

# IRREDUCIBILITY OF RANDOM HILBERT SCHEMES

by

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# Abstract

We prove that a random Hilbert scheme that parametrizes the closed subschemes with a fixed Hilbert polynomial in some projective space is irreducible and nonsingular with probability greater than 0.5. To consider the set of nonempty Hilbert schemes as a probability space, we transform this set into a disjoint union of infinite binary trees, reinterpreting Macaulay's classification of admissible Hilbert polynomials. Choosing discrete probability distributions with infinite support on the trees establishes our notion of random Hilbert schemes. To bound the probability that random Hilbert schemes are irreducible and nonsingular, we show that at least half of the vertices in the binary trees correspond to Hilbert schemes with unique Borel-fixed points.

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# Chapter 1

## Introduction

Hilbert schemes parametrizing the closed subschemes with a fixed Hilbert polynomial of some projective space are among the most important moduli spaces in algebraic geometry. Despite their importance, with the exception of Hilbert schemes parametrizing hypersurfaces [ACG11, E.g. 2.3], and points in the plane [Fog68], the basic geometric features of typical Hilbert schemes are still poorly understood. Moreover, techniques for producing arbitrarily pathological Hilbert schemes are known, generating Hilbert schemes with many irreducible components [Iar72], [FP96], with generically nonreduced components [Mum62], and with arbitrary singularity types [Vak06], showing that anything that can go wrong, ultimately does. We must proceed with caution when studying Hilbert schemes, but the questions of how prevalent the troublesome features are, and how to avoid them, remain unclear. What should we expect from a “random” Hilbert scheme? Our broad goals are to develop a method of randomly sampling from the set of all Hilbert schemes, and to study the pervasiveness of geometric features. We achieve our goals, by endowing the set of nonempty Hilbert schemes with the structure of a discrete probability space, and by establishing that irreducible Hilbert schemes are unexpectedly common.



More precisely, let  $\text{Hilb}[p, n]$  denote the Hilbert scheme parametrizing all closed subschemes of  $\mathbb{P}^n$  with Hilbert polynomial  $p$ . For each positive  $c \in \mathbb{Z}$ , we identify an infinite binary tree  $\mathcal{H}_c$  whose vertices correspond to Hilbert schemes  $\text{Hilb}[p, n]$  parametrizing codimension  $c = n - \deg p$  subschemes, by reinterpreting Macaulay's classification of admissible Hilbert polynomials. Assuming that the vertices at a fixed height in  $\mathcal{H}_c$  are equally likely, any choice of a probability distribution supported on  $\mathbb{N}$  for the height determines a probability distribution on the tree. Combining these choices with a probability distribution for the parametrized codimension  $c$  turns the set  $\mathcal{H}$  of all Hilbert schemes of the form  $\text{Hilb}[p, n]$  into the sample space of a discrete probability space, where  $p$  is an admissible Hilbert polynomial, and  $n$  is a positive integer greater than  $\deg p$ .

**Theorem 1.1.** *The probability that a random Hilbert scheme in  $\mathcal{H}$  is irreducible and nonsingular is greater than 0.5.*

This theorem demonstrates that the average Hilbert scheme is rather simple, in contrast with the many known pathological examples. Moreover, this theorem counterintuitively suggests that the geometry of the majority of Hilbert schemes is understandable.

We prove Theorem 1.1 in two broad steps. First, the set of rational polynomials that are Hilbert polynomials of homogeneous ideals is characterized by [Mac27], and we explore two forms of this characterization. We define two binary relations on the set of these 'admissible' Hilbert polynomials, and we show that the binary relations generate all such polynomials, and organize them into an infinite binary tree; see Theorem 2.13. We then generalize this binary tree structure to lexicographic ideals by defining two analogous mappings on the set of all lexicographic ideals, which partition the set by codimension into infinitely many copies of the previous binary tree; see Theorem 3.11. Every Hilbert scheme contains a unique lexicographic ideal, thus this infinite collection of binary trees

of lexicographic ideals can be viewed geometrically as an infinite collection of binary trees of Hilbert schemes; see Theorem 4.1, [Mac27], [Har66], [Ree95], and [PS05]. Endowing the binary trees of Hilbert schemes with basic structures of discrete probability spaces then allows us to study the probabilities that random Hilbert schemes have certain properties; Examples 4.7–4.10 consider some interesting random variables on the Hilbert forest.

The second broad step is to identify a family of irreducible Hilbert schemes that is sufficiently dense within the Hilbert trees and Hilbert forest. We do this by first studying saturated strongly stable ideals, which generalize lexicographic ideals and define Borel-fixed points on Hilbert schemes; see [Har66], [BS87a], [Ree95], [RS97], and [PS05]. An algorithm is known for generating saturated strongly stable ideals, and we explore the interplay between this algorithm and the numerators of Hilbert series, or  $K$ -polynomials, of saturated strongly stable ideals; see Algorithm 5.12, [Ree92], [Moo12], [CLMR11], and [MS05]. In particular, combining Proposition 6.6 and Proposition 6.8, we prove our second main result.

**Theorem 1.2.** *Let  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a saturated strongly stable ideal with Hilbert polynomial  $p$ , let  $L[p, n]$  be the corresponding lexicographic ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , and let  $K_I$  be the numerator of the Hilbert series of  $I$ . If  $I \neq L[p, n]$ , then we have  $\deg K_I < \deg K_{L[p, n]}$ .*

Using Theorem 1.2, we prove that every vertex in the Hilbert forest has a child corresponding to an irreducible Hilbert scheme. Specifically, Theorem 1.2 leads to a new proof of the main result of [Got89]. Moreover, our identification of the graph-theoretic and probabilistic nature of the set of all Hilbert schemes of projective spaces allows us to understand the density of this family of irreducible Hilbert schemes, and to prove Theorem 1.1.

## Synopsis

Our main result in Chapter 2 is Theorem 2.13, which reinterprets Macaulay’s classification of the numerical polynomials appearing as Hilbert functions of homogeneous ideals in the standard  $\mathbb{Z}$ -graded polynomial ring; see [Mac27], [Har66]. These admissible Hilbert polynomials have uniquely determined Macaulay–Hartshorne expressions and Gotzmann expressions as sums and differences of binomial coefficients, and we define two mappings  $\Phi$  and  $\Psi$  on the set of admissible Hilbert polynomials via the Gotzmann expressions. Observing that these two mappings generate all admissible Hilbert polynomials leads to Theorem 2.13.

Lexicographic ideals are of central importance, and the main result of Chapter 3 identifies repetitions of the infinite binary tree structure from the previous chapter in the set of lexicographic ideals. To prove the main result, we define two mappings on the set of all lexicographic ideals, and show in Proposition 3.7 and Proposition 3.9 that they act on lexicographic ideals analogously to the actions of  $\Phi$  and  $\Psi$  on admissible Hilbert polynomials. Theorem 3.11 then proves that these mappings organize the lexicographic ideals into a forest of infinitely many copies of the binary tree from Chapter 2, indexed by the codimensions of lexicographic ideals.

We begin Chapter 4 with a geometric version of the previous infinite binary trees, giving the set of all Hilbert schemes of projective spaces a graph structure isomorphic to the one on lexicographic ideals; see Theorem 4.1. This graph-theoretic structure allows us to study the set of all Hilbert schemes as a discrete probability space, and we define random variables for the height, parametrized dimension, radius, and parametrized degree on Hilbert trees and the Hilbert forest, and then we compute various probabilities and expected values associated with these random variables.

We review the fundamentals of saturated strongly stable ideals in Chapter 5. In particular, we need Algorithm 5.12, which computes all saturated strongly stable ideals with a given Hilbert polynomial, in a chosen polynomial ring.

The goal of Chapter 6 is to achieve a detailed understanding of aspects of the Hilbert functions of saturated strongly stable ideals. In Proposition 6.6, we describe how inequalities in the degrees of  $K$ -polynomials initialize with respect to Algorithm 5.12. Then, in Proposition 6.8, we give a condition on the minimal set of monomial generators of a saturated strongly stable ideal that ensures that an inequality of between degrees of  $K$ -polynomials persists. Combining these two propositions yields the main result of this section, Theorem 6.10, which in particular tells us about Hilbert functions of saturated strongly stable ideals.

Finally, in Chapter 7, we apply Theorem 6.10, proving in Theorem 7.3 that the only saturated strongly stable ideals with Hilbert polynomials of the form  $\Phi(q)$  are lexicographic, where  $q$  is an admissible Hilbert polynomial. From here, Corollary 7.5 states that every Hilbert scheme of the form  $\text{Hilb}[\Phi(q), n]$  has a unique point defined by a saturated strongly stable ideal. In particular, this provides a new proof of the main result of [Got89] via methods of combinatorial commutative algebra, and following our graph-theoretic interpretations of the classifications of admissible Hilbert polynomials and Hilbert schemes, we identify the density of this family in the Hilbert forest. Thus, in addition to the new proof of the result of [Got89], we provide a probabilistic interpretation for this family of Hilbert schemes, proving in Theorem 7.7 that the probability that a random Hilbert scheme is irreducible equals at least 0.5.

**Conventions.** Throughout this thesis,  $\mathbb{K}$  denotes an algebraically closed field,  $\mathbb{N}$  is the set of nonnegative integers, and  $\mathbb{K}[x_0, x_1, \dots, x_n]$  denotes the standard  $\mathbb{Z}$ -graded poly-

mial ring in  $n + 1$  variables. The Hilbert function, Hilbert polynomial, Hilbert series, and  $K$ -polynomial of the quotient  $\mathbb{K}[x_0, x_1, \dots, x_n]/I$  of the polynomial ring by a homogeneous ideal  $I$  are denoted  $h_I$ ,  $p_I$ ,  $H_I$ , and  $K_I$ , respectively.

# Chapter 2

## The Tree of Admissible Hilbert Polynomials

In this chapter, we identify a natural graph structure on the set of one-variable polynomials that determine nonempty Hilbert schemes of projective spaces. The pioneering work [Mac27] gives a classification of the numerical polynomials that are Hilbert polynomials of  $\mathbb{Z}$ -graded ideals as sums of binomial coefficients, and we show that there are two natural binary relations on these binomial sums; these binary relations generate all possible binomial sums and give rise to the Macaulay tree. The Macaulay tree is an infinite binary tree, whose vertices are exactly the admissible Hilbert polynomials, and whose edges correspond to the two mappings on the sets of admissible Hilbert polynomials.

To begin, let  $\mathbb{K}$  be an algebraically closed field, and let  $\mathbb{K}[x_0, x_1, \dots, x_n]$  denote the homogeneous coordinate ring of  $n$ -dimensional projective space  $\mathbb{P}^n$ . In other words, the ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$  is the standard  $\mathbb{Z}$ -graded polynomial ring in  $n + 1$  variables. Since we will not consider other gradings, we simply use the words ‘graded’ and ‘homogeneous’ to refer to this standard  $\mathbb{Z}$ -grading. Let  $M$  be a finitely generated graded  $\mathbb{K}[x_0, x_1, \dots, x_n]$ -module.

The **Hilbert function**  $h_M: \mathbb{Z} \rightarrow \mathbb{Z}$  of  $M$  is defined by  $h_M(i) := \dim_{\mathbb{K}}(M_i)$  for every  $i \in \mathbb{Z}$ . A famous theorem of Hilbert states that every  $M$  has a **Hilbert polynomial**  $p_M$ , that is, there exists a polynomial  $p_M(t) \in \mathbb{Q}[t]$  such that, for sufficiently large  $i \in \mathbb{N}$ , we have  $h_M(i) = p_M(i)$ ; see Theorem 4.1.3 of [BH93]. For a given homogeneous ideal  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ , we write  $h_I$  and  $p_I$  for the Hilbert function and Hilbert polynomial of the quotient module  $\mathbb{K}[x_0, x_1, \dots, x_n]/I$ , respectively, following the conventions in [GS]. If  $X \subseteq \mathbb{P}^n$  is a nonempty closed subscheme, then there is a unique saturated homogeneous ideal  $I_X \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  such that  $X = \text{Proj}(\mathbb{K}[x_0, x_1, \dots, x_n]/I_X)$ ; see Corollary II.5.16 of [Har77]. We define the **Hilbert function**  $h_X$  of  $X$  to be the Hilbert function  $h_{I_X} = h_{\mathbb{K}[x_0, x_1, \dots, x_n]/I_X}$ , and the **Hilbert polynomial**  $p_X$  of  $X$  to be the associated Hilbert polynomial  $p_{I_X} = p_{\mathbb{K}[x_0, x_1, \dots, x_n]/I_X}$ .

As a first example, we describe the Hilbert polynomial of  $\mathbb{P}^n$ .

**Example 2.1** (Hilbert polynomial of projective space). Fix a nonnegative integer  $n \in \mathbb{N}$ . The “stars-and-bars” argument, as in Section 1.2 of [Sta12], shows that the number of homogeneous polynomials of degree  $i \in \mathbb{Z}$  in  $n + 1$  variables is given by the binomial coefficient  $\binom{n+i}{n}$ . That is, letting  $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ , we have  $h_S(i) = \binom{n+i}{n}$ . The equality  $h_S(i) = p_S(i)$  is only valid for  $i \geq -n$ , because  $p_S$  only has the finitely many roots  $-n, -(n-1), \dots, -1$ , whereas  $h_S(i) = 0$  for every  $i < 0$ .

**Remark 2.2** (Binomial coefficients as polynomials). As in Example 2.1, we often encounter binomial coefficients that we wish to treat as polynomials. Following the definition in Section 5.1 of [GKP94], for a variable  $t$ , and for  $a, b \in \mathbb{Z}$ , we define the binomial coefficient  $\binom{t+a}{b}$  to be  $\frac{(t+a)(t+a-1)\cdots(t+a-b+1)}{b!} \in \mathbb{Q}[t]$  if  $b \geq 0$ , and  $\binom{t+a}{b} := 0$  otherwise. If  $b \geq 0$ , then this gives a degree  $b$  polynomial in  $t$ , with zeros  $-a, -(a-1), \dots, -(a-b+1)$ , so that  $\binom{t+a}{b}|_{t=j} \neq \binom{j+a}{b} = 0$  for all  $j < -a$ . Interestingly, page 533 in [Mac27] introduces distinct

notation for polynomial and integer binomial coefficients.

A polynomial is an **admissible** Hilbert polynomial if it is the Hilbert polynomial of some nonempty closed subscheme of a projective space. We are specifically interested in admissible Hilbert polynomials because they correspond to nonempty Hilbert schemes parametrizing closed subschemes with a fixed Hilbert polynomial of some projective space. A pioneering study in combinatorial commutative algebra, the paper [Mac27] gives the following classification of admissible Hilbert polynomials. Independently discovered in the study of connectedness of Hilbert schemes, this classification also appears in [Har66].

**Proposition 2.3.** *The following conditions are equivalent:*

(i) *The polynomial  $p(t) \in \mathbb{Q}[t]$  is an admissible Hilbert polynomial.*

(ii) *There exist integers  $e_0 \geq e_1 \geq \dots \geq e_d > 0$  such that*

$$p(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}.$$

(iii) *There exist integers  $b_1 \geq b_2 \geq \dots \geq b_r \geq 0$  such that*

$$p(t) = \sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}.$$

*Moreover, if  $p$  is admissible, then  $p$  has a unique corresponding sequence of  $e_i$ 's, and a unique corresponding sequence of  $b_j$ 's.*

*Proof.*

(i)  $\Leftrightarrow$  (ii) This is explained in Part I of [Mac27]; see the formula for “ $\chi(\ell)$ ” at the bottom of page 536 in [Mac27]. For a more recent account using modern terminology,



see Corollary 3.3 and Corollary 5.7 in [Har66].

(i)  $\Leftrightarrow$  (iii) This follows from Erinnerung 2.4 of [Got78]. See also Exercise 4.2.17 in [BH93].

The uniqueness of the sequences of integers attached to an admissible polynomial is also explained by the aforementioned sources.  $\square$

We define the **Macaulay–Hartshorne expression** of an admissible Hilbert polynomial  $p$  as its unique expression of the form  $p(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$ , for  $e_0 \geq e_1 \geq \dots \geq e_d > 0$ . Similarly, we define the **Gotzmann expression** of an admissible Hilbert polynomial  $p$  to be its unique expression of the form  $p(t) = \sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$ , where  $b_1 \geq b_2 \geq \dots \geq b_r \geq 0$ . From the Macaulay–Hartshorne expression, we easily identify the degree  $d$  of  $p$ , and its leading coefficient  $e_d/d!$ . From the Gotzmann expression, we also see the degree  $b_1$  of  $p$ , as well as the **Gotzmann number**  $r$ , which gives a certain regularity bound; see Remark 2.6.

The alternate classification of admissible Hilbert polynomials by Part (iii) of Proposition 2.3 leads to an especially simple description of the edges of the Macaulay tree. We obtain the following conjugacy between Macaulay–Hartshorne and Gotzmann expressions. The **conjugate** partition to a partition  $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_k)$  of an integer  $\ell = \sum_{i=1}^k \lambda_i$  is the partition of  $\ell$  obtained from the Ferrers diagram of  $\lambda$  by interchanging rows and columns, having  $\lambda_i - \lambda_{i+1}$  parts equal to  $i$ ; see Section 1.8 of [Sta12].

**Lemma 2.4.** *If  $p(t) \in \mathbb{Q}[t]$  is an admissible Hilbert polynomial with Macaulay–Hartshorne expression  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$ , for  $e_0 \geq e_1 \geq \dots \geq e_d > 0$ , and Gotzmann expression  $\sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$ , for  $b_1 \geq b_2 \geq \dots \geq b_r \geq 0$ , then we have  $r = e_0$  and the nonnegative partition  $(b_1, b_2, \dots, b_r)$  is conjugate to the partition  $(e_1, e_2, \dots, e_d)$ .*

*Proof.* We analyze the expressions by rewriting the Macaulay–Hartshorne expression of  $p$

in the form

$$\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} + \sum_{i=0}^{d-1} \binom{t+i-e_d}{i+1} - \binom{t+i-e_i}{i+1}.$$

Splitting off the first sum  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1}$ , we first prove that

$$\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d}.$$

Working by induction, if  $d = 0$ , then we obtain  $\binom{t}{1} - \binom{t-e_0}{1} = e_0$ . If  $d > 0$ , then we split the left-side sum into its degree  $d$  and degree less-than  $d$  parts, which gives the expression

$$\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \left[ \sum_{i=0}^{d-1} \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} \right] + \binom{t+d}{d+1} - \binom{t+d-e_d}{d+1}.$$

The induction hypothesis ensures that this equals  $\left[ \sum_{j=1}^{e_d} \binom{t+(d-1)-(j-1)}{d-1} \right] + \binom{t+d}{d+1} - \binom{t+d-e_d}{d+1}$ .

Examining the two main parts of this sum separately, we first apply the addition formula for binomial coefficients (see Section 5.1 of [GKP94]) to the right-hand side to obtain

$$\begin{aligned} \binom{t+d}{d+1} - \binom{t+d-e_d}{d+1} &= \binom{t+d}{d+1} - \binom{t+d-1}{d+1} + \binom{t+d-1}{d+1} - \binom{t+d-e_d}{d+1} \\ &= \binom{t+d-1}{d} + \binom{t+d-1}{d+1} - \binom{t+d-e_d}{d+1} \\ &= \dots \\ &= \binom{t+d-1}{d} + \binom{t+d-2}{d} + \dots + \binom{t+d-e_d}{d} \\ &= \left[ \sum_{j=2}^{e_d} \binom{t+d-(j-1)}{d} \right] + \binom{t+d-e_d}{d}. \end{aligned}$$

Next, applying the addition formula to  $\binom{t+d}{d}$  yields

$$\begin{aligned}
\binom{t+d}{d} &= \binom{t+d-1}{d-1} + \binom{t+d-1}{d} \\
&= \binom{t+d-1}{d-1} + \binom{t+(d-1)-1}{d-1} + \binom{t+(d-1)-1}{d} \\
&= \dots \\
&= \binom{t+d-1}{d-1} + \binom{t+(d-1)-1}{d-1} + \binom{t+(d-1)-2}{d-1} + \dots \\
&\quad \dots + \binom{t+(d-1)-(e_d-1)}{d-1} + \binom{t+(d-1)-(e_d-1)}{d} \\
&= \left[ \sum_{j=1}^{e_d} \binom{t+(d-1)-(j-1)}{d-1} \right] + \binom{t+d-e_d}{d}.
\end{aligned}$$

This latter expression shows that  $\sum_{j=1}^{e_d} \binom{t+(d-1)-(j-1)}{d-1} = \binom{t+d}{d} - \binom{t+d-e_d}{d}$ , and adding this to  $\binom{t+d}{d+1} - \binom{t+d-e_d}{d+1}$  gives  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d}$ , as desired.

We now prove the general claim by recursion. We rewrite the Macaulay–Hartshorne expression  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$  as

$$\begin{aligned}
p(t) &= \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} + \sum_{i=0}^{d-1} \binom{t+i-e_d}{i+1} - \binom{t+i-e_i}{i+1} \\
&= \sum_{j=1}^{e_d} \binom{t+d-(j-1)}{d} + \left[ \sum_{i=0}^{d-1} \binom{s+i}{i+1} - \binom{s+i-(e_i-e_d)}{i+1} \right]_{s:=t-e_d}.
\end{aligned}$$

Isolating the second part of this sum, we repeat the analogous decomposition. This yields

$$\begin{aligned}
\left[ \sum_{i=0}^{d-1} \binom{s+i}{i+1} - \binom{s+i-(e_i-e_d)}{i+1} \right]_{s:=t-e_d} &= \left[ \sum_{i=0}^{d-1} \binom{s+i}{i+1} - \binom{s+i-(e_{d-1}-e_d)}{i+1} \right] + \\
&\quad \sum_{i=0}^{d-2} \left( \binom{s+i-(e_{d-1}-e_d)}{i+1} - \binom{s+i-(e_i-e_d)}{i+1} \right) \Big|_{s:=t-e_d},
\end{aligned}$$

and the first part of the right-hand side equals  $\sum_{k=1}^{e_{d-1}-e_d} \binom{s+(d-1)-(k-1)}{d-1}$ , by the previous paragraph. Reindexing with  $j := k+e_d$ , and evaluating at  $s := t-e_d$ , gives  $\sum_{j=e_d+1}^{e_{d-1}} \binom{t+(d-1)-(j-1)}{d-1}$ . Therefore, in our original sum we have  $b_1 = b_2 = \dots = b_{e_d} = d$  for the first  $e_d$  parts, while  $b_{e_d+1} = b_{e_d+2} = \dots = b_{e_{d-1}} = d-1$ , for the next  $e_{d-1} - e_d$  parts. Continuing this process of decomposing our sum shows that the partition  $e_0 \geq e_1 \geq \dots \geq e_d > 0$  transforms uniquely into the partition  $b_1 = b_2 = \dots = b_{e_d} = d$ ,  $b_{e_d+1} = b_{e_d+2} = \dots = b_{e_{d-1}} = d-1$ ,  $\dots$ ,  $b_{e_1+1} = b_{e_1+2} = \dots = b_{e_0} = 0$ , and that  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_d}{i+1} = \sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$ . In other words,  $r = e_0$ , and exactly  $e_i - e_{i+1}$  parts in the partition associated to the Gotzmann expression of  $\mathfrak{p}$  are equal to  $i$ , for all  $0 \leq i \leq d$ , setting  $e_{d+1} := 0$ . Finally, we have

$$\sum_{j=1}^r b_j = (d)(e_d) + (d-1)(e_{d-1} - e_d) + \dots + (1)(e_1 - e_2) + (0)(e_0 - e_1) = \sum_{i=1}^d e_i,$$

and it follows from the definition that the nonnegative partition  $(b_1, b_2, \dots, b_r)$  is conjugate to the partition  $(e_1, e_2, \dots, e_d)$ .  $\square$

We repeat the above correspondence explicitly.

**Remark 2.5.** If  $\mathfrak{p}(t)$  is an admissible polynomial determined by the partition  $(e_0, e_1, \dots, e_d)$  and the nonnegative partition  $(b_1, b_2, \dots, b_r)$ , then we have

$$b_j = \begin{cases} d & \text{if } j \in \{1, 2, \dots, e_d\}, \\ d-1 & \text{if } j \in \{e_d+1, e_d+2, \dots, e_{d-1}\}, \\ \vdots & \\ 0 & \text{if } j \in \{e_1+1, e_1+2, \dots, e_0\}. \end{cases}$$

Conversely,  $e_i = \#\{j \mid b_j \geq i\}$ . Because the Macaulay–Hartshorne expression of  $\mathfrak{p}$  is determined by a partition (with *exactly*  $d+1$  parts) and the Gotzmann expression of  $\mathfrak{p}$  is

determined by a *nonnegative* partition (with *at most*  $r$  parts), the “conjugacy” between them is really a correspondence  $(e_0, e) \leftrightarrow (r, e')$  that remembers the Gotzmann number  $r = e_0$ , where  $e := (e_1, e_2, \dots, e_d)$  is a partition with  $e_0 \geq e_1$ , and  $e'$  is the partition conjugate to  $e$ .

We define the *Macaulay–Hartshorne partition* of an admissible Hilbert polynomial  $p(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$  to be the partition  $(e_0, e_1, \dots, e_d)$ . Similarly, we define the *Gotzmann partition* of an admissible Hilbert polynomial  $p(t) = \sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$  to be the nonnegative partition  $(b_1, b_2, \dots, b_r)$ .

**Remark 2.6.** Lemma 2.9 of [Got78] implies that if  $I$  is a homogeneous ideal whose Hilbert polynomial has Gotzmann expression  $\sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$ , then the saturation of  $I$  is  $r$ -regular; also see Theorem 4.3.2 in [BH93]. In other words, the Gotzmann number  $r$  of  $I$  (or of  $p$ ), puts an upper bound on the regularity of every saturated ideal with Hilbert polynomial  $p$ . This regularity bound is sharp, as it is attained by the lexicographic ideal; we show this in Remark 3.5. As an application, Remark 3.4 of [Got78] gives low-dimension projective embeddings of Hilbert schemes, improving the technique from Lecture 15 of [Mum66].

**Example 2.7** (Expressions for the twisted cubic). The twisted cubic curve  $X \subset \mathbb{P}^3$  is defined by the ideal  $I_X \subset \mathbb{K}[x_0, x_1, x_2, x_3]$  of  $(2 \times 2)$ -minors of the matrix  $\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$ . The Macaulay–Hartshorne and Gotzmann expressions for the Hilbert polynomial of the twisted cubic are

$$\begin{aligned} p_X(t) = 3t + 1 &= \left[ \binom{t+0}{0+1} - \binom{t+0-4}{0+1} \right] + \left[ \binom{t+1}{1+1} - \binom{t+1-3}{1+1} \right] \\ &= \binom{t+1}{1} + \binom{t+1-1}{1} + \binom{t+1-2}{1} + \binom{t+0-3}{0}, \end{aligned}$$

respectively. The associated partitions are  $(e_0, e_1) = (4, 3)$  and  $(b_1, b_2, b_3, b_4) = (1, 1, 1, 0)$ . Observe that  $(3)$  is conjugate to  $(1, 1, 1)$ , but that  $(1, 1, 1, 0)$  has  $r = e_0 = 4$ .

We next describe two binary relations on the set of admissible Hilbert polynomials. Let  $p$  be an admissible Hilbert polynomial, with Macaulay–Hartshorne partition  $(e_0, e_1, \dots, e_d)$  and Gotzmann partition  $(b_1, b_2, \dots, b_r)$ . We define a mapping  $\Phi$ , from the set of admissible Hilbert polynomials to itself, that takes  $p$  to the polynomial  $\Phi(p)$  whose corresponding Macaulay–Hartshorne partition is given by  $(e_0, e_0, e_1, \dots, e_d)$ , and whose Gotzmann partition is given by  $(b_1 + 1, b_2 + 1, \dots, b_r + 1)$ . Explicitly,  $\Phi(p)$  is the polynomial

$$\begin{aligned} [\Phi(p)](t) &= \binom{t+0}{0+1} - \binom{t+0-e_0}{0+1} + \sum_{i=1}^{d+1} \binom{t+i}{i+1} - \binom{t+i-e_{i-1}}{i+1} \\ &= \sum_{i=1}^r \binom{t+(b_i+1)-(i-1)}{b_i+1}, \end{aligned}$$

which is admissible, by Proposition 2.3.

To define the second binary relation on the set of admissible Hilbert polynomials, let  $\Psi: \mathbb{Q}[t] \rightarrow \mathbb{Q}[t]$  denote the mapping taking each polynomial  $p$  to  $1 + p$ . To be precise, if  $p(t)$  is an admissible Hilbert polynomial with Macaulay–Hartshorne partition  $(e_0, e_1, \dots, e_d)$  and Gotzmann partition  $(b_1, b_2, \dots, b_r)$ , then  $\Psi(p) = 1 + p$  is also an admissible Hilbert polynomial, with Macaulay–Hartshorne partition  $(e_0 + 1, e_1, e_2, \dots, e_d)$  and Gotzmann partition  $(b_1, b_2, \dots, b_r, 0)$ . Explicitly, we have

$$\begin{aligned} [\Psi(p)](t) &= \binom{t+0}{0+1} - \binom{t+0-(e_0+1)}{0+1} + \sum_{i=1}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} \\ &= \sum_{i=1}^r \binom{t+b_i-(i-1)}{b_i} + \binom{t+0-r}{0}. \end{aligned}$$

Therefore, the restriction of  $\Psi$  defines a mapping from the set of admissible Hilbert polynomials to itself, which we also denote by  $\Psi$ .

We consider a basic example.

**Example 2.8** ( $\Phi$  and  $\Psi$  on a point in projective space). The simplest admissible Hilbert polynomial is that of one reduced point in projective space, with Gotzmann expression  $1 = \binom{t+0-0}{0}$ . We have  $\Phi(1) = \binom{t+1-0}{1} = t + 1$ , which is the Hilbert polynomial of a reduced line in projective space. Also,  $\Psi(1) = \binom{t+0-0}{0} + \binom{t+0-1}{0} = 2$  is the Hilbert polynomial of two points in projective space.

**Remark 2.9** (Comparisons between  $\Phi$  and Macaulay representations). Given positive integers  $c, d \in \mathbb{Z}$ , the  $d$ -th **Macaulay representation** of  $c$  is the unique way of writing  $c$  as a sum of the form  $c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}$ , where  $k_d > k_{d-1} > \cdots > k_\delta \geq \delta > 0$ . We compare the mapping  $\Phi$  with the operations  $c \mapsto c^{(d)} := \binom{k_{d+1}}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_\delta+1}{\delta+1}$  and  $c \mapsto c_{(d)} := \binom{k_{d-1}}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_\delta-1}{\delta}$ ; see Definition 3.2 in [Gre10], or Proposition 4.2.8 and Theorem 4.2.10 in [BH93]. These operations work by adding or subtracting 1 to numerators or denominators in sums of binomial coefficients, and we wonder whether  $\Phi$  is derived from one of these operations. Suppose that  $p(t) = \sum_{i=1}^r \binom{t+b_i-(i-1)}{b_i}$  is an admissible Hilbert polynomial with Gotzmann partition  $(b_1, b_2, \dots, b_r)$ , and that  $d \in \mathbb{N}$  is nonzero. Observe that

$$p(d) = \sum_{i=1}^r \binom{d+b_i-(i-1)}{b_i} = \sum_{i=1}^r \binom{d+b_i-(i-1)}{d-(i-1)}$$

gives the  $d$ -th Macaulay representation of  $c := p(d)$ . Therefore, we have

$$c^{(d)} = \sum_{i=1}^r \binom{d+b_i+1-(i-1)}{d+1-(i-1)} \quad \text{and} \quad c_{(d)} = \sum_{i=1}^r \binom{d+b_i-1-(i-1)}{d-(i-1)}$$

where the former is a  $(d+1)$ -th Macaulay representation of  $c^{(d)}$ , and the latter is a  $d$ -th Macaulay representation of  $c_{(d)}$ . On the other hand, we have the similar but distinct expression

$$\sum_{i=1}^r \binom{d+b_i+1-(i-1)}{b_i+1} = \sum_{i=1}^r \binom{d+b_i+1-(i-1)}{d-(i-1)}$$

for the  $d$ -th Macaulay representation of  $[\Phi(\mathfrak{p})](d)$ .

To better understand the mapping  $\Phi$ , we utilize the **backwards difference operator**  $\nabla: \mathbb{Q}[t] \rightarrow \mathbb{Q}[t]$ , which takes each polynomial  $q$  to  $[\nabla(q)](t) := q(t) - q(t-1)$ ; compare with Lemma 4.1.2 of [BH93]. We collect some elementary properties that show the interplay between  $\Psi$ ,  $\Phi$  and  $\nabla$ . Backwards differences are discrete analogues of derivatives, and Part (ii) of Lemma 2.10 shows that  $\Phi$  gives indefinite sums, or antidifferences, of admissible polynomials. Part (iii) is then a discrete analogue of the fundamental theorem of calculus.

**Lemma 2.10.** *If  $\mathfrak{p}(t)$  is an admissible Hilbert polynomial with Macaulay–Hartshorne partition  $(e_0, e_1, \dots, e_d)$  and Gotzmann partition  $(b_1, b_2, \dots, b_r)$ , then we have the following:*

$$(i) \quad [\nabla(\mathfrak{p})](t) = \sum_{j=1}^r \binom{t+(b_j-1)-(j-1)}{b_j-1} = \sum_{i=0}^{d-1} \binom{t+i}{i+1} - \binom{t+i-e_{i+1}}{i+1};$$

$$(ii) \quad \nabla \Psi^a \Phi(\mathfrak{p}) = \mathfrak{p}, \text{ for all } a \in \mathbb{N};$$

(iii) *if  $\deg \mathfrak{p} > 0$  and  $k \in \{1, 2, \dots, r\}$  is the largest index such that  $b_k \neq 0$ , then we have*

$$\mathfrak{p} - \Phi \nabla(\mathfrak{p}) = r - k, \text{ but if } \deg \mathfrak{p} = 0, \text{ then } \nabla(\mathfrak{p}) = 0; \text{ and}$$

$$(iv) \quad [(\Phi \Psi - \Psi \Phi)(\mathfrak{p})](t) = t - r.$$

**Remark 2.11.** Setting  $a = 0$  in Part (ii) shows that  $\nabla \Phi(\mathfrak{p}) = \mathfrak{p}$ , so that Part (iii) shows that  $(\nabla \Phi - \Phi \nabla)(\mathfrak{p}) = r - k$ . In other words, performing  $\Phi$  and then  $\nabla$  returns  $\mathfrak{p}$ , but performing  $\nabla$  and then  $\Phi$  may result in a new polynomial that differs from  $\mathfrak{p}$  in its constant term.

*Proof.*

(i) Because  $\nabla$  is a linear operator on  $\mathbb{Q}[t]$ , it suffices to prove the statement on polynomials of the form  $\binom{t+b-i}{b}$  for  $b, i \in \mathbb{N}$ . By definition of  $\nabla$  and the addition formula for



binomial coefficients, we have

$$[\nabla(\mathfrak{p})](t) = \binom{t+b-i}{b} - \binom{t-1+b-i}{b} = \binom{t+b-1-i}{b-1}.$$

(ii) We have  $[\nabla \Phi(\mathfrak{p})](t) = \nabla \left( \sum_{i=1}^r \binom{t+b_i+1-(i-1)}{b_{i+1}} \right) = \sum_{i=1}^r \binom{t+b_i-(i-1)}{b_i} = \mathfrak{p}(t)$ , by part (i).

Further,  $\nabla \Psi^a \Phi(\mathfrak{p}) = \nabla(a + \Phi(\mathfrak{p}))$ , which equals  $\nabla \Phi(\mathfrak{p}) = \mathfrak{p}$ .

(iii) Because  $\deg \mathfrak{p} > 0$ , there exists a largest index  $k \in \{1, \dots, r\}$  such that  $b_k \neq 0$ .

Thus,  $[\Phi \nabla(\mathfrak{p})](t) = \Phi \left( \sum_{i=1}^k \binom{t+b_i-1-(i-1)}{b_{i-1}} \right) = \sum_{i=1}^k \binom{t+b_i-(i-1)}{b_i}$ . That is, the terms of the form  $\binom{t+0-(j-1)}{0}$  are dropped, and because they all equal 1, we are left with  $\mathfrak{p} - \Phi \nabla(\mathfrak{p}) = r - k$ .

(iv) We have  $[\Phi \Psi(\mathfrak{p})](t) = \sum_{j=1}^{r+1} \binom{t+b_j+1-(j-1)}{b_{j+1}}$ , where  $b_{r+1} := 0$ . On the other hand,  $[\Psi \Phi(\mathfrak{p})](t) = \sum_{j=1}^r \binom{t+b_j+1-(j-1)}{b_{j+1}} + \binom{t+0-r}{0}$ , and taking the difference yields the polynomial  $[(\Phi \Psi - \Psi \Phi)(\mathfrak{p})](t) = (t+1-r) - 1 = t-r$ .  $\square$

The following example illustrates Lemma 2.10.

**Example 2.12** (Hilbert polynomials of the twisted cubic and first Hirzebruch surface).

Example 2.7 shows that the Hilbert polynomial of the twisted cubic curve  $X$  in  $\mathbb{P}^3$  equals

$\mathfrak{p}_X(t) = 3t + 1 = \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-3}{0}$ , and we compute

$$\begin{aligned} [\Phi(\mathfrak{p}_X)](t) &= \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-2}{1} = \frac{3}{2}t^2 + \frac{5}{2}t - 1, \text{ and} \\ [\nabla \Phi(\mathfrak{p}_X)](t) &= \nabla \left( \frac{3}{2}t^2 + \frac{5}{2}t - 1 \right) = 3t + 1, \end{aligned}$$

which returns the original polynomial  $\mathfrak{p}_X$ .

Conversely, we find that

$$\begin{aligned} [\nabla(\mathfrak{p}_X)](t) &= \binom{t}{0} + \binom{t-1}{0} + \binom{t-2}{0} = 3, \text{ while} \\ [\Phi \nabla(\mathfrak{p}_X)](t) &= \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} = 3t, \end{aligned}$$

which differs from  $\mathfrak{p}_X$  by a constant.

Furthermore, if we let  $q(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$ , then we also obtain  $[\nabla(q)](t) = 3t + 1$ . In fact, the expression

$$q(t) = [\Psi^2 \Phi(\mathfrak{p}_X)](t) = \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-2}{1} + \binom{t-4}{0} + \binom{t-5}{0}$$

shows that the polynomial  $q$  is an admissible Hilbert polynomial. This polynomial is the Hilbert polynomial of the (minimally embedded) first Hirzebruch surface, also known as the blow-up of  $\mathbb{P}^2$  at a point.

These operations endow the set of admissible Hilbert polynomials with the structure of a graph. The *infinite binary tree*, or simply the *binary tree*, is the graph defined as follows:

- its vertex set is a countably infinite set partitioned into subsets indexed by nonnegative integers  $j \in \mathbb{N}$ , where the  $j$ -th subset contains exactly  $2^j$  vertices;
- every vertex in the  $j$ -th subset is connected by an edge to two distinct vertices in the  $(j+1)$ -th subset, and, for positive  $j \in \mathbb{N}$ , every vertex in the  $j$ -th subset is connected by an edge to exactly one vertex in the  $(j-1)$ -th subset.

The vertices in the  $j$ -th subset of the partition have *height*  $j$ . The *root* of the binary tree is the unique vertex at height 0, or equivalently, with degree 2. There is a unique *path*, i.e.

sequence of distinct adjacent vertices, between each pair of vertices in the binary tree; see Theorem 1.5.1 of [Die10]. Thus, in any binary tree, every vertex is uniquely determined by the path that links it to the root. The infinite binary tree is typically visualized as in Figure 2.1.

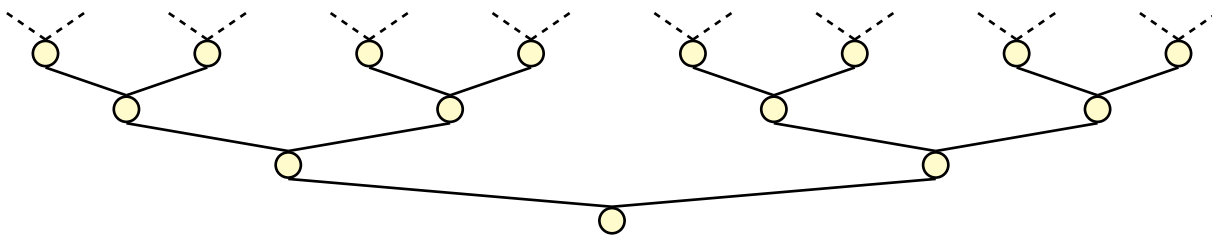


Figure 2.1: The infinite binary tree

**Theorem 2.13.** *The graph whose vertices correspond to admissible Hilbert polynomials, and whose edges correspond to pairs of the form  $(p, \Psi(p))$  and  $(p, \Phi(p))$ , for all admissible Hilbert polynomials  $p$ , forms an infinite binary tree. Moreover, the root of the tree corresponds to the constant polynomial 1.*

The *Macaulay tree*  $\mathcal{M}$  is defined to be the infinite binary tree of admissible Hilbert polynomials, described in the proof of Theorem 2.13.

*Proof.* We prove that the admissible Hilbert polynomial  $p(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$  equals

$$p = \Psi^{e_0-e_1} \Phi \Psi^{e_1-e_2} \Phi \dots \Phi \Psi^{e_{d-1}-e_d} \Phi \Psi^{e_d-1}(1),$$

where  $e_0 \geq e_1 \geq \dots \geq e_d > 0$ . We proceed by induction on the length  $d+1$  of the partition  $(e_0, e_1, \dots, e_d)$ . If  $d=0$ , then  $\binom{t+0}{0+1} - \binom{t+0-e_0}{0+1} = e_0 = \Psi^{e_0-1}(1)$  proves the claim. Supposing that  $d > 0$ , and applying the induction hypothesis to the partition  $(e_1, e_2, \dots, e_d)$ , shows

that

$$\sum_{i=0}^{d-1} \binom{t+i}{i+1} - \binom{t+i-e_{i+1}}{i+1} = \Psi^{e_1-e_2} \Phi \Psi^{e_2-e_3} \Phi \dots \Phi \Psi^{e_{d-1}-e_d} \Phi \Psi^{e_d-1}(1).$$

Applying  $\Phi$  to both sides, we obtain

$$e_1 + \sum_{i=1}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} = \Phi \Psi^{e_1-e_2} \Phi \Psi^{e_2-e_3} \Phi \dots \Phi \Psi^{e_{d-1}-e_d} \Phi \Psi^{e_d-1}(1),$$

and then applying  $\Psi^{e_0-e_1}$  to both sides yields the desired equality. Because every finite binary sequence of  $\Psi$ 's and  $\Phi$ 's has a unique such expression, we obtain the result.  $\square$

**Remark 2.14** (Alternate proof of Theorem 2.13). Another inductive proof of Theorem 2.13 can be given by using Part (iii) of Lemma 2.10 to establish existence of the binary expressions, and Part (ii) of Lemma 2.10 to establish uniqueness.

A portion of the Macaulay tree is displayed in Figure 2.2, with the polynomials given by their Gotzmann expressions. The same tree is depicted in Figure 2.3, with polynomials relative to the monomial basis.

**Remark 2.15** (Paths in the Macaulay tree via partitions). The path from the root 1 in  $\mathcal{M}$  to an admissible Hilbert polynomial  $p(t) := \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1} = \sum_{j=1}^r \binom{t+b_j-(j-1)}{b_j}$  is also encoded in the Gotzmann expression. If  $b_1 = b_2 = \dots = b_r$ , then Lemma 2.4 implies that the path from 1 to  $p$  is specified by applying  $\Psi^{r-1}$  to 1, and then applying  $\Phi^{b_r}$  to  $r = \Psi^{r-1}(1)$ . Similarly, if  $b_1 = b_2 = \dots = b_j > b_{j+1}$ , where  $1 \leq j \leq r-1$ , then Lemma 2.4 implies that the path from 1 to  $p$  is specified by applying  $\Psi^{j-1}$  to 1, then applying  $\Phi^{b_j-b_{j+1}}$  to  $j = \Psi^{j-1}(1)$ , and so on.

Further, from the explicit expression for the path given in the proof of Theorem 2.13,

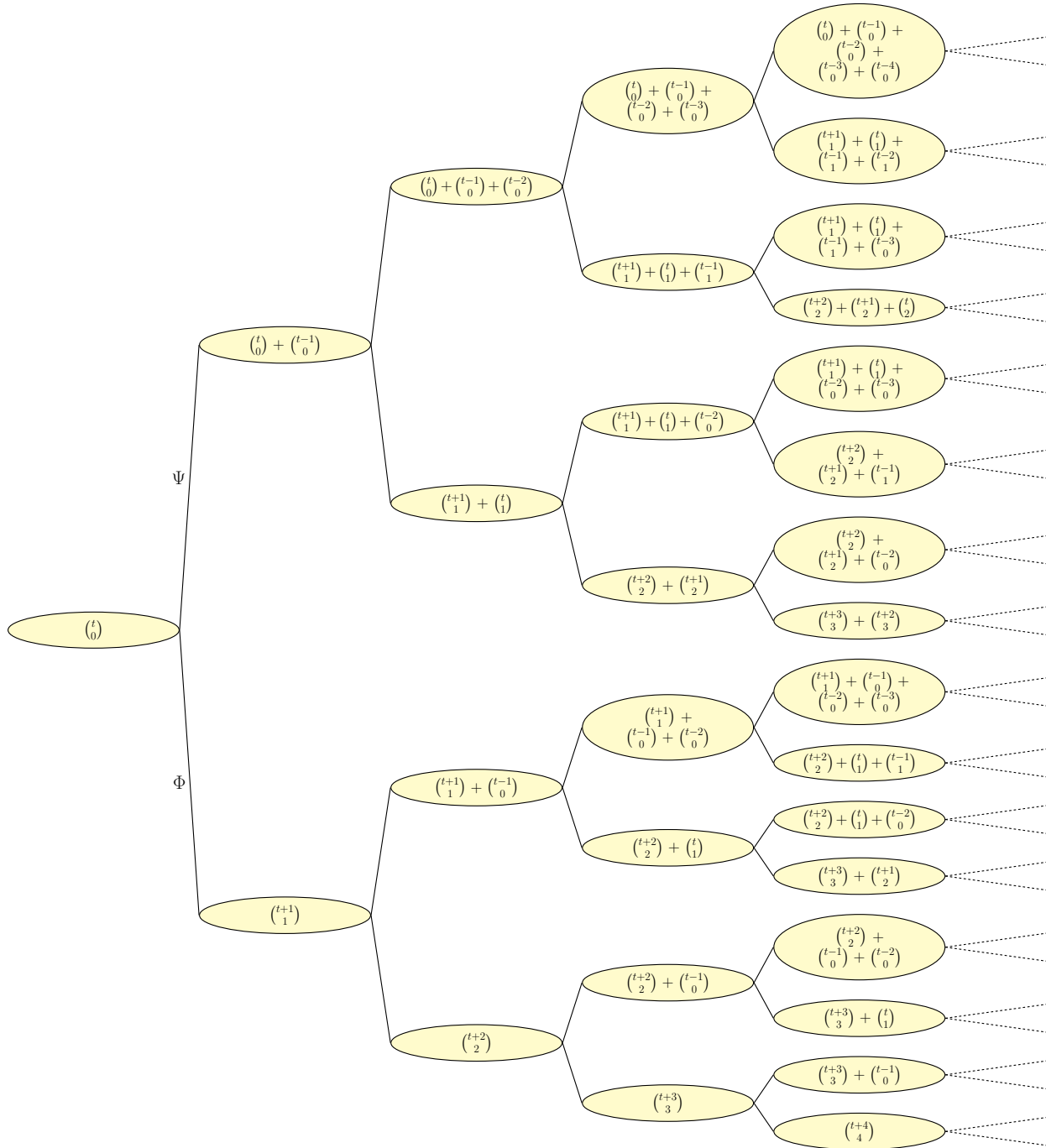


Figure 2.2: The Macaulay tree  $\mathcal{M}$  to height 4 with Gotzmann expressions

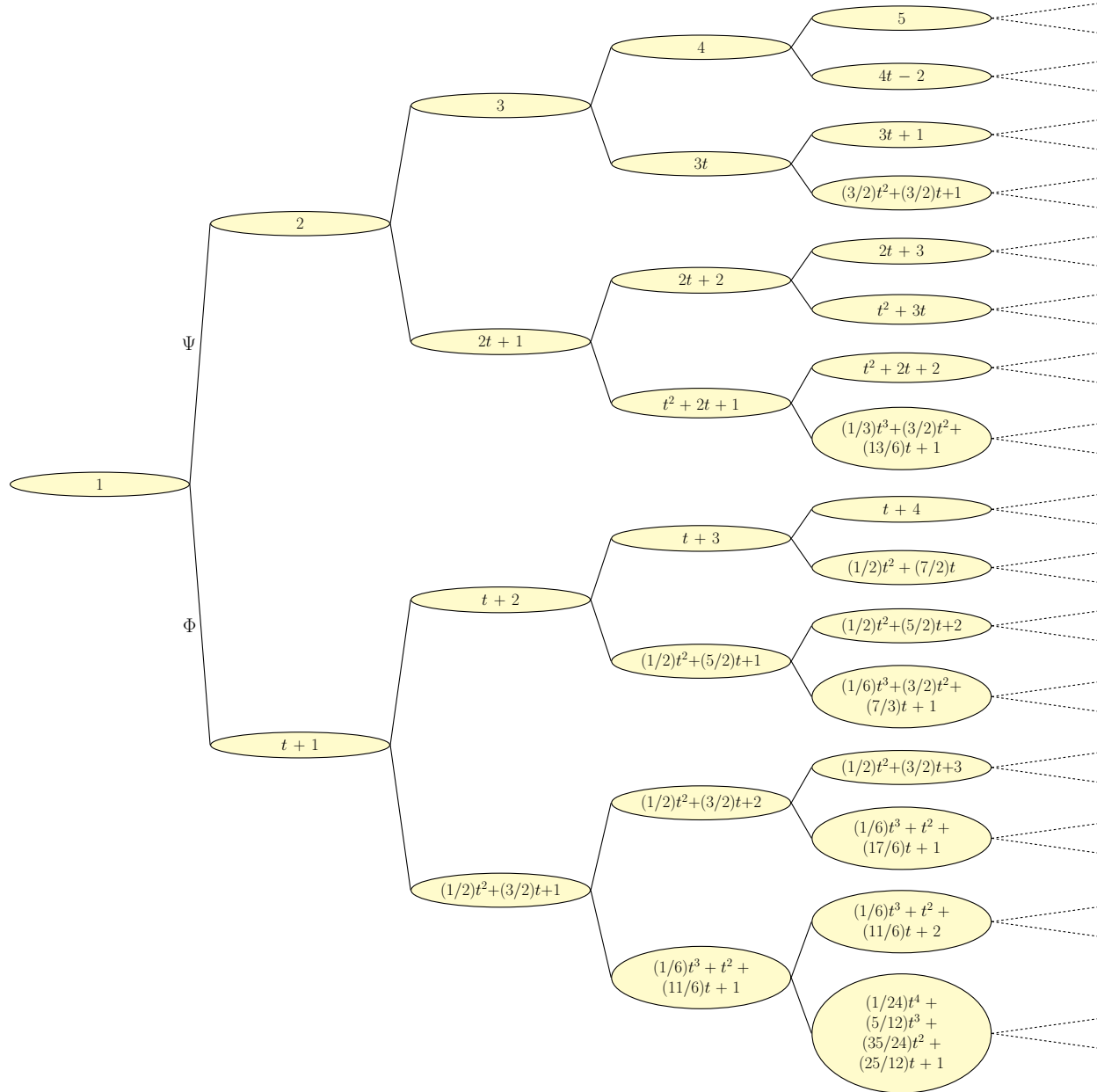


Figure 2.3: The Macaulay tree  $\mathcal{M}$  to height 4

we see that the height of the vertex  $p(t)$  is  $e_0 + d - 1 = r + b_1 - 1$ .

**Example 2.16** (Paths for the twisted cubic and first Hirzebruch surface). The Hilbert polynomial of the twisted cubic curve  $X \subset \mathbb{P}^3$  has associated partitions  $(e_0, e_1) = (4, 3)$  and

$(b_1, b_2, b_3, b_4) = (1, 1, 1, 0)$ ; see Example 2.7. The proof of Theorem 2.13 and the Macaulay–Hartshorne expression give  $\Psi^{4-3} \Phi \Psi^{3-1}(1) = 3t + 1$ , while Remark 2.15 and the Gotzmann expression equivalently give  $\Psi^1 \Phi^{1-0} \Psi^{3-1}(1) = 3t + 1$ ; this path is shown in Figure 2.4. From Example 2.12, we find that the unique path associated to the Hilbert polynomial of the minimal-degree embedding of the Hirzebruch surface  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subset \mathbb{P}^4$  is similarly given by  $\Psi^2 \Phi \Psi \Phi \Psi^2(1)$ ; see Figure 2.5.

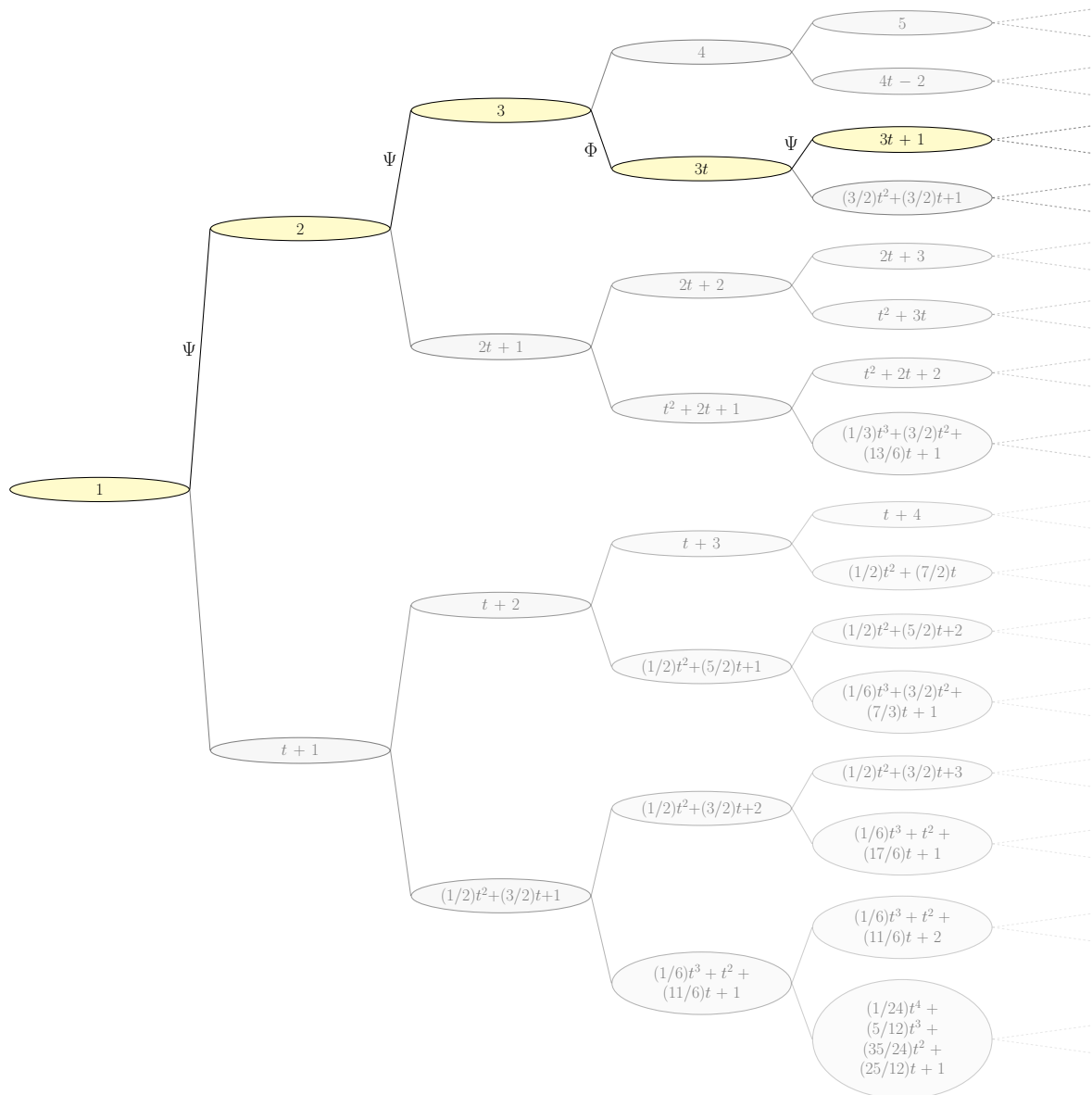


Figure 2.4: The path from 1 to  $p(t) := 3t + 1$  in the Macaulay tree

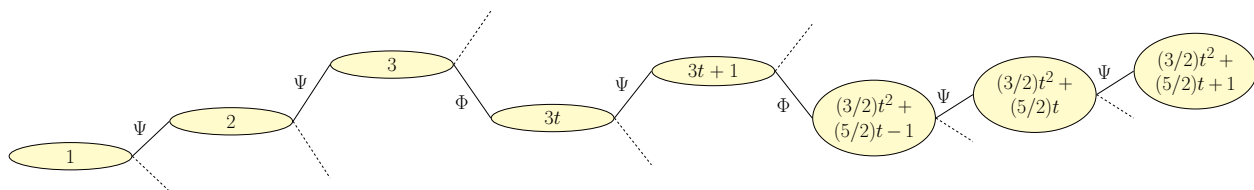


Figure 2.5: The path from 1 to  $p(t) := \frac{3}{2}t^2 + \frac{5}{2}t + 1$  in the Macaulay tree



# Chapter 3

## The Forest of Lexicographic Ideals

In this chapter, we connect lexicographic ideals with the Macaulay tree  $\mathcal{M}$  defined in Chapter 2. Specifically, in Theorem 3.11 we show that the tree of admissible Hilbert polynomials reappears infinitely many times as binary trees of saturated lexicographic ideals, with exactly one tree  $\mathcal{L}_c$  corresponding to each positive codimension  $c \in \mathbb{Z}$ . We achieve this by studying two binary relations of the set of all lexicographic ideals, defined in analogy with the mappings  $\Phi$  and  $\Psi$ . We list minimal sets of monomial generators for lexicographic ideals in terms of Macaulay–Hartshorne expressions in Lemma 3.4, and these allow us to understand the Hilbert polynomials of images of lexicographic ideals under our two mappings.

Lexicographic ideals, also known as lex-segment ideals, are monomial ideals whose homogeneous pieces are spanned by monomials that are as large as possible in the lexicographic ordering. These combinatorially defined homogeneous ideals play a central role in the classification of admissible Hilbert polynomials given by [Mac27]. The papers [Har66] and [PS05], as well as [Ree95], and [RS97] studying the connectedness, the radii, and the smoothness of Hilbert schemes, respectively, all show that the combinatorial nature of

lexicographic ideals conveniently captures essential geometric information about Hilbert schemes.

Monomial orderings are essential in computational commutative algebra, enabling us to extend polynomial division and row-reduction to polynomial rings in several variables. For any vector  $u := (u_0, u_1, \dots, u_n) \in \mathbb{N}^{n+1}$ , let  $x^u$  denote the monomial  $x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$ . The **lexicographic ordering**, or simply the **lex ordering**, is the relation  $>_{\text{lex}}$  on the monomials of the polynomial ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$  defined by  $x^u >_{\text{lex}} x^v$  if the first nonzero coordinate of  $u - v \in \mathbb{Z}^{n+1}$  is positive, where  $u, v \in \mathbb{N}^{n+1}$ . We consider a basic example.

**Example 3.1** (Lex-order of monomials). In the polynomial ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , the variables satisfy  $x_0 >_{\text{lex}} x_1 >_{\text{lex}} \cdots >_{\text{lex}} x_n$  in the lex ordering. Further, if we assume that  $n \geq 2$ , then  $x_0 x_2^2 >_{\text{lex}} x_1^4 >_{\text{lex}} x_1^3$ .

Given a homogeneous ideal  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ , the lexicographic ordering gives rise to two monomial ideals that are associated to  $I$ . First, if  $h_I$  is the Hilbert function of  $I$ , then the **lexicographic ideal** for the Hilbert function  $h_I$  in  $\mathbb{K}[x_0, x_1, \dots, x_n]$  is the monomial ideal  $L[h_I, n]$  whose  $i$ -th graded piece is spanned by the  $h_{\mathbb{K}[x_0, x_1, \dots, x_n]}(i) - h_I(i) = \dim_{\mathbb{K}} I_i$  lex-largest monomials in  $\mathbb{K}[x_0, x_1, \dots, x_n]_i$ , for every  $i \in \mathbb{Z}$ . By definition, we have the equality  $h_I = h_{L[h_I, n]}$ . Proposition 2.21 in [MS05] proves that  $L[h_I, n]$  is a homogeneous ideal of  $\mathbb{K}[x_0, x_1, \dots, x_n]$ ; also see Section II in [Mac27]. More importantly to us, if  $p_I$  is the Hilbert polynomial of the homogeneous ideal  $I$  in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , then the **saturated lexicographic ideal**  $L[p_I, n]$  for the Hilbert polynomial  $p_I$  is the monomial ideal

$$(L[h_I, n] : \langle x_0, x_1, \dots, x_n \rangle^\infty) := \bigcup_{j \geq 1} \{f \in \mathbb{K}[x_0, x_1, \dots, x_n] \mid f \langle x_0, x_1, \dots, x_n \rangle^j \subseteq L[h_I, n]\}.$$

Saturation with respect to the irrelevant ideal  $\langle x_0, x_1, \dots, x_n \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  does not

affect the Hilbert function in large degrees, hence the lexicographic ideal  $L[\mathfrak{p}_I, n]$  also has Hilbert polynomial  $\mathfrak{p}_I$ .

**Example 3.2** (Lexicographic ideals for three points). If  $X \subset \mathbb{P}^2$  consists of three distinct noncollinear points, then the Hilbert function of  $I_X \subset \mathbb{K}[x_0, x_1, x_2]$  is given by the values  $h_X(\mathbb{N}) = (1, 3, 3, 3, 3, \dots)$ . The lexicographic ideal in  $\mathbb{K}[x_0, x_1, x_2]$  for the Hilbert function  $h_X$  equals  $L(h_X; 2) = \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ . Therefore, the saturation of  $L[h_X, 2]$  with respect to the irrelevant ideal  $\langle x_0, x_1, x_2 \rangle$  is the lexicographic ideal  $L[3, 2] = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ , with Hilbert function given by  $h_{L[3,2]}(\mathbb{N}) = (1, 2, 3, 3, 3, 3, \dots)$ .

By definition, a saturated lexicographic ideal of the form  $L[\mathfrak{p}, n]$  exists if there is a homogeneous ideal  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  with Hilbert polynomial  $\mathfrak{p}$ . In particular, we must have  $n > \deg \mathfrak{p} = \dim \text{Proj}(\mathbb{K}[x_0, x_1, \dots, x_n]/I)$ ; see Theorem I.7.5 in [Har77]. However, it is not yet clear that for all choices of admissible Hilbert polynomials  $\mathfrak{p}$ , and all  $n > \deg \mathfrak{p}$ , there exists a lexicographic ideal  $L[\mathfrak{p}, n]$ . Remark 3.3 shows that if a lexicographic ideal  $L[\mathfrak{p}, n]$  exists, then  $L[\mathfrak{p}, n+1]$  also exists.

**Remark 3.3** (A mapping on lexicographic ideals). Suppose that  $n \in \mathbb{N}$  is fixed, and that  $L[\mathfrak{p}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is a proper lexicographic ideal with Hilbert polynomial  $\mathfrak{p}$ . We can explicitly describe the lexicographic ideal  $L[\mathfrak{p}, n+1] \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  via the following homomorphism: let  $\phi: \mathbb{K}[x_0, x_1, \dots, x_n] \rightarrow \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  be defined by  $x_j \mapsto x_{j+1}$ , for all  $j \in \{0, 1, \dots, n\}$ . Examining the monomials of  $L' := \langle x_0, \phi(L[\mathfrak{p}, n]) \rangle$  shows that it is a lexicographic ideal, and we further see that  $\mathbb{K}[x_0, x_1, \dots, x_n]/L[\mathfrak{p}, n] \cong \mathbb{K}[x_0, x_1, \dots, x_{n+1}]/L'$ , showing that  $h_{L[\mathfrak{p}, n]} = h_{L'}$ . Hence, because  $L'$  is saturated with respect to the irrelevant ideal  $\langle x_0, x_1, \dots, x_{n+1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ , we have  $L[\mathfrak{p}, n+1] = \langle x_0, \phi(L[\mathfrak{p}, n]) \rangle$ . For instance, starting from the lexicographic ideal  $L[3, 1] = \langle x_0^3 \rangle \subset \mathbb{K}[x_0, x_1]$ , we produce the lexicographic ideal  $L[3, 2] = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$  from Example 3.2, and then repeat-

ing the procedure produces the lexicographic ideals  $L[3, 3] = \langle x_0, x_1, x_2^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ ,  $L[3, 4] = \langle x_0, x_1, x_2, x_3^3 \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_4]$ ,  $L[3, 5] = \langle x_0, x_1, x_2, x_3, x_4^3 \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_5]$ ,  $L[3, 6] = \langle x_0, x_1, \dots, x_4, x_5^3 \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_6]$ , and so on, all for the admissible Hilbert polynomial  $p = 3$ . Geometrically, the homomorphism  $\phi$  corresponds to intersection with the hyperplane defined by  $x_0 = 0$ .

Given a finite sequence of nonnegative integers  $a_0, a_1, \dots, a_{n-1} \in \mathbb{N}$ , consider the monomial ideal  $L(a_0, a_1, \dots, a_{n-1})$  in  $\mathbb{K}[x_0, x_1, \dots, x_n]$  defined by

$$L(a_0, a_1, \dots, a_{n-1}) := \langle x_0^{a_{n-1}+1}, x_0^{a_{n-1}} x_1^{a_{n-2}+1}, \dots, x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, \\ x_0^{a_{n-1}} x_1^{a_{n-2}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle;$$

see Notation 1.2 in [RS97]. Lemma 3.4 shows how to use these ideals in conjunction with Macaulay–Hartshorne expressions to determine monomial generators of lexicographic ideals. Part (i) of the following Lemma appears in Theorem 2.23 of [Moo12].

**Lemma 3.4.** *Let  $p(t) := \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e_i}{i+1}$ , for integers  $e_0 \geq e_1 \geq \dots \geq e_d > 0$ , and let  $n \in \mathbb{N}$  satisfy  $n > d = \deg p$ .*

(i) *Define  $e_i := 0$ , for  $d+1 \leq i \leq n$ , and  $a_j := e_j - e_{j+1}$ , for all  $0 \leq j \leq n-1$ . We have*

$$L[p, n] = L(a_0, a_1, \dots, a_{n-1}) \\ = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d+1}, \\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1}, \dots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle.$$

(ii) *If there is an integer  $0 \leq \ell \leq d-1$  such that  $a_j = 0$  for all  $j \leq \ell$ , and  $a_{\ell+1} > 0$ , then the*

minimal monomial generators of  $L[\mathfrak{p}, n]$  are given by  $m_1, m_2, \dots, m_{n-(\ell+1)}$ , where

$$m_i := x_{i-1}, \text{ for every } 1 \leq i \leq n - (d + 1), \text{ and}$$

$$m_{n-d+k} := \left( \prod_{j=0}^{k-1} x_{n-(d+1)+j}^{a_{d-j}} \right) x_{n-(d+1)+k}^{a_{d-k+1}}, \text{ for all } 0 \leq k \leq d - (\ell + 1)$$

If  $a_0 \neq 0$ , then setting  $\ell = -1$  gives the minimal monomial generators, which are those listed in Part (i).

*Proof.*

- (i) Substituting the values  $a_j := e_j - e_{j+1}$  determined by the Macaulay–Hartshorne expression of  $\rho(t)$  in the definition of  $L(a_0, a_1, \dots, a_{n-1})$  gives the listed monomials. In degree  $a_d + a_{d-1} + \dots + a_{d-k} + 1$ , the monomial  $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \dots x_{n-(d+1)+k}^{a_{d-k+1}}$  is the lex-largest monomial lex-smaller than  $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \dots x_{n-(d+1)+k-1}^{a_{d-(k-1)+1}} x_n^{a_{d-k}}$ , for  $0 \leq k \leq d - 1$ . In degree  $a_d + a_{d-1} + \dots + a_0$ , the monomial  $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \dots x_{n-1}^{a_0}$  is similarly the lex-largest monomial lex-smaller than  $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \dots x_{n-2}^{a_1+1} x_n^{a_0-1}$ , thus,  $L := L(a_0, a_1, \dots, a_{n-1})$  is a lexicographic ideal. For any monomial  $g$  in the saturation  $(L : \langle x_0, x_1, \dots, x_n \rangle^\infty)$ , there exists  $j \in \mathbb{N}$  such that  $gx_n^j \in L$ . Because the generators of  $L$  are not divisible by  $x_n$ , this implies that  $g \in L$ , showing that  $L$  is saturated.

Before showing that  $L$  has the correct Hilbert polynomial, we first prove that the auxiliary ideal  $L' := L(0, 0, \dots, 0, a_d, 0, 0, \dots, 0) = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d} \rangle$  has Hilbert polynomial  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$ . Setting  $S := \mathbb{K}[x_{n-(d+1)}, x_{n-d}, \dots, x_n]$ , multiplication by  $x_{n-(d+1)}^{a_d}$  defines the first homomorphism in a short exact sequence  $0 \rightarrow S(-a_d) \rightarrow S \rightarrow S/\langle x_{n-(d+1)}^{a_d} \rangle \rightarrow 0$ . Additivity of Hilbert polynomials on short

exact sequences shows that  $\mathfrak{p}_{L'}(t) = \binom{t+d+1}{d+1} - \binom{t+d+1-a_d}{d+1}$ . Applying the summation formula on page 159 of [GKP94] then yields the result  $\mathfrak{p}_{L'}(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$ .

We prove the general case by induction on the degree  $d$  of  $\mathfrak{p}$ . Suppose that  $d = 0$ . The ideal  $L$  becomes  $L := \langle x_0, x_1, \dots, x_{n-2}, x_{n-1}^{a_0} \rangle$ , which has constant Hilbert polynomial equal to  $a_0 = \mathfrak{p}$ . Suppose that  $d > 0$ , let  $L' := L(0, 0, \dots, 0, a_d, 0, 0, \dots, 0)$ , and let  $L'' := L(a_0, a_1, \dots, a_{d-1}, 0, 0, \dots, 0)$ . Considering the short exact sequence

$$0 \rightarrow (\mathbb{K}[x_0, x_1, \dots, x_n]/L'')(-a_d) \rightarrow \mathbb{K}[x_0, x_1, \dots, x_n]/L \rightarrow \mathbb{K}[x_0, x_1, \dots, x_n]/L' \rightarrow 0,$$

whose first homomorphism takes  $1 \mapsto x_{n-(d+1)}^{a_d}$ , we find that  $\mathfrak{p}_L = \mathfrak{p}_{L'} + \mathfrak{p}_{L''}$ . The previous paragraph shows that  $\mathfrak{p}_{L'}(t) = \sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-a_d}{i+1}$ , and induction yields  $\mathfrak{p}_{L''}(t) = \sum_{i=0}^{d-1} \binom{t+i-a_d}{i+1} - \binom{t+i-e_i}{i+1}$ , so that adding these together yields  $\mathfrak{p}_L = \mathfrak{p}$  as desired. Hence,  $L := L(a_0, a_1, \dots, a_{n-1}) = L[\mathfrak{p}, n]$  is the lexicographic ideal for  $\mathfrak{p}$  in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ .

- (ii) We know that  $x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}}$ , because either  $a_0 = a_1 = \cdots = a_\ell = 0$ , or  $a_0 \neq 0$  and  $\ell = -1$ . If  $\ell \geq 0$ , then this implies that the monomial generators

$$\begin{aligned} x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}+1}, & \quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-(\ell+1)}^{a_{\ell+1}}, \quad \dots, \\ & \quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1} \end{aligned}$$

from Part (i) are redundant, as they are multiples of the last monomial generator. Removing these redundancies yields the aforementioned monomial generators  $m_1, m_2, \dots, m_{n-(\ell+1)}$ . For every  $i \in \{2, 3, \dots, n-(\ell+1)\}$  and every  $j \in \{1, 2, \dots, i-1\}$

there is a variable  $x_k$  dividing  $m_j$  to higher order than the order to which it divides  $m_i$ , and minimality follows.  $\square$

To show that the ideal  $L(a_0, a_1, \dots, a_{n-1})$  is saturated in the proof of Lemma 3.4, we could alternatively apply Lemma 5.2. The nonminimal list of monomial generators in Part (i) of Lemma 3.4 is useful for describing operations on lexicographic ideals, and for tracking the coefficients of their Macaulay–Hartshorne and Gotzmann expressions; see Proposition 3.9. Importantly, Lemma 3.4 shows that all pairs of an admissible Hilbert polynomial  $p$  and dimension  $n > \deg p$ , or equivalently all sequences  $a_0, a_1, \dots, a_{n-1}$  of nonnegative integers, determine a lexicographic ideal.

**Remark 3.5** (Sharpness of the Gotzmann regularity bound). Combining Lemma 2.4 with Lemma 3.4 shows that the degree  $e_{\ell+1} = a_d + a_{d-1} + \dots + a_{\ell+1}$  of the lex-smallest minimal monomial generator  $m_{n-(\ell+1)} \in L(a_0, a_1, \dots, a_{n-1})$  equals the Gotzmann number  $r$  of  $p$ . Via the Eliahou–Kervaire resolution, we can read off various invariants of any lexicographic ideal  $L[p, n]$ . Specifically, the regularity of  $L[p, n]$  equals the maximum degree of a minimal monomial generator of  $L[p, n]$ , which equals  $\deg m_{n-(\ell+1)} = e_{\ell+1}$ ; see Corollary 7.2.3 of [HH11]. Hence, the regularity of  $L[p, n]$  and the Gotzmann number of  $L[p, n]$  coincide. This proves that the upper bound given by the Gotzmann number on the regularity of saturated ideals with Hilbert polynomial  $p$  is sharp, as discussed in Remark 2.6.

The following example shows how to use Lemma 3.4 to identify minimal monomial generators from a Hilbert polynomial.

**Example 3.6** (Monomial generators of the lexicographic ideal for the twisted cubic). The Hilbert polynomial of the twisted cubic curve  $X$  in  $\mathbb{P}^3$  is

$$p_X(t) = 3t + 1 = \left[ \binom{t+0}{0+1} - \binom{t+0-4}{0+1} \right] + \left[ \binom{t+1}{1+1} - \binom{t+1-3}{1+1} \right],$$

with corresponding lexicographic ideal  $L[3t + 1, 3] \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ ; see Example 2.7. Computing monomial generators via Lemma 3.4, we have  $d = 1$ ,  $e_0 = 4$ ,  $e_1 = 3$ ,  $e_2 = 0$ , and  $e_3 = 0$ , so that  $a_0 = 1$ ,  $a_1 = 3$ , and  $a_2 = 0$ . Hence,  $L[3t + 1, 3] = L(1, 3, 0)$ , and we derive the list  $L(1, 3, 0) = \langle x_0, x_1^4, x_1^3x_2 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$  of minimal monomial generators.

Elaborating on Remark 3.5, the Gotzmann number of  $3t + 1$  equals 4, which can be realized as the sum  $a_0 + a_1 + a_2 = 1 + 3 + 0$ , as the number  $e_0 = 4$ , or as the degree of the minimal monomial generator  $x_1^3x_2 \in L[3t + 1, 3] = \langle x_0, x_1^4, x_1^3x_2 \rangle$ .

In analogy with the mapping  $\Psi$  defined in Chapter 2, we introduce the following binary relation on the set of all lexicographic ideals. We define the *lex-expansion* of any lexicographic ideal  $L[\mathfrak{p}, n] := L(a_0, a_1, \dots, a_{n-1}) \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  to be the lexicographic ideal  $\Psi(L[\mathfrak{p}, n]) := L(a_0 + 1, a_1, a_2, \dots, a_{n-1}) \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ . The following proposition explains our choice of notation for lex-expansion.

**Proposition 3.7.** *Let  $\mathfrak{p}$  be an admissible Hilbert polynomial, and let  $n$  be a positive integer satisfying  $n > \deg \mathfrak{p}$ . We have  $\Psi(L[\mathfrak{p}, n]) = L[\Psi(\mathfrak{p}), n]$ . In particular, the mapping  $\Psi$  on lexicographic ideals preserves the codimension.*

*Proof.* To find the relationship between the Hilbert polynomials of  $L[\mathfrak{p}, n]$  and  $\Psi(L[\mathfrak{p}, n])$ , we set  $a'_0 := a_0 + 1$  and  $a'_i := a_i$  for  $i \in \{1, 2, \dots, d\}$ , so that  $\Psi(L[\mathfrak{p}, n]) = L(a'_0, a'_1, \dots, a'_{n-1})$ . The Macaulay–Hartshorne expression of the Hilbert polynomial of  $\Psi(L[\mathfrak{p}, n])$  then equals  $\sum_{i=0}^d \binom{t+i}{i+1} - \binom{t+i-e'_i}{i+1}$ , where  $e'_0 := a'_0 - a'_1 = e_0 + 1$ , and  $e'_i := e_i$  for all  $i \in \{1, 2, \dots, d\}$ . Hence, the Hilbert polynomial of the lex-expansion equals  $1 + \mathfrak{p} = \Psi(\mathfrak{p})$ .

To show that the codimension is preserved, observe that the codimension of the lex-expansion  $\Psi(L[\mathfrak{p}, n]) = L[\Psi(\mathfrak{p}), n]$  is  $n - \deg \Psi(\mathfrak{p}) = n - \deg(1 + \mathfrak{p}) = n - \deg \mathfrak{p}$ .  $\square$

The following gives a concrete example of lex-expansions.



**Example 3.8** (Lex-expansion from the planar cubic to the twisted cubic). The Hilbert polynomial of a planar cubic curve  $C \subset \mathbb{P}^3$  is  $p_C(t) = 3t$ , with Macaulay–Hartshorne expression given by  $3t = \binom{t+0}{0+1} - \binom{t+0-3}{0+1} + \binom{t+1}{1+1} - \binom{t+1-3}{1+1}$ . Applying Lemma 3.4, we have  $a_0 = 0$ ,  $a_1 = 3$ , and  $a_2 = 0$ , so the corresponding lexicographic ideal  $L[3t, 3] = L(0, 3, 0)$  has minimal monomial generators  $\langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ . Proposition 3.7 and the definition of lex-expansion show that  $\Psi(L(0, 3, 0)) = L(1, 3, 0) = \langle x_0, x_1^4, x_1^3 x_2 \rangle$ ; compare with Example 3.6.

The next proposition describes our other mapping on lexicographic ideals, which mirrors the mapping  $\Phi$  on admissible Hilbert polynomials. Along with lex-expansion, this operation gives rise to the forest of lexicographic ideals. We introduce the following convenient notation: for every ideal  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ , we denote the extension ideal by  $\Phi(I) := I \cdot \mathbb{K}[x_0, x_1, \dots, x_{n+1}] \subseteq \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ . The following proposition explains our choice of notation for extension.

**Proposition 3.9.** *Let  $p$  be an admissible Hilbert polynomial, and let  $n$  be a positive integer satisfying  $n > \deg p$ . We have  $\Phi(L[p, n]) = L[\Phi(p), n + 1]$ . Equivalently, we have  $\Phi(L(a_0, a_1, \dots, a_{n-1})) = L(0, a_0, a_1, \dots, a_{n-1})$ , for nonnegative integers  $a_0, a_1, \dots, a_{n-1}$ . In particular, the mapping  $\Phi$  on lexicographic ideals preserves codimension.*

*Proof.* Let  $q$  denote the Hilbert polynomial of  $\Phi(L[p, n]) \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ . The extension  $\Phi(L[p, n])$  is generated by the images of the generators of  $L[p, n]$  under the inclusion mapping  $\mathbb{K}[x_0, x_1, \dots, x_n] \hookrightarrow \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ , namely

$$\begin{aligned} \Phi(L[p, n]) = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1}, \dots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-3}^{a_2} x_{n-2}^{a_1+1}, \\ x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-2}^{a_1} x_{n-1}^{a_0} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]. \end{aligned}$$

However, because  $\mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  has  $n + 2$  variables, we must reindex the powers in this list of generators to describe the Hilbert polynomial  $q$ . For every  $i \in \{1, 2, \dots, d + 1\}$ , set  $a'_i := a_{i-1}$ , and set  $a'_0 := 0$ . Rewriting the monomial generators using these reindexed exponents and adding the redundant generator  $x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-2}^{a'_2} x_{n-1}^{a'_1+1}$  to the list, we find that

$$\begin{aligned} \Phi(L[\mathfrak{p}, n]) = \langle x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a'_{d+1}+1}, x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d+1}, \dots, x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-2}^{a'_2} x_{n-1}^{a'_1+1}, \\ x_{n-(d+1)}^{a'_{d+1}} x_{n-d}^{a'_d} \cdots x_{n-1}^{a'_1} x_n^{a'_0} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}], \end{aligned}$$

giving monomial generators of  $\Phi(L[\mathfrak{p}, n])$  in the form of Part (i) of Lemma 3.4. Using Lemma 2.4, the first  $a'_{d+1} = a_d$  parts in the Gotzmann partition of  $q$  equal  $d + 1$ , the next  $a'_d = a_{d-1}$  parts in the Gotzmann partition equal  $d$ , and so on. That is, every part in the Gotzmann partition of  $\mathfrak{p}$  has increased by 1. Hence, we have  $q = \Phi(\mathfrak{p})$ .

To finish, the codimension of the extended ideal  $\Phi(L[\mathfrak{p}, n]) = L[\Phi(\mathfrak{p}), n]$  is given by  $n + 1 - \deg \Phi(\mathfrak{p}) = n + 1 - (1 + \deg \mathfrak{p}) = n - \deg \mathfrak{p}$ , showing that  $\Phi$  preserves codimension.  $\square$

We illustrate Proposition 3.9 through the behaviour of the lexicographic ideal associated to the twisted cubic curve in  $\mathbb{P}^3$ .

**Example 3.10** (Extension of the lexicographic ideal of the twisted cubic). Example 3.6 shows that  $L[3t + 1, 3] = \langle x_0, x_1^4, x_1^3 x_2 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ , so that Proposition 3.9 implies that  $\Phi(L[3t + 1, 3]) = \langle x_0, x_1^4, x_1^3 x_2 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3, x_4]$  is the lexicographic ideal with Hilbert polynomial  $\Phi(3t + 1) = \frac{3}{2}t^2 + \frac{5}{2}t - 1$ . That is,  $\Phi(L[3t + 1, 3])$  equals the lexicographic ideal  $L[\frac{3}{2}t^2 + \frac{5}{2}t - 1, 4]$ , and furthermore, both ideals have codimension 2, so that the first corresponds to curves in  $\mathbb{P}^3$ , while the second corresponds to surfaces in  $\mathbb{P}^4$ .

The following theorem is the main result of this chapter, describing a natural forest

structure on the set of lexicographic ideals.

**Theorem 3.11.** *For each positive codimension  $c \in \mathbb{Z}$ , the graph  $\mathcal{L}_c$  whose vertex set consists of all lexicographic ideals of codimension  $c$ , and whose edges are all possible pairs of the form  $(L[\mathfrak{p}, n], \Psi(L[\mathfrak{p}, n]))$  and  $(L[\mathfrak{p}, n], \Phi(L[\mathfrak{p}, n]))$ , where  $L[\mathfrak{p}, n]$  is a lexicographic ideal of codimension  $c$ , is an infinite binary tree. The root of the binary tree  $\mathcal{L}_c$  is the lexicographic ideal  $L[1, c] \subset \mathbb{K}[x_0, x_1, \dots, x_c]$ . In particular, this endows the set of lexicographic ideals with the structure of a forest.*

*Proof.* Proposition 3.7 and Proposition 3.9 show that both operations on lexicographic ideals, lex-expansion and extension, preserve codimension. Moreover, Lemma 3.4 implies that, for every positive codimension  $c \in \mathbb{Z}$  and every admissible Hilbert polynomial  $\mathfrak{p}$ , the lexicographic ideal  $L(a_0, a_1, \dots, a_d, a_{d+1}, \dots, a_{d+c-1})$  has Hilbert polynomial  $\mathfrak{p}$  and codimension  $c$ , where  $a_{d+1} = a_{d+2} = \dots = a_{d+c-1} = 0$  (specifying  $c$  is equivalent to specifying  $n = d + c$ ).

For each positive codimension  $c \in \mathbb{Z}$ , we define a binary tree  $\mathcal{L}_c$  whose vertices consist of all lexicographic ideals of codimension  $c$ , and whose edges consist of all pairs of the form  $(L[\mathfrak{p}, n], \Psi(L[\mathfrak{p}, n]))$  and  $(L[\mathfrak{p}, n], \Phi(L[\mathfrak{p}, n]))$ , where  $L[\mathfrak{p}, n]$  is the lexicographic ideal for  $\mathfrak{p}$  with codimension  $c$ . The root of the binary tree  $\mathcal{L}_c$  is the ideal  $L[1, c] := \langle x_0, x_1, \dots, x_{c-1} \rangle$  in  $\mathbb{K}[x_0, x_1, \dots, x_c]$ , which has Hilbert polynomial 1. Proposition 3.7 and Proposition 3.9 then show that the mapping  $\mathcal{M} \rightarrow \mathcal{L}_c$  defined by  $\mathfrak{p} \mapsto L[\mathfrak{p}, c + \deg(\mathfrak{p})]$ , which takes  $\mathfrak{p} \in \mathcal{M}$  to the unique lexicographic ideal for  $\mathfrak{p}$  of codimension  $c$  is a graph isomorphism.  $\square$

For positive  $c \in \mathbb{Z}$ , we call the tree  $\mathcal{L}_c$  of Theorem 3.11 the **lexicographic tree of codimension  $c$** , and we call the union  $\mathcal{L} := \bigsqcup_{c \in \mathbb{N}, c > 0} \mathcal{L}_c$  the **lexicographic forest**. The vertices of the lexicographic tree  $\mathcal{L}_c$  of codimension  $c$  are ideals living in infinitely many different polynomial rings. Figure 3.1 displays a portion of the lexicographic tree of codimension 2.

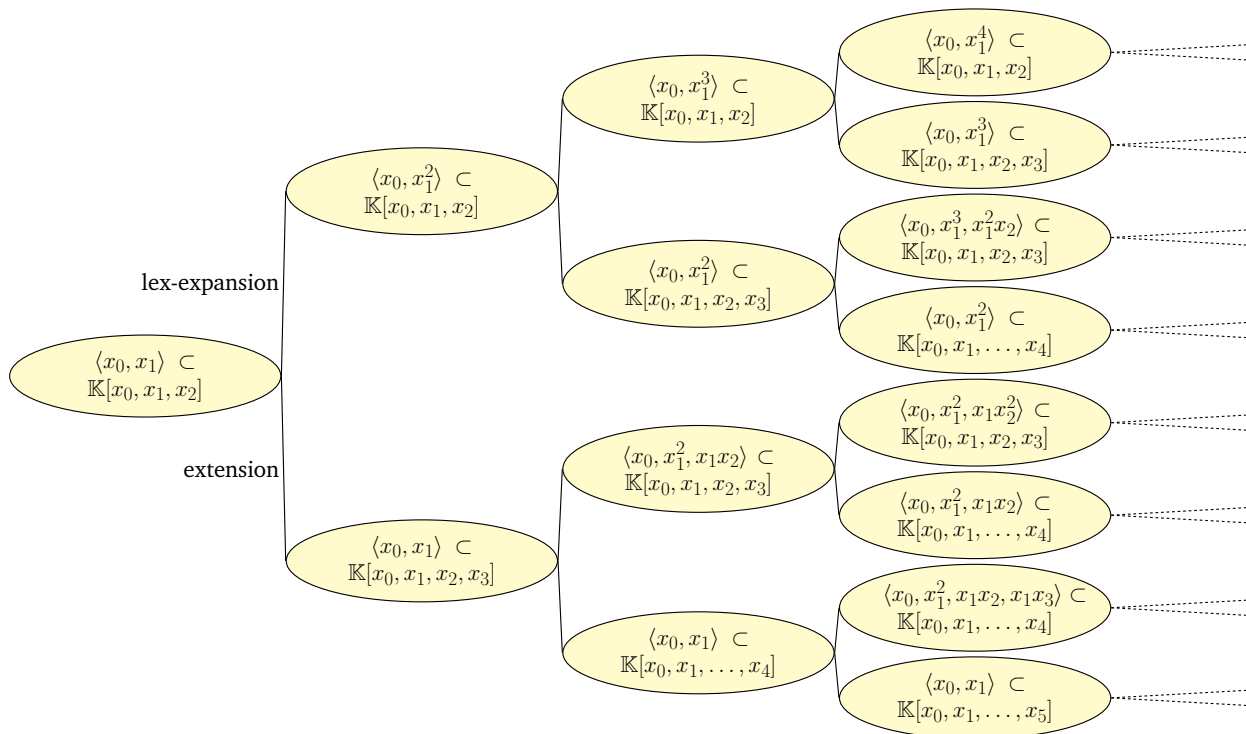


Figure 3.1: The lexicographic tree  $\mathcal{L}_2$  codimension  $c = 2$  to height 3

# Chapter 4

## Random Schemes in the Hilbert Forest

In this chapter, we endow the set of all Hilbert schemes, parametrizing closed subschemes with a fixed Hilbert polynomial of some projective space, with the structure of a discrete probability space. To accomplish this, we identify a natural graph structure on this set. We then use this forest structure to give precise probabilistic statements about the geometry of general Hilbert schemes. For example, treating any Hilbert tree individually, or the collective Hilbert forest, as a discrete probability space allows us to ask questions such as “What is the likelihood that a random Hilbert scheme has property  $P$ ?” The property  $P$  may be any attribute of Hilbert schemes. Specifically, we regard the dimensions and degrees of subschemes parametrized by Hilbert schemes, and the radii of Hilbert schemes, as random variables on the set of all Hilbert schemes, and we compute the expected values and variances of these random variables in Examples 4.7–4.10.

We denote by  $\text{Hilb}[p, n]$  the *Hilbert scheme* parametrizing all closed subschemes with a fixed Hilbert polynomial  $p$  in the projective space  $\mathbb{P}^n$ ; see Corollary 5.9 in [Har66], and Theorem 1.1 in [Har10].

**Theorem 4.1.** *For each positive codimension  $c \in \mathbb{Z}$ , the graph  $\mathcal{H}_c$  whose vertex set consists of*

every nonempty Hilbert scheme  $\text{Hilb}[\mathfrak{p}, n]$  that parametrizes codimension  $c$  subschemes of some projective space  $\mathbb{P}^n$ , and whose edges are all pairs of the form  $(\text{Hilb}[\mathfrak{p}, n], \text{Hilb}[\Psi(\mathfrak{p}), n])$  and  $(\text{Hilb}[\mathfrak{p}, n], \text{Hilb}[\Phi(\mathfrak{p}), n + 1])$ , where  $\mathfrak{p}$  is an admissible Hilbert polynomial and  $n := c + \deg \mathfrak{p}$ , is an infinite binary tree. The root of the tree  $\mathcal{H}_c$  is the Hilbert scheme  $\text{Hilb}[1, c]$ .

For positive  $c \in \mathbb{Z}$ , we call the tree  $\mathcal{H}_c$  of Theorem 4.1 the **Hilbert tree of codimension  $c$** , and we call the disjoint union  $\mathcal{H} := \bigsqcup_{c \in \mathbb{N}, c > 0} \mathcal{H}_c$  the **Hilbert forest**.

*Proof.* For any admissible Hilbert polynomial  $\mathfrak{p}$  and any positive codimension  $c \in \mathbb{Z}$ , there exist both a unique lexicographic ideal  $L[\mathfrak{p}, n]$  and a unique Hilbert scheme  $\text{Hilb}[\mathfrak{p}, n]$ , where  $n := c + \deg \mathfrak{p}$ . Thus, there is a bijective correspondence, defined by  $L[\mathfrak{p}, n] \mapsto \text{Hilb}[\mathfrak{p}, n]$ , between the vertices of the lexicographic forest  $\mathcal{L}$  and the set of all Hilbert schemes. Because  $L[\mathfrak{p}, n]$  has codimension  $c$  and determines a point  $[X_{L[\mathfrak{p}, n]}]$  on  $\text{Hilb}[\mathfrak{p}, n]$ , we know that  $\text{Hilb}[\mathfrak{p}, n]$  parametrizes codimension  $c$  subschemes of  $\mathbb{P}^n$ . Therefore, the correspondence between vertices of  $\mathcal{L}$  and Hilbert schemes restricts to a bijection between vertices of the lexicographic tree  $\mathcal{L}_c$  and Hilbert schemes parametrizing codimension  $c$  subschemes. By construction, the root of the tree  $\mathcal{H}_c$  is  $\text{Hilb}[1, c]$ , and these bijective correspondences respect the graph structures and define graph isomorphisms  $\mathcal{L}_c \rightarrow \mathcal{H}_c$  for every  $c \geq 1$ .  $\square$

The Hilbert forest  $\mathcal{H}$  organizes the set of all Hilbert schemes that parametrize subschemes of projective spaces, and the vertices of the Hilbert tree  $\mathcal{H}_c$  of codimension  $c$  are Hilbert schemes parametrizing subschemes in infinitely many different projective spaces. Figure 4.1 displays a portion of the Hilbert tree of codimension 2.

**Example 4.2** (Hilbert schemes of points in the Hilbert forest). Hilbert schemes parametrizing points in  $\mathbb{P}^n$  lie along one ray of the Hilbert tree  $\mathcal{H}_n$ . The root of the Hilbert tree  $\mathcal{H}_n$  is the Hilbert scheme  $\text{Hilb}[1, n]$  parametrizing single points in  $\mathbb{P}^n$ , and is connected via an

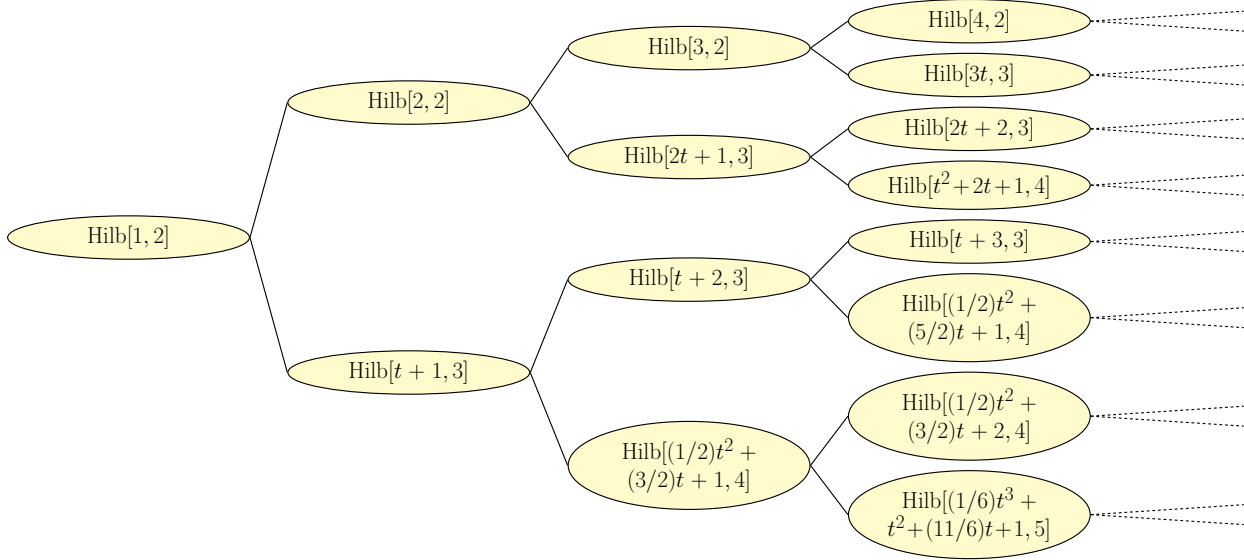


Figure 4.1: The Hilbert tree  $\mathcal{H}_2$  of codimension  $c = 2$  to height 3

edge to  $\text{Hilb}[\Psi(1), n] = \text{Hilb}[2, n]$ , which parametrizes pairs of points in  $\mathbb{P}^n$ . Continuing,  $\text{Hilb}[2, n]$  is connected to  $\text{Hilb}[\Psi(2), n] = \text{Hilb}[3, n]$  via an edge, and so on. Therefore, the vertices on the  $\Psi$ -ray of  $\mathcal{H}_n$  are exactly the Hilbert schemes  $\text{Hilb}[d, n]$  parametrizing collections of  $d$  points in  $\mathbb{P}^n$ .

**Example 4.3** (Grassmannians in the Hilbert forest). The *Grassmannian*  $\mathbb{G}(k, n)$  is the parameter space for  $k$ -planes in  $\mathbb{P}^n$ . The Hilbert polynomial of a linear  $k$ -dimensional subspace of  $\mathbb{P}^n$  is  $\binom{t+k}{k} \in \mathbb{Q}[t]$ , so we have  $\mathbb{G}(k, n) = \text{Hilb}[\binom{t+k}{k}, n] = \text{Hilb}[\Phi^k(1), n]$ . In other words, the root of the Hilbert tree  $\mathcal{H}_c$  is  $\text{Hilb}[1, c] = \mathbb{G}(0, c)$ , which is connected via an edge to the Grassmannian  $\text{Hilb}[\Phi(1), c+1] = \mathbb{G}(1, c+1)$ , and so on, so that the vertices of the  $\Phi$ -ray in  $\mathcal{H}_c$  are all Grassmannians  $\mathbb{G}(n-c, n)$  parametrizing codimension  $c$  linear subspaces of  $\mathbb{P}^n$ , for  $n \geq c$ .

**Remark 4.4** (Geometry of the edges of Hilbert trees). The edges of the Macaulay tree and of the lexicographic forest arise from mappings on their sets of vertices, so we consider

natural candidates for geometric mappings corresponding to the edges of the Hilbert forest.

Examining edges of the Hilbert forest joining Hilbert schemes of the form  $\text{Hilb}[\mathfrak{p}, n]$  and  $\text{Hilb}[\Psi(\mathfrak{p}), n]$ , for any  $x \in \mathbb{P}^n$  there exists a rational mapping  $\text{Hilb}[\mathfrak{p}, n] \dashrightarrow \text{Hilb}[\Psi(\mathfrak{p}), n]$ , defined by  $[X] \mapsto [X \cup \{x\}]$  for every  $X$  such that  $x \notin X$ . However, this mapping can not be extended to a regular mapping, even in the most elementary cases. For instance, let  $\varphi: \text{Hilb}[1, 2] \dashrightarrow \text{Hilb}[2, 2]$  denote such a mapping, where  $x := [0 : 0 : 1]$  is the “origin,” and consider the values  $\varphi(x')$ , for  $x' \in \mathbb{P}^2 \cong \text{Hilb}[1, 2]$  approaching  $x$  along different lines through the origin. To complete the morphism, the value  $\varphi(x)$  must parametrize a double point at  $x$ , which is a subscheme of the line of approach. Hence,  $\varphi$  has no well-defined value at  $x$ , and similar difficulties arise when we attempt to define  $\varphi$  via a varying point  $x$ .

Every homogeneous ideal  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  has an associated closed subscheme  $X_I \subseteq \mathbb{P}^n$ , and any closed subscheme  $X \subset \mathbb{P}^n$  with Hilbert polynomial  $\mathfrak{p}$  determines an associated point  $[X] \in \text{Hilb}[\mathfrak{p}, n]$ . The **lexicographic point** of  $\text{Hilb}[\mathfrak{p}, n]$  is the point  $[X_{L[\mathfrak{p}, n]}]$  on  $\text{Hilb}[\mathfrak{p}, n]$ , which lies on a unique irreducible component called the **lexicographic component**; see [RS97]. Considering edges joining Hilbert schemes of the form  $\text{Hilb}[\mathfrak{p}, n]$  and  $\text{Hilb}[\Phi(\mathfrak{p}), n + 1]$ , if  $I$  is a homogeneous ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$  defining  $X_I \subset \mathbb{P}^n$ , then the extension ideal  $\Phi(I) := I \cdot \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  determines the join  $\text{Join}(X_I, x)$  of  $X_I$  and  $x := [0 : 0 : \dots : 0 : 1] \in \mathbb{P}^{n+1}$ . Thus, we reinterpret Proposition 3.9 as showing that the join  $\text{Join}(X_{L[\mathfrak{p}, n]}, x) = X_{\Phi(L[\mathfrak{p}, n+1])}$  has Hilbert polynomial  $\Phi(\mathfrak{p})$ . By change of coordinates, the same is true for general points on the lexicographic component, thus, letting  $\text{Hilb}[\mathfrak{p}, n]^{lex} \subseteq \text{Hilb}[\mathfrak{p}, n]$  denote the lexicographic component, there is a rational mapping  $\text{Hilb}[\mathfrak{p}, n]^{lex} \dashrightarrow \text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  defined by  $X_I \mapsto X_{\Phi(I)}$ . This mapping cannot be extended to other irreducible components of  $\text{Hilb}[\mathfrak{p}, n]$ , or even to the intersection of the



lexicographic component with other irreducible components, due to the presence of non-lexicographic strongly stable points on these components; see Remark 2.2 of [Ree95], and Theorem 7.3.

**Remark 4.5** (Subschemes of  $\mathbb{P}^n$  for fixed  $n$ ). We identify the vertices of the Hilbert forest which correspond to nonempty Hilbert schemes of the form  $\text{Hilb}[\mathfrak{p}, n]$ , for a fixed positive  $n \in \mathbb{Z}$ . These vertices are found in the Hilbert trees  $\mathcal{H}_c$ , for all  $1 \leq c \leq n$ ; given  $c$  such that  $1 \leq c \leq n$ , we identify all such vertices in  $\mathcal{H}_c$  as those whose corresponding admissible Hilbert polynomials have degree  $d := n - c$ .

For example, if  $n = 3$ , then the vertices corresponding to Hilbert schemes parametrizing closed subschemes of  $\mathbb{P}^3$  are contained in  $\mathcal{H}_1$ ,  $\mathcal{H}_2$ , and  $\mathcal{H}_3$ . In  $\mathcal{H}_3$ , we set  $d = 0$ , obtaining the vertices corresponding to Hilbert schemes of points  $\text{Hilb}[k, 3]$ , for positive  $k \in \mathbb{Z}$ . In  $\mathcal{H}_2$ , we have  $d = 1$ , and we identify the Hilbert schemes of the form  $\text{Hilb}[\Psi^{e_0 - e_1} \Phi \Psi^{e_1 - 1}(1), 3]$ , where  $e_0 \geq e_1 > 0$ . In  $\mathcal{H}_1$ , we have  $d = 2$ , and we identify those Hilbert schemes of the form  $\text{Hilb}[\Psi^{e_0 - e_1} \Phi \Psi^{e_1 - e_2} \Phi \Psi^{e_2 - 1}(1), 3]$ , where  $e_0 \geq e_1 \geq e_2 > 0$ .

Exploiting the discrete graph structure, we can study the set of nonempty Hilbert schemes as a probability space. To form a probability space, we let the sample space of elementary outcomes be the set of vertices of  $\mathcal{H}_c$ , and the  $\sigma$ -algebra of events be the power set  $2^{\mathcal{H}_c}$ . A countably additive probability measure  $\text{Pr}: 2^{\mathcal{H}_c} \rightarrow [0, 1]$  is determined by specifying a nonnegative function  $\text{pr}: \mathcal{H}_c \rightarrow \mathbb{R}$  satisfying  $\sum_{\text{Hilb}[\mathfrak{p}, n] \in \mathcal{H}_c} \text{pr}(\text{Hilb}[\mathfrak{p}, n]) = 1$ . Specifically, we obtain  $\text{Pr}$  from  $\text{pr}$  by defining  $\text{Pr}(A) := \sum_{\text{Hilb}[\mathfrak{p}, n] \in A} \text{pr}(\text{Hilb}[\mathfrak{p}, n])$ , for every subset  $A \subseteq \mathcal{H}_c$ ; see Examples 2.8–2.9 in [Bil95]. Thus, the graph  $\mathcal{H}_c$  is given the structure of a probability space by specifying the likelihood  $\text{Pr}(\{\text{Hilb}[\mathfrak{p}, n]\}) := \text{pr}(\text{Hilb}[\mathfrak{p}, n])$  of each elementary outcome  $\text{Hilb}[\mathfrak{p}, n] \in \mathcal{H}_c$ . To form a discrete probability space from the entire Hilbert forest, we replace  $\mathcal{H}_c$  with  $\mathcal{H}$  in the definitions of the sample space, the  $\sigma$ -algebra

of events, and the probability measure.

There are many options for the function  $\text{pr}$  defining the discrete probability measure, and we study the fundamental candidates. To mimic the uniform distribution, we consider every vertex of  $\mathcal{H}_c$  at a fixed height to be equally likely, so that given any probability mass function  $f_c: \mathbb{N} \rightarrow [0, 1]$ , a discrete probability measure is determined on  $2^{\mathcal{H}_c}$  by defining  $\text{pr}(\text{Hilb}[\mathfrak{p}, n]) := f_c(k)/2^k$ , for every  $\text{Hilb}[\mathfrak{p}, n] \in \mathcal{H}_c$  having height  $k$ . Elementary candidates for the mass function  $f_c$  include the **geometric** mass function  $f_c(k) := p_c(1 - p_c)^k$ , for a real number  $0 < p_c < 1$ , and the **Poisson** mass function  $f_c(k) := e^{-\lambda_c} \lambda_c^k / k!$ , for a real number  $\lambda_c > 0$ . If  $f_c$  is geometric, then  $p_c = f_c(0)/2^0$  represents the likelihood of choosing the root of  $\mathcal{H}_c$ . For both of these distributions, every vertex of the tree has nonzero probability. To specify a discrete probability measure on  $2^{\mathcal{H}_c}$ , we first specify a probability mass function  $f_c: \mathbb{N} \rightarrow [0, 1]$  for each Hilbert tree  $\mathcal{H}_c$ , along with another probability mass function  $f: \mathbb{N} \setminus \{0\} \rightarrow [0, 1]$  for the codimension, and we then define  $\text{pr}(\text{Hilb}[\mathfrak{p}, n]) := f(c)f_c(k)/2^k$ , for every Hilbert scheme  $\text{Hilb}[\mathfrak{p}, n] \in \mathcal{H}_c$  having height  $k$ .

A **random variable** is a measurable function from a sample space to some measure space. We wish to treat the appearances of various geometric properties of Hilbert schemes as values of random variables. Specifically, this allows us to study the prevalence of certain properties of random Hilbert schemes. The following remark gives a nongeometric random variable with associated probability mass function  $f_c$ .

**Remark 4.6** (Height as a random variable). The **height** random variable  $\text{hgt}_c: \mathcal{H}_c \rightarrow \mathbb{N}$  is defined by sending a Hilbert scheme to its height as a vertex in the Hilbert tree  $\mathcal{H}_c$ . We have  $\Pr(\text{hgt}_c = k) = \sum_{\text{hgt}_c(\text{Hilb}[\mathfrak{p}, n])=k} \text{pr}(\text{Hilb}[\mathfrak{p}, n]) = 2^k (f_c(k)/2^k) = f_c(k)$ , so that the probability mass function of  $\text{hgt}_c$  is exactly the underlying mass function  $f_c$  on  $\mathbb{N}$ . If  $f_c$  is Poisson, then  $\lambda_c$  represents the mean distance of a random Hilbert scheme from the root

of its tree. Treating the height as a random variable  $\text{hgt}: \mathcal{H} \rightarrow \mathbb{N}$  on the entire Hilbert forest, we similarly obtain  $\Pr(\text{hgt} = k) = \sum_{c>0} f(c)f_c(k)$ . If the probability distributions on the trees are independent of  $c$ , then the probability that the height equals  $k$  reduces to  $\Pr(\text{hgt} = k) = \sum_{c>0} f(c)f_1(k) = f_1(k)$ , and is independent of the mass function  $f$ .

As in Remark 4.6, if the probability distributions on the Hilbert trees are equal, then the expected values and variances of the following random variables defined on the Hilbert forest can be computed through the restricted random variables on any chosen Hilbert tree. The following examples examine some random variables derived from geometric properties of Hilbert schemes.

**Example 4.7** (Dimension of parametrized schemes as a random variable on  $\mathcal{H}_c$ ). For any positive  $c \in \mathbb{Z}$ , consider the **parametrized dimension** random variable  $\text{pdm}_c: \mathcal{H}_c \rightarrow \mathbb{N}$ , defined by  $\text{pdm}_c(\text{Hilb}[\mathfrak{p}, n]) := \deg \mathfrak{p}$ . In other words,  $\text{pdm}_c$  is the random variable on  $\mathcal{H}_c$  representing the dimension of the subschemes of projective space parametrized by a random Hilbert scheme. Suppose that the probability measure on  $\mathcal{H}_c$  is determined by a mass function  $f_c: \mathbb{N} \rightarrow [0, 1]$ . We determine the expected parametrized dimension  $E[\text{pdm}_c] := \sum_{d \in \mathbb{N}} d \Pr(\text{pdm}_c = d)$  by first computing the probability  $\Pr(\text{pdm}_c = d)$ , for each  $d \in \mathbb{N}$ . At each height  $k$ , there are exactly  $\binom{k}{d}$  vertices in  $\mathcal{H}_c$  of parametrized dimension  $d$ ; these correspond to the paths with  $k$  edges, originating from the root, such that  $\Phi$  appears exactly  $d$  times. Each vertex at height  $k$  has likelihood  $f_c(k)/2^k$ , so the likelihood that the parametrized dimension is  $d$  and the height is  $k$  equals  $\binom{k}{d} f_c(k)/2^k$ ; countable additivity then implies that  $\Pr(\text{pdm}_c = d) = \sum_{k \in \mathbb{N}} \binom{k}{d} f_c(k)/2^k$ . Hence, the expected parametrized dimension of a Hilbert scheme is  $\sum_{d \in \mathbb{N}} d \sum_{k \in \mathbb{N}} \binom{k}{d} f_c(k)/2^k$ .

To explicitly compute the expected parametrized dimension, we consider diagonal par-

tial sums of this double series. For any positive integer  $K \in \mathbb{Z}$ , we have

$$\begin{aligned} \sum_{d=0}^K d \sum_{k=0}^K \binom{k}{d} \frac{f_c(k)}{2^k} &= \sum_{k=1}^K \sum_{d=1}^k d \binom{k}{d} \frac{f_c(k)}{2^k} = \sum_{k=1}^K 2^{k-1} k \frac{f_c(k)}{2^k} \\ &= \frac{1}{2} \sum_{k=1}^K k f_c(k), \end{aligned}$$

using the identity  $\sum_{d=0}^k d \binom{k}{d} = 2^{k-1} k$ . The double series converges to half the expected value of the distribution described by  $f_c$ . For instance, if  $f_c$  is geometric, then we obtain  $E[\text{pdm}_c] = \frac{1}{2} E[\text{hgt}_c] = (1 - p_c)/2p_c$ , where  $p_c = f_c(0) \in (0, 1)$ , and where  $\text{hgt}_c$  is the height random variable with probability mass function  $f_c$ . If  $f_c$  is a Poisson mass function with mean  $\lambda_c > 0$ , then the expected parametrized dimension is  $E[\text{pdm}_c] = \frac{1}{2} E[\text{hgt}_c] = \lambda_c/2$ .

To compute the variance of the parametrized dimension  $\text{pdm}_c$ , we first compute the second moment of  $\text{pdm}_c$ , similarly examining diagonal partial sums of the double series  $E[\text{pdm}_c^2] := \sum_{d \in \mathbb{N}} d^2 \sum_{k \in \mathbb{N}} \binom{k}{d} f_c(k)/2^k$ . This leads to

$$\begin{aligned} \sum_{k=1}^K k \frac{f_c(k)}{2^k} \sum_{d=0}^{k-1} (d+1) \binom{k-1}{d} &= \sum_{k=1}^K k \frac{f_c(k)}{2^k} 2^{k-2} (k+1) \\ &= \sum_{k=1}^K k(k+1) \frac{f_c(k)}{4}, \end{aligned}$$

which converges to  $\frac{1}{4} (E[\text{hgt}_c^2] + E[\text{hgt}_c])$ , where the first equality follows from the identity  $\sum_{d=0}^k d \binom{k}{d} = 2^{k-1} k$ . Applying this, we compute the variance in the parametrized dimensions of Hilbert schemes to be

$$\begin{aligned} \text{var}(\text{pdm}_c) &:= E[\text{pdm}_c^2] - E[\text{pdm}_c]^2 = \frac{1}{4} (E[\text{hgt}_c^2] + E[\text{hgt}_c]) - \left( \frac{E[\text{hgt}_c]}{2} \right)^2 \\ &= \frac{1}{4} (\text{var}(\text{hgt}_c) + E[\text{hgt}_c]). \end{aligned}$$

For instance, if  $\text{hgt}_c$  has a geometric distribution, then  $\text{var}(\text{pdm}_c) = (1 - p_c^2)/4p_c^2$ , while if  $\text{hgt}_c$  is Poisson with mean  $\lambda_c$ , then  $\text{var}(\text{pdm}_c) = \lambda_c/2$ .

The next example generalizes Example 4.7 to the entire Hilbert forest.

**Example 4.8** (Dimension of parametrized schemes as a random variable on  $\mathcal{H}$ ). More generally, let the *parametrized dimension* random variable  $\text{pdm}: \mathcal{H} \rightarrow \mathbb{N}$  be defined by  $\text{pdm}(\text{Hilb}[\mathfrak{p}, n]) := \deg \mathfrak{p}$ , so that  $\text{pdm}|_{\mathcal{H}_c} = \text{pdm}_c$ . Letting  $f: \mathbb{N} \setminus \{0\} \rightarrow [0, 1]$  be a mass function for the codimension, we compute the expected parametrized dimension  $\mathbb{E}[\text{pdm}]$  by first deriving the probability  $\Pr(\text{pdm} = d) = \sum_{c>0} f(c) \Pr(\text{pdm}_c = d)$  in a similar fashion to the calculation of  $\Pr(\text{pdm}_c = d)$  in Example 4.7. The likelihood that the codimension is  $c$ , the parametrized dimension is  $d$ , and the height is  $k$  equals  $\binom{k}{d} f(c) f_c(k)/2^k$ . Hence, we obtain  $\mathbb{E}[\text{pdm}] = \sum_{c>0} f(c) \mathbb{E}[\text{pdm}_c]$ , which equals  $\sum_{c>0} \frac{1}{2} f(c) \mathbb{E}[\text{hgt}_c]$ , where  $\text{hgt}_c$  denotes the height random variable on the tree  $\mathcal{H}_c$ . This series might not converge, depending on the probability distributions on the Hilbert trees, as would be the case if the expected value of each  $\text{hgt}_c$  were  $1/f(c)$ . Conversely, if all Hilbert trees were to have equivalent probability distributions, then we would obtain  $\mathbb{E}[\text{pdm}] = \mathbb{E}[\text{pdm}_c]$ , for all positive  $c \in \mathbb{Z}$ . One can similarly derive the expression  $\text{var}(\text{pdm}) = \sum_{c>0} f(c) \mathbb{E}[\text{pdm}_c^2] - (\sum_{c>0} f(c) \mathbb{E}[\text{pdm}_c])^2$  for the variance.

The next example highlights the interdependence between probability distributions on Hilbert trees and likelihoods of random Hilbert schemes having certain basic geometric properties. In particular, the way irreducible components of random Hilbert schemes tend to intersect depends on the method of sampling.

**Example 4.9** (Radius as a random variable). Consider  $\text{rad}_c: \mathcal{H}_c \rightarrow \mathbb{N}$  and  $\text{rad}: \mathcal{H} \rightarrow \mathbb{N}$  mapping any Hilbert scheme  $\text{Hilb}[\mathfrak{p}, n]$  to its *radius*, which is the radius of the incidence

graph of irreducible components of  $\text{Hilb}[\mathfrak{p}, n]$ ; see page 650 of [Ree95]. Theorem 7 of [Ree95] translates to the inequality  $\text{rad} \leq 1 + \text{pdm}$ , and in particular, for any Hilbert scheme  $\text{Hilb}[\mathfrak{p}, n]$  and any  $r \in \mathbb{N}$ , if we have  $\text{pdm}(\text{Hilb}[\mathfrak{p}, n]) \leq r - 1$ , then  $\text{rad}(\text{Hilb}[\mathfrak{p}, n]) \leq r$ , which implies that  $\Pr(\text{pdm} \leq r - 1) \leq \Pr(\text{rad} \leq r)$ . Because the radius bounds of [Ree95] are measured from the lexicographic component, Example 4.7 allows us to estimate the likelihood that every irreducible component intersects the lexicographic component on a random Hilbert scheme in  $\mathcal{H}_c$  as follows:

$$\Pr(\text{rad}_c \leq 1) \geq \Pr(\text{pdm}_c \leq 0) = \Pr(\text{pdm}_c = 0) = \sum_{k \in \mathbb{N}} \frac{f_c(k)}{2^k}.$$

Similarly,  $\Pr(\text{rad} \leq 1) \geq \sum_{c > 0} \sum_{k \in \mathbb{N}} f(c) f_c(k) / 2^k$  gives the corresponding estimate on  $\mathcal{H}$ . If  $f_c(k) := p_c(1 - p_c)^k$  is geometric, then the first estimate is  $\Pr(\text{rad}_c \leq 1) \geq 2p_c / (1 + p_c)$ ; for instance,  $p_c := 1/2$  implies that  $\Pr(\text{rad}_c \leq 1) \geq 2/3$ . If  $f_c(k) := e^{-\lambda_c} \lambda_c^k / k!$  is Poisson, then the latter series equals  $e^{-\lambda_c/2}$ . Probability estimates for larger radii are much higher, specifically, if  $f_c$  is geometric, then we similarly derive  $\Pr(\text{rad}_c \leq 2) \geq 4p_c / (1 + p_c)^2$ , which demonstrates that  $\Pr(\text{rad}_c \leq 2) \geq 8/9$  when  $p_c := 1/2$ . If  $f_c$  is a Poisson mass function, then  $\Pr(\text{rad}_c \leq 2) \geq (1 + \lambda_c/2)e^{-\lambda_c/2}$ . Hence, randomly sampled Hilbert schemes may have small radii with high probability, but this depends on the underlying distribution.

In Examples 4.7, 4.8, and 4.9, the resulting probabilities and expected values are dependent on the parameters of the underlying probability distributions. In Example 4.10, we discover a random variable whose expectation is bounded independently of the underlying distribution on  $\mathbb{N}$ .

**Example 4.10** (Degree of parametrized schemes as a random variable). We define the *parametrized degree* random variables  $\text{pdg}_c: \mathcal{H}_c \rightarrow \mathbb{N}$  and  $\text{pdg}: \mathcal{H} \rightarrow \mathbb{N}$  as follows: let

$LC: \mathcal{H} \rightarrow \mathbb{Q}$  be the random variable on  $\mathcal{H}$  that maps a Hilbert scheme  $\text{Hilb}[p, n]$  to the leading coefficient  $LC(p)$  of  $p(t) \in \mathbb{Q}[t]$ ; we then let  $\text{pdg} := (\text{pdm!})(LC)$  and  $\text{pdg}_c := \text{pdg}|_{\mathcal{H}_c}$ , so that the random variable  $\text{pdg}$  takes a Hilbert scheme to the degree of the subschemes it parametrizes. To compute the expected value  $E[\text{pdg}_c] := \sum_{d>0} d \Pr(\text{pdg}_c = d)$ , we observe that the set of vertices such that  $\text{pdg}_c = 1$  consists of the root and the “ $\Phi$ -subtree” of  $\mathcal{H}_c$ , the set of vertices such that  $\text{pdg}_c = 2$  consists of the vertex  $\text{Hilb}[2, c]$  and the the “ $\Phi$ -subtree of the  $\Psi$ -subtree” of  $\mathcal{H}_c$ , and so on. In other words, the vertices with  $\text{pdg}_c = d$  are  $\text{Hilb}[d, c]$  and its  $\Phi$ -subtree. Thus, at any given height  $k \in \mathbb{N}$ , for  $1 \leq d \leq k$  there are  $2^{k-d}$  vertices such that  $\text{pdg}_c = d$ , and there is one vertex such that  $\text{pdg}_c = k + 1$ .

Using this, we compute

$$\Pr(\text{pdg}_c = d) = \left( \frac{f_c(d-1)}{2^{d-1}} + \sum_{k \geq d} \frac{2^{k-d} f_c(k)}{2^k} \right) = \frac{1}{2^{d-1}} \left( f_c(d-1) + \sum_{k \geq d} \frac{f_c(k)}{2} \right) < \frac{1}{2^{d-1}}.$$

This shows that

$$E[\text{pdg}_c^2] = \sum_{d>0} d^2 \Pr(\text{pdg}_c = d) \leq \sum_{d>0} \frac{d^2}{2^{d-1}} = 12.$$

Appropriately modifying this to the Hilbert forest case also yields  $\Pr(\text{pdg} = d) < 1/2^{d-1}$  and  $E[\text{pdg}^2] \leq 12$ . Because  $0 \leq \text{var}(\text{pdg}) = E[\text{pdg}^2] - E[\text{pdg}]^2$ , we obtain the estimate  $E[\text{pdg}] \leq \sqrt{12} \approx 3.46$ , and the closer the expected value  $E[\text{pdg}]$  is to this upper bound, the smaller its variance. Hence, the expected degree of the subschemes parametrized by a random Hilbert scheme satisfies  $1 \leq E[\text{pdg}] \leq \sqrt{12}$ , and moreover, this result does not depend on the underlying mass functions  $f_c$  for the trees  $\mathcal{H}_c$ , or  $f$  for the codimension.

# Chapter 5

## Strongly Stable Ideals

In this chapter, we work with strongly stable monomial ideals in polynomial rings, which broaden the class of lexicographic ideals. We begin with some useful basic facts about these ideals, which then enable us to consider mappings on the set of strongly stable ideals that generalize the mappings  $\Phi$  and  $\Psi$  on lexicographic ideals. We investigate the behaviour of Hilbert polynomials of saturated strongly stable ideals under these mappings, finding that their interaction with the Macaulay tree is more nuanced than that of lexicographic ideals. We finish by reviewing an important algorithm that generates all saturated strongly stable ideals.

Strongly stable ideals capture fundamental information about the geometry of Hilbert schemes, in particular, every irreducible component, and every intersection of irreducible components, of a Hilbert scheme  $\text{Hilb}[p, n]$  contains at least one point  $[X_I]$  corresponding to a strongly stable ideal  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ ; see [Har66], [Ree92], [Ree95], [RS97], and [PS05].

A monomial ideal  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  is **strongly stable** if, for every monomial  $m \in I$ , for every variable  $x_j$  dividing  $m$ , and for every  $x_i >_{\text{lex}} x_j$ , we have  $mx_ix_j^{-1} \in I$ . Proposi-



tion 2.7 of [BS87a] proves that, in characteristic 0, strongly stable ideals are equivalent to **Borel-fixed** ideals, those ideals that are fixed under the canonical action on  $\mathbb{K}[x_0, x_1, \dots, x_n]$  of the Borel subgroup of upper triangular matrices in  $\mathrm{GL}_{n+1}(\mathbb{K})$ . For any given monomial  $m \in \mathbb{K}[x_0, x_1, \dots, x_n]$ , let  $\max m$  denote the maximum integer  $j$  such that the variable  $x_j$  divides  $m$ , and  $\min m$  denote the minimum such integer.

**Example 5.1.** The monomial ideal  $I = \langle x_0^2, x_0x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$  is strongly stable. The monomial  $m := x_1^5x_2x_7^2 \in \mathbb{K}[x_0, x_1, \dots, x_{13}]$  has maximum index  $\max m = 7$ , and minimum index  $\min m = 1$ .

The following basic properties of strongly stable ideals are used repeatedly; Part (i) of Lemma 5.2 appears as Lemma 2.11 of [MS05], and Part (iv) of Lemma 5.2 appears in Lemma 3.17 of [Moo12].

**Lemma 5.2.** *Let  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  be a monomial ideal.*

- (i) *The ideal  $I$  is strongly stable if and only if, for every minimal monomial generator  $g \in I$ , for every variable  $x_j$  dividing  $g$ , and for every  $x_i >_{\mathrm{lex}} x_j$ , we have  $gx_ix_j^{-1} \in I$ .*
- (ii) *If, for every minimal monomial generator  $g' \in I$ , for every variable  $x_j$  dividing  $g'$ , and for every  $x_i >_{\mathrm{lex}} x_j$ , we have  $g'x_ix_j^{-1} \in I$ , then, for every monomial  $m \in I$ , there exists a unique minimal monomial generator  $g \in I$  and unique monomial  $m' \in \mathbb{K}[x_0, x_1, \dots, x_n]$  such that  $m = gm'$  and  $\max g \leq \min m'$ .*
- (iii) *If  $I$  is a strongly stable ideal, then  $I$  is saturated with respect to the irrelevant ideal  $\langle x_0, x_1, \dots, x_n \rangle$  if and only if the minimal monomial generators of  $I$  are not divisible by the variable  $x_n$ .*
- (iv) *If  $I$  is strongly stable with constant Hilbert polynomial  $p_I \in \mathbb{N}$ , then there exists an integer  $k \in \mathbb{N}$  such that  $x_{n-1}^k \in I$ .*

*Proof.*

- (ii) To discern uniqueness, suppose that two such expressions exist,  $m := g_0 m'_0 = g_1 m'_1$ . If  $\max g_0 < \max g_1$ , then  $g_0$  divides  $g_1$ , a contradiction. By symmetry,  $\max g_0 > \max g_1$  also gives a contradiction, so that  $\max g_0 = \max g_1$  and either  $g_0$  divides  $g_1$ , or vice versa. Hence,  $g_0 = g_1$  and  $m'_0 = m'_1$ . For existence, suppose that  $m = gm'$ , but that  $\max g > \min m'$ . The condition on generators guarantees that  $gx_{\min m'} x_{\max g}^{-1} \in I$ , and we set  $g' \in I$  to be any minimal monomial generator dividing  $gx_{\min m'} x_{\max g}^{-1}$ ; either  $\max g' < \max g$ , or  $\max g' = \max g$  but  $x_{\max g}$  divides  $g'$  to lower order than it divides  $g$ . We set  $m''$  to be the monomial such that  $m = g'm''$ , and repeat this process until we reach the desired inequality between indices.
- (i) If  $I$  is strongly stable, then the given condition on generators holds for every  $g \in I$ . In the opposite direction, let  $m \in I$  be any monomial. By Part (ii), there exists a unique factorization of the form  $m = gm'$ , where  $g \in I$  is a minimal monomial generator and  $m' \in \mathbb{K}[x_0, x_1, \dots, x_n]$  is a monomial such that  $\max g \leq \min m'$ . Let  $x_j$  divide  $m$  and let  $x_i >_{\text{lex}} x_j$ . Either  $x_j$  divides  $g$ , in which case  $gx_i x_j^{-1} \in I$  and so  $mx_i x_j^{-1} \in I$ , or  $x_j$  divides  $m'$ , in which case  $mx_i x_j^{-1}$  is a multiple of  $g$ .
- (iii) If  $x_n$  divides a minimal monomial generator  $g \in I$ , then for every variable  $x_j$  we have  $(gx_n^{-1})x_j \in I$ , while  $gx_n^{-1} \notin I$ . Conversely, any monomial  $m \in (I : \langle x_0, x_1, \dots, x_n \rangle) \setminus I$  yields a minimal monomial generator  $mx_n \in I$ , for example by applying Part (ii).
- (iv) Suppose that no such integer  $k$  exists. Because  $I$  is strongly stable, no monomial of the form  $x_{n-1}^{j-i} x_n^i$  is contained in  $I$ , for any  $j \in \mathbb{N}$  and  $i \in \{0, 1, \dots, j\}$ . This implies  $\rho_I \geq j + 1$ , which is a contradiction, as  $j$  is arbitrary. Hence, there exists  $k \in \mathbb{N}$  such that  $x_{n-1}^k \in I$ .  $\square$

**Example 5.3** (Lexicographic ideals are strongly stable). If  $L[\mathbf{p}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is lexicographic, with minimal monomial generators  $\{m_1, m_2, \dots, m_{n-(\ell+1)}\}$  as in Lemma 3.4, then applying Part (i) of Lemma 5.2 shows that  $L[\mathbf{p}, n]$  is strongly stable. Indeed, if  $x_j$  is a variable dividing the minimal monomial generator  $m_k$ , then, for every  $x_i >_{\text{lex}} x_j$ , we have  $m_k x_i x_j^{-1} \in \langle m_1, m_2, \dots, m_{k-1} \rangle$ . We know that  $L[\mathbf{p}, n]$  is saturated, by definition, and examining the minimal monomial generators of  $L[\mathbf{p}, n]$  verifies this.

Our first key operation is expansion of saturated strongly stable ideals, which is important for explicitly generating saturated strongly stable ideals. If  $I$  is a saturated strongly stable ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , then a minimal monomial generator  $g \in I$  is **expandable** if the set  $\{g x_{i+1} x_i^{-1} \mid x_i \text{ divides } g \text{ and } 0 \leq i < n-1\}$  contains no minimal monomial generators of  $I$ . The **expansion** of  $I$  at an expandable generator  $g$  is the monomial ideal  $I' \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  generated by

$$I' := \langle I \setminus \{g x_n^i \mid i \geq 0\} \rangle + \langle g x_j \mid \max g \leq j \leq n-1 \rangle;$$

see Definition 3.4 of [Moo12]. The monomial  $1 \in \langle 1 \rangle$  is vacuously expandable, with expansion  $\langle x_0, x_1, \dots, x_{n-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ . Part (i) of Lemma 5.2 ensures that the expansion of a saturated strongly stable ideal is again strongly stable. Moreover, avoiding the variable  $x_n$  in the definition of expansion ensures that expansions of saturated strongly stable ideals are saturated, by Part (iii) of Lemma 5.2.

The following example relates the lex-expansions defined by the mapping  $\Psi$  on the lexicographic forest with expansions of saturated strongly stable ideals.

**Example 5.4** (Lex-expansion is expansion at the last minimal monomial generator). Let  $L[\mathbf{p}, n] = L(a_0, a_1, \dots, a_{n-1}) \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a lexicographic ideal with minimal mono-

mial generators  $L[\mathfrak{p}, n] = \langle m_1, m_2, \dots, m_{n-(\ell+1)} \rangle$ , as in Lemma 3.4. The lex-smallest minimal monomial generator is  $m_{n-(\ell+1)} := x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}}$ . If  $a_i > 0$ , then  $x_{n-(i+1)}$  divides  $m_{n-(\ell+1)} x_{n-i} x_{n-(i+1)}^{-1}$  to order  $a_i - 1$ , which is not the case for any minimal monomial generator of  $L[\mathfrak{p}, n]$ . Therefore, the monomial  $m_{n-(\ell+1)}$  is expandable.

By definition of expansion, the list of minimal monomial generators of the expansion at  $m_{n-(\ell+1)}$  is

$$\begin{aligned} & \left\{ m_1, m_2, \dots, m_{n-(\ell+2)}, \right. \\ & \quad \left. m_{n-(\ell+1)} x_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+1)}, \dots, m_{n-(\ell+1)} x_{n-1} \right\} = \\ & \left\{ x_0, x_1, \dots, x_{n-(d+2)}, x_{n-(d+1)}^{a_d+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}+1}, \dots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+4)}^{a_{\ell+3}} x_{n-(\ell+3)}^{a_{\ell+2}+1}, \right. \\ & \quad x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}+1}, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-(\ell+1)}, \dots \\ & \quad \left. \dots, x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}} x_{n-1} \right\}. \end{aligned}$$

By inspection, we see that this is exactly the list of minimal monomial generators of the ideal  $L(a_0 + 1, a_1, a_2, \dots, a_{n-1})$ . Hence, the expansion of  $L[\mathfrak{p}, n]$  at  $m_{n-(\ell+1)}$  is equal to  $L(a_0 + 1, a_1, a_2, \dots, a_{n-1}) = \Psi(L[\mathfrak{p}, n])$ , in other words, every lex-expansion is expansion at the lex-smallest minimal monomial generator.

Generalizing Proposition 3.7, the following lemma relates expansion operations on saturated strongly stable ideals to the Macaulay tree. In particular, Lemma 5.5 explains the relationship between Hilbert polynomials of monomial ideals and those of their expansions.

**Lemma 5.5.** *If  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is a saturated strongly stable ideal, and  $I'$  is any expansion of  $I$ , then we have  $\mathfrak{p}_{I'} = \Psi(\mathfrak{p}_I)$ .*

*Proof.* Let  $d := \deg g$ , where  $I'$  is the expansion of  $I$  at  $g$ . It is clear that  $h_{I'}(d) = 1 + h_I(d)$ , because  $g$  is the only monomial in  $I_d \setminus I'_d$ . In any higher degree  $d > \deg g$ , applying Part (ii)

of Lemma 5.2 shows that the only monomial in  $I_d \setminus I'_d$  is  $gx_n^{d-\deg g}$ , so that  $h_{I'}(d) = 1 + h_I(d)$ . Hence, we have  $p_{I'} = 1 + p_I = \Psi(p_I)$ .  $\square$

We demonstrate this lemma with an example.

**Example 5.6.** The homogeneous ideal  $I := \langle x_0^2, x_0x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$  is not lexicographic, but is strongly stable by Part (i) of Lemma 5.2, and is saturated by Part (iii) of Lemma 5.2. The Hilbert polynomial of  $I$  equals  $p_I(t) = 3$ . Because  $x_0^2 \cdot x_1x_0^{-1} = x_0x_1$  and  $x_0x_1 \cdot x_1x_0^{-1} = x_1^2$  are minimal monomial generators of  $I$ , the only expandable minimal monomial generator of  $I$  is  $x_1^2$ . The expansion of  $I$  at  $x_1^2$  is the saturated strongly stable ideal  $I' := \langle x_0^2, x_0x_1, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ , which has Hilbert polynomial  $p_{I'}(t) = 4 = \Psi(3)$ .

To develop a similar theory for the mapping  $\Phi$ , we introduce the following function on saturated strongly stable ideals: for a saturated strongly stable ideal  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ , let  $\nabla(I) \subseteq \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$  denote the image of  $I$  under the evaluation map

$$\mathbb{K}[x_0, x_1, \dots, x_n] \rightarrow \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$$

sending  $x_j \mapsto x_j$  for  $0 \leq j \leq n-2$ , and  $x_k \mapsto 1$  for  $k \in \{n-1, n\}$ . Equivalently,  $\nabla(I)$  is the saturation with respect to  $\langle x_0, x_1, \dots, x_{n-1} \rangle$  of the restricted ideal  $I \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$  in  $\mathbb{K}[x_0, x_1, \dots, x_{n-1}]$ , and is  $(\text{sat}_{x_{n-1}, x_n} I) \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$ , where  $\text{sat}_{x_{n-1}, x_n} I$  is the **double saturation** of  $I$  in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , defined by setting  $x_{n-1} = x_n = 1$  in the monomials of  $I$ ; see page 642 of [Ree95].

**Lemma 5.7.** *If  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is a saturated strongly stable ideal, then we have the equality  $p_{\nabla(I)} = \nabla(p_I)$ .*

*Proof.* Let  $S := \mathbb{K}[x_0, x_1, \dots, x_n]$  be the graded polynomial ring. Because  $I$  is saturated, the variable  $x_n$  is a nonzerodivisor, and the homomorphism  $(S/I)(-1) \rightarrow S/I$  defined by

multiplication by  $x_n$  is injective; this gives a short exact sequence

$$0 \longrightarrow (S/I)(-1) \longrightarrow S/I \longrightarrow \mathbb{K}[x_0, x_1, \dots, x_{n-1}]/I \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}] \longrightarrow 0.$$

Thus, for every  $i \in \mathbb{Z}$ , the Hilbert function of the restricted ideal  $I \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$  satisfies  $h_{I \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}]}(i) = h_I(i) - h_I(i-1)$ . Hence, by saturating  $I \cap \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$  with respect to  $\langle x_0, x_1, \dots, x_{n-1} \rangle$ , we find that  $p_{\nabla(I)}(t) = p_I(t) - p_I(t-1) = [\nabla(p_I)](t)$ .  $\square$

**Example 5.8.** The homogeneous ideal  $I := \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$  is strongly stable by Part (i) of Lemma 5.2, and is saturated by Part (iii) of Lemma 5.2. The Hilbert polynomial of  $I$  equals  $p_I(t) = 3t+1$ ; the versal deformation space of this ideal is calculated in Lemma 6 of [PS85] to analyze the component structure of the Hilbert scheme of twisted cubics  $\text{Hilb}[3t+1, 3]$ . We compute

$$\nabla(I) = \nabla(\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle) = \langle x_0^2, x_0x_1, x_0, x_1^3 \rangle = \langle x_0, x_1^3 \rangle \subset \mathbb{K}[x_0, x_1, x_2],$$

which has Hilbert polynomial  $p_{\nabla(I)}(t) = 3 = \nabla(3t+1)$ .

The following lemma generalizes Proposition 3.9 to extensions of arbitrary saturated strongly stable ideals.

**Lemma 5.9.** *If  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is a saturated strongly stable ideal, then the Hilbert polynomial of the extension  $\Phi(I) \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  of  $I$  satisfies  $p_{\Phi(I)} - \Phi(p_I) \in \mathbb{N}$ . That is, for every such  $I$ , there exists a unique  $j \in \mathbb{N}$  such that  $p_{\Phi(I)} = \Psi^j \Phi(p_I)$ .*

*Proof.* The extension  $\Phi(I)$  has the same minimal monomial generators as  $I$ , so Part (i) and Part (iii) of Lemma 5.2 show that  $\Phi(I)$  is a saturated strongly stable ideal. Because  $I$  is saturated and strongly stable, none of its minimal monomial generators are divisible by  $x_n$ ,

so the definition of  $\nabla$  on ideals implies that  $\nabla(\Phi(I)) = I$ . Lemma 5.7 then implies that  $\nabla(\mathfrak{p}_{\Phi(I)}) = \mathfrak{p}_I$ , therefore we have  $\Phi \nabla(\mathfrak{p}_{\Phi(I)}) = \Phi(\mathfrak{p}_I)$ . To finish, Part (iii) of Lemma 2.10 shows that  $\mathfrak{p}_{\Phi(I)} - \Phi \nabla(\mathfrak{p}_{\Phi(I)}) \in \mathbb{N}$ , hence we have  $\mathfrak{p}_{\Phi(I)} - \Phi(\mathfrak{p}_I) \in \mathbb{N}$ , as desired. Letting  $j := \mathfrak{p}_{\Phi(I)} - \Phi(\mathfrak{p}_I)$ , we may also write  $\mathfrak{p}_{\Phi(I)} = j + \Phi(\mathfrak{p}_I) = \Psi^j \Phi(\mathfrak{p}_I)$ .  $\square$

**Example 5.10.** The homogeneous ideal  $I := \langle x_0^2, x_0x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$  from Example 5.6 is saturated and strongly stable, with Hilbert polynomial  $\mathfrak{p}_I(t) = 3$ . The extension ideal  $\Phi(I) := \langle x_0^2, x_0x_1, x_1^2 \rangle \subset \mathbb{K}[x_0, x_1, x_2, x_3]$  has Hilbert polynomial  $3t + 1 = \Psi \Phi(3)$ .

The heart of the algorithm in [Ree92] for generating saturated strongly stable ideals translates to the following theorem.

**Theorem 5.11.** *If  $I \neq \langle 1 \rangle$  is a saturated strongly stable ideal, with codimension  $c > 0$ , then there exists a finite binary sequence of expansions and extensions that begins at the homogeneous ideal  $\langle x_0, x_1, \dots, x_{c-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_c]$  and ends at  $I$ .*

The following definition helps us prove this: a monomial  $x^u := x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$  in the ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$  is **contractible** for the saturated strongly stable ideal  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ , where  $u := (u_0, u_1, \dots, u_n) \in \mathbb{N}^{n+1}$ , if  $\{x^u x_{i-1} x_i^{-1} \mid x_i \text{ divides } x^u \text{ and } 0 < i \leq n-1\} \subset I$  and  $x^u x_{n-1} \in I$  is a minimal monomial generator; the **contraction** of  $I$  at the contractible monomial  $x^u$  is the ideal  $I + \langle x^u \rangle$ . It follows from the definitions that, if  $x^u$  is contractible for  $I$ , then  $x^u \in I + \langle x^u \rangle$  is an expandable minimal monomial generator, with associated expansion  $(I + \langle x^u \rangle)' = I$ , thus one can undo a contraction via an expansion; for details, see Lemma 3.8 of [Moo12].

*Proof of Theorem 5.11.* We begin by proving that, for all saturated strongly stable ideals  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$ , there exists a sequence of expansions that begins at  $\text{sat}_{x_{n-1}, x_n} I$  and ends at  $I$ . If  $I \neq \text{sat}_{x_{n-1}, x_n} I$ , then some minimal monomial generator of  $I$  is divisible by

$x_{n-1}$ . Amongst all monomials  $x^v$  such that  $x^v x_{n-1} \in I$  is a minimal monomial generator, the lex-largest monomial, say  $x^u$ , is contractible. To prove this, let  $x_i$  divide  $x^u$ , so that strong stability implies  $(x^u x_{i-1} x_i^{-1}) x_{n-1} \in I$ ; this is not a minimal monomial generator, because  $x^u x_{i-1} x_i^{-1} >_{\text{lex}} x^u$ , so Part (ii) of Lemma 5.2 implies  $x^u x_{i-1} x_i^{-1} \in I$ . By construction, the contraction of  $I$  at this monomial has the same double saturation as  $I$ , but has fewer minimal monomial generators divisible by  $x_{n-1}$ . Continuing, we reach  $\text{sat}_{x_{n-1}, x_n} I$  by a sequence of contractions, and reversing the sequence gives a sequence of expansions beginning at  $\text{sat}_{x_{n-1}, x_n} I$  and ending at  $I$ .

To prove the full result, we use induction on the degree of  $\mathfrak{p}_I$ . If  $\deg \mathfrak{p}_I = 0$ , so that  $\mathfrak{p}_I$  is a constant, then Part (iv) of Lemma 5.2 implies that  $\text{sat}_{x_{n-1}, x_n} I = \langle 1 \rangle$ . By the first paragraph, there is a sequence of contractions from  $I$  to  $\langle 1 \rangle$ , where the final contraction must be the contraction of  $\langle x_0, x_1, \dots, x_{c-1} \rangle$  at 1. Ignoring the last contraction, and reversing the sequence gives the desired sequence of expansions from  $\langle x_0, x_1, \dots, x_{c-1} \rangle$  to  $I$ .

If  $\deg \mathfrak{p}_I > 0$ , then Lemma 5.7 implies that  $\deg \mathfrak{p}_{\nabla(I)} = (\deg \mathfrak{p}_I) - 1$ , and the induction hypothesis guarantees that there is a finite binary sequence of expansions and extensions that begins at  $\langle x_0, x_1, \dots, x_{c-1} \rangle$  and ends at  $\nabla(I)$ . Observing that  $\Phi \nabla(I) = \text{sat}_{x_{n-1}, x_n} I$ , the beginning of the proof shows that there exists a sequence of expansions beginning at  $\Phi \nabla(I)$  and ending at  $I$ . Hence, we obtain the desired finite binary sequence of expansions and extensions that begins at  $\langle x_0, x_1, \dots, x_{c-1} \rangle$  and ends at  $I$  by concatenating these two sequences, with a  $\Phi$  in between.  $\square$

In general, uniqueness is not guaranteed for the binary sequences from Theorem 5.11 ending in a given ideal  $I$ . However, if  $I := L[\mathfrak{p}, n]$  is a lexicographic ideal, then uniqueness is ensured, by the uniqueness of paths joining distinct vertices in the lexicographic tree  $\mathcal{L}_c$ .

Theorem 5.11 also ensures that there is an algorithm for generating every saturated



strongly stable ideal; this algorithm first appears in Appendix A of [Ree92], and is further studied in Chapter 3 of [Moo12], and in Section 5 of [CLMR11]. We describe that algorithm as follows:

**Algorithm 5.12.** *To generate all saturated strongly stable ideals with Hilbert polynomial  $p$  in a polynomial ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$  with  $n > d := \deg p$ , we proceed as follows:*

- (i) *compute the backwards difference polynomials  $\nabla(p), \nabla^2(p), \dots, \nabla^d(p)$ ;*
- (ii) *compute all saturated strongly stable ideals in  $\mathbb{K}[x_0, x_1, \dots, x_c]$  with Hilbert polynomial  $\nabla^d(p)$ , where  $c := n - d$ , by constructing all sequences of expansions that begin at  $\langle x_0, x_1, \dots, x_{c-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_c]$  and have length  $\nabla^d(p) - 1$ ;*
- (iii) *for each  $j \in \{0, 1, \dots, d-1\}$ , and for every ideal  $J$  computed at the  $j$ -th step with Hilbert polynomial  $\nabla^{d-j}(p)$ , do the following:*
  - *compute the Hilbert polynomial  $p_{\Phi(J)}$ ;*
  - *if  $k := \nabla^{d-(j+1)}(p) - p_{\Phi(J)} \in \mathbb{N}$ , then compute all sequences of expansions that begin with  $\Phi(J)$  of length  $k$ ;*
  - *if  $k := \nabla^{d-(j+1)}(p) - p_{\Phi(J)} < 0$ , then drop  $\Phi(J)$ ;*
- (iv) *return the ideals remaining after the  $d$ -th step.*

*Proof.* To prove that the algorithm works, suppose that  $I$  is a saturated strongly stable ideal with Hilbert polynomial  $p$ . By Theorem 5.11, there exists a finite binary sequence of expansions and extensions that begins at  $\langle x_0, x_1, \dots, x_{c-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_{c-1}]$  and ends at  $I$ . The algorithm works by evaluating every possible binary sequence beginning at  $\langle x_0, x_1, \dots, x_{c-1} \rangle$  and ending at a saturated strongly stable ideal with Hilbert polynomial

p. Each monomial ideal has finitely many expandable generators, and the Hilbert polynomials of the binary sequences follow the unique finite path from the root 1 to the vertex p in the Macaulay tree. Hence, the algorithm terminates, and  $I$  is obtained.  $\square$

We conclude this chapter with a demonstration of the algorithm.

**Example 5.13.** Suppose that we wish to compute all saturated strongly stable ideals with Hilbert polynomial  $p(t) := 3t + 1$ . We start by computing  $\nabla(p) = \nabla(3t + 1) = 3$ . We then compute all sequences of expansions that begin with  $\langle x_0, x_1 \rangle \subset \mathbb{K}[x_0, x_1, x_2]$ , and have length 2. The only expandable generator of  $\langle x_0, x_1 \rangle$  is  $x_1$ , with expansion  $\langle x_0, x_1^2 \rangle$ . Both  $x_0$  and  $x_1^2$  are expandable in  $\langle x_0, x_1^2 \rangle$ , with respective expansions  $\langle x_0^2, x_0x_1, x_1^2 \rangle$  and  $\langle x_0, x_1^3 \rangle$  in  $\mathbb{K}[x_0, x_1, x_2]$ . Extending each of these ideals to  $\mathbb{K}[x_0, x_1, x_2, x_3]$ , we find that  $\langle x_0^2, x_0x_1, x_1^2 \rangle$  has Hilbert polynomial  $3t + 1$ , while  $\langle x_0, x_1^3 \rangle$  has Hilbert polynomial  $3t$ . Thus, we perform all possible expansions on the latter; expansion at  $x_0$  results in  $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ , and expansion at  $x_1^3$  results in  $\langle x_0, x_1^4, x_1^3x_2 \rangle$ . Hence, there are three saturated strongly stable ideals with Hilbert polynomial  $3t + 1$  in  $\mathbb{K}[x_0, x_1, x_2, x_3]$ , namely,  $\langle x_0^2, x_0x_1, x_1^2 \rangle$ ,  $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ , and  $\langle x_0, x_1^4, x_1^3x_2 \rangle$ .

# Chapter 6

## Klimbing Trees

This chapter studies  $K$ -polynomials of saturated strongly stable ideals. The goal is to attain a precise understanding of where Hilbert functions and Hilbert polynomials of saturated strongly stable ideals coincide. We prove that amongst all saturated strongly stable ideals  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  with a fixed Hilbert polynomial  $p$ , the degree of the  $K$ -polynomial of the lexicographic ideal  $L[p, n]$  is *strictly* the largest. In essence, the proof of the main result works by tracking the genesis of minimal monomial generators along binary sequences of expansions and extensions as they trace the unique path from 1 to  $p$  in the Macaulay tree, and terminate in  $I$  and  $L[p, n]$ . In practice, Proposition 6.6 identifies the point in the binary sequences where the desired inequality between  $K$ -polynomials first occurs. Proposition 6.8 then shows that the inequality between  $K$ -polynomials persists along the rest of the binary sequences. These results combine and culminate in Theorem 6.10.

The **Hilbert series** of a finitely generated graded  $\mathbb{K}[x_0, x_1, \dots, x_n]$ -module  $M$  is the formal power series  $H_M(T) := \sum_{i \in \mathbb{Z}} h_M(i) T^i \in \mathbb{Z}[T^{-1}][[T]]$ . The Hilbert series of  $M$  can be expressed as a rational function  $H_M(T) = K_M(T)/(1 - T)^{n+1}$ , where  $K_M$  may be divisible by  $1 - T$ , and the  **$K$ -polynomial** of  $M$  is defined to be the polynomial numerator  $K_M$  of the

Hilbert series  $H_M$ ; see Theorem 8.20 of [MS05]. As in previous chapters, we are primarily interested in situations where  $M$  is a quotient  $\mathbb{K}[x_0, x_1, \dots, x_n]/I$  for some homogeneous ideal  $I$  of  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , and, in these cases, we use the notation  $H_I := H_{\mathbb{K}[x_0, x_1, \dots, x_n]/I}$  and  $K_I := K_{\mathbb{K}[x_0, x_1, \dots, x_n]/I}$ . We consider a fundamental example.

**Example 6.1** (*K*-polynomials of free modules). The Hilbert series of the polynomial ring  $S := \mathbb{K}[x_0, x_1, \dots, x_n]$  is given by  $H_S(T) = (1 - T)^{-(n+1)}$ , so that the *K*-polynomial of the polynomial ring is  $K_S(T) = 1$ . If  $d \in \mathbb{N}$ , then the Hilbert series of the twisted *S*-module  $S(-d) \cong S$  with generator in degree  $d$  is  $H_{S(-d)}(T) = T^d(1 - T)^{-(n+1)}$ , and the *K*-polynomial of  $S(-d)$  equals  $K_{S(-d)}(T) = T^d$ .

The following lemma shows that *K*-polynomials of strongly stable ideals can be derived directly from lists of minimal monomial generators; see Proposition 2.12 of [MS05] for the  $\mathbb{N}^{n+1}$ -graded case.

**Lemma 6.2.** *Let  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  be a strongly stable ideal.*

- (i) *The *K*-polynomial of  $I$  equals  $K_I(T) = 1 - \sum_g T^{\deg g} (1 - T)^{\max g}$ , where the sum is over all the minimal monomial generators  $g$  of  $I$ , and  $\max g$  is the maximum index amongst the variables dividing  $g$ .*
- (ii) *The *K*-polynomial of  $I$  satisfies the bound  $\deg K_I \leq \max_g \{\deg g + \max g\}$ , where the maximum is over all the minimal monomial generators  $g$  of  $I$ .*

*Proof.*

- (i) By Part (ii) of Lemma 5.2, the set of monomials in  $I$  consists of the disjoint union over all minimal monomial generators  $g \in I$  of the products of  $g$  with the monomials in  $\mathbb{K}[x_{\max g}, x_{\max g+1}, \dots, x_n]$ . For each minimal monomial generator  $g \in I$ ,

the number of such products is encoded in the Hilbert series of the twisted module  $\mathbb{K}[x_{\max g}, x_{\max g+1}, \dots, x_n](-\deg g) \cong g \cdot \mathbb{K}[x_{\max g}, x_{\max g+1}, \dots, x_n]$ , which equals  $T^{\deg g}(1-T)^{-(n+1)+\max g}$ , by Example 6.1. Hence, the Hilbert series of  $I$  equals

$$H_I(T) = \frac{1}{(1-T)^{n+1}} - \sum_g \frac{T^{\deg g}(1-T)^{\max g}}{(1-T)^{n+1}} = \frac{1 - \sum_g T^{\deg g}(1-T)^{\max g}}{(1-T)^{n+1}},$$

where the sum is over the minimal monomial generators  $g$  of  $I$ , and the  $K$ -polynomial of  $I$  equals  $K_I(T) = 1 - \sum_g T^{\deg g}(1-T)^{\max g}$ , as desired.

- (ii) The bound is immediately derived from Part (i) by examining the top degree summands after expanding the products  $(1-T)^{\max g}$ .  $\square$

We show how to apply Lemma 6.2 in an example.

**Example 6.3** ( $K$ -polynomial of the lexicographic ideal of the twisted cubic). Example 3.6 shows that the minimal monomial generators of the lexicographic ideal of the twisted cubic are given by  $L[3t+1, 3] = \langle x_0, x_1^4, x_1^3x_2 \rangle$ . The  $K$ -polynomial of the lexicographic ideal  $L[3t+1, 3]$  is computed from Part (i) of Lemma 6.2 to be

$$K_{L[3t+1,3]}(T) = 1 - [T^1(1-T)^0 + T^4(1-T)^1 + T^4(1-T)^2] = 1 - T - 2T^4 + 3T^5 - T^6.$$

Therefore,  $\deg K_{L[3t+1,3]} = 6$ , which is certainly less than or equal to the upper bound  $\max\{1+0, 4+1, 4+2\} = 6$  obtained from Part (ii) of Lemma 6.2.

Degrees of  $K$ -polynomials are rather useful invariants for studying Hilbert functions of ideals. Lemma 6.4 establishes that  $\deg H_I := (\deg K_I) - (n+1)$  encodes the minimal value after which the Hilbert function and Hilbert polynomial of  $I$  coincide; see Corollary 11.10 in [Kem11] for the case of filtrations of affine rings.

**Lemma 6.4.** *Let  $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$  be a homogeneous ideal with rational Hilbert series*

$$H_I(T) = \sum_{i \in \mathbb{Z}} h_I(i) T^i = \frac{K_I(T)}{(1-T)^{n+1}}$$

and Hilbert polynomial  $p_I$ . We have  $h_I(i) = p_I(i)$  for all  $i > \deg H_I$ , while  $h_I(i) \neq p_I(i)$  for  $i = \deg H_I$ .

*Proof.* Let  $K_I(T) = c_0 + c_1 T + \dots + c_d T^d \in \mathbb{Z}[T]$ , so that the Hilbert series of  $I$  is equal to  $H_I(T) = (c_0 + c_1 T + \dots + c_d T^d) \sum_{k \in \mathbb{N}} \binom{n+k}{n} T^k$ . Collecting like powers, we rewrite this Hilbert series as  $\sum_{i \in \mathbb{N}} (c_0 \binom{n+i}{n} + c_1 \binom{n+i-1}{n} + \dots + c_d \binom{n+i-d}{n}) T^i$ , showing that the Hilbert function is equal to  $h_I(i) = c_0 \binom{n+i}{n} + c_1 \binom{n+i-1}{n} + \dots + c_d \binom{n+i-d}{n}$ , for  $i \in \mathbb{Z}$ . From this, we see that the Hilbert polynomial of  $I$  equals  $p_I(t) = c_0 \binom{t+n}{n} + c_1 \binom{t+n-1}{n} + \dots + c_d \binom{t+n-d}{n}$ , where  $\binom{t+n-j}{n}$  denotes the polynomial  $\frac{(t+n-j)(t+n-j-1)\dots(t-j+1)}{n!} \in \mathbb{Q}[t]$ , for each  $j \in \{0, 1, \dots, d\}$ . The smallest root of the polynomial  $\binom{t+n-j}{n}$  is  $t = -(n-j)$ , which implies that the equality  $\binom{n+i-j}{n} = \binom{t+n-j}{n} \Big|_{t=i}$  holds if and only if  $i \geq -(n-j)$ . Hence, we find that  $h_I(i) = p_I(i)$  for all  $i \in \mathbb{Z}$  such that  $i \geq -(n-d) = d-n = 1 + \deg H_I$ . This proves the first statement. To finish, set  $i := d-n-1 = \deg H_I$ . We then have

$$\begin{aligned} p_I(i) &= \sum_{j=0}^d c_j \binom{t+n-j}{n} \Big|_{t=i} = \sum_{j=0}^{d-1} c_j \binom{t+n-j}{n} \Big|_{t=i} + c_d \binom{t+n-d}{n} \Big|_{t=i} \\ &= \sum_{j=0}^{d-1} c_j \binom{t+n-j}{n} \Big|_{t=i} + c_d (-1)^n, \end{aligned}$$

whereas  $h_I(i) = \sum_{j=0}^{d-1} c_j \binom{i+n-j}{n} + c_d \cdot 0$ . Since  $c_d \neq 0$ , we have  $p_I(i) \neq h_I(i)$ , as desired.  $\square$

We further examine Lemma 6.4 through a concrete example.

**Example 6.5** (Hilbert function of the lexicographic ideal of the twisted cubic). Exam-

ple 6.3 shows that the lexicographic ideal  $L[3t+1, 3]$  of the twisted cubic has  $K$ -polynomial  $K_{L[3t+1,3]}(T) = 1 - T - 2T^4 + 3T^5 - T^6$ , so that the degree of the Hilbert series of  $L[3t+1, 3]$  is given by  $\deg H_{L[3t+1,3]} = 6 - 4 = 2$ . Therefore, Lemma 6.4 implies that the Hilbert function satisfies  $h_{L[3t+1,3]}(2) \neq p_{L[3t+1,3]}(2)$ , and  $h_{L[3t+1,3]}(i) = p_{L[3t+1,3]}(i)$ , for all  $i > 2$ . Indeed, computing the Hilbert functions directly yields  $h_{L[3t+1,3]}(\mathbb{N}) = (1, 3, 6, 10, 13, 16, \dots)$  and  $p_{L[3t+1,3]}(\mathbb{N}) = (1, 4, 7, 10, 13, 16, \dots)$ .

The two propositions that follow capture the essential behaviour of  $\deg K_I$ , for a saturated strongly stable ideal  $I$ , and allow us to prove the main theorem of this chapter. In particular, Proposition 6.6 shows that expansions of lexicographic ideals at non-lex-smallest minimal monomial generators result in lower-degree  $K$ -polynomials than do lex-expansions.

**Proposition 6.6.** *Let  $L[\mathfrak{p}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be any lexicographic ideal.*

- (i) *If  $m \in L[\mathfrak{p}, n]$  is a minimal monomial generator, then  $m$  is expandable if and only if  $m$  is the lex-smallest minimal monomial generator of its degree in  $L[\mathfrak{p}, n]$ .*
- (ii) *Let  $m_{n-(\ell+1)}$  denote the lex-smallest minimal monomial generator of  $L[\mathfrak{p}, n]$ . If we let  $m \neq m_{n-(\ell+1)}$  be any other expandable minimal monomial generator of  $L[\mathfrak{p}, n]$ , and  $L[\mathfrak{p}, n]'$  denotes the expansion of  $L[\mathfrak{p}, n]$  at  $m$ , then every minimal monomial generator  $g \in L[\mathfrak{p}, n]'$  satisfies the inequality  $\deg g < 1 + \deg m_{n-(\ell+1)}$ .*
- (iii) *Moreover, in Part (ii), we have  $\deg K_{\Psi(L[\mathfrak{p}, n])} > \deg K_{L[\mathfrak{p}, n]'}$ , and equivalently, we have  $\deg H_{\Psi(L[\mathfrak{p}, n])} > \deg H_{L[\mathfrak{p}, n]'}$ .*

*Proof.*

- (i) Let  $\{m_1, m_2, \dots, m_{n-(\ell+1)}\}$  be the minimal set of monomial generators of  $L[\mathfrak{p}, n]$ , as in Lemma 3.4. These generators are listed in descending lex-order, but in nondecreasing degree. This is immediately verified for the linear generators, and the only expandable linear generator is  $m_{n-(d+1)} = x_{n-(d+2)}$ . For  $\ell + 3 \leq j \leq d$ , the two consecutive minimal monomial generators  $m_{n-j} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(j+2)}^{a_{j+1}} x_{n-(j+1)}^{a_j+1}$  and  $m_{n-(j-1)} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(j+1)}^{a_j} x_{n-j}^{a_{j-1}+1}$  have equal degree if and only if  $a_{j-1} = 0$ . In this case,  $m_{n-(j-1)} \in \{m_{n-j} x_{i+1} x_i^{-1} \mid x_i \text{ divides } m_{n-j} \text{ and } 0 \leq i < n-1\}$ , showing that  $m_{n-j}$  is not expandable. Similarly,  $m_{n-(\ell+2)} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+4)}^{a_{\ell+3}} x_{n-(\ell+3)}^{a_{\ell+2}+1}$  has the same degree as the last generator  $m_{n-(\ell+1)} = x_{n-(d+1)}^{a_d} x_{n-d}^{a_{d-1}} \cdots x_{n-(\ell+3)}^{a_{\ell+2}} x_{n-(\ell+2)}^{a_{\ell+1}}$  if and only if  $a_{\ell+1} = 1$ . If these last two generators have the same degree, then we find that  $m_{n-(\ell+1)} \in \{m_{n-(\ell+2)} x_{i+1} x_i^{-1} \mid x_i \text{ divides } m_{n-(\ell+2)} \text{ and } 0 \leq i < n-1\}$ , so that  $m_{n-(\ell+2)}$  is not expandable. Hence, for any minimal monomial generator  $m \in L[\mathfrak{p}, n]$ , we find that  $m$  is expandable if and only if  $m$  is lex-minimal in its degree.
- (ii) By Part (i), we must have  $\deg m < \deg m_{n-(\ell+1)}$ . But the minimal monomial generators of  $L[\mathfrak{p}, n]'$  are given by

$$(\{m_1, m_2, \dots, m_{n-(\ell+1)}\} \setminus \{m\}) \cup \{m x_{\max m}, m x_{\max m+1}, \dots, m x_{n-1}\},$$

where  $\deg m_{n-(\ell+1)}$  is maximal among minimal monomial generators of  $L[\mathfrak{p}, n]$ . Hence, we obtain the desired inequality.

- (iii) The lex-expansion  $\Psi(L[\mathfrak{p}, n])$  is the expansion of  $L[\mathfrak{p}, n]$  at  $m_{n-(\ell+1)}$ , by Example 5.4. The minimal monomial generators of  $\Psi(L[\mathfrak{p}, n])$  are

$$\{m_1, m_2, \dots, m_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+2)}, m_{n-(\ell+1)} x_{n-(\ell+1)}, \dots, m_{n-(\ell+1)} x_{n-1}\}.$$



Because  $m_{n-(\ell+1)}$  has maximal degree in the minimal set of monomial generators of  $L[\mathfrak{p}, n]$ , Part (i) of Lemma 6.2 shows that the degree of the  $K$ -polynomial of  $\Psi(L[\mathfrak{p}, n])$  satisfies

$$\deg \mathbf{K}_{\Psi(L[\mathfrak{p}, n])} = \deg(m_{n-(\ell+1)}x_{n-1}) + \max(m_{n-(\ell+1)}x_{n-1}) = \deg m_{n-(\ell+1)} + n.$$

On the other hand, Part (ii) of Lemma 6.2 and Part (ii) yield

$$\begin{aligned} \deg \mathbf{K}_{L[\mathfrak{p}, n]'} &\leq \max \{ \deg g + \max g \mid g \text{ is a minimal monomial generator of } L[\mathfrak{p}, n]' \} \\ &< 1 + \deg m_{n-(\ell+1)} + n - 1 = \deg \mathbf{K}_{\Psi(L[\mathfrak{p}, n])}, \end{aligned}$$

hence, the inequality holds for  $K$ -polynomials, and the inequality for Hilbert series is obtained by subtracting  $n + 1$  on both sides.  $\square$

We work this out in an example.

**Example 6.7** ( $K$ -polynomials of expansions of the lexicographic ideal of a planar cubic). Consider the lexicographic ideal  $L[3t, 3] = \langle x_0, x_1^3 \rangle$  of a planar cubic curve in  $\mathbb{P}^3$ . Both  $x_0$  and  $x_1^3$  are the lex-smallest generators in their respective degrees, and Part (i) of Proposition 6.6 shows that they are expandable. Example 3.8 demonstrates that the expansion at  $x_1^3$  equals  $L[3t+1, 3]$ , and Example 6.3 shows that  $\deg \mathbf{K}_{L[3t+1, 3]} = 6$ . On the other hand, the expansion at  $x_0$  is the saturated strongly stable ideal  $L[3t, 3]' := \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ , which also has Hilbert polynomial  $3t + 1$ , and we may compute its  $K$ -polynomial via Part (i) of Lemma 6.2 as follows:

$$\mathbf{K}_{L[3t, 3]'}(T) = 1 - [T^2(1 - T)^0 + T^2(1 - T)^1 + T^2(1 - T)^2 + T^3(1 - T)^1] = 1 - 3T^2 + 2T^3.$$

Hence, we have  $\deg K_{L[3t,3]'} = 3 < 6$ , and equivalently,  $\deg H_{L[3t,3]'} = -1 < 2 = \deg H_{L[3t+1,3]}$ . In particular, the Hilbert function  $h_{L[3t,3]'}$  and Hilbert polynomial  $p_{L[3t,3]'}$  coincide, that is, they satisfy  $h_{L[3t,3]'}(i) = p_{L[3t,3]'}(i)$ , for all  $i \in \mathbb{N}$ .

Proposition 6.8 explains the persistence of inequalities of the sort described by Proposition 6.6.

**Proposition 6.8.** *Let  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a saturated strongly stable ideal with Hilbert polynomial  $p := p_I$ , and let  $L[p, n]$  be the corresponding lexicographic ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ . Let  $m_{n-(\ell+1)}$  denote the lex-smallest minimal monomial generator of  $L[p, n]$ . Consider the following condition on  $I$ :*

*every minimal monomial generator  $g \in I$  satisfies the inequalities*

$$\deg g < \deg m_{n-(\ell+1)} \text{ and } \max g \leq \max m_{n-(\ell+1)}. \quad (\star)$$

*If  $I$  satisfies  $(\star)$ , then the following are true:*

- (i)  $\deg K_{L[p,n]} > \deg K_I$ , or equivalently,  $\deg H_{L[p,n]} > \deg H_I$ ;
- (ii) if  $I'$  denotes any expansion of  $I$ , then  $I'$  satisfies  $(\star)$  with respect to its corresponding lexicographic ideal  $\Psi(L[p, n]) = L[\Psi(p), n]$ ;
- (iii) the extension  $\Phi(I)$  satisfies  $(\star)$  with respect to its corresponding lexicographic ideal  $\Psi^j \Phi(L[p, n]) = L[\Psi^j \Phi(p), n + 1]$ , where  $j \in \mathbb{N}$  is given by Lemma 5.9; and
- (iv) if  $I_{(0)}, I_{(1)}, \dots, I_{(i)}$  denotes any finite binary sequence of expansions and extensions that begins at  $I_{(0)} := I$ , then  $I_{(i)}$  satisfies  $(\star)$ .

*Proof.*

(i) Part (ii) of Lemma 6.2 and the condition  $(\star)$  give

$$\begin{aligned} \deg K_I &\leq \max \{ \deg g + \max g \mid g \text{ is a minimal monomial generator of } I \} \\ &< \deg m_{n-(\ell+1)} + \max m_{n-(\ell+1)} \\ &= K_{L[p,n]}. \end{aligned}$$

(ii) By definition of lex-expansion, the condition  $(\star)$  for the expansion  $I'$  becomes that every minimal monomial generator  $g' \in I'$  satisfies  $\deg g' < 1 + \deg m_{n-(\ell+1)}$  and  $\max g' \leq n - 1$ . Both inequalities hold, by definition of the minimal monomial generators of  $I'$ , and because  $I$  satisfies  $(\star)$ .

(iii) By definition of extension, a condition analogous to  $(\star)$  already holds between  $\Phi(I)$  and  $\Phi(L[p, n])$ . Replacing  $\Phi(L[p, n])$  with the iterated lex-expansion  $\Psi^j \Phi(L[p, n])$  results in increased degree and maximum index of the lex-smallest minimal monomial generator of  $\Psi^j \Phi(L[p, n])$ . Hence,  $\Phi(I)$  satisfies  $(\star)$ .

(iv) We work by induction on the length  $i$  of the binary sequence  $I_{(0)}, I_{(1)}, \dots, I_{(i)}$ . The case  $i = 1$  is resolved by Part (ii) and Part (iii). If  $i > 1$ , then Part (ii) and Part (iii) ensure that  $I_{(1)}$  satisfies  $(\star)$ , and we then obtain the result by applying the induction hypothesis to the binary subsequence  $I_{(1)}, I_{(2)}, \dots, I_{(i)}$ .  $\square$

The following example demonstrates the persistence of the inequalities described by Proposition 6.8.

**Example 6.9** (Persistence of the condition  $(\star)$ ). The ideal  $I := L[3t, 3]' = \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$  in  $\mathbb{K}[x_0, x_1, x_2, x_3]$  from Example 6.7, obtained by expanding  $x_0$  in  $L[3t, 3] = \langle x_0, x_1^3 \rangle$ , satisfies the condition  $(\star)$  with respect to its lexicographic ideal  $L[3t + 1, 3] = \langle x_0, x_1^4, x_1^3x_2 \rangle$ ,

because the generators of  $I$  have degrees 2 or 3, and maximum indices 0, 1, or 2, while the lex-smallest generator  $x_1^3x_2$  in  $L[3t+1, 3]$  has degree 4 and maximum index 2. The expandable monomials in  $I$  are  $x_0x_2$  and  $x_1^3$ , and their respective expansion ideals are  $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$  and  $\langle x_0^2, x_0x_1, x_0x_2, x_1^4, x_1^3x_2 \rangle$ , whose minimal monomial generators have degrees less than 5 and maximum indices less than or equal to 2. Comparing these against the corresponding lexicographic ideal  $\langle x_0, x_1^4, x_1^3x_2 \rangle$  shows that both expansions also satisfy  $(\star)$ , as the lex-smallest minimal monomial generator  $x_1^3x_2^2$  has degree 5 and maximum index 2.

In the other direction, the extension of  $I$  to  $\mathbb{K}[x_0, x_1, \dots, x_4]$  has Hilbert polynomial  $p_I(t) = (3/2)t^2 + (5/2)t + 1$ , while  $\Phi(3t+1) = (3/2)t^2 + (5/2)t - 1$ . Thus, the lexicographic ideal corresponding to  $\Phi(I)$  is derived from  $L[(3/2)t^2 + (5/2)t - 1, 4] = \langle x_0, x_1^4, x_1^3x_2 \rangle$  by performing two lex-expansions, and equals  $\Psi^2(L[(3/2)t^2 + (5/2)t - 1, 4]) = \langle x_0, x_1^4, x_1^3x_2^2, x_1^3x_2x_3^2 \rangle$ . Because the lex-smallest minimal monomial generator  $x_1^3x_2x_3^2$  of  $L[(3/2)t^2 + (5/2)t + 1, 4]$  has degree 6 and maximum index 3, the ideal  $\Phi(I) = \Phi(L[3t, 3]') = \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$  in  $\mathbb{K}[x_0, x_1, \dots, x_4]$  also satisfies the condition  $(\star)$ .

With Proposition 6.6 and Proposition 6.8 at hand, we can prove the main result of this chapter.

**Theorem 6.10.** *Let  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a saturated strongly stable ideal with Hilbert polynomial  $p := p_I$ , and let  $L[p, n]$  be the corresponding lexicographic ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ . If  $I \neq L[p, n]$ , then  $\deg K_{L[p, n]} > \deg K_I$ , and equivalently,  $\deg H_{L[p, n]} > \deg H_I$ .*

*Proof.* The ideals  $I$  and  $L[p, n]$  are saturated and strongly stable, so they are both generated by Algorithm 5.12. In particular, if the codimension of  $I$  is denoted by  $c \in \mathbb{N}$ , then there exists a finite binary sequence of expansions and extensions  $I_{(0)}, I_{(1)}, \dots, I_{(i)}$  such that  $I_{(0)} = \langle x_0, x_1, \dots, x_{c-1} \rangle \subset \mathbb{K}[x_0, x_1, \dots, x_c]$  and  $I_{(i)} = I$ . Chapter 3 implies that to

attain a nonlexicographic saturated strongly stable ideal, the binary sequence must contain an expansion that is not a lex-expansion. Thus, if  $I \neq L[\mathfrak{p}, n]$ , then there must exist an index  $k \in \{1, 2, \dots, i\}$  such that  $I_{(j)}$  equals either  $\Psi(I_{(j-1)})$  or  $\Phi(I_{(j-1)})$ , for every  $j \in \{1, 2, \dots, k-1\}$ , but  $I_{(k)} \neq \Psi(I_{(k-1)})$  and  $I_{(k)} \neq \Phi(I_{(k-1)})$ . By Part (i) of Proposition 6.6, the ideal  $I_{(k)}$  is the expansion of  $I_{(k-1)}$  at a generator of lower degree than that of the lex-smallest minimal monomial generator of  $I_{(k-1)}$ . Applying Part (ii) of Proposition 6.6 then shows that  $I_{(k)}$  satisfies the condition  $(\star)$  from Proposition 6.8. Therefore, we may apply Part (iv) of Proposition 6.8 to the binary subsequence  $I_{(k)}, I_{(k+1)}, \dots, I_{(i)}$ , showing that  $I_{(i)} = I$  satisfies the condition  $(\star)$ . Hence, applying Part (i) of Proposition 6.8 finishes the proof, showing that  $\deg \mathcal{K}_{L[\mathfrak{p}, n]} > \deg \mathcal{K}_I$ , and equivalently,  $\deg \mathcal{H}_{L[\mathfrak{p}, n]} > \deg \mathcal{H}_I$ .  $\square$

*Proof of Theorem 1.2.* Theorem 6.10 proves the claim.  $\square$

**Example 6.11.** As shown in Example 5.13, there are exactly three saturated strongly stable ideals in  $\mathbb{K}[x_0, x_1, x_2, x_3]$  with Hilbert polynomial  $\mathfrak{p} := 3t + 1$ , namely,  $\langle x_0^2, x_0x_1, x_1^2 \rangle$ ,  $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$ , and the lexicographic ideal  $L[3t + 1, 3] = \langle x_0, x_1^4, x_1^3x_2 \rangle$ . Example 6.3 shows that the degree of the  $K$ -polynomial of the lexicographic ideal  $L[3t + 1, 3]$  is 6, whereas Example 6.7 shows that the degree of the  $K$ -polynomial of  $\langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$  is 3. We compute the  $K$ -polynomial of  $\langle x_0^2, x_0x_1, x_1^2 \rangle$  via Part (i) of Lemma 6.2 to be

$$\mathcal{K}_{\langle x_0^2, x_0x_1, x_1^2 \rangle}(T) = 1 - [T^2(1 - T)^0 + T^2(1 - T)^1 + T^2(1 - T)^1] = 1 - 3T^2 + 2T^3,$$

which shows that the degree of the  $K$ -polynomial of  $\langle x_0^2, x_0x_1, x_1^2 \rangle$  also equals 3. Hence, both of these nonlexicographic saturated strongly stable ideals have  $K$ -polynomials of strictly smaller degree than that of the lexicographic ideal.

Reformulating Theorem 6.10 for Hilbert functions using Lemma 6.4, we obtain the

following result:

**Corollary 6.12.** *Let  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a saturated strongly stable ideal, and let  $L[\mathfrak{p}, n]$  be the corresponding lexicographic ideal in  $\mathbb{K}[x_0, x_1, \dots, x_n]$ , with  $\mathfrak{p} := \mathfrak{p}_I$ . If  $I \neq L[\mathfrak{p}, n]$ , then  $h_I$  coincides with  $\mathfrak{p}$  strictly earlier than  $h_{L[\mathfrak{p}, n]}$  does, in the sense that there exists  $k \in \mathbb{Z}$  such that  $h_I(j) = \mathfrak{p}(j)$ , for all  $j \geq k$ , but  $h_{L[\mathfrak{p}, n]}(k) \neq \mathfrak{p}(k)$ .*

*Proof.* Lemma 6.4 and Theorem 6.10 show that this is the case for  $k := 1 + \deg H_I$ .  $\square$

# Chapter 7

## Irreducibility of Random Hilbert Schemes

In this chapter, we revisit the probabilistic interpretation of the Hilbert forest from Chapter 4 to study irreducibility of general Hilbert schemes of projective spaces. We consider the numbers of irreducible components of Hilbert schemes to be the outcomes of a random variable on the forest of Hilbert schemes, in which case the question of which and how many Hilbert schemes are irreducible translates to the problem of determining the likelihood that this random variable equals 1. Theorem 7.7 provides the lower bound 0.5 for the probability that a random Hilbert scheme is irreducible. To prove the bound, we apply our main result on  $K$ -polynomials, Theorem 6.10, to prove Theorem 7.3, which is interpreted as saying that every node in the Hilbert forest has at least one child node that corresponds to an irreducible Hilbert scheme. This family of irreducible Hilbert schemes, described in Corollary 7.5, also appears through entirely different methods in Proposition 1 of [Got89], whose goal is to generalize the classes of Hilbert schemes that parametrize planar curves in some larger projective space, and that parametrize hypersurfaces.

The following lemma presents two elementary aspects of Hilbert functions, which we use to prove Theorem 7.3.

**Lemma 7.1.** *Let  $I \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  be a homogeneous ideal, and let  $\mathfrak{p} := \mathfrak{p}_I$  denote the Hilbert polynomial of  $I$ .*

(i) *If  $L[\mathfrak{p}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  denotes the lexicographic ideal corresponding to  $I$ , then*

$$h_I(i) \geq h_{L[\mathfrak{p}, n]}(i), \text{ for every } i \in \mathbb{Z}.$$

(ii) *The Hilbert function of the extension ideal  $\Phi(I) \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$  is given by the sum*

$$h_{\Phi(I)}(i) = \sum_{0 \leq j \leq i} h_I(j).$$

*Proof.*

(i) As defined in Chapter 3, let  $L[\mathfrak{h}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  denote the lexicographic ideal for the Hilbert function  $\mathfrak{h} := h_I$  of  $I$ . By definition,  $L[\mathfrak{p}, n]$  is the saturation of  $L[\mathfrak{h}, n]$  with respect to  $\langle x_0, x_1, \dots, x_n \rangle$ , and therefore contains  $L[\mathfrak{h}, n]$ . In any given degree  $i \in \mathbb{Z}$ , the value  $h(i)$  equals the  $\mathbb{K}$ -dimension of  $\mathbb{K}[x_0, x_1, \dots, x_n]_i / I_i$ , which equals the  $\mathbb{K}$ -dimension of  $\mathbb{K}[x_0, x_1, \dots, x_n]_i / L[\mathfrak{h}, n]_i$ , which is at least as large as that of  $\mathbb{K}[x_0, x_1, \dots, x_n]_i / L[\mathfrak{p}, n]_i$ . Hence,  $h(i) \geq h_{L[\mathfrak{p}, n]}(i)$  follows.

(ii) The homogeneous piece  $(\Phi(I))_i$  has decomposition

$$(\Phi(I))_i = \bigoplus_{j \in \mathbb{N}, j \leq i} I_j \cdot x_{n+1}^{i-j} \subset \bigoplus_{j \in \mathbb{N}, j \leq i} \mathbb{K}[x_0, x_1, \dots, x_n]_j \cdot x_{n+1}^{i-j},$$

where the latter is a direct sum decomposition for  $(\mathbb{K}[x_0, x_1, \dots, x_{n+1}])_i$ , and the desired equality follows directly.  $\square$

We demonstrate this with an example.



**Example 7.2.** Let  $I := L[3t, 3]' = \langle x_0^2, x_0x_1, x_0x_2, x_1^3 \rangle$  be the nonlexicographic expansion in  $\mathbb{K}[x_0, x_1, x_2, x_3]$  described in Example 6.7. Because  $\deg H_I < 0$ , Lemma 6.4 implies that the nonzero values of  $h_I$  are given by  $h_I(\mathbb{N}) = p_I(\mathbb{N}) = (1, 4, 7, 10, 13, 16, \dots)$ . This determines the corresponding lexicographic ideal  $L[h_I, 3] = \langle x_0^2, x_0x_1, x_0x_2, x_0x_3^2, x_1^4, x_1^3x_2 \rangle$ , and its saturation  $L[3t+1, 3]$  with respect to  $\langle x_0, x_1, x_2, x_3 \rangle$  has Hilbert function given by  $h_{L[3t+1, 3]}(\mathbb{N}) = (1, 3, 6, 10, 13, 16, \dots)$ ; these values clearly satisfy the inequality in Part (i) of Lemma 7.1. Moreover, the Hilbert functions of the extension ideals  $\Phi(I)$  and  $\Phi(L[h_I, 3])$  take values  $h_{\Phi(I)}(\mathbb{N}) = h_{\Phi(L[h_I, 3])}(\mathbb{N}) = (1, 5, 12, 22, 35, 51, \dots)$ , and the Hilbert function of  $\Phi(L[3t+1, 3])$  takes values  $h_{\Phi(L[3t+1, 3])}(\mathbb{N}) = (1, 4, 10, 20, 33, 49, \dots)$ . The two extension ideals cannot have equal Hilbert polynomials, in fact, Part (ii) of Lemma 7.1 and the values of the Hilbert functions  $h_I$  and  $h_{L[3t+1, 3]}$  imply that  $p_{\Phi(I)} = 2 + p_{\Phi(L[3t+1, 3])} = p_{\Psi^2 \Phi(L[3t+1, 3])}$ , which equals  $(3/2)t^2 + (5/2)t + 1$ .

Example 7.2 gives a saturated strongly stable ideal  $I$  such that the Hilbert polynomial of  $\Phi(I)$  is obtained by adding a positive integer to the Hilbert polynomial of the extension of the corresponding lexicographic ideal. The following theorem generalizes this observation.

**Theorem 7.3.** *Let  $p$  be an admissible Hilbert polynomial. For each positive codimension  $c \in \mathbb{Z}$ , the lexicographic ideal is the unique saturated strongly stable ideal of codimension  $c$  with Hilbert polynomial  $\Phi(p)$ .*

*Proof.* Saturated strongly stable ideals are generated by Algorithm 5.12. The procedure is recursive, and particularly, Phase (iii) of Algorithm 5.12 shows that it generates the codimension  $c$  saturated strongly stable ideals with Hilbert polynomial  $\Phi(p)$  by extending all codimension  $c$  saturated strongly stable ideals with Hilbert polynomial  $p = \nabla \Phi(p)$ , and then selecting the resulting extended ideals whose Hilbert polynomial equals  $\Phi(p)$ .

By Proposition 3.9, the unique lexicographic ideal  $L[\mathfrak{p}, n] \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  extends to  $L[\Phi(\mathfrak{p}), n+1] \subset \mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ , where  $n = c + \deg \mathfrak{p}$ . Therefore, to prove the theorem, it suffices to prove the following statement:

*If  $J \subset \mathbb{K}[x_0, x_1, \dots, x_n]$  is a saturated and strongly stable, but not lexicographic, ideal with Hilbert polynomial  $\mathfrak{p}$ , then  $\mathfrak{p}_{\Phi(J)} \neq \Phi(\mathfrak{p})$ .*

Let  $I := \Phi(J)$  denote the extension of such an ideal  $J$ . By Lemma 5.9, we must equivalently show that  $\mathfrak{p}_I - \Phi(\mathfrak{p})$  is a positive integer. Setting  $d_{\mathfrak{p}} := \deg H_{L[\mathfrak{p}, n]}$ , we achieve this by showing that  $h_I(i) > h_{L[\Phi(\mathfrak{p}), n+1]}(i)$ , for every integer  $i$  satisfying  $i \geq d_{\mathfrak{p}}$ . Indeed, Part (ii) of Lemma 7.1 implies that we can write

$$\begin{aligned} h_I(i) &= \sum_{0 \leq j \leq i} h_J(j) = \sum_{0 \leq j \leq d_{\mathfrak{p}}} h_J(j) + \sum_{d_{\mathfrak{p}} < j \leq i} h_J(j), \text{ and} \\ h_{L[\Phi(\mathfrak{p}), n+1]}(i) &= \sum_{0 \leq j \leq i} h_{L[\mathfrak{p}, n]}(j) = \sum_{0 \leq j \leq d_{\mathfrak{p}}} h_{L[\mathfrak{p}, n]}(j) + \sum_{d_{\mathfrak{p}} < j \leq i} h_{L[\mathfrak{p}, n]}(j). \end{aligned}$$

Theorem 6.10 implies that  $\deg H_J < d_{\mathfrak{p}}$ , so that applying Lemma 6.4 to both  $J$  and  $L[\mathfrak{p}, n]$  shows that  $\sum_{d_{\mathfrak{p}} < j \leq i} h_J(j) = \sum_{d_{\mathfrak{p}} < j \leq i} \mathfrak{p}(j) = \sum_{d_{\mathfrak{p}} < j \leq i} h_{L[\mathfrak{p}, n]}(j)$ . Thus, to prove the desired inequality  $h_I(i) > h_{L[\Phi(\mathfrak{p}), n+1]}(i)$ , we must prove that  $\sum_{0 \leq j \leq d_{\mathfrak{p}}} h_J(j) > \sum_{0 \leq j \leq d_{\mathfrak{p}}} h_{L[\mathfrak{p}, n]}(j)$ . Part (i) of Lemma 7.1 guarantees that  $h_J(j) \geq h_{L[\mathfrak{p}, n]}(j)$ , for all  $0 \leq j \leq d_{\mathfrak{p}}$ , so that the inequality  $\sum_{0 \leq j \leq d_{\mathfrak{p}}} h_J(j) \geq \sum_{0 \leq j \leq d_{\mathfrak{p}}} h_{L[\mathfrak{p}, n]}(j)$  holds, and further, strict inequality fails if and only if  $h_J(j) = h_{L[\mathfrak{p}, n]}(j)$ , for all  $0 \leq j \leq d_{\mathfrak{p}}$ . However, if this were true, then we would have  $h_J = h_{L[\mathfrak{p}, n]}$ , which would contradict Corollary 6.12. Therefore, we obtain the strict inequality  $\sum_{0 \leq j \leq d_{\mathfrak{p}}} h_J(j) > \sum_{0 \leq j \leq d_{\mathfrak{p}}} h_{L[\mathfrak{p}, n]}(j)$ , showing that  $h_I(i) > h_{L[\Phi(\mathfrak{p}), n+1]}(i)$ , and in particular, that  $\mathfrak{p}_I - \Phi(\mathfrak{p})$  is a positive integer. Hence, the unique saturated strongly stable ideal with Hilbert polynomial  $\Phi(\mathfrak{p})$  is the lexicographic ideal  $L[\Phi(\mathfrak{p}), n+1]$ .  $\square$

Although Theorem 7.3 is framed entirely algebraically, it manifests in the geometry of Hilbert schemes. The following example relates the above family of admissible polynomials with unique saturated strongly stable ideals to well-known Hilbert schemes.

**Example 7.4.** Suppose that  $p = d$  is a positive integer, considered as a constant polynomial. The Gotzmann expression of  $p$  is supplied by  $p(t) = d = \binom{t+0}{0} + \binom{t+0-1}{0} + \cdots + \binom{t+0-(d-1)}{0}$ , and the extended Hilbert polynomial equals  $[\Phi(p)](t) := \binom{t+1}{1} + \binom{t+1-1}{1} + \cdots + \binom{t+1-(d-1)}{1}$ , which equals  $dt + 1 - \frac{(d-1)(d-2)}{2}$ . Therefore, any Hilbert scheme of the form  $\text{Hilb}[\Phi(p), n+1]$  with  $p = d$  parametrizes curves with arithmetic genus  $p_a := \frac{(d-1)(d-2)}{2}$ , and it is well-known that these are exactly the planar curves of degree  $d$ , which have irreducible and nonsingular Hilbert schemes. Hence, the Hilbert schemes of the form  $\text{Hilb}[\Phi(p), n+1]$  for constant  $p$  are nonsingular and irreducible.

The following corollary generalizes Example 7.4 and rediscovers the family of irreducible Hilbert schemes described in Proposition 1 of [Got89].

**Corollary 7.5.** *If  $p$  is any admissible Hilbert polynomial, and  $n \in \mathbb{Z}$  satisfies  $n > \deg p$ , then the Hilbert scheme  $\text{Hilb}[\Phi(p), n+1]$  is irreducible and nonsingular.*

*Proof.* Every irreducible component, and every intersection of irreducible components, of  $\text{Hilb}[\Phi(p), n+1]$  contains at least one point  $[X_I]$  that is defined by a saturated strongly stable ideal  $I$ ; see Remark 2.1 of [Ree95]. However, Theorem 7.3 proves that  $\text{Hilb}[\Phi(p), n+1]$  only has a single point defined by a saturated strongly stable ideal, which must be the lexicographic point  $[X_{L[\Phi(p), n+1]}]$ . Thus, if there were multiple irreducible components of  $\text{Hilb}[\Phi(p), n+1]$ , then the lexicographic point  $L[\Phi(p), n+1]$  would lie on all of their pairwise intersections. However, Theorem 1.4 of [RS97] proves that lexicographic points are always nonsingular, so that  $[X_{L[\Phi(p), n+1]}]$  cannot lie on an intersection of distinct irreducible com-

ponents. Therefore,  $\text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  has a unique irreducible component, which must be generically nonsingular, as it contains a nonsingular point.

Suppose that  $\text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  has a singular point that is defined by an ideal  $I$  in  $\mathbb{K}[x_0, x_1, \dots, x_{n+1}]$ . For any  $G \in \text{GL}_{n+2}(\mathbb{K})$ , the point  $[X_{G \cdot I}] \in \text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  is also singular, and for generic  $G \in \text{GL}_{n+2}(\mathbb{K})$ , the initial ideal of  $G \cdot I$  with respect to any chosen monomial ordering is saturated and strongly stable; see Theorem 2 of [Gal74] and Proposition 1 of [BS87b]. Thus, there exists a one-parameter family of singular points of  $\text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  degenerating to the lexicographic ideal; see Proposition I.2.12 in [Bay82], or Theorem 15.17 in [Eis95]. By upper semicontinuity of the dimension of the cohomology of the normal sheaf, we conclude that the lexicographic ideal is singular, which gives a contradiction; see Theorem III.12.8 in [Har77], Part (b) of Theorem 1.1 in [Har10], and Theorem 1.4 in [RS97]. Hence,  $\text{Hilb}[\Phi(\mathfrak{p}), n + 1]$  is nonsingular and irreducible.  $\square$

**Remark 7.6.** Proposition 1 of [Got89] also shows that the Hilbert schemes described in Corollary 7.5 are irreducible. To prove that the two families are the same, it is necessary to translate between Gotzmann expressions of Hilbert polynomials of quotients  $\mathbb{K}[x_0, x_1, \dots, x_n]/I$  by homogeneous ideals  $I$ , and another type of expression for Hilbert polynomials of homogeneous ideals (not of quotients) as sums of binomials, which appears in Equation 1 of [Got89]. The information needed to make the translation can be mined out of Section 2 of Appendix C in [IK99].

Following Chapter 4, we endow Hilbert trees and the Hilbert forest with natural discrete probability distributions arising from probability mass functions on the nonnegative integers. A benefit of our method of discovery of the family of irreducible Hilbert schemes described in Corollary 7.5 is that we immediately see that at least half of the vertices at any given height in any given Hilbert tree correspond to irreducible and nonsingular Hilbert

schemes; each vertex at a fixed height  $k \in \mathbb{N}$  is connected by an edge to at least one vertex at height  $k + 1$  corresponding to an irreducible and nonsingular Hilbert scheme. This leads to the following probabilistic interpretation of Theorem 7.3.

**Theorem 7.7.** *Let  $\text{irr}: \mathcal{H} \rightarrow \mathbb{N}$  denote the random variable on the Hilbert forest  $\mathcal{H}$  taking a Hilbert scheme to the number of its irreducible components, and for every positive  $c \in \mathbb{Z}$ , let  $\text{irr}_c := \text{irr}|_{\mathcal{H}_c}$  be the restriction to the Hilbert tree  $\mathcal{H}_c$ . The probability that a random Hilbert scheme is irreducible satisfies  $\Pr(\text{irr} = 1) > 0.5$ , and moreover,  $\Pr(\text{irr}_c = 1) > 0.5$  for every positive  $c \in \mathbb{Z}$ .*

*Proof.* As in Chapter 4, the probability distribution  $\Pr$  on the Hilbert forest  $\mathcal{H}$  is determined by a function  $f: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$  satisfying  $\sum_{c>0} f(c) = 1$ , and a collection of functions  $f_c: \mathbb{N} \rightarrow \mathbb{N}$  satisfying  $\sum_{k \in \mathbb{N}} f_c(k) = 1$ , by setting  $\Pr(\text{Hilb}[\mathbf{p}, n]) := f(c)f_c(k)/2^k$  for every Hilbert scheme  $\text{Hilb}[\mathbf{p}, n]$  at height  $k$  in the tree  $\mathcal{H}_c$ . Letting  $A$  denote the set of irreducible Hilbert schemes, we compute

$$\begin{aligned}
\Pr(\text{irr} = 1) &= \sum_{\text{Hilb}[\mathbf{p}, n] \in A} \Pr(\{\text{Hilb}[\mathbf{p}, n]\}) \\
&\geq \sum_{c>0} \left( f(c)f_c(0) + \sum_{k \geq 1} \frac{2^k}{2} \frac{f(c)f_c(k)}{2^k} \right) \\
&= \sum_{c>0} \left( \frac{f(c)f_c(0)}{2} + \sum_{k \in \mathbb{N}} \frac{f(c)f_c(k)}{2} \right) \\
&= \sum_{c>0} \left( \frac{f(c)f_c(0)}{2} + \frac{f(c)}{2} \right) \\
&= \sum_{c>0} \left( \frac{f(c)f_c(0)}{2} \right) + \frac{1}{2} \\
&> \frac{1}{2},
\end{aligned}$$

where the first inequality follows because in each Hilbert tree  $\mathcal{H}_c$ , at each height  $k \geq 1$ , there are at least  $2^k/2$  vertices corresponding to irreducible Hilbert schemes, by Corollary 7.5. Hence, the probability that a random Hilbert scheme is irreducible is greater than 0.5.

In any chosen Hilbert tree  $\mathcal{H}_c$ , we similarly compute

$$\Pr(\text{irr}_c = 1) = \sum_{\text{Hilb}[\mathbf{p}, n] \in A \cap \mathcal{H}_c} \Pr(\{\text{Hilb}[\mathbf{p}, n]\}) \geq f_c(0) + \sum_{k \geq 1} \frac{2^k}{2} \frac{f_c(k)}{2^k} = \frac{f_c(0)}{2} + \sum_{k \in \mathbb{N}} \frac{f_c(k)}{2} > \frac{1}{2}.$$

Hence, the probability that a random Hilbert scheme parametrizing subschemes of a chosen codimension is irreducible is also at least 0.5.  $\square$

*Proof of Theorem 1.1.* Giving the set of all Hilbert schemes that parametrize closed subschemes with a fixed Hilbert polynomial of some projective space a structure of a discrete probability space, as we do for  $\mathcal{H}$ , letting  $\text{irr}$  be the random variable counting the number of irreducible components of a Hilbert scheme, and taking into account that the Hilbert schemes described in Corollary 7.5 are both irreducible and nonsingular, we find that Theorem 7.7 proves that a random Hilbert scheme is irreducible and nonsingular with probability greater than 0.5.  $\square$

The likelihood that a random Hilbert scheme in the Hilbert forest is irreducible and nonsingular is certainly higher than what we find by considering only the family in Corollary 7.5. For instance, every Hilbert scheme of the form  $\text{Hilb}[k, 2]$ , for positive  $k \in \mathbb{Z}$ , is irreducible and nonsingular, and these are not accounted for by Theorem 7.7; see Theorem 8.11 of [Har10]. Moreover, we conjecture that all Hilbert schemes of the form  $\text{Hilb}[\Psi \Phi^d(1), n]$ , for  $d \in \mathbb{N}$ , also have a unique point determined by a saturated strongly stable ideal. It would be interesting to know whether there is some universal bound  $k$  such

that we are guaranteed that a Hilbert scheme of the form  $\text{Hilb}[\Psi^{j+k} \Phi(\mathbf{q}), n]$  has distinct components, for  $j \in \mathbb{N}$ , or in the other direction, such that  $\text{Hilb}[\Psi \Phi(\mathbf{q}), n]$ ,  $\text{Hilb}[\Psi^2 \Phi(\mathbf{q}), n]$ ,  $\dots$ ,  $\text{Hilb}[\Psi^k \Phi(\mathbf{q}), n]$  all have some suitably small number of components.

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