’s work on Kummer three-folds in [66]. We then analyze the construction of the intermediate Jacobians as complex varieties more closely to determine a $\mathbb{Q}$-structure in each case.

5.1 Complex torus structure

For any $\tau = \alpha + \beta i$ in the upper half plane we have the elliptic curve $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$, with uniformizing parameter $z = x + iy$. Thus, as complex tori, $E_i = E_4$ and $E_\zeta = E_3$ where $\zeta = e^{2\pi i/3}$.

Translations by 1 and $\tau$ in $\mathbb{C}$ give rise to a basis $e, f \in H_1(E_\tau, \mathbb{Z})$ so that

$$
\begin{pmatrix}
e \\
f
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
\alpha & \beta
\end{pmatrix} \begin{pmatrix}
\partial_x \\
\partial_y
\end{pmatrix},
$$

where $\partial_x$ and $\partial_y$ are a basis for $H_1(E_\tau, \mathbb{C})$ corresponding to the uniformizing parameter. Taking duals in cohomology then gives a basis $e^*, f^* \in H^1(E_\tau, \mathbb{Z})$ such that

$$
\begin{pmatrix}
e^* & f^*
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
\alpha & \beta
\end{pmatrix} = \begin{pmatrix}
dx \\
dy
\end{pmatrix},
$$

where $dz = dx + idy$ is the holomorphic 1-form on $E$ corresponding to the uniformizing parameter. Thus, we have

$$
2e^* = \left(1 + \frac{\alpha}{\beta}i\right)dz + \left(1 - \frac{\alpha}{\beta}\right)d\bar{z},
$$

$$
2f^* = \frac{i}{\beta}(d\bar{z} - dz),
$$
so using $d\bar{z}/2$ as a generator for $H^{0,1}(E_\tau)$ we have

$$H^{0,1}(E_\tau)/H^1(E_\tau, \mathbb{Z}) = \mathbb{C} \left/ \left( \left( 1 - \frac{\alpha}{\beta} i \right) \mathbb{Z} \oplus \frac{i}{\beta} \mathbb{Z} \right) \right..$$

Applying the homothety given by multiplication by $\beta/i$ we have

$$H^{0,1}(E_\tau)/H^1(E_\tau, \mathbb{Z}) \simeq \mathbb{C}/(\mathbb{Z} \oplus (\alpha + i\beta)\mathbb{Z}) = E_\tau.$$

To mimic this procedure for the intermediate Jacobians of our rigid Calabi-Yau threefolds we need to find a basis for the integral cohomology and find a period relation with a basis for the complex cohomology.

As $H^3(E^3_\tau/G_\tau, \mathbb{Z}) \simeq H^3(E^3_\tau, \mathbb{Z})^{G_\tau}$, we may start with the simpler $H^3(E^3_\tau, \mathbb{Z})$. As above, let $(z_1, z_2, z_3) \in \mathbb{C}^3$ be uniformizing coordinates for $E^3_\tau$, with $z_k = x_k + iy_k$ corresponding to the $k$-th coordinate. Let $\{e_k, f_k\}$ be a basis for $H_1(E_\tau, \mathbb{Z})$, for $k = 1, 2, 3$, with dual bases $e^*_k, f^*_k$. This naturally gives bases for both the homology $H_3(E^3_\tau, \mathbb{Z})$ and the cohomology $H^3(E^3_\tau, \mathbb{Z})$, with the relations

$$\begin{pmatrix} e_k \\ f_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \partial_{x_k} \\ \partial_{y_k} \end{pmatrix} \quad \begin{pmatrix} e^*_k \\ f^*_k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} dx_k & dy_k \end{pmatrix}$$

for each $k$. The holomorphic three-form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ can be written using the dual basis to give the desired period relation for the intermediate Jacobian. Indeed,
as

\[ dz_1 \wedge dz_2 \wedge dz_3 = dx_1 \wedge dx_2 \wedge dx_3 + i(dx_1 \wedge dx_2 \wedge dy_3 + dx_1 \wedge dy_2 \wedge dx_3 + dy_1 \wedge dx_2 \wedge dx_3) - i(dx_1 \wedge dy_2 \wedge dy_3 + dy_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dx_3), \]

we may write \( \Omega = \text{Re}(\Omega) + i\text{Im}(\Omega) \), where

\[
\text{Re}(\Omega) = dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dy_2 \wedge dy_3 - dy_1 \wedge dx_2 \wedge dy_3 - dy_1 \wedge dy_2 \wedge dx_3
\]

\[ = e_1^* \wedge e_2^* \wedge e_3^* + \alpha(e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^*) + (\alpha^2 - \beta^2)(e_1^* \wedge f_2^* \wedge f_3^* + f_1^* \wedge e_2^* \wedge f_3^* + f_1^* \wedge f_2^* \wedge e_3^*)
\]

\[ + (\alpha^3 - 3\alpha\beta^2)f_1^* \wedge f_2^* \wedge f_3^*, \]

and

\[
\text{Im}(\Omega) = dx_1 \wedge dx_2 \wedge dy_3 + dx_1 \wedge dy_2 \wedge dx_3 + dy_1 \wedge dx_2 \wedge dx_3 - dy_1 \wedge dy_2 \wedge dy_3
\]

\[ = \beta(e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^*) + (3\alpha^2\beta - \beta^3)f_1^* \wedge f_2^* \wedge f_3^* + 2\alpha\beta(e_1^* \wedge f_2^* \wedge f_3^* + f_1^* \wedge e_2^* \wedge f_3^* + f_1^* \wedge f_2^* \wedge e_3^*). \]

With this we can now compute the intermediate Jacobians of our rigid Calabi-Yau threefolds. We start with the simpler case of \( X_4 \), a crepant resolution of \( E_4^3/G_4 \).

To compute the intermediate Jacobian of \( X_4 \), we have each underlying elliptic curve in the product having complex period \( \tau = i \), so

\[
\text{Re}(\Omega) = e_1^* \wedge e_2^* \wedge e_3^* - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^*.
\]
Im(Ω₄) = e₁^* ∧ e₂^* ∧ f₃^* + e₁^* ∧ f₂^* ∧ e₃^* + f₃^* ∧ e₂^* ∧ e₃^* − f₃^* ∧ f₂^* ∧ f₃^*,

where Ω₄ = dz₁ ∧ dz₂ ∧ dz₃ ∈ H³,₀(X₄). These give a basis for H³(X₄, C). For the left hand side of the period relation, we choose classes A₄ = Re(Ω₄) and B₄ = Im(Ω₄) and note that A₄, B₄ ∈ H³(X, Z). We claim this is a basis for H³(X₄, Z).

Indeed, suppose we have a basis C, D of H³(X₄, Z). As A₄, B₄ ∈ H³(X₄, Z), we know there is some matrix M with integral entries so that

\[
\begin{pmatrix}
A₄ \\
B₄
\end{pmatrix} = M \begin{pmatrix}
C \\
D
\end{pmatrix}.
\]  

(5.1)

Write

\[
M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

If ad = bc then A₄ must be a multiple of B₄, which we know is not true from our explicit expressions above, so a, b, c, d ∈ Z and ad ≠ bc, i.e., M ∈ GL₂(Q). Moreover, as C, D ∈ H³(X₄, Z) we may write

\[
C = qe₁^* ∧ e₂^* ∧ e₃^* + re₁^* ∧ e₂^* ∧ f₃^* + \ldots,
\]

\[
D = se₁^* ∧ e₂^* ∧ e₃^* + te₁^* ∧ e₂^* ∧ f₃^* + \ldots,
\]

for some q, r, s, t ∈ Z. Multiplying all this out in (5.1) then gives

\[
A₄ = (aq + bs)e₁^* ∧ e₂^* ∧ e₃^* + (ar + bt)e₁^* ∧ e₂^* ∧ f₃^* + \ldots,
\]

\[
B₄ = (cq + ds)e₁^* ∧ e₂^* ∧ e₃^* + (cr + dt)e₁^* ∧ e₂^* ∧ f₃^* + \ldots,
\]
and so, comparing with our expressions for \( A_4 \) and \( B_4 \) above, we have integral matrices such that

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
q & r \\
s & t
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

i.e., \( M \in \text{GL}_2(\mathbb{Z}) \). Hence, \( A_4, B_4 \) do indeed form a basis for \( H^3(X_4, \mathbb{Z}) \).

Thus, the period relation for the intermediate Jacobian of \( X_4 \) is

\[
\begin{pmatrix}
A_4 & B_4
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega_4) & \text{Im}(\Omega_4)
\end{pmatrix}
\]

so that

\[
2A_4 = \Omega_4 + \Omega_4
\]

\[
2B_4 = i(\Omega_4 - \Omega_4),
\]

and using \( \Omega_4/2 \) as a basis for \( H^{0,3}(X_4) \), we have

\[
J(X_4) \simeq \mathbb{C} / (\mathbb{Z} \oplus i\mathbb{Z}) = E_4.
\]

5.2 A model for the intermediate Jacobian \( J(X_4) \) over \( \mathbb{Q} \)

With this in place, we now choose a \( \mathbb{Q} \)-structure for the threefolds, via the underlying elliptic curves, and see if we can distinguish this structure and compute models of their respective intermediate Jacobians defined over \( \mathbb{Q} \). We are then able to compare the \( L \)-series of the intermediate Jacobians and their respective threefolds and check the conjecture.

As we are not just interested in the complex torus structure of the intermediate
5.2. MODEL FOR $J(X_4)$ OVER $\mathbb{Q}$

Jacobian, we may not simply work up to homothety, and we must be more careful in how to recover not just the torus structure from the period relation, but the exact model defined over $\mathbb{Q}$. By the Uniformization Theorem, we know for any elliptic curve $E = \mathbb{C}/\Lambda$, there is a $\lambda \in \mathbb{C}^\times$ such that any particular model of $E$ corresponds uniquely to the torus $\mathbb{C}/\lambda \Lambda$, the correspondence being

$$E : y^2 = 4x^3 - \lambda^{-4}g_2(\Lambda)x - \lambda^{-6}g_3(\Lambda) \longleftrightarrow \mathbb{C}/\lambda \Lambda,$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$, with $G_{2k}$ the Eisenstein series of weight $2k$.

We are interested in using the computation above to recover a particular model, thus suppose we have some $E = \mathbb{C}/\lambda \langle 1, \tau \rangle$. Translation by 1 and $\tau = \alpha + i\beta$ no longer gives a basis for $H_1(E, \mathbb{Z})$. We now get a basis using translation by $\lambda$ and $\lambda \tau$. Similarly, our integral classes are no longer $e^*$ and $f^*$, and our period relation is

$$
\begin{pmatrix}
\frac{e^*}{\lambda} \\
\frac{f^*}{\lambda}
\end{pmatrix}
= 
\begin{pmatrix}
\lambda & 0 \\
\lambda \alpha & \lambda \beta
\end{pmatrix}
\begin{pmatrix}
dx \\
dy
\end{pmatrix}.
$$

Hence, we recover the same relations as before

$$2e^* = \left(1 + \frac{\alpha}{\beta} i\right) dz + \left(1 - \frac{\alpha}{\beta} i\right) d\bar{z},$$

$$2f^* = \frac{i}{\beta}(d\bar{z} - dz).$$

Using $d\bar{z}/2$ as a basis for $H^{0,1}(E)$, like above, gives

$$H^{0,1}(E)/H^1(E, \mathbb{Z}) = \mathbb{C} \left/ \left( \left(1 - \frac{\alpha}{\beta} i\right) \mathbb{Z} \oplus \frac{i}{\beta} \mathbb{Z} \right) \right..$$
which is not \( \mathbb{C}/\lambda(1, \tau) \), the torus we started with. Instead, we must use the basis \( d\bar{z}/(2\beta \lambda i) \) so that
\[
\begin{align*}
e^* &= \lambda(\alpha - \beta i)\frac{dz}{2\beta \lambda i} - \lambda(\alpha + \beta i)\frac{d\bar{z}}{2\beta \lambda i}, \\
f^* &= \lambda\frac{dz}{2\beta \lambda i} - \lambda\frac{d\bar{z}}{2\beta \lambda i}.
\end{align*}
\]
Then we have
\[
H^{0,1}(E)/H^1(E, \mathbb{Z}) = \mathbb{C}/\lambda(\mathbb{Z} \oplus (\alpha + i\beta)\mathbb{Z}) = E.
\]
Thus for any \( E = \mathbb{C}/\lambda(\mathbb{Z} \oplus \tau\mathbb{Z}) \) we can recover the model using the period relation and basis \( d\bar{z}/(2\beta \lambda i) \) of \( H^{0,1}(E) \).

In our case of interest with \( E_4 : y^2 = x^3 - x \) we have (see [78]) \( E_4 = \mathbb{C}/\lambda(1, i) \) with
\[
\lambda = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}.
\]

*Remark.* We note that \( \lambda \) is transcendental, as \( \Gamma(1/4) \) and \( \sqrt{\pi} = \Gamma(1/2) \) are algebraically independent, *loc. cit.*

On the threefold we have
\[
\left(\begin{array}{cc}
e_k^* & f_k^*
\end{array}\right) \left(\begin{array}{cc}
\lambda & 0 \\
0 & \lambda
\end{array}\right) = \left(\begin{array}{cc}
dx_k & dy_k
\end{array}\right)
\]
for the \( k \)-th component of \( E_4^3 \). Writing \( \Omega_4 = \text{Re}(\Omega_4) + i\text{Im}(\Omega_4) \) and using the period relations for each of the underlying \( E_4 \) to write these in terms of the \( e_k^* \) and \( f_k^* \) we
again find

\[ \text{Re}(\Omega_4) = A_4, \]
\[ \text{Im}(\Omega_4) = B_4. \]

These classes are no longer integral, but as above our integral classes are \( A_4/\lambda^3 \) and \( B_4/\lambda^3 \) so that the period relation is

\[
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix}
\begin{pmatrix}
\lambda^3 & 0 \\
0 & \lambda^3
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\Omega_4) \\
\text{Im}(\Omega_4)
\end{pmatrix}.
\]

To get the correct model of the elliptic curve described by this relation, we must use the basis

\[ \frac{\Omega_4}{2\beta \lambda^3 i} = \frac{\Omega_4}{2\lambda^3 i}. \]

for \( H^{0,3}(X_4) \). Thus, writing

\[ A_4 = i\lambda^3 \left( \frac{\Omega_4}{2\lambda^3 i} \right) + i\lambda^3 \left( \frac{\bar{\Omega}_4}{2\lambda^3 i} \right), \]
\[ B_4 = \lambda^3 \left( \frac{\Omega_4}{2\lambda^3 i} \right) - \lambda^3 \left( \frac{\bar{\Omega}_4}{2\lambda^3 i} \right), \]

we find

\[ J(X_4) = \mathbb{C}/\lambda^3(\mathbb{Z} \oplus i\mathbb{Z}) \neq E_4. \]

Even worse, the intermediate Jacobian is not even defined over a number field! However, using the basis \( \bar{\Omega}_4/(2\lambda i) \) instead, we recover the model of \( E_4 \) for the intermediate Jacobian, which is still quite natural. Thus, we have the following.
Theorem 5.2. Let $D$ be a square-free integer, and $E_4(D^2)$ the elliptic curve with affine equation $y^2 = x^3 - D^2x$, a quadratic twist of $E_4$. Let $Y_4(D^2)$ be a crepant resolution of 

$$(E_4(D^2) \times E_4(D^2) \times E_4(D^2))/G_4.$$ 

Then 

$$J(Y_4(D^2)) = E_4(D^2),$$ 

and hence there is a model for the intermediate Jacobian $J(Y_4(D^2))$ such that 

$$L(J(Y_4(D^2)), s) = L(s, \chi)$$ 

where $\chi$ is the Hecke character of $E_4(D^2)$, and 

$$L(Y_4(D^2), s) = L(s, \chi^3).$$ 

In particular, Conjecture 5.1 is true for $Y_4(D^2)$.

Remark. As our $\mathbb{Q}$-model for $J(Y_4(D^2))$ requires choosing a scaled basis for the complex cohomology, one could argue the result above is ad hoc and simply designed to get the result we desire. However, note that having fixed the underlying elliptic curves on the threefold, we have fixed the integral cohomology classes, and so the period relation cannot be scaled by a non-integer. The Uniformization Theorem tells us integral twists require non-integral homotheties, and so our period relation cannot be scaled in some way to give another model for $J(Y_4(D^2))$, defined over $\mathbb{Q}$. Choosing a different uniformizing parameter to get a different model defined over $\mathbb{Q}$ is then what appears ad hoc. Hence, while the particular model for $J(Y_4(D^2))$ depends on
the choice of basis, the *natural* \( \mathbb{Q} \)-model does not. Now, as any CM elliptic curve has a model defined over a number field, our \( \mathbb{Q} \)-model seems the best fit.

Given this, one can ask if a period relation

\[
\begin{pmatrix}
\frac{A_4}{\mu} & \frac{B_4}{\mu} \\
\end{pmatrix}
\begin{pmatrix}
\mu & 0 \\
0 & \mu \\
\end{pmatrix}
= 
\begin{pmatrix}
\Re(\Omega_4) & \Im(\Omega_4) \\
\end{pmatrix}
\]

for the intermediate Jacobian of a rigid Calabi-Yau threefold always gives the \( \mathbb{Q} \)-model if we use \( \Omega_4/(2\sqrt{\kappa_1\kappa_2\kappa_3}\lambda^3i) \) as a basis, which is still quite natural. This, however, does not work in, e.g., the case where each underlying elliptic curve has a distinct twist. Indeed, consider real \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) such that \( \mathbb{C}/\kappa_i\lambda(1, \tau) \) are distinct quadratic twists \( E_4(D_1^2), E_4(D_2^2) \) and \( E_4(D_3^2) \). Let \( X \) be a crepant resolution of

\[
(E_1(D_1^2) \times E_2(D_2^2) \times E_3(D_3^2))/G_4.
\]

The period relation for \( J(X) \) is then

\[
\begin{pmatrix}
\frac{A_4}{\kappa_1\kappa_2\kappa_3\lambda^3} & \frac{B_4}{\kappa_1\kappa_2\kappa_3\lambda^3} \\
\kappa_1\kappa_2\kappa_3\lambda^3 & \kappa_1\kappa_2\kappa_3\lambda^3 \\
0 & 0 \\
0 & \kappa_1\kappa_2\kappa_3\lambda^3 \\
\end{pmatrix}
= 
\begin{pmatrix}
\Re(\Omega_4) & \Im(\Omega_4) \\
\end{pmatrix}
\]

and using \( \Omega_4/(2\sqrt{\kappa_1\kappa_2\kappa_3}\lambda^3i) \) as a basis for \( H^{0,3}(X)/H^3(X, \mathbb{Z}) \) then gives a model not defined over \( \mathbb{Q} \), while the model

\[
J(X) : y^2 = x^3 - (D_1D_2D_3)^2x
\]

comes from using \( \Omega_4/(2\kappa_1\kappa_2\kappa_3\lambda i) \) as a basis instead. To reconcile these differences,
we move away from working entirely with the underlying elliptic curves, and instead move to focusing on the threefold. Note that

\[(E_4(D_1^2) \times E_4(D_2^2) \times E_4(D_3^2))/G_4\]

is birational to

\[(E_4((D_1D_2D_3)^2) \times E_4 \times E_4)/G_4.\]

Hence, we may shift our computations from using the underlying elliptic curves and exploit the twists of the threefold itself. In this setting, writing the quadratic twist of

\[(E_4(D_1^2) \times E_4(D_2^2) \times E_4(D_3^2))/G_4\]

as \(D = D_1D_2D_3\), with \(E_4(D) = \mathbb{C}/\kappa\lambda(1,i)\), we have that \(d\bar{\Omega}/(2\kappa\lambda i) \in H^{0,3}(X(D))\) used as a basis recovers

\[E(D^2) : y^2 = x^3 - D^2x = x^3 - (D_1D_2D_3)^2x,\]

the appropriate model for the conjecture.

Unfortunately, when allowing biquadratic twists, the conjecture is not as naturally true. If we have a biquadratic character \(\psi\), then \(\psi^3 = \overline{\psi}\), i.e., the character

\[\overline{\psi}(z) = \overline{\psi(z)}\]

and so if we have a rigid Calabi-Yau threefold \(Y\), with CM, satisfying the conjecture, and take a biquadratic twist by \(\psi\), (e.g., twist only the first elliptic curve by \(\psi\) then
by the work above, its intermediate Jacobian has a model with

\[ L(J(Y_\psi), s) = L(s, \chi \otimes \psi) \]

while

\[ L(Y_\psi, s) = L(s, \chi^3 \otimes \psi) \neq L(s, \chi^3 \otimes \bar{\psi}) = L(s, (\chi \otimes \psi)^3). \]

As \( J(Y_\psi) \) is a complex variety, we may use the weaker hypothesis of the conjecture to our advantage. We need only show there exists a model of the intermediate Jacobian giving the desired relation with the \( L \)-series. Hence, we take a quadratic twist of our (natural) model for \( J(Y_\psi) \) by \( \psi^2 \). With this model instead, we have

\[ L(J(Y_\psi) \otimes \psi^2, s) = L(s, \chi \otimes \psi^3) \]

and so

\[ L(Y_\psi, s) = L(s, \chi^3 \otimes \psi) = L(s, (\chi \otimes \psi^3)^3), \]

as desired.

**Theorem 5.3.** Let \( Y(D) \) be a crepant resolution of

\[ (E_4(D) \times E_4 \times E_4)/G_4 \]

where \( D \) is a cube free integer that is not a square. Then the intermediate Jacobian \( J(Y(D)) \) is isomorphic to \( E_4(D) \) as a complex variety, and hence the model

\[ E_4(D^3) : y^2 = x^3 - D^3 x \]
for the intermediate Jacobian satisfies

\[ L(J(Y(D)), s) = L(\chi, s) \]

and

\[ L(Y(D), s) = L(\chi^2, s). \]

5.3 A model for the intermediate Jacobian \( J(X_6) \) over \( \mathbb{Q} \)

Similarly for \( E_6 \), we have our exact model corresponding to the complex torus \( \mathbb{C}/\mu \Gamma \) where

\[ \mu = \frac{\Gamma(\frac{1}{3})^3}{2^{4/3} \pi} \]

and \( \Gamma = \mathbb{Z} \oplus \zeta_3 \mathbb{Z} \) with \( \zeta_3 = e^{2\pi i/3} \).

Remark. Again, we have that \( \mu \) is transcendental by [78].

Abusing notation, let \( z_k = x_k + iy_k \) be the uniformizing parameter of the \( k \)-th elliptic curve on \( E_6^3 \). The period relation for the (complex) intermediate Jacobian of \( X_6 \) is

\[
\begin{pmatrix}
A_6 & B_6
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega_6) & \text{Im}(\Omega_6)
\end{pmatrix}
\]

where \( \Omega_6 = dz_1 \wedge dz_2 \wedge dz_3 \). This time we find

\[ \text{Re}(\Omega_6) = A_6 - \frac{1}{2} B_6, \]

\[ \text{Im}(\Omega_6) = \frac{\sqrt{3}}{2} B_6, \]
with integral classes

\[ A_6 = e_1^* \wedge e_2^* \wedge e_3^* - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^* - f_1^* \wedge f_2^* \wedge f_3^* \]

\[ B_6 = e_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge e_3^* + f_1^* \wedge e_2^* \wedge e_3^* \]

\[ - e_1^* \wedge f_2^* \wedge f_3^* - f_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge e_3^* \]

The method above shows

\[ H^0,3(X_6)/H^3(X_6, \mathbb{Z}) \simeq E_6 \]

as complex tori.

We can apply the same steps above to moreover get an exact model for \( J(X_6) \) over \( \mathbb{Q} \), and putting everything together we find that, as above, if the conjecture is true for one of our threefolds, it is not true for any cubic or sextic twists of that threefold.

**Theorem 5.4.** Let \( D_1, D_2 \) and \( D_3 \) be square-free integers, and \( E_6(D_j^{k_j}) \) the elliptic curve with affine equation \( y^2 = x^3 - D_j^{k_j} \), where \( k_j \in \{1, 2, 3\} \) for each \( j \). Let \( Y_6(D_1^{k_1}, D_2^{k_2}, D_3^{k_3}) \) be a crepant resolution of

\[ (E_6(D_1^{k_1}) \times E_6(D_2^{k_2}) \times E_6(D_3^{k_3}))/S, \]

with \( S = G_6, H_6, L_6 \) or \( M_6 \). Then

\[ J(Y_6(D_1^{k_1}, D_2^{k_2}, D_3^{k_3})) = E_6(D_1^{k_1} D_2^{k_2} D_3^{k_3}). \]
There is a model for the intermediate Jacobian such that

\[ L(J(Y_6(D_1^{k_1}, D_2^{k_2}, D_3^{k_3})), s) = L(s, \chi), \]

where \( \chi \) is the Hecke character of \( E_6(D_1^{k_1}D_2^{k_2}D_3^{k_3}) \), and

\[ L(Y_6(D_1^{k_1}, D_2^{k_2}, D_3^{k_3}), s) = L(s, \chi^3). \]

if and only if \( D_1^{k_1}D_2^{k_2}D_3^{k_3} \) is the cube of an integer. In particular, Conjecture 5.1 is true for \( Y_6(D_1^{k_1}, D_2^{k_2}, D_3^{k_3}) \) if and only if it is a quadratic twist of \( \tilde{E}_6^3/S \).

**Proof.** If a twist of \( \tilde{E}_6^3/S \) has corresponding \( L \)-series \( L(s, \chi^3 \otimes \psi) \), then the intermediate Jacobian has a \( \mathbb{Q} \)-model with \( L \)-series \( L(s, \chi \otimes \psi) \). For the conjecture to be true, we must have

\[ (\chi \otimes \psi)^3 = \chi^3 \otimes \psi, \]

and this is only possible if \( \psi \) is quadratic. \( \square \)

Thus, we have verified Yui’s conjecture for many examples, but it is not always the case. A somewhat sour irony here is that the construction of our rigid threefolds requires the CM automorphisms, and it is precisely this presence of CM that causes the conjecture to fail.

### 5.4 Intermediate Jacobians of higher dimensional Calabi-Yaus

Note that, while the CM is what causes problems for the conjecture above, the problem is the dimension of the Calabi-Yau being 3. If \( \psi \) is a biquadratic character, then \( \psi^5 = \psi \), and if \( \varphi \) is a sextic character, then \( \varphi^7 = \varphi \). This inspires an investigation of a
generalization of the rigid threefold construction, to higher dimensional Calabi-Yau, in the spirit of [22]. One may consider an $n$-fold product

\[ E_j \times E_j \times \cdots \times E_j \]

and $G$, the maximal group of automorphisms given by products of the form

\[ \iota_j^{a_1} \times \iota_j^{a_2} \times \cdots \times \iota_j^{a_n} \]

preserving the holomorphic $n$-form, i.e., such that $\sum a_i \equiv 0 \pmod{j}$. When $n$ is odd, only the holomorphic and anti-holomorphic $n$-forms are preserved by the entire group. For any fixed (odd) $n$, we can find all subgroups of $G$ such that a crepant resolution of the quotient by $G$ is a Calabi-Yau $n$-fold $Z_n$ with $h^n(Z_n) = 2$.

We can repeat the computation of the $L$-series, and using the notation above one finds the action of Frobenius on $Z_n$ at good primes $p$, given by matrices

\[
\begin{pmatrix}
\alpha_p^n & 0 \\
0 & \beta_p^n
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
0 & h_p^n \\
k_p^n & 0
\end{pmatrix}
\]

when $p$ is a quadratic residue or non-residue respectively, so that

\[ L(Z_n, s) = L(s, \chi^n) \]

where $L(E_j, s) = L(s, \chi)$. Similarly, we may twist each of the underlying elliptic curves to get a birational rigid Calabi-Yau that is a twist of $Z_n$.

The computation of the middle intermediate Jacobian can also be extended.
Again, let the $k$-th component in the product $E^n_r$ have uniformizing parameter $z_k = x_k + iy_k$, and period $\tau = \alpha + i\beta$, such that the period relation

$$\begin{pmatrix} e^*_k & f^*_k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} dx_k & dy_k \end{pmatrix}$$

holds. The holomorphic $n$-form is

$$\Omega_n = \bigwedge_{k=1}^n dz_k = \bigwedge_{k=1}^n dx_k + idy_k$$

$$= \bigwedge_{k=1}^n (e^*_k + \alpha f^*_k) + i\beta f^*_k$$

$$= \bigwedge_{k=1}^n e^*_k + \tau f^*_k.$$ 

Hence, we have

$$\Omega_n = e^*_1 \wedge e^*_2 \wedge \cdots \wedge e^*_n + \tau(e^*_1 \wedge \cdots \wedge e^*_{n-1} \wedge f^*_n + \cdots + f^*_1 \wedge e^*_2 \wedge \cdots \wedge e^*_n)$$

$$+ \tau^2(e^*_1 \wedge \cdots \wedge e^*_{n-2} \wedge f^*_{n-1} \wedge f^*_n + \cdots + f^*_1 \wedge f^*_2 \wedge e^*_3 \wedge \cdots \wedge e^*_n) + \cdots$$

$$\cdots + \tau^n(f^*_1 \wedge f^*_2 \wedge \cdots \wedge f^*_n).$$

Our particular choices of $\tau$ are roots of unity of small order, and we have

$$\zeta^n \in \{1, \zeta, \bar{\zeta}\} \quad \text{and} \quad i^n \in \{\pm 1, \pm i\}$$

for all $n$, where $\zeta$ is a primitive third root of unity. For each of our $n$-folds, this allows
us to find integral classes $A_n, B_n \in H^n(Z_n, \mathbb{Z})$ such that

$$
\begin{pmatrix}
A_n & B_n
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\alpha & \beta
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\Omega_n) & \text{Im}(\Omega_n)
\end{pmatrix}
$$

and so the intermediate Jacobian $J^{n-1}(Z_n) \simeq E_\tau$ as complex varieties.

*Remark.* We are interested only in $n$ odd, not only because $h^n(Z_n) > 2$ when $n$ is even, so conjecture 5.1 is not relevant, but because there is no intermediate Jacobian associated to even cohomology, which is the middle cohomology when $n$ is even!

In the arithmetic setting we now have

$$
\text{Re}(\Omega_n) = \lambda^n A_n
$$

$$
\text{Im}(\Omega_n) = \lambda^n B_n
$$

so that

$$
\begin{pmatrix}
\lambda^n & 0 \\
\lambda^n \alpha & \lambda^n \beta
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\Omega_n) & \text{Im}(\Omega_n)
\end{pmatrix}
$$

and the $\mathbb{Q}$-model satisfies

$$
L(J^{n-1}(Z_n), s) = L(s, \chi) = L(E_\tau, s).
$$

If we twist the underlying elliptic curves in $Z_n$ so that the $L$-series of the threefold is $L(s, \chi^n \otimes \psi)$, then the intermediate Jacobian has a $\mathbb{Q}$-model with $L$-series $L(s, \chi \otimes \psi)$. Thus, if $n \equiv 1 \pmod{4}$, the conjecture is true for all rigid threefolds in our construction coming from $E_4$, and if $n \equiv 1 \pmod{6}$, the conjecture is true for all
rigid threefolds in our construction coming from $E_6$.

Let $n$ be an odd positive integer. The natural question is then as follows. Let $Z$ is a rigid Calabi-Yau $n$-fold of CM-type, defined over a number field $F$, having intermediate Jacobian $J^{n-1}(Z)$ with CM by a number field $K$, where the CM automorphism of $J^{n-1}(Z)$ has order $m$. Then, if $n \equiv 1 \pmod{m}$ and

$$L(J^{n-1}(Z), s) = \begin{cases} L(\chi, s)L(\overline{\chi}, s) & \text{if } K \subset F, \\ L(\chi, s) & \text{otherwise} \end{cases}$$

then must we have

$$L(Z, s) = \begin{cases} L(\chi^n, s)L(\overline{\chi^n}, s) & \text{if } K \subset F, \\ L(\chi^n, s) & \text{otherwise}. \end{cases}$$

Also, if $n \not\equiv 1 \pmod{m}$, does the above only fail if we exploit the CM automorphism of $J^{m-1}(Z)$?

### 5.5 Intermediate Jacobians of our non-rigid Calabi-Yau threefolds

As before, the next step is trying to generalize the process above, on the rigid Calabi-Yau threefolds, to our non-rigid examples. Again, we wish to start with the simplest of the non-rigid cases, which in this case turns out to be when $h^{2,1} = 1$ comes from the resolution.

We start with the simplest of these cases, a crepant resolution of

$$E_6^3/I_6 = E_6^3/(\iota_6^2 \times \iota_6^4 \times \text{id}, \iota_6^4 \times \iota_6 \times \iota_6),$$
from Theorem 4.1, in which \( h^{2,1} = 1 \) is a class coming from resolving a fixed \( E_6 \) on the quotient. As such, we have

\[
H^3(\tilde{E}^3_6/I_6, \mathbb{Z}) \simeq H^3(E^3_6, \mathbb{Z})^{I_6} \oplus H^1(E_6, \mathbb{Z})
\]

and so, letting \( dz_k = x_k + iy_k \) be the uniformizing parameter of the \( k \)-th component of \( E^3_6 \) and \( dz = x + iy \) the uniformizing parameter of the \( E_6 \) being blown up, we have, up to isomorphism, the period relation

\[
\begin{pmatrix}
A_6 & B_6 & C_6 & D_6
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{pmatrix}
= \begin{pmatrix}
\Omega_6 & \overline{\Omega}_6 & dz & d\bar{z}
\end{pmatrix}
\]

where \( A_6, B_6 \) the integral classes in rigid case above, and

\[
\Omega_6 = dz_1 \wedge dz_2 \wedge dz_3 = \text{Re}(\Omega_6) + i \text{Im}(\Omega_6)
\]

satisfies

\[
\text{Re}(\Omega_6) = A_6 - \frac{1}{2}B_6 \quad \text{Im}(\Omega_6) = \frac{\sqrt{3}}{2}B_6.
\]

Similarly, \( C_6, D_6 \in H^1(E_6, \mathbb{Z}) \) are a basis such that

\[
\begin{pmatrix}
C_6 & D_6
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{pmatrix}
= \begin{pmatrix}
dz & d\bar{z}
\end{pmatrix}.
\]
Hence, the intermediate Jacobian is, as a complex variety, isomorphic to the product

\[ E_6 \times E_6 \]

and hence has a model defined over \( \mathbb{Q} \).

When the \((2,1)\)-class comes from the Künneth component instead of the resolution, we have a little more work to do to study the integral cohomology. Consider a crepant resolution of

\[ E_4^3/H_4 = E_4^3/\langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^3 \times \text{id} \rangle \]

in Theorem 4.2. As \( h^3(E_4^3/H_4) = 4 \), our intermediate Jacobian is a 2-dimensional torus, and the work above gives a natural guess that our intermediate Jacobians should be isomorphic to \( E_4 \times E_4 \). We may approach the reconstruction of the torus structure as before. Let \( z_j = x_j + iy_j \) be a uniformizing parameter for the \( j \)-th elliptic curve in the product \( E_4^3 \), and let \( e_j, f_j \in H_1(E_4, \mathbb{Z}) \) be elements corresponding to translations by 1 and \( \tau = i \) for the \( j \)-th elliptic curve. Then we may choose cohomology classes

\[ \Omega = dz_1 \wedge dz_2 \wedge dz_3, \quad \Psi = dz_1 \wedge dz_2 \wedge d\bar{z}_3, \]
so that, writing things out as before, we have

\[
\begin{pmatrix}
A_1 & B_1 & A_2 & B_2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega) & \text{Im}(\Omega) & \text{Re}(\Psi) & \text{Im}(\Psi)
\end{pmatrix}
\]

where

\[
\text{Re}(\Omega) = e^*_1 \wedge e^*_2 \wedge e^*_3 - f^*_1 \wedge f^*_2 \wedge e^*_3 - f^*_1 \wedge e^*_2 \wedge f^*_3 - e^*_1 \wedge f^*_2 \wedge f^*_3,
\]

\[
\text{Im}(\Omega) = e^*_1 \wedge e^*_2 \wedge f^*_3 + e^*_1 \wedge f^*_2 \wedge e^*_3 + f^*_1 \wedge e^*_2 \wedge e^*_3 - f^*_1 \wedge f^*_2 \wedge e^*_3 + f^*_1 \wedge e^*_2 \wedge f^*_3 - e^*_1 \wedge f^*_2 \wedge f^*_3,
\]

\[
\text{Re}(\Psi) = e^*_1 \wedge e^*_2 \wedge e^*_3 - f^*_1 \wedge f^*_2 \wedge e^*_3 + f^*_1 \wedge e^*_2 \wedge f^*_3 + e^*_1 \wedge f^*_2 \wedge f^*_3,
\]

\[
\text{Im}(\Psi) = -e^*_1 \wedge e^*_2 \wedge f^*_3 + e^*_1 \wedge f^*_2 \wedge e^*_3 + f^*_1 \wedge e^*_2 \wedge e^*_3 + f^*_1 \wedge f^*_2 \wedge f^*_3.
\]

As these classes are all integral we also denote the integral representatives differently to avoid confusion:

\[
A_1 = \text{Re}(\Omega), \quad B_1 = \text{Im}(\Omega), \quad A_2 = \text{Re}(\Psi) \quad \text{and} \quad B_2 = \text{Im}(\Psi).
\]

If \( A_1, B_1, A_2, B_2 \) form a basis for the integral cohomology, then we have that the intermediate Jacobian of \( X \) is indeed a 2-torus isomorphic to \( E_4 \times E_4 \).

Unfortunately, this is where trouble occurs. In the 2-dimensional case, linear independence of elements is just a matter of seeing if one is a scalar multiple of the other, and in the non-rigid case above we were given independence as a byproduct of the resolution. In this case however, linear independence is not immediate, and in
fact our elements are not a basis for $H^3(X, \mathbb{Z})$.

To see why, it is easier to go to a more general case. Let $E = \mathbb{C}/\langle 1, \sigma \rangle$ be an elliptic curve with $\sigma$ in the upper half plane, and (abusing notation) let $H_4$ from above act on $E_4 \times E_4 \times E$. Call $X_\sigma$ a crepant resolution of $(E_4 \times E_4 \times E)/H_4$. Abusing notation some more, let $\Omega \in H^{3,0}(X_\sigma)$ and $\Psi \in H^{2,1}(X_\sigma)$ be as before. Expanding things out we find

\[
\begin{align*}
\text{Re}(\Omega) &= -\delta A + \gamma B + C, \\
\text{Im}(\Omega) &= \delta B + \gamma A + D, \\
\text{Re}(\Psi) &= \delta A + \gamma B + C, \\
\text{Im}(\Psi) &= -\delta B + \gamma A + D.
\end{align*}
\]

where

\[
\begin{align*}
A &= f_1^* \wedge e_2^* \wedge f_3^* + e_1^* \wedge f_2^* \wedge f_3^* = \frac{1}{2\delta}(\text{Re}(\Psi) - \text{Re}(\Omega)), \\
B &= e_1^* \wedge e_2^* \wedge f_3^* - f_1^* \wedge f_2^* \wedge f_3^* = \frac{1}{2\delta}(\text{Im}(\Omega) - \text{Im}(\Psi)), \\
C &= e_1^* \wedge e_2^* \wedge e_3^* - f_1^* \wedge f_2^* \wedge e_3^* = \frac{1}{2}(\text{Re}(\Omega) + \text{Re}(\Psi)) - \frac{\gamma}{2\delta}(\text{Im}(\Omega) - \text{Im}(\Psi)), \\
D &= f_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge f_2^* \wedge e_3^* = \frac{1}{2}(\text{Im}(\Omega) + \text{Im}(\Psi)) - \frac{\gamma}{2\delta}(\text{Re}(\Psi) - \text{Re}(\Omega)),
\end{align*}
\]

are integral classes. Hence, we have a period relation

\[
\begin{pmatrix}
A & B & C & D
\end{pmatrix}
\begin{pmatrix}
-\delta & \gamma & \delta & \gamma \\
\gamma & \delta & \gamma & -\delta \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\text{Re}(\Omega) & \text{Im}(\Omega) & \text{Re}(\Psi) & \text{Im}(\Psi)
\end{pmatrix}
\]
Note that, as \(\mathbb{Z}\)-modules,

\[
\langle \text{Re}(\Omega), \text{Im}(\Omega), \text{Re}(\Psi), \text{Im}(\Psi) \rangle \subset \langle A, B, C, D \rangle
\]

and so our \(A_1, B_1, A_2, B_2\) cannot be a basis for \(H^3(X_\sigma, \mathbb{Z})\). Fortunately, in this generality, our integral classes \(A, B, C\) and \(D\) do form a basis.

**Lemma 5.5.** With notation as above, \(A, B, C\) and \(D\) form a basis for \(H^3(X_\sigma, \mathbb{Z})\) for any \(\sigma\) in the upper half plane.

**Proof.** While this is more complicated when dealing with more than two vectors, we may approach this in the same spirit as in the rigid case.

Suppose we have a basis \(\{P, Q, R, S\}\) of \(H^3(X_\sigma, \mathbb{Z})\). As \(A, B, C, D \in H^3(X_\sigma, \mathbb{Z})\) there must be an integral matrix

\[
M = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}
\]

such that

\[
\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = M \begin{pmatrix} P \\ Q \\ R \\ S \end{pmatrix} \quad (5.2)
\]

We know the elements in the basis are integral linear combinations of the elementary
integral 3-forms in $H^3(E^3_4)$, so write

$$P = b_{11}f^*_1 \wedge e^*_2 \wedge f^*_3 + b_{12}e^*_1 \wedge e^*_2 \wedge f^*_3 + b_{13}e^*_1 \wedge e^*_2 \wedge e^*_3 + b_{14}f^*_1 \wedge e^*_2 \wedge e^*_3 + \cdots,$$

$$Q = b_{21}f^*_1 \wedge e^*_2 \wedge f^*_3 + b_{22}e^*_1 \wedge e^*_2 \wedge f^*_3 + b_{23}e^*_1 \wedge e^*_2 \wedge e^*_3 + b_{24}f^*_1 \wedge e^*_2 \wedge e^*_3 + \cdots,$$

$$R = b_{31}f^*_1 \wedge e^*_2 \wedge f^*_3 + b_{32}e^*_1 \wedge e^*_2 \wedge f^*_3 + b_{33}e^*_1 \wedge e^*_2 \wedge e^*_3 + b_{34}f^*_1 \wedge e^*_2 \wedge e^*_3 + \cdots,$$

$$S = b_{41}f^*_1 \wedge e^*_2 \wedge f^*_3 + b_{42}e^*_1 \wedge e^*_2 \wedge f^*_3 + b_{43}e^*_1 \wedge e^*_2 \wedge e^*_3 + b_{44}f^*_1 \wedge e^*_2 \wedge e^*_3 + \cdots.$$

Expanding (5.2) with these, we find

$$a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} = 1,$$

$$a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} = 0,$$

$$a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} + a_{14}b_{43} = 0,$$

$$a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} = 0.$$

Similarly, one computes that

$$\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & b_{13} & b_{14} \\
  b_{21} & b_{22} & b_{23} & b_{24} \\
  b_{31} & b_{32} & b_{33} & b_{34} \\
  b_{41} & b_{42} & b_{43} & b_{44}
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix},$$

and so $M \in \text{GL}_4(\mathbb{Z})$. Hence $A, B, C$ and $D$ indeed form a basis for $H^3(X_\sigma, \mathbb{Z})$. \qed
With a period relation in hand, we may now compute the structure of the intermediate Jacobian of $X_\sigma$. Note that, as we may only apply integral linear change of coordinates to $H^3(X_\sigma, \mathbb{Z})$, no significant simplification of the period matrix is possible if $\gamma, \delta \notin \mathbb{Z}$. Even if we restrict to the case where $\gamma, \delta$ are integers, we may only change (integral) bases to get integral $A', B', C', D' \in H^3(X_\sigma, \mathbb{Z})$ such that

$$
\begin{pmatrix}
A' & B' & C' & D'
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2\delta & 0 \\
0 & 0 & 0 & 2\delta
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\Omega) & \text{Im}(\Omega) & \text{Re}(\Psi) & \text{Im}(\Psi)
\end{pmatrix}.
$$

The only way this can be (integrally) reduced to block diagonal matrix, is if $2\delta = 1$, which is not possible if $\delta \in \mathbb{Z}$! Hence, the intermediate Jacobian is not a product of two elliptic curves in this case.

On the other hand, as $\mathbb{Z}$-modules we have

$$T := \langle \text{Re}(\Omega), \text{Im}(\Omega), \text{Re}(\Psi), \text{Im}(\Psi) \rangle \subset \langle A, B, C, D \rangle =: \Lambda$$

and the natural map on the tori gives an exact sequence

$$0 \rightarrow \Lambda/T \rightarrow \mathbb{C}^2/T \rightarrow \mathbb{C}^2/\Lambda \rightarrow 0.$$

Hence we have

$$\mathbb{C}^2/\Lambda \simeq (\mathbb{C}^2/T)/(\Lambda/T).$$

Recall $\mathbb{C}^2/T \simeq E_4 \times E_4$, so the intermediate Jacobian is a finite quotient of the
5.5. NON-RIGID CALABI-YAU THREEFOLDS

product of two copies of $E_4$.

With this insight, we may now show the intermediate Jacobians and the threefolds in the family $X_\sigma$ are defined over $\mathbb{Q}$ only in the original CM case with $E = E_4$ and $\sigma = i$. To see this, note that each point in $\ker(T \hookrightarrow \Lambda)$ corresponds to a torsion point on $E_4 \times E_4$. For example, the class $A = \frac{1}{2 \delta} (\Re(\Psi) - \Re(\Omega))$ in $\Lambda$ corresponds to a $2\delta$-torsion point on $\mathbb{C}/T = E_4 \times E_4$. As torsion points must have integral order, this means we must have $\delta \in \mathbb{Z}$. Similarly, the torsion point corresponding to the integral class $C$ forces $1/\gamma \in \mathbb{Z}$, and hence $E$ must have both periods $1$ and $\sigma$ algebraic. However, Schneider proves the following on the transcendence of elliptic curves defined over $\mathbb{Q}$, discussed in Theorem 7 and the discussion thereafter in [78].

**Theorem 5.6.** Let $E = \mathbb{C}/\langle \omega_1, \omega_2 \rangle$ be an elliptic curve defined over $\overline{\mathbb{Q}}$, and let $\tau = \omega_j/\omega_k$, for $j, k \in \{1, 2\}$, be the quotient in the upper half plane.

- If $E$ does not have CM, then $\tau$ is transcendental over $\mathbb{Q}$.

- If $E$ has CM, then $\mathbb{Q}(\tau)$ is an imaginary quadratic field.

In particular, if $E = \mathbb{C}/\langle 1, \sigma \rangle$ does not have CM, and $\sigma$ is algebraic, then $E$ is not defined over $\mathbb{Q}$. Otherwise, if $E$ has CM by $\mathbb{Q}(\sqrt{-D})$ it is known that $\tau = i$ if $D = 1$, $\tau = \sqrt{2}i$ if $D = 2$ and $\tau = -\frac{1}{2} + i\frac{\sqrt{-D}}{2}$ for all other $D$. Hence the only algebraic choice of $\sigma$ in the upper half plane that corresponds to an elliptic curve defined over $\mathbb{Q}$ such that the quotient

$$(E_4 \times E_4 \times E)/H_4$$

is defined over $\mathbb{Q}$ is $\sigma = i$. 

Theorem 5.7. Consider $E_4 : y^2 = x^3 - x$ and let $X$ be a crepant resolution of

$$(E_4 \times E_4 \times E_4)/\langle \iota_4 \times \iota_4 \times \iota_4^2, \iota_4 \times \iota_4^3 \times \text{id} \rangle.$$

The intermediate Jacobian $J(X)$ has a model defined over $\mathbb{Q}$.

Proof. From above, we know that the intermediate Jacobian of $X$ is isomorphic to $(\mathbb{C}^2/T)/(\ker(T \hookrightarrow \Lambda))$. As $\mathbb{C}^2/T \simeq E_4 \times E_4$, we note that each point in $\ker(T \hookrightarrow \Lambda)$ corresponds to a torsion point on $E_4 \times E_4$. These torsion points may require we consider the product $E_4 \times E_4$ over a number field and not just $\mathbb{Q}$, but as the torsion points of $E_4$ are Galois invariant, the quotient has a model defined over $\mathbb{Q}$. 

As the intermediate Jacobian $J(X)$ has a model defined over $\mathbb{Q}$, we may address arithmetic questions similar to those in the conjecture of Yui. In particular, we can try to compute the $L$-function of $J(X)$ and compare it with $L(X, s)$ computed above.

Writing the intermediate Jacobian as $J(X) = E_4^2/G$, where $G$ is a finite group of translations, we know $J(X)$ is an abelian surface, isogenous (over $\mathbb{Q}$) to $E_4 \times E_4$. Hence

$$L(J(X), s) = L(E_4 \times E_4, s).$$

To expand this as above, note the Hodge numbers for $E_4 \times E_4$ are completely determined by the Künneth formula. We have

$$H^2_\ell(E_4 \times E_4) \simeq (H^0_\ell(E_4) \otimes H^2_\ell(E_4))^2 \oplus H^1_\ell(E_4)^{\otimes 2},$$
so may explicitly study the Galois representation

$$\rho : G_Q \to \text{Aut} \left( V_\ell(E_4)^{\otimes 2} \oplus \bigwedge^2 V_\ell(E_4) \right)$$

where $V_\ell(E_4)$ is the Tate module of $E_4$ extended to $\mathbb{Q}_\ell$.

Let $\chi \in G_Q$ be the non-trivial character associated to $\mathbb{Q}(i)$. If $v, w$ is an eigenbasis for $[i]$ in $V_\ell(E_4)$, recall the action of $\text{Frob}_p$, where $p \neq 2, \ell$, is given by

$$\begin{pmatrix}
\alpha_p & 0 \\
0 & \beta_p
\end{pmatrix}$$

if $\chi(\text{Frob}_p) = 1$, and

$$\begin{pmatrix}
0 & h_p \\
k_p & 0
\end{pmatrix}$$

if $\chi(\text{Frob}_p) = -1$. Hence on $V_\ell(E_4)^{\otimes 2} \oplus \bigwedge^2 V_\ell(E_4)$ the action is given by

$$\begin{pmatrix}
\alpha_p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_p\beta_p & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_p\beta_p & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_p^2 & 0 & 0 \\
0 & 0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p
\end{pmatrix}$$
if \( \chi(\text{Frob}_p) = 1 \), and
\[
\begin{pmatrix}
0 & 0 & 0 & h_p^2 & 0 & 0 \\
0 & 0 & -p & 0 & 0 & 0 \\
0 & -p & 0 & 0 & 0 & 0 \\
k_p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p \\
0 & 0 & 0 & 0 & p & 0
\end{pmatrix}
\]
otherwise. Hence
\[
\text{tr}(\rho(\text{Frob}_p)) = \begin{cases} 
\alpha_p^2 + 2\alpha_p\beta_p + \beta_p^2 + 2p & \text{if } \chi(\text{Frob}_p) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]
As \( \alpha_p, \beta_p = \pm i\sqrt{p} \) when \( \chi(\text{Frob}_p) = -1 \), we may write
\[
\text{tr}(\rho(\text{Frob}_p)) = \left( (\alpha_p + \beta_p)^2 - 2p \right) + 2p\chi(\text{Frob}_p) + 2p.
\]
Hence
\[
L(J(X_i), s) = L(s, \chi_4^2)L(s - 1, \chi)^2\zeta(s - 1)^2
= L(\text{Sym}^2 f_4, s)L(s - 1, \chi)^2\zeta(s - 1)
\]
where \( f_4 \) is the modular form associated to \( E_4 \), and \( \zeta \) is the Riemann zeta function.

On the other hand, as \( J(X) \) is an abelian surface one can study its cohomology entirely via \( H^1_\ell(E_4 \times E_4) \), and so we can also see what pertinent information comes
from the $L^1$-function, the $L$-series associated to $H^1_{\ell}(E_4 \times E_4)$. We now have

$$H^1_{\ell}(E_4 \times E_4) \cong (H^0_{\ell}(E_4) \otimes H^1_{\ell}(E_4))^2$$

and so study the Galois representation

$$\eta: G_\mathbb{Q} \rightarrow \text{Aut}(V_{\ell}(E_4)^2).$$

For a prime $p \neq 2, \ell$, if $\chi(\text{Frob}_p) = 1$ the action of $\text{Frob}_p$ is given by

$$\begin{pmatrix}
\alpha_p & 0 & 0 & 0 \\
0 & \beta_p & 0 & 0 \\
0 & 0 & \alpha_p & 0 \\
0 & 0 & 0 & \beta_p
\end{pmatrix}$$

and otherwise by

$$\begin{pmatrix}
0 & h_p & 0 & 0 \\
k_p & 0 & 0 & 0 \\
0 & 0 & 0 & h_p \\
0 & 0 & k_p & 0
\end{pmatrix}$$

Thus

$$\text{tr}(\eta(\text{Frob}_p)) = \begin{cases} 
2(\alpha_p + \beta_p) & \text{if } \chi(\text{Frob}_p) = 1, \\
0 & \text{otherwise}
\end{cases}$$

and so, using the CM condition,

$$\text{tr}(\eta(\text{Frob}_p)) = 2(\alpha_p + \beta_p).$$
From this we see the $L$-function of $H^1_1(E_1 \times E_4)$ is rather simple,

$$L^1(E_4 \times E_4, s) = L(E_4, s)^2.$$ 

Both the $L$-function and the $L^1$-function have components coming from the elliptic curve, so a similar conjecture could be made. However, without more evidence we refrain from making any guesses based on our examples – coming from triple product of elliptic curves – that may be too simple!

As the classes coming from the resolution are much easier to work with when computing the respective intermediate Jacobians, we note the work above immediately implies the following, as each $(2,1)$-class coming from resolving a fixed elliptic curve is independent from all the other $(2,1)$-classes.

**Theorem 5.8.** With notation as in Theorem 4.1, the intermediate Jacobians of the following respective Calabi-Yau threefolds are, as complex varieties,

$$J(\widetilde{E}_3^6/I_6) \simeq E_6^2,$$

$$J(\widetilde{E}_3^6/O_6) \simeq E_6^6,$$

In particular, they all have models defined over $\mathbb{Q}$.

We may combine the case where we only have $(2,1)$-classes coming from the resolution as well as one from the Künneth component, as before.

**Theorem 5.9.** With notation as in Theorems 4.1 and 4.2, the intermediate Jacobian $J(\widetilde{E}_3^3/J_6)$ is isogenous to $E_6^4$ over $\mathbb{Q}$, the intermediate Jacobian $J(\widetilde{E}_3^3/N_6)$ is isogenous to $E_6^{12}$ over $\mathbb{Q}$, and $J(\widetilde{E}_4^3/J_4)$ is isogenous to $E_4^8$ over $\mathbb{Q}$. 
We may also compute the $L$-functions of the above varieties, but note that when the powers get larger, e.g., $E_6^{12}$, the $L$-functions are rather messy, so we may again use the fact that the $L$-function of an abelian variety is completely determined by the $L^1$-function. The above gives

**Theorem 5.10.** With notation as above, we have

\[
L^1(J(E_6^3/\Gamma_6), s) = L(E_6, s)^2,
\]

\[
L^1(J(E_6^3/\mathcal{N}_6), s) = L(E_6, s)^{12},
\]

\[
L^1(J(E_6^3/O_6), s) = L(E_6, s)^6,
\]

\[
L^1(J(E_6^3/J_6), s) = L(E_6, s)^4,
\]

and

\[
L^1(J(E_4^3/J_4), s) = L(E_4, s)^8.
\]

Comparing with the $L$-functions computed in Section 4.3 one may ask a question extending the conjecture of Yui.

**Question.** If $X$ is a Calabi-Yau threefold of CM-type defined over $\mathbb{Q}$ then does the intermediate Jacobian $J(X)$ have a model defined over $\mathbb{Q}$? If so, and the $L^1$-function satisfies

\[
L^1(J(X), s) = L(s, \chi)^k
\]

for some positive integer $k$, then is

\[
L(X, s) = L(s, \chi^3)L(s - 1, \chi)^{k-1}
\]
or the problem with the exponent $3$ occurs, and there exists a non-quadratic twist $\psi$ of $J(X)$ and $X$ such that

$$L^1(J(X) \otimes \psi, s) = L(s, \kappa)^k$$

and

$$L(X \otimes \psi, s) = L(s, \kappa^3)L(s - 1, \kappa)^{k-1}.$$  

With this, the Borcea examples are the only ones left. Starting with $E_4^3/I_4$ we again use the period relation

$$\begin{pmatrix} e^* & f^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} dx & dy \end{pmatrix}$$

to find the following period relation for the threefold

$$\begin{pmatrix} A_1 & B_1 & A_2 & B_2 & A_3 & B_3 & A_4 & B_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \text{Re}(\Omega_4) & \text{Im}(\Omega_4) & \text{Re}(\Psi_1) & \text{Im}(\Psi_1) & \text{Re}(\Psi_2) & \text{Im}(\Psi_2) & \text{Re}(\Psi_3) & \text{Im}(\Psi_3) \end{pmatrix}$$
where

\[ A_1 = e_1^* \wedge e_2^* \wedge e_3^*, \quad B_1 = f_1^* \wedge e_2^* \wedge e_3^*, \]
\[ A_2 = e_1^* \wedge f_2^* \wedge e_3^*, \quad B_2 = e_1^* \wedge e_2^* \wedge f_3^*, \]
\[ A_3 = f_1^* \wedge f_2^* \wedge e_3^*, \quad B_3 = f_1^* \wedge e_2^* \wedge f_3^*, \]
\[ A_4 = e_1^* \wedge f_2^* \wedge f_3^*, \quad B_4 = f_1^* \wedge f_2^* \wedge f_3^*, \]

while \( \Omega_4 = dz_1 \wedge dz_2 \wedge dz_3 \) and \( \Psi_k \) is the \((2, 1)\)-class \( dz_1 \wedge \cdots \wedge d\bar{z}_k \wedge \cdots \wedge dz_3 \). There is an integral change of basis with an upper triangular period matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

and so there exist an integral basis (abusing notation) such that

\[ \text{Re}(\Omega_4) = A_1, \quad \text{Im}(\Omega_4) = B_1, \]
\[ \text{Re}(\Psi_1) = A_1 + 2A_2, \quad \text{Im}(\Psi_1) = B_1 + 2B_2, \]
\[ \text{Re}(\Psi_2) = A_1 + 2A_3, \quad \text{Im}(\Psi_2) = B_1 + 2B_3, \]
\[ \text{Re}(\Psi_3) = A_1 + 2A_2 + 2A_3 + 4A_4, \quad \text{Im}(\Psi_3) = 3B_1 + 2B_2 + 2B_3 + 4B_4. \]

Hence

\[ A_1 = \text{Re}(\Omega_4), \quad B_1 = \text{Im}(\Omega_4), \]

\[ A_2 = \frac{1}{2}(\text{Re}(\Psi_1) - \text{Re}(\Omega_4)), \quad B_2 = \frac{1}{2}(\text{Im}(\Psi_1) - \text{Im}(\Omega_4)), \]

\[ A_3 = \frac{1}{2}(\text{Re}(\Psi_2) - \text{Re}(\Omega_4)), \quad B_3 = \frac{1}{2}(\text{Im}(\Psi_2) - \text{Im}(\Omega_4)), \]

\[ A_4 = \frac{1}{4}(\text{Re}(\Psi_3) + \text{Re}(\Omega_4) - \text{Re}(\Psi_1) - \text{Re}(\Psi_2)), \]

\[ B_4 = \frac{1}{4}(\text{Im}(\Psi_3) - \text{Im}(\Omega_4) - \text{Im}(\Psi_1) - \text{Im}(\Psi_2)). \]

By the Künneth formula, as before we find that

\[ H^3(\widetilde{E_3^3/I_4}, \mathbb{Z}) = \langle A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 \rangle \]

\[ \neq \langle \text{Re}(\Omega_4), \text{Re}(\Psi_1), \text{Re}(\Psi_2), \text{Re}(\Psi_3), \text{Im}(\Omega_4), \text{Im}(\Psi_1), \text{Im}(\Psi_2), \text{Im}(\Psi_3) \rangle \]

as \( \mathbb{Z} \)-modules. Hence, the intermediate Jacobian is a finite quotient of \( E_4^4 \), has a model defined over \( \mathbb{Q} \), and we find

\[ L(J(\widetilde{E_3^3/I_4}, s)) = L(E_4^4, s) \]

and

\[ L^1(J(\widetilde{E_3^3/I_4}, s)) = L(E_4, s)^4. \]

Note this satisfies the conclusion of the question above as well.

Similarly, when working with the Borcea construction and elliptic curve \( E_6 \) one gets a period relation where the period matrix, under an integral change of basis, is
upper triangular. Hence, the intermediate Jacobian is a finite quotient of $E_6^4$ and so isogenous to $E_6^4$. The respective $L$-functions are

$$L(J(\tilde{E}_6^3/K_6, s)) = L(E_6^4, s),$$

and

$$L^1(J(\tilde{E}_6^3/K_6, s)) = L(E_6, s)^4.$$

We note that, as before, considering the automorphic examples coming from $E_6 \times E_6 \times E$ with the quotient by either $J_6$ or $N_6$, does not give a Calabi-Yau threefold (defined over $\mathbb{Q}$) with a model of the intermediate Jacobian defined over $\mathbb{Q}$ unless $E = E_6$. We can, however, get a few more examples over Calabi-Yau threefolds defined over $\mathbb{Q}$, whose intermediate Jacobians have models defined over $\mathbb{Q}$.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, and let $\iota$ be a non-symplectic involution on $E$. The Borcea group

$$B = \langle \iota \times \iota \times \text{id}, \iota \times \text{id} \times \iota \rangle$$

acts on the threefold $E^3$ and a crepant resolution of the threefold $E^3/B$ is a Calabi-Yau threefold defined over $\mathbb{Q}$.

**Theorem 5.11.** Let $E$ be one of the 9 elliptic curves with CM by the full ring of integers of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$, with a model defined over $\mathbb{Q}$. The intermediate Jacobian of $\tilde{E}^3/B$ has a model defined over $\mathbb{Q}$. Moreover, if we have $L(E, s) = L(s, \chi)$, then

$$L(J(\tilde{E}^3/B, s)) = L(E^4, s),$$
and

\[ L^1(J(\widetilde{E^3/B}), s) = L(E, s)^4 = L(s, \chi)^4. \]

In particular, we see these examples also satisfy our natural generalization of Conjecture 5.1.

**Proof.** If \( E = \mathbb{C}/(1, \alpha + i\beta) \), then there are integral classes \( e^*, f^* \in H^1(E, \mathbb{Z}) \) such that

\[
\begin{pmatrix} e^* & f^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}.
\]

Hence, for \( \widetilde{E^3/B} \) we have a period relation

\[
\begin{pmatrix} A_1 & B_1 & A_2 & B_2 & A_3 & B_3 & A_4 & B_4 \end{pmatrix} P
\]

\[
= \begin{pmatrix} \text{Re}(\Omega) & \text{Im}(\Omega) & \text{Re}(\Psi_1) & \text{Im}(\Psi_1) & \text{Re}(\Psi_2) & \text{Im}(\Psi_2) & \text{Re}(\Psi_3) & \text{Im}(\Psi_3) \end{pmatrix}
\]

where \( P \) is the matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha & -\beta \\
\alpha & \beta & \alpha & \beta & \alpha & -\beta & \alpha & \beta \\
\alpha & \beta & \alpha & -\beta & \alpha & \beta & \alpha & \beta \\
\alpha^2 - \beta^2 & 2\alpha\beta & \alpha^2 - \beta^2 & 2\alpha\beta & \alpha^2 + \beta^2 & 0 & \alpha^2 + \beta^2 & 0 \\
\alpha^2 - \beta^2 & 2\alpha\beta & \alpha^2 + \beta^2 & 0 & \alpha^2 - \beta^2 & 2\alpha\beta & \alpha^2 + \beta^2 & 0 \\
\alpha^2 - \beta^2 & 2\alpha\beta & \alpha^2 + \beta^2 & 0 & \alpha^2 + \beta^2 & 0 & \alpha^2 - \beta^2 & 2\alpha\beta \\
\alpha^3 + \alpha\beta^2 & \alpha^2\beta + \beta^3 & \alpha^3 + \alpha\beta^2 & \alpha^2\beta + \beta^3 & \alpha^3 + \alpha\beta^2 & \alpha^2\beta + \beta^3 & \alpha^3 + \alpha\beta^2 & \alpha^2\beta + \beta^3
\end{pmatrix}
\]
and

\[
A_1 = e_1^* \wedge e_2^* \wedge e_3^*, \quad B_1 = f_1^* \wedge e_2^* \wedge e_3^*,
\]

\[
A_2 = e_1^* \wedge f_2^* \wedge e_3^*, \quad B_2 = e_1^* \wedge e_2^* \wedge f_3^*,
\]

\[
A_3 = f_1^* \wedge f_2^* \wedge e_3^*, \quad B_3 = f_1^* \wedge e_2^* \wedge f_3^*,
\]

\[
A_4 = e_1^* \wedge f_2^* \wedge f_3^*, \quad B_4 = f_1^* \wedge f_2^* \wedge f_3^*,
\]

while \( \Omega = dz_1 \wedge dz_2 \wedge dz_3 \) and \( \Psi_k \) is the \((2, 1)\)-class \( dz_1 \wedge \cdots \wedge d\bar{z}_k \wedge \cdots \wedge dz_3 \).

As \( E \) has CM, \( \mathbb{Q}(\alpha + i\beta) \) is a quadratic extension, and these are the only such extensions by [73] or the discussion around Schneider’s work above. Hence, there is an integral change of basis giving an upper triangular period matrix, so that the respective \( L \)-functions are

\[
L(\widetilde{E^3}/B, s) = L(s, \chi^3)L(s - 1, \chi)^3,
\]

where \( L(E, s) = L(s, \chi) \),

\[
L(J(\widetilde{E^3}/B), s) = L(E^4, s),
\]

and

\[
L^1(J(\widetilde{E^3}/B), s) = L(E, s)^4 = L(s, \chi)^4.
\]
Bibliography


