DYNAMIC MODELLING OF GEAR TRANSMISSION SYSTEMS WITH
AND WITHOUT LOCALIZED TOOTH DEFECTS

by

Wennian Yu

A thesis submitted to the Department of Mechanical and Materials Engineering
In conformity with the requirements for
the degree of Doctor of Philosophy

Queen’s University
Kingston, Ontario, Canada
July, 2017

Copyright ©Wennian Yu, 2017
Abstract

Gear transmission systems are widely used in many industry applications. Increased demand for higher-speed, improved performance and longer-lived machinery, makes the prediction and control of gear vibration and noise, as well as the early detection and diagnosis of gear defects important considerations. Many researchers use dynamic modelling of gear vibration to increase knowledge about the vibration generating mechanisms in gear transmission systems and the dynamic behaviour of gear transmission systems in the presence of some kinds of localized tooth defects.

This project aims to advance the current understanding of gear dynamics by introducing more accurate and realistic gear dynamic modelling strategies for cylindrical gear (spur gear and helical gear) transmission systems with and without localized tooth defects (tooth fillet crack and spalling defect). A series of studies have been conducted to reflect different aspects of gear dynamics. The main conclusions can be as summarized below:

1. Corner contact effect should not be neglected in the gear dynamic analysis if no or an insufficient amount of tooth profile modification is applied when the gears are working under heavy load.

2. The dynamic coupling behaviour is obvious in the direction of off-line of action when a gear pair with gear eccentricities is running at relatively low-speed range where the resonances of gear torsional vibration are most likely to be excited.

3. The addendum modification can affect gear dynamics through the back-side mesh stiffness, especially when the gears are working under light load or idling conditions.

4. Special attention should be paid to the tooth inclination deformations for the early detection of the initial crack damage. Besides, the spatial crack with non-uniform crack depth will lead to unevenness of
the dynamic load distribution on the cracked tooth flank. Tilting motions can therefore be excited whenever the cracked tooth comes into the mesh.

5. The nonlinear elliptical contact patterns do affect the excitations due to spalls especially for spalls with small size and gear pairs running under heavy load. The proposed model considering the effect of the nonlinear elliptical contact patterns by introducing modification coefficients can yield more realistic and accurate results.

**Keywords:** Cylindrical gear; gear mesh stiffness; gear transmission error; gear dynamic model; secondary effects; localized tooth defects.
Co-Authorship

This thesis is mainly written by the author, Wennian Yu. It conforms to the Manuscript format. Most contents in this thesis are adapted from multi-authored (Wennian Yu, Chris Mechske (PhD supervisor), Markus Timusk (PhD co-supervisor at Laurentian University), and Yimin Shao (MSc supervisor at Chongqing University, China)) publications in peer-reviewed journals and at several international conferences. The author’s primary supervisor, Dr. Chris Mechske, contributed to the revisions of contents, as well as the refinements of English writing of this thesis.
Acknowledgements

I would like to first thank China Scholarship Council (CSC: 201306050004) for covering my living expenses during my PhD study at Queen’s, without which my international doctoral studies would not have been possible. I would also like to thank Queen’s University and the Natural Science and Engineering Research Council of Canada (NSERC: 203023-06) for providing me several forms of awards covering my tuitions, and facility resources supporting my research. I sincerely appreciate the experimental set-up and support provided by the Bharti School of Engineering, Laurentian University.

For individuals, I would like to express my deepest gratitude to my supervisor, Dr. Chris Mechefske for his continuous support, guidance, expertise, and countless reviews on all my draft papers before I submitted to a journal or a conference committee for publication. Without his help, I couldn’t have published my work so smoothly. I would like to thank my co-supervisor, Dr. Markus Timusk, Laurentian University, for his guidance and technical support with the experimental set-up, reviews on some of my publications, and once warm invitation to his house for dinner during my stay at Sudbury last year. I would also like to express my sincere gratitude to Dr. Diane Wowk, Royal Military College, for her expertise and guidance in the Finite Element work in this thesis. I would be remiss to not thank Greg Lakanen, Dustin Helm, and some staff at Machine Shop and Physics Department for their assistance during the conducting of the experiments.

I would like to thank my office mates and fellow lab mates for the fun and stimulating discussions we had over my time at Queen’s. I would also like to thank student assistants and professors at McLaughlin Hall, with special thanks to Jane Davis and Gabrielle Whan for always being nice and willing to help.

Last, but not least, I thank my family and my friends for having my back, and all the people who always care about me.
Statement of Originality

I hereby certify that all of the work described within this thesis is the original work of the author. Any published (or unpublished) ideas and/or techniques from the work of others are fully acknowledged in accordance with the standard referencing practices.

Wennian Yu

June, 2017
Table of Contents

Abstract .............................................................................................................................. ii
Co-Authorship .................................................................................................................. iv
Acknowledgements ......................................................................................................... v
Statement of Originality .................................................................................................. vi
List of Figures .................................................................................................................. xi
List of Tables ................................................................................................................... xvii
List of Abbreviations ...................................................................................................... xviii
List of Symbols ............................................................................................................... xix

Chapter 1 General Introduction ......................................................................................... 1
1.1 Background and Motivation ....................................................................................... 1
1.2 Literature Review ...................................................................................................... 3
  1.2.1 Gear Mesh Stiffness ......................................................................................... 3
  1.2.2 Gear Dynamic Models ...................................................................................... 12
  1.2.3 Summary .......................................................................................................... 21
1.3 Research Objective ................................................................................................... 23
1.4 Organization of Thesis ............................................................................................. 24
1.5 References ............................................................................................................... 26

Chapter 2 Theoretical Background ...................................................................................... 39
2.1 Introduction .............................................................................................................. 39
2.2 Involute Tooth Profile ............................................................................................ 39
2.3 Gear Mesh Stiffness ................................................................................................ 41
  2.3.1 Three Commonly-used Approaches ................................................................. 42
  2.3.2 Comparisons and Discussions ......................................................................... 49
2.4 Gear Transmission Error ....................................................................................... 53
  2.4.1 Measurement of Transmission Error ............................................................... 53
  2.4.2 Gear Tooth Profile Modifications ................................................................. 56
2.5 Gear Dynamic Models ........................................................................................... 58
  2.5.1 SDOF Model ..................................................................................................... 58
Chapter 5 Influence of the Addendum Modification on Spur Gear Back-side Mesh Stiffness and Dynamics ................................................................. 131
  5.1 Introduction .............................................................................. 131
  5.2 Derivation of the Time-varying Asymmetric Mesh Stiffness Model ................................................................. 132
    5.2.1 Standard Gear Pair ................................................................. 134
    5.2.2 Addendum Modified Gear Pair .............................................. 137
  5.3 Dynamic Simulation .................................................................. 141
    5.3.1 Acceleration Excitation on the Driving Gear ....................... 141
    5.3.2 Speed Sweep under Constant Light Load ......................... 150
  5.4 Conclusions ............................................................................. 155
  5.5 References ............................................................................... 156

Chapter 6 Effects of Gear Tooth Spatial Crack on Gear Mesh Stiffness, Inclination Deformation and Dynamics ......................................................... 159
  6.1 Introduction .............................................................................. 159
  6.2 Mesh Stiffness Model for Gear Tooth with Spatial Crack ............ 160
    6.2.1 Tooth Model with a Uniform-Depth Plane Crack .................. 160
    6.2.2 Tooth Model with a Spatial Crack ........................................ 161
    6.2.3 Spatial Crack Propagation Scenario and Time-Varying GMS .... 165
  6.3 Inclination Model for Gear Tooth with Spatial Crack .................. 170
    6.3.1 Gear Tooth Inclination Model .............................................. 170
    6.3.2 Gear Tooth Inclination Deformation ...................................... 173
  6.4 Dynamic Analysis .................................................................... 178
    6.4.1 Stiffness Cell ..................................................................... 178
    6.4.2 Dynamic Model .................................................................. 179
List of Figures

Figure 1.1: Typical types of gear tooth defects [9]: (a) fillet crack, (b) spalling ......................................................... 1
Figure 1.2: Schematic of the mesh behaviour of a spur gear pair (1<CR<2): (a) mesh process, (b) gear mesh stiffness curve .................................................................................................................. 4
Figure 1.3: GMS for various crack depth: (a) geometry of a localized plan crack, (b) GMS curves [74]... 9
Figure 1.4: GMS for various spall width: (a) geometry of a rectangular-shaped spall, (b) GMS curves [49] .................................................................................................................................................. 9
Figure 1.5: Stiffness changes in the gear mesh model [13]: (a) phase change, (b) magnitude change...... 10
Figure 1.6: Crack propagation depth [79]: (a) actual path, (b) assumed path .......................................................... 11
Figure 1.7: Coulomb friction effect [131]: (a) friction forces, (b) various formulations of $\mu$................. 17
Figure 2.1: Geometry of rack [5] (Note: $f_r$ is the addendum coefficient, $c_r$ is the tip clearance coefficient and $\alpha_r$ is the pressure angle) ........................................................................................................... 40
Figure 2.2: Generation of tooth profile: (a) involute region, (b) fillet region ......................................................... 40
Figure 2.3: Model of spur gear tooth: (a) geometric parameters for gear body, (b) geometric parameters for single tooth .......................................................................................................................... 44
Figure 2.4: FE models for a meshed gear pair: (a) 2D model, (b) 3D model ......................................................... 50
Figure 2.5: Mesh stiffness calculated by different methods: (a) mesh stiffness curves, (b) order spectrum ................................................................................................................................................... 51
Figure 2.6: Mesh stiffness calculated by FE methods under various input torques: (a) 3D model, (b) 2D Model .................................................................................................................................................. 52
Figure 2.7: Main sources of unloaded static transmission error: (a) profile manufacturing errors, (b) profile modifications, (c) surface defects (Keys: The solid green line represents the actual tooth profile, and the dashed black line represents the standard involute profile) ..................... 54
Figure 2.8: The arrangement of linear accelerometers on a gear wheel [31] .......................................................... 55
Figure 2.9: Tip and root relief ......................................................................................................................................... 56
Figure 2.10: Three different types of tip relief .................................................................................................................. 57
Figure 2.11: Typical gear rotary model: (a) pure torsional model, (b) equivalent SDOF model.................. 59
Figure 2.12: 6 DOF model .............................................................................................................................................. 60
Figure 2.13: 3D cylindrical gear mesh model: (a) the 3D gear mesh model, (b) projection on the plane of action ............................................................................................................................................... 62
Figure 3.1: Spur gear tooth pairs in mesh at the beginning (B) and end (E) of a meshing cycle and the separation distance in: (a) approach (b) recess (Note: the dashed tooth profiles are the un-
deformed (theoretical) profiles, whereas the solid tooth profiles are the deformed (actual) profiles due to neighboring loaded teeth). ................................................................. 74

Figure 3.2: Variation of VVMS, LSR and LSTE under different torques without gear errors ........... 82

Figure 3.3: Variation of contact ratio versus torque ....................................................................... 82

Figure 3.4: Variation of VVMS, LSR and LSTE under different amounts of tip relief: (a) $T_1 = 0$, (b) $T_1 = 340$ Nm ................................................................................................................... 83

Figure 3.5: Rotary model of a meshing spur gear pair [26] ................................................................. 84

Figure 3.6: Comparison of measured [25] and predicted $A_{rms}$ versus speed for an unmodified gear pair at 340 Nm ................................................................. 89

Figure 3.7: Experimentally measured [30] and predicted $A_{rms}$ versus $f_m$ for an unmodified gear under three different torques: (a) Experiment, (b) VVMS model, (c) LSTE model, (d) FVMS model (Note (+) 100, (◊) 200 and (•) 300 Nm). .................................................................................. 91

Figure 3.8: Experimental measured [24] and predicted $A_f$ versus $A$ for two modified gear pairs with $\delta_n = 10 \mu m$ at 340 Nm: (a) $\alpha_m = 22.2$ degrees, (b) $\alpha_m = 20.9$ degrees ................................................................................................................ 92

Figure 3.9: Simulated $A_{rms}$ versus $A$ for various amounts of tip relief with $L_n = 1$ using 3 models: (a) $A = 0$, (b) $A = 0.2$, (c) $A = 0.5$, (d) $A = 0.8$, (e) $A = 1$, (f) $A = 1.2$ .................................................................................................................. 94

Figure 4.1: The Timoshenko beam element ...................................................................................... 103

Figure 4.2: The 3D cylindrical gear mesh model: (a) the local (U-V-W) and global (X-Y-Z) coordinate systems, (b) the 3D gear mesh model, (c) projection drawing in W-direction, (d) projection drawing in V-direction .................................................................................. 105

Figure 4.3: Gear mounting errors: (a) misalignments, (b) eccentricities ........................................ 109

Figure 4.4: One stage geared rotor bearing system from [10] .......................................................... 115

Figure 4.5: Campbell diagrams of the system with and without a gear mesh: (a) with gear mesh (b) without gear mesh .................................................................................. 115

Figure 4.6: Schematic of the experimental helical gear system in [15] ............................................ 119

Figure 4.7: Comparisons of the simulated and measured DTE of the gear pair: (a) $CR = 1.4$ at 150 Nm, (b) $CR = 1.6$ at 250 Nm .................................................................................. 119

Figure 4.8: Relative difference between the transverse responses of the driving gear with and without the dynamic coupling terms in the LOA and OLOA direction: (a) original, (b) close-up ........ 121

Figure 4.9: Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various profile error cases ($e_f = 0, 10, 20, 50, 100 \mu m$) when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................................ 122

xii
Figure 4.10: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \) or \( X_1 \)) among various profile error cases \((\varepsilon_1 = 0, 10, 20, 50, 100 \, \mu\text{m})\) when rotating speed is 2400 rpm: (a) original, (b) close-up ................................................................. 122

Figure 4.11: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \) or \( X_1 \)) among various tooth number cases when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................. 124

Figure 4.12: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \) or \( X_1 \)) among various helix angle cases when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................. 125

Figure 4.13: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \) or \( X_1 \)) among various magnitudes of driving gear eccentricity \( e_1 \) when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................. 126

Figure 4.14: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \) or \( X_1 \)) among various driven gear eccentricity \( e_2 \) and \( \theta_2 \) when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................. 127

Figure 4.15: Comparisons of the transverse dynamic response in the OLOA direction of driven gear \( (V_2 \) or \( X_2 \)) among various driven gear eccentricity \( e_2 \) and \( \theta_2 \) when rotating speed is 420 rpm: (a) original, (b) close-up ................................................................. 128

Figure 5.1: Several critical contact positions of a mesh tooth pair (Keys: The red and blue line represents the LOA in forward direction and backward direction respectively; \( A, P, S, D - \) contact points on forward LOA; \( a, p, s, d - \) contact points on backward LOA) ......................................................... 133

Figure 5.2: Geometric relationship of the various critical mesh positions .................................................. 135

Figure 5.3: Drive-side and back-side mesh stiffness function versus time \( t \) and for \( \tau \) unmodified gear pair ........................................................................................................ 137

Figure 5.4: Various addendum modified gear tooth profiles ......................................................................... 139

Figure 5.5: Drive-side and back-side mesh stiffness function versus time \( t \) and \( \tau \) for addendum modified gear pairs: (a) \( x_1 = -0.5, x_2 = 0.5 \), (b) \( x_1 = 0.5, x_2 = -0.5 \) ................................................................. 140

Figure 5.6: Six degrees of freedom spur gear pair ...................................................................................... 141

Figure 5.7: Gear mesh stiffness for various addendum modification cases: (a) \( k(t) \), (b) \( k_b(t) \) when \( x_1 = -0.5 \), (c) \( k_b(t) \) when \( x_1 = -0.3 \), (d) \( k_b(t) \) when \( x_1 = 0 \), (e) \( k_b(t) \) when \( x_1 = 0.3 \), (f) \( k_b(t) \) when \( x_1 = 0.5 \) ................................................................................................................................. 145

Figure 5.8: Simulated gear dynamic transmission error for various addendum modification cases under input angular acceleration excitation \( (A_{c1} = 141.4 \, \text{rad/s}^2, f_{c1} = 20 \, \text{Hz}, \omega_1 = 0 \) and 3600 rpm) ................................................................................................................................. 147
Figure 5.9: Simulated gear dynamic mesh force for various addendum modification cases under input angular acceleration excitation \((Ae_1 = 141.4 \text{ rad/s}^2, fe_1 = 20 \text{ Hz}, \omega_1 = 0 \text{ and } 3600 \text{ rpm})\) ...... 149

Figure 5.10: Simulated driving gear lateral response (LOA direction) for various addendum modification cases under input angular acceleration excitation \(............... 150

Figure 5.11: Simulated \(A_1\) response versus rotating speed of driving gear \(\omega_1\) for various addendum modification cases \((\tilde{F}_0 = 0.3, \zeta = \zeta_b = 0.008)\) (Keys: — speed-increasing, \(\cdots\) speed-decreasing) ........................................................................................................ 153

Figure 5.12: Simulated \(A_1\) response versus rotating speed of driving gear \(\omega_1\) for various addendum modification cases \((\tilde{F}_0 = 0.3, \zeta = \zeta_b = 0.03)\) (Keys: — speed-increasing, \(\cdots\) speed-decreasing) ........................................................................................................ 153

Figure 5.13: Model of spur gear tooth with a plane crack of uniform crack depth ........................................................................................................ 161

Figure 5.14: Model of spur gear tooth with a spatial crack and non-uniform crack depth: \(a\) spatial crack in the tooth fillet region, \(b\) gear tooth thin piece, \(c\) projection of the crack path on the \(x-z\) plane, \(d\) projection of the crack path on the \(x-y\) plane ........................................................................................................ 162

Figure 5.15: Tooth crack propagation scenario ........................................................................................................ 166

Figure 5.16: GMS and LSR for the proposed scenario: \(a\) GMS versus angular displacement of the driving gear \(b\) LSR versus angular displacement of the driving gear ............................... 168

Figure 5.17: Crack growth path in tooth surface: \(a\) linear, \(b\) monotonous parabolic, \(c\) non-monotonous parabolic ........................................................................................................ 168

Figure 5.18: Maximum reduction in GMS among three crack growth paths in the tooth surface........... 169

Figure 5.19: Schematic for inclination deformation of the gear tooth thin piece due to root crack \([12]\) .......................... 170

Figure 5.20: Crack model at gear tooth fillet region: \(a\) Case A, \(b\) Case B ........................................................................................................ 174

Figure 5.21: Distributions of the gear tooth inclination deformations on the cracked tooth flank: \(a\) Case A, \(b\) Case B ........................................................................................................ 175

Figure 6.2: Crack model at gear tooth fillet region: \(a\) Case C, \(b\) Case D ........................................................................................................ 177

Figure 6.3: Distributions of the gear tooth inclination deformations on the cracked tooth flank: \(a\) Case C, \(b\) Case D ........................................................................................................ 177

Figure 6.4: Mesh stiffness model based on stiffness cells: \(a\) Discretization of the contact line, \(b\) Network of springs to model the gear mesh ........................................................................................................ 178

Figure 6.5: Dynamic load factor at the middle tooth width surface for Case A: \(a\) TC-A1, \(b\) TC-A2, \(c\) TC-A3, \(d\) TC-A4 ........................................................................................................ 183

Figure 6.6: Dynamic load factor at the middle tooth width surface for Case B: \(a\) TC-B1, \(b\) TC-B2, \(c\) TC-B3, \(d\) TC-B4 ........................................................................................................ 184

xv
Figure 6.15: Dynamic load distributions on the cracked tooth flank for Case A: (a) TC-A1, (b) TC-A2, (c) TC-A3, (d) TC-A4................................................................................................................. 186

Figure 6.16: Dynamic load distributions on the cracked tooth flanks for Case B: (a) TC-B1, (b) TC-B2, (c) TC-B3, (d) TC-B4................................................................................................................. 187

Figure 6.17: Tilting motion of the driven gear $\theta_y$ for Case A: (a) TC-A1, (b) TC-A2, (c) TC-A3, (d) TC-A4................................................................................................................. 188

Figure 6.18: Tilting motion of the driven gear $\theta_y$ for Case B: (a) TC-B1, (b) TC-B2, (c) TC-B3, (d) TC-B4................................................................................................................. 188

Figure 7.1: 3D cylindrical gear mesh model: (a) the 3D gear mesh model, (b) projection on the plane of action.................................................................................................................................................. 194

Figure 7.2: A rectangular-shaped spall on a cylindrical gear tooth .......................................................................................................................... 198

Figure 7.3: Spall model on the plane of action: (a) shallow spall, (b) deep spall.............................. 199

Figure 7.4: Tooth surface contact region under various loads: (a) light load, (b) intermediate load, (c) heavy load (Note the red area represents the contact region between a mating tooth pair) . 200

Figure 7.5: Tooth surface contact region with the same length of the spall: (a) heavy load (b) light load (Note the red area represents the contact region) .................................................................................................................. 201

Figure 7.6: Tooth surface contact region under same load: (a) small size of spall (b) large size of spall (Note the red area represents the contact region) .................................................................................................................. 201

Figure 7.7: Schematic diagram of the experimental set-up (1 − 3:1 ratio speed reduction gearbox; 2 − universal coupling; 3 − encoder on driving shaft; 4 − driving gear; 5 − accelerometer; 6 − bearing; 7 − 1:1 ratio test gearbox; 8 − driven gear; 9 − encoder on driven shaft; 10 − load sensor)............................................................................................................................................... 203

Figure 7.8: Photos of experimental equipment: (a) sensors, (b) test gear pair (1 − accelerometer; 2 − encoder on driving shaft; 3 − encoder on driven shaft; 4 − driving gear; 5 − driven gear).... 204

Figure 7.9: Gear #1: (a) tooth #1, (b) tooth #7, (c) tooth #14 ............................................................................................................................ 205

Figure 7.10: Gear #2: (a) tooth #1, (b) tooth #7, (c) tooth #14 ............................................................................................................................ 205

Figure 7.11: Photos of the biggest defects: (a) shallow spall on 14th tooth of gear #1, (b) deep spall on 14th tooth of gear #2............................................................................................................................................... 205

Figure 7.12: Comparisons of experimentally measured response $\ddot{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm ......... 208

Figure 7.13: Comparisons of simulated dynamic response $\ddot{y}_j$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm: (a) model in [5], (b) model in this chapter........................................................................................................................................ 208
Figure 7.14: Comparisons of the power spectrum of $\tilde{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm: (a) experimental results, (b) simulation results based on the proposed model.................................................210

Figure 7.15: Comparisons of experimentally measured response $\tilde{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and load is: (a) 21 Nm, (b) 5 Nm....211

Figure 7.16: Comparisons of the simulated dynamic response $\tilde{y}_1$ without considering the modification coefficients between the gear pairs with shallow spalls and deep spalls when rotating speed is 600 rpm and the load is: (a) 21 Nm (b) 5 Nm .................................................................211

Figure 7.17: Comparisons of the simulated dynamic response $\tilde{y}_1$ considering the modification coefficients between the gear pairs with shallow spalls and deep spalls when rotating speed is 600 rpm and the load is: (a) 21 Nm (b) 5 Nm.................................................................212

Figure 7.18: Comparisons of the statistical values for the gear pair with deep spalls under different cases: (a) Kurt, (b) $S_r$, (c) $C$, (d) $CK_1$ ........................................................................................................215

Figure 7.19: Comparisons of the statistical values for the gear pair with shallow spalls under different cases: (a) Kurt, (b) $S_r$, (c) $C$, (d) $CK_1$........................................................................................................216
List of Tables

Table 2.1: Values of the coefficients in Equation (2.11) [20] ................................................................. 45
Table 2.2: Values of coefficients in Equation (2.23) [22] ............................................................................. 49
Table 2.3: Parameters of the gear pair for simulation ...................................................................................... 50
Table 2.4: Time cost for the calculation of GMS in one mesh cycle for each method ................................. 51
Table 2.5: Transmission error in each case .......................................................................................................... 54
Table 3.1: Static transmission error with corner contact effect for a NCR spur gear pair ......................... 79
Table 3.2: Static transmission error with the corner contact effect for a HCR spur gear pair ................... 79
Table 3.3: Parameters of the gear pair in [23, 24, 25] ....................................................................................... 81
Table 4.1: Parameters of the geared rotor system [10, 12] .............................................................................. 116
Table 4.2: Comparisons of the first 13 natural frequencies when rotating speed is 0 .............................. 117
Table 4.3: Design parameters of the helical gear pair used in [15] ............................................................... 118
Table 4.4: Other gear, bearing and shaft parameters for simulation ......................................................... 118
Table 4.5: Nominal values of some main parameters ...................................................................................... 120
Table 5.1: Design parameters of the spur gear system for simulation ......................................................... 144
Table 6.1: Parameters of gear pairs set from [4] ............................................................................................ 166
Table 6.2: Data for the propagation cases of the proposed scenario ........................................................... 167
Table 6.3: Crack parameter (mm) ................................................................................................................ 174
Table 6.4: Parameter of the cracks (mm) ........................................................................................................ 176
Table 6.5: Design parameters of the spur gear system for simulation ........................................................ 182
Table 7.1: Dimensions of the spalls ............................................................................................................... 206
Table 7.2: Parameters of the spur gear pair .................................................................................................... 206
Table 7.3: Rotating speed and load for each case .......................................................................................... 214
### List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D</td>
<td>Two-Dimensional</td>
</tr>
<tr>
<td>3D</td>
<td>Three-Dimensional</td>
</tr>
<tr>
<td>BR</td>
<td>Backup Ratio</td>
</tr>
<tr>
<td>CR</td>
<td>Contact Ratio</td>
</tr>
<tr>
<td>DMF</td>
<td>Dynamic Mesh Force</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree of Freedom</td>
</tr>
<tr>
<td>DTE</td>
<td>Dynamic Transmission Error</td>
</tr>
<tr>
<td>EHL</td>
<td>Elasto-Hydrodynamic Lubrication</td>
</tr>
<tr>
<td>ETC</td>
<td>Extended Tooth Contact</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>FVMS</td>
<td>Fixed-Variable Mesh Stiffness</td>
</tr>
<tr>
<td>GMS</td>
<td>Gear Mesh Stiffness</td>
</tr>
<tr>
<td>HCR</td>
<td>High Contact Ratio</td>
</tr>
<tr>
<td>HP2DTC</td>
<td>Highest Point of Second Double Tooth Contact</td>
</tr>
<tr>
<td>HPSTC</td>
<td>Highest Point of Single Tooth Contact</td>
</tr>
<tr>
<td>IA</td>
<td>Improved Analytical</td>
</tr>
<tr>
<td>LOA</td>
<td>Line of Action</td>
</tr>
<tr>
<td>LSR</td>
<td>Load Sharing Ratio</td>
</tr>
<tr>
<td>LSTE</td>
<td>Loaded Static Transmission Error</td>
</tr>
<tr>
<td>MDOF</td>
<td>Multi-Degrees of Freedom</td>
</tr>
<tr>
<td>NCR</td>
<td>Normal Contact Ratio</td>
</tr>
<tr>
<td>OLOA</td>
<td>Off-line of Action</td>
</tr>
<tr>
<td>RMS</td>
<td>Root Mean Square</td>
</tr>
<tr>
<td>RTE</td>
<td>Residual Transmission Error</td>
</tr>
<tr>
<td>SDOF</td>
<td>Single Degree of Freedom</td>
</tr>
<tr>
<td>STE</td>
<td>Static Transmission Error</td>
</tr>
<tr>
<td>TA</td>
<td>Traditional Analytical</td>
</tr>
<tr>
<td>TC</td>
<td>Tooth Crack</td>
</tr>
<tr>
<td>USTE</td>
<td>Unloaded Static Transmission Error</td>
</tr>
<tr>
<td>VVMS</td>
<td>Variable-Variable Mesh Stiffness</td>
</tr>
</tbody>
</table>
List of Symbols

Chapter 1

d_s, l_s, w_s depth, length and width of a rectangular-shaped spall
f_s gear rotation frequency
k_0 tooth stiffness density per unit length
f_m gear mesh frequency
q_c crack depth

Greek symbols

α_c crack angle
μ_c coulomb friction coefficient

Chapter 2

A_i, B_i, C_i, D_i, E_i, F_i (i = 1, 2, 3 4) parameters shown in Equation (2.11) for calculations of L^*(i = 1), M^*(i = 2), P^*(i = 3) and Q^*(i = 4) respectively
α_r, b_r, r dimensions of the rack in Figure 2.1
b gear backlash
C_M, C_R, C_B correction factor, gear blank factor, and basic rack factor when calculating k_0 in Equation (2.22)
C_B(0), C_B(i) bearing dampings supporting the i-th gear along the U-axis and V-axis
(i = 1 for driving gear, and i = 2 for driven gear)
c viscous damping of the gear mesh behaviour
E_i, E_j(q) structure vector of the 6 DOF, and structure vector at the contact point M_j along the LOA of the 3D model
E, G Young’s Modulus, and shear modulus of the gear material
e(t) time-varying gear tooth profile deviations (gear errors)
e_j gear error at M_j
F, F_b, F_m, M mesh force, the resultant axial compressive force, shear force, and bending moment as shown in Figure 2.3(b)
F_0, F_1(t), F_2(t, q) excitation vectors due to static load, gear global mounting errors, and localized tooth profile errors
addendum coefficient, and tip clearance coefficient
g(·)
tooth contact function
$H_j$
contact function at $M_j$ shown in Figure 2.13(b)
$h_x$
half of the tooth thickness at the integral section in Figure 2.3(b)
$h_f$
ratio of $R_f$ and $R_{int}$
$I_x, A_x$
effective area moment of inertia, and area of the integral section in Figure 2.3(b).
$I_i, I_{pi} (i = 1, 2)$
transverse, and polar moment of inertia of the $i$th gear of the 3D model
$K_g(t, q)$
stiffness matrix related to the time-varying GMS
$K(t)$ (or $k(t)$)
time-varying gear mesh stiffness
$k_0$
tooth stiffness density per unit length
$k_i^t, C_i^t (i = 1, 2, \ldots n)$
equivalent stiffness, and compliance of the $i$th tooth pair according to TA method
$k_T$
total mesh stiffness for $N$ tooth pairs according to IA method
$k_i^T, C_i^T (i = 1, 2, \ldots n)$
equivalent stiffness, and compliance of the $i$th tooth pair according to TA method
$k_j$
stiffness cell at $M_j$ shown in Figure 2.13(b)
$k_{Biv}, k_{Biu} (i = 1, 2)$
bearing stiffness supporting the $i$th gear along the $U$-axis, and $V$-axis
$L$
contact length between the two meshing teeth of a mating tooth pair
$L(t)$
time-varying length of contact line
$L^*, M^*, P^*, Q^*$coefficients in Equation (2.10)
l, x, hvariables in Figure 2.3(b)
$M, C_b, K_b$
mass matrix of gears, damming and stiffness matrices of bearings in the 3D model
$m$
gear module
$m_e, f_0$
equivalent mass, and static load of the SDOF model
$m_i, J_i (i = 1, 2)$mass, and polar moment of inertia of $i$th gear of the 6 DOF model
$N$
number of tooth pairs in mesh simultaneously
$n$
maximum number of tooth pairs in the mesh zone of a gear pair (nearest integer that is larger than its CR)
$q$
DOF vector
$R, R_b$
radii of the pitch circle, and base circle
$R_{bi} (i = 1, 2)$base radius of the $i$th gear
\( R, R_m \) outer radius, and inter radius of gear body as shown in Figure 2.3
\( T_i (i = 1, 2) \) torque applied on the \( i \)th gear
\( T_n \) mesh period
\( t \) time
\( u_i, s_i, \theta_i \) variables shown in Figure 2.3
\( W \) face width of the gear
\( x(t) \) gear transmission error
\( x_p, y_p \) coordinates of the point \( p \) in the \( X \)-axis and \( Y \)-axis
\( x_i (i = 1, 2) \) profile shift coefficient on the \( i \)th gear
\( Z_i (i = 1, 2) \) number of teeth of the \( i \)th gear
\( z_i(t) (i = 1, 2, 3, 4) \) signals of the four accelerometers mounted in Figure 2.8

**Greek symbols**

\( \alpha, \alpha_r \) pressure angle of the gear, and the rack
\( \alpha_p \) mesh angle of the contact point \( p \) in Figure 2.3(b)
\( \beta \) helix angle of the gear pair
\( \gamma \) mesh angle of point \( p \) in Figure 2.2(a)
\( \gamma_0 \) angle between the tooth centre line and the line \( OQ \) in Figure 2.2(a)
\( \Delta, L_n \) non-dimensional tip relief amount, and length shown in Figure 2.9
\( \delta, \delta_h, \delta_s, \delta_f \) tooth total deformation along LOA, and deformations along LOA due to Hertzian contact, tooth beam and tooth foundation respectively
\( \delta_b, \delta_s, \delta_a \) tooth deformations along LOA due to bending moment, shear force, and axial compressive force respectively
\( \varepsilon_\alpha, \varepsilon_\beta \) transverse contact ratio, and overlap contact ratio of the gear pair
\( \theta_i (i = 1, 2) \) angular displacement of the \( i \)th gear along its axis
\( \lambda_{IN} (i = 1, 2) \) correction coefficient of the tooth foundation induced stiffness for the \( i \)th gear
\( \mu \) Poisson’s ratio of the gear material
\( \mu_c \) coulomb friction coefficient
\( \phi, \alpha' \) angles illustrated in Figure 2.2(b)

**Chapter 3**

\( A_{rms}, A_1 \) the RMS value, and the \( i \)th harmonic amplitude of DTE
b
gear backlash

c
viscous damping of the gear mesh behaviour

$C^i, e^i \ (i = 1, 2, \ldots n)$
compliance, and unloaded static transmission error of the $i$th tooth pair at the contact point $j$

$E, \mu$
Young’s Modulus and Poisson’s ratio of the gear material

e$(t)$
synthetic gear tooth profile errors

$e_0, e, \theta_i$
mean value of $e(t)$, amplitude, and phase of the $i$th harmonic of $e(t)$

$F, F^i_j \ (i = 1, 2, \ldots n)$
total contact force, and contact force between the two meshing teeth of the $i$th tooth pair at point $j$

$F(t), G(t)$
elastic, and damping forces in Equation (3.19)

$f_m$
excitation frequency (or gear mesh frequency)

$h$
tooth contact function

$J_k \ (k = 1, 2)$
polar moment of inertia of $k$th gear ($k = 1$ for driving gear, $k = 2$ for driven gear)

$j$
contact point along tooth involute profile

$K^i_j, K^i_{Vj}\ (i = 1, 2, \ldots n)$
FVMS and VVMS of the $i$th tooth pair at point $j$

$K_{Fj}$ (or $K_{F}(t), k_{F}(t)$)
FVMS of the gear pair at point $j$ (or time instant $t$)

$K_{Vj}$ (or $K_{V}(t), k_{V}(t)$)
VVMS of the gear pair at point $j$ (or time instant $t$)

$k_F(t), e(t), h^i \ (i = 1, 2, \ldots n)$
individual FVMS, tooth error and contact function for the $i$th tooth pair respectively

$k_{F0}, k_{F0}, \phi_{F0}$
mean value of $k_F(t)$, amplitude, and phase of the $i$th harmonic of $k_F(t)$

$LSR^i_j \ (i = 1, 2, \ldots n)$
load sharing ratio of the $i$th tooth pair at point $j$

$m$
gear module

$m_e, f_0$
equivalent mass, and static load of the SDOF model

$n$
maximum number of tooth pair in the mesh zone of a gear pair (nearest integer that is larger than its CR)

$R_{bk} \ (k = 1, 2)$
base radius of the $k$th gear

$S_{a^i_j}, S_{r^i_j} \ (i = 1, 2, \ldots n)$
separation distance of the $i$th tooth pair at point $j$ during approach and recess respectively

$T_k \ (k = 1, 2)$
torque applied on the $k$th gear

$t$
time

$W$
face width of the gear

$x (or x(t))$
gear transmission error

xxii
\((\hat{x}_j)_r\) loaded static transmission error at point \(j\) without consideration of corner contact effect

\((x_j)_r\) (or \(x_r(t)\)) loaded static transmission error at point \(j\) (or time instant \(t\)) with the consideration of corner contact effect

\(Z_k\ (k = 1, 2)\) number of teeth of the \(k\)th gear

**Greek symbols**

\(\theta_k\ (k = 1, 2)\) angular displacement of the \(k\)th gear

\((\delta_j^i)_d, (\delta_j^p)_{m_s}, (\delta_j^p)_{p_\perp}\) (\(i = 1, 2, \ldots n\)) deflections, manufacturing errors and profile modifications of the \(i\)th tooth pair at point \(j\)

\(\Sigma, \Pi\) notations for summation, and product of a sequence of terms

\(\alpha_0\) pressure angle of the gear pair

\(\Delta, L_n\) non-dimensional tip relief amount, and length

\(\delta_m, \alpha_m\) actual tip relief amount, and starting angle

\(\zeta\) viscous damping ratio

\(\tau, \lambda, \hat{x}(\tau), \hat{x}_s(\tau), \hat{k}_f(\tau), \hat{e}(\tau), \hat{f}_0, \hat{h}\) non-dimensional \(t, f_m, x(t), x_s(t), k_f(t), e(t), f_0\) and \(h\) respectively

\(\omega_n\) natural frequency of the gear system

---

**Chapter 4**

\(E(M_j)\) local structure vector at contact point \(M_j\)

\(e(M_j)\) displacement excitation at \(M_j\) in the form of the unloaded static transmission error

\(e_{tx}, e_{ty}\) (\(i = 1, 2\)) perturbations along X- and Y-axis due to the eccentricity of the \(i\)th gear (\(i = 1\) for driving gear, \(i = 2\) for driven gear)

\(e_j\) (\(i = 1, 2\)) magnitude of the eccentricity of the \(i\)th gear

\(e_{f_i}(M_j), e_f(M_j)\) (\(i = 1, 2\)) tooth local scale error at \(M_j\) of the \(i\)th gear, and the synthetic local scale error at \(M_j\)

\(e_{m_i}(M_j), e_{c}(M_j)\) mounting errors at \(M_j\) due to misalignments and gear eccentricities respectively

\(F_0, F_1(t), F_2(t, u)\) excitation vectors due to static load, gear global mounting errors and localized tooth profile errors of the gear pair respectively

\(\tilde{F}_0, \tilde{F}_1(t), \tilde{F}_2(t, u)\) extensions of \(F_0, F_1(t), F_2(t, u)\) to the total amount of DOF (completed by zeros)
\(K_b\) bearing stiffness matrix
\(k_{g}(t, u)\) time-varying gear mesh stiffness
\(k_{ij}, k_g\) cell stiffness at \(M_j\), and the mesh stiffness of the gear pair
\(k_{xx}, k_{yy}, k_{zz}\) bearing lateral stiffnesses along \(X-, Y-, Z\)-axis respectively
\(k_{\theta x\theta x}, k_{\theta y\theta y}, k_{\theta z\theta z}\) bearing rotational stiffnesses along \(X-, Y-, Z\)-axis respectively
\(M_s, K_s, G_s\) shaft mass, stiffness, and gyroscopic moment matrices
\(M_g, K_g(t, u)\) gear mass, and mesh stiffness matrices
\(M, C, G, K\) the overall system mass, viscous damping, gyroscopic, and stiffness matrices
\(M_j\) contact point along the contact line
\(\bar{n}_{ij} (i = 1, 2)\) outer unit vector normal to the \(i\)th gear tooth profile at \(M_j\)
\(q\) local DOF vector of the gear pair
\(R_{bi} (i = 1, 2)\) base radius of the \(i\)th gear
\(T\) transform matrix between the global coordinate system \(X-Y-Z\) and
local coordinate system \(U-V-W\) shown in Figure 4.2(a)
\(T_i (i = 1, 2)\) torque applied on the \(i\)th gear
\(T_s\) mesh period
\(t\) time
\(u\) global DOF vector of the gear pair
\(u^e\) global DOF vector of the shaft node
\(u^m\) subset of \(u\)
\(u, v, w\) translational motions along \(U-, V-, W\)-axis respectively
\(V(M_j)\) global structure vector at \(M_j\)
\(V_0\) the average structure vector in a mesh period \(T_s\)
\(V_m(M_j)\) subset of \(V(M_j)\)
\(W_0\) face width of the gear
\(x, y, z\) translational motions along \(X-, Y-, Z\)-axis respectively

**Greek symbols**
\(\alpha\) mounting angle of the gear pair
\(\beta\) helix angle of the cylindrical gear
\(\Gamma_U, \Gamma_V\) relative difference between the responses with and without the inclusion of the dynamic coupling terms in the OLOA and LOA direction respectively
\(\Delta(M_j)\) contact deflection at \(M_j\)
\( \delta(M_i) \)  
\( \theta_x, \theta_y, \theta_z \)  
\( \theta_u, \theta_v, \theta_w \)  
\( \theta_{ui}, \theta_{vi} \)  
\( \xi \)  
\( \varphi \)  
\( \psi \)  
\( \Omega_i \)  
\( A, P, S, D \)  
\( A_1 \)  
\( A_{el}, f_{el} \)  
\( C_{ui} \)  
\( c(t), c_b(t) \)  
\( E \)  
\( F_0 \)  
\( F_k, F_s, F_t \)  
\( f_m \)  
\( k_0 \)  
\( k(t), k_b(t) \)  
\( k_{Bi}, C_{Bi} \)  
\( M, C_B, K_B \)  
\( m \)  
\( m_e, F_0 \)
mass, and polar moment of inertia of the \( i \)th gear

circular pitch of the \( i \)th gear

DOF vector

pitch radius, and base radius of the \( i \)th gear

torques applied on the \( i \)th gear

mesh period

time variables starting from the initial mesh point \( A \), and the symmetric point \( C \) shown in Figure 5.1

time durations for a tooth pair moving from the initial mesh point \( A \) to the pitch point \( P \), symmetric point \( S \) and the end point \( D \) shown in Figure 5.1, respectively

time duration for a tooth pair moving from the pitch point \( P \) to the symmetric point \( S \)

translational motions along \( U \)-, \( V \)-axis respectively

addendum modification coefficient of the \( i \)th gear

number of teeth of the \( i \)th gear

pressure angle at the pitch point shown in Figure 5.2

contact-type coefficient

normal approach (transmission error)

unloaded static transmission error

drive-side, and back-side damping ratio of the SDOF gear model

half of the nominal backlash in the gear pair

rotational motions along \( W \)-axis

mesh angle at the initial point \( A \), and end point \( D \) shown in Figure 5.2

non-dimensional \( \delta, e, \bar{F}_0, \bar{t}, \bar{k}, \bar{k}_b, \bar{A} \)

nominal rotating speed of the \( i \)th gear

natural frequency of SDOF gear model

**Chapter 6**

\( A, B, C, D, E, E', E'' \) point labels defined in Figure 6.7
\( c_i \) distance of the contact point \( M_i \) to the tooth center along \( Z \)-axis in Figure 6.12(a)

\( e_w(E) \) distance between points \( E' \) and \( E'' \) along LOA

\( e_{ji} (j = 1, 2) \) tooth profile deviation error at the contact point \( M_i \) of \( j \)th gear \( (j = 1 \) for driving gear, \( j = 2 \) for driven gear)

\( F \) total tooth contact force

\( F_0, F_i(q) \) external excitations due to static load and tooth profile deviations

\( f_i \) contact force at \( M_i \)

\( H_i \) contact function at \( M_i \)

\( h_x, h_y, l_e, l_t \) variables shown in Figure 6.1

\( K_g(t, q), K_b \) stiffness matrices regarding to gear pair, and bearings

\( k_{Bxj}, k_{B0yj}, k_{B0zj} (j = 1, 2) \) radial stiffness along \( X \)-axis, and rotational stiffnesses along \( Y \)- and \( Z \)-axis of the bearing supporting the \( j \)th gear respectively

\( k_i, k_{ci} \) cell stiffness, and local contact stiffness at \( M_i \)

\( k_g \) global stiffness

\( l_0, l_2, l_w \) crack locations at the two ends of tooth, and variation of crack location along tooth width shown in Figure 6.10

\( M, C, K \) overall system mass, viscous damping, and stiffness matrices

\( M \) moment about \( Y \)-axis due to total contact force \( F \)

\( M_i \) contact point along the contact line

\( m \) gear module

\( m_j, I_j, I_{pj} (j = 1, 2) \) mass, transverse moment of inertia, and polar moment of inertia of the \( j \)th gear

\( q \) DOF vector of the gear pair

\( q_0, q_2, q_w \) crack depth at the two ends of tooth and variation of crack depth along tooth width as shown in Figure 6.8

\( R_b, R_f \) radii of the gear base circle and reference circle respectively

\( T \) transform matrix between the \( X-Y \) coordinate and \( U-V \) coordinate system shown in Figure 6.7

\( T_j (j = 1, 2) \) torques applied on the \( j \)th gear

\( t \) time

\( V_i \) structure vector at \( M_i \)

\( W, W_c \) length of the tooth width, and the crack along tooth width direction
translational motions along $X$, $Y$, $Z$-axis respectively in Figure 6.12(a)

**Greek symbols**

$\alpha$ gear pressure angle  
$\alpha_c$ crack angle  
$\Delta w$ width of tooth thin piece shown in Figure 6.2  
$\delta_i$ normal approach at $M_i$  
$\varepsilon_i$ elastic deformation at $M_i$  
$\theta_x, \theta_y, \theta_z$ rotational motions along $X$, $Y$, $Z$-axis respectively in Figure 6.12(a)  
$\theta_{p0}, \theta_p$ Initial tooth inclination angle, and inclination angle along tooth width  
$\theta_T$ acute angle between the $X$-axis and $U$-axis in Figure 6.7

**Chapter 7**

$CK_1$ correlated kurtosis with the shift value being 1  
$d_s, w_s, l_s$ depth, width, and length of the rectangular-shaped spall shown in Figure 7.2  
$E_j(q)$ structure vector at $M_j$ of the 3D model  
$E_0$ average structure vector in a mesh period  
$e_{p}(t), e_{m}(t), e_{d}(t)$ tooth profile modifications, tooth manufacturing errors and tooth surface defect at the contact point $M_j$ respectively  
$e_j(t)$ synthetic gear error at $M_j$  
$F_0, F_1(t), F_2(t, q)$ excitation vectors due to static load, gear global mounting errors, and localized tooth profile errors  
$f_p, f_g, f_e$ rotating frequency of the driving gear, driven gear, and gear mesh frequency  
$H_j$ contact function at $M_j$ shown in Figure 7.1  
$Kurt$ kurtosis  
$k_0$ tooth stiffness density per unit length  
$k_j$ cell stiffness at $M_j$  
$k_g(t, q)$ time-varying gear mesh stiffness  
$L(t, q)$ time-varying, position-dependent length of contact line  
$M, C$ overall system mass, viscous damping matrices of the 3D model
contact point along the contact line

number of individual cells along the contact line

stiffness matrices regarding to gear pair and bearings

DOF vector defined by $U$-$V$-$W$ coordinate system shown in Figure 7.1

base radius of the $i$th gear ($i = 1$ for driving gear, $i = 2$ for driven gear)

skewness

torque applied on the gear with spalls

torques applied on the $i$th gear

mesh period

reference torque, spall length, and width in Equation (7.12)

time

translational motions along $U$-, $V$-, $W$-axis respectively

variables shown in Figure 7.1(b)

face width of gear

displacement response of the $i$th gear in the vertical direction

number of teeth of the $i$th gear

contact coefficients in Equation (7.12)

helix angle of the gear pair

normal approach at $M_j$

modification coefficients for deep spalls and shallow spalls respectively

rotational motions along $U$-, $V$-, $W$-axis respectively

gear pressure angle

nominal rotating speed of the $i$th gear
Chapter 1

General Introduction

1.1 Background and Motivation

Gears are essential components in many machines. Their configurations range from simple single-mesh gear pairs to complex planetary gear sets. Gear transmission systems are widely used in a variety of industry applications, such as automotive, wind turbines, mining, marine and industrial power transmission [1]. However, due to increasingly intense competition in the market for industrial machinery, the prediction and the control of vibration and noise in gear transmission systems have become important considerations. Typically these factors are considered at the design stage for the majority of practical applications using gear transmission systems [2]. Analytical studies on the gear transmission systems are thus necessary as they can provide fundamental understanding of the dynamics and guide vibration and noise reduction strategies [3].

Figure 1.1: Typical types of gear tooth defects [9]: (a) fillet crack, (b) spalling

In addition, gear defects are significant factors causing machinery failure. Statistics show gear failures make up about 10% of all rotating machinery failures [4, 5]. Therefore, gearbox monitoring for the early detection and diagnosis of gear faults is an important task in industrial maintenance.
Two typical types of localized tooth defects that receive the most attention in the literature are the tooth fillet crack and tooth surface spalling [6, 7, 8], as shown in Figure 1.1. Gear tooth fillet cracks normally develop in the tooth fillet region and are often caused by deficiencies in the gear tooth which result in a high stress concentration at a particular location. Tooth spallings tend to develop in the region near the pitch circle of the tooth surface where the meshing tooth pair is subjected to a higher mesh force. Tooth spallings are usually caused by excessively high localized contact stresses generated at the tooth contact area [9, 10].

The early and accurate detection and diagnosis of the initial gear damage is important since it can contribute significantly to the prevention of catastrophic failures. It has still to be addressed from both an experimental and theoretical viewpoint [11]. Up to now, work on gear fault detection and diagnosis has been mainly focused on two aspects: signal processing and dynamic analysis [4]. To improve the current techniques of gearbox vibration monitoring and diagnosis, many researchers are actively working on gear dynamic modelling to ascertain the effect of different types of gear damage on gear vibration [4, 12]. Dynamic analysis of gear systems with and without gear faults can also help to populate vibration signature databases which is necessary for the development of an effective pattern recognition scheme, so that a condition-based maintenance strategy can be implemented to avoid unnecessary cost and downtime caused by the traditional uneconomical maintenance strategies such as regular visual inspections and preventive maintenance [13].

Numerous gear dynamic models with and without the consideration of localized tooth defects have been established, especially for the cylindrical type of gears (spur gears and helical gears), which is the most common type of gears used in reality. Many vibration reduction strategies and vibration-based fault detection techniques have been put forward based on the information provided by the dynamic simulation results [11, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. However, due to the complexity of gear transmission configurations and the interactions of numerous effects, including the time-varying gear
mesh stiffness, backlash, lubrication, friction, manufacturing errors, assembling errors, profile modifications, back-side impacts, corner contact, torque fluctuations, and tooth defects etc., accurately and effectively modelling the gear dynamic behaviour with and without the consideration of tooth defects is still a challenging task to date. In fact, many aspects of gear dynamics are still not satisfactorily understood. This constitutes the primary motivation of this project.

1.2 Literature Review

1.2.1 Gear Mesh Stiffness

The time-varying gear mesh stiffness (GMS) is one of the main excitations that cause unwanted vibration and noise in gear transmission systems. Its magnitude is not constant mainly due to the periodic change of the number of tooth pairs in the mesh zone. Figure 1.2(a) shows the mesh behaviour of a spur gear pair with contact ratio (CR) between 1 and 2. The mesh zone (along the line of action) A-E is divided into three sections: A-B, B-D and D-E section. There are double tooth pairs in mesh simultaneously during the A-B and D-E sections, whereas there is only one tooth pair in mesh during the B-D section. Therefore, there will be a higher magnitude of GMS at the double mesh zone, and lower magnitude of GMS at the single mesh zone, as shown in Figure 1.2(b). The periodic change of the number of tooth pairs in mesh leads to the periodic change of the GMS. This constitutes an inherent excitation to the gear system, meaning gear vibration problem always exists even when the gears are perfectly machined and assembled. This is the distinct dynamic feature of gears compared with other common machinery components (bearings and shafts) [25, 26, 27].
Figure 1.2: Schematic of the mesh behaviour of a spur gear pair (1<CR<2): (a) mesh process, (b) gear mesh stiffness curve

1.2.1.1 For the Gear Pair without Tooth Profile Deviations

The GMS is directly related to the stiffness of each single tooth. The definition of tooth stiffness is, according to ISO 6336-1-2006, “The requisite load over 1 mm face width, directed along the line of action to produce, in line with the load, the deformation amounting to 1 μm of one or more pairs of deviation-free teeth in contact” [28]. This standard also provides the formulas for the evaluation of maximum stiffness $c'$ of a single pair of spur teeth, as well as corrected $c'$ for helical gears. An important assumption within this standard is that the mesh stiffness per unit length $k_0$ ($c'$ in the standard) is constant along the contact line. Based on this, Velex [29] proposed an approximation method to obtain the GMS of a cylindrical gear pair by directly multiplying the stiffness per unit length $k_0$ with the time-varying periodic length of the contact line, which can be expressed by Fourier series [30, 31].

The analytical method to calculate the mesh stiffness follows to a great extent that developed by Weber [32]. He divided the deformation of a loaded tooth into 3 factors: 1) the local deformation caused by the contact between two teeth; 2) the basic deflection caused by the tooth acting as a cantilever beam; and 3) the deflection caused by the flexibility of the fillet foundation (gear body). Cornell [33] used a method
which largely parallels Weber’s works but using O’Donnel’s [34] foundation factors to derive the gear
tooth deflection expression due to the foundation effect, which differs slightly from those given by
Weber.

Based on the potential energy principle, Yang and Lin [35] derived new expressions which takes only
Hertzian energy, bending energy and axial compressive energy into consideration, and calculated the
tooth stiffness related to these components separately. Tian [36] refined his model by taking shear energy
into consideration. However, both Yang’s and Tian’s work neglect the tooth foundation effect. Weber
[32], Attia [37] and Cornell [33] all derived equations based on a half plane hypothesis to account for this
effect. Their equations are of the same form and differ only slightly in terms of coefficients. Sainsot and
Velex [38] derived an improved analytical formula based on the theory of Muskhelishvili [39] applied to
circular elastic rings to calculate the tooth foundation flexibility induced deflection. Charri et al. [40],
Chen and Shao [41], and Yu et al. [42] summarized the analytical method to calculate tooth stiffness
based on the potential energy principle, and used this approach to calculate the mesh stiffness of gear pair
with a localized tooth crack.

In all of the above analytical studies, the gear mesh stiffness was obtained by simply summing the
stiffnesses of all the tooth pairs in a mesh. By comparing with finite element (FE) results, Kiekbusch et al.
[43] and Ma et al. [44] pointed out that this strategy will overestimate the tooth foundation induced effect
during the meshing of multiple tooth pairs as all the mating tooth pairs are connected to the same gear
body. Ma et al. [44] proposed an improved analytical method which uses the correction coefficients to
account for the tooth foundation effects when multiple tooth pairs are in the mesh zone.

From the 1970s, the quick development of the computer science has led to the possibility of large
numerical computation featuring finite difference methods, boundary element methods and finite element
methods [26, 27]. The FE methods are gradually becoming another commonly-used approach to obtain
the GMS. They can yield realistic and accurate results due to their unique formulation in combining
detailed contact modelling between elastic teeth with a combined surface integral/finite element solution to accurately capture tooth deformations [45]. Therefore, the FE calculation results are often used as a “benchmark” to verify the correctness of other methods used for the calculation of GMS. A significant deficiency related with the FE methods is that sufficient mesh refinement is required at the contact region in the FE models making FE methods usually computationally expensive. Therefore, two-dimensional (2D) or three-dimensional (3D) FE models with a few teeth or a single tooth are common modelling strategies in the literature. Kiekbusch et al. [43] and Wang [46] used the ANSYS parametric design language to build the detailed 2D and 3D FE models respectively, from which the torsional mesh stiffness of a spur gear pair were obtained. To find the singular stiffness of a tooth with crack in its fillet region, Chaari et al. [47] built a 3-dimensional FE model with only one tooth considered, whereas the FE contact between the driving gear and driven gear was not considered. To improve the computational efficiency, Ma et al. [44, 48, 49, 50] proposed a 2-dimensional plane strain FE modelling strategy with only a few teeth considered by means of ANSYS software. Contact elements with sufficient refinement were used in the model. They used this FE modelling strategy to obtain the GMS of the gear pair with the consideration of different factors, such as tip relief, extended tooth contact (corner contact, or tip contact), spalling defects and fillet cracks.

Hybrid methods combining the FE models and analytical or semi-analytical formulas were also widely used in the literature to evaluate GMS, with the purpose of significantly reducing computation time as well as keeping computation accuracy. Vijayakar [51] proposed a combined (hybrid) modelling strategy: the FE model was built for the outer region (away from the contact zone) with a relative coarse mesh, and the analytical surface integral approach was employed for the analysis in the inner region. A commercial finite element/contact mechanics (FE/CM) software named Calyx was thus developed, which can not only precisely calculate the gear transmission error and GMS through static analysis [18, 52, 53, 54, 55, 56], but also be used for the dynamic simulation of gear systems with comparatively simple structure [17, 45, 57]. Chang et al. [58] also introduced a hybrid model for determining the GMS of cylindrical gear pairs.
by combining the 3-dimensional FE analysis for the global term of the deformation and analytical line-contact formula deduced from Hertzian contact theory for the local contact term. They proved that “the mesh stiffness calculated from the proposed method is in close agreement with that from published formulae, but with less time consumption and improved steadiness compared with conventional FE models using contact elements.”

The literature also reports a few investigations using experimental methods to measure the stiffness of a single spur gear tooth, or the mesh stiffness of a tooth pair. Munro et al. [59] obtained the mesh stiffness of a spur gear tooth pair through the measurement of the gear transmission error by a pair of Heidenhain 360 000 line encoders. One of the challenges in the measurement is that the large force needed to deflect the teeth by a significant amount will also deflect other components (shafts, bearings) in the test rig, which will cause non-tooth deflections in the measurement. Due to the difficulties in directly measuring the gear tooth stiffness, Yesilyurt et al. [60] proposed an alternative vibration-based experimental procedure to estimate the stiffness reductions of spur gear tooth with tooth damage using the frequency response function (FRF) of the system. Pandya and Parey [61] used the conventional photoelasticity technique to measure the mesh stiffness of a gear pair, and calculate the mesh stiffness changes of a pinion under different crack depths.

1.2.1.2 For the Gear Pair with Tooth Profile Deviations

It has already been pointed out by the Japan Society of Mechanical Engineers (JSME) that the definition of tooth stiffness is different with the definition of spring stiffness (spring constant), as the definition of tooth stiffness is not certain [62]. Depending on the actual objective of calculating GMS (e.g. find the optimum amount of tip relief, find the resonance point of gear system, etc.), the definition of tooth stiffness may be different under different cases.

In the literature, there are indeed discrepancies in the concept of GMS. The main discrepancy comes from the treatment of gear tooth profile deviations [63]. In some papers, tooth profile deviations including the
undesirable gear tooth profile manufacturing errors and the intentional profile modifications, are treated as displacement excitations (Kahraman et al. [18], Parker et al. [17, 64] and Velex et al. [65, 66, 67, 68]), while in other papers, instead of being treated as a displacement excitations, the smoothing effect coming from the profile modifications was implicitly reflected in the calculation of the GMS (Pellicano et al. [2, 14], Lin et al. [15, 16, 69, 70], Chen and Shao [71], and He. et al. [72]). In other words, the mesh stiffness used in the former model is only determined by gear macro-geometries (tooth face width and contact ratio, etc.) and material properties (Young modulus and poison ratio, etc.), whereas in the latter model, the micro-geometries (the profile modifications) are also considered in the evaluation of mesh stiffness. This confusion was first noticed by Kasuba and Evans [73]. In their paper, the mesh stiffness that is totally independent of the gear error and the load is defined as the fixed-variable mesh stiffness (FVMS), whereas the mesh stiffness that is influenced by the gear error and load is defined as the variable-variable mesh stiffness (VVMS). He et al. [72] referred to it as the realistic mesh stiffness.

Currently, the VVMS of a gear pair with tooth profile modifications is mainly obtained through FE methods [2, 14, 44, 48, 49, 50, 72], as it can easily incorporate the tooth profile modifications and some secondary effects (e.g. corner contact effect) into the FE models and accurately capture the nonlinear contact behaviour between mating teeth. Lin et al. [15, 16, 69, 70], and Chen and Shao [71] proposed several analytical formulas to obtain the VVMS for the gear pair with tooth profile modifications. But their models did not consider the corner contact effect. Currently, researchers [44, 63] are trying to propose improved analytical formulas for the calculation of VVMS that can include several secondary effects and provide consistent results with those yielded through FE methods.

1.2.1.3 For the Gear Pair with Localized Tooth Defects

The above discussions deal with the gear mesh stiffness of a healthy gear pair without localized tooth defects. For gears with localized tooth defects (fillet crack and spalling defect), the GMS is considered to be an important parameter reflecting the fault status, as it is well known that the existence of tooth defects
can reduce the tooth flexibility and thus cause a reduction in the GMS whenever the faulty tooth enters the mesh zone. However different types of tooth defects may result in different features of the reduction in GMS. Figure 1.3 shows the GMS curves for a gear pair with various plane crack depths provided in [74], whereas Figure 1.4 shows the GMS curves for a gear pair with various rectangular-shaped spalls provided in [49]. Through 3D FE modelling of gears with localized cracks and spalling defects, Jia and Howard [7] found out that the crack will affect the GMS during the whole mesh cycle of the cracked tooth, whereas the spall will affect the GMS only when the spall affects the contact between mating teeth.

![Figure 1.3: GMS for various crack depth: (a) geometry of a localized plan crack, (b) GMS curves [74]](image1)

![Figure 1.4: GMS for various spall width: (a) geometry of a rectangular-shaped spall, (b) GMS curves [49]](image2)
In the 1980s, when researchers just started to work on the dynamic modelling of gears with defects, Randall [75] and McFadden et al. [76, 77] had already pointed out that local tooth defects will produce “short-term changes in the amplitude and phase of the meshing vibration as the defect goes through the mesh”, which can be described by the amplitude and phase modulation functions (periodic with the gear rotation frequency \( f_s \)) superimposed on the normal meshing vibration function (periodic with the gear mesh frequency \( f_m \), same with that of the GMS function). Based on this point, Choy et al. [13] built a dynamic model for the gear transmission system with several types of tooth defect, in which the amplitude and phase changes in the vibration were represented by qualitative magnitude and phase changes in the GMS (as shown in Figure 1.5). The more severe the defect, the more the GMS changes. This appears to be the first attempt in the literature to model the effects of tooth defects on gear vibration through the modifications of mesh stiffness. The methods for qualitative determination of the influence of tooth defects on the GMS were therefore required for an accurate dynamic modelling of the gear system. FE methods are the most used technique to do this. Recently analytical methods based on the potential energy principle are becoming more and more popular to obtain GMS for the gear pair with localized tooth defects as they are computationally efficient compared with the FE methods which need certain mesh refinements for the defects and then significant computation time [78].

![Figure 1.5](image)

**Figure 1.5**: Stiffness changes in the gear mesh model [13]: (a) phase change, (b) magnitude change
For the tooth fillet crack defect, Tian [36] and Wu et al. [79] assumed the tooth fillet crack propagation path to be a straight line (as shown in Figure 1.6), and derived the analytical equations to calculate the GMS for the tooth with different crack sizes. However, their models did not consider the tooth foundation effect. Charri et al. [40] refined their models by including the fillet foundation effect into the model through the formula provided by Sainsot and Velex [38]. They calculated the GMS for the gear pair with various sizes of uniform-depth plane cracks in the fillet region of a pinion tooth, and compared the analytical results with the FE results. Chen and Shao [41] derived an analytical model to calculate the GMS for the gear pair with non-uniform-depth plane cracks based on the slicing principle (“thin-slice” approach). Based on Chen and Shao’s work, Mohammed et al. [23, 80] studied a crack propagation scenario with the crack extending in the depth direction and tooth width direction simultaneously, and analysed its effect on the GMS as well as the dynamic response of a one-stage cracked spur gear system. By revising the crack propagation path and limiting lines related to the tooth thickness to parabolic curves, Mohammed et al. [81], Chen et al. [74] and Ma et al. [82] proposed improved analytical mesh stiffness models for cracked spur gear pairs and researched their influence on the dynamic response, respectively. Howard et al. [83] presented a 2D FE model for the gear pair with tooth fillet crack. In this model, the contact elements are used to model the contact effects between the mating teeth, and the singular elements

Figure 1.6: Crack propagation depth [79]: (a) actual path, (b) assumed path
around the tip are utilized to realistically simulate the influence of the crack tip. This strategy was later adopted by Ma et al. [50, 84] who built a similar 2D FE model using ANSYS software for the cracked gear pair to study the GMS with different crack depths and the case when the crack propagates into the rim of a perforated gear, considering the effects of the extended tooth contact (corner contact). Mohammed et al. [81], Chaari et al. [40] and Zouari et al. [85] established 2D and 3D FE models for a spur gear with only one cracked tooth in order to improve the computational efficiency. The stiffness of the cracked tooth was obtained by applying a concentrated force on the theoretical contact point (2D models) or along the tooth width (3D models). Detailed review on the calculation methods of GMS for gear pairs with localized tooth fillet cracks can be found in [86].

For the spalling defect, Chaari et al. [78] proposed an analytical method based on the potential energy principle to quantify the reductions of GMS due to various sizes of rectangular-shaped spalls and tooth breakage. The vibration signature of each tooth fault is identified through a dynamic analysis of one-stage spur gear transmission system. Ma et al. [49] proposed similar analytical methods for the GMS calculation of spur gears with rectangular-shaped spalling defects. The extended tooth contact effect and revised fillet foundation effect during the meshing of multiple tooth pairs were included in their model. Shao et al. [87] updated Chaari’s model by taking the edge contact (contact at the spall edges) effect into consideration. Badaoui et al. [21, 11] and Jiang et al. [88] used another strategy to quantify the reductions of the GMS through the reduction of the length of contact line caused by the spalling defect. A constant tooth stiffness density per unit length \( k_0 \) along the contact line was assumed, whose value can be found in the ISO 6336 standard [28]. Jia and Howard [7] built a 3D FE model to obtain the reductions of GMS resulting from a localized round spall. They found that the spall that is not across the whole tooth affects the GMS only marginally, and the spall depth has little influence on the GMS.

1.2.2 Gear Dynamic Models
The mathematical modelling of the dynamic behaviour of gears has attracted a significant amount of attention in the literature. Numerous mathematical models have been developed for different purposes in the past decades. The first systematic studies in gear dynamics can be dated back to the 1920s and 1930s by Ross [89] and Buckingham [90] respectively. At that time, the primary objective in the dynamic analysis of a gear system was to predict tooth dynamic loads for designing gears for use at high speeds. The first simple mass-spring models (lumped parameter models) for gear dynamics was suggested by Tuplin in the 1950s [91]. The major aim was still the estimation of tooth dynamic loads. In the mid-1950s, more involved models including the consideration of some other effects (tooth profile errors, time-varying mesh stiffness, helical gear) were introduced to represent the dynamic behaviour of gears in mesh [92, 93, 94]. In the late 1950s and 1960s, several theoretical and experimental studies were reported on gear dynamics [95, 96]. Since the 1970s, the possibility of large numerical computations induced two distinct types of extension to the original models [12]: first, more global flexural-torsional models with a large number of degrees of freedom (DOF) aimed at simulating the complete gear system (including shafts and bearings) [97, 98, 99]; and secondly, more refined local models with fewer DOF, but including some nonlinear and secondary effects (gear backlash, sliding friction, lubrication damping, corner contact) [35, 73, 101] and gear tooth defects or faults [13, 21, 11]. The research objectives of gear dynamic analysis have gradually become more and more diverse, including the prediction and control of gear lateral and axial vibrations, rattle noise, stability regions, dynamic stresses, and the detection and diagnosis of gear defects. Numerical techniques have usually been adopted to solve the complex differential equations built for the lumped parameter models of gear systems. Investigations on the continuous system or finite element models also appeared and quickly developed. Some systematic experimental studies have been conducted in the 1990s [102, 103, 104, 105] and 2000s [106, 107, 108, 109] to verify the capability of the established dynamic models in predicting the actual gear dynamic behaviour (dynamic transmission error and dynamic tooth root stresses). Ozguven and Houser [110] carried out probably the first comprehensive review of mathematical models used for gear dynamic
analysis up to the late 1980s. In 2003, Wang et al. [111] reviewed the progress in nonlinear dynamics (mainly due to the time varying gear mesh stiffness and gear backlash) of gear transmission systems since the 1980s, and provided several critical areas and unsettled issues for future research. Meanwhile, Parey and Tandon [12] mainly reviewed the work on dynamic modelling of spur gears including defects (wear, spalls) up to 2003. Gear dynamic models with various nonlinearities, such as the time varying gear mesh stiffness, mesh damping, gear errors, backlash, and sliding friction were also discussed.

This project does not intend to present a comprehensive and complete review of all the papers and research dealing with the dynamic behaviour of gear transmission systems with various forms of configurations for all kinds of purposes or aims, but will focus on the literature that is closely related to this project (dynamic vibration responses of cylindrical gear pair transmission systems with/without localized tooth defects), especially those papers since 2003 that have not been reviewed in [12, 110, 111].

1.2.2.1 Models Considering Major Effects

For a cylindrical gear pair transmission system, the literature reports on a vast number of investigations (mostly analytical, and a few experimental and FE studies) dealing with effects of multiple types of excitations on gear dynamic behaviour. However, experimental studies [96, 102, 103, 104, 105] have demonstrated that the time-varying gear mesh stiffness and gear backlash should be included in the gear dynamic analysis, especially when the research purpose is to determine the tooth dynamic loads, dynamic factors or dynamic transmission errors. Based on these two major effects, Kahraman and Singh [112] classified the dynamic models of gear pair transmission systems into four main groups: 1) linear time invariant (LTI) models; 2) linear time varying (LTV) models which include time-varying GMS; 3) nonlinear time invariant (NTI) which include gear backlash; and 4) nonlinear time varying (NTV) models which include time-varying GMS and gear backlash simultaneously. In the following work, Kahraman et al. [113, 114, 115] did a series of analytical studies of the interactions among the time varying GMS, gear backlash and external forcing excitations (due to gear transmission errors, TE), and their effects on the
gear dynamic performance. Theodossiades and Natsiavas [116] investigated the dynamics of a spur gear pair system involving backlash and time-dependent mesh stiffness. Instead of using the traditional numeric techniques (direct integration, trigonometric colocation, harmonic balance, shooting method), a new hybrid methodology was adopted to yield the steady-state response. Parker et al. [45] used the FE/CM software [51] to simulate the dynamic responses of a spur gear pair, and compared the FE results with the experimental results [102], and the analytical results from two NTV models (load-dependent and load-independent mesh stiffness models) which consider the backlash non-linearity and time-varying GMS. However, they mistakenly found that “the expectedly better model produces poorer agreement with experiments”.

Apart from the above two effects, gear static transmission error (STE) is also a significant factor affecting gear dynamic behaviour. It is widely accepted that the dynamic loading and noise generated by a gear pair is strongly related to its STE [117]. Normally, gear static transmission error consists of inevitable tooth manufacturing errors and intentional tooth profile modifications. The concept of gear tooth profile modifications (modifying involute gear tooth flanks) to reduce dynamic loading and noise has been an accepted practice for many years as suitable profile modifications can compensate for tooth deflections under load and tooth profile manufacturing errors, so that gear transmission error (fluctuation component) can be minimized [118]. Lin et al. [15, 16, 70] conducted a series of research analytically dealing with the dynamic loading of spur gears with linear and parabolic tooth profile modifications (tip relief), and concluded that linear tip relief is superior to parabolic relief when the gears operate at a nearly constant load, whereas parabolic relief is preferred when gears operate over a range of loading conditions. Kahraman and Blankenship [103] experimentally demonstrated the effect of involute tip relief on the dynamic performance of a spur gear pair. Faggioni et al. [14] introduced a global optimization method focused on gear vibration reduction by means of profile modifications, which considers different regimes (the amount and length of tip relief) and torque levels. Bonori and Pellicano [2] conducted analytical research on the nonlinear vibrations of a spur gear pair in the presence of manufacturing errors. The
manufacturing errors were treated stochastically, starting from the knowledge of the gear tolerance class. They found that the presence of manufacturing errors magnifies the vibration amplitude and may lead to chaotic vibrations (high amplitude of oscillation with a wide band spectrum response).

1.2.2.2 Models Considering Minor Effects

There were also many analytical investigations dealing with dynamics of a cylindrical gear pair with the focus on some other nonlinearities (mesh damping, friction, partial contact) and secondary effects (corner contact, back-side impact). These effects are not deterministic factors affecting gear dynamic behaviour. But under some special occasions, they can be influential and should not be neglected in the gear dynamic analysis.

Viscous damping is normally required in the gear dynamic models to account for the energy loss (due to lubrication, friction) at the gear mesh interface. Normally, a user-defined constant viscous damping ratio (ranging from values as low as 1-2% [105] to those as high as 10% [119]) is employed to represent the damping element. Amabili and Rivola [120] proposed a single degree of freedom (SDOF) model of a pair of low contact ratio spur gears. The main differences of this model with previous SDOF models include the consideration of time-varying mesh damping and the inclusion of gear errors for each meshing tooth pair (to simulate the partial contact phenomenon). The time-varying mesh damping was obtained by assuming a viscous damping proportional to the mesh stiffness. Huang et al. [121] presented a gear dynamic model with the consideration of time-varying lubrication damping. The lubricant effect was derived based on the elastohydrodynamic-lubrication (EHL) and squeezed-film theories. Li and Kahraman [122] built a two degree of freedom purely torsional model for a spur gear pair with the viscous damping provided by the instantaneous tribological behaviour of the tooth contacts, which is based on a study on a transient, non-Newtonian mixed EHL model of involute spur gear tooth contacts [123]. Guilbault et al. [124] also built a two degree of freedom purely torsional model for a spur gear pair, where the mesh damping consists of three sources: the surrounding element contribution, hysteresis of the
teeth and oil squeeze damping. The resulted nonlinear damping ratio calculated at different mesh frequencies and torque amplitudes presented average values between 5.3 percent and 8 percent, which is comparable to the constant 8 percent ratio used in most of published numerical simulations.

![Diagram of gear meshing](image)

**Figure 1.7**: Coulomb friction effect [131]: (a) friction forces, (b) various formulations of $\mu_c$

Typically, the sliding friction phenomenon between mating teeth is modelled by assuming Coulomb formulation with a constant coefficient ($\mu_c$, normally in the range of 0.04-0.06 as measured in [125]) of friction. A distinct feature of the friction force generated between mating teeth is that its direction will be reversed at the pitch point $P$ ($F_{p1}$ and $F_{p0}$ as shown in Figure 1.7(a)) meaning a discontinuity should happen on $\mu_c$. Vaishya and Singh [126, 127] presented a new analytical model of a gear pair with time-varying mesh stiffness, viscous damping and the periodically varying sliding friction parameters. The coulomb friction model was considered for the friction effect and the forced vibration response was found with the application of Floquet theory. Velex et al. [128, 129] conducted an original analytical and experimental study of tooth friction excitation (based upon the Coulomb model) in errorless spur and helical gears. The simulated and measured quasi-static bearing forces were compared and showed satisfactory consistency which largely validates the theoretical developments. Their results revealed the
potentially significant contribution of tooth friction to gear vibration and noise when gear tooth profiles are modified to minimize STE fluctuation and working at low-medium speeds. He et al. [55, 72, 130, 131, 132] did a series of studies about the sliding frictional effects on the dynamics of spur gear and helical gears. He also investigated the dynamic friction forces in spur gear pairs using alternative sliding friction formulas in which several kinds of friction models (as shown in Figure 1.7(b)) based on the Coulomb formulation and tribological theories were compared and discussed.

When gears with no tip relief are running under heavy load, extended tooth contact (ETC, also referred to as corner contact, tip contact, tip interference, off-line-of-action contact) may occur due to gear tooth flexibility under load, which will affect gear normal mesh behaviour. Normally, this secondary effect is included into the gear dynamic model by modifying the GMS. FE methods are convenient to obtain the modified GMS due to ETC as this effect can be easily incorporated into the FE models [48, 49, 50]. Umezawa et al. [133] suggested several kinds of gear mesh stiffness function to accommodate this effect. An experimental test was conducted to see which stiffness function better depicts the gear dynamic behaviour. Lin et al. [134], Munro et al. [135], and Seager [136] have derived exact or approximate versions of analytical formulae for the calculation of the transmission error outside the normal path of contact (also referred to as the gear separation distance), which can be used to determine whether a mating tooth pair is in corner contact or not. Han et al. [137] researched the parametric stability of the spur gear pair with and without the consideration of the effect of extended tooth contact (corner contact, or tip contact) through the Floquet theory for stability analysis. The influence of the ETC on the GMS was obtained by modifying the rectangular-shaped GMS to trapezoidal-shaped GMS.

Gear back-side tooth impact may happen under light load or idling conditions due to the inevitable existence of gear backlash. This will often cause a significant increase of gearbox noise level which can appear either in the driving condition (“rattle” noise) or in the neutral condition (“idle” noise) [138]. The time-varying symmetric mesh stiffness model that assumes identical mesh stiffness variation in both the
forward (drive-side) and backward (back-side) contact direction was usually used in most published work [105, 113, 114, 115]. Guo et al. [139] pointed out that the number of tooth pairs in contact along the back-side line of action is not always equal to that along the drive-side line of action. They also derived the time-varying back-side mesh stiffness and its correlation with backlash in terms of the known drive-side mesh stiffness for any standard spur gear pair. Chen et al. [140] proposed a time-varying asymmetric mesh stiffness model and studied its effect on gear dynamic responses.

1.2.2.3 Models Including More DOF

In the above discussions, the research objective is a single-stage, single-mesh cylindrical gear pair, and fewer DOF were considered in the dynamic models. There were also many global models with more degrees of freedom but fewer considerations of nonlinear and secondary effects aimed at simulating the complete gear system. Lin et al. [141, 142] built a model for a simple parallel-shaft, spur gear transmission, where the stiffness and damping of the shafts as well as the frictional effect were considered. Velex and Maatar [24], Eritenel and Parker [54, 143], and He et al. [130] proposed general three-dimensional (3D) models considering the full degrees of freedom (12 DOF) of the cylindrical gear pair system based on the classic “thin-slice” approach (slice the tooth into a series of independent thin pieces along the contact line as the elastic coupling effects are usually negligible for narrow-faced gears with low helix angles [144]), which can be used to solve the tooth load distributions along the contact line. In another study, Ajimi and Velex [145] built a similar model for the solid wide-faced cylindrical gear pair based on the Pasternak’s elastic foundations model. They found that the convective effects (elastic coupling effects) among the stiffness cells along contact lines are normally negligible and the model based on the “thin-slice” approach may give sufficient approximations for dynamic simulations. There were some other studies attempting to model the whole gear set (geared rotor system) including the shaft flexibilities, ranging from simple equivalent lumped springs to finite elements [146, 147, 148, 149], for the purpose of studying the complex couplings amongst the transverse, torsional, axial and rocking motions of the geared rotor system, especially in the presence of gear mounting errors (gear eccentricities,
misalignments). There were also many significant efforts mainly from Parker et al. [57, 64, 150, 151, 152, 153], Kahraman et al. [154, 155, 156, 157] and Velex et al. [158, 159] on the dynamic studies of multi-mesh gear systems and planetary gear sets, in terms of parametric stability, natural frequencies and mode shapes, load sharing characteristics, and steady state responses.

1.2.2.4 Models Including Localized Tooth Defects

In the above discussions, the dynamic models of gears without localized tooth defects have mainly been reviewed. In the past 20 years, to improve the detection techniques of localized tooth defects, researchers have actively worked on the dynamic modelling of gears with defects. In 1996, Choy et al. [13] presented a comprehensive procedure to simulate and study the vibrations of a gear transmission system in the presence of tooth surface pitting, wear, and partial tooth fracture, which were achieved mainly by qualitatively modifying the magnitude and phase of the gear mesh stiffness. Howard et al. [83] explored the effect of a single tooth plane crack (two-dimensional crack) on the GMS and studied the resulting dynamic behaviour of a single stage reduction gearbox. Later, they [7] compared the changes in GMS due to the localized spalling and crack damage using the FE method. The dynamics simulation results due to these two types of tooth defect were compared. Endo et al. [6, 22] presented a technique to differentially diagnose these two types of localized tooth defects. Through static and dynamic simulation model, they found that the effect of a tooth crack depends on the change in stiffness of the tooth while the effect of a spall is predominantly determined by the geometry of faults. This conclusion is applicable only for shallow spalls. Experimental results were also provided to verify the effectiveness of the proposed technique for differential diagnosis. Chen and Shao [41] derived an analytical model to calculate the GMS for the gear pair with non-uniform-depth plane cracks based on the slicing principle (“thin-slice” approach). The effects of crack size on the gear dynamics were thus simulated and the corresponding changes in statistical indicators (RMS and kurtosis) were investigated. Aside from the changes of the GMS, Mark et al. [160, 161] recently found, through a series of experimental studies, that tooth bending fatigue can also lead to whole-tooth plastic deformation which will contribute the static transmission error
(STE). Consequently, Shao and Chen [162] proposed a theoretical method to approximate the tooth inclination deformations due to a plane crack, and included this into the dynamic model to study the effects on the dynamic performance of a planetary gear set.

For spalling defects, most researchers studied their effects on the gear dynamic behaviour through the changes in GMS [7, 49, 78, 87, 88]. However, Endo et al. [6, 22] found that a shallow spall will modify the STE whenever the mating tooth rolls into and makes contact at the bottom of the spall. Based on the 3D model built by Velex and Maatar [24], Badaoui et al. [11, 21] proposed a comprehensive model to simulate the dynamic response of a cylindrical gear system with a spall defect, in which the contribution of the spall to the gear dynamics was determined by the depth of the spall (with respect to the average static deflection): small depth spalls generate an external displacement excitation whereas deeper spalls modify the mesh stiffness. One of the main assumptions made by Badauiit is that the type of contact between the meshing surfaces of a mating tooth pair is an ideal linear line of contact, so that the mating tooth can never bridge over a spall. Ma et al. [4, 8] researched the dynamic mechanism of the gear system with localized crack and spalling defects theoretically and experimentally. Some useful results were derived except that they did not consider the influence of spall dimensions (depth and length) and load on the influencing mechanism of the spalls to the gear vibration.

1.2.3 Summary

Although numerous research efforts have been made toward a better understanding of gear dynamic behaviour, there are still some aspects of gear dynamics that are not satisfactorily understood. Some of them can be summarized as follows:

1. The analytical formulas determining the corner contact effect for a gear pair with no profile modifications have been reported in the literature [134, 135, 136], but the analytical method to determine the corner contact effect with the inclusion of tip relief has not been found. Besides, there is still some basic confusion in regard to the concept of mesh stiffness (FVMS and VVMS). SDOF models of a spur
22

2. In almost all previous dynamic models for a gear pair with eccentricities, the inertial force due to gear eccentricity is simplified as an uncoupled standard centrifugal inertial force, whereas the dynamic coupling effect on the inertial force due to gear torsional vibrations is normally ignored [146, 147, 148, 149]. Under some conditions, this dynamic coupling effect may play an important role in the gear translational motions along the off-line of action (OLOA).

3. Although several time-varying asymmetric mesh stiffness models (different mesh stiffness functions in the drive-side and back-side contact direction) have already been proposed, they were only applicable to standard spur gears [139, 140]. For a spur gear pair with addendum modifications, the analytical equation describing the relationship between back-side mesh stiffness and drive-side mesh stiffness has not been established so far. Besides, the influence of the asymmetric mesh stiffness model on the dynamic performance of a spur gear pair transmission system is rarely studied in the literature.

4. Most former crack models take only plane cracks into consideration, which only considers cracks propagating either in the depth direction or the tooth width direction [23, 36, 79, 40, 41, 80, 162] and neglect the more typical spatial crack (also referred to as a 3D crack in some references) that will propagate not only in the depth direction and tooth width direction but also in the tooth profile direction. The effects of localized tooth spatial cracks on gear mesh stiffness, tooth inclination deformations and gear dynamics have not yet been determined.

5. Currently, the most advanced dynamic model of a cylindrical gear pair with localized spalling defect is based on the assumption that the type of contact between the meshing surfaces of a mating tooth pair is an ideal linear line of contact [11, 21], so that the mating tooth can never bridge over a spall. However, in reality, gear tooth contact behaviour is nonlinear in nature and the contact pattern is normally elliptical
depending on the load transmitted as well as the amount of tooth surface crowning. Therefore, a more advanced model needs to be introduced to account for this issue.

1.3 Research Objective

Accurate dynamic modelling of gear transmission systems can provide us with not only the vibration generation mechanisms in gear transmissions, but also the dynamic properties of various types of gear faults. The summary made in the last section after the literature review clearly points out several limitations in the current research work about the dynamic modelling of the cylindrical gear pair system with and without localized tooth defects. Therefore, the research objectives of this project include:

1. Introduce an analytical method to evaluate the corner contact effect for a spur gear pair with tip relief; study the effect of the corner contact on the gear dynamic response; generalize and compare several types of commonly used SDOF models in the literature.

2. Investigate the dynamic coupling effect on the inertial force due to gear eccentricity; analyse its influence on the dynamic behaviour of a cylindrical geared rotor system.

3. Derive the analytical equations describing the relationship between back-side mesh stiffness and drive-side mesh stiffness for a spur gear pair with addendum modifications; analyse the influence of the asymmetric mesh stiffness model on the dynamic performance of the spur gear pair system.

4. Extend the current plane crack model into a spatial crack model; introduce modified expressions to calculate the equivalent GMS and tooth inclination deformations for a spur gear pair with a localized spatial crack; analyse the influence of a spatial crack on the dynamic performance of the gear system.

5. Propose a more advanced model for a cylindrical gear pair with a localized spalling defect that can address the issue of the nonlinear elliptical contact pattern; provide experimental work to validate the superiority of the new model.
The first and third objectives focus on the dynamic modelling of a spur gear pair without localized tooth defects, but with the consideration of some secondary effects (corner contact and back-side impact) under some special conditions. The second objective focuses on the dynamic modelling of a cylindrical geared rotor system with the consideration of a secondary effect (dynamic coupling) induced by gear eccentricities. The fourth objective focuses on the dynamic modelling of a spur gear pair system with a localized spatial crack. The fifth objective focuses on the dynamic modelling of a cylindrical gear pair system with a localized spalling defect. Even though the research objectives are slightly independent from each other, they represent different aspects of the gear dynamics. The ultimate purpose of this project is to propose more accurate and realistic gear dynamic models suitable to different situations, and to advance the current understanding of the gear dynamic behaviour.

1.4 Organization of Thesis

This thesis mainly focuses on the dynamic modelling of cylindrical gear transmission systems with and without localized tooth defects. Each main chapter (except for chapter 1, chapter 2 and chapter 8) is written in the journal paper style, and deals with one major objective outlined in the last section:

1. Chapter 1 introduces background information of this project, stresses the significance and contributions of this work to the development of theoretical gear dynamic modelling strategies and practical engineering applications, and systematically reviews current research status in the literature, from which the research objectives of this project are summarized.

2. Chapter 2 presents some fundamental theories about gear dynamics, which include the involute tooth profiles, gear mesh stiffness, gear transmission error as well as some typical gear dynamic models. A part of the work in this chapter is from a conference paper:
W. Yu, C.K. Mechefske and M. Timusk, A comparison of several methods for the calculation of gear mesh stiffness, *International Conference on Sensing, Diagnostics, Prognostics and Control* SDPC 2017, Shanghai, China. (*Accepted*)

3. Chapter 3 mainly studies the corner contact effects with the inclusion of tip relief, and compares several SDOF models that have been widely used in the literature. This chapter is directly adapted from publications in the *Mechanism and Machine Theory* and at a conference in Charlotte, USA:


4. Chapter 4 investigates the dynamic coupling behaviour between the torsional vibration and the translational vibration for a geared rotor system with eccentricities. Parameter studies are also conducted. This chapter is directly adapted from publications in the *Mechanism and Machine Theory* and at a conference in Chongqing, China:


5. Chapter 5 introduces an analytical method to determine the back-side mesh stiffness for a spur gear pair with addendum modifications. The effects of different amounts of addendum modification on the gear back-side impacts are discussed in two cases. This chapter is directly adapted from a publication in the *Journal of Sound and Vibration*:

6. Chapter 6 derives modified expressions to calculate equivalent GMS and tooth inclination deformations for a spur gear pair with a localized spatial crack. The influence of the spatial crack on the dynamic performance of a spur gear pair system is also studied. This chapter is directly adapted from publications in the *Engineering Failure Analysis* and the *Nonlinear Dynamics*:


7. Chapter 7 proposes a new dynamic model for a cylindrical gear pair system with a localized spalling defect. Experimental results are also provided and compared with simulation results yielded by the proposed models and previous model in order to validate the superiority of the proposed model. This chapter is directly adapted from a draft paper just submitted to the *Nonlinear Dynamics*:


8. Chapter 8 presents the general conclusions based on all the research work in this project. Contributions of this thesis to the literature are clearly illustrated. Future research directions are also pointed out.

**1.5 References**

1. T. Eriitenel, Three-dimensional nonlinear dynamics and vibration reduction of gear pairs and planetary gears. PhD. Thesis. The Ohio State University, Columbus, Ohio, USA, 2011.


3. Y. Guo, Analytical study on compound planetary gear dynamics. PhD. Thesis. The Ohio State University, Columbus, Ohio, USA, 2011.


55. S. He, Effect of sliding friction on spur and helical gear dynamics and vibro-acoustics. PhD. Thesis. The Ohio State University, Columbus, Ohio, USA, 2008.


Chapter 2

Theoretical Background

2.1 Introduction

The inevitable fluctuation of the total number of tooth pairs in the mesh zone and tooth profile geometric errors are the two main excitation sources that cause gear vibrations. In mathematical terms, the former is represented by time-varying gear mesh stiffness (GMS), which serves as parametric excitation, whereas the latter manifests itself as gear transmission error (TE), which is usually modelled as displacement excitation. Therefore, determination of GMS and TE has always been the priority issue in gear dynamic analysis. This chapter will provide theoretical background knowledge regarding gear dynamics which is necessary for the topics of following chapters, starting from the analytical expressions of the involute tooth profile, calculation methods of gear mesh stiffness, measurements of transmission error, and the introduction of three typical types of gear dynamic models that will be used in the following chapters.

2.2 Involute Tooth Profile

The most frequently used cylindrical gear tooth profile in reality is involute due to several advantageous characteristics [1]: 1) the angular velocity ratio between two gears of a conjugated gear pair remain constant throughout mesh; 2) the rotation transmission is smooth, and independent of small error in centre distance; 3) the sum of the contact forces is constant, and they always act in the same direction (line of action); 4) an involute gear can work together with mating gears with a different number of teeth; 5) manufacturing is relatively easy and the same tools can be used to machine gears with different numbers of teeth.

In this project, we consider the standard spur gear tooth whose involute profile is generated by using a standard rack \( (f_r = 1, c_r = 0.25, \alpha_r = 20^\circ) \) with double rounded tooth tip at each side as shown in Figure
2.1. The generation of spur gear tooth shape during the cutting process is equivalent to pure rolling of the pitch line of the rack against the pitch circle of the gear blank [2]. The involute region of the tooth profile is directly generated by the straight line AB of the rack tooth (as shown in Figure 2.2(a)), whereas the root fillet of gear tooth is defined by the rounded corner at the tip BC (as shown in Figure 2.2(b)).

**Figure 2.1:** Geometry of rack [5] (Note: \( f_r \) is the addendum coefficient, \( c_r \) is the tip clearance coefficient and \( \alpha_r \) is the pressure angle)

**Figure 2.2:** Generation of tooth profile: (a) involute region, (b) fillet region

The involute and fillet region of the spur gear tooth profile can be expressed by two separate parametric equations, which means that the coordinate of any point on the tooth profile can be exclusively determined [3, 4]. We first define an X-Y coordinate system fixed to the gear, with the origin \( O \) at the gear centre, and Y-axis coinciding with the centre line of the tooth being cut. For the involute profile as shown in Figure 2.2(a), the coordinate for any point \( P \) at the involute region is given as [3, 4]:
\[
\begin{align*}
    x_p &= R_b[(\gamma + \gamma_0)\cos\gamma - \sin\gamma] \\
    y_p &= R_b[(\gamma + \gamma_0)\sin\gamma + \cos\gamma]
\end{align*}
\] (2.1)

where \( R_b \) is the gear base radius, \( \gamma \) is the mesh angle of point \( P \), which is defined as the angle between the line \( OT \) (line connecting gear centre \( O \) and the tangent point \( T \)) and the tooth centre line (\( Y \)-axis). \( \gamma_0 \) indicates the angle between the tooth centre line and the line \( OQ \), where point \( Q \) is the intersection point of the involute curve and the base circle.

For the fillet region, the coordinate of any specific point \( P \) at the fillet region is given as [3, 4, 5]:

\[
\begin{align*}
    x_p &= R\sin\varphi - \left( \frac{a_r}{\sin a'} + r \right)\cos(a' - \varphi) \\
    y_p &= R\cos\varphi - \left( \frac{a_r}{\sin a'} + r \right)\sin(a' - \varphi)
\end{align*}
\] (2.2)

where \( \varphi \) is computed as:

\[
\varphi = \frac{a_r\tan a' + b_r}{r}
\] (2.3)

where \( a_r, b_r \) and \( r \) are the dimensions of the rack as shown in Figure 2.1, \( R \) is the radius of the pitch circle, \( \varphi \) and \( a' \) are illustrated in Figure 2.2(b). Detailed discussion about this can be found in [5]. Therefore, the standard spur gear tooth shape is completely defined by the 2 parametric equations, which is essential when determining GMS with and without the consideration of localize tooth defects.

### 2.3 Gear Mesh Stiffness

The time-varying gear mesh stiffness is one of the main excitations that cause unwanted vibration and noise of gear transmission systems. Its magnitude is not constant due to the periodic change of the number of tooth pairs in the mesh zone. Numerous methods for the calculation of gear mesh stiffness have been introduced by different researchers. This section generalizes three commonly-used approaches that have been used in literature to yield gear mesh stiffness. They are based on finite element (FE) methods, analytical methods using the potential energy principle, and an approximation method based on
the ISO standard. Each approach has its own advantages and disadvantages. Systematic comparisons among them have not been found in the literature, which constitutes the main objective of this research. It should also be noted that there are some other analytical expressions of GMS in the literature [6, 7, 8, 9]. However, these have not received significant subsequent attention in the literature, therefore, will not be discussed here.

2.3.1 Three Commonly-used Approaches

2.3.1.1 FE Approach

FE models are the primary tools used to obtain GMS due to their significant advantage in representing the crucial tooth contact behaviour. Two-dimensional (2D) models and three-dimensional (3D) models are both common in the literature. Compared with the 3D models, 2D models with a plain strain assumption for a thick gear body, or a plain stress assumption for a thin gear body, can yield comparatively accurate results with significantly fewer nodes and reduced computation times. 3D models can overcome the limitations of 2D models in simulating helical gears, spur gears with modifications along the face width direction, or gears with misalignments [9].

Although FE models of a conjugating (mating) gear pair built by different researchers are slightly different from each other, some common modelling strategies should be noted, such as: the nodes at the inner hub (or ring) of the driven gear are completely constrained from motion; the nodes at the inner hub of the driving gear are only allowed to rotate around the centre; the static torque $T_1$ (or moment) is applied on the driving gear by applying suitable tangential forces on the nodes of the driving gear hub [4, 9]. By adjusting the angular positions of the gears, the time-varying rotational deformation $\theta_1(t)$ of the driving gear’s hub at each roll angle can be determined once the gear deformation field is evaluated. Gear mesh stiffness is then calculated by:

$$K(t) = \frac{(T_1/R_{b1})}{R_{b1}\theta_1(t)} = \frac{T_1}{R_{b1}^2\theta_1(t)}$$

(2.4)
where $R_{b1}$ is the base radius of the driving gear. It should be noted that the contact between the conjugated gear pair is the most important and critical part. Therefore, significant mesh refinement should be applied at the contact region between contacting teeth in FE models.

### 2.3.1.2 Analytical Approach

According to [10], three contributing factors need to be considered when analytically calculating the tooth deformation (or deflection) in the line of action (LOA) at a contact point $j$ subjected to a certain mesh force $F$: 1) the local deformation caused by the Hertzian contact; 2) the beam deflection caused by tooth considered as a cantilever beam; and 3) the deflection caused by the flexibility of the foundation (gear body).

#### 1) Hertzian Contact Deformation $\delta_h$

Generally, the Hertzian contact deformation $\delta_h$ between the meshing tooth surfaces of a mating tooth pair is nonlinear. Various approximate formulas for the calculation of $\delta_h$ can be found in [9, 10, 11]. A simplified nonlinear contact deformation based on the semi-empirical equation developed by Palmgren [12] had been adopted by many researchers [11, 13, 14]:

$$
\delta_h = \frac{1.275 F^{0.9}}{E^{0.9} L^{0.8}}
$$  \hspace{1cm} (2.5)

where $F$ is the mesh force, $E$ is the Young’s modulus of the gear material, and $L$ is the contact length between the two meshing teeth of a mating tooth pair.

Nevertheless, Yang [15] derived a linear contact deformation function and claimed that the error of linearization approximation is less than 0.5 percent for steel gears. Therefore, Yang’s formula has been widely used in literature [3, 16, 17, 18, 19], which is written as:

$$
\delta_h = \frac{4F(1-\mu^2)}{\pi EL}
$$  \hspace{1cm} (2.6)
where $\mu$ is the Poisson’s ratio of the gear material.

2) Tooth Beam Induced Deformation $\delta_t$

![Figure 2.3: Model of spur gear tooth: (a) geometric parameters for gear body, (b) geometric parameters for single tooth](image)

The tooth beam induced deformation $\delta_t$ can be determined by considering it as a non-uniform cantilever beam while the tooth foundation is assumed as perfect rigid. The potential energy method is widely used to derive the tooth beam deflection under load. In general, when a tooth beam is under the action of the mesh force $F$ at point $j$ (as shown in Figure 2.3), there will be potential energy stored in the tooth beam due to the bending moment $M$, the shear force $F_b$ and the axial compressive force $F_a$. Therefore, the total tooth deformation consists of bending moment induced $\delta_b$, shear force induced $\delta_s$, and axial compressive force induced $\delta_a$, which can be expressed as:

$$\delta_t = \delta_b + \delta_s + \delta_a$$  \hspace{1cm} (2.7)

The calculation for $\delta_b$, $\delta_s$, and $\delta_a$ can be found in [3, 17, 18, 19]. They are given by:

$$\delta_b = F \int_0^l \left( \frac{(l-x)\cos\alpha_p - \sin\alpha_p}{EI_x} \right)^2 dx, \quad \delta_s = F \int_0^l \frac{1.2\cos^2\alpha_p}{GA_x} dx, \quad \delta_a = F \int_0^l \frac{\sin^2\alpha_p}{EA_x} dx$$  \hspace{1cm} (2.8-a, b, c)
where \( l, x, h \) and \( \alpha_p \) are shown in Figure 2.3(b). \( G \) is the shear modulus of the tooth material, and can be calculated as \( G = E / 2(1 + \mu) \). \( I_x \) and \( A_x \) are the effective area moment of inertia and area of the integral section as shown in Figure 2.3(b). For a tooth with no crack, \( I_x \) and \( A_x \) are calculated on the whole section:

\[
I_x = \frac{1}{12} (h_x + h_x)^3 L, \quad A_x = (h_x + h_x)L \tag{2.9-a, b}
\]

3) Tooth Foundation Induced deformation \( \delta_f \)

Various expressions are given by previous researchers to calculate the foundation induced deflection. However, they are in the same form [11]:

\[
\delta_f = \frac{E \cos^2 \alpha_p}{L E} \left[ L^* \left( \frac{u_f}{s_f} \right)^2 + M^* \left( \frac{u_f}{s_f} \right) + P^* (1 + Q^* \tan^2 \alpha_p) \right] \tag{2.10}
\]

where \( u_f \) and \( s_f \) are shown in Figure 2.3. Different researchers use slightly different values for the coefficients: \( L^* \), \( M^* \), \( P^* \) and \( Q^* \).

| Table 2.1: Values of the coefficients in Equation (2.11) [20] |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( A_i \)       | \( B_i \)       | \( C_i \)       | \( D_i \)       | \( E_i \)       | \( F_i \)       |
| \( L^* \)       | -5.574×10^{-5} | -1.9986×10^{-3} | -2.3015×10^{-4} | 4.7702×10^{-3} | 0.0271          | 6.8045          |
| \( M^* \)       | 60.111×10^{-5} | 28.100×10^{-3}  | -83.431×10^{-4} | -9.9256×10^{-3} | 0.1624          | 0.9086          |
| \( P^* \)       | -50.952×10^{-5} | 185.50×10^{-3}  | 0.0538×10^{-4}  | 53.3×10^{-3}    | 0.2895          | 0.9236          |
| \( Q^* \)       | -6.2042×10^{-3} | 9.0889×10^{-3}  | -4.0964×10^{-4} | 7.8297×10^{-3}  | -0.1472         | 0.6904          |

Sainsot et al. [20] derived the tooth foundation induced deflection by assuming linear and constant stress at the root circle. A polynomial function for the calculation of each coefficient in Equation (2.10) was provided:

\[
X_i^* (h_f, \theta_f) = A_i \frac{h_f}{\theta_f} + B_i h_f^2 + C_i \frac{h_f}{\theta_f} + D_i \frac{h_f}{\theta_f} + E_i h_f + F_i \tag{2.11}
\]
where $X_i^*$ denotes the coefficients $L^*, M^*, P^*$ and $Q^*$; $h_f = R_f / R_{int}$, where $R_f$ is the outer radius of the gear body, and $R_{int}$ is the inner radius of the gear body. The related geometric parameters in Equation (2.10) and (2.11) are defined in Figure 2.3. The values of $A_i$, $B_i$, $C_i$, $E_i$ and $F_i$ are given in Table 2.1.

4) GMS Model based on the Traditional Analytical (TA) Methods

Traditionally, the equivalent mesh stiffness of a mating gear pair is calculated by direct summation of equivalent mesh stiffness of tooth pairs in contact:

$$K(t) = \sum_{i=1}^{N} k_i^j$$

(2.12)

where $N$ denotes the number of tooth pairs in mesh simultaneously; $k_i^j$ is the equivalent stiffness of the $i$th tooth pair at the contact point $j$ (at time instant $t$), which is expressed as:

$$k_i^j = \frac{F}{\delta f_1 + \delta t_1 + \delta h_1 + \delta t_2 + \delta f_2}$$

(2.13)

where subscripts 1 and 2 in $\delta_t$ and $\delta_f$ represent the tooth beam induced deformation and tooth foundation induced deformation for the driving gear and driven gear respectively. The equivalent compliance of the $i$th tooth pair at the contact point $j$ (at time instant $t$) can be expressed as:

$$C_i^j = \frac{\delta f_1 + \delta t_1 + \delta h_1 + \delta t_2 + \delta f_2}{F}$$

(2.14)

5) GMS Model based on the Improved Analytical (IA) Methods

Kiekbusch et al. [9] and Ma et al. [13] pointed out that traditional analytical models will overestimate the tooth foundation induced deformation during the meshing of multiple tooth pairs as all the mating tooth pairs are connected to the same gear body. The TA method will yield a gear mesh stiffness higher than the actual value in the multi-tooth engagement. Therefore, Ma et al. [13] proposed the improved analytical method which is described as:
\[ K(t) = \frac{F}{(\lambda_1 N \delta_{f1} + \frac{F}{k_T} + \lambda_2 N \delta_{f2})} \]  

(2.15)

where:

\[ k_T = \sum_{i=1}^{N} k_T^i, \quad k_T^i = \frac{F}{\delta_{s1} + \delta_{s2}} \]  

(2.16-a, b)

where \( k_T^i \) denotes the equivalent mesh stiffness of the \( i \)th tooth pair (note the difference between Equation (2.16-b) and Equation (2.13)). Similarly, the equivalent compliance of the \( i \)th tooth pair is defined as:

\[ C_T^i = \frac{\delta_{s1} + \delta_{s2}}{F} \]  

(2.17)

In Equation (2.15), \( \lambda_{1N} \) and \( \lambda_{2N} \) are the correction coefficients of the tooth foundation induced stiffness for the driving gear and driven gear respectively. Their values should be 1 when only one tooth pair is in mesh (\( N = 1 \)). When there are multiple tooth pairs in mesh simultaneously (\( N > 1 \)), their values can be determined using FE methods [4, 9].

### 2.3.1.3 Approximation Method based on ISO Standard

From Equations (2.6) - (2.17), we can find that the GMS is approximately proportional to the contact length \( L \). Based on this, a simplified approximation method based on the ISO standard 6336 [22] was proposed by Velex [23]. An important assumption within ISO standard 6336 is that the mesh stiffness density per unit length \( k_0 \) along the contact line is considered as approximately constant so that the following formula can be used [23, 24]:

\[ K(t) = k_0 L(t) \]  

(2.18)

where \( L(t) \) is the time-varying length of the contact line. The analytical expression of the time-varying contact length \( L(t) \) for a cylindrical gear pair can be found in [25, 26]. They decomposed the periodic time-varying contact length \( L(t) \) into Fourier series as:
\[ L(t) = W_n \cdot \left[ 1 + \sum_{k=1}^{\infty} A_k \cdot \cos \left( \frac{2\pi kt}{T_n} \right) + B_k \cdot \sin \left( \frac{2\pi kt}{T_n} \right) \right] \quad (2.19) \]

with \( W_n = W \cdot \epsilon_\alpha / \cos \beta \) where \( W \) is the face width (supposing the length of the contact line between a mating tooth pair is the face width \( W \)), \( \epsilon_\alpha \) is the transverse contact ratio of the gear pair, \( \beta \) is the helix angle of the gear pair, \( T_n \) is the mesh period, and:

(a) for spur gears:

\[
A_k = \frac{1}{\pi k \epsilon_\alpha} \sin(2\pi k \epsilon_\alpha) \quad B_k = \frac{1}{\pi \epsilon_\alpha k} [1 - \cos(2\pi k \epsilon_\alpha)] \quad (2.20-a, b)
\]

(b) for helical gears:

\[
A_k = \frac{1}{2\pi^2 k^2 \epsilon_\alpha \epsilon_\beta} \left[ \cos(2\pi k \epsilon_\beta) + \cos(2\pi k \epsilon_\alpha) - \cos \left( 2\pi k (\epsilon_\alpha + \epsilon_\beta) \right) \right] - 1
\]

\[
B_k = \frac{1}{2\pi^2 k^2 \epsilon_\alpha \epsilon_\beta} \left[ \sin(2\pi k \epsilon_\beta) + \sin(2\pi k \epsilon_\alpha) - \sin \left( 2\pi k (\epsilon_\alpha + \epsilon_\beta) \right) \right] \quad (2.21-a, b)
\]

where \( \epsilon_\beta \) is the overlap contact ratio of the cylindrical gear pair (\( \epsilon_\beta = 0 \) for spur gear pair).

The ISO standard 6336 provides some expressions to derive \( k_0 \) [22, 23], one of which gives:

\[
k_0 = \frac{C_M C_R C_{\beta \cos \beta}}{q} \quad (2.22)
\]

where \( C_M \) is the correction factor accounting for the difference between the measured values and the theoretical calculated values, \( C_R \) is the gear blank factor accounting for the flexibility of gear rims and webs, \( C_\beta \) is the basic rack factor accounting for the deviations of the actual basic rack profile of the gear from the standard basic rack profile (ISO 53) with \( f_r = 1 \), \( c_r = 0.25 \), \( \alpha_r = 20^\circ \) shown in Figure 2.1. Their values can be determined by expressions and graphs provided in [22]. In addition:

\[
q = C_1 + \frac{C_2}{2n_1} + \frac{C_3}{2n_2} + C_4 x_1 + C_5 \frac{x_1}{2n_1} + C_6 x_2 + C_7 \frac{x_2}{2n_2} + C_8 x_1^2 + C_9 x_2^2 \quad (2.23)
\]
where \( Z_n = Z_i / (\cos^3\beta) \), \( Z_i \) are the number of teeth of the driving gear \((i=1)\) and driven gear \((i=2)\), \( x_i \) are the profile shift coefficients on the driving gear and driven gear. Coefficients \( C_1, \ldots, C_9 \) have been tabulated and listed in Table 2.2.

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( C_7 )</th>
<th>( C_8 )</th>
<th>( C_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04723</td>
<td>0.15551</td>
<td>0.25791</td>
<td>-0.00635</td>
<td>-0.11654</td>
<td>-0.00193</td>
<td>-0.24188</td>
<td>0.00529</td>
<td>0.00182</td>
</tr>
</tbody>
</table>

It should be noted that the mesh stiffness formulae (e.g. Equation (2.22)) in the ISO standard 6336 stem from Weber’s analytical formulae [10] which were modified to bring the analytical results into closer agreement with the experimental results [22]. In [27], the authors considered a time-dependent mesh stiffness density per unit length \( k_0(t) \), where a small variable \( a(a<1) \) is introduced to represent the relative variation in amplitude, and a time dependent function \( \Phi(t) \) is used to account for the variations of mesh stiffness density per unit length depending on the contact point position on the tooth profiles.

### 2.3.2 Comparisons and Discussions

In this section, the calculation results of each method will be compared and discussed. The main parameters of the gear pair for simulation are shown in Table 2.3. The 2D FE model with plane stress assumption and 3D FE model for the gear pair shown in Table 2.3 were built in the ANSYS Workbench, as shown in Figure 2.4. The total number of nodes for the 2D model and 3D model are 29765 and 172007 respectively. The values of the correction coefficients in Equation (2.15) were suggested as \( \lambda_{11} = \lambda_{21} = 1 \), and \( \lambda_{12} = \lambda_{22} = 1.1 \) [9, 21]. Figure 2.5(a) shows the mesh stiffness curves calculated by each method when the input torque is 100 N.m. In the single contact region (mesh cycle between 0.6 and 1.0 in Figure 2.5(a)), the mesh stiffness curves yielded by different methods are close to each other. However, in the double contact regions (mesh cycle between 0 and 0.6 or 1.0 and 1.6), there are obvious discrepancies. Compared with other methods, the mesh stiffness curve yielded by the approximation method has the
largest fluctuation, seconded by the TA method. This demonstrates that these two methods overestimate the mesh stiffness during the meshing of multiple tooth pairs. The IA method proposed in [13] can yield consistent results compared with those from FE methods.

<table>
<thead>
<tr>
<th>Table 2.3: Parameters of the gear pair for simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driving gear</td>
</tr>
<tr>
<td>Teeth number $Z_i$</td>
</tr>
<tr>
<td>Mass $m_g$ (kg)</td>
</tr>
<tr>
<td>Module $m$ (inch/mm)</td>
</tr>
<tr>
<td>Helix angle $\beta$ (degree)</td>
</tr>
<tr>
<td>Pressure angle $\alpha$ (degree)</td>
</tr>
<tr>
<td>Face width $W$ (inch/mm)</td>
</tr>
<tr>
<td>Material</td>
</tr>
</tbody>
</table>

**Figure 2.4:** FE models for a meshed gear pair: (a) 2D model, (b) 3D model
Figure 2.5: Mesh stiffness calculated by different methods: (a) mesh stiffness curves, (b) order spectrum

Figure 2.5(b) shows the corresponding dimensionless order spectrums (with respect to the mean value of GMS) for the mesh stiffness calculated by each method (only the first 5 harmonics are shown). The harmonic amplitudes yielded by the approximation method and TA method are noticeably larger than those yielded by the other methods, whereas the harmonic amplitudes yielded by the IA are in good agreement with those of the FE methods.

The difference in the calculation results between the 2D and 3D models is less than 10% in GMS curve, and less than 5% in the order spectrum, which is acceptable for the dynamic analysis. This validates the suitability of using the 2D FE method to model the gear pair when calculating GMS.

Table 2.4: Time cost for the calculation of GMS in one mesh cycle for each method

<table>
<thead>
<tr>
<th>Approximation Method</th>
<th>Analytical Methods</th>
<th>2D FE Method</th>
<th>3D FE Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time cost</td>
<td>&lt;1s</td>
<td>2-3 s</td>
<td>10-20 minutes</td>
</tr>
</tbody>
</table>

Table 2.4 lists the time cost for the calculations of GMS in one mesh cycle for each method, which were carried out in a personal computer equipped with Intel Core i7 processor, 2.4 GHz clock speed and 8 GB
RAM. The approximation method and analytical methods are much faster than the FE methods. The 2D FE method can provide sufficiently quick computation compared to the 3D FE method. The latter is usually computationally expensive depending on the mesh refinement applied on the contact regions of the meshing tooth pairs.

![Graphs showing mesh stiffness under various torques for 3D and 2D models](image)

**Figure 2.6:** Mesh stiffness calculated by FE methods under various input torques: (a) 3D model, (b) 2D Model

It should be noted that the calculation results from FE methods are traditionally assumed as more realistic and accurate than those from other methods, as FE models can easily incorporate some secondary effects, which are normally neglected in the analytical methods. For example, Figure 2.6 shows the mesh stiffness curves yielded by the FE methods under various input torques. It can be found that GMS is load dependent (as the input torque increases, the amplitudes of the GMS increase slightly). This is due to the nonlinear contact behaviour between the meshing tooth surfaces of a mating tooth pair, which is normally linearized in analytical methods for simplification (as shown in Equation (2.6)). Besides, the duration of the meshing of a single tooth pair slightly reduces as the load increases, which is due to the occurrence of extended tooth contact (ETC) under higher loads when an incoming tooth pair is about to enter the contact zone, or an outgoing tooth pair is about to exit from the contact zone [13, 28]. However, this secondary effect is ignored in most existing analytical methods and approximation methods.
In summary, the FE methods typically yield the most accurate and realistic results of the GMS but are normally computationally expensive. The 2D FE method is therefore usually used in literature to simplify the gear model without losing too much accuracy. Analytical methods have significant advantages in terms of computational speed and modelling efficiency. However, traditional analytical models will overestimate the mesh stiffness during the meshing of multiple tooth pairs. The improved analytical method proposed in [13] can provide consistent results compared with the FE results. The approximation method based on the ISO standard can be used when only the order of magnitude of the GMS is of interest (for example, in the design stage of a gear transmission system). It should also be noticed that, since the fluctuation of GMS of a helical gear pair is significantly smaller than that of a spur gear pair, the approximation method can yield quite accurate results in this case.

2.4 Gear Transmission Error

Gear transmission error (TE) is considered to be an important internal displacement excitation to gear vibration and noise. According to Welbourn [29], the definition of TE is “The difference between the actual position of the output gear and the position it would occupy if the gear drive were perfectly conjugate.” It can be expressed as angular displacement or as linear displacement along the line of action [30]:

$$x = R_{b1} \theta_1 - R_{b2} \theta_2$$  \hspace{1cm} (2.24)

where $\theta_1$ and $\theta_2$ are the angular displacements of the driving gear and driven gear respectively, $R_{b1}$ and $R_{b2}$ are the base radii of the driving gear and driving gear respectively. There are three typical sources of TE. They are tooth deflections $\delta_d$ due to contact forces or mesh forces (as described in Section 2.3.1.2), gear manufacturing and mounting errors $\delta_m$ (such as profile undulations, spacing errors, run-out and misalignment), and tooth profile modifications $\delta_p$ (such as tooth tip relief and lead crowning).

2.4.1 Measurement of Transmission Error
TE can be measured statically or dynamically (low or high speed), unloaded or loaded (light or heavy load), as shown in Table 2.5 [1]. Unloaded static transmission error (USTE) is the most frequently used concept for gear quality inspection since it gives information of tooth profile deviations relative to the perfect geometry in an unloaded and static condition. On the local scale, gear tooth profile manufacturing errors, profile modifications and tooth surface defects (wear, spall) are the main sources of the USTE (as shown in Figure 2.7). On the global scale, gear mounting errors (misalignments, eccentricities) will also contribute.

**Table 2.5:** Transmission error in each case

<table>
<thead>
<tr>
<th>Transmission Error</th>
<th>Load (or torque)</th>
<th>Low</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>Unloaded static transmission error</td>
<td></td>
<td>Loaded static transmission error</td>
</tr>
<tr>
<td>High</td>
<td>Unloaded dynamic transmission error</td>
<td></td>
<td>Loaded dynamic transmission error</td>
</tr>
</tbody>
</table>

![Figure 2.7](image)

**Figure 2.7:** Main sources of unloaded static transmission error: (a) profile manufacturing errors, (b) profile modifications, (c) surface defects (Keys: The solid green line represents the actual tooth profile, and the dashed black line represents the standard involute profile)

Transmission error is often measured with optical encoders, which gives typically several thousands of impulse per revolution. The transmission error is thus acquired by comparing the signals from the two
encoders on each shaft [1]. Another measurement technique is to measure the torsional accelerations on both shafts of the mating gear pair, and take the difference of the two accelerations (corrected by speed ratio). The resulting signal is then integrated twice to obtain the transmission error. In order to get accurate torsional vibration, Blankenship and Kahraman [31] devised a measurement strategy by attaching four tangentially mounted piezoelectric accelerometers to the gear wheel as shown in Figure 2.8. Slip rings were used to connect the rotating accelerometers to the “fixed” signal collecting system. The torsional vibration of the gear was obtained by vectorally adding signals from all four accelerometers $z_i(t), i = 1, 2, 3, 4$ to cancel gravitational effects as well as the any translation vibrations.

![Figure 2.8](image)

**Figure 2.8:** The arrangement of linear accelerometers on a gear wheel [31]

It should be noted that the encoder-based methods are only capable of measuring static transmission error (STE) or dynamic transmission error (DTE) at lower rotating speeds as the limited number of impulses generated by encoders per second at higher speeds results in poor resolution in TE. On the other hand, the accelerometer-based methods are more accurate at higher speeds due to the noise involved with double integrating the torsional accelerations to rotations at lower speeds. A hybrid approach has been used [32] that combines both encoder- and accelerometer-based methods to measure the DTE for a helical gear pair.
running at a wide speed range (0 to 4000 rpm). The high precision optical encoders (18,000 pulses/revolution) used allowed measurement of DTE with speed range of 0 to 600 rpm, whereas two diametrically opposed accelerometers were employed within 500 to 4000 rpm.

2.4.2 Gear Tooth Profile Modifications

Intentionally modifying the gear tooth profiles has been proved theoretically and also experimentally to significantly affect the static transmission error, dynamic transmission error, dynamic load and the gear operation life [33, 34, 35]. Proper profile modification will greatly minimize the gear vibration and corresponding noise generated during operation and also increase its working life.

Among all kinds of profile modification methods, gear tooth tip relief and root relief are the most commonly-used modification strategies. They are basically an intentional removal of the material from the perfect involute profile, as shown in Figure 2.9. Normally, the same amount and length of profile modification are applied to the tooth tips of both gear and pinion. Since modifying the root of a tooth is equivalent to modifying the tip of its counterpart, only tip relief will be considered in this study. In practice, modifying the root of a gear tooth will be much more difficult than modifying the tip especially for some extremely low contact ratio gears, making it preferable to give only tip modification [34, 35].
The conventional amount of tip relief, as stated in [34], is equal to the sum of the combined deflection evaluated at the highest point of a single tooth pair contact region (HPSTC, this is for normal contact ratio gear pairs with a contact ratio between 1 and 2. For high contact ratio (HCR) gear pairs with contact ratio between 2 and 3, it should be evaluated at the highest point of the second double tooth pair contact region, HP2DTC) and twice the maximum spacing error.

![Diagram of tip relief types](image)

**Figure 2.10:** Three different types of tip relief

The conventional amount of tip relief is usually chosen as a reference value. The non-dimensional modification amount is designated as $\Delta$, and $\Delta$, therefore, equals 1 for the conventional amount of tip relief. The conventional length of tip relief starts from the HPSTC (for HCR gear pairs, it should start at the HP2DTC) and ends at the tooth tip point. This value is usually set as the reference. The non-dimensional modification length is designated as $L_n$. As a result, by varying $\Delta$ and $L_n$, one can define any amount and length of tip relief.

The last key point that needs to be ascertained for tip relief is how it is going to apply at the tooth tip. In [36, 37], three different types of profile modification are introduced. They are described as linear, parabolic I and parabolic II, as shown in Figure 2.10. The linear modification has a linear tip relief trace on a profile chart. The parabolic I modification has a parabolic trace with zero slope at the start of the modification (tangent to the involute profile) in the profile chart, whereas the parabolic II modification
has an infinite slope (vertical) at the end of the modification (the tooth tip point). In the following chapters, only linear tip relief will be considered.

2.5 Gear Dynamic Models

The study of gear dynamics is not only essential to the design of reliable power transmission system with acceptable levels of gear vibration and noise, but also advantageous to our understanding of the dynamic behaviour of gear transmission systems in the presence of localized defects. It is obvious that the type of mathematical model that should be used for a reliable dynamic analysis depends on the object of the study as well as the relative dynamic properties of different elements in the system [38]. For instance, if the object of the study is to predict the dynamic mesh and tooth load, dynamic factors and dynamic transmission errors, or if the stiffnesses of shafts and bearings are relatively high or low compared to the gear mesh stiffness, it is a reasonable approximation to uncouple the torsional vibrations of the gears due to mesh stiffness from the other vibration modes of the system. In such cases, a single degree of freedom (SDOF) model can yield accurate results. Otherwise, a multi-degree of freedom (MDOF) model should be employed to study other vibration modes, such as the transverse motions, rocking motions of the gears, or even the possible couplings between these motions with the torsional motions of the gears. Three typical types of mathematical model for a mating gear pair will be introduced in this section. They are SDOF, 6 DOF and 3D gear dynamic models. It needs to be noted that throughout the thesis, the model parameters (rotating speeds, torques, etc.) of the driving gear and the driven gear are designated with subscripts of 1 and 2 representing the driving gear and driven gear respectively.

2.5.1 SDOF Model

SDOF model is the simplest gear dynamic model, which can be used to study torsional motion of a gear pair. Some important effects that should be included in the dynamic model for a gear pair are the time-varying mesh stiffness, the backlash and the excitation due to the gear profile errors.
Figure 2.11: Typical gear rotary model: (a) pure torsional model, (b) equivalent SDOF model

Figure 2.11(a) shows a typical rotary model for a mating spur gear pair [39, 40, 41, 42]. It consists of two disks ($J_1$ and $J_2$ representing the moment inertia of the driving gear and driven gear respectively) coupled by time-varying gear mesh stiffness $k(t)$, viscous damping $c$ (normally in the range of 0.01-0.2 for a steel structure [43, 44]), backlash $b$, and displacement excitation due to gear profile deviations (or errors) $e(t)$. $R_{bi}$, $T_i$ and $\theta_i$ represent the base radii, torques and rotation motions of the driving gear ($i=1$) and driven gear ($i=2$) respectively. The corresponding governing equations can be expressed as:

$$
\begin{align*}
J_1 & \ddot{\theta}_1(t) + R_{b1}c \left( R_{b1} \dot{\theta}_1(t) - R_{b2} \dot{\theta}_2(t) \right) + R_{b1} k(t) g(R_{b1} \theta_1(t) - R_{b2} \theta_2(t) - e(t)) = T_1 \\
J_2 & \ddot{\theta}_2(t) - R_{b2}c \left( R_{b1} \dot{\theta}_1(t) - R_{b2} \dot{\theta}_2(t) \right) - R_{b2} k(t) g(R_{b1} \theta_1(t) - R_{b2} \theta_2(t) - e(t)) = -T_2 
\end{align*}
$$

(2.25)

where $g(\cdot)$ is a contact function. It is usual to introduce the dynamic transmission error $x(t) = R_{b1} \theta_1(t) - R_{b2} \theta_2(t)$, which reduces Equation (2.25) to a single degree of freedom model:

$$
m_e \ddot{x}(t) + c \dot{x}(t) + k(t) g(x(t) - e(t)) = f_0
$$

(2.26)

where the equivalent mass $m_e$ and the static load $f_0$ are defined as:

$$
m_e = \frac{J_1 J_2}{J_1 R_{b2}^2 + J_2 R_{b1}^2}, \quad f_0 = \frac{T_1}{R_{b1}} = \frac{T_2}{R_{b2}}
$$

(2.27)

The contact function $g(\cdot)$ is usually modelled as a piecewise linear phenomenon.
\[ g(x(t) - e(t)) = \begin{cases} 
 x(t) - e(t) - b; & x(t) - e(t) > b \\
 0; & \text{Otherwise} \\
 x(t) - e(t) + b; & x(t) - e(t) < -b 
\end{cases} \] (2.28)

It should be noted that there are some other SDOF models in the literature that include several other nonlinear and secondary effects (tooth friction [45, 46], corner contact [28], lubrication between teeth [47, 121, 49]), depending on their object of study. Although SDOF models built by different researchers may slightly differ in some aspects, the experimental results provided by Gregory et al. [50] and Kahraman et al. [51, 52, 53, 54], clearly demonstrated that the unavoidable fluctuation of the number of meshing tooth pairs and the gear backlash should be included in gear dynamic analysis so that an accurate prediction of gear dynamic behaviour can be obtained.

2.5.2 6 DOF Model

The 6 DOF model for a mating gear pair is also widely used in the literature when the translational motions of the gears are also of interest, or the flexibilities of the bearings supporting the gear pair are commensurate with the gear mesh stiffness.

![6 DOF model](image)

*Figure 2.12: 6 DOF model*
Figure 2.12 shows a typical 6 DOF gear dynamic model [55]. The gear pair is modelled as rigid cylinders \((m_i \text{ and } J_i \text{ representing the mass and moment of inertia of the } i\text{th gear})\) connected by a spring and a viscous damper representing the time-varying gear mesh stiffness \(k_g(t)\) and viscous damping \(c_g\) respectively. A Cartesian coordinate system \((U-V)\) is built with the \(U\)-axis in the direction of the line of action (LOA), and the \(V\)-axis in the direction of the off-line of action (OLOA). The bearing stiffness and damping in the LOA and OLOA directions are modelled as linear springs \((k_{Biu} \text{ and } k_{Biw})\) and dampers \((c_{Biu} \text{ and } c_{Biw})\). The corresponding governing equations can be expressed as:

\[
\begin{align*}
    m_1\ddot{u}_1 + c_{B1u}\dot{u}_1 + k_{B1u}u_1 &= F_t \\
    m_1\dot{v}_1 + c_{B1v}\dot{v}_1 + k_{B1v}v_1 &= 0 \\
    J_1\ddot{\theta}_1 + F_tR_1 &= T_1 \\
    m_2\ddot{u}_2 + c_{B2u}\dot{u}_2 + k_{B2u}u_2 &= -F_t \\
    m_2\dot{v}_2 + c_{B2v}\dot{v}_2 + k_{B2v}v_2 &= 0 \\
    J_2\ddot{\theta}_2 + F_tR_2 &= T_2
\end{align*}
\]

where \(F_t\) is the net contact force which is the sum of the elastic force \(F_k\) and damping force \(F_c\):

\[
F_t = F_k + F_c = k_g\delta + c_g\dot{\delta}
\]

where \(\delta = R_{o1}\dot{\theta}_1 + R_{o2}\dot{\theta}_2 + u_2 - u_1\) is the transmission error (also referred to as normal approach in [24]). The above equations can also be expressed in the matrix form:

\[
M\ddot{q} + C_b\dot{q} + K_bq + F_tE = F_0
\]

where:

\[
\begin{align*}
    q^T &= \{u_1, v_1, \theta_1, u_2, v_2, \theta_2\} \\
    M &= \text{diag}(m_1, m_1, J_1, m_2, m_2, J_2) \\
    C_b &= \text{diag}(c_{B1u}, c_{B1v}, 0, c_{B2u}, c_{B2v}, 0) \\
    K_b &= \text{diag}(k_{B1u}, k_{B1v}, 0, k_{B2u}, k_{B2v}, 0) \\
    E^T &= \{-1, 0, R_{o1}, 1, 0, R_{o2}\} \\
    F_0^T &= \{0, 0, T_1, 0, 0, T_2\}
\end{align*}
\]

\[\text{(2.32-a, b, c, d)}\]
It should be noted that the friction force $F_f$ caused by the sliding between mating teeth was included in the model by some researchers [56, 57, 58, 59] as its direction is normal to the contact force $F_t$ and will influence the OLOA motion directly. A Coulomb friction model with a constant or time-varying coefficient $\mu_c$ (normally in the range of 0.04-0.06 [16, 60]) is usually employed. Discussion of some other advanced friction models can be found in [59].

2.5.3 3D Model

The above-mentioned models are mainly for gear pairs that maintain only in-plane motions. For helical gear pairs, or spur gear pairs with misalignments, Blankenship and Singh [61] pointed out that additional DOF should be included into the model, which are the axial and rocking (rotation about an axis parallel to the LOA) motion. Therefore, 3D models considering full degrees of freedom for each gear body should be built to analyse the axial, twisting and rocking motions of the gear pair.

![3D cylindrical gear mesh model](image)

**Figure 2.13:** 3D cylindrical gear mesh model: (a) the 3D gear mesh model, (b) projection on the plane of action

Figure 2.13(a) shows a general 3D model of a cylindrical gear pair [62]. The classic “thin-slice” approach is applied in this model. The mesh behaviour of the gear pair is modelled by two rigid disks connected by a series of stiffness cells $k_j$ along the instantaneous contact line on the base plane in the direction determined by the helix angle $\beta$ [63]. A localized Cartesian coordinate system ($U$-$V$-$W$) is established.
The \( U \)-axis is in the direction of the line of action, and the \( V \)-axis is in the off line of action direction. The \( W \)-axis is along the axial direction and can be determined by following the right-hand rule. Lagrange’s equations were used to derive the following equations of motion:

\[
M\ddot{q} + C\dot{q} + (K_b + K_g(t, q))q = F_0 + F_1(t) + F_2(t, q)
\] (2.33)

where:

\[
q^T = \{u_1, v_1, w_1, \theta_{u1}, \theta_{v1}, \theta_{w1}, u_2, v_2, w_2, \theta_{u2}, \theta_{v2}, \theta_{w2}\}
\]

\[
M_g = \text{diag}(m_1, m_1, I_1, I_1, I_p, m_2, m_2, I_2, I_2)
\]

\[
K_b = \text{diag}(k_{Bu1}, k_{Bv1}, k_{Bw1}, k_{B\theta u1}, k_{B\theta v1}, k_{B\theta w1}, k_{Bu2}, k_{Bv2}, k_{Bw2}, k_{B\theta u2}, k_{B\theta v2}, k_{B\theta w2})
\]

\[
K_g(t, q) = \sum_{j=1}^{N_c(t)} k_j(t)H_j(q)E_j(q)^T
\]

\[
F_0^T = \{0, 0, 0, 0, 0, T_1, 0, 0, 0, 0, T_2\}
\]

\[
H_j = \begin{cases} 
1, & E_j(q)^Tq > e_j(t) \\
0, & E_j(q)^Tq \leq e_j(t)
\end{cases}
\] (2.34-a, b, c, d, e, f)

where \( m_i, I_i, I_{pi} \) are the mass, transverse moment of inertia and the polar moment of inertia of the \( i \)th gear \((i = 1, 2 \) representing the driving and driven gear respectively), respectively; \( k_{Bu}, k_{Bv}, k_{Bw}, k_{B\theta u}, k_{B\theta v}, k_{B\theta w} \) are the linear stiffnesses of the \( i \)th bearing in 6 principle directions; \( k_j \) is the cell stiffness at the contact point \( M_j \). \( N_c(t) \) is the number of individual cells along the contact line. \( H_j \) is the tooth contact function that determines the contact condition at \( M_j \), which is either 0 (contact loss) or 1 (contact). \( E_j(q) \) is the so-called structure vector used to relate the degree of freedom \( q \) to the normal approach at \( M_j \). \( F_0, F_1(t), \) and \( F_2(t, q) \) are the excitations due to static load, gear global mounting errors, and gear localized tooth profile errors. \( C \) is the viscous damping matrix of the system. Detailed discussions of these vectors and matrices can be found in [24, 62, 63, 64, 65]. Depending on the research objective as well as the relative dynamic properties of the system, different researchers may use slightly different modelling strategies regarding these vectors and matrices.
2.6 Summary

This chapter presented basic theories about gear dynamic analysis, including the involute tooth profiles, gear mesh stiffness, gear transmission error as well as some typical gear dynamic models. Two parametric equations were introduced to analytically determine the involute tooth profile generated by a standard rack. Classifications and measurement strategies of gear transmission error were briefly explained. Three typical mathematical models for a mating gear pair were introduced. Their corresponding governing equations of motion were also given.

More attention was paid to the calculation methods of gear mesh stiffness as it has great influence on the gear dynamic performance. In fact, the inclusions of some secondary effects or localized tooth defects that will be discussed in the following chapters are mainly achieved by modifying the gear mesh stiffness. Therefore, intensive study of gear mesh stiffness is of significant importance to gear dynamic analysis.

2.7 References


43. J.D. Stevenson, Structural damping values as a function of dynamic response stress and deformation levels, *Nuclear Engineering and Design* 60(1980) 211 -237.


Chapter 3

Analytical Modelling of Spur Gear Corner Contact Effect

3.1 Introduction

Corner contact (tip interference), which happens due to the tooth flexibility when subjected to a heavy load, is normally neglected in the dynamic analysis of gear transmission systems with profile modifications when using analytical methods. It is commonly believed that corner contact can be avoided if sufficient tip and/or root relief on the teeth is used, which is the main reason that this effect is neglected in gear dynamics studies [1, 2, 3]. However, when insufficient tip and/or root relief is applied, especially for the gears working under heavily-loaded conditions, corner contact might still exist. Analytical methods to determine the effect of corner contact on the static transmission error (STE) and gear mesh stiffness (GMS) of heavily-loaded spur gears have been initiated to some degree. Several kinds of stiffness functions have been discussed by Umezawa et al. [4] to accommodate tip interference, and an experimental test was conducted to see which stiffness function better depicts the behaviour of transmission error under static load and dynamic meshing conditions. Lin et al. [5] derived exact versions of analytical formulae for the calculation of the transmission error outside the normal path of contact (also referred to as the gear separation distance), whereas Munro et al. [6] and Seager [7] developed approximate formulae that are simple to use and accurate. However, in their discussions, the effect of gear tooth profile errors on corner contact is not included. Analytical studies of this case are limited. To fill this gap, an analytical method to study corner contact effect on gear mesh stiffness, static transmission error and the dynamic response of a spur gear pair with tip relief will be introduced in this chapter.

The literature reports on a large number of investigations dealing with the effect of multiple types of excitations on gear dynamic behaviour. However, gear models established by different researchers are slightly different from each other. The main discrepancies include the treatment of gear profile errors and
corner contact. In some literature, both the undesirable gear manufacturing errors and the intentional profile modifications are treated as displacement excitation [1, 2, 8, 9, 10, 11, 12], while in other papers, instead of being treated as a displacement excitation the smoothing effect coming from the profile modifications is implicitly reflected in the mesh stiffness [13, 14, 15, 16, 17, 18, 19, 20]. These confusion was first noticed by Kasuba and Evans [21]. In their paper, the GMS that is totally independent of the gear error and the load is defined as the fixed-variable mesh stiffness (FVMS), whereas the GMS that is influenced by the gear error and load is defined as the variable-variable mesh stiffness (VVMS).

Based on the two different definitions of mesh stiffness, two different types of gear model can be established. In the FVMS model: 1) gear tooth errors have negligible effect or no effect on mesh stiffness; 2) the contact ratio and/or mesh stiffness are not affected by transmitted load, premature or delayed engagement (corner contact); 3) dynamic simulations are based on uninterrupted periodic mesh stiffness functions and error displacement strips. In the VVMS model, gear tooth errors and corner contact effect are reflected in the calculation of the VVMS in the static analysis, and therefore eliminated from the governing equation but reflected in the VVMS.

The differences between these two models have been noticed for a while, but they are both widely used. Systematic and direct comparisons between these two models have not yet been made. Liu and Parker [1] have compared the dynamic response predicted by some existing discrete parameter models against a FE benchmark. However, the VVMS model was not included in their analysis. Which model is the most effective and efficient for gear dynamic analysis remains an open question. This constitutes another focus of this study.

3.2 Gear Mesh Stiffness Model

3.2.1 Gear Static Transmission Error and Load Sharing Ratio
The transmission error, $x$, which is defined as the difference between the actual and ideal positions of the driven gear, is usually expressed as the linear displacement along the line of action [3]:

$$x = R_{b1} \theta_1 - R_{b2} \theta_2$$  \hspace{1cm} (3.1)

where $\theta_k$ and $R_{bk}$ are the angular displacement and base radius of the $k$th gear ($k = 1$ for driving gear, and $k = 2$ for driven gear), respectively. The sign convention used for the transmission error $x$ is positive behind the ideal position of the driven gear. Considering a spur gear pair with a normal contact ratio (NCR, $1 < CR < 2$) running at a low speed subjected to the load $F$, the transmission error of the gear pair working under such circumstances is usually called the loaded static transmission error (LSTE), $(x)_s$, which normally includes the tooth deflections $(\delta_j^d)$, gear manufacturing errors $(\delta_j^m)$ and profile modifications $(\delta_j^p)$ at the contact position $j$, and should be equal for every tooth pair in mesh at $j$:

$$\begin{align*}
(\hat{x})_s &= (\delta_j^d)_d + (\delta_j^i)_p + (\delta_j^i)_m \\
(\hat{x})_s &= (\delta_j^d)_d + (\delta_j^i)_p + (\delta_j^i)_m \\
\end{align*}$$  \hspace{1cm} (3.2)

In these equations, corner contact effect is not included and will be considered in the next section (hence a ‘hat’ symbol is added on LSTE for differentiation). Substituting Equation (2.14) into the above equations and noting that the sum of the load shared by every tooth pair in mesh equals the total static load $F$, gives:

$$\begin{align*}
(\hat{x})_s &= C_1^j F_j^1 + e_j^1 \\
(\hat{x})_s &= C_2^j F_j^2 + e_j^2 \\
F &= F_j^1 + F_j^2 \\
\end{align*}$$  \hspace{1cm} (3.3)

where $F_j^i$ is the static load shared by the $i$th tooth pair (since $1 < CR < 2$, $i = 1, 2$) at the contact position $j$; $e_j^i = (\delta_j^i)_p + (\delta_j^i)_m$ is the equivalent gear profile error regarding the $i$th tooth pair in mesh, which includes the gear manufacturing errors and tooth profile modifications. It is interesting to note that the static transmission error can be experimentally measured if the driving gear runs at a low speed. When experimentally measuring unloaded static transmission error, the gear pair should be worked under a light
load so that the gear tooth deflection will be negligible. In Equations (3.3), \( F, C_j^1, C_j^2, e_j^1 \) and \( e_j^2 \) can be either calculated or measured directly, while \( F_j^1, F_j^2 \) and \( (\bar{x}_j) \) are the three unknowns. Solving Equations (3.3) simultaneously gives:

\[
\begin{cases}
(\bar{x}_j)_s = \frac{c_j^1 C_j^2 + c_j^1 e_j^1 + c_j^2 e_j^1}{c_j^1 + c_j^2} \\
F_j^1 = \frac{c_j^2 F + e_j^2}{c_j^1 + c_j^2} \\
F_j^2 = \frac{c_j^1 F - e_j^1}{c_j^1 + c_j^2}
\end{cases}
\]

(3.4)

where \( e_j^{pq} = e_j^p - e_j^q \) \((p, q = 1, 2)\). The load sharing ratio (LSR) is defined as the ratio of the load shared by one tooth pair in mesh to the total static load. According to Equations (3.4):

\[
\begin{cases}
LSR_j^1 = \frac{F_j^1}{F} = \frac{c_j^2 F + e_j^2}{c_j^1 + c_j^2} = \frac{c_j^2 + e_j^2}{c_j^1 + c_j^2}/F \\
LSR_j^2 = \frac{F_j^2}{F} = \frac{c_j^1 F - e_j^1}{c_j^1 + c_j^2} = \frac{c_j^1 - e_j^1}{c_j^1 + c_j^2}/F
\end{cases}
\]

(3.5)

As to the high contact ratio (HCR with \( CR > 2 \)) gear pair, the loaded static transmission error, the static load shared by a tooth pair and the corresponding load sharing ratio can be obtained by following the same procedure described above:

\[
\begin{cases}
(\bar{x}_j)_s = \frac{\left(\prod_{p=1}^n c_j^p \right) F + \sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q) e_j^p}{\sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q)} \\
F_j^1 = \frac{\left(\prod_{q=1}^n c_j^q \right) F - \sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q) e_j^p}{\sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q)} \\
LSR_j^1 = \frac{\left(\prod_{q=1}^n c_j^q \right) - (\sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q) e_j^p)/F}{\sum_{p=1}^n (\prod_{q=1,q \neq p}^n c_j^q)}
\end{cases}
\]

(3.6)

where \( n \) is the nearest integer that is larger than \( CR \).

### 3.2.2 Corner Contact
Corner contact, or contact outside the normal path of contact, can occur in spur as well as helical gear transmission systems owing to the elastic deflection of the loaded teeth, which will lead to premature and delayed engagement. This can be easily explained by observing Figure 3.1. The load-carrying tooth pair #2 deform elastically, which cause the incoming tooth pair #3 to enter contact earlier than the theoretical start of contact $B$. Similarly, the loaded outgoing tooth pair #1 will leave contact later than the theoretical end of contact $E$. This extends the tooth contact zone and increases the contact ratio. Obviously, the degree of this increased contact is dependent on the torque applied, since the higher the transmitted load, the greater the increase in elastic deflection due to the loaded teeth. The analysis by Lin et al. [5] shows that neglecting corner contact effect results in underestimating resonant speed and overestimating the maximum dynamic load. Lin et al. used the concept of separation distance to analytically analyse the influence of corner contact on transmission error and produce the final LSTE curve.

3.2.2.1 Gear Tooth Separation Distance
Gear tooth separation distance is defined as the distance between a pair of teeth just out of contact, during approach or recess, if there is no elastic deformation [5]. This distance, expressed along the line of action (\(S_a\) and \(S_r\) as shown in Figure 3.1), will be compared with the LSTE to determine the contact condition. Lin et al. [5] have derived an exact version of analytical formulae for the calculation of gear tooth separation distance, whereas Munro et al. [6] and Seager [7] developed several approximate formulae that are simple to use and accurate.

3.2.2.2 LSTE Including Corner Contact Effect

The influence of corner contact on transmission error has been thoroughly discussed by Lin et al. [5] with gear tooth profile errors neglected, which may be a reasonable assumption for high-quality, heavily-loaded gears. However, in this study, a general form of LSTE including corner contact effect will be derived.

Considering a spur gear pair with a normal contact ratio, corner contact mainly happens when an incoming tooth pair is just coming into the theoretical starting point of the engagement and an outgoing tooth pair is just leaving the theoretical end point of the engagement. In this study, we assume: 1) corner contact affects the gear mesh only at the start and end of each single tooth pair mesh zone, and does not influence the double mesh zone; and 2) in the single mesh zone where tooth pair #2 is theoretically the only tooth pair in mesh as shown in Figure 3.1, profile errors on this tooth pair are not sufficient to lead to loss of contact. The first assumption is reasonable since according to FEA results [22], the hand-over region between single and double mesh zones will move slightly with increasing load so that the single zone reduces when corner contact happens whereas the double zone is relatively stable. The second assumption is introduced for the purpose of disregarding profile irregularities due to local spalls, pits or even teeth breakage. As a result, there will be three distinct cases when corner contact happens.

Case #1: Tooth pair #3 is in corner contact, and tooth pair #1 is not in corner contact
In this case, tooth pair #2 carries the most load. Tooth pair #3 comes into contact earlier and gradually increases its share of the total transmitted load. Tooth pair #1 is out of contact and carries no load. Since the transmission error for each tooth pair should be equal, thus:

\[
\begin{align*}
(x_j)_s &= C_j^3 F^3_j + S a^3_j + e_j^3 \\
(x_j)_s &= C_j^2 F^2_j + e_j^2 \\
F &= F^2_j + F^3_j
\end{align*}
\] (3.7)

where \( S a^3_j \) is the separation distance of tooth pair #3 during approach when tooth pair #2 contacts at point \( j \). Solving these equations will give:

\[
\begin{align*}
(x_j)_s &= \frac{c_j^3 F + S a^3_j + C_j^3 e^3_j}{c_j^3 + c_j^1} \\
F^2_j &= \frac{c_j^2 F + S a^3_j + e^3_j}{c_j^2 + c_j^1} \\
F^3_j &= \frac{c_j^3 F - S a^3_j - e^3_j}{c_j^3 + c_j^1}
\end{align*}
\] (3.8)

However, under some circumstances, the introduction of tooth profile errors will prevent the corner contact of tooth pair #3 during approach and only the tooth pair #2 carries the load. Therefore, whenever a non-positive value is yielded for the load carried by the tooth pair #3 \( F^3_j \leq 0 \):

\[
\begin{align*}
(x_j)_s &= C_j^2 F + e_j^2 \\
F^2_j &= F \\
F^3_j &= 0
\end{align*}
\] (3.9)

Case #2: Tooth pair #3 is not in corner contact, and tooth pair #1 is in corner contact
In this case, elastic deflection causes the tooth pair #1 to remain in contact even after the theoretical end point of engagement. Its load share gradually decreases to zero. Meanwhile, tooth pair #3 does not come into contact. Thus:

\[
\begin{align*}
(x_j)_s &= C^1_j F^1_j + S_{r_j} + e^1_j \\
(x_j)_s &= C^2_j F^2_j + e^2_j \\
F &= F^1_j + F^2_j
\end{align*}
\]  

(3.10)

where \( S_{r_j} \) is the separation distance of tooth pair #1 during recess when tooth pair #2 contacts at point \( j \).

Solving these equations will give:

\[
\begin{align*}
\left( x_j \right)_s &= \frac{c_j^j (c_j^j F + S_{r_j}^j) + c_j^j e_j^j + c_j^j e_j^1}{c_j^j + c_j^1} \\
F^1_j &= \frac{c_j^j F - S_{r_j}^j - e_j^j}{c_j^j + c_j^1} \\
F^2_j &= \frac{c_j^j F + S_{r_j}^j + e_j^j}{c_j^j + c_j^2}
\end{align*}
\]

(3.11)

Similarly, whenever a non-positive value is yielded for the load carried by the tooth pair #1 \((F^1_j \leq 0)\):

\[
\begin{align*}
\left( x_j \right)_s &= C^2_j F + e^2_j \\
F^1_j &= 0 \\
F^2_j &= F
\end{align*}
\]

(3.12)

**Case #3: Tooth pair #3 and tooth pair #1 are both in corner contact**

In this case, elastic deflection causes tooth pair #1 to remain in contact and tooth pair #3 to come into contact earlier. This is a triple contact zone where tooth pair #2 carries most of the load but tooth pairs #1 and #3 are both in contact. Thus:
\[
\begin{align*}
(x_j)_s &= C_j^1 F_j^1 + S r_j^1 + e_j^1 \\
(x_j)_s &= C_j^2 F_j^2 + e_j^2 \\
(x_j)_s &= C_j^3 F_j^3 + S a_j^3 + e_j^3 \\
F &= F_j^1 + F_j^2 + F_j^3
\end{align*}
\]

(3.13)

Solving these equations will give:

\[
\begin{align*}
(x_j)_s &= \frac{c_j^1 c_j^2 F + c_j^1 s r_j^1 + c_j^1 s a_j^1 + c_j^2 c_j^3 e_j^1 + c_j^1 c_j^3 e_j^2}{c_j^1 c_j^2 + c_j^1 c_j^3 + c_j^2 c_j^3} \\
F_j^1 &= \frac{c_j^1 c_j^2 e_j^1 - c_j^2 e_j^1 - c_j^1 e_j^3 - (c_j^1 + c_j^3) s r_j^1 + c_j^3 s a_j^3}{c_j^1 c_j^2 + c_j^1 c_j^3 + c_j^2 c_j^3} \\
F_j^2 &= \frac{c_j^1 c_j^3 e_j^1 - c_j^2 e_j^2 - c_j^2 e_j^2 - c_j^3 s r_j^1 + c_j^3 s a_j^3}{c_j^1 c_j^2 + c_j^1 c_j^3 + c_j^2 c_j^3} \\
F_j^3 &= \frac{c_j^1 c_j^3 e_j^1 - c_j^2 e_j^2 - c_j^2 e_j^2 - c_j^3 s r_j^1 + (c_j^1 + c_j^3) s a_j^3}{c_j^1 c_j^2 + c_j^1 c_j^3 + c_j^2 c_j^3}
\end{align*}
\]

(3.14)

Similarly, whenever a non-positive value is yielded for the load carried by the tooth pair #1 \((F_j^1 \leq 0)\), meaning tooth pair #1 is not in corner contact, then return to Case #1. Whenever a non-positive value is yielded for the load carried by the tooth pair #3 \((F_j^3 \leq 0)\), meaning tooth pair #3 is not in corner contact, then return to Case #2. However, when non-positive values are yielded for both tooth pair #1 and tooth pair #3 \((F_j^1 \leq 0 \text{ and } F_j^3 \leq 0)\):

\[
\begin{align*}
(x_j)_s &= C_j^2 F + e_j^2 \\
F_j^1 &= 0 \\
F_j^2 &= 0 \\
F_j^3 &= 0
\end{align*}
\]

(3.15)

Table 3.1 shows the formulae of the LSTE with corner contact effect \((\langle x_j \rangle_s)\) under various conditions. The unique way to determine whether a tooth pair is in corner contact is through its shared load calculated based on the corresponding equations mentioned above.
Table 3.1: Static transmission error with corner contact effect for a NCR spur gear pair

<table>
<thead>
<tr>
<th>Tooth pair #1</th>
<th>In corner contact</th>
<th>Not in corner contact</th>
</tr>
</thead>
<tbody>
<tr>
<td>In corner contact</td>
<td>[ C_j^3(C_j^1 C_j^3 F + C_j^2 S r_j^1 + C_j^1 S a_j^3) ] + [ C_j^1 C_j^2 e_j^3 + C_j^2 C_j^3 e_j^1 + C_j^1 C_j^3 e_j^2 ] [ C_j^1 C_j^2 C_j^3 + C_j^1 C_j^3 ]</td>
<td>[ C_j^2(C_j^3 F + S a_j^3) ] + [ C_j^2 e_j^3 + C_j^3 e_j^2 ] [ C_j^2 + C_j^3 ]</td>
</tr>
<tr>
<td>Tooth pair #3</td>
<td>Not in corner contact</td>
<td>[ C_j^2(C_j^1 F + S r_j^1) + C_j^1 e_j^2 + C_j^2 e_j^1 ] [ C_j^1 + C_j^2 ]</td>
</tr>
</tbody>
</table>

Table 3.2: Static transmission error with the corner contact effect for a HCR spur gear pair

<table>
<thead>
<tr>
<th>Tooth pair #1</th>
<th>In corner contact</th>
<th>Not in corner contact</th>
</tr>
</thead>
<tbody>
<tr>
<td>In corner contact</td>
<td>[ (\Pi_{p=1}^{n+1} C_j^p) F + \sum_{p=1}^{n+1} \left( (\Pi_{q=1,q\neq p}^n C_j^q) e_j^p \right) ] + [ (\Pi_{q=2}^{n+1} C_j^q) S r_j^1 ] + [ (\Pi_{q=1}^n C_j^q) S a_j^{n+1} ] [ \sum_{p=1}^{n+1} (\Pi_{q=1,q\neq p}^n C_j^q) ]</td>
<td>[ (\Pi_{p=1}^{n+1} C_j^p) F + \sum_{p=2}^{n+1} \left( (\Pi_{q=2,q\neq p}^n C_j^q) e_j^p \right) ] + [ (\Pi_{q=2}^{n+1} C_j^q) S a_j^{n+1} ] [ \sum_{p=2}^{n+1} (\Pi_{q=2,q\neq p}^n C_j^q) ]</td>
</tr>
<tr>
<td>Tooth pair # (n+1)</td>
<td>Not in corner contact</td>
<td>[ (\Pi_{p=1}^n C_j^p) F + \sum_{p=1}^n \left( (\Pi_{q=1,q\neq p}^n C_j^q) e_j^p \right) ] + [ (\Pi_{q=2}^n C_j^q) S r_j^1 ] [ \sum_{p=1}^n (\Pi_{q=1,q\neq p}^n C_j^q) ]</td>
</tr>
</tbody>
</table>

Regarding the HCR gear pair, the loaded static transmission error and the static load shared by a tooth pair can be obtained by following the same procedure described above but will not be detailed here. Table 3.2 shows the formulae of static transmission error with the corner contact effect for a HCR spur gear.
pair. The corner contact happens when the tooth pair \( #(n+1) \) is just about to enter the mesh zone and the tooth pair \#1 just leaves the mesh zone, where \( n \) is nearest integer that is larger than CR.

### 3.2.3 FVMS and VVMS

The FVMS of the \( i \)th tooth pair \( K^i_{Fj} \) at a contact position \( j \), is directly defined as the reciprocal of its equivalent compliance:

\[
K^i_{Fj} = \frac{1}{c^i_j}
\]

Therefore, as with the compliance, FVMS is determined by the gear macro-geometries and the material properties, and independent of the gear error and the static load applied. In other words, FVMS is “fixed” for a given gear tooth pair and totally independent of the gear micro-geometries and the load. Traditionally, the FVMS of the gear pair is defined as the sum of the FVMS of every tooth pair in mesh:

\[
K_{Fj} = \sum_{i=1}^{n} K^i_{Fj}
\]

The VVMS of the gear \( K_{Vj} \) and the \( i \)th tooth pair \( K^i_{Vj} \) at a contact position \( j \) are defined as:

\[
\begin{align*}
K^i_{Vj} &= F^i_j / (x^i_j)_s \\
K_{Vj} &= F / (x)_s
\end{align*}
\]

where \( (x^i_j)_s \) is the loaded static transmission error calculated in the previous section considering corner contact effects. Therefore, unlike FVMS, VVMS is determined not only by the gear macro-geometries and the material properties, but also by the gear micro-geometries and the transmitted load.

It should be noted that \( K_{Fj} \), \( K_{Vj} \), and \( (x^i_j)_s \), are all position-dependent, and normally not constant along the contact positions. This indicates their time-varying characteristics. Hence, they can be also written as \( K_F(t) \), \( K_V(t) \) and \( x_i(t) \).

### 3.2.4 Effect of Toque and Linear Tip Relief on LSTE and VVMS

80
In the following discussion, only linear tip relief will be considered. The spur gear pair described in Table 3.3 will be used for simulation.

Table 3.3: Parameters of the gear pair in [23, 24, 25]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driven gear</th>
<th>Driving gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teeth number (Z_k)</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>Module (m) (mm)</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Face width (W) (mm)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Pressure angle (\alpha)</td>
<td>20°</td>
<td></td>
</tr>
<tr>
<td>Young modulus (E) (N/mm(^2))</td>
<td>(2.06 \times 10^5)</td>
<td></td>
</tr>
<tr>
<td>Pitch diameter (mm)</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>Root diameter (mm)</td>
<td>140.68</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio (\mu)</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Theoretical contact ratio</td>
<td>1.75</td>
<td></td>
</tr>
<tr>
<td>Backlash (2(b)) on line of action (mm)</td>
<td>0.136</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.2 shows the variation of the VVMS, LSR and LSTE of the spur gear pair in one theoretical mesh cycle under different torques when there are no gear profile errors \((e_j = 0)\). It should be noted that there will be no corner contact when the applied torque \(T_1 = 0\) (on driving gear). In this case \((T_1 = 0 \text{ and } e_j = 0)\), the FVMS is the same as the VVMS. Compared with the FVMS (when \(T_1 = 0\)), VVMS is load-dependent, and the abrupt change of mesh stiffness in the transition regions between the single and double mesh zone is smoothed. Since the fluctuation of mesh stiffness can considerably affect the dynamics of gear transmission systems, one can expect that the corner contact effect may play a significant role in gear dynamics, especially for heavily-loaded gear transmission systems.
Figure 3.2: Variation of VVMS, LSR and LSTE under different torques without gear errors

Figure 3.3: Variation of contact ratio versus torque

The LSR of the three meshing tooth pairs (the middle sub-figure of Figure 3.2) shows, in the theoretical single mesh zone, there may be double tooth pairs in mesh, or even triple tooth pairs in mesh (when $T_1 = 340$ Nm), since the corner contact admits the early engagement of the incoming tooth pair and delayed...
contact of the outgoing tooth pair. As a result, the actual tooth contact zone is extended, meaning the contact ratio is increased compared with the theoretical value. The variation of CR when the load increases from 0 to 340 Nm is shown in Figure 3.3. It is obvious that the larger the transmitted load, the more the contact ratio increases. This means an increasing transmitted load will increase the average mesh stiffness (in terms of VVMS), and therefore, increase the natural frequency. This is another significant difference between the FVMS and VVMS models.

![Figure 3.4](image)

**Figure 3.4:** Variation of VVMS, LSR and LSTE under different amounts of tip relief: (a) $T_1 = 0$, (b) $T_1 = 340$ Nm

Figure 3.4 shows the variation of the VVMS, LSR (only for the middle tooth pair #2) and LSTE of the spur gear pair in one theoretical mesh cycle with different amounts of $\Delta$ (just profile modifications, no other types of tooth errors), but the same length ($L_r = 1$) of linear tip relief when the applied torques are 0 and 340 Nm respectively. By comparing Figure 3.4(a) and Figure 3.4(b), one can find that as the amount of tip relief increases, the corner contact effect is gradually reduced. When the tip relief amount exceeds a
certain value (in this case \( \Delta = 1 \)), the corner contact effect can be completely avoided. Chen and Shao [15] pointed out that the existence of corner contact can be estimated by observing whether there is an abrupt ‘jump’ in the transmission regions of mesh stiffness. In most cases, gears are working under varying loads, which means a certain amount of tip relief may be sufficient to smooth the abrupt change of mesh stiffness in the transition region to avoid corner contact for a specific load, but may be insufficient for another load. Therefore, corner contact effect should not be neglected even when a certain amount of tip relief is applied, and the dynamic analysis based on the load-independent mesh stiffness (FVMS) model may not be accurate to predict the gear dynamic behaviour.

### 3.3 Three Types of SDOF Model

![Rotary model of a meshing spur gear pair](image)

**Figure 3.5**: Rotary model of a meshing spur gear pair [26]

Some important effects that should be included in the dynamic model for a gear pair are the time-varying mesh stiffness, the clearance (backlash) and the excitation due to the gear error. In this study, the friction force developed between the gear tooth mesh faces is not considered. However, to account for the
frictionless model additional damping is introduced into the analysis [8, 9, 10, 11], as shown in Figure 3.5 [26]. In this situation the dynamic equations can be written as [15]:

\[
\begin{align*}
J_1 \ddot{\theta}_1(t) &= T_1 - (F(t) + G(t))R_{b1} \\
J_2 \ddot{\theta}_2(t) &= -T_2 + (F(t) + G(t))R_{b2}
\end{align*}
\tag{3.19}
\]

where \( F(t) \) and \( G(t) \) represent the elastic and damping forces during the contact, respectively; \( J_i, \theta_i, T_i, R_{bi} \) are the moment of inertia, angular displacement from the nominal position, the applied torque, and the base radius of the \( i \)th gear \( (i = 1, 2 \) representing the driving and driven gear respectively) as shown in Figure 3.5.

### 3.3.1 FVMS Model

In this model: 1) the gear error \( e(t) \) has no effect on mesh stiffness, and is treated as a displacement excitation, 2) corner contact effect is not considered and have no effect on mesh stiffness. Therefore, the elastic and damping forces can be represented as:

\[
F(t) = hk_x(t) (x(t) - e(t))
\]

\[
G(t) = hc(\dot{x}(t) - \dot{e}(t))
\tag{3.20-a, b, c}
\]

\[
h = [\text{sgn}(x(t) - e(t) - b) + \text{sgn}(x(t) - e(t) + b)]/2
\]

where \( x(t) = R_{b1}\theta_1(t) - R_{b2}\theta_2(t) \) is the dynamic transmission error; \( e(t) \) is the synthetic tooth profile error; \( h \) \( \in \{1, 0, -1\} \) is the tooth contact function that determines drive-side contact (1), contact loss (0), or backside contact (-1); \( c \) is the viscous damping coefficient; and \( b \) is the tooth backlash. Substituting Equations (3.20-a, b, c) into Equation (3.19) can reduce Equation (3.19) to a single degree of freedom model:

\[
m_e \ddot{x}(t) + c \dot{x}(t) + hk_f(t)(x(t) - e(t)) = f_0
\tag{3.21}
\]

where the equivalent mass \( m_e \) and the static load \( f_0 \) are defined as:

\[
m_e = \frac{J_1 J_2}{J_1 R_{b2} + J_2 R_{b1}} , \quad f_0 = \frac{T_1}{R_{b1}} = \frac{T_2}{R_{b2}}
\tag{3.22-a, b}
\]

85
The influence of gear error on the damping term is neglected as they have minimum effect on the response \([9, 10, 11, 12, 25]\). The time-varying mesh stiffness \(k_F(t)\) and the gear error \(e(t)\) can both be considered as periodic and therefore can be expanded in Fourier series:

\[
k_F(t) = k_{F0} + \sum_{i=1}^{\infty} k_{Fi} \cos(2\pi f_m t + \varphi_{Fi}), \quad e(t) = e_0 + \sum_{i=1}^{\infty} e_i \cos(2\pi f_m t + \theta_i) \quad (3.23-a, b)
\]

where \(t\) is the time [s]; \(f_m\) is the fundamental excitation frequency (mesh frequency) [Hz]; \(k_{F0}\) and \(e_0\) are the mean value of \(k_F\) [N/m] and \(e\) [m]; \(k_{Fi}\) and \(\varphi_{Fi}\) are the amplitude and phase of the \(i\)th component of \(k_F(t)\); \(e_i\) and \(\theta_i\) are the amplitude and phase of the \(i\)th component of \(e(t)\). Equation (3.21) can be rewritten in a non-dimensional form [13, 14]:

\[
\ddot{x}(\tau) + 2\zeta \dot{x}(\tau) + \bar{h} k_F(\tau)(\ddot{x}(\tau) - \dot{e}(\tau)) = \ddot{f}_0
\]

where \(\tau\) is the non-dimensional time; \(\zeta\) is the damping ratio; \(\ddot{x}(\tau), \dot{\bar{h}}(\tau), \ddot{e}(\tau), \ddot{f}_0\) and \(\bar{h}\) are the non-dimensional dynamic transmission error, mesh stiffness, gear error, load and contact function respectively. Moreover:

\[
\begin{cases}
\bar{x}(\tau) = \frac{x(\tau)}{b}; \quad \bar{e}(\tau) = \frac{e(\tau)}{b}; \quad \ddot{f}_0 = \frac{f_0}{k_{F0}}; \quad \tau = \omega_n t; \\
\bar{h} = \frac{\text{sgn}(\ddot{x}(\tau) - \dot{\bar{e}}(\tau) - 1) + \text{sgn}(\ddot{x}(\tau) - \dot{\bar{e}}(\tau) + 1)}{2} \\
\bar{k}_F(\tau) = \frac{k_F(\tau)}{k_{F0}} = 1 + \sum_{i=1}^{\infty} \bar{k}_{Fi} \cos(i\Lambda \tau + \varphi_{Fi}) \\
\Lambda = \frac{2\pi f_m}{\omega_n}; \quad \bar{k}_{Fi} = \frac{k_{Fi}}{k_{F0}}; \quad \omega_n = \frac{k_{F0}}{\sqrt{m_c}}; \quad \zeta = \frac{e}{2m_c\omega_n};
\end{cases}
\]

where \(\Lambda\) is the non-dimensional excitation frequency; \(\omega_n\) is the natural frequency of the system.

FVMS model has been widely used in the literature \([1, 2, 8, 9, 10, 11, 12]\). Since FVMS is fixed for a given gear tooth pair, some researchers built the FVMS model based on the individual loads of each tooth pair in order to consider the partial contact loss \([1, 2]\):
\[
\begin{align*}
F(t) &= \sum_{i=1}^{n} h^i k_i^j(t)(x(t) - e_i^i(t)) \\
G(t) &= \sum_{i=1}^{n} h^i c(\dot{x}(t) - \dot{e}_i^i(t)) \\
h^i &= \left[\text{sgn}(x(t) - e_i^i(t) - b) + \text{sgn}(x(t) - e_i^i(t) + b)\right]/2
\end{align*}
\]  

where \(k_i^j(t), e_i^i(t), h^i\) are the individual FVMS, tooth error and contact function for the \(i\)th tooth pair in mesh respectively. \(n\) is the maximum number of tooth pairs in mesh.

### 3.3.2 VVMS Model

Some researchers have used the VVMS of gear pairs to analyse gear dynamics, especially for some FE models [13, 14, 15, 16, 17, 18, 19, 20] used for dynamic analysis. The gear error \(e(t)\), instead of being treated as displacement excitation, was used to evaluate the mesh stiffness of the gear pair under the static analysis. Besides, corner contact effect is normally incorporated in the calculation of mesh stiffness.

According to Equations (3.18), the elastic and damping forces can be represented as:

\[
\begin{align*}
F(t) &= h k_{v^i}(t) x(t) \\
G(t) &= h c \dot{x}(t) \\
h &= \left[\text{sgn}(x(t) - e(t)) + \text{sgn}(x(t) - e(t))\right]/2
\end{align*}
\]  

Still, if we follow the same procedure described in the previous section, we can get the non-dimensional form:

\[
\ddot{x}(\tau) + 2\zeta \dot{x}(\tau) + \ddot{\phi} x(\tau) = \ddot{f}_0
\]  

The VVMS model has been recognized or initiated to some degree in [13, 14, 15, 16, 17, 18, 19, 20]. Kasuba and Evan [21] developed a large scale digitized model extending gear modelling to include the VVMS to analyse spur gearing dynamics. Compared with their approach, the proposed VVMS model is much more concise and easy to solve analytically since it averts the coupling of parametric excitation and displacement excitation. In fact, a generalized solution methodology was introduced in [10] based on the harmonic balance method and Newton-Raphson procedure to analytically solve the differential equation with combined parametric excitation and clearance non-linearity. Another significant merit of the VVMS
model over the FMVS model is that the corner contact effect can be easily incorporated in the dynamic analysis through the VVMS. However, since the VVMS is dependent on gear micro-geometries and transmitted load, the dynamic tooth load division between the individual tooth pairs in mesh is neglected, meaning partial contact loss cannot be simulated through this model.

### 3.3.3 LSTE Model

Both FVMS equations and VVMS equations incorporate the time-varying meshing stiffness acting as a parametric excitation to the gear system. Ozguven and Houser [27], and Cai and Hayashi [28] both introduced a linearly approximated equation, in which a constant mesh stiffness is assumed. However, the self-excitation effect of mesh stiffness variation is indirectly included into this approximate equation by using the loaded static transmission error as the input. The non-dimensional form of this model is:

\[
\ddot{x}(\tau) + 2\zeta\dot{x}(\tau) + \ddot{h} = \ddot{h}_s(\tau)
\]  

(3.29)

In the LSTE model, a constant mesh stiffness assumption with a displacement excitation is used to represent the basic characteristic of actual mesh stiffness. Although they are derived based on the deletion of the alternating component of mesh stiffness, the self-excitation effect of time-varying mesh stiffness is included in the analysis with approximate terms. It has been proved that, under some circumstances, this approximate method based on LSTE can show good agreement with that including time-varying mesh stiffness [27]. Besides, corner contact effect can be easily incorporated (see Section 3.2.2.2). However, the LSTE model includes the fluctuating mesh stiffness only indirectly and also neglects the dynamic tooth load division between the individual tooth pair and partial contact loss [1].

### 3.4 Verification with the Experimental Results

Verification of the analytical method to determine corner contact effect with profile modifications introduced in Section 3.2, and the proposed SDOF models in Section 3.3 is given in this section by comparing the simulation results against experimental results provided in the literature.
Kahraman and Blankenship have conducted a series of experiments to investigate the dynamics of a gear pair with backlash clearance, parametric and internal displacement excitation due to gear manufacturing errors [23, 24, 25]. Their experimental studies guided many modelling efforts and have been used extensively for modal verification. Different sets of spur test gear pairs are considered in the experimental study representing different modification parameters and involute contact ratios. However, some common parameters of the spur gear sets are described in Table 3.3. Their tests were conducted over a speed range from 600 to 4100 rpm which corresponds to a gear mesh frequency \( f_m \) from 500 to 3400 Hz. Applied torque \( T \) was varied from 0 to 340 Nm. The measured DTE values are given in terms of the RMS (root mean square) \( A_{rms} = \sqrt{\sum_{i=1}^{3} A_i^2} \), which include only the first three gear mesh harmonic amplitudes.

### 3.4.1 Corner Contact Effect

In this section, we will research whether corner contact effect affects gear dynamics by comparing the dynamic response predicted by the proposed models to that of the measured value for an unmodified spur gear pair.

![Figure 3.6: Comparison of measured [25] and predicted \( A_{rms} \) versus speed for an unmodified gear pair at 340 Nm](image)
In Figure 3.6, the experimental measured $A_{rms}$ for a gear pair with no modification ($e = 0$) at $T = 340$ Nm [25], are compared to predictions of the three proposed models. For the VVMS and LSTE model, a damping ratio of 0.02 is used as indicated in [25] through the experimental investigation. However, for the FVMS model, such a low damping ratio will yield such an intense response that the double-sided impact (DSI) solution will be obtained in the primary resonance region, which did not happen in the experiment. In order to be consistent with the measured value, Parker et al. [29] suggested a damping ratio of 0.07 for the load-independent mesh stiffness model (FVMS model).

The primary resonance, super-harmonic resonances, softening non-linearity and the classical jump phenomena near resonance regions that appeared in the experimental measured response are also accurately predicted by the proposed models, as shown in Figure 3.6. However, it seems that dynamic response results predicted by the VVMS and LSTE model are more consistent with the experimental measured results compared with those of the FVMS model. This means the load-dependent mesh stiffness model appears more representative of the physical system, and corner contact effect should not be neglected in the gear dynamic analysis.

As a further comparison, Figure 3.7(a) from [30] presents the experimental measured $A_{rms}$ versus $f_m$ for an unmodified gear pair under three different torques $T = 100$, 200, and 300 Nm. The amplitude of the resonance response is noticeably larger for increasing torque. In addition, it appears that the jump frequencies are load-dependent. There is a nearly 20% difference in natural frequency when the torque changes from 100 Nm to 300 Nm in the experiment. Figure 3.7(b), (c) and (d) show the predicted responses from VVMS, LSTE and FVMS model respectively. Compared with the FVMS model, the LSTE and VVMS models agree well with the experimental results in terms of not only the response amplitude but also the load-dependent jump frequencies. One exception is that the difference in natural frequencies between the 100 Nm and 300 Nm torque is not as obvious as the experimental result. However, the load-dependent mesh stiffness model is clearly more appropriate to capture the gear
dynamic behaviour than the load-independent mesh stiffness model, which again shows that corner contact effect should not be neglected in the dynamic analysis. This is contrary to the conclusion drawn in [29] which stated that the expectedly more accurate load-dependent stiffness model yields conflicting results with the experiments. The reason is that Parker et al. [29] used a universal damping ratio of 0.07 in their discrete SDOF models which was too large for the load-independent mesh stiffness model and that the non-linearity behaviour of the gear pair working at high loads was suppressed.

Figure 3.7: Experimentally measured [30] and predicted $A_{rms}$ versus $f_m$ for an unmodified gear under three different torques: (a) Experiment, (b) VVMS model, (c) LSTE model, (d) FVMS model (Note (+) 100, (◊) 200 and (*) 300 Nm)
3.4.2 The Effect of Tip Relief

Figure 3.8 shows the comparison of the experimentally measured $A_1$ (the first harmonic response) from [24] and the predicted $A_1$ over the non-dimensional frequency range $0.6 < A < 1.2$ (this is the frequency range in which $A_1$ dominates the dynamic response) for two tip relieved gear pairs with the same tip relief amount $\delta_m = 10 \mu m (A = 0.62)$ but different tip relief starting angles, $\alpha_m = 22.2$ degrees and $20.9$ degrees ($L_m = 0.92$ and $1.15$). The reason that the comparisons are made based on the non-dimensional frequency $A$ is because the natural frequencies of the gear pairs are dependent on the tip relief and applied torque. Besides, in order to provide valid comparisons among the three SDOF models, the same damping ratio of $0.02$ is used for all the discrete SDOF models.

![Graph](image)

**Figure 3.8:** Experimental measured [24] and predicted $A_1$ versus $A$ for two modified gear pairs with $\delta_m = 10 \mu m$ at $340$ Nm: (a) $\alpha_m = 22.2$ degrees, (b) $\alpha_m = 20.9$ degrees

Compared with the response for the unmodified gear pair (Figure 3.7), the introduction of tip relief can substantially decrease the dynamic response amplitude especially in the primary resonance region when $\alpha_m = 20.9$ degrees. The VVMS and LSTE model can relatively accurately capture the experimentally measured results, whereas the FVMS model presents a much more intense softening nonlinearity than that of the experimental results. A number of computational studies have been done and it was found that the
corner contact effect is the main factor leading to this discrepancy. This proves that the corner contact effect is still in play, and therefore should not be neglected even though a certain amount of tip relief is applied. Besides, the proposed analytical method to calculate the LSTE and VVMS considering corner contact effect with tip relief is valid and can provide relatively consistent results compared with the experimental results. In addition, it seems that the dynamic response predicted by the VVMS model is close to that of the LSTE model in most cases. A detailed analysis will be provided in the next section.

3.5 Comparisons and Discussion

In this section, detailed comparisons of the simulated steady-state responses of the gear pair described in Table 3.3 with varying amounts of linear tip relief by using the proposed 3 SDOF models will be provided. The advantages and disadvantages of each model will be discussed.

3.5.1 Comparisons of the Three SDOF Models

In order to make valid dynamic performance comparisons among these models, all the response curves in this section are based on non-dimensionless frequency $\Lambda$. Figure 3.9 shows the steady-state response in terms of $A_{rms}$ of the DTE versus $\Lambda$ with $\Lambda = 0, 0.2, 0.5, 0.8, 1$ and $1.2$ ($\delta_m = 0, 3.2, 8.0, 12.9, 16$ and $19.2$ $\mu$m) when $L_n = 1$ ($\alpha_m = 21.7$ degrees) at $T_1 = 340$ Nm. The corresponding mesh stiffness curves for each modification amount have already been shown in Figure 3.4(a). Three interesting phenomena can be noticed.

1) The introduction of linear tip relief can significantly affect the dynamic behaviour of the gear pair. In fact, a proper modification can significantly minimize the DTE fluctuation of the gear pair working at a specific design load, and this point has been proved theoretically and also experimentally in many previous published papers.
Figure 3.9: Simulated $A_{rms}$ versus $\Lambda$ for various amounts of tip relief with $L_n = 1$ using 3 models: (a) $\Delta = 0$, (b) $\Delta = 0.2$, (c) $\Delta = 0.5$, (d) $\Delta = 0.8$, (e) $\Delta = 1$, (f) $\Delta = 1.2$
2) As the amount of relief increases, the corner contact effect is gradually alleviated (as shown in Figure 3.4(b)) resulting in the closer and closer agreement between the load-independent mesh stiffness model and the load-dependent mesh stiffness model especially in the non-resonance region. When $\Delta = 1$, there will be no corner contact effect. However, in Figure 3.9(e), there still exist some discrepancies between the FVMS model and VVMS model in the resonance region, which is mainly due to different treatments of tip relief in these two models.

3) Steady-state responses of the VVMS model become closer and closer to those of the LSTE model. Especially when $\Delta = 1$, the amplitudes of the responses given by these two models are entirely the same at each non-dimensioned excitation frequency $\Lambda$. In fact, a simple analytical calculation can prove that the VVMS equation (Equation (3.28)) is approximately equivalent to the LSTE equation (Equation (3.29)) as long as the fluctuation of LSTE (or VVMS) is comparatively small [31].

3.5.2 Discussion

From the comparisons made above, it is obvious that corner contact is still in play when an insufficient amount of tip relief is applied. Therefore, the FVMS model that uses a load-independent mesh stiffness curve may yield inconsistent results compared with those of the VVMS model and LSTE model, especially in the non-resonance region. As the amount of tip relief increases, the corner contact effect is gradually reduced, and the dynamic response predicted by the load-independent mesh stiffness model moves closer to that of the load-dependent mesh stiffness model, which justifies the employing of a load-independent mesh stiffness model in the dynamic analysis for the multi-mesh gear set [1] and the planetary gear set [2] with sufficient amount of profile modifications. However, due to the different treatments of the tooth profile errors, there are still some discrepancies between the FVMS model and VVMS model especially in the resonance regions. The FVMS model presents a much softer nonlinearity in these regions than the VVMS model. One of the biggest advantages of the FVMS model is that it allows for the partial contact loss by dividing the dynamic tooth load into individual loads for each tooth.
pair in mesh as shown in Equations (3.26), which can be called the ITMS (individual tooth mesh stiffness) model. Liu and Parker [1] found that this model best agrees with the FE benchmark for dynamic predictions regardless of different loads, profile modifications and bearings.

It has been shown in this study that the LSTE model, which is initially proposed by Ozguven and Houser [27], is approximately equivalent to the VVMS model proposed in this study as long as the fluctuation of the LSTE (or VVMS) is comparatively small. The LSTE model can easily incorporate corner contact effect into analysis based on the analytical method proposed in this study, which simplifies the problem and provides reasonable estimates of the gear dynamics under certain conditions [1, 27]. However, it neglects the dynamic tooth load division between individual tooth pairs in mesh. Besides, since constant mesh stiffness is used in this model, the Mathieu-Hill type stability characteristic of the system cannot be studied.

The VVMS model proposed in this study can include corner contact effect into the analysis, and shows good agreement with the experimental results. Unlike the FVMS model that treats the tooth profile error as a displacement excitation to the system, the VVMS model includes the tooth profile error in the VVMS during the static analysis. However, like the LSTE model, the dynamic tooth load division between the individual tooth pair in mesh is neglected. Probably, those are the reasons why the dynamic response predicted by the VVMS model still shows some discrepancies with the experimental measured results.

3.6 Conclusions

This chapter proposed an analytical method to calculate the LSTE considering corner contact effect for the spur gear pair with tip relief, based on which the effect of the corner contact on the dynamic response of the spur gear pair has been studied. Two types of the gear mesh stiffness model used in the literature have been differentiated, and three types of commonly used SDOF models (the FVMS model, the VVMS model and the LSTE model) were generalized and their corresponding non-dimensional governing equations were given. Comparisons of the predicted response from the proposed three models with
experimentally measured results provided in literature were made. It was found that the load-dependent mesh stiffness model (VVMS model in this study) yields more consistent results, which proves that corner contact effect should not be neglected in the gear dynamic analysis when no or an insufficient amount of profile modification is applied. Besides, the proposed analytical method used to calculate the LSTE and VVMS considering corner contact effect with tip relief is valid and can provide relatively consistent results compared with experimental results. Detailed comparisons of the steady state responses predicted by these three types of SDOF models were made by introducing different amounts of linear tip relief into the analysis, and the advantages and disadvantages of each model were generalized which can be concluded as:

1) The FVMS model disregards the corner contact effect. However, this model can yield consistent results when a sufficient amount of tip relief is applied. Besides, this model allows for load division between the individual tooth pairs in mesh. Therefore, partial contact loss can be simulated through this analysis.

2) The LSTE model is approximately equivalent to the VVMS model as long as the fluctuation of the LSTE (or VVMS) is comparatively small. Plus, this model can easily incorporate corner contact effect into the analysis. However, partial contact loss and Mathieu-Hill type stability analysis cannot be studied using this model.

3) The VVMS model proposed in this study can include corner contact effect into the analysis, and shows good agreement with experimental results. However, the dynamic tooth load division between individual tooth pairs in mesh is neglected, which may explain some discrepancies between the predicted results from this model and experimental results.

It should be noted that even though this study mainly focuses on the spur gear pair, the conclusions also apply to the other types of gears. The corner contact effect should not be neglected in the gear dynamic analysis when no or insufficient tip relief is applied especially under heavily-loaded conditions.
3.7 References


Chapter 4

The Dynamic Coupling Behaviour due to Gear Eccentricities

4.1 Introduction

Currently, dynamic analysis of gear transmission systems remains an essential and important method to simulate gear dynamic behaviour [1, 2, 3, 4, 5, 6]. It can help researchers to implement suitable solutions to reduce gear vibration and noise. Most early research focused on the modelling of a single spur gear pair supported by flexible shafts and bearings. Ozguven and Houser [7], and Wang et al. [8] have carried out a comprehensive review of mathematical models used in this case, where gear torsional vibrations are the main concern. Although neglecting lateral vibrations might provide a reasonable approximation for systems having shafts with small compliances, the dynamic coupling between the transverse (translational) and torsional vibrations due to the gear mesh affects the system behaviour considerably when the shafts have high compliances (geared rotor systems). In the literature, various formulations are used to represent the shaft flexibilities, ranging from simple equivalent lumped springs to finite elements. As a result, lumped mass and finite element analysis are both widely used to couple the lateral and torsional dynamics typical of geared rotor systems [9, 10, 11]. In some other research, the transfer matrix method is also commonly used to study the flexural-torsional coupling behaviour [12]. However, with the growth of computing power, the finite element method has become dominant as the most efficient and accurate modelling method for rotor dynamics studies. The dynamic investigations of a helical gear pair can be found in a variety of studies, where a three-dimensional (3D) model was normally built to study the complex coupling amongst the transverse, torsional, axial and rotational motions of gears [13, 14].

Gear geometric eccentricity is one of the most common gear mounting errors. Therefore, it should be an important consideration in the process of dynamic simulation. Research regarding it can be found in [9, 10, 11, 12, 13, 14]. Kubur et.al [15] developed a finite element model for spur geared rotor systems and
analysed the influence of geometric eccentricities, mass unbalances, static transmission error, and mesh stiffness variation on the response. Velex and Maatar [14] built a comprehensive 3D model for analyzing the influence of tooth shape deviations and mounting errors on gear dynamics. Their work forms the basis of the 3D gear model used in this research for the gear dynamic analysis with gear eccentric errors. Choi and Mau [12] used the transfer matrix method to study the coupling motion of the lateral and torsional vibration of spur geared rotor systems with geometric eccentricity. Based on the previous work, Zhang et al. [11] proposed a dynamic model of a multi-shaft geared rotor system which consists of a finite element model of shaft structures and a helical gear model with gear eccentricity and static transmission error. The steady-state responses due to these two excitations were studied. In all of the above studies, the inertial force due to gear eccentricity is simplified as an uncoupled standard centrifugal inertial force, whereas the dynamic coupling effect on the inertial force due to gear torsional vibrations is normally ignored.

Therefore, the objective of this chapter is mainly to investigate the role of the coupling terms in the gear eccentricity induced inertial force, and analyse their influence on the dynamic behaviour of the cylindrical geared rotor system. In order to achieve this, a general dynamic model for the helical geared rotor system with local tooth profile errors and global mounting errors was developed. This model includes a finite element (FE) model for the shaft structure based on the work in [10, 11, 12], a lumped-parameter bearing model and a 3D gear mesh model mainly based on work of Velex and Maatar [14]. This combined model was verified firstly by comparing natural frequencies of a one-stage spur geared rotor system against those presented in the literature, and secondly by comparing the simulated dynamic transmission error of a helical gear pair with previous experimental results [15]. Furthermore, the dynamic coupling effect in the gear eccentricity induced inertial force on the gear dynamic behaviour of a helical geared rotor system is intensively investigated, and some conclusions are drawn.
4.2 Dynamic Model

A typical geared rotor system consists of the following elements: (1) shafts, (2) rigid disks, (3) flexible bearings, and (4) gears. When two shafts are not coupled, each gear can be modelled as a rigid disk. However, when they are in mesh, these rigid disks are connected by a spring-damper element representing the mesh stiffness and damping.

4.2.1 Shaft Model

In the literature, the finite element formulation is widely used to model the shafts including either as Euler beam model [10] or Timoshenko beam model [12, 15]. Here, a Timoshenko beam formulation is employed as it can include the effects of the translational and rotary inertia, gyroscopic moments as well as the shear deformation, which are expected to be significant especially in high-speed cases.

![Figure 4.1: The Timoshenko beam element](image)

There are 2 nodes for each Timoshenko beam element (as shown in Figure 4.1), and six degrees of freedom at each node:

\[
\mathbf{u}^e = \{x_A, y_A, z_A, \theta_{xA}, \theta_{yA}, \theta_{zA}, x_B, y_B, z_B, \theta_{xB}, \theta_{yB}, \theta_{zB}\}^T
\]  

(4.1)

The mass \(M^e_{si}\), stiffness \(K^e_{si}\) and gyroscopic \(G^e_{si}\) matrices for the \(i\)th finite shaft element can be found in [12, 15, 16], and will not be detailed here. These matrices are assembled to form the mass \(M^j_s\), stiffness \(K^j_s\) and gyroscopic \(G^j_s\) matrices for the \(j\)th shaft (\(j = 1\) to \(N\) where \(N\) is the number of the shafts considered; \(i = 1\) to \(m_j\) where \(m_j\) is the number of finite elements used to define the \(j\)th shaft). Therefore, the overall shaft mass \(M_S\), stiffness \(K_S\) and gyroscopic \(G_S\) can be expressed as:
\[ M_s = \begin{bmatrix} M^1_s & & \\ & M^2_s & \\ & & \ddots \end{bmatrix}, \quad K_s = \begin{bmatrix} K^1_s & & \\ & K^2_s & \\ & & \ddots \end{bmatrix}, \quad G_s = \begin{bmatrix} G^1_s & & \\ & G^2_s & \\ & & \ddots \end{bmatrix} \] (4.2-a, b, c)

All of these matrices are symmetric and square, of dimension \( q \) where \( q = 6 \sum_{j=1}^{N} (m_j + 1) \) is the total number of degrees of freedom of the system in hand.

### 4.2.2 Bearing Model

Typically, each shaft is supported by at least two rolling element bearings with varying types and parameters. Normally, they are modelled as a stiffness form and the cross terms and damping are ignored [12, 16], that is six spring stiffnesses in the \( x, y, z, \theta_x, \theta_y, \theta_z \) directions:

\[
K^c_b = \begin{bmatrix}
  k_{xx} & k_{yy} & k_{zz} \\
  & k_{\theta_x\theta_x} & k_{\theta_y\theta_y} \\
  & & k_{\theta_z\theta_z}
\end{bmatrix}
\] (4.3)

These bearing stiffness matrices are then assembled to the overall shaft stiffness matrix \( K_s \) at the nodes corresponding to their locations on the shafts.

### 4.2.3 Cylindrical Gear Mesh Model

A general three-dimensional (3D) model of a cylindrical gear pair is shown in Figure 4.2. In this model, the mesh behaviour of the gear pair is represented by two rigid cylinders whose radii are the gear base circles \( R_{b1} \) and \( R_{b2} \), connected by a series of stiffness cells \( k_j \) along the contact line in the direction of the tooth normal determined by the helix angle \( \beta \). \( \Omega_1 \) and \( \Omega_2 \) are the nominal rotating speeds of the driving gear and driven gear, respectively. \( T_1 \) and \( T_2 \) are the toques applied on the driving gear and driven gear, respectively. \( X-Y-Z \) is the global Cartesian coordinate system as defined in the shaft model. The helix angle \( \beta \) of the cylindrical gear pair is defined as:
\[ \beta = \begin{cases} > 0 & \text{if the driving gear has left hand teeth} \\ = 0 & \text{if the driving gear is a spur gear} \\ < 0 & \text{if the driving gear has right hand teeth} \end{cases} \] (4.4)

Figure 4.2: The 3D cylindrical gear mesh model: (a) the local \((U-V-W)\) and global \((X-Y-Z)\) coordinate systems, (b) the 3D gear mesh model, (c) projection drawing in \(W\)-direction, (d) projection drawing in \(V\)-direction

For the purpose of illustration, a localized Cartesian coordinate system \((U-V-W)\) is established (as shown in Figure 4.2(a)). The origin is at the driving gear centre \(O_1\). The \(U\)-axis is in the direction of the line of action (LOA) and positive from the driving gear to the driven gear. The \(V\)-axis points from the driving gear centre \(O_1\) to the tangent point \(A\) at the base circle of the driving gear (off line of action (OLOA)). The \(W\)-axis is along the axial direction and can be determined by following the right-hand rule. It should be
noted that the $U$-$V$ plane is in the same plane with the $X$-$Y$ plane. Therefore, the dynamic motions of the two gear centres consist of three translations and three rotations such that [14, 17]:

$$\{D_i\} = \begin{cases} \ddot{\eta}_i(O_i) = u_i\ddot{U} + v_i\ddot{V} + w_i\ddot{W} \\ \ddot{\omega}_i = \theta_{ui}\ddot{U} + \theta_{vi}\ddot{V} + \theta_{wi}\ddot{W} \end{cases}, \quad i = 1, 2. \quad (4.5)$$

In Figure 4.2(c), the mounting angle $\alpha$ ($0 \leq \alpha < 2\pi$) describes the relative position of the gears, which is defined as the angle that the line connecting the gear centres makes with the positive $X$-axis. Therefore, the angle between the positive $U$-axis and positive $Y$-axis becomes $\psi$. Since the plane of action changes direction depending on the rotation direction of the driving gear, $\psi$ is defined as:

$$\psi = \begin{cases} \alpha - \varphi & \Omega_1: \text{positive } \bar{Z} \text{ direction} \\ \alpha + \varphi - \pi & \Omega_1: \text{negative } \bar{Z} \text{ direction} \end{cases} \quad (4.6)$$

where $\varphi$ is the transverse operating pressure angle of the gear pair. With $\psi$ defined, the transform matrix $T$ between the global coordinate system $X$-$Y$-$Z$ and the local coordinate system $U$-$V$-$W$ can be determined as:

$$T = \begin{bmatrix} -\sin \psi & \cos \psi & 0 \\ \cos \psi & \sin \psi & 0 \\ 0 & 0 & -1 \end{bmatrix} \Omega_1: \text{positive } \bar{Z} \text{ direction}$$

$$\begin{bmatrix} -\sin \psi & \cos \psi & 0 \\ -\cos \psi & -\sin \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \Omega_1: \text{negative } \bar{Z} \text{ direction} \quad (4.7)$$

### 4.2.3.1 Normal Approach

For a contact point $M_j$ with local coordinates $(u_j, R_{bi}, w_j)$, the dynamic motions of the two gear centres will give a normal approach on the base plane relative to rigid body positions:

$$\delta(M_j) = \sum_{i=1}^2 \ddot{\eta}_i(M_j) \ast \ddot{n}_{ij} = \sum_{i=1}^2 \{\ddot{\eta}_i(O_i) + \ddot{\omega}_i \times \ddot{O}_iM_j\} \ast \ddot{n}_{ij} \quad (4.8)$$
where \( \mathbf{n}_{ij} \) is the outer unit vector normal to gear \( i \) at contact point \( M_j \). It is assumed that the base plane is unchanged and the outer normal vector \( \mathbf{n}_{ij} \) is not modified by defects, thus \( \mathbf{n}_{ij} = \mathbf{n}_i \). Besides,

\[
\mathbf{O}_1 M_j = u_j \mathbf{U} + R_{b1} \mathbf{V} + w_j \mathbf{W}, \quad \mathbf{O}_2 M_j = -(u_g - u_j) \mathbf{U} - R_{b2} \mathbf{V} + w_j \mathbf{W}
\]  

(4.9-a, b)

where \( u_g \) is the length of the LOA as shown Figure 4.2(d). Equation (4.8) can be rewritten in matrix form:

\[
\delta(M_j) = E(M_j)q^\top
\]  

(4.10)

where \( q = \{u_1, v_1, w_1, \theta_{x1}, \theta_{y1}, u_2, v_2, \theta_{x2}, \theta_{y2}\} \) is the local degree of freedom vector of the gear pair considered, and \( E(M_j) \) is the so-called structure vector which relates the local degree of freedom vector \( q \) to the normal approach \( \delta(M_j) \), which depends on gear geometry and the position of \( M_j \) on the base plane: namely,

\[
E^\top(M_j) = \begin{bmatrix}
-v_j \cos \beta \\
0 \\
\sin \beta \\
R_{b1} \sin \beta \\
-w_j \cos \beta - u_j \sin \beta \\
R_{b1} \cos \beta \\
\cos \beta \\
0 \\
-\sin \beta \\
R_{b2} \sin \beta \\
w_j \cos \beta - (u_g - u_j) \sin \beta \\
R_{b2} \cos \beta 
\end{bmatrix}
\]  

(4.11)

The normal approach \( \delta(M_j) \) can be also expressed in terms of the global degree of freedom vector:

\[
\delta(M_j) = E(M_j) \Xi u^\top
\]  

(4.12)

where \( u = \{x_1, y_1, z_1, \theta_{x1}, \theta_{y1}, x_2, y_2, z_2, \theta_{x2}, \theta_{y2}\} \) is the global degree of freedom vector of the gear pair considered. \( \Xi \) is the transform matrix:
\[ \mathbf{Z} = \begin{bmatrix} T & T & T & T \end{bmatrix} \] (4.13)

Therefore, the structure vector \( \mathbf{V}(M_j) \) that relates the global degree of freedom vector \( \mathbf{u} \) to normal approach \( \delta(M_j) \) is defined as:

\[ \mathbf{V}(M_j) = E(M_j) \mathbf{Z} \] (4.14)

The contact deflection at \( M_j \) is:

\[ \Delta(M_j) = \delta(M_j) - e(M_j) \] (4.15)

where \( e(M_j) \) is the displacement excitation in the form of the unloaded static transmission error at contact point \( M_j \), which describes the tooth shape deviations relative to the perfect geometry in an unloaded, static condition. The sign convention used for the unloaded static transmission error \( e \) is positively defined in the outer normal direction and \( e \) is supposed to be small enough so that tooth contacts remain on the theoretical base plane [14]. In the local scale, gear tooth undesirable manufacturing errors (flank undulations, pitch errors, etc.), intentional modifications (tip reliefs, lead modifications, etc.) and defects (wear, spalls, etc.) are the main sources of the unloaded static transmission error. In the global scale, gear mounting errors (misalignments, eccentricities, etc.) will also contribute.

4.2.3.2 Unloaded Static Transmission Error

(a) Local scale errors

Gear tooth local profile errors and defects can adversely affect gear dynamic behaviour. It will not only lead to a sudden increase of the vibration and noise level, but also a shortened service life of the machine. On the other hand, it is well known that suitable tooth profile modifications can significantly reduce gear vibrations [2, 4].
In most of the previous research work employing a two-dimensional (2D) gear model, gear unloaded static transmission error excitation $e$ is expressed as a simple harmonic excitation as a function of time. However, for a 3D model, it is a function of the contact point $M_j$ which depends not only on time but also on the contact position along the tooth face width direction. A detailed discussion can be found in [14]. In this study, gear tooth local scale errors are denoted as $e_{fl}(M_j)$ and $e_{f2}(M_j)$ for the driving gear and driven gear, respectively (as shown in Figure 4.2(d)). The overall profile error of the gear pair $e_f(M_j)$ is defined as the sum of $e_{fl}(M_j)$ and $e_{f2}(M_j)$.

(b) Global scale errors

Figure 4.3: Gear mounting errors: (a) misalignments, (b) eccentricities

Two typical types of mounting errors of a gear pair are misalignments and eccentricities. Misalignments of gears are modelled by small constant angles of $\theta_{x1}^m, \theta_{y1}^m$ relative to the X-axis and the Y-axis [14] as shown in Figure 4.3(a). Therefore, the corresponding normal deviations in $M_j$ due to misalignments can be expressed as:

$$e_m(M_j) = V^m(M_j) \mathbf{u}^m$$  \hspace{1cm} (4.16)
where $V^m(M_j)$ is a subset of $V(M_j)$ which includes only the components related to bending slopes $\theta_{zi}$ and $\theta_{zi}^n$, and $u^m$ is the misalignment vector:

$$u^m = \{\theta_{x1}^m, \theta_{y1}^m, \theta_{x2}^m, \theta_{y2}^m\}^T \quad (4.17)$$

Eccentricity of an $i$th gear is defined as the distance $e_i$ between the centre of rotation and the centre of inertia (as shown in Figure 4.3(b)) with an initial phase angle $\theta_i^e$ with regard to the $X$-axis. Therefore, the perturbations along the $X$-axis and $Y$-axis due to eccentricities will be $e_i^x$ and $e_i^y$, namely:

$$\begin{cases}
e_i^x = e_i \cos(\Omega_it + \theta_{zi} + \theta_i^e) \\
e_i^y = e_i \sin(\Omega_it + \theta_{zi} + \theta_i^e)
\end{cases} \quad (4.18)$$

where $\theta_{zi}$ is the rotational perturbation of the $i$th gear along the $Z$-axis. According to geometric relationships in Figure 4.3(b), the corresponding normal deviation at $M_j$ due to eccentricities can be expressed as:

$$e_e(M_j) = \cos\beta \cdot \{\sin(\Omega_1t + \theta_{z1} + \theta_1^e - \psi) \sin(\Omega_2t + \theta_{z2} + \theta_2^e - \psi}\} \{e_1 \quad e_2\} \quad (4.19)$$

(c) **Total error**

The total error at the contact point $M_j$ is the sum of the abovementioned individual errors, namely:

$$e(M_j) = e_f(M_j) + e_f^2(M_j) + e_m(M_j) + e_e(M_j) \quad (4.20)$$

### 4.2.3.3 Equations of Motion

Lagrange’s equations were used in [14] to derive the un-damped equations of motion for the gear pair, namely:

$$M_g \ddot{u} + (K_p + K_g(t,u))u = F_0 + F_1(t) + F_2(t,u) \quad (4.21)$$

where:
\[ M_g = \text{diag}(m_1, m_1, l_1, l_1, l_p, m_2, m_2, l_1, l_2, l_p) \]

\[ K_b = \text{diag}(K_{b1}^e, K_{b2}^e) \]

\[ K_g(t, u) = \sum_{j=1}^{N_{eq}} k_j V(M_j) V(M_j)^T \]

\[ F_0^e = \{0, 0, 0, 0, 0, T_1, 0, 0, 0, 0, 0, T_2\} \]

\[ F_1^e(t) = \{-m_1 \ddot{e}_x^1, -m_1 \ddot{e}_y^1, 0, 0, 0, -m_2 \ddot{e}_x^2, -m_2 \ddot{e}_y^2, 0, 0, 0, 0\} \]

\[ F_2(t, u) = \sum_{j=1}^{n_c} k_j e(M_j) V(M_j) \quad (4.22-\text{a, b, c, d, e, f}) \]

where \( m_i, l_i, l_p \) are the mass, transverse moment of inertia and the polar moment of inertia of the \( i \)th gear, respectively \((i = 1, 2\) representing the driving and driven gear respectively); \( K_b \) is the constant bearing stiffness matrix; \( K_{b1}^e \) and \( K_{b2}^e \) are the constant stiffness matrices corresponding to the bearings at the driving gear and driven gear respectively; \( k_j \) is the cell stiffness at the contact point \( M_j \); \( K_g(t, u) \) is the nonlinear gear stiffness matrix which depends not only on time \( t \), but also on \( M_j \); \( F_0^e \) is the force vector produced by the nominal input and output torques; \( F_1^e(t) \) is the force excitation coming from the centrifugal and tangential inertial force due to gear eccentricities; \( F_2(t, u) \) is the force excitation due to the unloaded static transmission error, which is also time and position dependent.

**4.2.4 Global Equations of Motion**

Normally, a geared rotor system consists of the following part: shafts, bearings and gear pairs. Therefore, the overall system mass, stiffness, damping and gyroscopic matrices may be readily assembled by placing pure translational and rotational vibrations diagonally and the matrices of the coupled vibrations off-diagonally. The equation of motion of the geared rotor system is:

\[ M\ddot{u} + (C + G)\dot{u} + K(t, u)u = \tilde{F}_0 + \tilde{F}_1(t) + \tilde{F}_2(t, u) \quad (4.23) \]

where \( M \) and \( G \) are the overall system mass matrix and gyroscopic matrix respectively, including the mass of shafts and gear pairs; \( K \) is the overall system stiffness matrix, including the stiffness of shafts, bearings and gear pairs; \( \tilde{F}_0, \tilde{F}_1(t) \) and \( \tilde{F}_2(t, u) \) are extensions of \( F_0, F_1(t) \) and \( F_2(t, u) \) to the total amount
of DOF (completed by zeros). $C$ is the viscous damping matrix of the system, which is normally defined as Rayleigh-type damping [12], namely,

$$C = \alpha_c M + \beta_c K$$  (4.24)

where

$$\alpha_c = \frac{2(\omega_2 \zeta_1 - \omega_1 \zeta_2) \omega_1 \omega_2}{(\omega_2^2 - \omega_1^2)} , \; \beta_c = \frac{2(\omega_2 \zeta_1 - \omega_1 \zeta_2)}{(\omega_2^2 - \omega_1^2)}$$  (4.25)

where $\omega_1$ and $\omega_2$ are the first and second un-damped natural frequencies (rad/s) respectively, and $\zeta_1$ and $\zeta_2$ are the first and second modal damping ratios, respectively.

4.2.4.1 Averaged Structure Vector

The parameters $K(t, u)$ and $\bar{F}_2(t, u)$ in Equation (4.23) are time dependent, and also contact position dependent, which makes the solving of this equation complex. A simplification can be made as we noted that most of the components of $V(M_i)$ are independent of the position of contact $M_i$ except those related to bending slopes $\theta_{xi}$ and $\theta_{yi}$. Their influence is usually disregarded especially for narrow-faced gears [18]. Besides, in order to get a constant structure vector $V_0$, the contributions of bending angles are usually averaged over one mesh period [19, 20]. Therefore,

$$K_g(t, u) = \sum_{i=1}^{N_c(t, u)} k_i V(M_j)V(M_j)^T = V_0 V_0^T \sum_{i=1}^{N_c(t, u)} k_j = V_0 V_0^T k_g(t, u)$$  (4.26)

where

$$V_0 = \frac{1}{T_s W_0} \iint V(M_j)dt dz$$  (4.27)

is the averaged structural vector, $T_s$ is the mesh period and $W_0$ is the tooth face width along the axial direction (as shown in Figure 4.2); $N_c(t, u)$ is the number of stiffness cells $k_j$ considered; $k_g(t, u)$ is the time-varying, nonlinear, mesh stiffness of the gear pair, which can be acquired either by finite element
analysis (FEA) methods or by analytical methods [18, 21, 22, 23]. In this study, a much simpler method to predict gear mesh stiffness based on the ISO standard 6336 will be used. An important simplification brought by the ISO formulae is that the mesh stiffness per unit contact length $k_0$ is considered as approximately constant so that the following approximation can be used:

$$k_g(t, u) = k_0 L(t, u)$$

(4.28)

where $L(t, u)$ is the time-varying (possibly non-linear) contact length [24]. Accurate expressions to derive $k_0$ can be found in the ISO standard. The analytical expressions for the time-varying contact length $L(t)$ in perfect cylindrical gears can be found in [25].

### 4.2.4.2 Dynamic Coupling Terms

When Equation (4.18) is substituted into Equation (4.22-e), the inertial forces caused by the $i$th gear eccentricity are:

$$\begin{align*}
F^c_{x_i} &= -m_i \ddot{e}_i \Omega_i^2 \cos(\Phi_i) + 2m_i e_i \Omega_i \dot{\theta}_{zl} \cos(\Phi_i) + m_i e_i \dot{\theta}_{zl}^2 \cos(\Phi_i) + m_i e_i \dot{\theta}_{zl} \sin(\Phi_i) \\
F^c_{y_i} &= -m_i \ddot{e}_i \Omega_i^2 \sin(\Phi_i) + 2m_i e_i \Omega_i \dot{\theta}_{zl} \sin(\Phi_i) + m_i e_i \dot{\theta}_{zl}^2 \sin(\Phi_i) + m_i e_i \dot{\theta}_{zl} \cos(\Phi_i)
\end{align*}$$

(4.29)

where $\Phi_i = \Omega_i t + \theta_{zl} + \dot{\theta}_i$. If Equations (4.29) are expanded:

$$\begin{align*}
F^c_{x_i} &= m_i e_i \Omega_i^2 \cos(\Phi_i) + 2m_i e_i \Omega_i \dot{\theta}_{zl} \cos(\Phi_i) + m_i e_i \dot{\theta}_{zl}^2 \cos(\Phi_i) + m_i e_i \dot{\theta}_{zl} \sin(\Phi_i) \\
F^c_{y_i} &= m_i e_i \Omega_i^2 \sin(\Phi_i) + 2m_i e_i \Omega_i \dot{\theta}_{zl} \sin(\Phi_i) + m_i e_i \dot{\theta}_{zl}^2 \sin(\Phi_i) + m_i e_i \dot{\theta}_{zl} \cos(\Phi_i)
\end{align*}$$

(4.30)

According to Equations (4.30), the inertial force due to gear eccentricity is constituted by 4 terms [26]. The first term $m_i e_i \Omega_i^2 \cos(\Phi_i)$ in $F^c_{x_i}$ (or $m_i e_i \Omega_i^2 \sin(\Phi_i)$ in $F^c_{y_i}$) is the standard centrifugal inertial force, whereas the latter 3 terms are dynamic coupling terms as the torsional vibrations ($\theta_{zl}$) come into play. However, these terms have usually been neglected in previous work [10, 11, 12] since they are usually smaller in magnitude than the standard centrifugal inertial force and the corresponding translational
inertial force $m_i \ddot{x}_i$ (or $m_i \ddot{y}_i$). Therefore, the inertial forces due to $i$th gear eccentricity can be simplified as:

$$
\begin{align*}
F^e_{xi} &\approx m_i e_i \Omega_i^2 \cos(\Phi_i) \\
F^e_{yi} &\approx m_i e_i \Omega_i^2 \sin(\Phi_i)
\end{align*}
$$

(4.31)

Although in most cases, Equations (4.31) provide a good simplification to account for the inertial forces excited by the gear eccentricities, the dynamic coupling terms may play an important role if several criteria are satisfied. For example, when a resonance related to the gear rotational vibration $\dot{\theta}_z$ happens and is excited by the gear mesh frequency, its derivatives $\ddot{\theta}_z$ and $\dddot{\theta}_z$ may be so large that the dynamic coupling terms in Equations (4.30) are commensurate with the standard centrifugal inertial force or the corresponding translational inertial force. In this case, the dynamic coupling terms should not be neglected. Several parameters can influence the dynamic coupling terms and they will be discussed in the following section.

4.3 Validation of the Model

In this section, two examples are chosen to validate the FE modelling technique and the proposed cylindrical gear mesh model respectively. In the first example, the natural frequencies of a one-stage spur geared rotor system are compared with previous results by using either the FE methods [10, 12] or the transfer matrix method [11]. In the second example, the simulated dynamic transmission error of a helical gear pair is directly compared with previous experimental results [15].

4.3.1 Case #1
Figure 4.4: One stage geared rotor bearing system from [10]

A one-stage spur geared rotor bearing system from [10] is shown in Figure 4.4. The values of the main system parameters are listed in Table 4.1. It is assumed that the spur gear system has a constant mesh stiffness $k_g = 1.0 \times 10^8$ N/m, and the bearings are isotropic with stiffness $k_{xx} = k_{yy} = 1.0 \times 10^9$ N/m.
Table 4.1: Parameters of the geared rotor system [10, 12]

<table>
<thead>
<tr>
<th>Segment</th>
<th>Outer diameter $d_o$ (mm)</th>
<th>Inner Diameter $d_i$ (mm)</th>
<th>Length $l$ (mm)</th>
<th>Outer Diameter $d_o$ (mm)</th>
<th>Inner Diameter $d_i$ (mm)</th>
<th>Length $l$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>37</td>
<td>0</td>
<td>127</td>
<td>37</td>
<td>0</td>
<td>127</td>
</tr>
<tr>
<td>2</td>
<td>37</td>
<td>0</td>
<td>127</td>
<td>37</td>
<td>10</td>
<td>127</td>
</tr>
</tbody>
</table>

**Shaft dimensions (mm)**

<table>
<thead>
<tr>
<th>Gear</th>
<th>$m$ (kg)</th>
<th>$I$ (kg*m$^2$)</th>
<th>$I_p$ (kg*m$^2$)</th>
<th>Teeth number $N$</th>
<th>Pitch diameter $R$ (m)</th>
<th>Mesh stiffness $k_g$ (N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driving gear</td>
<td>1.84</td>
<td>0.0009</td>
<td>0.0018</td>
<td>28</td>
<td>0.0445</td>
<td>$1.0 \times 10^8$</td>
</tr>
<tr>
<td>Driven gear</td>
<td>1.84</td>
<td>0.0009</td>
<td>0.0018</td>
<td>28</td>
<td>0.0445</td>
<td></td>
</tr>
</tbody>
</table>

**Gear parameters**

<table>
<thead>
<tr>
<th>Bearing</th>
<th>$k_{xx}$ (N/m)</th>
<th>$k_{yy}$ (N/m)</th>
<th>$k_{zz}$ (N/m)</th>
<th>$k_{\phi_0\phi_0}$ (N*m/rad)</th>
<th>$k_{\phi\phi_0}$ (N*m/rad)</th>
<th>$k_{\phi\phi_0}$ (N*m/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>$1.0 \times 10^9$</td>
<td>$1.0 \times 10^9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Material parameters**

<table>
<thead>
<tr>
<th>Modulus of elasticity $E$ (Gpa)</th>
<th>Density of material $\rho$ (kg/m$^3$)</th>
<th>Poisson’s ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>207.8</td>
<td>7806.0</td>
<td>0.3</td>
</tr>
</tbody>
</table>

The first 13 natural frequencies and their corresponding mode descriptions of the system are listed in Table 4.2. Results from some other researchers are also listed for comparison. It is obvious that these sets of results are in reasonably good agreement, which validates the FE modelling techniques. The Campbell diagrams for this system with ($k_g = 1.0 \times 10^8$ N/m) and without ($k_g = 0$ N/m) the gear mesh are shown in Figure 4.5. It is observed that the first and third lateral modes in the LOA direction of both the driving gear and the driven gear in the non-geared system are replaced by the corresponding coupled lateral-torsional modes in the geared system. Besides, these coupled lateral-torsional natural frequencies remain comparatively constant regardless of the rotating speed, whereas the lateral natural frequencies, especially
the 2\textsuperscript{nd} order, are sensitive to the rotational speed. Some of these frequencies increase as the operating speed increases, while the others decrease as a result of the increase in the rotational speed. These phenomena are consistent with the conclusions draw by previous works [9, 11].

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Natural frequencies (Hz)</th>
<th>Mode description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>559.57</td>
<td>580.92</td>
</tr>
<tr>
<td>2</td>
<td>675.08</td>
<td>686.91</td>
</tr>
<tr>
<td>3</td>
<td>676.67</td>
<td>688.98</td>
</tr>
<tr>
<td>4</td>
<td>678.24</td>
<td>691.05</td>
</tr>
<tr>
<td>5</td>
<td>2516.11</td>
<td>2524.04</td>
</tr>
<tr>
<td>6</td>
<td>3296.55</td>
<td>3386.98</td>
</tr>
<tr>
<td>7</td>
<td>3296.55</td>
<td>3386.98</td>
</tr>
<tr>
<td>8</td>
<td>3327.48</td>
<td>3421.04</td>
</tr>
<tr>
<td>9</td>
<td>3327.59</td>
<td>3421.04</td>
</tr>
<tr>
<td>10</td>
<td>6066.69</td>
<td>6447.05</td>
</tr>
<tr>
<td>11</td>
<td>6080.97</td>
<td>6539.04</td>
</tr>
<tr>
<td>12</td>
<td>6133.85</td>
<td>6830.93</td>
</tr>
<tr>
<td>13</td>
<td>6157.50</td>
<td>6840.00</td>
</tr>
</tbody>
</table>

4.3.2 Case #2
Table 4.3: Design parameters of the helical gear pair used in [15]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driving gear</th>
<th>Driven gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teeth number</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>Transverse module ( m ) (mm)</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Transverse pressure angle ( \varphi ) (°)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Helix angle ( \beta ) (°)</td>
<td>25.323</td>
<td></td>
</tr>
<tr>
<td>Face width ( L ) (mm)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Mesh stiffness per unit contact length ( k_a ) (N/m²)</td>
<td>1.2 \times 10^{10}</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Other gear, bearing and shaft parameters for simulation

<table>
<thead>
<tr>
<th>Gear</th>
<th>Addendum modification coefficient when CR = 1.4</th>
<th>Addendum modification coefficient when CR = 1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>-0.1</td>
<td>-0.21</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shaft</th>
<th>( l_1 ) (mm)</th>
<th>( l_2 ) (mm)</th>
<th>( l_3 ) (mm)</th>
<th>( d_1 ) (mm)</th>
<th>( d_2 ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>100</td>
<td>100</td>
<td>50</td>
<td>37.6</td>
<td>43</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bearing</th>
<th>( k_{xx} ) (N/m)</th>
<th>( k_{yy} ) (N/m)</th>
<th>( k_{zz} ) (N/m)</th>
<th>( k_{\theta x\theta x} ) (N*m/rad)</th>
<th>( k_{\theta y\theta y} ) (N*m/rad)</th>
<th>( k_{\theta y\theta y} ) (N*m/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>( 1 \times 10^8 )</td>
<td>( 1 \times 10^8 )</td>
<td>( 0.5 \times 10^6 )</td>
<td>( 1 \times 10^6 )</td>
<td>( 1 \times 10^6 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

A one-stage helical geared rotor system was used for the experimental study in [15]. The main design parameters of the test helical gear pair are listed in Table 4.3. Several separate gear pairs of the same design parameters were considered in their study. Gear teeth surfaces were ultra-precisely ground and no surface modifications were applied. Gear mounting errors were negligible. Therefore, the unloaded static transmission of the gears considered can be assumed as 0. Several separate gear pairs of the same design parameters were considered in [15]. The main difference among them is that they have different outside...
diameters resulting in different involute contact ratios (CR). The detailed information of the helical geared rotor system is not given in [15]. In this study, some other necessary parameter values for simulation are listed in Table 4.4. Their values are chosen in order to make the simulated results consistent with the experimental results.

**Figure 4.6**: Schematic of the experimental helical gear system in [15]

**Figure 4.7**: Comparisons of the simulated and measured DTE of the gear pair: (a) $CR = 1.4$ at 150 Nm, (b) $CR = 1.6$ at 250 Nm
Figure 4.7(a) presents the comparisons between the simulated DTE forced response based on the gear mesh model proposed in this study and the measured values for the helical gear pair with $CR = 1.4$ at 150 Nm. Similarly, Figure 4.7(b) shows the comparisons for the helical gear pair with $CR = 1.6$ at 250 Nm. It can be observed that the simulated values agree well with the measured data in both cases, especially in the range near the primary resonance speed. Therefore, the proposed 3D gear mesh model is effective to predict the dynamic behaviour of cylindrical gear pairs.

4.4 Effect of Dynamic Coupling Terms

In order to effectively illustrate the role of the dynamic coupling terms in the dynamic behaviour of the gear pair subjected to eccentricities, the transverse responses in two directions (LOA and OLOA) of the driving gear with and without the inclusion of dynamic coupling terms are simulated using the Matlab ODE45 subroutine. This simulation is performed on the helical geared rotor systems as described in Table 4.3 and Table 4.4. Gear profile error and eccentricities are introduced. The nominal values of some main parameters are described in Table 4.5. It should be noted that the rotational direction of the driving gear and the value of the gear mounting angle are set to fulfil a specific condition where the global $Y$-axis coincides with the local $U$-axis ($\psi = 0$), and the global $X$-axis coincides with the local $V$-axis.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driving gear</th>
<th>Driven gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gear profile error $e_f$ (mm)</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>gear teeth number $N$</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Helix angle $\beta$ (degrees)</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Pressure angle $\phi$ (degrees)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Magnitude of eccentricities (mm)</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>Initial phase of eccentricities (rad)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.5: Nominal values of some main parameters
The root mean square values (RMS) of the transverse responses are employed to characterize relative difference $\Gamma$ between the responses with and without the inclusion of the dynamic coupling terms in the two directions:

\[
\Gamma_U = \frac{U'_{RMS} - U_{RMS}}{U_{RMS}}, \quad \Gamma_V = \frac{V'_{RMS} - V_{RMS}}{V_{RMS}}
\]  

(4.32)

where $\Gamma_U$ and $\Gamma_V$ are the relative difference in the LOA direction and OLOA direction respectively. $U'_{RMS}$ and $V'_{RMS}$ are the transverse responses with consideration of the dynamic coupling terms, whereas $U_{RMS}$ and $V_{RMS}$ are the responses without consideration of the dynamic coupling terms. The larger the relative difference, the more influence of the dynamic coupling terms have on the transverse responses.

![Figure 4.8](image)

**Figure 4.8:** Relative difference between the transverse responses of the driving gear with and without the dynamic coupling terms in the LOA and OLOA direction: (a) original, (b) close-up

Figure 4.8 shows the speed sweep of the relative difference between the transverse responses of the driving gear in the two directions. The relative difference in the LOA direction is nearly 0 at any speed, which means that the dynamic coupling terms have no influence on the transverse response in the LOA direction. However, in the OLOA direction, there are two resonance peaks in the low speed region, which is because the mesh frequency of the driving gear is closer to the natural frequencies corresponding to the first and second transverse modes ($f_1$ and $f_2$). It is clear from Figure 4.8 that the dynamic coupling
behaviour between the transverse motion and the rotational motion exists mainly in the OLOA direction when the gears are running in the low speed region. In the following section, the influence of some parameters on the gear dynamic coupling is investigated.

4.4.1 Gear Overall Profile Error

![Graphs showing transverse dynamic response](image)

**Figure 4.9:** Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various profile error cases ($e_f = 0, 10, 20, 50, 100 \mu m$) when rotating speed is 420 rpm: (a) original, (b) close-up

![Graphs showing transverse dynamic response](image)

**Figure 4.10:** Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various profile error cases ($e_f = 0, 10, 20, 50, 100 \mu m$) when rotating speed is 2400 rpm: (a) original, (b) close-up
Figure 4.10 shows similar comparisons when the rotating speed is 2400 rpm. It demonstrates that the dynamic coupling between gear transverse and rotation motions becomes weaker as the rotating speed increases. This happens because in a relatively higher rotating speed range, instead of the mesh frequency components, the rotational frequency components in the rotational motion are excited.

4.4.2 Number of Teeth

Figure 4.11 shows the comparisons of the dynamic response of the driving gear in the OLOA direction ($V_1$ or $X_1$) among various number of gear teeth cases. Note that the speed ratio of the gear pair is 1, meaning that the two gears have the same number of teeth. In Figure 4.11(a), the dynamic coupling when the number of teeth is 30 is the strongest. This is because the mesh frequency in this case ($420 \times 30 / 60 = 210$ Hz) is closer to the first natural frequency $f_1$, which is about 201 Hz. The dynamic coupling when the number of teeth is 80 is also stronger. This is due to the same reason that the mesh frequency in this case ($420 \times 80 / 60 = 560$) is closer to the eighth natural frequency $f_8$, which is about 560 Hz. Figure 4.11(b) shows the zoomed-in comparisons among the remaining three case where the resonance is not obviously excited. It is found that the number of teeth has some influence on the dynamic coupling between gear transverse and rotational motions. The larger the number of teeth, the stronger the coupling. The reason is that in the low speed range, the mesh frequency components are dominant in the rotational vibration, that is $\theta = \Theta \sin(N \Omega t + \Phi)$, where $\Theta$, $\Omega$ and $\Phi$ are the amplitude (rad), nominal rotation frequency (rad/s) and the phase (rad) respectively. Therefore, the rotational velocity and acceleration are dependent on the number of teeth $N$. A higher $N$ will lead to a larger dynamic coupling force as shown in Equations (4.30). It can be also concluded that in the middle or high speed range where the mesh frequency components are not dominant, this dependence on $N$ will gradually disappear.
Figure 4.11: Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various tooth number cases when rotating speed is 420 rpm: (a) original, (b) close-up

4.4.3 Gear Helix Angle

Figure 4.12: Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various helix angle cases when rotating speed is 420 rpm: (a) original, (b) close-up

Figure 4.12 shows the comparisons of the dynamic response of the driving gear in the OLOA direction ($V_1$ or $X_1$) among various helix angle cases. Figure 4.12(b) demonstrates that the helix angle of the
cylindrical gear pair researched does have some influence on the dynamic coupling among gear transverse and rotational motions, and the smaller the helix angle, the stronger the coupling. This phenomenon can be explained by noting that gear helix angle will directly determine the magnitude of fluctuation of gear mesh stiffness during operation. A smaller helix angle means a more abrupt change of the number of tooth pairs in mesh, and as a result a more severe variation of gear mesh stiffness, which excites stronger mesh frequency components in the gear rotational motions.

4.4.4 Gear Eccentricity and Initial Phase

Figure 4.13: Comparisons of the transverse dynamic response in the OLOA direction of driving gear \( (V_1 \text{ or } X_1) \) among various magnitudes of driving gear eccentricity \( e_1 \) when rotating speed is 420 rpm: (a) original, (b) close-up

Figure 4.13 shows the comparisons of the dynamic response of the driving gear in the OLOA direction \( (V_1 \text{ or } X_1) \) among various magnitudes of driving gear eccentricity cases. It was found that a larger magnitude of eccentricity will excite larger magnitudes of both the standard vibration response (uncoupled vibration response) and the dynamic coupling of the driving gear, and it seems that the dynamic coupling remains “fixed” with regard to the standard vibration response. It was revealed that, although the magnitude of eccentricity alone can directly affect the magnitude (large or small) of the
dynamic coupling, it has negligible influence on the level (strong or weak) of this coupling with regard to the standard vibration response.

Figure 4.14 shows the comparisons of the dynamic response of the driving gear in the OLOA direction ($V_1$ or $X_1$) among various driven gear eccentricity cases. The results show that driven gear eccentricity, including the magnitude $e_2$ and initial phase $\theta_2$, have no influence on the driving gear vibration as well as the dynamic coupling, in the OLOA direction. In other words, the standard vibration and the dynamic coupling in the OLOA direction of one gear are independent of its counterpart’s eccentricity. Figure 4.15 shows the comparisons of the dynamic response of the driven gear in the OLOA direction ($V_2$ or $X_2$) among various driven gear eccentricity cases. The initial phase $\theta_2$ will affect the phase of the standard vibration (the big oscillation at the rotating frequency as shown in Figure 4.15) but has no influence on the dynamic coupling in the OLOA direction.

Figure 4.14: Comparisons of the transverse dynamic response in the OLOA direction of driving gear ($V_1$ or $X_1$) among various driven gear eccentricity $e_2$ and $\theta_2$ when rotating speed is 420 rpm: (a) original, (b) close-up

To sum up, the standard vibration and the dynamic coupling in the OLOA direction of one gear are independent of its counterpart’s eccentricity. The magnitude of one gear’s eccentricity alone can directly
affect the magnitude of the dynamic coupling, but it has negligible influence on the level (strong or weak) of this coupling with regard to its standard vibration response. The initial phase will directly affect the phase of the standard vibration but has no influence on the dynamic coupling in the OLOA direction.

![Graph](image)

**Figure 4.15:** Comparisons of the transverse dynamic response in the OLOA direction of driven gear ($V_2$ or $X_2$) among various driven gear eccentricity $e_2$ and $\theta_e$ when rotating speed is 420 rpm: (a) original, (b) close-up

### 4.5 Conclusions

In this chapter, a general dynamic model for the cylindrical geared rotor system with local tooth profile errors and global mounting errors was developed based on previous work. This model includes an FE model for the shaft structure, a lumped parameter bearing model and a 3D gear mesh model. The verification of this combined model was conducted by first comparing the natural frequencies of a one-stage spur geared rotor system with previous results by using either the FE method or the transfer matrix method. In addition to that, the simulated dynamic transmission error of a helical gear pair is directly compared with previous experimental results, which proves that this combined model is capable of simulating the dynamic response of a cylindrical geared rotor system. Finally, an intensive parameter study was performed on this model to research the influence of various parameters including gear overall
profile error, number of teeth, helix angle and gear eccentricities on the dynamic coupling behaviour between gear transverse and rotational motions, which can be concluded as follows:

1) The dynamic coupling term, which was neglected in previous work regarding dynamic response of a cylindrical gear pair subjected to eccentricities, is obvious in the OLOA direction when a gear pair is running in the low-speed range where the mesh frequency components are dominant in the dynamic response. In the middle and high speed range where the rotating frequency components are dominant, this dynamic coupling term is small compared with the standard centrifugal inertial force and can be completely neglected.

2) Gear overall profile error, number of teeth and helical angle directly affect the dynamic coupling behaviour in the OLOA direction. The larger the profile error and number of teeth, the stronger the dynamic coupling. The helix angle will adversely affect the dynamic coupling.

3) The standard vibration and the dynamic coupling in the OLOA direction of one gear are independent of its counterpart’s eccentricity. A larger magnitude of one gear’s eccentricity can excite a larger magnitude of the standard vibration and dynamic coupling of that gear, therefore making the dynamic coupling “fixed” with regard to the standard vibration. The initial phase of one gear’s eccentricity directly affects the phase of the standard vibration but has no influence on the dynamic coupling at all.

4.6 References


Chapter 5

Influence of the Addendum Modification on Spur Gear Back-side Mesh

Stiffness and Dynamics

5.1 Introduction

Gear back-side tooth impact may happen under light load or idling conditions due to the inevitable existence of gear backlash. This will often cause a significant increase of gearbox noise which can appear either in the driving condition (“rattle” noise) or in the neutral condition (“idle” noise) [1, 2, 3]. As the restrictions on the noise annoyance problem from gearboxes become increasingly tight, gear rattle or idle noise is more and more a concern for car manufacturers and gear researchers.

Gear rattle or idle noise is often associated with the back-side tooth impact (or contact) in mesh, which refers to the contact on the tooth surfaces that are not intended to transmit power. For a better understanding, it is necessary to build a model that accurately describes the back-side contact mesh stiffness. Although there is a significant amount of research in the literature on mesh stiffness variation and gear dynamics, little has been reported regarding the back-side mesh stiffness variation and its influence on gear dynamics. The time-varying symmetric mesh stiffness model that assumes identical mesh stiffness variation in both the forward (drive-side) and backward (back-side) contact direction was usually used in previous research work [3, 4, 5, 6, 7, 8, 9]. This model ignores the phase shift between the drive-side and back-side mesh stiffness. To cope with this challenge, the time-varying asymmetric mesh stiffness model was proposed by Guo et al. [1] and Chen et al. [2]. They have derived analytical equations to calculate the back-side mesh stiffness based on the drive-side mesh stiffness respectively. However, their methods were only applicable to standard spur gears. For the spur gear with addendum modification, the phase shift between the drive-side and back-side mesh stiffness will be affected.
Addendum modification is done by increasing or decreasing the height of the tooth addendum. The amount by which the addendum is increased or decreased is known as “Addendum Modification” [10, 11, 12, 13]. Gear teeth may have modified addenda in order to avoid undercut, to balance the bending stresses in the gear, or to vary the relative amounts of approach and recess action. It is achieved by shifting the generating rack outward or inward of the material of the generated gear. Addendum modified gears have different sizes of tooth shape, which may influence the phase shift between the drive-side and back-side mesh stiffness. As a result, a more general equation is required, which constitutes one of the focuses of this study.

The effect of tooth mesh stiffness asymmetric nonlinearity on the dynamics of hypoid gears has been intensively studied by Wang and Lim [14]. Specifically, the effects of asymmetric mesh stiffness parameters including mean mesh stiffness ratio, mesh stiffness variation and mesh stiffness phase shift on the dynamic mesh force were examined. It should be noted that in their work, the asymmetric characteristic is introduced by the inherent curvilinear tooth shape and pinion offset in a hypoid set. However, in this study, the research objective is a spur gear which has exactly the same mean mesh stiffness and variation in forward and backward mesh behaviour. The asymmetric characteristic is introduced by the addendum modification which can alter the phase shift between the drive-side and back-side mesh stiffness. Therefore, addendum modification may affect gear back-side tooth impact and gear dynamic behaviour whenever the double-sided tooth impact happens through the back-side mesh stiffness. This constitutes another focus of this study.

5.2 Derivation of the Time-varying Asymmetric Mesh Stiffness Model

The drive-side mesh stiffness is the stiffness of contacting teeth at a mesh when running in the forward direction along the line of action (LOA) [1]. It varies as the number of contacting tooth pairs fluctuates with gear rotation. The mesh stiffness can be determined either by using finite element tools or semi-empirical analytical expressions.
Similarly, the back-side mesh stiffness is defined as the stiffness of the contacting teeth at a mesh along the backward direction of the LOA. The average of the back-side mesh stiffness is identical to that of the drive-side mesh stiffness for a spur gear pair. However, their instantaneous values at any contacting point except the symmetric position (as shown in Figure 5.1(c)) are normally different. For example, in Figure 5.1(a), when an incoming tooth of the driving gear contacts with its counterpart at initial mesh point A on the LOA, there are simultaneously 2 contacting tooth pairs in the forward direction while only 1 contacting tooth pair in the backward direction. In Figure 5.1(b), when a tooth pair contacts at the pitch point B or the symmetric position, the contact points are on the forward LOA, and the contact points are on the backward LOA. Therefore, the instantaneous mesh stiffnesses are different. In Figure 5.1(c), the pitch point C and the symmetric point S are on the forward LOA, and the pitch point B and the symmetric point S are on the backward LOA. The instantaneous mesh stiffnesses at these points are different.
point \( P \), there is only 1 contacting tooth pair in the forward direction and 2 contacting tooth pairs in the backward direction. These phenomena demonstrate that the mesh stiffness function of a spur gear pair is asymmetric in the two directions. In most cases, especially when there is actually no contact between the back-side surfaces of the tooth pair, this asymmetry in mesh stiffness has no influence on the gear dynamic behaviour. But when the back-side contact happens and this asymmetry is too strong to be negligible, it is necessary to calculate the backside mesh stiffness and take it into consideration.

### 5.2.1 Standard Gear Pair

We first investigate a standard gear pair that has no addendum modifications on both gears. At \( t = 0 \), the incoming tooth pair begins to contact at the initial mesh point \( A \). At this instant, the number of tooth pairs in mesh suddenly changes from 1 to 2 (if the contact ratio is between 1 and 2). Suppose the drive-side mesh stiffness function and back-side mesh stiffness function are \( k(t) \) and \( k_b(t) \) respectively. At \( t = t_p \), the mesh point of the above-mentioned contacting tooth pair moves to the pitch point \( P \). At \( t = t_s \), the gear pair is in a specific mesh position such that both gears are symmetric about the centreline (suppose the influence of backlash is negligible), and the number of tooth pairs in contact along the drive-side LOA equals that along the back-side LOA [1]. This means that the back-side mesh stiffness function \( k_b(t) \) should equal the drive-side mesh stiffness function \( k(t) \) at this moment \( (t = t_s) \), and the drive-side mesh stiffness function after this moment is identical to the back-side mesh stiffness function before this moment. At \( t = t_d \), the tooth pair loses contact at the end point \( D \) and leaves the contact zone. We define a new time variable \( \tau \), and at \( \tau = 0 \), the tooth pair makes a contact at the symmetric point \( S \). Therefore, the mesh stiffness functions satisfy:

\[
k_b(\tau) = k(-\tau) \tag{5.1}
\]

Inserting \( \tau = t - t_s \) into Equation (5.1) gives:

\[
k_b(t - t_s) = k(-t + t_s) \tag{5.2}
\]
Therefore:

\[ k_b(t) = k(-t + 2t_s) \quad (5.3) \]

**Figure 5.2:** Geometric relationship of the various critical mesh positions

Equation (5.3) illustrates the relationship between two mesh stiffness functions in terms of \( t \), which starts at the initial mesh point \( A \). \( t_s \) is the amount of time spent by the contacting tooth pair that moves from the initial mesh point \( A \) to symmetric mesh point \( S \) along the drive-side LOA. Supposing the rotating speed of the driving gear \( \omega_1 \) remains constant, thus:

\[ t_s = \frac{AS}{R_{b1} \omega_1} \quad (5.4) \]

where \( AS \) is the distance between point \( A \) and \( S \) along the drive-side LOA, and \( R_{b1} \) is the base radius of the driving gear. According to the geometric relationship illustrated in Figure 5.2:

\[ AS = TP - AP + PS \quad (5.5) \]

and,

\[ TP = R_{b1} \times \tan \alpha, \text{ and } AP = R_{b1} \times \theta_A \quad (5.6) \]
where \( \alpha \) is the pressure angle at the pitch point, and \( \theta_A \) is the initial mesh angle as illustrated in Figure 5.2. Therefore, the amount of time spent by the contacting tooth pair moving from the initial mesh point \( A \) to pitch point \( P \) is:

\[
t_p = R_{b1} \frac{(\tan \alpha - \theta_A)}{\omega_1} = \frac{\tan \alpha - \theta_A}{\omega_1} \tag{5.7}
\]

\( \overline{PS} \) is the distance between pitch point \( P \) and the symmetric point \( S \) along the LOA. According to Figure 5.2, the amount of time spent by the contacting tooth pair moving from pitch point \( P \) to the symmetric point \( S \) along the LOA is equivalent to that of the driving gear rotating from \( P' \) to \( S' \) along the pitch circle, thus,

\[
t_{ps} = \frac{\overline{PS'}}{R_1 \omega_1} \tag{5.8}
\]

where \( R_1 \) is the pitch radius of the driving gear, and \( \overline{PS'} \) is the arc length along the driving gear pitch circle, which is actually half of the circular tooth thickness at the pitch circle. For a standard gear pair:

\[
\overline{PS'} = \frac{p_1}{4} = \frac{2 \pi R_1}{2 Z_1} = \frac{2 \pi}{2 Z_1}
\]

where \( p \) is the driving gear circular pitch, and \( Z_1 \) is the number of driving gear teeth. Substituting Equations (5.5) - (5.9) into Equation (5.4) gives:

\[
t_s = t_p + t_{ps} = R_{b1} \frac{(\tan \alpha - \theta_A)}{R_{b1} \omega_1} + \frac{p_1}{2 \omega_1} = \frac{\tan \alpha - \theta_A}{\omega_1} + \frac{\pi}{2 Z_1 \omega_1} \tag{5.10}
\]

It should be noted that for a standard gear pair, the driving gear will rotate one circular pitch \( p \) in one mesh period \( T_s \). Thus Equation (5.10) can be simplified as:

\[
t_s = \frac{\tan \alpha - \theta_A}{\omega_1} + \frac{T_s}{4} \tag{5.11}
\]
In the above discussion, we assume that the influence of backlash on the mesh stiffness functions is negligible. Detailed discussion about this can be found in [1].

Figure 5.3 shows the time-history mesh stiffness plot of the gear pair considered in Figure 5.1, where the drive-side and back-side mesh stiffness at various critical points (point A, P, S and D as shown in Figure 5.1) are illustrated. Note that the drive-side mesh stiffness after the point $S$ is symmetrical to the back-side mesh stiffness before the point $s$.

**Figure 5.3**: Drive-side and back-side mesh stiffness function versus time $t$ and for $r$ unmodified gear pair

### 5.2.2 Addendum Modified Gear Pair

A profile shift or addendum modification can be made to have an influence on tooth shape and tooth thickness [10]. It is one of the most important and influential values for tooth profile [11]. Positive addendum modification will lead to higher load capacity of each tooth but decreased contact ratio (poor transmission). On the contrary, negative modification will lead to higher contact ratio but poor load capacity of each tooth. A suitable choice of the addendum modification can balance these two factors, and
guarantee good load capacity for each tooth without losing too much transmission reliability. The addendum modification of the tooth profile is defined as the product of the addendum modification coefficient value $x$ and the gear pair module $m$. Supposing $x_1$ and $x_2$ represent the addendum modification coefficient of the driving gear and the driven gear respectively. In practice, $x$ is normally selected between -1 and 1. Figure 5.4 illustrates the tooth profiles of the driving gear with various addendum modification coefficient values ($x_1 = -0.5, 0$ and $0.5$). It can be seen that, except for the radii of the pitch circle and base circle, all the other radii of a gear (root circle and addendum circle) with addendum modification are different from those of a gear without addendum modification. Since the outside radii (addendum radii) of a gear pair affect the initial contact point $A$ and final contact point $D$ along the LOA, gear pairs with different amounts of addendum modifications will have different contact ratio, initial mesh angle $\theta_A$ and drive-side and back-side mesh stiffness functions compared with those for the standard gear pairs without addendum modification ($x = 0$).

On the other hand, tooth thickness at the pitch circle is also significantly changed. Compared with the standard gears without addendum modification, gears with a positive addendum modification coefficient ($x > 0$) have a larger tooth thickness at the pitch circle. Similarly, gears with a negative addendum modification coefficient ($x < 0$) have a smaller tooth thickness at the pitch circle. According to Equation (5.8), tooth thickness at the pitch circle determines the time span ($t_{ps}$) for a tooth pair moving from the pitch point to the symmetric point along the LOA. Therefore, addendum modification can directly affect the phase shift of the back-side mesh stiffness function with respect to the drive-side mesh stiffness function.
In this chapter, we consider the most general case of addendum modification, where the sum of the addendum modification coefficients for the driving gear and driven gear is 0 ($x_1 + x_2 = 0$). This is the most widely used addendum modification strategy as the gear pair can operate on the same standard centre distance as the unmodified gear pair. The pitch circles of both gears are concentric with their reference circles. The half of the circular thickness at the pitch circle of the driving gear will be:

$$\bar{P}'S' = \frac{\pi m}{4} + x_1 m \tan \alpha$$  \hspace{1cm} (5.12)

Compared with the Equation (5.9) for the unmodified gear, there is an additional term $x_1 m \tan \alpha$ that reflects the influence of the addendum modification. Figure 5.5 shows mesh stiffness functions for different amounts of addendum modified gear pairs. Compared with the unmodified gear pair (as shown in Figure 5.3), there are two main changes. One is the distance of symmetric point S relative to the initial point A. When $x_1 = -0.5$, $x_2 = 0.5$, the distance is longer than that of an unmodified gear pair, and when $x_1 = 0.5$, $x_2 = -0.5$, the distance is shorter than that of an unmodified gear pair. This is because an increased addendum of a driven gear results in an earlier contact of the incoming tooth pair (smaller $\theta_A$), and a decreased addendum of driven gear results in late contact of the incoming tooth pair (larger $\theta_A$). Similarly, the addendum of the driving gear will determine the position of the end point D along the LOA ($\theta_D$). The second change is the distance between the point S and P. This phenomenon is well explained by Equation
(5.12), that a positively addendum modified gear has a larger circular thickness at the pitch circle than that of an unmodified gear, and a negatively addendum modified gear has a smaller circular thickness at the pitch circle.

Figure 5.5: Drive-side and back-side mesh stiffness function versus time $t$ and $\tau$ for addendum modified gear pairs:
(a) $x_1 = -0.5, x_2 = 0.5$, (b) $x_1 = 0.5, x_2 = -0.5$
5.3 Dynamic Simulation

5.3.1 Acceleration Excitation on the Driving Gear

In this section, the research objective is a spur gear pair system as shown in Figure 5.6. The gear set is supported by identical flexible bearings which can be modelled as linear springs and viscous dampers. The gear pair is modelled as rigid cylinders connected by a spring and a damper representing the time-varying mesh stiffness and damping respectively. It is assumed that the gear set maintains only in-plane motions. This means that all motions occur within the two-dimensional plane. A localized Cartesian coordinate system (U-V-W) is established. The origin is at the driving gear centre. The U-axis is in the direction of the LOA, and the V-axis is in the off line of action (OLOA) direction. The W-axis is along the axial direction and can be determined by following the right-hand rule. Therefore, the motion of the gear set can be defined with six coordinates:

\[ \mathbf{q}^T = \{ u_1, v_1, \theta_{w1}, u_2, v_2, \theta_{w2} \} \]  

(5.13)
According to [15, 16], the corresponding structure vector $E$ that relates the local degree of freedom vector $q$ to the normal approach $\delta$ (transmission error) is:

$$E^T = \{-1, 0, R_{b1}, 1, 0, R_{b2} \} \quad (5.14)$$

Such that:

$$\delta = E^T q \quad (5.15)$$

In Equation (5.14), $R_{b1}$ and $R_{b2}$ are the base radii of the driving gear and driven gear respectively. The gear tooth contact force model proposed by Chen et al. [2] and Dion et al. [3] is used in this study. Depending on the transmission error $\delta$ compared to the unloaded static transmission error $\varepsilon$ (tooth local profile errors or shape deviations) and the backlash $2\eta$, the explicit expressions of the elastic force $F_k$ and damping force $F_c$ are:

① When the gears are in contact on the tooth back-side $F-$: $\delta < -\varepsilon-\eta$:

$$F_k = k_p(t) \cdot (\delta + \eta) \cdot \left| \frac{\delta + \eta}{2\eta} \right|^{\Delta-1} \quad , \quad F_c = c_p(t) \cdot \dot{\delta} \quad (5.16-a, b)$$

② When the gears are near contact on the tooth back-side $F-$: $-\varepsilon-\eta \leq \delta < -\eta$:

$$F_k = \frac{k_p(t)}{2} \cdot \left( 1 - \left( \frac{\varepsilon}{2} + \delta + \eta \right) \right) \cdot \sin \left( \frac{\pi}{\varepsilon} \right) \cdot (\delta + \eta) \cdot \left| \frac{\delta + \eta}{2\eta} \right|^{\Delta-1}$$

$$F_c = \frac{c_p(t)}{2} \cdot \left( 1 - \left( \frac{\varepsilon}{2} + \delta + \eta \right) \right) \cdot \sin \left( \frac{\pi}{\varepsilon} \right) \cdot \dot{\delta} \quad (5.17-a, b)$$

③ When the gears are separated: $-\eta \leq \delta < \eta$:

$$F_k = 0 \quad , \quad F_c = 0 \quad (5.18-a, b)$$

④ When the gears are near contact on the tooth drive-side $F+$: $\eta \leq \delta < \eta+\varepsilon$: 


\[ F_k = \frac{k(t)}{2} \left( 1 - \left( \frac{\varepsilon}{2} + \delta - \eta \right) \right) \sin \left( \frac{\pi}{\varepsilon} \right) \left( \delta - \eta \right) \frac{\delta \eta}{2\eta}^{\Delta - 1} \]
\[ F_c = \frac{c(t)}{2} \left( 1 - \left( \frac{\varepsilon}{2} + \delta - \eta \right) \right) \sin \left( \frac{\pi}{\varepsilon} \right) \delta \quad (5.19\text{-a, b}) \]

5) When the gears are in contact on the tooth drive-side \( F^+ : \eta + \varepsilon \leq \delta \):

\[ F_k = k(t) \left( \delta - \eta \right) \frac{\delta \eta}{2\eta}^{\Delta - 1}, \quad F_c = c(t) \delta \quad (5.20\text{-a, b}) \]

where \( \eta \) is half of the nominal backlash existing in the gear pair; \( c_b(t) \) and \( c(t) \) are the time-varying damping coefficients of the contact on the back-side and drive-side respectively, \( \Delta \) is the contact-type coefficient (3/2 for point contact and 10/9 for line contact). The actual contact of a tooth pair is neither linear nor punctual. A value of 4/3 is suggested by Dion et al. [3] to minimize the discrepancy between the experimental measures and numerical simulations. The actual tooth contact force is the sum of the elastic force and damping force:

\[ F_t = F_k + F_c \quad (5.21) \]

Therefore, the equations of motion governing the vibrations of the 6 DOF gear model can be represented by:

\[ m_1 \dddot{u}_1 + c_{b1} \dddot{u}_1 + k_{b1} u_1 - F_t = 0 \]
\[ m_1 \dddot{v}_1 + c_{b1} \dddot{v}_1 + k_{b1} v_1 = 0 \]
\[ I_{p1} \dddot{\theta}_{w1} + c_{s1} \dddot{\theta}_{w1} + F_t R_{b1} = T_1 \]
\[ m_2 \dddot{u}_2 + c_{b2} \dddot{u}_2 + k_{b2} u_2 + F_t = 0 \]
\[ m_2 \dddot{v}_2 + c_{b2} \dddot{v}_2 + k_{b2} v_2 = 0 \]
\[ I_{p2} \dddot{\theta}_{w2} + c_{s2} \dddot{\theta}_{w2} + F_t R_{b2} = T_2 \quad (5.22\text{-a, b, c, d, e, f}) \]

where \( m_1, I_{p1}, m_2 \) and \( I_{p2} \) are the mass and polar moment of inertia of the driving gear and driven gear respectively; \( c_{b1} \) and \( c_{b2} \) are the radial damping coefficients of the bearings supporting the driving gear and driven gear respectively; \( k_{b1} \) and \( k_{b2} \) are the radial stiffness of the bearings; \( c_{s1} \) and \( c_{s2} \) are the torsional damping coefficients accounting for the dissipative behaviour introduced by lubricant between the gears.
and shafts; \( T_1 \) and \( T_2 \) are the torques applied on the driving gear and driven gear respectively. The above equations can also be expressed in matrix form which is easier to be programed in Matlab for solving:

\[
M \ddot{q} + C_B \dot{q} + K_B q + F_T E = F_0
\]  

(5.23)

where:

\[
M = \text{diag}(m_1, m_1, I_{p1}, m_2, m_2, I_{p2})
\]

\[
C_B = \text{diag}(c_{B1}, c_{B1}, c_{s1}, c_{B2}, c_{B2}, c_{s2})
\]

\[
K_B = \text{diag}(k_{B1}, k_{B1}, 0, k_{B2}, k_{B2}, 0)
\]

\[
F_0^T = \{0, 0, T_1, 0, 0, T_2\}
\]

(5.24-a, b, c, d)

Equation (5.23) was solved in Matlab software by using the ODE 45 subroutine. The design parameters of the spur gear system for simulation are shown in Table 5.1. It should be noted that the parameters regarding the bearings and gear-shaft interface are cited from [2], whereas the parameters regarding the gear pair are mainly from [7, 8, 9].

**Table 5.1: Design parameters of the spur gear system for simulation**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driven gear</th>
<th>Driving gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth ( Z_i )</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>Backlash (2( \eta )) on line of action (( \mu )m)</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Module ( m ) (mm)</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Pressure angle ( \alpha_0 ) (degree)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Mass ( m_i ) (kg)</td>
<td>2.52</td>
<td></td>
</tr>
<tr>
<td>Mass moment of inertia ( I_{pi} ) (kg( \cdot )m(^2))</td>
<td>7.4\times10(^{-3})</td>
<td>7.4\times10(^{-3})</td>
</tr>
<tr>
<td>Bearing radial stiffness ( K_B ) (N/m)</td>
<td>6.56\times10(^7)</td>
<td>6.56\times10(^7)</td>
</tr>
<tr>
<td>Bearing damping coefficient ( C_B ) (N( \cdot )s/m)</td>
<td>1.8\times10(^5)</td>
<td>1.8\times10(^5)</td>
</tr>
<tr>
<td>Damping of gear-shaft interface ( C_{si} ) (N( \cdot )m( \cdot )s/rad)</td>
<td>0.025</td>
<td>0.025</td>
</tr>
</tbody>
</table>
Figure 5.7: Gear mesh stiffness for various addendum modification cases: (a) $k(t)$, (b) $k_b(t)$ when $x_1 = -0.5$, (c) $k_b(t)$ when $x_1 = -0.3$, (d) $k_b(t)$ when $x_1 = 0$, (e) $k_b(t)$ when $x_1 = 0.3$, (f) $k_b(t)$ when $x_1 = 0.5$

Drive-side mesh stiffness and back-side mesh stiffness in one mesh period for various addendum modification cases ($x_1 = -0.5$, -0.3, 0, 0.3 and 0.5) are displayed in Figure 5.7. They are obtained by using analytical models developed in [17, 18, 19, 20, 21, 22]. When calculating the drive-side and back-side gear mesh stiffness, it is assumed that the vibration angular motions are small in comparison to the nominal angular motions. Consequently, the contact position on the LOA depends only on the nominal angular motions.

The friction forces developed between the gear tooth mesh faces are not considered. However, additional damping is introduced to account for the frictionless model [8, 9]. The suggested value of $\varepsilon$ is 1.0 in [2, 3]. The correctness of the tooth contact force model has already been verified in [3] by comparing the simulated results against experimental measured results. The objective of this section is to simulate the gear dynamic responses with various addendum modification cases when an angular acceleration
excitation is applied on the driving gear so that the mesh tooth pair is forced to contact at both sides, in exactly the same manner as the tests in [3]:

\[ \ddot{\theta}_{p1} = A_{e1} \sin(2\pi f_{e1} t) \]  

(5.25)

where \( A_{e1} \) and \( f_{e1} \) are the excitation amplitude and frequency applied on the driving gear respectively. In this study, the simulation is performed when \( A_{e1} \) is 141.4 rad/s\(^2\) (100 rad/s\(^2\) RMS) and \( f_{e1} \) is 20 Hz.

Figure 5.8 shows the simulated dynamic transmission error (DTE) for various addendum modification cases, which shows similar features as observed in previous experimental and simulated results [2, 3]. There are several tooth impacts in both the drive-side (single-sided tooth impact or contact) and back-side direction (double-sided tooth impact or contact). This is because the first tooth impact is not strong enough to push the tooth to the other side. More impacts happen before the contact direction changes. Figure 5.8(a) and (b) describe the DTE when the nominal rotating speed of driving gear is zero, and the initial mesh position when applying the angular acceleration excitation is at \( t = 0 \). This means the drive-side and back-side mesh stiffness remain constant (time-invariant mesh stiffness model, \( k(0) \) and \( k_b(0) \)) as we assume that vibration angular motions are negligible compared to the nominal angular motions. It was found that the amplitudes and intervals of the tooth impacts in the drive-side direction for various addendum modifications are all the same. However, in the back-side direction, the tooth impacts when \( x_1 = -0.5 \) and 0.3 are slightly different with those of others. This phenomenon is expected as \( k_b(0) \) when \( x_1 = -0.5 \) and 0.3 (Figure 5.7(b) and (e)) are at minimum, \( K_{\text{min}} \), whereas the others are at the maximum, \( K_{\text{max}} \).
Figure 5.8: Simulated gear dynamic transmission error for various addendum modification cases under input angular acceleration excitation ($A_{e1} = 141.4$ rad/s$^2$, $f_{e1} = 20$ Hz, $\omega_1 = 0$ and 3600 rpm)

Compared to the time-invariant mesh stiffness model, there are more distinct differences in the tooth impacts among various addendum modification cases when the rotating speed of the driving gear is nonzero, which can be observed in Figure 5.8(c)-(d). This is consistent with the conclusion made by Chen et al. [2] that the time-varying mesh stiffness model and the phase shift between the drive-side and back-
side mesh stiffness induce more abundant high frequency components. It was also found that the amplitudes and frequencies of the tooth impacts for various addendum modification cases are almost similar in the drive-side direction, whereas significantly different in the back-side direction. In other words, the addendum modification has negligible influence on the drive-side tooth impacts, whereas significant effects on the back-side tooth impacts. This is expected as the addendum modification has slight influence on the drive-side mesh stiffness $k(t)$. However, it will alter the phase shift ($2t_i$ in Equation (5.3)) of the back-side mesh stiffness $k_b(t)$ with respect to $k(t)$. Another phenomenon is that the back-side tooth impacts at $x_1 = -0.5$ are similar to those at $x_1 = 0.3$. This is due to the similar shape of $k_b(t)$ between $x_1 = -0.5$ and $x_1 = 0.3$ as shown in Figure 5.7(b) and (e). Likewise, the tooth impacts at $x_1 = 0.5$ and $x_1 = -0.3$ share similar amplitudes and intervals which is also caused by the similar shape of $k_b(t)$ between these two cases as shown in Figure 5.7(c) and (f).

Figure 5.9 shows the simulated dynamic mesh force (DMF) for various addendum modification cases when the rotating speed is 0 and 3600 rpm respectively. It can be found that the addendum modification has little influence on the DMF generated during the drive-side tooth impacts, whereas in the back-side, the DMFs for various cases are slightly different from each other in terms of the amplitudes, frequencies and positions.

Figure 5.10 shows the simulated driving gear lateral displacement (LOA direction, $u_1$) for various addendum modification cases when the rotating speed is 0 and 3600 rpm respectively. It reveals similar results that the displacements for various addendum modification cases are nearly identical when the gear tooth pair is meshing in the drive-side direction, whereas different in the back-side direction. This demonstrates that the addendum modification does not affect gear lateral response unless the back-side tooth impact happens. Therefore, the time-varying asymmetric mesh stiffness model should be used to analyse gear dynamic performance whenever the back-side tooth impact happens [2].
Figure 5.9: Simulated gear dynamic mesh force for various addendum modification cases under input angular acceleration excitation ($A_{e1} = 141.4 \text{ rad/s}^2, f_{e1} = 20 \text{ Hz}, \omega_1 = 0 \text{ and } 3600 \text{ rpm}$)
5.3.2 Speed Sweep under Constant Light Load

In this section, we will build a single degree of freedom (SDOF) model to simulate gear dynamic response during speed-increasing and speed-decreasing sweeps to capture the effect of the asymmetric mesh stiffness model on the classic jump phenomenon. The torque applied on the gear system is held constant and there is no external angular acceleration excitation. The time-varying mesh stiffness serves
as the only internal excitation to the system. A typical SDOF model can be found in [8, 9], which is usually expressed in terms of the dynamic transmission error $\delta(t)$:

$$m_c\ddot{\delta}(t) + F_c(t) + F_k(t) = F_0$$

(5.26)

where the equivalent mass $m_c$ and the static load $F_0$ are defined as:

$$m_c = \frac{i_{p1}i_{p2}}{i_{p1}R_{b2} + i_{p2}R_{b1}}, \quad F_0 = \frac{T_1}{R_{b1}} = \frac{T_2}{R_{b2}}$$

(5.27-a, b)

$F_c(t)$ and $F_k(t)$ are the damping force and elastic force respectively. Instead of using the complex tooth contact force model developed in [2, 3], a simple model based on the symmetric backlash type function is considered without loss of generality [8, 9]:

$$F_c(t) = \begin{cases} c(t)\left(\dot{\delta}(t) - \dot{\epsilon}(t)\right) & \delta(t) - \epsilon(t) \geq \eta \\ 0 & |\delta(t) - \epsilon(t)| < \eta \\ c_b(t)\left(\dot{\delta}(t) - \dot{\epsilon}(t)\right) & \delta(t) - \epsilon(t) \leq -\eta \end{cases}$$

(5.28)

$$F_k(t) = \begin{cases} k(t)(\delta(t) - \epsilon(t) - \eta) & \delta(t) - \epsilon(t) \geq \eta \\ 0 & |\delta(t) - \epsilon(t)| < \eta \\ k_b(t)(\delta(t) - \epsilon(t) + \eta) & \delta(t) - \epsilon(t) \leq -\eta \end{cases}$$

(5.29)

For simplicity, the dimensionless form of Equation (5.26) can be obtained:

$$\ddot{\delta}(\bar{t}) + \bar{F}_c(\bar{t}) + \bar{F}_k(\bar{t}) = \bar{F}_0$$

(5.30)

where the dimensionless damping force and elastic force function is:

$$\bar{F}_c(\bar{t}) = \begin{cases} 2\zeta\left(\dot{\delta}(\bar{t}) - \dot{\epsilon}(\bar{t})\right) & \delta(\bar{t}) - \epsilon(\bar{t}) \geq 1 \\ 0 & |\delta(\bar{t}) - \epsilon(\bar{t})| < 1 \\ 2\zeta_b\left(\dot{\delta}(\bar{t}) - \dot{\epsilon}(\bar{t})\right) & \delta(\bar{t}) - \epsilon(\bar{t}) \leq -1 \end{cases}$$

(5.31)

$$\bar{F}_k(\bar{t}) = \begin{cases} \bar{k}(\bar{t})(\delta(\bar{t}) - \epsilon(\bar{t}) - 1) & \delta(\bar{t}) - \epsilon(\bar{t}) \geq 1 \\ 0 & |\delta(\bar{t}) - \epsilon(\bar{t})| < 1 \\ \bar{k}_b(\bar{t})(\delta(\bar{t}) - \epsilon(\bar{t}) + 1) & \delta(\bar{t}) - \epsilon(\bar{t}) \leq -1 \end{cases}$$

(5.32)
Moreover:

\[
\begin{align*}
\delta(t) &= \frac{\delta(t)}{\eta}; \quad \ddot{\epsilon}(t) = \frac{\epsilon(t)}{\eta}; \quad \bar{F}_0 = \frac{F_0}{k_0}; \quad \bar{\epsilon} = \omega_n t; \\
\zeta &= \frac{c(t)}{2m_2\omega_n}; \quad \zeta_b = \frac{c_b(t)}{2m_2\omega_n}; \quad \bar{k}(t) = \frac{k(t)}{k_0}; \quad \bar{k}_b(t) = \frac{k_b(t)}{k_0}; \\
\Lambda &= \frac{2\pi f_m}{\omega_n}; \quad \omega_n = \sqrt{\frac{k_0}{m}}; \quad f_m = \frac{Z_1\omega_1}{60} = \frac{Z_2\omega_2}{60}
\end{align*}
\]  

(5.33)

where \(Z_1, Z_2\) are the number of teeth of the driving gear and driven gear respectively; \(\omega_1, \omega_2\) are the rotating speed of the driving gear and driven gear respectively [rpm]; \(f_m\) is the fundamental excitation frequency (mesh frequency, Hz); \(k_0\) is the mean value of the gear mesh stiffness and \(\omega_n\) is the system natural frequency. The same design parameters of the gear pair are used for the speed sweep simulation, as listed in Table 5.1. In order to induce the double-sided tooth impacts, a light load \(\bar{F}_0 = 0.4\) is chosen in the simulation. Only the first gear mesh harmonic amplitude \(A_1\) in the simulated DTE response is given as \(A_1\) is dominant in the primary resonance region where the double-sided tooth impacts are most likely to happen.

Figure 5.11(a) shows the simulated \(A_1\) response versus rotating speed of driving gear \(\omega_1\) for various addendum modification cases altogether when the damping ratio is 0.008 (supposing \(\zeta = \zeta_b\)). The primary resonance, softening and hardening non-linearity and classic jump discontinuities can be clearly observed as expected. In the single-value region and the softening jump region where either the no-impact or single-sided tooth impact happens, the responses are similar for various addendum modification cases. However, in the hardening jump region where the double-sided tooth impact happens, the responses are significantly different for different addendum modification cases. It seems that when \(x_1 = 0\), the hardening nonlinearity is stronger than that of other cases.
Figure 5.11: Simulated A1 response versus rotating speed of driving gear $\omega_1$ for various addendum modification cases ($\bar{F}_0 = 0.3$, $\zeta = \zeta_b = 0.008$) (Keys: — speed-increasing, ⋯⋯ speed-decreasing)
\[ x_1 = -0.5 \]
\[ x_1 = -0.3 \]
\[ x_1 = 0 \]
\[ x_1 = 0.3 \]
\[ x_1 = 0.5 \]

Figure 5.12: Simulated \( A_1 \) response versus rotating speed of driving gear \( \omega_1 \) for various addendum modification cases \( (F_0 = 0.3, \zeta = \zeta_b = 0.03) \) (Keys: — speed-increasing, … speed-decreasing)
Figure 5.12(a) shows the simulated $A_1$ response versus rotating speed of driving gear $\omega_1$ for various addendum modification cases altogether when the damping ratio is 0.03. In this case, there is no hardening jump in the primary region due to the large damping, meaning no double-sided tooth impact happens. The responses for various addendum modification cases are quite similar. Figure 5.12(b)-(f) show the responses for each modification case separately. There are slight differences regarding the width of the softening jump region and the amplitude of response in the single-value region. This is due to the slight influence of addendum modifications on the drive-side mesh stiffness.

The above phenomenon demonstrates that the addendum modification can significantly affect gear dynamics in the primary resonance region when the double-sided tooth impact happens. Therefore, the time-varying asymmetric mesh stiffness model should be used to analyse gear dynamic performance when the double-sided tooth impact happens. For example, when the gears are working under light load, small damping and large amplitude excitations including mesh stiffness variation and torque fluctuation.

5.4 Conclusions

In this chapter, the influence of the addendum modification on the spur gear back-side mesh stiffness was investigated. The relationship between the drive-side and back-side mesh stiffness for spur gear pairs with various addendum modifications (the centre distance remains unchanged) was analytically determined. It was found that the phase shift between these two mesh stiffness functions is mainly affected by the tooth thickness at the pitch circle. Since gears with different amounts of addendum modification have different length of the tooth thickness at the pitch circle, their back-side mesh stiffness also varies in terms of the phase shift with regard to their corresponding drive-side mesh stiffness.

Two typical cases where the back-side tooth impact may happen were discussed. In the first case, a 6 DOF gear dynamic model excited by an angular acceleration excitation was built. A complex tooth contact force model which was proved to produce consistent results with experimental measured results was employed. It was found that the addendum modification affects gear DTE and lateral vibration...
response (LOA direction) through back-side mesh stiffness when the back-side tooth impact happens. In the second case, a conventional SDOF gear dynamic model based on the symmetric backlash type function was used to examine the effect of the addendum modification on the classic jump phenomenon during speed-increasing and speed-decreasing sweeps. The simulated results reveal that addendum modification can significantly affect gear dynamics in the primary resonance region when the double-sided tooth impact happens. Therefore, the proposed time-varying asymmetric mesh stiffness model should be used to analyse gear dynamics when the gears are working under light load or idling conditions.

5.5 References


156


Chapter 6

Effects of Gear Tooth Spatial Crack on Gear Mesh Stiffness, Inclination Deformation and Dynamics

6.1 Introduction

Gear teeth mesh models with fatigue cracks in the fillet region have been the subject of a large amount of research. Lewicki et al. [1, 2] analysed the tooth crack propagation path using the FEA method with principles of linear elastic fracture mechanics and found that the crack propagation path tends to be smooth, continuous and rather straight with only a slight curve. Tian [3] and Chaari et al. [4] implemented a crack along the whole tooth width with uniform depth into their mesh models. Wu et al. [5] assumed that the crack only propagates in crack depth direction and simplified the crack model by considering the crack path to be straight lines which are assumed to be exactly symmetric around the tooth central line. Chen and Shao [6] built a model assuming that crack depth is non-uniform along the tooth width and the crack propagates along the tooth width as well. Mohammed et al. [7] studied a crack propagation scenario with the crack extending in the depth direction and tooth width direction simultaneously.

All of the former crack models proposed take only plane cracks into consideration, which only considers the cracks propagating either in the crack depth direction or the tooth face width direction and neglect the more typical spatial crack (also named as three-dimensional (3D) crack in some references) that will propagate not only in the depth direction and tooth width direction but also in the tooth profile direction [2, 8, 9]. Besides, the expressions developed to calculate gear mesh stiffness (GMS) with plane crack assume that the initial crack location is always at the involute tooth root circle, which totally neglects the influence of initial crack location on GMS.
Aside from the changes of the mesh stiffness, Mark et al. [10, 11] recently found, through a series of precision measurements made on gear teeth that failed in a tooth-bending-fatigue test, that tooth bending fatigue can also lead to whole-tooth plastic inclination deformation (yielding). They suggested that such deformations are geometric deviation STE (static transmission error) contributions and are much larger than the deformations resulting from the reductions of the mesh stiffness. Consequently, Shao and Chen [12] proposed a theoretical method to approximate the tooth inclination deformation due to crack and included them into the dynamic model to study their effects on the dynamic performance of a planetary gear set. However, both Mark’s and Shao’s work assumed a uniform-depth planar crack into their analysis. All axial locations on the tooth (along the tooth face width direction) have the same inclination deformations at each tooth profile location. When a non-uniform-depth spatial crack occurs, some part of the cracked gear tooth may experience larger inclination deformations while the others have lesser due to the non-uniform depth of the crack along the tooth width direction. In addition, a non-uniform distribution of the tooth inclination deformation will lead to an uneven distribution of the dynamic load on the cracked tooth flanks, which will further affect gear dynamics.

So, the purpose of this chapter is to fill the gap in the literature regarding the calculation of gear mesh stiffness and inclination deformation with the consideration of localized tooth spatial crack, and to investigate the effect of spatial crack on the gear dynamics.

6.2 Mesh Stiffness Model for Gear Tooth with Spatial Crack

6.2.1 Tooth Model with a Uniform-Depth Plane Crack

Considering a plane crack which propagates along the tooth width in a straight line with uniform crack depth, the tooth crack model can be decomposed into a two-dimensional (2D) plane, as shown in Figure 6.1.
When the tooth is subjected to a mesh force $F$, compared with the healthy tooth case (as discussed in Section 2.3.1.2), the existence of a crack will not influence the local Hertzian contact deformation $\delta_h$ and tooth foundation induced deformation $\delta_l$. Nor will it influence the axial compressive force induced deformation $\delta_a$ [3]. However, a crack will affect tooth bending moment induced $\delta_b$ and shear force induced $\delta_s$ significantly as it changes the effective moment of inertia and area of integral section during integration (Equation (2.9-a, b)):

$$I_{xc} = \frac{1}{12} H_{xc} W = \begin{cases} \frac{1}{12} (h_x + h_q)^3 W, & x \geq l_e \\ \frac{1}{12} (h_x + h_q)^3 W, & l_e > x \geq l_t \\ \frac{1}{12} (h_x + h_q)^3 W, & l_t > x \geq 0 \end{cases}$$  \hspace{1cm} (6.1)$$

$$A_{xc} = H_{xc} W = \begin{cases} (h_x + h_q) W, & x \geq l_e \\ (h_x + h_q) W, & l_e > x \geq l_t \\ (h_x + h_q) W, & l_t > x \geq 0 \end{cases}$$  \hspace{1cm} (6.2)$$

where $h_x, h_q, l_e, l_t$ and $x$ are illustrated in Figure 6.1, $W$ is the length of tooth width. In this case, $I_{xc}$ and $A_{xc}$ are used instead of $I_x$ and $A_x$ when calculating $\delta_b$ and $\delta_s$ through Equations (2.8-a, b).

**6.2.2 Tooth Model with a Spatial Crack**

The tooth model with a plane crack and non-uniform crack depth has already been intensively researched in [6]. For spatial cracks, Lewicki et al. [2] and Kramberger et al. [8] have studied the 3D gear crack

![Figure 6.1: Model of spur gear tooth with a plane crack of uniform crack depth](image.png)

Figure 6.1: Model of spur gear tooth with a plane crack of uniform crack depth
(spatial crack) growth path by using boundary element modelling and linear elastic fracture mechanics. Moreover, Shao et al. [9] has simulated the 3D crack propagation path at the tooth root and obtained the dynamic characteristics of the cracked gear by using the theory of fracture mechanics and FEM method. However, the analytical method to establish the tooth model with a spatial crack has not yet been established.

Figure 6.2: Model of spur gear tooth with a spatial crack and non-uniform crack depth: (a) spatial crack in the tooth fillet region, (b) gear tooth thin piece, (c) projection of the crack path on the x-z plane, (d) projection of the crack path on the x-y plane

In this chapter, we consider a spatial crack which propagates along the tooth width through a non-straight curve with a non-uniform crack depth, as shown in Figure 6.2(a). The projections of the crack path on the x-z plane (plane I) and the x-y plane (plane II) are shown in Figure 6.2(c) and Figure 6.2(d), which can be described as functions of \( w \) in the tooth width direction:
\[
\begin{align*}
q_w &= q(w) \\
l_w &= l(w)
\end{align*}
\]  \(w \in [0, W]\) \tag{6.3}

These two equations completely describe how the spatial crack propagates in both the crack depth direction (CD direction as shown in Figure 6.2(b)) and the tooth profile direction (TP direction in Figure 6.2(b)) along the tooth width direction (TW direction in Figure 6.2(b)) simultaneously. In this case, we cannot directly use Equations (2.7), (6.1) and (6.2) to calculate \(\delta\), since the crack depth \(q_w\) and location \(l_w\) along the tooth width are changing.

However, if we divide the gear tooth into several independent thin pieces [6], as shown in Figure 6.2(b), so that the individual width for each thin slice is \(\Delta w\). The crack depth \(q_w\) and crack location \(l_w\) along the sliced tooth width can be regarded as constant if \(\Delta w\) is small enough, which could thus be reduced to the crack model with plane crack and uniform crack depth as discussed in the previous section. The gear tooth with a localized spatial crack can be considered as a series of staggered tooth pieces with uniform-depth plane cracks with no coupling effect since they are usually negligible for narrow-faced gears [13].

Since gears with small size (pinions) have relatively low fatigue strength and load carrying capacity, and therefore tend to develop fatigue fracture at their fillet region more quickly than their counterparts. If the crack is on the pinion fillet, then, according to Equation (2.8-a, b, c):

\[
\delta_{tp} = \delta_{bp} + \delta_{sp} + \delta_{ap} = \frac{F}{W} \int_0^l f_{bp} + f_{sp} + f_{ap} \, dx
\]

where:

\[
\begin{align*}
f_{bp}(x) &= \frac{12[(l - x) \cos \alpha_p - h \sin \alpha_p]^2}{EH_x^3} \\
f_{sp}(x) &= \frac{1.2 \cos^2 \alpha_p}{GH_x} \\
f_{ap}(x) &= \frac{\sin^2 \alpha_p}{2EH_x}
\end{align*}
\] \(6.5\)-a, b, c
Close attention must be paid to the difference between $H_{xc}$ and $2h_c$, where the former represents the effective tooth thickness of the integral section under bending moment and shear force, and the latter represents the whole tooth thickness of the integral section, since the cracked part can still bear an axial compressive force as if no crack exists. Therefore:

$$\frac{1}{\delta_{tp}} = \frac{W}{F} \int_0^l (f_{bp} + f_{sp} + f_{ap}) dx$$  \hspace{1cm} (6.6)

As a result:

$$\frac{1}{\delta_{tp}} = \frac{1}{F} \int_0^W dw \int_0^l (f_{bp}(x) + f_{sp}(x) + f_{ap}(x)) dx$$

$$= \int_0^W \frac{1}{F} \int_0^l \left( f_{bp}(x) \frac{1}{\Delta K_{bp}} dx + f_{sp}(x) \frac{1}{\Delta K_{sp}} dx + f_{ap}(x) \frac{1}{\Delta K_{ap}} dx \right)$$

$$\approx \Sigma_0^W \frac{1}{F} \left( \frac{1}{\Delta K_{bp}} + \frac{1}{\Delta K_{sp}} + \frac{1}{\Delta K_{ap}} \right)$$  \hspace{1cm} (6.7)

where:

$$\frac{1}{\Delta K_{bp}} = \int_0^l f_{bp}(x) \frac{1}{\Delta w} dx, \quad \frac{1}{\Delta K_{sp}} = \int_0^l f_{sp}(x) \frac{1}{\Delta w} dx, \quad \frac{1}{\Delta K_{ap}} = \int_0^l f_{ap}(x) \frac{1}{\Delta w} dx$$  \hspace{1cm} (6.8-a, b, c)

The equivalent mesh stiffness of one tooth pair with a spatial crack at the pinion fillet can be calculated by:

$$k_e = \frac{F}{\delta_{fp} + \delta_{tp} + \delta_{tg} + \delta_{fg}}$$  \hspace{1cm} (6.9)

where $\delta_{tg}$ and $\delta_{fg}$ are the tooth beam induced deformation and tooth foundation induced deformation of the gear tooth (healthy) mating with the cracked pinion tooth, and are calculated through Equations (2.7) and (2.10); $\delta_n$ is the contact deformation of the mating tooth pair, and is calculated through Equation (2.6); $\delta_{tp}$ is the tooth foundation induced deformation of cracked pinion tooth, is still calculated through Equation...
\( \delta_{tp} \) is the tooth beam induced deformation of the cracked pinion tooth, and is calculated through Equation (6.4) (note the crack will only influence the cracked tooth beam induced deformation).

The equivalent gear mesh stiffness of a gear pair with a spatial crack at the pinion fillet can therefore be obtained through Equation (2.12).

### 6.2.3 Spatial Crack Propagation Scenario and Time-Varying GMS

In previous publications, three different crack propagation scenarios have been researched: 1) the crack is assumed to extend through the whole tooth width and propagates only in the crack depth direction \([3, 4, 5]\); 2) the crack propagates through the tooth width direction with the crack depth unchanged \([6]\); and 3) the crack propagates through the crack depth direction and tooth width direction simultaneously \([7]\).

In this chapter, we will consider the development of spatial crack which will propagate not only in the crack depth direction and tooth width direction but also the tooth profile direction. Experimental tests and the practical fatigue crack failure samples show that spatial cracks are more frequent in reality due to a number of factors such as non-uniform load distribution, local non-homogeneity of material during manufacturing and misalignment of gear shafts or bearings\([14]\). Therefore, it is more reasonable and practical to assume the crack is propagating through three directions (CD, TP and TW as shown in Figure 6.2(b)). The main parameters of the gear set for simulation are from \([4]\) and are shown in Table 6.1.
Table 6.1: Parameters of gear pairs set from [4]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driving gear</th>
<th>Driven gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teeth number $Z_i$</td>
<td>30</td>
<td>25</td>
</tr>
<tr>
<td>Module $m$ (mm)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Face width $W$ (mm)</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Pressure angle $\alpha$ (degree)</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Young modulus (N/mm$^2$)</td>
<td>$2 \times 10^5$</td>
<td>$2 \times 10^5$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Contact ratio ($CR$)</td>
<td>1.63</td>
<td>1.63</td>
</tr>
<tr>
<td>Backup ratio ($BR$)</td>
<td>$\approx 4$</td>
<td>$\approx 4$</td>
</tr>
</tbody>
</table>

Figure 6.3: Tooth crack propagation scenario
It is difficult to analytically analyse the actual growth of a spatial crack with time which depends on load distribution, tooth texture and material homogeneity and manufacturing and installation errors of the gear-shaft-bearing system, and we are more concerned about the effect of spatial crack propagation on the reduction of GMS. Therefore, we assume the spatial crack propagates in a quasi-parabolic way as described in Figure 6.3. Seven cases regarding to this type of propagation scenario have been assumed as shown in Table 6.2. All the parameters involved are shown in Figure 6.3. It should be noted that $q_2$ and $l_2$ are defined as the crack length and crack location developed at the other end surface of the tooth.

<table>
<thead>
<tr>
<th>Case</th>
<th>$q_0$ (mm)</th>
<th>$w_c$ (mm)</th>
<th>$q_2$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>①</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>②</td>
<td>0.2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>③</td>
<td>0.4</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>④</td>
<td>0.6</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>⑤</td>
<td>0.8</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>⑥</td>
<td>1.0</td>
<td>20</td>
<td>0.45</td>
</tr>
<tr>
<td>⑦</td>
<td>1.2</td>
<td>20</td>
<td>0.7</td>
</tr>
</tbody>
</table>

With the parameter values provided in Table 6.2, three growth paths of the spatial crack on the tooth surface are investigated to determine their effect on the reduction of GMS of a gear pair. They are linear, monotonous parabolic and non-monotonous parabolic respectively as shown in Figure 6.5. The equations describing the projections of these paths on plane $\Pi$ (Figure 6.3(c)) are:

\[
l_w = \begin{cases} 
Aw + B & w \in [0, W] \\
Aw^2 + B & \frac{A(w-b)^2 + 0.5}{W-0.5} \\
A(w-b)^2 + 0.5 & \end{cases}
\]  

(6.10)
The value 0.5 in the third equation (non-monotonous parabolic) is added to ensure that only tooth fracture will be considered in this research.

![Graphs showing GMS and LSR for the proposed scenario](image)

**Figure 6.4:** GMS and LSR for the proposed scenario: (a) GMS versus angular displacement of the driving gear (b) LSR versus angular displacement of the driving gear

![Crack growth paths in tooth surface](image)

**Figure 6.5:** Crack growth path in tooth surface: (a) linear, (b) monotonous parabolic, (c) non-monotonous parabolic
Figure 6.4 shows the corresponding GMS and LSR (Load Sharing Ratio) of the gear pair when the crack growth path in the tooth surface is described as Figure 6.5(b). It is obvious that GMS continues decreasing as the crack propagates. The load shared by the cracked tooth is also reduced especially when the tooth enters the meshing zone. However, the load sharing changes only a little when the cracked tooth is about to exit from meshing.

Figure 6.6 shows the maximum reduction in GMS versus the propagating cases of the crack among the proposed three crack growth paths in the tooth surface. The gear pair with the crack growing in a linear way (Figure 6.5(a)) always has the smallest reduction in mesh stiffness among the three paths during the propagation of the crack. The gear pair with the crack growing in a non-monotonous parabolic way (Figure 6.5(c)) will have the largest reduction in GMS especially during the late stages of propagation. This interesting phenomenon happens because cracks growing in a linear way tend to be more distant from the tooth root and therefore result in a smaller reduction in GMS, while cracks growing in a non-monotonous way tend to be located closer to the tooth root and therefore lead to a larger reduction in GMS.
6.3 Inclination Model for Gear Tooth with Spatial Crack

6.3.1 Gear Tooth Inclination Model

The gear tooth inclination model due to the planar crack with uniform crack depth along the tooth width direction has already been investigated in [12]. For this type of crack, a 2-dimensional (2D) model is capable of defining the crack dimensions. However, for the planar crack with non-uniform crack depth or spatial crack, a 3D model is necessary.

Figure 6.2(a) shows a spatial crack which propagates along the tooth width through a non-straight curve with a non-uniform crack depth. Similarly, if we divide the gear tooth into several independent thin pieces [6], a 2D model with a planar crack and a uniform crack depth can be used to investigate the inclination deformation for those thin pieces.

Figure 6.7: Schematic for inclination deformation of the gear tooth thin piece due to root crack [12]

Figure 6.7 shows the schematic for the inclination deformation of the gear tooth thin piece (Figure 6.2(b)) due to the tooth fillet crack (also referred to as root crack in [12]), which is based on the model proposed in [12]. The solid curve represents the original tooth profile whereas the dashed curve represents the deformed tooth profile. A global X-Y coordinate system is built with its origin at the gear centre O, and
the Y-axis coinciding with the centre line of the cracked tooth. Point A is the intersection point between the involute profile and fillet profile. Point C is the crack position at the tooth profile. Point D and point C are symmetric with the tooth centre line (Y-axis). Point B is the crack tip. The crack depth $q_w$ and the crack inclination angle $\alpha_c$ (with respect to the V-axis) are illustrated in Figure 6.7.

When the tooth inclination happens, it can be assumed that the tooth part above the crack is a cantilever beam with its fixed foundation at straight line $BD$, and rotates around the middle point $o$ between points $B$ and $D$. In order to determine the inclination deformation, a local $U$-$V$ coordinate system is established with its origin at point $o$ and $V$-axis coinciding with the line $BD$. Suppose a mesh position $E$ at the original tooth profile, at a given whole tooth inclination angle $\theta_p$, it will rotate to the point $E'$ at the deformed tooth profile. $E''$ is the mesh position at the original tooth profile that shares the same line of action (LOA) with $E'$ at the deformed tooth profile. Therefore, the distance between points $E'$ and $E''$ ($\delta_w(E)$) along the LOA represents the inclination deformation at point $E$ that acts as the displacement excitation on the gear dynamics. This distance can be calculated based on the geometric relationship shown in Figure 6.7.

The global coordinates of the points $A$, $C$, $E$ and $D$ on the tooth profiles can be obtained based on the two parametric equations proposed in the Section 2.2. Suppose they are: $A(x_A, y_A)$, $C(x_C, y_C)$, $E(x_E, y_E)$ and $D(x_D, y_D)$. Therefore, the global coordinate of point $B (x_B, y_B)$ can be calculated as:

$$
\begin{align*}
(x_B &= x_C - q_w \sin(\alpha_c)) \\
y_B &= y_C - q_w \cos(\alpha_c)
\end{align*}
$$

As a result, the global coordinate of the middle point $o (x_o, y_o)$ is:

$$
\begin{align*}
(x_o &= \frac{(x_B+x_D)}{2}) \\
y_o &= \frac{(y_E+y_D)}{2}
\end{align*}
$$

The acute angle $\theta_f$ between $X$-axis and $U$-axis can be expressed as [12]:

171
\[ \theta_T = \arctan\left(\frac{x_B - x_D}{y_D - y_B}\right) \]  

(6.13)

This angle can be used to transform the global coordinate to the local coordinate through the transform matrix \( T \):

\[
T = \begin{bmatrix}
\cos(\theta_T) & -\sin(\theta_T) \\
\sin(\theta_T) & \cos(\theta_T)
\end{bmatrix}
\]  

(6.14)

Therefore, the local coordinate of point \( E \) \((u_E, v_E)\) can be established as:

\[
\begin{bmatrix}
  u_E \\
  v_E 
\end{bmatrix} = T^{-1} \ast \begin{bmatrix}
  x_E - x_o \\
  y_E - y_o 
\end{bmatrix}
\]  

(6.15)

When the tooth is inclined at an angle of \( \theta_p \), the local coordinate of point \( E'(u_E', v_E') \) is:

\[
\begin{align*}
  u_E' &= \sqrt{u_E'^2 + v_E'^2} \cos(\arctan(v_E'/u_E') + \theta_p) \\
  v_E' &= \sqrt{u_E'^2 + v_E'^2} \sin(\arctan(v_E'/u_E') + \theta_p)
\end{align*}
\]  

(6.16)

Transforming the local coordinate of point \( E'(u_E', v_E') \) back to a global coordinate \( E'(x_E, y_E) \) through the transform matrix \( T \) gives:

\[
\begin{bmatrix}
  x_E' \\
  y_E'
\end{bmatrix} = T^{-1} \ast \begin{bmatrix}
  u_E' \\
  v_E'
\end{bmatrix} + \begin{bmatrix}
  x_o \\
  y_o
\end{bmatrix}
\]  

(6.17)

Point \( E'' \) on the original tooth profile shares the same LOA with the point \( E' \) on the deformed tooth profile, meaning point \( E'' \) is just on the line \( TE' \) as shown in Figure 6.7. Therefore, the mesh angle for point \( E'' \) is:

\[
\alpha_{E''} = \arccos\left(\frac{R_b}{\sqrt{x_E'^2 + y_E'^2}}\right) - \arctan\left(x_E'/y_E'\right)
\]  

(6.18)

Substitute this mesh angle into the Equation (2.1) gives the global coordinate of point \( E'' \) \((x_{E''}, y_{E''})\). Consequently, the distance between points \( E' \) and \( E'' \) along the LOA is obtained as:
\[ e_w(E) = \sqrt{(x_{E'} - x_E)^2 + (y_{E'} - y_E)^2} \]  
(6.19)

Based on Equations (6.11) - (6.19), one can easily deduce the gear tooth inclination deformation along the tooth profile for each thin piece resulting from the spatial crack described by Equation (6.3). It should be noted that the proposed method to calculate the inclination deformation by dividing the gear tooth into thin pieces is used based on the assumption that the coupling between the adjacent points of contact is negligible [15], which is a reasonable assumption for narrow-faced cylindrical gears with low helix angles [13]. For wide-faced and thin-rimmed gears, this assumption may not be valid, and further analysis tools are needed to take into account the coupling effect between adjacent contact points [15, 16].

### 6.3.2 Gear Tooth Inclination Deformation

In this section, we will simulate the gear tooth inclination deformation when a spatial crack exists on one of the driven gear teeth. The main parameters of the spur gear set are shown in Table 6.1. All the gear teeth except the cracked tooth are considered as ideal without any profile errors. Firstly, we assume that the crack is initiated at the position \( l_0 \) determined by the 30° tangential method defined in BS ISO 6336-3:2006 [17], and propagates only in the tooth width and crack depth directions,

\[ l_w = l_0, \quad w \in [0, W] \]  
(6.20)

Also, suppose the crack depth is distributed as a parabolic function along the tooth width as shown in Figure 6.8. If the crack extends only a part of the tooth width (crack length \( W_c < W \)), as shown in the Figure 6.8(a):

\[
\begin{align*}
q_w &= -\frac{q_0}{w_c^2} \times w^2 + q_0, \quad w \in [0, W_c] \\
q_w &= 0, \quad w \in [W_c, W]
\end{align*}
\]  
(6.21)

If the crack extends through the whole tooth (\( W_c = W \)), as shown in the Figure 6.8(b):

\[ q_w = \frac{q_2 - q_0}{w_c^2} \times w^2 + q_0, \quad w \in [0, W] \]  
(6.22)
where $q_2$ is the crack depth at the other end surface of the cracked tooth. All the crack parameters are shown in Table 6.3.

![Crack model at gear tooth fillet region](image_url)

**Figure 6.8:** Crack model at gear tooth fillet region: (a) Case A, (b) Case B

<table>
<thead>
<tr>
<th>Tooth Crack (TC)</th>
<th>Case A</th>
<th>Case B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(q_0, q_2, W_c)$</td>
<td>$(q_0, q_2, W_c)$</td>
</tr>
<tr>
<td>0</td>
<td>TC-A1</td>
<td>TC-B1</td>
</tr>
<tr>
<td>inclination angle $\theta_{p0}$ (degree)</td>
<td>0.2</td>
<td>TC-A2</td>
</tr>
<tr>
<td>0.4</td>
<td>TC-A3</td>
<td>TC-B3</td>
</tr>
<tr>
<td>0.6</td>
<td>TC-A4</td>
<td>TC-B4</td>
</tr>
</tbody>
</table>

Two groups of tooth damage are considered where the crack inclination angle $\alpha_c$ is kept at a constant $60^\circ$. Group A represents the case where the crack extends through only a part of the tooth width, whereas group B represents the case where the crack extends through the whole tooth. Suppose the inclination
angle $\theta_p$ along the tooth width direction is proportional to the crack depth, and the initial inclination angle $\theta_{p0}$ is increasing from 0 to 0.2, 0.4 and 0.6 degrees as shown in Table 6.3.

Figure 6.9: Distributions of the gear tooth inclination deformations on the cracked tooth flank: (a) Case A, (b) Case B

Figure 6.9 shows the tooth inclination deformations at each discrete point ranging from the base circle radius to the tip radius for each independent tooth thin piece. For Case A, only the cracked tooth part experiences the inclination deformation whereas the un-cracked part has no deformation at all. It can also be seen that the inclination deformation increases nearly linearly from the base radius to the tip radius, which is consistent with the experimental measured results in [10] and simulation results in [12]. Besides, the larger the inclination angle $\theta_p$, the larger the deformation at each point. In case B, the crack extends through the whole tooth with a non-uniform crack depth along the tooth width direction, which results in the non-uniform distributions of the inclination deformations on the cracked tooth flank. These non-uniformly distributed inclination deformations will serve as geometric deviation static transmission error (STE) contributions and lead to non-uniform dynamic load distributions on the cracked tooth flank, and will therefore, significantly affect the gear dynamic performance.
We then assume that the crack propagates only in the tooth profile direction with uniform crack depth along the tooth width,

\[ q_w = q_0, \quad w \in [0, W] \]  \hspace{1cm} (6.23)

Two growth paths of the spatial crack on the tooth flank are investigated. They are linear and monotonous parabolic respectively as shown in Figure 6.10. If the crack growth path is linear, as shown in Figure 6.10(a):

\[ l_w = \frac{l_2 - l_0}{W} w + l_0, \quad w \in [0, W] \]  \hspace{1cm} (6.24)

If the crack growth path is parabolic, as shown in Figure 6.10(b):

\[ l_w = \frac{l_2 - l_0}{W^2} w^2 + l_0, \quad w \in [0, W] \]  \hspace{1cm} (6.25)

where \( l_2 \) is the distance from the crack position at the other end surface to the tooth root as shown in Figure 6.10. The crack inclination angle \( \alpha_c \) is still kept at a constant 60°. The initial crack position \( l_0 \) is still determined by the 30° tangential method. The crack depth \( q_0 \) is kept at 0.4 mm, and the tooth inclination angle \( \theta_p \) is assumed as 0.6 degrees. Three different crack growth distances along the tooth flank \((l_2 - l_0)\) are shown in Table 6.4.

<table>
<thead>
<tr>
<th>Table 6.4: Parameter of the cracks (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tooth Crack (TC)</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Crack growth distance along tooth surface (l_2-l_0) (mm)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Figure 6.10: Crack model at gear tooth fillet region: (a) Case C, (b) Case D

Figure 6.11: Distributions of the gear tooth inclination deformations on the cracked tooth flank: (a) Case C, (b) Case D

Figure 6.11 shows the 3D distributions of the gear tooth inclination deformations on the cracked tooth flank. It is obvious from these figures, that as the crack propagates away from the tooth root, the inclination deformation is decreasing along the tooth width direction. A linear crack growth path leads to a linear decreasing trend whereas a parabolic growth path leads to a parabolic decreasing trend. These demonstrate that aside from the crack depth and inclination angle, the crack position in the fillet region will also affect the inclination deformation distributions.
6.4 Dynamic Analysis

6.4.1 Stiffness Cell

In Section 6.2, we derived the equivalent mesh stiffness of one tooth pair with spatial crack at the pinion fillet (Equation (6.9)). However, in order to analyse the load distribution on the crack tooth flank, we need to discretize the contact line into a series of segments (as shown in Figure 6.12(a)), and each contact line segment is assigned a linear stiffness $k_i$ at its centre $M_i$. The value of $k_i$ for each stiffness cell can be obtained based on the work by Eritenel and Parker [18], which is calculated as:

$$k_i = \frac{k_g k_{ci} H_i}{k_g + \sum_{i=1}^{n} k_{ci} H_i} \quad (6.26)$$

where $k_g$ is the global stiffness accounting for all stiffness except the local contact stiffness ($k_{1g}$ and $k_{2g}$ as shown in Figure 6.12(a)), and is assumed to be the same for all contact segments; $k_{ci}$ is the local contact stiffness of the $i$th segment (related to $\delta_i$ in Equation (6.9)), and is nonlinear based on the contact function $H_i$.
where $e_i$ is the profile deviation error of the $i$th segment ($e_{1i}$ and $e_{2i}$ as shown in Figure 6.12(a)) which includes the tooth profile manufacturing errors, profile modifications and the inclination deformations discussed above, $\delta_i$ is the normal approach at the $i$th segment. For a spur gear pair, $\delta_i$ is the same for each segment. $H_i$ represents the contact condition at the $i$th segment. When $H_i$ is 0, it means that there is a contact loss at the $i$th segment, and the local contact stiffness $k_{ci}$ will not contribute to the effective mesh stiffness $k_i$ at the $i$th segment. $k_g$ and $k_{ci}$ can be obtained from finite element analysis of gears [18, 20]. They can also be estimated using analytical approaches [3, 4, 5, 6, 19], as described in Equations (6.8-a, b, c). The tooth fillet crack does not affect the local contact stiffness $k_{ci}$. However, it will directly influence the global stiffness $k_g$ as it can weaken the cracked tooth bending and shearing strength. Detailed discussion about this can be found in Section 6.2.

6.4.2 Dynamic Model

The research focus is a spur gear pair system as shown Figure 6.12 (a). A Cartesian coordinate system ($X$-$Y$-$Z$) is established. The $X$-axis is in the direction of the LOA, and the $Y$-axis is in the off line of action (OLOA) direction. The $Z$-axis is along the axial direction and can be determined by following the right-hand rule. The frictional forces developed between the contact tooth pair are neglected. Therefore, the only translational degree of freedom (DOF) considered is along the $X$ axis ($x_1$ and $x_2$). Since the inclination deformations resulting from the spatial crack are non-uniformly distributed on the cracked tooth flank, there will be a tilting moment about the $Y$-axis. Therefore, the rotational DOF about the $Y$-axis should be also taken into consideration ($\theta_{y1}$ and $\theta_{y2}$). The torsional vibration $\theta_{c1}$ and $\theta_{c2}$ should also be considered. Therefore, a 6 DOF model was built to analyse the influence of the inclination deformations on the dynamics of a spur gear pair. It should be noted that for a helical gear pair set, and/or when it is subjected to misalignments or eccentricities, more DOF are needed in order to fully study its dynamic behaviour. The normal approach at the contact point $M_i$ of the $i$th segment is calculated as:
\[ \delta_i = V_i^T q \]  

(6.28)

where \( q^T = \{x_1, \theta_{z1}, x_2, \theta_{z2} \} \) is the degree of freedom vector of the gear pair considered, and \( V_i \) is the structure vector defined as:

\[ V_i^T = \{-1, -c_i, R_{b1}, 1, c_i, R_{b2} \} \]  

(6.29)

where \( c_i \) is the distance of \( M_i \) to the tooth centre along the Z-axis, and its value should be between \(-W/2\) to \(W/2\). Suppose tooth profile deviations relative to the perfect geometry are positively defined in the direction of the outer normal and they are small enough to allow the contact to remain on the theoretical base plan [15]. The elastic deformation \( \varepsilon_i \) at \( M_i \) is:

\[ \varepsilon_i = \delta_i - e_i \]  

(6.30)

where profile deviation error \( e_i \) at \( M_i \) consists of the profile deviation errors of the driving and driven gears \((e_i = e_{1i} + e_{2i})\). Therefore, the contact force \( f_i \) at \( M_i \) can be expressed as:

\[ f_i = k_i \varepsilon_i \]  

(6.31)

The total contact force \( F \) and the resultant moment about the Y-axis acting at the driving gear are:

\[ F = \sum_{i=1}^{n} f_i, \quad M = \sum_{i=1}^{n} f_i c_i \]  

(6.32)

The un-damped governing equation can be expressed as:

\[
\begin{align*}
    m_1 \ddot{x}_1 + k_{Bx1} x_1 - F &= 0 \\
    I_{1} \ddot{\theta}_{y1} + k_{B\theta_{y1}} \theta_{y1} - M &= 0 \\
    I_{p1} \ddot{\theta}_{x1} + k_{B\theta_{x1}} \theta_{x1} + FR_{b1} &= T_1 \\
    m_2 \ddot{x}_2 + k_{Bx2} x_2 + F &= 0 \\
    I_{2} \ddot{\theta}_{y2} + k_{B\theta_{y2}} \theta_{y2} + M &= 0 \\
    I_{p2} \ddot{\theta}_{x2} + k_{B\theta_{x2}} \theta_{x2} + FR_{b2} &= T_2
\end{align*}
\]  

(6.33)
where $m_j, I_j$ and $I_{pj}$ are the mass, the transverse moment of inertia and the polar moment of inertial of the $j$th gear ($j = 1, 2$ representing the driving gear and driven gear respectively), respectively; $K_{Bj}, K_{B0j}$ and $K_{B0j}$ are the radial stiffness, and rotational stiffness along the $Y$-axis and $Z$-axis of the bearing supporting the $j$th gear, respectively; $T_j$ is the external torque applied on the $j$th gear. Equation (6.33) can be also expressed in matrix form if we substitute Equation (6.28) and (6.29) into it:

$$M\ddot{q} + [K_g(t, q) + K_b]q = F_0 + F_1(q)$$ (6.34)

where:

$$M = \text{diag}(m_1, I_1, I_{p1}, m_2, I_2, I_{p2})$$

$$K_g(t, q) = \sum_{i=1}^{n} k_i V_i V_i^T$$

$$K_b = \text{diag}(k_{Bx1}, k_{B\theta1}, k_{Bz1}, k_{Bx2}, k_{B\theta2}, k_{Bz2})$$

$$F_0^T = \{0, 0, T_1, 0, 0, T_2\}$$

$$F_1(q) = \sum_{i=1}^{n} k_i e_i V_i$$ (6.35-a, b, c, d, e)

where the dependence of $K_g(t, q)$ and $F_1(q)$ on $q$ is from $k_i$ in Equation (6.26). If the damping effect is included, the equations of motion would be:

$$M\ddot{q} + C\dot{q} + K(t, q)q = F_0 + F_1(q)$$ (6.36)

where $K = K_g(t, q) + K_b$, and $C$ is the viscous damping matrix of the system, which is normally defined by the Rayleigh-type damping [21], namely,

$$C = \alpha M + \beta K$$ (6.37)

where

$$\alpha = \frac{2(\omega_2^2 - \omega_1^2)\omega_1 \omega_2}{(\omega_2^2 - \omega_1^2)}, \quad \beta = \frac{2(\omega_2^2 - \omega_1^2)\omega_1 \omega_2}{(\omega_2^2 - \omega_1^2)}$$ (6.38)

181
where $\omega_1$ and $\omega_2$ are the first and second un-damped natural frequencies (rad/s) respectively, and $\zeta_1$ and $\zeta_2$ are the first and second modal damping ratios, respectively.

Table 6.5: Design parameters of the spur gear system for simulation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driving gear</th>
<th>Driven gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>30</td>
<td>25</td>
</tr>
<tr>
<td>Mass $m_j$ (kg)</td>
<td>1.78</td>
<td>1.23</td>
</tr>
<tr>
<td>Applied torque (N·m)</td>
<td>120</td>
<td>100</td>
</tr>
<tr>
<td>Transverse moment of inertia $I_j$ (kg·m²)</td>
<td>1.6×10⁻²</td>
<td>7.68×10⁻³</td>
</tr>
<tr>
<td>Polar moment of inertia $I_{pj}$ (kg·m²)</td>
<td>3.2×10⁻³</td>
<td>1.54×10⁻²</td>
</tr>
<tr>
<td>Bearing radial stiffness $K_{Bj}$ (N/m)</td>
<td>2×10⁸</td>
<td></td>
</tr>
<tr>
<td>Bearing rotational stiffness $K_{B\theta j}$ (N/rad)</td>
<td>1.8×10⁶</td>
<td></td>
</tr>
<tr>
<td>Bearing rotational stiffness $K_{B\phi j}$ (N/rad)</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The main parameter values of the spur gear system are given in Table 6.1 and Table 6.5. The spatial cracks described in Table 6.3 and Table 6.4 are assumed on one of the driven gear teeth, whereas all the other driven gear teeth and driving gear teeth are totally normal. Supposing there are no manufacturing errors and profile modifications, the only possible geometric deviations come from the deformations $e_{2w}$ ($E$) of the cracked tooth on the driven gear. It should be noted that $e_{2w}$ ($E$) is position-dependent (along the tooth width direction) and also time-dependent (along the tooth profile direction). Hence it can be also written as $e_{2w}(t)$, and,

$$ e_i(t) = e_{2w}(t) \quad (6.39) $$

6.4.3 Effect of the Tooth Inclination Deformations on the Gear Dynamics

In this section, the effect of the tooth inclination deformations on the gear dynamic load factor (ratio of the dynamic tooth load to the static tooth load) is investigated. For simplicity, only the tooth cracks in
Case A and Case B described in Table 6.3 are analysed. The cracks are assumed on the driven gear teeth. Figure 6.13 and Figure 6.14 display the dynamic load factor at the middle tooth width surface for the abovementioned 8 tooth crack cases. The rotating speed of the driving gear for simulation is 2000 rpm. Therefore, the rotating period of the driven gear for one revolution is 0.025 s.

Figure 6.13: Dynamic load factor at the middle tooth width surface for Case A: (a) TC-A1, (b) TC-A2, (c) TC-A3, (d) TC-A4
In Figure 6.13(a) where the tooth inclination angle $\theta_p$ is 0, only the reductions of the mesh stiffness caused by the tooth crack are considered in the analysis. In Figure 6.13(b), (c) and (d), $\theta_p$ is 0.2, 0.4 and 0.6 degrees respectively, which means that the reductions of the mesh stiffness together with the tooth inclination deformations both come into play. Compared with the no tooth crack induced inclination deformation case, the tooth impact impulse per revolution is more and more obvious as the tooth inclination angle $\theta_p$ increases. Similar results appear in Figure 6.14, except that the dynamic load factor is slightly larger than that of Figure 6.13, which is due to the larger crack size in Case B than that of Case A.

From the results shown in Figure 6.13 and Figure 6.14, it can be concluded that, instead of the reductions of the mesh stiffness, tooth inclination deformations caused by a tooth crack play a dominant role in the
dynamic response of the gear system. This conclusion is consistent with the theoretical findings made by Shao and Chen [12] and the experimental results provided by Mark et al. [10]. In fact, when a tooth fillet crack happens, if only the reductions of mesh stiffness are considered, it can be difficult to identify the defect features by observing the dynamic response shown in Figure 6.13(a) and Figure 6.14(a), especially when the crack size is small. As a result, for the early detection of the initial crack damage, special attention should be paid on the tooth inclination deformations, instead of the reductions of the tooth mesh stiffness, as the latter are likely to cause significantly smaller changes in the dynamic response [10].

6.4.4 Effect of the Tooth Inclination Deformation on the Dynamic Load Distributions

As explained earlier, the non-uniform distributions of the inclination deformations due to spatial crack will lead to the non-uniform dynamic load distributions on the cracked tooth flank. This point can be verified by Figure 6.14 and Figure 6.15, which display the dynamic load distributions on the cracked tooth flank when subjected to the spatial cracks illustrated in Case A and Case B. The rotating speed of the driving gear is 6000 rpm.

In Figure 6.15(a) and Figure 6.16(a) where there is no tooth inclination deformation, the dynamic load is nearly uniformly distributed along the tooth width direction, just like the case where no tooth crack occurs. However, when the tooth inclination deformation is taken into consideration, it can be found that the dynamic load is more and more concentrated on the other end of the tooth width where the inclination deformations are smaller. The larger the tooth inclination angle $\theta_p$, the more non-uniformity of the dynamic load distributed along the tooth width direction. This is expected as the tooth thin pieces having smaller inclination deformations will support more load compared with tooth thin pieces having larger inclination deformations. Comparing Case A and Case B, it can also be found that the dynamic load distributions in Case A are more uneven than those in the Case B.
A non-uniformly distributed dynamic load along the tooth width direction will cause a tilting moment of the gears along the Y-axis. Therefore, the tilting motion of the driven gear $\theta_{y2}$ (or the driving gear $\theta_{y1}$) can best indicate the non-uniformity of the dynamic load distribution along the tooth width direction. Figure 6.17 and Figure 6.18 present the tilting motion of the driven gear $\theta_{y2}$ for various tooth crack cases.
Figure 6.16: Dynamic load distributions on the cracked tooth flanks for Case B: (a) TC-B1, (b) TC-B2, (c) TC-B3, (d) TC-B4

It can be seen that the tilting motion of the driven gear when there is no inclination motion (TC-A1 and TC-B1) is always 0, which demonstrates that the dynamic load is always uniformly distributed along the tooth width direction in this case. When the tooth inclination deformation is taken into consideration, the tilting motion is excited whenever the cracked tooth comes into the mesh, whose amplitude quickly decays to 0 as the cracked tooth exits from the mesh zone. The larger the tooth inclination angle $\theta_p$, the higher the amplitude of the tilting motion being excited. In addition, comparing Case A and Case B, it can also be found that the tilting motion of Case A is higher in amplitude than that of Case B, which
demonstrates again that the dynamic load distribution along the tooth width direction in Case A is more uneven that that of Case B.

**Figure 6.17:** Tilting motion of the driven gear $\theta_{y2}$ for Case A: (a) TC-A1, (b) TC-A2, (c) TC-A3, (d) TC-A4

**Figure 6.18:** Tilting motion of the driven gear $\theta_{y2}$ for Case B: (a) TC-B1, (b) TC-B2, (c) TC-B3, (d) TC-B4
6.5 Conclusions

In this chapter, modified expressions to calculate the equivalent GMS for the gear teeth with spatial cracks were developed based on the slicing principle. A spatial crack propagating scenario, which is more realistic and reasonable in the real-world setting, was assumed to investigate its effects on the change of GMS. Besides, a tooth inclination model to analytically calculate the 3D distributions of the tooth inclination deformations resulting from a non-uniform-depth spatial crack was introduced. A 6 DOF model was built for a spur gear pair to investigate their influence on the dynamic performance of the gear system, mainly including the dynamic load distributions on the cracked tooth flank and the tilting motions of the gears. The main conclusions can be generalized as follow:

1) The existence of a crack will lead to a reduction in GMS and the load shared by the cracked tooth pair, especially when the cracked tooth pair begins to mesh. Besides, a crack that tends to propagate further away from the tooth root along the tooth profile direction usually causes a smaller reduction in GMS and LSR when compared to the normal condition.

2) Simulation results demonstrated the predominant influence played by the tooth inclination deformations over the reduction of the mesh stiffness resulting from the tooth fillet crack. It was also found that the spatial crack with non-uniform crack depth will lead to unevenness of the dynamic load distribution on the cracked tooth flank. Tilting motions can therefore be excited whenever the cracked tooth comes into the mesh. The larger the tooth inclination, the higher the amplitude of the tilting motion being excited. As a result, special attention should be paid to the tooth inclination deformations for the early detection of the initial crack damage.

It should be noted that the proposed tooth inclination model was proposed based on the assumption that the coupling between the adjacent points of contact is negligible. Such an assumption was made for the simplicity of the calculation, and may not hold valid for the wide-faced and thin-rimmed gears. In addition, the relationship between the tooth inclination angle and the tooth fillet crack size is rather
complex which mainly depends on the gear materials, materials-processing and operating conditions. Study of these limitations can be some of the necessary extensions of the present work.

6.6 References

6. Z. Chen and Y. Shao, Dynamic simulation of spur gear with tooth root crack propagating along tooth width and crack depth, Engineering Failure Analysis 18(2011) 2149-2164.


Chapter 7

A New Dynamic Model of a Cylindrical Gear Pair with Localized Spalling Defects

7.1 Introduction

For gears with localized tooth defects, the gear mesh stiffness is considered to be an important parameter reflecting the fault status, as it is well known that the existence of tooth defects can reduce the tooth flexibility and affect the gear train output dynamic performance [1, 2, 3, 4, 5, 6, 7, 8, 9]. Litak and Friswell [3], Mohammed et al. [10, 11], Chaari et al. [12, 13], Chen and Shao [14, 15], Jia and Howard [16], Yu et al. [17], Ma et al. [18] and Wu et al. [19] proposed various analytical and finite element (FE) models to determine the tooth stiffness reductions due to tooth fillet crack and tooth surface spalling defects to assess gear tooth failures. Various dynamic models have also been built for different gear transmission systems, ranging from simple spur/helical (cylindrical) gear pairs [1, 2, 5, 6, 10, 11, 12, 13] to complex planetary gear sets [4, 15]. The reduced mesh stiffness was then input into the governing equations, to accurately simulate the dynamic performance of gear transmission systems with localized tooth defects.

Modelling the effects of gear tooth defects on gear dynamics through the reduction of mesh stiffness has been demonstrated to be an acceptable strategy for gears with tooth fillet crack defects [19]. However, for gears with shallow tooth surface defects (wear, pitting and spalling), Endo et al. [7, 8] found that in some instances, the mating teeth may make contact at the bottom of the spall (a cavity or a dent on the tooth surface that has a size larger than a pit), and the spall thus acted as a source of displacement excitation to the gear system. In some other similar studies conducted by Badaoui et al. [5, 6], the authors proposed a comprehensive model to simulate the dynamic response of a gear system with a spalling defect, in which
the contribution of the spall to the gear dynamics was determined by the depth of the spall (with respect to the average static deflection): small depth spalls generate an external displacement excitation whereas deeper spalls modify the mesh stiffness.

However, based on the three-dimensional (3D) finite element analysis (FEA) study of the residual transmission error (RTE) of gears with various sizes of shallow spall defects, Endo et al. [7, 8] classified the spalls mainly into two cases: spalls with small size over which the crowned mating tooth bridges, and spalls which are wide enough for the mating tooth to roll into and make contact at the bottom. Whether the mating tooth contacts at the bottom of a spall or bridges over a spall is totally dependent on the size of the spall (wide or narrow) and the amount of crowing on the gear tooth. The transmitted load was not considered as a deterministic factor in their work. Therefore, Baduit’s model is only applicable to spalls belonging to the second case. It cannot accurately model the case when the mating tooth bridges over a shallow, as well as a deep spall. One of the main assumptions made by Baduit et al. [5, 6] is that the type of contact between the meshing surfaces of a mating tooth pair is an ideal linear line of contact, so that the mating tooth can never bridge over a spall. However, in reality, gear tooth contact behaviour is nonlinear in nature and the contact pattern is normally elliptical depending on the load transmitted as well as the amount of tooth surface crowning. Therefore, Baduit’s model needs to be modified in order to address this issue. This constitutes the main objective of this research.

In this study, we provide a new model of a cylindrical gear pair with localized spalling defects which is mainly based on the work done by Badoui et al. [5, 6] and Endo et al. [7, 8]. Modification coefficients are introduced to account for the influence of the transmitted load and the sizes of the spall on the contributions (displacement excitation or stiffness reduction) of the spall to the gear dynamics. Experimental work has been done by Ma et al. [1] to research the dynamics and fault mechanism of a spur gear pair with spalling defects. A dynamic model incorporating the mesh stiffness reduction and displacement excitation simultaneously was also established. Some useful results were derived except that
they did not consider the influence of spall dimensions (mainly depth and length) and load on the influencing mechanism of the spalls to the gear vibration. In this research, experimental work was also conducted to demonstrate the validity of the proposed model. The simulated results and the experimentally measured results will be directly compared in terms of the time-history acceleration, power spectrum and some statistical indicators.

7.2 Gear Dynamic Model with Spalling Defects

7.2.1 General Model for a Cylindrical Gear Pair

Figure 7.1: 3D cylindrical gear mesh model: (a) the 3D gear mesh model, (b) projection on the plane of action

Figure 7.1 shows a general three-dimensional (3D) model of a cylindrical gear pair. The classic “thin-slice” approach is applied in this model. The mesh behaviour of the gear pair is modelled by two rigid disks connected by a series of stiffness cells $k_j$ along the instantaneous contact line on the base plane in the direction determined by the helix angle $\beta$ [20, 21]. Ajmi and Velex [22] proved that such “thin-slice” model is sufficient for dynamic calculations as long as local disturbances can be ignored. A localized Cartesian coordinate system ($U$-$V$-$W$) is established. The $U$-axis is in the direction of the line of action (LOA), and the $V$-axis is in the off line of action (OLOA) direction. The $W$-axis is along the axial
direction and can be determined by following the right-hand rule. Lagrange’s equations were used to derive the following equations of motion:

\[ M \ddot{q} + C \dot{q} + (K_0 + K_g(t, q)) q = F_0 + F_1(t) + F_2(t, q) \] (7.1)

where \( q^T = \{u_1, v_1, w_1, \theta_{u1}, \theta_{v1}, \theta_{w1}, u_2, v_2, w_2, \theta_{u2}, \theta_{v2}, \theta_{w2}\} \) is the degree of freedom vector of the gear pair, as shown in Figure 7.1(a). \( M, C, K_b \) are the mass, damping and bearing stiffness matrices, respectively. Detailed information about these matrices can be found in [20]. \( F_0 \) is the external excitation from the applied torques, and \( F_1(t) \) includes all inertial effects produced by unsteady rigid-body rotations (gear eccentricities). The analytical expressions of the two quantities that largely determine the gear dynamics are given as [5, 20]:

\[ K_g(t, q) = \sum_{j=1}^{N_c(t)} k_j(t) H_j E_j(q) E_j(q)^T \]

\[ F_2(t, q) = \sum_{j=1}^{N_c(t)} k_j(t) H_j e_j(t) E_j(q) \]

\[ H_j = \begin{cases} 1, & E_j(q)^T > e_j(t) \\ 0, & E_j(q)^T \leq e_j(t) \end{cases} \] (7.2-a, b, c)

where \( k_j \) is the cell stiffness at the contact point \( M_j \). \( N_c(t) \) is the number of individual cells along the contact line. \( H_j \) is the tooth contact function that determines the contact condition at \( M_j \), which is either 0 (contact loss) or 1 (contact). \( E_j(q) \) is the so-called structure vector which is defined as:

\[ E_j(q) = \begin{pmatrix} -\cos \beta \\ 0 \\ \sin \beta \\ R_{b1} \sin \beta \\ -w_j \cos \beta - u_j \sin \beta \\ R_{b1} \cos \beta \\ \cos \beta \\ 0 \\ -\sin \beta \\ R_{b2} \sin \beta \\ w_j \cos \beta - (u_g - u_j) \sin \beta \\ R_{b2} \cos \beta \end{pmatrix} \] (7.3)
where $R_{b1}$ and $R_{b2}$ are the base radii of the driving gear and driven gear respectively. $w_j$ and $u_j$ are the coordinates of $M_j$, and $u_g$ is the centre distance between the two conjugated gears (as shown in Figure 7.1(b)). $E_j(q)$ is used to relate the degree of freedom $q$ to the normal approach at $M_j$:

$$
\delta(M_j) = E(M_j)^T q
$$  \hspace{1cm} (7.4)

e_j(t)$ in Equation (7.2-b) is the unloaded static transmission error (USTE) at contact point $M_j$, which describes the tooth profile deviation relative to the perfect geometry in an unloaded, static condition. Normally it includes the intentional tooth profile modifications (tip relief, root relief, lead crown, etc.), inevitable manufacturing errors, and tooth surface defects [20]:

$$
e_j(t) = e_{pj}(t) + e_{mj}(t) + e_{dj}(t)
$$  \hspace{1cm} (7.5)

where $e_{pj}(t)$, $e_{mj}(t)$ and $e_{dj}(t)$ are the tooth profile modifications, tooth manufacturing errors and tooth surface defects at $M_j$ respectively. The distribution of the profile deviations ($e_j(t)$), is time-dependent, as well as position-dependent, therefore it can also be expressed as $e(\eta, t)$ in terms of time $t$ and position $\eta$ along the contact line.

The structure vector $E(q)$ is contact position-dependent, which makes the solving of the Equation (2.33) complex. Normally, a simplified constant structure vector $E_0$ is used as we notice that most of the components in $E(M_j)$ are independent of the contact position $M_j$ except those related to bending slopes $\theta_{ui}$ and $\theta_{vi}$ [20]. Their contributions are usually averaged over one mesh period [5, 6, 20]:

$$
K_g(t, q) = \sum_{j=1}^{Nc(t)} k_j(t) H_j E_j(q)E_j(q)^T \approx E_0 E_0^T \sum_{i=1}^{Nc(t)} k_j(t) H_j = E_0 E_0^T k_g(t, q)
$$  \hspace{1cm} (7.6)

where

$$
E_0 = \frac{1}{T_{x}W_0} \int \int E_j(q) dt dw
$$  \hspace{1cm} (7.7)
is the averaged structural vector, $T_s$ is the mesh period and $W_0$ is the tooth face width along the axial direction. It should be noted that this simplification is especially valid for narrow-faced gears and spur gears, so that gear bending slopes can be neglected [21, 22, 23].

In Equation (7.6), $k_g(t, q)$ is the time-varying, nonlinear mesh stiffness function of the gear pair. The values can be acquired either by FEA based methods [24, 25] or by analytical methods [14, 17, 18]. A simplified method based on the ISO standard 6336 [26] will be used in this chapter. An important assumption brought by the ISO standard 6336 is that the mesh stiffness density per unit length $k_0$ along the contact lines is considered as approximately constant so that the following formula can be used [9, 20]:

$$k_g(t, q) = k_0 L(t, q) \quad (7.8)$$

where $L(t, q)$ is the time-varying length of the contact line [21]. The analytical expressions of the time-varying contact length $L(t)$ for a healthy cylindrical gear pair can be found in [27, 28]. The ISO standard 6336 provides some expressions to derive $k_0$ [26]. In [29], the authors considered a time-dependent mesh stiffness density per unit length $k_0(t)$, where a small variable $\alpha (\alpha \leq 1)$ is introduced to represent the relative variation in amplitude, and a time dependent function $\varphi(t)$ is used to account for the variations of mesh stiffness density per unit length depending on the contact point position on the tooth profiles. More details can be found in [29].

### 7.2.2 Incorporation of the Spalling Defects

Without loss of generality, we consider a rectangular-shaped spalling defect, which is parallel to the contact line, on one of the driving gear teeth as shown in Figure 7.2. Its width, length (in the direction of gear tooth roll) and depth are represented by $w_s$, $l_s$ and $d_s$ respectively.
For a shallow tooth surface defect (with respect to the average static deflection), it can be assumed that the mesh stiffness variation along the defect width is negligible. Tooth flexibility is negligibly affected [5, 6, 16]. Therefore, $k_f(t) \approx k_0$ and $H_f = 1$ at all points along the contact lines for reasonably small defects. If the gear pair is loaded, the corresponding mating tooth on the driven gear will roll into the spall and contact at the bottom. In this case, the localized spall can be modelled by a set of normal deviations at a given location on the base plane. The distributions of the deviations caused by the spall can be denoted as the displacement excitation $\epsilon(\eta, t)$, which depends on the position $\eta$ along the contact line, and time $t$ in order to simulate variations of the spall depth in the profile direction (as shown in Figure 7.3(a)). For the rectangular-shaped spall shown in Figure 7.2, the excitation $F_2(t, q)$ in Equation (7.2-b) due to the spall is directly affected whenever the mating tooth rolls into it as the instantaneous defect depth $d_s$ will alter the unloaded static transmission error $e_{dj}(t)$ through $e_{dj}(t)$ in Equation (7.5):

$$e_{dj}(t) = \begin{cases} 
  d_s, & \text{when } M_j \text{ is within the spall} \\
  0, & \text{when } M_j \text{ is outside the spall}
\end{cases} \quad (7.9)$$
On the other hand, for a deep tooth surface defect, contact losses within the defect area may happen, therefore $H_f = 0$ at all points within the defect area whereas $H_f = 1$ at points outside the defect area (as shown in Figure 7.3(b)). This will reduce the length of the contact line directly, and thus affect the mesh stiffness matrix $K(t, q)$ in Equation (7.2-a). For the rectangular-shaped spall shown in Figure 7.2, the length of the contact line will be reduced whenever the mating tooth rolls across the spall. Therefore, the updated gear mesh stiffness will be:

$$k_g(t, q) = \begin{cases} 
  k_0(L(t) - w_s), & \text{when } M_j \text{ is within the spall} \\
  k_0L(t), & \text{when } M_j \text{ is outside the spall}
\end{cases} \quad (7.10)$$

where $L(t)$ is the ideal length of the contact line when no tooth surface defect is involved. It should be noted that even though the rectangular-shaped spalls are chosen as the research objective, the proposed model is also suitable for any other shapes of spalls (circular, oval, etc.) as long as their excitations to the system can be expressed by functions like Equations (7.9) and (7.10).

In a word, defects with shallow and deep depths contribute differently to the equations of motion since small depth tooth defects generate external excitations whereas mesh stiffness is modified by deeper defects [5]. It should be stressed that Equation (7.9) and Equation (7.10) are derived based on the assumption that the contact pattern between a conjugated tooth pair is a linear line contact so that the mating tooth can never bridge over a spall.
Figure 7.4: Tooth surface contact region under various loads: (a) light load, (b) intermediate load, (c) heavy load
(Note the red area represents the contact region between a mating tooth pair)

However, in practical cases, the gear surface contact behaviour is usually nonlinear (as shown in Figure 7.4). The contact area of a conjugated tooth pair is normally elliptical depending on the load transmitted [18, 30, 31]. Besides, gear teeth surfaces are usually shaped to ensure a contact area limited to the central part of the tooth [31]. The direct outcome is that the mating tooth may bridge over the spall as shown in Figure 7.2. This point has been partially demonstrated by Endo et al. [7, 8] who found that the mating tooth may bridge over a shallow spall with small size through a three-dimensional (3D) FEA study of the residual transmission error (RTE) of the gears with spall defects. For deep spalls, the above point should also be true. Taking the spur gear pair as an example, Figure 7.5 shows that a larger transmitted load will lead to a more elliptical contact area [31], and therefore less effective reduction in the length of the contact line due to the spall compared with the length reduction due to the same size of the spall based on the line contact assumption. Figure 7.6 shows that the difference in the length reduction of the contact line between these two contact behaviours is more significant for a smaller size of the spall. The amount of tooth crowing should also affect the contact area, but will not be discussed in this research.
Therefore, the effective reduction in the mesh stiffness due to a deep spall will be significantly influenced by this nonlinear elliptical contact behaviour, meaning that Equation (7.10) needs to be modified in order to compensate for this nonlinear elliptical contact behaviour. In order to do that, a contact coefficient is introduced:

$$ k_g(t, q) = k_0(L(t) - \eta_d(t, T, l_s, w_s)w_s), \text{ when } M_i \text{ is within the spall} $$ \hspace{1cm} (7.11)

where $\eta_d(t, T, l_s, w_s)$ is the modification coefficient in the reduction of mesh stiffness for deep spalls, whose value should be between 0 and 1, and dependent on time, transmitted load and the geometries of
the spall. Generally speaking, the larger the transmitted load $T$, and the smaller the spall size ($l_s$ and $w_s$), the smaller the coefficient $\eta_d$. A semi-empirical expression for $\eta_d$ is introduced based on the experimentally measured results provided in the following section:

$$
\eta_d = \eta_0 + (1 - \eta_0)(1 - \frac{T}{T_n})(\alpha^{-1})(\gamma^{-1})(1 + (20 + \frac{T}{T_n})^{-1})
$$

(7.12)

where $T_n$, $l_n$ and $w_n$ are the reference torque, length and width of the spall, whose values are dependent on the actual gear working condition and the spall. $\eta_0$ determines the lower limit of $\eta_d$, and is normally between 0.85 and 1 for deep spalls. $\alpha$ and $\gamma$ are the contact coefficients [31, 32], and $\alpha = 1$, $\gamma = 1$ for line contact, $\alpha = 4/3$, $\gamma = 8/3$ for elliptical contact. When line contact is assumed, $\eta_d$ will be 1, and Equation (7.11) will be reduced to Equation (7.10). It should be noted that $\eta_d$ should be different for different contact positions within the spall. However, in this work, we will use a time-independent $\eta_d$ during simulation for the sake of simplicity.

Similar modification should also be considered for the USTE due to the shallow spall whenever the mating tooth rolls across it:

$$
e_{dj}(t) = \eta_s(t, T, l_s, w_s)ds
$$

(7.13)

where $\eta_s(t, T, l_s, w_s)$ is the modification coefficient in the displacement excitation for the shallow spall. The same (or similar) semi-empirical expression described in Equation (7.12) can also be used, except that $\eta_0$ should be much lower for shallow spalls.

In the following section, we will compare the experimentally measured gear response with the simulated response based on the proposed model to show its superiority over the previous model without considering the modification coefficient (the model in [5]).
7.3 Experimental Set-up

In order to validate the superiority of the proposed model against previous similar models, a series of experiments were conducted to measure the vibration response of a gear system with different dimensions of spalling defects working under various load and speed conditions. The schematic diagram of the experimental test rig is shown in Figure 7.7, which consists of a 3:1 ratio speed reduction gearbox, a 1:1 ratio test gearbox, a driving motor, a load motor, and sufficient mounting locations for instrumentation.

![Figure 7.7: Schematic diagram of the experimental set-up (1 – 3:1 ratio speed reduction gearbox; 2 – universal coupling; 3 – encoder on driving shaft; 4 – driving gear; 5 – accelerometer; 6 – bearing; 7 – 1:1 ratio test gearbox; 8 – driven gear; 9 – encoder on driven shaft; 10 – load sensor)](image)

The spalling defects were implemented near the pitch line of the teeth on the driving gear, since that is the position where the tooth surface tends to spall most frequently [1]. Three sets of steel spur gears (Gear #1, 2 and 4) with different defect dimensions were considered and assembled as the driving gear in the 1:1 gearbox. In gear #1, three rectangular-shaped spall defects with different dimensions but the same shallow depth (0.05 mm) were implemented approximately evenly on the driving gear teeth (1st, 7th and 14th tooth). In gear #2, three rectangular-shaped spall defects with different dimensions but the same deep depth (1 mm) were implemented approximately evenly on the driving gear teeth. Gear #4 is healthy without any intentional manufactured defect. The dimensions of each defect on each driving gear are
shown in Table 7.1. The schematic of each defect is shown in Figure 7.9 and Figure 7.10. The photos of the biggest shallow spall (on the 14\textsuperscript{th} tooth on gear #1) and deep spall (on the 14\textsuperscript{th} tooth on gear #2) are shown in Figure 7.11. Gear #5 is also healthy and assembled as the driven gear meshing with each of the driving gears described above during the experiment. It should be noted that all the gears used in this experiment have the same design parameters, as shown in Table 7.2.

![Figure 7.8: Photos of experimental equipment: (a) sensors, (b) test gear pair (1 – accelerometer; 2 – encoder on driving shaft; 3 – encoder on driven shaft; 4 – driving gear; 5 – driven gear)](image)

Three loading conditions were tested, which are approximately equal to 5 Nm (light load), 21 Nm (intermediate load) and 45 Nm (heavy load) respectively. Five running speeds were applied on the driving gear for each loading condition and each meshing gear pair, which were approximately 60 rpm, 120 rpm, 300 rpm, 600 rpm and 800 rpm respectively. Taking the rotating speed of 600 rpm as an example, the rotating frequencies for the driving gear and driven gear should be $f_p = f_p = 10$ Hz, and the mesh frequency should be $f_e = 200$ Hz. The accelerometer was attached on the outer surface of the test gearbox near the faulty gear, in the upright direction (as shown in Figure 7.8). The sampling frequency used was 10 kHz. Each test gear pair was run-in under load for an appropriate amount of time before collecting the vibration data.
Figure 7.9: Gear #1: (a) tooth #1, (b) tooth #7, (c) tooth #14

Figure 7.10: Gear #2: (a) tooth #1, (b) tooth #7, (c) tooth #14

Figure 7.11: Photos of the biggest defects: (a) shallow spall on 14th tooth of gear #1, (b) deep spall on 14th tooth of gear #2
Table 7.1: Dimensions of the spalls

<table>
<thead>
<tr>
<th>Gear number</th>
<th>Tooth number</th>
<th>#1</th>
<th>#7</th>
<th>#14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gear #1 (shallow spalls)</td>
<td></td>
<td>$l_s = 1$ mm</td>
<td>$l_s = 1$ mm</td>
<td>$l_s = 3$ mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_s = 0.05$ mm</td>
<td>$d_s = 0.05$ mm</td>
<td>$d_s = 0.05$ mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$w_s = 0.25$ inch</td>
<td>$w_s = 0.5$ inch</td>
<td>$w_s = 0.5$ inch</td>
</tr>
<tr>
<td>Gear #2 (deep spalls)</td>
<td></td>
<td>$l_s = 1$ mm</td>
<td>$l_s = 1$ mm</td>
<td>$l_s = 3$ mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_s = 1$ mm</td>
<td>$d_s = 1$ mm</td>
<td>$d_s = 1$ mm</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$w_s = 0.25$ inch</td>
<td>$w_s = 0.5$ inch</td>
<td>$w_s = 0.5$ inch</td>
</tr>
<tr>
<td>Gear #4</td>
<td></td>
<td>No defect</td>
<td>No defect</td>
<td>No defect</td>
</tr>
<tr>
<td>Gear #5</td>
<td></td>
<td>No defect</td>
<td>No defect</td>
<td>No defect</td>
</tr>
</tbody>
</table>

Table 7.2: Parameters of the spur gear pair

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Driving gear</th>
<th>Driven gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teeth number $Z_i$</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Mass $m_g$ (kg)</td>
<td></td>
<td>0.753</td>
</tr>
<tr>
<td>Polar moment of inertia $I_p$ (kg·m²)</td>
<td></td>
<td>1.17×10⁻³</td>
</tr>
<tr>
<td>Module $m$ (inch/mm)</td>
<td></td>
<td>0.2/5.1</td>
</tr>
<tr>
<td>Helix angle $\beta$ (degree)</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Pressure angle $\phi$ (degree)</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Face width $W_o$ (inch/mm)</td>
<td></td>
<td>0.5/12.7</td>
</tr>
<tr>
<td>Material</td>
<td></td>
<td>steel</td>
</tr>
</tbody>
</table>

7.4 Comparisons and Discussions

The model has a number of user-specified parameters. In order to correlate with experimental results, the reference $T_n$, $l_n$ and $w_n$ are set as 200 Nm, 3 mm and 0.5 inch (12.7 mm). $\eta_0$ is 0.9 and 0.5 for the deep spall and shallow spall respectively. The mesh stiffness per unit length $k_0$ is $1.197\times10^{10}$ N/m/m [26].
bearing radial stiffness $k_b$ is one of the main parameters determining the primary resonance frequency of the gear system, and was found to be approximately $2 \times 10^8$ N/m. A random uniform distribution of the tooth profile error ($e_m$ in Equation (7.5)) on the interval [0, 50 μm] was applied on the gears during the simulation accounting for the surface roughness of tooth profiles as the test gears are not ultra-precision ground.

The dynamic response $q$ of the test gear pairs was obtained by solving Equation (2.33) using the Matlab ODE 45 subroutine. The velocities and accelerations of the gear system in the LOA ($U$-axis) and OLOA ($V$-axis) directions were achieved by integrating the corresponding displacements in $q$. The vertical acceleration $\ddot{y}_1$ of the driving gear was obtained by using:

$$\ddot{y}_1 = \dot{u}_1 \cos \varphi - \dot{v}_1 \cos \varphi$$

(7.14)

where $\dot{u}_1$ and $\dot{v}_1$ are the accelerations of the driving gear in the LOA and OLOA direction. $\varphi$ is the pressure angle of the test gear pair. All values (including displacement, velocity and acceleration) are normalized with respect to the healthy gear results in order to make them comparable [1]. This was done by dividing the values relating to the faulty gears results with the corresponding RMS value of the healthy gear results so that the influence of the gearbox structure on the vibration signal collected from the accelerometer can be minimized as this influence was not included in the simulation.

### 7.4.1 Influence of the Spall Size

Figure 7.12 shows the comparisons of the measured time-history acceleration $\ddot{y}_1$ of the driving gear in the vertical direction between the gear pairs with shallow spalls (gear #4 and gear #1) and deep spalls (gear #4 and gear #2) when the rotating speed is 600 rpm and the load is 45 Nm. It can be found that the time domain signal, for both deep and shallow spall cases, is characterized by the presence of periodic impulses (or shocks) whenever the mating tooth pair rolls over the biggest defects on tooth #14. The amplitudes of the impulses for the deep spalled gear pair are about 2 times larger than those for the
shallow spalled gear pair. The impulses due to the intermediate spalls on tooth #7 are slightly present within the signal, and at significantly lower amplitude than the large spall impulses. The impulses due to the smallest defects on tooth #1 are completely immersed into the background noise.

Figure 7.12: Comparisons of experimentally measured response $\ddot{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm

Figure 7.13: Comparisons of simulated dynamic response $\ddot{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm: (a) model in [5], (b) model in this chapter
Figure 7.13(a) shows the comparisons of the simulated dynamic response $\ddot{y}_i$ without considering the modification coefficients (the model built in [5]) between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm. It can be found that there are 2 distinct impulses every rotation of the driving gear with shallow spalls and deep spalls, which are due to the defects on tooth #7 and #14 respectively. The reason for the obvious impulses due to tooth #7 (compared with the slight impulses due to tooth #7 in the experimental results as shown in Figure 7.12) is that the defect model built in [5] did not consider the influence of the practical nonlinear tooth surface contact behaviour on the defect excitations. Taking the shallow spall case for example, the smallest size of the spall on tooth #1 make the mating tooth most easily bridge over it so that its excitation to the gear dynamic response is negligible. The small length of the spall on tooth #7 may make the mating tooth barely contact at the bottom, whereas the biggest size of the spall on the tooth #14 make the mating tooth pair directly contact at the its bottom so that its excitation is maximum among all. Therefore, the defect model in [5] will exaggerate the defect excitation when the mating tooth pair bridges over a spall with a small size.

Figure 7.13(b) shows the simulated dynamic response $\ddot{y}_i$ considering the modification coefficients (the model built by the author). It can be found that there are slight impulses due to the defects on tooth #7, even though the defects on tooth #7 and tooth #14 have the same depth and width. This indicates that the sizes of the spalls on tooth #1 and #7 are so small compared with those of the spalls on tooth #14 that the excitations (either mesh stiffness reduction or displacement excitations) due to the larger sized spalls is more significant than those due to the smaller sized spalls as proved by Figure 7.6.

Figure 7.14 shows the power spectrum density (PSD) of the measured acceleration $\ddot{y}_i$ and the simulated responses based on the proposed model. It can be found that that there are broad-band frequency components near 2 kHz, which is the primary resonance frequency of the gear system. For the gear pair with shallow spalls, the interval of the primary sidebands is approximate 200 Hz, which is the mesh
frequency. For the gear pair with deep spalls, the sidebands with the interval of mesh frequency still exist. However, the close-up plot near the primary resonance region clearly shows multiple sidebands near the primary resonance frequency, and the interval of the sidebands is approximate 10 Hz, which is the rotating frequency of the driving gear $f_p$. This demonstrates that the spalling defects induce on the gear system periodical short impact shocks which stimulate the structure resonances [33], especially for the gear pair with deep spalls. The above phenomenon is also obvious in the PSD of the simulated results based on the proposed model as shown in Figure 7.14(b), which also have the broad-band frequency components near 2 kHz. The PSD for the gear pair with deep spalls are dominated by the sidebands with the interval of the rotating frequency of the driving gear, which proves that the huge periodic impacts due to the deep spalls strongly excites the structure resonance of the gear system.

![Normalized PSD for shallow spall](image1) ![Normalized PSD for deep spall](image2)

**Figure 7.14:** Comparisons of the power spectrum of $\ddot{y}_1$ between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 45 Nm: (a) experimental results, (b) simulation results based on the proposed model

### 7.4.2 Influence of the Load

Figure 7.15 shows the measured time-history acceleration $\ddot{y}_1$ of the driving gear for the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 21 Nm and 5 Nm respectively. Together with the measured results when the load is 45 Nm as shown in Figure 7.15(a), it
can be found that as the load decreases, the periodic impulses due to the smaller spalls on tooth #1 and #7 are more and more obvious. This demonstrates that a light load between the mating teeth alleviates the influence of the nonlinear tooth surface contact behaviour on the defect excitations as shown in Figure 7.15, especially for the small dimensional defects.

Figure 7.15: Comparisons of experimentally measured response $\ddot{y}_1$ between the gear pairs with shallow spalls and deep spalls when rotating speed is 600 rpm and load is: (a) 21 Nm, (b) 5 Nm

Figure 7.16: Comparisons of the simulated dynamic response $\ddot{y}_1$ without considering the modification coefficients between the gear pairs with shallow spalls and deep spalls when rotating speed is 600 rpm and the load is: (a) 21 Nm (b) 5 Nm
Figure 7.17: Comparisons of simulated dynamic response $y_1$ considering the modification coefficients between the gear pairs with shallow spalls and deep spalls when rotating speed is 600 rpm and the load is: (a) 21 Nm (b) 5 Nm

Figure 7.16 shows the simulated response $y_1$ without considering the modification coefficients (the model built in [5]) for the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 21 Nm and 5 Nm respectively. There are two distinct impulses every rotation of the driving gear for both cases, which are due to the defects on the tooth #7 and #14 respectively, the amplitudes of the impulses due to tooth #7 are only slightly smaller than those due to tooth #14, which contradicts the measured results.

Figure 7.17 shows the simulated dynamic response $y_1$ considering the modification coefficients (the model built by the author) between the gear pairs with shallow spalls and deep spalls when the rotating speed is 600 rpm and the load is 21 Nm and 5 Nm respectively. It can be found that as the load decreases, the smaller impulses due to the defects on tooth #7 become more and more obvious. Therefore compared with the model built in [5] where the linear line contact was assumed, the proposed model that takes the nonlinear contact pattern into consideration can yield more realistic and accurate results.
There are still some discrepancies between the simulated results based on the proposed model and the experimentally measured results. For example, the amplitudes of the simulated response are generally larger than those of the measured results. This may be partially attributed to the vibration transmission loss from gear meshes to the upper surface of the gearbox housing where the sensor was mounted. In addition, the impact impulses in the measured results are noisier and unsteady compared with those shown in the simulated results. Apart from the factors including the load, speed fluctuation and some other gear assembling errors which are not considered during simulation, another important reason is because the proposed model using time-invariant modification coefficients is still limited to account for the interactions between the nonlinear contact pattern and the tooth surface defects. In general, the simulation results based on the proposed model agree with the experimentally measured results.

7.4.3 Statistical Study

It is well known that potential gear or bearing defects induce a periodical impulsive nature on the vibration signal of rotating machines [33]. Therefore, some simple signal metrics like Kurtosis can be used for the detection of the mechanical faults within a gear transmission system. In this section, several statistical indicators are evaluated based on the normalized time domain waveforms of the measured results, as well as the simulated results for various rotating speeds and loads. They are RMS, Kurtosis ($Kurt$), third statistical moment parameter ($Skewness, S_r$), index of amplitude ($C$, or Crest Factor) and the Correlated Kurtosis ($CK$) proposed in [34], which are expressed as:

$$RMS = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2}, \quad \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$Kurt = \frac{\left(\frac{1}{N}\right) \sum_{i=1}^{N} (x_i - \bar{x})^4}{\left[\left(\frac{1}{N}\right) \sum_{i=1}^{N} (x_i - \bar{x})^2\right]^2}$$

$$S_r = \frac{\left(\frac{1}{N}\right) \sum_{i=1}^{N} [(x_i - \bar{x})^2]^{3/2}}{\left[\left(\frac{1}{N}\right) \sum_{i=1}^{N} (x_i - \bar{x})^2\right]^{3/2}}$$

$$C = \frac{\max(x_i)}{RMS}$$

$$CK_M(T) = \frac{\sum_{i=1}^{N} (\prod_{m=0}^{M} x_{i+mT})^2}{\sum_{i=1}^{N} x_i^{M+1}}$$

(7.15-a, b, c, d, e, f)
where \( N \) is the number of sample data points, \( x_i \) is the value of the \( i \)th data point, \( T \) is the period of interest (the period of the fault signature that needs to be identified), and \( M \) is the shift [34, 35]. In this study, the first shift \( M = 1 \) is studied.

All the statistics described above except RMS are extremely sensitive to the impulses within a signal. \textit{Kurt} has higher sensitivity to incipient faults in gear systems. However, it also shows higher susceptibility to spurious vibration and noise [36]. \( S_r \) is superior to kurtosis from the viewpoint of robustness. \( C \) is mainly used to indicate the intensity of impactive power [1]. \( CM_i(T) \) is quite suitable for repetitive impulses when \( T \) matches with the period of the impulses.

<table>
<thead>
<tr>
<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotating speed ( \Omega_i ) (rpm)</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>600</td>
<td>600</td>
<td>600</td>
<td>800</td>
<td>800</td>
<td>800</td>
</tr>
<tr>
<td>Load ( T_i ) (Nm)</td>
<td>5</td>
<td>21</td>
<td>45</td>
<td>5</td>
<td>21</td>
<td>45</td>
<td>5</td>
<td>21</td>
<td>45</td>
</tr>
</tbody>
</table>

In the following section, the variations of these statistical indicators under 3 different rotating speeds (300, 600 and 800 rpm) and 3 different loads (5, 21 and 45 Nm) for the experimental signal, and simulated signal based on the previous model [5] and the proposed model are compared. There will be nine cases of the working condition as shown in Table 7.3. The cases when the rotating speeds are 60 and 120 rpm are not considered due to the highly expensive computation in the simulation for extremely low speed.
Figure 7.18: Comparisons of the statistical values for the gear pair with deep spalls under different cases: (a) Kurt, (b) S, (c) C, (d) CK$_1$

Figure 7.18 shows the variations of the four statistical values for the gear pair with deep spalls. It can be found that under the same load, the Kurt, S, and C normally decrease as the speed increases for all the signal sources. Under the same speed, the above three indicators are relatively stable with the load variation for the simulated signals whereas varied significantly for the experimental signal. One of the reasons may be due to the existence of spurious impacts and vibration in the experimental signal. There are significant differences in the values of CK$_1$ between the simulation signals and experimental signal. The reason is that the slight fluctuation in speed makes the period of the experimental impact impulses not constant, which significantly affects CK$_1$ as it will reach maximum whenever $T$ in Equation (7.15-f)
matches the period of the repetitive impulses perfectly. It should also be noted that $CK_1$ is highly sensitive to the number of data points in the signal for calculation ($N$ in Equation (7.15-f)). The longer the number of data points for calculation, the smaller the $CK_1$. Therefore, from the viewpoint of a robust condition monitoring strategy, $CK_1$ is not recommended. In general, the trend of the statistical indicators with load and speed variations for the simulated signal (especially for the simulated signal based on the proposed model) agrees with that of the statistical indicators for the experimental signal.

![Graphs showing variations in statistical values for the gear pair with shallow spalls under different cases](image)

**Figure 7.19**: Comparisons of the statistical values for the gear pair with shallow spalls under different cases: (a) $Kurt$, (b) $S_r$, (c) $C$, (d) $CK_1$

Figure 7.19 shows the variations of the four statistical values for the gear pair with shallow spalls. It shows some similar phenomenon as shown in Figure 7.18, but there are more differences. It seems that
the impact impulses due to smaller spalls are more sensitive to the load variations than the impact impulses due to the larger spalls. One noticeable phenomenon is that, for all signal sources, the values of these four indicators for the gear pair with shallow spalls are smaller than those for the gear pair with deep spalls. It seems that, in the case of the gear pair with shallow spalls, the trend of the statistical indicators with load and speed variations for the simulated signal are not consistent with that of the statistical indicators for the experimental signal, which suggests the necessity of a further improvement of the proposed model in order to accurately model the dynamic response of the gear system with shallow spalls.

7.5 Conclusions

In this chapter, according to the different excitation mechanisms of the spalls to the gear vibration, two types of spalling defects were classified: small depth spalls that generate external displacement excitation, and deep depth spalls that modify the gear mesh stiffness. A new dynamic model of the general cylindrical gear pair with localized spalling defects was then established. The main improvement of this model against previous similar models is that it takes the effect of the nonlinear elliptical contact pattern on the excitations due to spalling defects into consideration by introducing modification coefficients, which are mainly dependent on the transmitted load and the geometries of the spall. A semi-empirical expression for this coefficient was suggested. The experiment using several 1:1 ratio spur gear pair sets with different dimensions of spalling defects was conducted under various load and rotating speed conditions. By comparing the experimentally measured responses with the simulated responses based on the proposed model as well as a previous model in terms of the time-history acceleration, power spectrum and some statistical indicators, it was found that the nonlinear elliptical contact patterns do affect the excitations due to spalls especially with small size and running under heavy load. Therefore, the proposed model can yield more realistic and accurate results.
However, there are still some discrepancies between the simulated results based on the proposed model and the experimental results, which suggest the necessity of a further improvement of the proposed model. This should be the focus of our future research on the same matter.

7.6 References


Chapter 8

Conclusions

8.1 Summary

This thesis mainly focuses on the dynamic modelling of the cylindrical gear transmission system with and without localized tooth defects. A series of studies have been conducted to reflect different aspects of gear dynamics, as summarized below.

In Chapter 3, an analytical method to calculate the loaded static transmission error (LSTE) and variable-variable mesh stiffness (VVMS) considering the corner contact effect for a spur gear pair with tip relief was established. Two types of gear mesh stiffness model (fix-variable mesh stiffness (FVMS) and VVMS) and three types of commonly used single degree of freedom (SDOF) models (the FVMS model, the VVMS model and the LSTE model) were identified. The experimental results reported in the literature were used to validate the correctness of each model, which proves that the corner contact effect should not be neglected in the gear dynamic analysis when no or an insufficient amount of profile modification is applied under heavy load. Detailed comparisons of the steady-state responses based on these models under different amounts of tip relief were made. Advantages and disadvantages of each model were generalized. It was found that the proposed VVMS model yields the most consistent results when compared with the experimental results. The FVMS model disregarding the corner contact effect, can also produce consistent results when a sufficient amount of tip relief is applied on the gears. The LSTE model is approximately equivalent to the VVMS model as long as the fluctuation of the LSTE (or VVMS) is comparatively small.

In Chapter 4, a general dynamic model for a cylindrical geared rotor system with local tooth profile errors and global mounting errors was developed. The experimental results reported in the literature were
employed to validate the capability of the proposed model in simulating the dynamic response of a cylindrical geared rotor system. The generation mechanism of the dynamic coupling terms (between gear translational and rotational motions) in the inertial force due to gear eccentricity was explained. Finally, an intensive parameter study was performed to research the influence of various parameters on the dynamic coupling behaviour between gear translational and rotational motions. It was found that the dynamic coupling term is obvious in the direction of off-line of action (OLOA) when a gear pair (with gear eccentricities) is running at relatively low-speed range where the resonances of gear torsional vibration are most likely to be excited. Gear tooth profile errors, number of teeth, helical angle, and the magnitudes of gear eccentricities will all affect the dynamic coupling behaviour between the translational motion in OLOA direction and torsional motion.

In Chapter 5, the influence of the addendum modification on the spur gear back-side gear mesh stiffness (GMS) was investigated. The analytical relationship between the back-side and drive-side GMS for a spur gear pair with various amounts of addendum modification was determined. It was found that the tooth thickness at the pitch circle will affect the phase shift between these two forms of GMS. In order to study gear dynamic behaviour using the proposed asymmetric GMS model, two typical cases where the back-side tooth impact may most likely happen were analysed and discussed. In the first case, a dynamic model with 6 degrees of freedom (DOF) for a spur gear pair with different amounts of addendum modification excited by an angular acceleration excitation was built. A complex tooth contact force model was adopted as it was proved to produce consistent results with the experimental measured results in the literature. In the second case, a conventional SDOF model based on the symmetric backlash type function was used to examine the effect of the addendum modification on the classic jump phenomenon during speed-increasing and speed-decreasing sweeps. It was found that the addendum modification can affect gear dynamics through the back-side mesh stiffness. Thus, traditional symmetric mesh stiffness model assuming identical GMS in the back-side and drive-side directions, cannot precisely capture some dynamic effects when the gears are working under light load or idling conditions.
In Chapter 6, the traditional plane crack model used to calculate the GMS and tooth inclination deformations of a spur gear pair with a localized plane crack was extended to the spatial crack case based on the slicing principle. Modified expressions were thus derived for the evaluation of the GMS and the inclination deformation for a spur gear pair with a localized spatial crack which is more typical and common in practical situations. A 6 DOF model was built to study the dynamic performance of a spur gear pair with a localized spatial crack, mainly including the dynamic load distributions on the cracked tooth flank and the tilting motions of the gears. Simulation results demonstrated the predominant influence played by the tooth inclination deformations over the reduction of the mesh stiffness resulting from the tooth fillet crack. It was found that special attention should be paid to the tooth inclination deformations for the early detection of the initial crack damage. Besides, the spatial crack with non-uniform crack depth will lead to unevenness of the dynamic load distribution on the cracked tooth flank. Tilting motions can therefore be excited whenever the cracked tooth comes into the mesh.

In Chapter 7, a new dynamic model of a general cylindrical gear pair with a localized spalling defect was introduced. The main improvement of this model over previous similar models is that it takes the effect of the nonlinear elliptical contact patterns on the excitations due to spalling defects into consideration by introducing modification coefficients, which are mainly dependent on the transmitted load and the spall size. A semi-empirical expression for this coefficient was suggested. The experiment of several 1:1 ratio spur gear pair sets with different dimensions of spalling defects was conducted under various load and rotating speed conditions. The experimental measured responses were directly compared with the simulated responses based on the proposed model as well as a previous model in terms of the time-history acceleration, power spectrum and some statistical indicators. It was found that the nonlinear elliptical contact patterns do affect the excitations due to spalls especially with small size and running under heavy load. Therefore, the proposed model can yield more realistic and accurate results.
8.2 Contributions

The studies made in this project enrich the current literature on the dynamic modelling of cylindrical gear transmission systems with the considerations of some secondary effects and localized tooth defects. The main contributions of this thesis to the literature can be summarized as:

1. An analytical method to study corner contact effect for a gear pair with tip relief was established. The remaining confusion regarding to the usage of two distinct types of GMS in the literature was clarified. Systematic comparisons among several types of SDOF model used in the literature were made. It has been shown that all of the SDOF models used in the literature can generally provide consistent results with the experimental results. These models are basically equivalent to each other if specific conditions are satisfied.

2. The generation mechanism of the dynamic coupling behaviour between the torsional vibration and translational vibration of a cylindrical gear pair with gear eccentricities was explained. The conditions that the dynamic coupling behaviour should not be neglected have been pointed out.

3. An analytical formula describing the relationship between the back-side and drive-side GMS for a spur gear pair with addendum modifications was derived. It has been proved that traditional symmetric mesh stiffness models cannot precisely capture some dynamic effects when the gears are working under light load or idling conditions.

4. The traditional plane crack model was extended to the more typical spatial crack model. Modified expressions for the evaluation of GMS and tooth inclination deformations of a spur gear pair with a localized spatial crack were derived. It was found that the spatial crack affects gear dynamic behaviour in terms of tilting motion and dynamic load distributions on the cracked tooth flank.

5. A more advanced model for a cylindrical gear pair with a localized spalling defect was introduced. A semi-empirical expression was suggested to modify the excitations due to the spall considering the
nonlinear contact pattern. Experimental vibration data (acceleration) from several 1:1 ratio spur gear sets with different dimensions of spalling defects under various load and rotating speed conditions were provided to verify the superiority of the proposed model.

8.3 Future Directions

There are three main directions that can be followed for the future research based on the present study:

1. Analyse the encoder data to study the influence of the spalling defects on the transmission error, and validate the simulation results from the new model proposed in Chapter 7. Follow-up tests are required, since it was found that currently, the encoder on the driving shaft counted many more impulses than those from the encoder on the driven shaft.

2. The tests on a gear (gear #3) with localized spatial cracks have already been conducted. The measured results (acceleration and transmission error) can be analysed and compared with the simulation results from the model proposed in Chapter 6. More follow-up tests may be required as the fault features in the measured signature due to the spatial crack are much less obvious that those due to spalling defects.

3. Currently, the research focus of this project is cylindrical gear pair transmission systems. The proposed new models with the consideration of some secondary effects and localized tooth defects can be extended to multi-stage, multi-mesh, or even planetary gear sets.
Appendix A: Publications

A.1 Peer-reviewed Journal


A.2 International Conference

