Controllability of underactuated coupled parabolic systems

by

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Abstract

This thesis studies the null controllability of a system of coupled parabolic PDEs. Moreover, our work specializes to an important subclass of these control problems, where systems are underactuated and are coupled by first and zero-order couplings. We pose our control problem in a fairly new framework which divides the problem into interconnected parts: we refer to the first part as the analytic control problem, where we use slightly non-classical techniques to prove null controllability by means of internal controls appearing on every equation; we refer to the second part as the algebraic control problem, where we use an algebraic method to invert a linear partial differential operator that describes our system, which allows us to recover null controllability by means of internal controls which appear on only a few of the equations. By solving these control problems concurrently, we resolve the original problem (after some technical verifications on the regularity of the controls in the analytic system).

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Chapter 1

Introduction

1.1 Motivation and literature review

In recent years, problems concerning the controllability of coupled parabolic PDEs have received much interest from the mathematical control community (see [4] and references therein). One classification of these numerous control problems is into problems with zero-order couplings between equations (i.e., the reaction term in a usual parabolic PDE is now replaced with terms which couple the evolution of the solution with the solutions to other PDEs in the system) and problems with first-order couplings between equations (i.e., the advection term is now replaced with terms which couple the evolution of the solution with the gradient of the solutions to other PDEs in the system). The applications of such control problems are ubiquitous: zero-order couplings arise in engineering problems modelled by reaction-diffusion equations, such as [8, 16, 30]; whereas first-order couplings arise in engineering problems modelled by reaction-advection-diffusion equations, such as [13, 26, 27, 31].

For systems of several coupled parabolic equations, an important problem is to establish their controllability with reduced number of controls; we refer to such systems with reduced controls as underactuated systems of coupled parabolic PDEs. For the case of zero-order couplings and with internal controls, this control problem has been studied extensively in [2, 3]. In [3], a necessary and sufficient condition for exact controllability (similar to the Kalman rank condition for finite-dimensional systems) is proved for a system of m equations with constant coupling coefficients. In [2], some results similar to the Silverman-Meadows condition are obtained for time-varying coefficients.

General conditions for controllability of systems with first and zero-order couplings and internal controls have proven to be more elusive. In [22], a system of $n+1$ coupled heat equations with constant couplings and with one underactuation is studied, and a sufficient condition for null controllability is given under some restrictions on the controls. In [6], a system of three parabolic equations coupled by (time and space) varying coefficients is studied for two underactuations. The authors were able to recover a null controllability condition under some technical restrictions on the control domain and the coupling terms. In [15], a necessary and sufficient condition for null controllability is given for a system of m equations with one underactuation and constant coupling coefficients; furthermore, the authors study the case of (time and space) varying coupling coefficients and prove a sufficient controllability condition for a system of two equations with one underactuation, under some technical conditions.

1.2 Statement of contributions

This work has three main contributions. The first contribution is Theorem 3.1.5, which partially generalizes [15, Theorem 1]. In particular, our result gives a sufficient condition for the null and approximate controllability of an underactuated system of coupled parabolic PDEs, with constant first and zero-order couplings, when more than half of the equations are actuated. This controllability condition, which is the non-singularity of a matrix containing the coupling coefficients (and possibly zeros) as entries, is generic in most cases (where in this context, by “generic condition” we mean that “a generic matrix is invertible”). The technique used to prove this result is adapted from [13], where it was first introduced.

Secondly, in the cases where this controllability condition may not be generic (hence, we recover a “non-generic matrix” which may be singular due to sparsity), we characterize precisely why these non-generic conditions arise in our treatment. At the end of Section 4.4, we demonstrate the technical nature of these non-generic cases and show how they are artifacts of our treatment.

Our final contribution is Proposition 5.3.1, which is an extension of [15, Proposition 2.2]. Specifically, our Carleman estimate contains higher differential order terms on the lefthand side of (5.14), which allows us to construct very regular controls in Proposition 6.2.1. Importantly, these highly regular controls may be necessary when applying Theorem 3.1.5 to engineering problems with many underactuations, as discussed Remark 4.4.5.

1.3 Organization

This work is organized as follows: in Chapter 2, foundational mathematical preliminaries, such as Sobolov space theory, existence and uniqueness of solutions for coupled parabolic PDEs, parabolic regularity and sparse matrix theory are presented; one familiar with PDE theory can easily skip this chapter and refer to it as necessary.

In Chapter 3, a general coupled parabolic PDE is posed as a control system with internal controls, and the usual notions of controllability are presented. Furthermore, this work's main result, Theorem 3.1.5, is stated here.

Chapter 4 introduces two coupled parabolic control problems related to the original one, which we call the *analytic* and *algebraic* control problems. This chapter gives details on the treatment we employ to prove this work's main controllability result. Furthermore, this chapter specializes to studying the algebraic control problem, and we pose this problem under a fairly new framework which has the goal of *algebraically inverting* a linear partial differential operator describing our algebraic problem. This algebraic inversion allows us to construct a solution to the algebraic control problem. One assumption made here is that the controls for the analytic problem be highly regular.

In Chapter 5, a *Carleman estimate* is established for solutions to the adjoint of the analytic control problem, and this estimate is used to prove a *weighted* observability inequality for this control problem in Proposition 5.1.1. Importantly, this Carleman estimate contains many higher-order derivative terms on the lefthand side of the inequality, which will allow us to construct highly regular controls for the analytic problem.

Chapter 6 employs the weighted observability inequality established in Chapter 5 and a well-known optimal control result to construct a solution to the analytic control problem satisfying $\tilde{y}(T, \cdot) = 0$. Moreover, the Carleman estimate proved in Chapter 5 is utilized to verify the necessary regularity on the controls in the analytic problem. We conclude this chapter by proving Theorem 3.1.5.

Chapter 7 presents closing remarks for this work and possible future works, and

Chapter 8 contains some proofs of technical results stated in Chapter 5 that were omitted from the main body of this work.

Chapter 2

Mathematical Preliminaries

2.1 Notation and motivation

In this chapter, we introduce some notational conventions and present some mathematical background that we utilize throughout this thesis.

Throughout this work, we define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and similarly, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $n, k \in \mathbb{N}^*$, we denote the set of $n \times k$ matrices with real-valued entries by $\mathcal{M}_{n\times k}(\mathbb{R})$, and we denote the set of $n \times n$ matrices with real-valued entries by $\mathcal{M}_n(\mathbb{R})$. We denote the set of linear maps from a vector space U to a vector space V by $\mathscr{L}(U; V)$. For $(X, \mathcal{T}_X)$ a topological space, we denote the closure of X by $\bar{X}$.

In most fields of engineering, equations which describe the conservation of physical quantities are paramount. Among these *conservation equations*, the heat equation can be used to model (among many other diffusive quantities) the evolution in time of the distribution of heat in a given region. Let $\Omega \subset \mathbb{R}^n$ open and bounded, $t > 0$ and $x \in \Omega$. The *homogeneous* heat equation is given as

$$\partial_t y - \Delta y = 0, \tag{2.1}$$

whereas the *nonhomogeneous* heat equation is given as

$$\partial_t y - \Delta y = r, \tag{2.2}$$

where $r : [0,\infty) \times \Omega \to \mathbb{R}$ is given in (2.2), $y : [0,\infty) \times \Omega \to \mathbb{R}$ is the unknown, and each equation is subject to initial and boundary conditions. Important behaviours of the solutions to these types of diffusion equations are captured by their classification as *parabolic* PDEs. For demonstration purposes, we discuss one of these behaviours for $n = 1$; its counterpart for a generalized parabolic system when $n \geq 1$ is stated and utilized below. To this end, we first define a parabolic PDE for $n = 1$.

Definition 2.1.1. Let $x \in \mathbb{R}$, $t \in [0,\infty)$ and consider the second-order nonhomogeneous PDE given by

$$d\partial_{xx}y + e\partial_{xt}y + f\partial_{tt}y + g\partial_x y + h\partial_t y + ay = r, \tag{2.3}$$

where d, e, f, g, h, a and r are functions of t and x. The general linear PDE (2.3) is said to be parabolic at (t,x) if $\lambda_1(t,x) = 0$ and $\lambda_2(t,x) \in \mathbb{R}^*$, where λ_i denotes the i-th eigenvalue of

$$D := \begin{pmatrix} d & e/2 \\ e/2 & f \end{pmatrix}.$$

Remark 2.1.2. For $n = 1$, an equivalent condition for a PDE to be parabolic at (t,x) is that its discriminant, $e^2 - 4df$, equal zero at (t,x). Indeed, the characteristic polynomial of D is

$$\lambda^2 - (d+f)\lambda + \left(\frac{4df - e^2}{4}\right),$$

and hence one must have $e^2 - 4df = 0$ for 0 to be an eigenvalue of D. It is immediate, then, that (2.1) and (2.2) are parabolic PDEs in $[0, \infty) \times \Omega$. Our study will focus on the case where $e = f = 0$ and $d > 0$ in $[0, \infty) \times \Omega$ (i.e., typical forward diffusion equations). •

Remark 2.1.3 (Parabolic smoothing on a 1-D bar). An important behaviour of parabolic diffusion equations which will be consequential in the work to follow is the parabolic smoothing effect. With the help of formal calculations, the smoothing effect is demonstrated for a 1-D parabolic equation with $g = a = 0$ and $h = d = 1$, i.e., the nonhomogeneous heat equation (2.2).

For $T > 0$, let $\Omega = (0, l)$ and $\Gamma := (0, T) \times \partial\Omega$; consider the solution to (2.2), where we additionally require $y(t, x) = 0$ on Γ and $y(0, x) = y^0(x)$ in Ω. We multiply (2.2) by y and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^l (y(t,x))^2 dx + \int_0^l (\partial_x y(t,x))^2 dx = \int_0^l y(t,x) r(t,x) dx.$$

Integrating now over $[0, t]$ and applying Cauchy–Schwarz and Young's inequalities with $\epsilon > 0$ yields

$$\begin{aligned}
\frac{1}{2}\int_0^l (y(t,x))^2 dx - \frac{1}{2}\int_0^l (y^0(x))^2 dx + \int_0^t \int_0^l (\partial_x y(t,x))^2 dx dt & \qquad (2.4)\\
= \int_0^t \int_0^l y(t,x) r(t,x) dx dt & \\
\leq \left(2\epsilon \int_0^t \int_0^l (y(t,x))^2 dx dt\right)^{1/2} \left(\frac{1}{2\epsilon}\int_0^t \int_0^l r^2(t,x) dx dt\right)^{1/2} & \\
\leq \epsilon \int_0^t \int_0^l (y(t,x))^2 dx dt + \frac{1}{4\epsilon}\int_0^t \int_0^l r^2(t,x) dx dt & \\
\leq T\epsilon \sup_{t\in[0,T]} \int_0^l (y(t,x))^2 dx dt + \frac{1}{4\epsilon}\int_0^T \int_0^l r^2(t,x) dx dt. &
\end{aligned}$$

Choosing $\epsilon = \frac{1}{4T}$ and taking the supremum of (2.4) over $t \in [0,T]$ gives

$$\frac{1}{4}\sup_{t\in[0,T]}\int_0^l (y(t,x))^2dx + \int_0^T\int_0^l (\partial_x y(t,x))^2 dxdt \leq T\int_0^T\int_0^l r^2(t,x)dxdt + \frac{1}{2}\int_0^l (y^0(x))^2dx.$$

This last inequality demonstrates the parabolic smoothing effect: the L^2-norm of the spatial derivative of the solution y is bounded by the L^2-norms of its initial condition and of the forcing function r. Hence, for regular enough initial condition and forcing function, one can ascertain higher spatial regularity of the solution to (2.2). These formal calculations can be adapted for any parabolic diffusion equation given by (2.3) for $e = f = 0$ in $(0,T)\times\Omega$ and, importantly, $d \neq 0$ for all $(t,x) \in (0,T)\times\Omega$. •

We return to the case of arbitrary $n \geq 1$ and study a generalized heat equation. Let $Q_T := (0,T)\times\Omega$ and $\Sigma_T := (0,T)\times\partial\Omega$ for some $T > 0$; consider the second-order PDE

$$\begin{cases} \partial_t y + \mathcal{L}y = r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $r : Q_T \to \mathbb{R}$ and $y^0 : \Omega \to \mathbb{R}$ are known, $y : \bar{Q}_T \to \mathbb{R}$ is the unknown, and for each $t \in (0,T)$, $\mathcal{L}$ denotes the second-order linear differential operator given by

$$\mathcal{L}y = -\sum_{i,j=1}^{n} \partial_{x_j}\left(d^{ij}(t,x)\partial_{x_i}y\right) + \sum_{i=1}^{n} g^i(t,x)\partial_{x_i}y + a(t,x)y, \tag{2.6}$$

for given coefficients d^{ij}, g^i, a, for $i,j \in \{1,\dots,n\}$. Equation (2.5) can be used to describe the evolution in time of the distribution of some quantity y, where the second-order term models diffusion, the first-order term models advection, the zero-order term models linear generation or depletion, and the forcing function accounts for external heat sources or sinks. As we will see, the qualitative properties of (2.5) (e.g., smoothing) are very similar to those of (2.2). Let us first give some definitions that help us classify (2.5).

Definition 2.1.4. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index and denote $\partial_{\alpha_1} \cdots \partial_{\alpha_n} y$ by $\partial_\alpha y$. For $k,\ l \in \mathbb{N}$ and $(d_\alpha)_\alpha$ coefficients depending on α, where $d_\alpha : Q_T \to \mathbb{R}$, a linear time-variant differential operator of order $l = 2k$ on Ω given by

$$\mathcal{L}y = \sum_{|\alpha| \leq l} d_\alpha(t,x)\partial_\alpha y$$

satisfies the uniform ellipticity condition if there exists $C > 0$ such that,

$$\sum_{|\alpha|=l} d_\alpha(t,x)\xi^\alpha \geq C|\xi|^l, \qquad \forall\, \xi \in \mathbb{R}^n, \forall (t,x) \in Q_T, \tag{2.7}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Remark 2.1.5. Suppose $n = 1$. If $\mathcal{L}$ satisfies the uniform ellipticity condition, then (2.5) is parabolic in Q_T. Indeed, if (2.7) is satisfied (where $l = 2$), then we have that $\exists\, C > 0$ such that

$$d(t,x)\xi^2 \geq C\xi^2, \qquad \forall\, \xi \in \mathbb{R}, \forall\, (t,x) \in Q_T,$$

and hence we have that $d(t,x) > 0\ \forall\, (t,x) \in Q_T$. Since $e = f = 0$ in Q_T, $\lambda_1(x,t) = 0$ and $\lambda_2(x,t) = d(x,t) > 0$. For $n > 1$ and $l = 2$, the uniform ellipticity condition is equivalent to the eigenvalues of $(d_{\alpha_1,\alpha_2})_{1 \leq \alpha_1, \alpha_2 \leq n}$ being bounded below by zero uniformly in x and t. This motivates the following definition. •

Definition 2.1.6. A partial differential operator $\partial_t + \mathcal{L}$ is (uniformly) parabolic if $\mathcal{L}$ satisfies the uniform ellipticity condition.

Of greater interest in many areas of engineering is the study a *system of second-order parabolic PDEs* (e.g., [28], [32]). We express systems consisting of m equations in vector form as

$$\begin{cases} \partial_t y + \mathcal{L}y = r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.8}$$

where $y^0 := (y_1, \ldots, y_m)$ and $r := (r_1, \ldots, r_m)$ are known, $y := (y_1, \ldots, y_m)$ are the unknowns, and the differential operator $\mathcal{L}$ is now defined as

$$\mathcal{L}y = \sum_{k=1}^{m} \left(- \sum_{i,j=1}^{n} \partial_{x_j} \left(d_k^{ij}(t,x) \partial_{x_i} y_k \right) + \sum_{i=1}^{n} g_k^i(t,x) \partial_{x_i} y_k + a_k(t,x) y_k \right) \mathbf{e_k},$$

where $\mathbf{e_k}$ is the k-th canonical basis vector in $\mathbb{R}^m$. Yet another very practical extension of this system of second-order PDEs is when the *equations within the system of parabolic PDEs are coupled* (e.g., [4, 23, 30]): denoting the p-th entry of $\mathcal{L}y$ as $\mathcal{L}_p y$ for $p \in \{1, \ldots, m\}$, we now have

$$\mathcal{L}_p y = \sum_{k=1}^{m} \left(- \sum_{i,j=1}^{n} \partial_{x_j} \left(d_{pk}^{ij}(t,x) \partial_{x_i} y_k \right) + \sum_{i=1}^{n} g_{pk}^i(t,x) \partial_{x_i} y_k + a_{pk}(t,x) y_k \right). \tag{2.9}$$

When $p \neq k$, we call d_{pk}^{ij} the *second-order coupling coefficients*, g_{pk}^i the *first-order coupling coefficients*, and a_{pk} the *zero-order coupling coefficients*. This work studies a particular case of first and zero-order constant coupling coefficients, where for δ_{ij}

denoting the Kronecker delta function, $d_{pk}^{ij}(t,x) = d_p^{ij}\delta_{pk} \in \mathbb{R}$, $g_{pk}^i(t,x) = -g_{pk}^i \in \mathbb{R}$ and $a_{pk}(t,x) = -a_{pk} \in \mathbb{R}$, for $i,j \in \{1,\dots,n\}$ and $p \in \{1,\dots,m\}$. Additionally, we study the case where $d_p^{ij} = d_p^{ji}$, for $i,j \in \{1,\dots,n\}$ and $p \in \{1,\dots,m\}$. Hence, we can write $\mathcal{L}y$ as

$$\mathcal{L}y = \sum_{p=1}^{m}\left(-\mathrm{div}(d_p\nabla y_p) - \sum_{k=1}^{m} g_{pk}\cdot\nabla y_k - \sum_{k=1}^{m} a_{pk}y_k\right)\mathbf{e_p}, \tag{2.10}$$

where $g_{pk} := (g_{pk}^1,\dots,g_{pk}^n) \in \mathbb{R}^n$, $d_p \in \mathcal{M}_n(\mathbb{R})$ is symmetric and $\mathbf{e_p}$ is the p^{th} canonical basis vector in $\mathbb{R}^m$, for $p \in \{1,\dots,m\}$. With these choices of coefficients, system (2.8) becomes

$$\begin{cases} \partial_t y = \mathrm{div}(D\nabla y) + G\cdot\nabla y + Ay + r, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{2.11}$$

where $D := \mathrm{diag}(d_1,\dots,d_m)$, $G := (g_{pk})_{1\le p,k\le m} \in \mathcal{M}_m(\mathbb{R}^n)$ and $A := (a_{pk})_{1\le p,k\le m} \in \mathcal{M}_m(\mathbb{R})$. In the study of such PDEs, it is important to define the regularity of the boundary of the spatial domain Ω.

Definition 2.1.7. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $k \in \mathbb{N}^*$. We say U is of class C^k if for each point $x^0 \in \partial U$, there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$U \cap B(x^0,r) = \{x \in B(x^0,r) : x_n > \gamma(x_1,\dots,x_{n-1})\}.$$

Likewise, U is of class C^∞ if it is of class C^k for all $k \in \mathbb{N}^*$.

Prior to studying a system of coupled parabolic PDEs, we recall some notions from Sobolev space theory.

2.2 Sobolev space theory

We denote the space of infinitely differentiable and compactly supported *test functions* on Ω by $C_c^\infty(\Omega)$, and we denote the space of locally integrable functions on spatial domain Ω by $L^1_{\text{loc}}(\Omega)$. To study systems such as (2.11), we need to weaken our notion of partial derivatives.

Definition 2.2.1. Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ and α a multi-index of length n. We call v the α-th weak partial derivative of u provided that

$$\int_\Omega u\partial_\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\phi dx,$$

for all $\phi \in C_c^\infty(\Omega)$. We write $v = \partial_\alpha u$.

The notion of weak derivatives allows us to define Sobolev spaces.

Definition 2.2.2. For $k \in \mathbb{N}$ and $p \in \mathbb{N}^*$, the Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α of length n with $|\alpha| \leq k$, $\partial_\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

In this work, we mainly use the Sobolev space $W^{k,2}(\Omega)$, and we denote this space by $H^k(\Omega)$. The next definition concerns the closure of the space of test functions in Sobolev spaces.

Definition 2.2.3. We denote by $H_0^k(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^k(\Omega)$.

For the multi-index α of length n (i.e., $\alpha = (\alpha_1, \ldots, \alpha_n)$), we interpret $H_0^k(\Omega)$ as the space of $u \in H^k(\Omega)$ such that $\partial_\alpha u = 0$ on $\partial\Omega$ for all $|\alpha| \leq k - 1$. We now define the dual space to $H_0^k(\Omega)$, which will be instrumental in our construction of a solution to system (2.11).

Definition 2.2.4. We denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. That is, $f \in H^{-1}(\Omega)$ if and only if f is a bounded linear functional on $H_0^1(\Omega)$. Denoting the duality pairing as $\langle \cdot, \cdot \rangle$, the norm on $H^{-1}(\Omega)$ is taken to be the dual norm:

$$||f||_{H^{-1}(\Omega)} = \sup \left\{ \langle f, y \rangle : y \in H_0^1(\Omega) \text{ and } ||y||_{H^1(\Omega)} \leq 1 \right\}.$$

Next, we present a very practical theorem that characterizes functionals in $H^{-1}(\Omega)$.

Theorem 2.2.5. *[17, Theorem 1, Subsection 5.9.1] For $\Omega \subset \mathbb{R}^n$ open, assume that $f \in H^{-1}(\Omega)$. Then there exists functions $f^0, \dots, f^n \in L^2(\Omega)$ such that for $v \in H_0^1(\Omega)$,*

$$\langle f, v \rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^n f^i \partial_{x_i} v \right) dx. \tag{2.12}$$

Furthermore,

$$||f||_{H^{-1}(\Omega)} = \inf \left\{ \left(\int_\Omega \sum_{i=0}^n |f^i|^2 dx \right)^{1/2} : f \text{ satisfies } (2.12) \text{ for } f^0, \dots, f^n \in L^2(\Omega) \right\}. \tag{2.13}$$

The so-called Gagliardo-Nirenberg interpolation inequality is stated next.

Theorem 2.2.6. *[25] For $\Omega \subset \mathbb{R}^n$ open, for $q, r \in \mathbb{R}$ such that $1 \leq q, r \leq \infty$ and for $m \in \mathbb{N}$, let $u : \Omega \to \mathbb{R}$ such that $u \in L^q(\Omega) \cap W^{m,r}(\Omega)$. For $0 \leq j \leq m$, we have*

$$||u||_{W^{j,p}(\Omega)} \leq C ||u||_{W^{m,r}(\Omega)}^{\alpha} ||u||_{L^q(\Omega)}^{1-\alpha}, \tag{2.14}$$

where p satisfies

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}$$

for all α in the interval $\frac{j}{m} \leq \alpha \leq 1$, where $C := C(n, m, j, q, r, \alpha)$, with the following exceptional assumptions:

(i) if $j = 0$, $rm < n$, $q = \infty$, then we require $u \to 0$ at infinity, and;

(ii) if $1 < r < \infty$ and $m - j - \frac{n}{r}$ a nonnegative integer, then (2.14) only holds for α satisfying $\frac{j}{m} \leq \alpha < 1$.

Next, we present the Poincaré inequality.

Theorem 2.2.7. *[17, Theorem 1, Subsection 5.8.1] Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected and of class C^1. There exists a constant $C := C(\Omega)$ such that*

$$||y||_{L^2(\Omega)} \leq C||\nabla y||_{L^2(\Omega)},$$

for each $y \in H_0^1(\Omega)$.

Let X be a Banach space; for reasons that will become apparent, it is convenient to denote Banach-space valued functions $u : [0, T] \to X$ by $\mathbf{u} := [\mathbf{u}(t)]$.

Definition 2.2.8. The Sobolev space $L^p((0, T); X)$ consists of all strongly measurable functions $\mathbf{u} : [0, T] \to X$ with

1. $||\mathbf{u}||_{L^p((0,T);X)} := \left(\int_0^T ||\mathbf{u}(t)||^p dt\right)^{1/p} < \infty$, for $1 \leq p < \infty$, and;
2. $||\mathbf{u}||_{L^\infty((0,T);X)} := \operatorname{ess\,sup}_{0\leq t\leq T} ||\mathbf{u}(t)|| < \infty$,

where $|| \cdot ||$ is the norm associated to X.

The above definition gives the framework to study evolution equation. Next, we state a regularity theorem concerning an important Sobolev space involving time.

Theorem 2.2.9. *[17, Theorem 3, Subection 5.9.2] Suppose* $\mathbf{u} \in L^2((0,T); H_0^1(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$*, where* $\mathbf{u} := [\mathbf{u}(t)](x)$*. Then*

(i) $\mathbf{u} \in C([0,T]; L^2(\Omega))$ *(after possibly being redefined on a set of measure zero);*

(ii) the mapping

$$t \mapsto ||\mathbf{u}(t)||^2_{L^2(\Omega)}$$

is absolutely continuous, with

$$\frac{d}{dt}||\mathbf{u}(t)||^2_{L^2(\Omega)} = 2\left\langle \frac{d}{dt}\mathbf{u}(t), \mathbf{u}(t) \right\rangle,$$

for almost every $0 \leq t \leq T$*, and;*

(iii) we have the estimate

$$\max_{0 \leq t \leq T} ||\mathbf{u}(t)||_{L^2(\Omega)} \leq C||\mathbf{u}||_{L^2((0,T);H_0^1(\Omega)) \cap H^1((0,T);H^{-1}(\Omega))}.$$

2.3 Well-posedness results for parabolic systems

We state existence and uniqueness of solution results for system (2.11). To adapt existing well-posedness results to a system of coupled parabolic PDEs such as in system (2.11), one can follow the treatment, for example, in [17, Section 7] but write all intermediary results for a system of solutions (rather than for a single solution). For completeness, we sketch the steps of this treatment. From now on, we assume that $\mathcal{L}$ satisfies (2.7). Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. For $u, v \in H_0^1(\Omega)^m$, we define a bilinear form

$$B[u,v] := \int_\Omega \sum_{p,k=1}^{m} \left(\sum_{i,j=1}^{n} d_p^{ij} (\partial_{x_i} u_p)(\partial_{x_j} v_p) - \sum_{i=1}^{n} g_{pk}^i (\partial_{x_i} u_k) v_p - a_{pk} u_k v_p \right) \mathbf{e_p} dx.$$

As before, we associate with y the mapping

$$\begin{aligned} \mathbf{y} :&[0,T] \to H_0^1(\Omega)^m \\ &[\mathbf{y}(t)](x) := y(t,x), \end{aligned}$$

for $(t,x) \in Q_T$, and similarly for r, we associate the mapping

$$\begin{aligned} \mathbf{r} :&[0,T] \to H_0^1(\Omega)^m \\ &[\mathbf{r}(t)](x) := r(t,x). \end{aligned}$$

for $(t,x) \in Q_T$. Fixing $v \in H_0^1(\Omega)^m$, we multiply system (2.11) by v and integrate the divergence term by parts to get

$$\int_\Omega \left(\frac{d}{dt} \mathbf{y} \right)^T v dx + B[\mathbf{y}, v] = \int_\Omega \mathbf{r}^T v dx, \quad \forall\, t \in (0,T), \tag{2.15}$$

and hence, defining $q^0 := \sum_{p=1}^m \left(r_p - \sum_{k=1}^m \left(\sum_{i=1}^n g_{pk}^i \partial_{x_i} y_k - a_{pk} y_k \right) \right) \mathbf{e_p}$ and $q^j := \sum_{p=1}^m \left(\sum_{i=1}^n d_p^{ij} (\partial_{x_i} y_p) \right) \mathbf{e_p}$ for $j \in \{1, \dots, n\}$, we have

$$\partial_t y = q^0 + \sum_{j=1}^{n} \partial_{x_j} q^j \quad \text{in } Q_T.$$

Hence, by (2.12) and (2.13) in Theorem 2.2.5, we have that $\partial_t y \in H^{-1}(\Omega)^m$. Furthermore, using (2.13), we have

$$||\partial_t y||_{H^{-1}(\Omega)^m} \leq \left(\sum_{i=0}^{n} ||q^i||^2_{L^2(\Omega)^m} \right)^{1/2} \leq C \left(||y||_{H^1_0(\Omega)^m} + ||r||_{L^2(\Omega)^m} \right),$$

for $C := C(D, G, A) > 0$, where D, G and A are given in (2.11). This allows us to express (2.15) as

$$\left\langle \frac{d}{dt}\mathbf{y}, v \right\rangle + B[\mathbf{y}, v] = \int_\Omega \mathbf{r}^T v dx,$$

and motivates the following definition.

Definition 2.3.1. Suppose $r \in L^2(Q_T)^m$, $y^0 \in L^2(\Omega)^m$. A function $\mathbf{y} \in L^2((0,T); H^1_0(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$ is said to be a weak solution of system (2.11) provided that for every $v \in H^1_0(\Omega)^m$ and almost every $t \in [0,T]$

(i) $\langle \frac{d}{dt}\mathbf{y}, v\rangle + B[\mathbf{y}, v] = \int_\Omega \mathbf{r}^T v dx$, and;

(ii) $\mathbf{y}(0) = y^0$,

where the second equality makes sense thanks to Theorem 2.2.9.

We follow the so-called Galerkin method to state existence of a weak solution. We begin by constructing a sequence of weak solutions. For $k \in \mathbb{N}$, we assume the existence of smooth functions $w_k := w_k(x)$, where $\{w_k\}_{k=1}^\infty$ is an orthogonal basis for $H^1_0(\Omega)^m$ and furthermore, $\{w_k\}_{k=1}^\infty$ is an orthonormal basis for $L^2(\Omega)^m$. This can be deduced, for example, from [17, Theorem 1, Subection 6.5.1].

Theorem 2.3.2. *[17, Theorem 1, Section 7.1.2] For each $i \in \mathbb{N}$, there exists a unique*

$\mathbf{y}_i : [0,T] \to H_0^1(\Omega)$ *of the form*

$$\mathbf{y}_i(t) := \sum_{k=1}^{i} d_i^k(t) w_k,$$

for $t \in [0,T]$ *and* $k \in \{1, \dots, i\}$*, where*

$$d_i^k(0) = \int_\Omega (y^0)^T w_k dx$$

and

$$\int_\Omega \left(\frac{d}{dt}\mathbf{y}_i\right)^T w_k dx + B[\mathbf{y}_i, w_k] = \int_\Omega \mathbf{r}^T w_k dx, \tag{2.16}$$

for $t \in [0,T]$ *and* $k \in \{1, \dots, i\}$*.*

Similarly to the energy estimate derived formally in (2.4), we have the following result concerning the above sequence of weak solutions.

Theorem 2.3.3. *[17, Theorem 2, Section 7.1.2] There exists* $C := C(\Omega, T) > 0$ *such that*

$$\max_{0 \le t \le T} ||\mathbf{y}_i(t)||_{L^2(\Omega)^m} + ||\mathbf{y}_i||_{L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m}$$
$$\le C \left(||\mathbf{r}||_{L^2((0,T);L^2(\Omega))^m} + ||y^0||_{L^2(\Omega)^m} \right),$$

for $m \in \mathbb{N}$*.*

Thanks to Theorems 2.3.2 and 2.3.3, one can construct a weak solution to system (2.11).

Theorem 2.3.4. *[17, Theorem 3, Section 7.1.2] There exists a weak solution to system* (2.11)*.*

Owing to Theorem 2.2.9 and Grönwall's integral inequality, the following is ensured.

Theorem 2.3.5. *[17, Theorem 4, Section 7.1.2] A weak solution of system* (2.11) *is unique.*

2.4 Higher parabolic regularity

Next, we state regularity results for the weak solution of system (2.11) which will be essential in the work to follow (cf. Chapter 5).

Theorem 2.4.1. *[17, Theorem 5, Subsection 7.1.3] Assume $y^0 \in H_0^1(\Omega)^m$ and $\mathbf{r} \in L^2(Q_T)^m$; suppose that $\mathbf{y} \in L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m$ is the weak solution of system* (2.11)*. Then in fact*

$$\mathbf{y} \in L^2((0,T);H^2(\Omega))^m \cap L^\infty((0,T);H_0^1(\Omega))^m \cap H^1((0,T);L^2(\Omega))^m,$$

and we have the estimate

$$\begin{aligned}&\|\mathbf{y}\|_{L^2((0,T);H^2(\Omega))^m \cap L^\infty((0,T);H_0^1(\Omega))^m \cap H^1((0,T);L^2(\Omega))^m}\\ &\leq C\left(||\mathbf{r}||_{L^2((0,T);L^2(\Omega))^m} + ||y^0||_{H_0^1(\Omega)^m}\right),\end{aligned}$$

where $C := C(\Omega, T)$. If, in addition, $y^0 \in H^2(\Omega)^m$ and $\mathbf{r} \in H^1((0,T);L^2(\Omega))^m$, then

$$\mathbf{y} \in L^\infty((0,T);H^2(\Omega))^m \cap H^2((0,T);H^{-1}(\Omega))^m$$

and

$$\frac{d}{dt}\mathbf{y} \in L^\infty((0,T);L^2(\Omega))^m \cap L^2((0,T);H_0^1(\Omega))^m,$$

with the estimate

$$\operatorname*{ess\,sup}_{0\le t\le T}\left(||\mathbf{y}||_{H^2(\Omega)^m}+\left\|\frac{d}{dt}\mathbf{y}\right\|_{L^2(\Omega)^m}\right)+||\mathbf{y}||_{H^1((0,T);H_0^1(\Omega))^m\cap H^2((0,T);H^{-1}(\Omega))^m}$$
$$\le C\left(||\mathbf{r}||_{L^2((0,T);L^2(\Omega))^m}+||y^0||_{H^2(\Omega)^m}\right).$$

Under certain conditions, one can expect an even higher regularity for the weak solutions of system (2.11). We have the following regularity result.

Theorem 2.4.2. *[17, Theorem 6, Subsection 7.1.3] For $d\in\mathbb{N}$, assume $y^0\in H^{2d+1}(\Omega)^m$, $\mathbf{r}\in L^2((0,T);H^{2d}(\Omega))^m\cap H^d((0,T);L^2(\Omega))^m$, and assume that $\mathbf{y}\in L^2((0,T);H_0^1(\Omega))^m\cap H^1((0,T);H^{-1}(\Omega))^m$ is the weak solution of system* (2.11)*. Suppose also that the following compatibility conditions hold:*

$$\begin{cases} g^0 := y^0\in H_0^1(\Omega)^m;\\ g^1 := \mathbf{r}(0)-Lg^0\in H_0^1(\Omega)^m;\\ \vdots\\ g^d := \frac{d^{d-1}\mathbf{r}}{dt^{d-1}}(0)-Lg^{d-1}\in H_0^1(\Omega)^m. \end{cases}$$

Then $\mathbf{y}\in L^2((0,T);H^{2d+2}(\Omega))^m\cap H^{d+1}((0,T);L^2(\Omega))^m$ and we have the estimate

$$||\mathbf{y}||_{L^2((0,T);H^{2d+2}(\Omega))^m\cap H^{d+1}((0,T);L^2(\Omega))^m}\le C\left(||\mathbf{r}||_{L^2((0,T);H^{2d}(\Omega))^m\cap H^d((0,T);L^2(\Omega))^m}\right.$$
$$\left.+||y^0||_{H^{2d+1}(\Omega)^m}\right). \tag{2.17}$$

2.5 Some sparse matrix theory

When studying the invertibility of certain linear operators of interest, we are faced with studying the invertibility of matrices associated to coupled parabolic PDEs of interest (cf. Section 4.4). By nature of their construction, these matrices are *sparse*. In this section, we describe an algorithm that can be used to decompose a sparse matrix into block triangular form. Importantly, this algorithm can be applied to matrices with symbolic entries as it only makes use of the placement of zero entries in the matrix.

Given a matrix $P \in \mathcal{M}_{q\times r}(\mathbb{R})$, consider the bipartite graph associated to P given by the triple $G(P) := (R, C, E)$, where $R := \{r_1, \dots, r_q\}$ is the set of row vertices associated to P, $C := \{c_1, \dots, c_r\}$ is the set of column vertices associated to P, and E denotes the set the edges (r_i, c_j) associated to every nonzero entry p_{ij} of P, for $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, r\}$. We have the following definitions, as in [7].

Definition 2.5.1. A matching $M \subset E$ in $G(P)$ is such that the edges in M have no common vertices. We define the cardinality of M as the number of edges in M. A maximum matching is a matching with maximum cardinality. Furthermore, M is said to be column-perfect if every column vertex in C is matched; it is said to be row-perfect if every row vertex in R is matched; and it is said to be perfect if it is both column-perfect and row-perfect. A vertex v_i is said to be matched with respect to M if there exists $(v_i, v_j) \in M$ for appropriate indices i, j.

Definition 2.5.2. The structural rank of a matrix $P \in \mathcal{M}_{q\times r}(\mathbb{R})$ is the cardinality of a maximum matching $M \subset E$ in $G(P)$.

Definition 2.5.3. For an appropriate index i, let either $v_i = r_i$ or $v_i = c_i$. Fix a maximum matching M in $G(P)$. For $k \in \mathbb{N}^*$, a walk is a sequence of (possibly repeated) vertices $(v_i)_{i=0}^{k}$ such that (v_i, v_{i+1}) is an edge for $i \in \{1, \ldots, k-1\}$. An alternating walk is a walk with every second edge belonging to M. An alternating path is an alternating walk with no repeated vertices.

Definition 2.5.4. Let M be a maximum matching in $G(P)$ with row set R and column set C. We define the following sets of vertices with respect to M:

(i) $VR :=$ {row vertices reachable by alternating paths from an unmatched row};

(ii) $HR :=$ {row vertices reachable by alternating paths from an unmatched col.};

(iii) $VC :=$ {column vertices reachable by alternating paths from an unmatched row};

(iv) $HC :=$ {column vertices reachable by alternating paths from an unmatched col.};

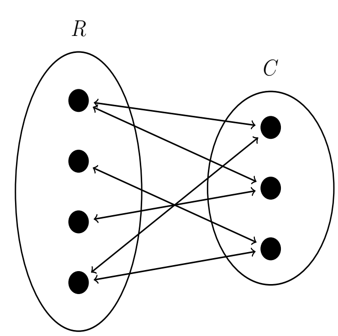
(v) $SR := R \setminus (VR \cup HR)$, and;

(vi) $SC := C \setminus (VC \cup HC)$.

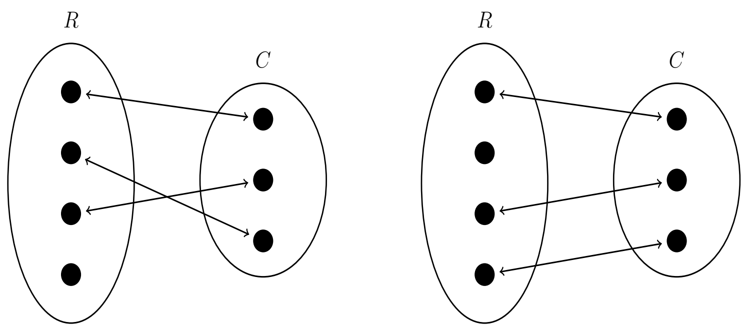
It was proven in [14] that VR, HR and SR are pairwise disjoint, and also that VC, HC and SC are pairwise disjoint. We demonstrate these definitions on an example.

Example 2.5.5. *Consider the matrix $P \in \mathcal{M}_{4\times 3}(\mathbb{R})$ and its bipartite graph $G(P)$ given by*

$$P = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & a_{43} \end{pmatrix}$$



Let $M_1 := \{(r_1, c_1), (r_2, c_3), (r_3, c_2)\}$ *and* $M_2 := \{(r_1, c_1), (r_3, c_2), (r_4, c_3)\}$ *be two matchings in* $G(P)$. *Note that these are maximal matchings.*



Maximum matching M_1 Maximum matching M_2

Hence, M_1 *and* M_2 *are column-perfect, and the structural rank of* A *is 3. Note that the structural rank is independent of the weights of the edges (i.e., the structural rank is independent of every nonzero entry of* P*, where each nonzero entry corresponds to an edge in* E*). For matching* M_1*, an alternating path is given by the sequence* r_4, c_1, r_1, c_2, r_3*. Furthermore, for matching* M_1*, we have* $VR := \{r_1, r_2, r_3, r_4\}$ *and* $VC := \{c_1, c_2, c_3\}$.

•

In the above example, the structural rank of P is equal to the rank of P. It is easily deduced that the structural rank of a matrix in $\mathcal{M}_{q,r}(\mathbb{R})$ is an upper bound on the rank of that matrix, and is never greater than $\min\{q, r\}$. We arrive at the following important result, which is identified in literature as the *Dulmage-Mendelsohn decomposition*, which can be deduced from [14, 29].

Theorem 2.5.6. *Let* $P \in \mathcal{M}_{q\times r}(\mathbb{R})$*, and let* M *be a maximum matching in* $G(P)$.

Then, one can permute the rows and columns of P to obtain the following block-triangular form (which we refer to as coarse decomposition):

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & 0 & P_{23} & P_{24} \\ 0 & 0 & 0 & P_{34} \\ 0 & 0 & 0 & P_{44} \end{pmatrix},$$

where

(i) (P_{11}, P_{12}) *is the underdetermined part of the matrix (i.e., more rows than columns), is generated by* $(r_i, c_i) \in HR \times HC$*, and has row-perfect matching;*

(ii) $\begin{pmatrix} P_{34} \\ P_{44} \end{pmatrix}$ *is the overdetermined part of the matrix (i.e., more columns than rows), is generated by* $(r_i, c_i) \in VR \times VC$*, and has column-perfect matching;*

(iii) P_{23} *is generated by* $(r_i, c_i) \in SR \times SC$*, and;*

(iv) P_{12}, P_{23}, P_{34} *are square matrices with nonzero diagonal, and hence have perfect matchings (i.e., they are of maximal structural rank).*

Moreover, P_{12}, P_{23}, P_{34} *can be further decomposed into block-triangular form with nonzero diagonal (which we refer to as fine decomposition). The structural rank of* P *is given by the sum of the structural ranks of* P_{12}, P_{23}, P_{34}.

Remark 2.5.7. If P is overdetermined, then (P_{11}, P_{12}) will be present only if P does not have a column-perfect matching. Similarly, if P is underdetermined, then (P_{34}, P_{44}) will appear only if P does not have a row-perfect matching. In both of these cases, the presence of P_{23} depends on the nonzero structure of P. If P is square, non-symmetric and has a perfect maximum matching, then its coarse decomposition will consist only of P_{23}. •

Remark 2.5.8. It was proven in [14] that the Dulmage-Mendelsohn decomposition is independent of the choice of maximum matching in $G(P)$. •

We are now ready to study system (2.11) under the framework of control systems, in the sense that we "select" the forcing term r to drive the system to a desired final state in some time T.

Chapter 3

Problem Statement

For $Q_T := (0,T) \times \Omega$ and $\Sigma_T := (0,T) \times \partial\Omega$, we revisit the system consisting of m second-order parabolic PDEs given by system (2.11). In Theorem 2.3.4, it was stated that for any initial condition $y^0 \in L^2(\Omega)^m$ and $r \in L^2(Q_T)^m$, system (2.11) admits a weak solution in $L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$.

3.1 The control problem

We now recast system (2.11) as a *control system*, where $r = Bu$ with $u \in L^2(Q_T)^c$ being control inputs to be chosen, and $B \in \mathcal{M}_{m\times c}(\mathbb{R})$, with $0 < c \leq m$, yielding

$$\begin{cases} \partial_t y = \operatorname{div}(D\nabla y) + G\cdot\nabla y + Ay + Bu, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0(\cdot), & \text{in } \Omega. \end{cases} \tag{3.1}$$

We associate to the control system (3.1) the operator

$$\mathcal{L}(y) := -\left(\operatorname{div}(D\nabla) + G\cdot\nabla + A\right)(y). \tag{3.2}$$

Let us now introduce our objectives that we aim to achieve by selecting appropriate control inputs. We have the following notions of *controllability* for system (3.1).

Definition 3.1.1. We say that system (3.1) is null controllable in time T if for every initial condition $y^0 \in L^2(\Omega)^m$, there exists a control $u \in L^2(Q_T)^c$ such that the solution $y \in L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m$ to (3.1) satisfies

$$y(T) = 0 \quad \text{in} \quad \Omega.$$

Definition 3.1.2. We say that system (3.1) is approximately controllable in time T if for every $\epsilon > 0$, for every initial condition $y^0 \in L^2(\Omega)^m$ and for every $y_T \in L^2(\Omega)^m$, there exists a control $u \in L^2(Q_T)^c$ such that the solution $y \in L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m$ to (3.1) satisfies

$$||y(T) - y_T||^2_{L^2(\Omega)^m} \leq \epsilon.$$

In the theory of control of PDEs, one encounters two prominent types of control: internal (or distributed) control; and boundary control. As the names suggest, distributed control refers to the case where $\text{supp}(u) \subset \Omega$ which is nonempty, whereas boundary control refers to the case where $\text{supp}(u) \subset \partial\Omega$, and hence the controls are defined via the boundary conditions of the PDE. Our work specializes to the former case: for $\omega \subset \Omega$ nonempty and open, we study the case where $r = \mathbb{1}_\omega Bu$, and henceforth, we denote by q_T the set $(0,T) \times \omega$.

An interesting control problem that arises in many engineering applications is underactuation, that is, when $c < m$. Our work will further specialize to this case, where there are currently very few results for *first and zero-order couplings*, for arbitrary m

and $c < m-1$ (even for the case of constant coefficients).

Since we treat the particular case of a system of linear parabolic PDEs with constant coefficients (constant in space and *time*), we are easily able to ascertain approximate controllability of system (3.1) from its null controllability.

Theorem 3.1.3. *[11, Theorem 2.45] Assume that for every $T > 0$, the control system* (3.1) *is null controllable in time T. Then, for every $T > 0$, system* (3.1) *is approximately controllable in time T.*

Proof. For $t \geq 0$, we associate to (3.1) the one-parameter family $S(t)$ of continuous linear operators from $L^2(\Omega)^m$ into $L^2(\Omega)^m$ which verify

$$\begin{cases} S(0) = \mathrm{Id}; \\ S(t_1 + t_2) = S(t_1) \circ S(t_2), \quad \forall\, t_1, t_2 \in [0, \infty); \\ \lim\limits_{t \to 0^+} S(t)x = x, \quad \forall\, x \in L^2(\Omega)^m, \text{ and;} \\ y(t, \cdot) = S(t)y^0(\cdot) + \displaystyle\int_0^t S(t-\tau)Bu(\tau, \cdot)d\tau, \quad \forall\, y^0 \in L^2(\Omega)^m, \end{cases} \tag{3.3}$$

where y satisfies system (3.1). When (3.3) is verified, we call $S(t)$ a *strongly continuous semigroup of continuous linear operators on $L^2(\Omega)^m$ associated to $\mathcal{L}$*. The existence of such an S associated to $\mathcal{L}$ is shown, for example, in [17, Theorem 5, Subection 7.4.3]. Let $T > 0$, $y^0, y^1 \in L^2(\Omega)^m$ and $\epsilon > 0$. By the third property of $S(t)$ in (3.3), there exists $\eta \in (0, T)$ such that

$$||S(\eta)y^1 - y^1||_{L^2(\Omega)^m} \leq \epsilon. \tag{3.4}$$

We assume (3.1) is null controllable in time η; then, for $\bar{y}$ satisfying (3.1) with initial condition $\bar{y}(0,\cdot) = S(T-\eta)y^0(\cdot)$, there exists $\bar{u} \in L^2((0,\eta);L^2(\Omega))^c$ such that

$$\bar{y}(\eta,\cdot) = S(\eta)y^1(\cdot).$$

Indeed, by assuming (3.1) is null controllable in time η, we have that for initial condition $y(0,\cdot) = S(T-\eta)y^0(\cdot) - y^1(\cdot)$, there exists a control $\bar{u} \in L^2([0,T];L^2(\Omega))^c$ such that

$$S(\eta)\circ S(T-\eta)y^0(\cdot) - S(\eta)y^1(\cdot) + \int_0^\eta S(\eta-t)B\bar{u}(t,\cdot)dt = 0. \tag{3.5}$$

We have from (3.5) that

$$\begin{aligned}
\bar{y}(\eta,\cdot) &= S(\eta)\circ S(T-\eta)y^0(\cdot) + \int_0^\eta S(\eta-t)B\bar{u}(t,\cdot)dt \\
&= S(T)y^0(\cdot) + S(\eta)y^1(\cdot) - S(T)y^0(\cdot) \\
&= S(\eta)y^1(\cdot).
\end{aligned}$$

Set

$$u(t,\cdot) = \begin{cases} 0 & \text{for } t \in (0,T-\eta]; \\ \bar{u}(t-T+\eta,\cdot) & \text{for } t \in (T-\eta,T), \end{cases}$$

and let y be the solution to (3.1) with initial condition $y^0 \in L^2(\Omega)^m$. With this choice of control u, we obtain

$$\begin{cases} y(t,\cdot) = S(t)y^0(\cdot), & \forall\, t \in [0,T-\eta], \\ y(t,\cdot) = \bar{y}(t-T+\eta,\cdot), & \forall\, t \in (T-\eta,T], \end{cases}$$

and as claimed, we have that the solution to (3.1) satisfies

$$||y(T,\cdot) - y^1(\cdot)||_{L^2(\Omega)^m} \leq \epsilon,$$

for any $y^1 \in L^2(\Omega)^m$. □

Remark 3.1.4. For the case where D, G and A are not constant in time, a strongly continuous semigroup associated to $\mathcal{L}$ does not necessarily exist. However, one can use a backward uniqueness result for linear parabolic PDEs stated in [20] to arrive at the same conclusion.

The main controllability theorem of this work is stated next, where we assume that more than half of the equations in system (3.1) are actuated. Furthermore, we assume throughout this work that the every equation in system (3.1) is distinct, i.e., the matrix associated to system (3.1) (which is constant in space and time) has full row rank.

Theorem 3.1.5. *For a fixed m, suppose $\Omega \subset \mathbb{R}^n$ nonempty, open and bounded. Furthermore, suppose Ω is of class C^r and connected for r large enough, which is qualified in Remark 5.2.2. For $\lfloor \frac{m}{2} \rfloor + 1 \leq c \leq m$, if*

(i) $c \geq h$, where $h := (m-c)(n+1)$, and;

(ii) the matrix $C \in \mathcal{M}_h(\mathbb{R})$ given by

$$C := \begin{pmatrix} a_{(m-c)\alpha_1} & \cdots & a_{m\alpha_1} & g^1_{(m-c)\alpha_1} & \cdots & g^1_{m\alpha_1} & \cdots & g^n_{(m-c)\alpha_1} & \cdots & g^n_{m\alpha_1} \\ a_{(m-c)\alpha_2} & \cdots & a_{m\alpha_2} & g^1_{(m-c)\alpha_1} & \cdots & g^1_{m\alpha_1} & \cdots & g^n_{(m-c)\alpha_1} & \cdots & g^n_{m\alpha_1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{(m-c)\alpha_h} & \cdots & a_{m\alpha_h} & g^1_{(m-c)\alpha_h} & \cdots & g^1_{m\alpha_h} & \cdots & g^n_{(m-c)\alpha_h} & \cdots & g^n_{m\alpha_h} \end{pmatrix}$$

is non-singular for any $\{\alpha_1, \dots, \alpha_h\} \subseteq \{1, \dots, c\}$ *with* $\alpha_1 \neq \dots \neq \alpha_h$*, where* g_{ij}^k *is the* k*-th component of* g_{ij}*, for* $k \in \{1, \dots, n\}$ *and for* $i, j \in \{1, \dots, m\}$*,*

then the system (3.1) *is null (and hence approximately) controllable in time* T*.*

Remark 3.1.6. The invertibility of C is not a necessary condition for null controllability of (3.1), as will become apparent in Lemma 4.4.2. However, the non-singularity condition above is generic (in the sense that "a generic matrix is invertible"). Furthermore, we have derived a similar null controllability condition for when $c < h$ and $c \geq \lfloor \frac{m}{2} \rfloor + 1$: in this case, the condition becomes to verify the invertibility of a sparse matrix $\tilde{C}$ which contains a block along its diagonal that is identical to the entries of C, but with size $c \times c$. For example, for $m = 5$, $c = 3$ and $n = 1$, the sufficient condition for null controllability is that the following matrix be non-singular:

$$\tilde{C} := \begin{pmatrix} a_{41} & a_{51} & g_{41} & g_{51} & 0 & 0 \\ a_{42} & a_{52} & g_{42} & g_{52} & 0 & 0 \\ a_{43} & a_{53} & g_{43} & g_{53} & 0 & 0 \\ 0 & 0 & a_{41} & a_{51} & g_{41} & g_{51} \\ 0 & 0 & a_{42} & a_{52} & g_{42} & g_{52} \\ 0 & 0 & a_{43} & a_{53} & g_{43} & g_{53} \end{pmatrix}.$$

This condition becomes more complicated for larger systems in higher spatial dimensions and, given its sparsity, it is more difficult to conclude that it is generic. We will discuss this in greater detail in Chapter 4.

The rest of this thesis is devoted to proving the above result.

Chapter 4

Fictitious control method

This chapter presents a technique that can be used to prove the null controllability of the coupled system (3.1) with possibly multiple underactuations (i.e., when $c \leq m - 1$). We first introduce the so-called *fictitious control method*, developed in [13], which allows one to bifurcate the null controllability problem into interconnected problems: an analytic control problem, where *fictitious* controls act on every equation in the coupled system (3.1); and an algebraic control problem, where there are possibly many underactuations. For the analytic problem, one can prove a so-called *weighted observability inequality* which will help deduce null controllability of the analytic system. For the algebraic problem, one can pose this underactuated control problem as an underdetermined system involving differential operators, and, under some conditions, "invert" one of these operator algebraically. This chapter focuses on the latter treatment (cf. Chapter 5 for the presentation of a method to solve the analytic problem).

4.1 Definitions

Recall that we denote our control domain by q_T, which is the set $(0,T)\times\omega$. We begin with some definitions.

Definition 4.1.1. For $n \in \mathbb{N}^*$, let α be a multi-index of length $n+1$. For $k,l \in \mathbb{N}^*$, a linear map $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^l$ is called a linear partial differential operator in q_T of order $m \in \mathbb{N}$ if for every α verifying $|\alpha| \leq m$, there exists $A_\alpha \in C^\infty(q_T;\mathscr{L}(\mathbb{R}^k;\mathbb{R}^l))$ such that for all $\phi \in C^\infty(q_T)^k$ and $(t,x) \in q_T$,

$$(\mathcal{B}\phi)(t,x) = \sum_{|\alpha|\leq m} A_\alpha(t,x)\partial_\alpha\phi(t,x). \tag{4.1}$$

If $n = 0$ and the above is satisfied, then $\mathcal{B}$ is called a linear ordinary differential operator. Note that for a linear ordinary differential operator $\mathcal{B}$, one may not be able to associate it to a linear ODE: indeed, for such an association, one may have to additionally impose an invertibility condition on one of the A_α's.

Let $c,m,k \in \mathbb{N}$ and consider the linear partial differential operators

$$\begin{cases} \mathcal{L} : C^\infty(q_T)^{m+c} \to C^\infty(q_T)^m, \\ \mathcal{N} : C^\infty(q_T)^k \to C^\infty(q_T)^m. \end{cases}$$

Suppose that for $(\hat{y} \quad \hat{u})^T \in C^\infty(q_T)^{m+c}$ and $\tilde{u} \in C^\infty(q_T)^k$, the linear equation

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) = \mathcal{N}(\tilde{u}) \tag{4.2}$$

is of interest, where $\tilde{u}$ is given and $(\hat{y} \quad \hat{u})^T$ are the unknowns. We characterize the solvability of (4.2).

Definition 4.1.2. We say that the linear equation (4.2) is algebraically solvable in q_T if there exists a linear partial differential operator $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^{m+c}$ such that

$$\mathcal{L} \circ \mathcal{B} = \mathcal{N}, \tag{4.3}$$

that is, $\mathcal{B}(\tilde{u})$ is a solution to (4.2) for every $\tilde{u} \in C^\infty(q_T)^k$. If $k = m$ and $\mathcal{N} = \mathrm{Id}_{C^\infty(q_T)^m}$, then we call $\mathcal{B}$ the right inverse of $\mathcal{L}$. For $\mathcal{L}$ and $\mathcal{N}$ linear ordinary differential operators, if there exists a linear ordinary differential operator $\mathcal{B}$ satisfying (4.3), then we say that (4.2) is algebraically solvable.

In other words, we wish to find $\mathcal{B}$ such that the following diagram is commutative:

$$\begin{array}{ccc} C^\infty(q_T)^{m+c} & \xrightarrow{\mathcal{L}} & C^\infty(q_T)^m \; . \\ \uparrow{\scriptstyle \mathcal{B}} & \nearrow{\scriptstyle \mathcal{N}} & \\ C^\infty(q_T)^k & & \end{array}$$

4.2 Motivation: the linear-time-varying example

An illuminating example of an algebraically solvable equation is the general linear time-variant ordinary control system under the so-called *Silverman-Meadows (controllability) condition.*

Example 4.2.1. *For times* $0 < T_0 < T_1$*, consider the linear time-varying control system given by*

$$\dot{x} = A(t)x + B(t)u, \quad t \in [T_0, T_1], \tag{4.4}$$

where $A : (T_0, T_1) \to \mathcal{M}_m(\mathbb{R})$ *denotes an element of* $C^\infty((T_0, T_1); \mathcal{M}_m(\mathbb{R}))$*,* B :

$(T_0, T_1) \to \mathcal{M}_{m\times c}(\mathbb{R})$ *denotes an element of* $C^\infty((T_0,T_1);\mathcal{M}_{m\times c}(\mathbb{R}))$*, and the state and control at a time* $t \in [T_0, T_1]$ *are* $x(t) \in \mathbb{R}^m$ *and* $u(t) \in \mathbb{R}^c$*, respectively. We define by induction on* i *a sequence of maps* $B_i \in C^\infty([T_0,T_1];\mathcal{M}_{m\times c}(\mathbb{R}))$ *by*

$$\begin{cases} B_0(t) := B(t), \\ B_i(t) := \dot{B}_{i-1}(t) - A(t)B_{i-1}(t). \end{cases} \tag{4.5}$$

Regarding the algebraic solvability of (4.4)*, we have the following theorem, which is a reformulation of [11, Theorem 1.18].*

Theorem 4.2.2. *Assume that, for some* $\bar{t} \in [T_0, T_1]$*,*

$$\mathrm{Span}\{B_i(\bar{t})u : u \in \mathbb{R}^c, i \in \mathbb{N}\} = \mathbb{R}^m, \tag{4.6}$$

which is called the Silverman-Meadows condition for (4.4)*. Then, there exists* $\epsilon > 0$ *and* $[t_0, t_1] := [T_0, T_1] \cap [\bar{t} - \epsilon, \bar{t} + \epsilon]$ *such that the linear ordinary differential operator*

$$\begin{aligned} \mathcal{L} : C^\infty([t_0,t_1];\mathbb{R}^{m+c}) &\to C^\infty([t_0,t_1];\mathbb{R}^m) \\ (x,u) &\mapsto \dot{x} - A(t)x - B(t)u \end{aligned} \tag{4.7}$$

has a right inverse in $[t_0, t_1]$*, which we denote by* $\mathcal{B}$*. That is, for any* $q \in C^\infty([t_0,t_1];\mathbb{R}^m)$*,*

$$(\mathcal{L} \circ \mathcal{B})q = q.$$

Proof. Let $p \in \mathbb{N}$ be such that

$$\mathrm{Span}\{B_i(\bar{t})u : u \in \mathbb{R}^c, i \in \{0, \dots, p\}\} = \mathbb{R}^m.$$

By lower-semicontinuity of rank, there exists $\epsilon > 0$ such that for $[t_0, t_1] = [T_0, T_1] \cap [\bar{t} - \epsilon, \bar{t} + \epsilon]$,

$$\mathrm{Span}\{B_i(t)u : u \in \mathbb{R}^c, i \in \{0, \dots, p\}\} = \mathbb{R}^m, \quad \forall\, t \in [t_0, t_1].$$

It follows that $\sum_{i=0}^{p} B_i(t)B_i^T(t)$ is invertible for every $t \in [t_0, t_1]$. Hence, for $j \in \{0, \dots, p\}$, one can define $Q_j \in C^\infty([t_0, t_1]; \mathcal{M}_{c\times m}(\mathbb{R}))$ by

$$Q_j(t) := B_j^T(t) \left(\sum_{i=0}^{p} B_i(t)B_i^T(t) \right)^{-1}.$$

One has

$$\sum_{i=0}^{p} B_i(t)Q_i(t) = \mathrm{Id}_{\mathbb{R}^m}, \quad \forall\, t \in [t_0, t_1]. \tag{4.8}$$

Let $x^0, x^1 \in \mathbb{R}^m$, and let $\gamma^0, \gamma^1 \in C^\infty([T_0, T_1]; \mathbb{R}^m)$ be the solution of

$$\begin{cases} \dot{\gamma}^0 = A(t)\gamma^0, \\ \gamma^0(T_0) = x^0, \end{cases}$$

and

$$\begin{cases} \dot{\gamma}^1 = A(t)\gamma^1, \\ \gamma^0(T_1) = x^1, \end{cases}$$

respectively. Let $d \in C^\infty(T_0, T_1])$ be such that

$$\begin{cases} d = 1 \text{ on a neighbourhood of } [T_0, t_0] \in [T_0, T_1], \\ d = 0 \text{ on a neighbourhood of } [t_1, T_1] \in [T_0, T_1]. \end{cases}$$

Let $\Gamma \in C^{\infty}([T_0, T_1]; \mathbb{R}^m)$ be defined by

$$\Gamma(t) := d(t)\gamma^0(t) + (1 - d(t))\gamma^1(t), \quad \forall\, t \in [T_0, T_1].$$

By construction of Γ, one has

$$\Gamma(T_0) = x^0, \quad \Gamma(T_1) = x^1. \tag{4.9}$$

Next, define $q \in C^{\infty}([T_0, T_1]; \mathbb{R}^m)$ by

$$\begin{aligned} q(t) &:= -\dot{\Gamma}(t) + A(t)\Gamma(t), \quad \forall\, t \in [T_0, T_1] & (4.10)\\ &= -\dot{d}(t)\gamma^0(t) + \dot{d}(t)\gamma^1(t). & (4.11)\end{aligned}$$

It follows that $q = 0$ on a neighbourhood of $[T_0, t_0] \cup [t_1, T_1] \in [T_0, T_1]$. We now define a sequence of $(u_i)_{i \in \{0, \ldots, p-1\}} \subset C^{\infty}([t_0, t_1]; \mathbb{R}^c)$ by decreasing induction on i:

$$\begin{cases} u_{p-1} := Q_p(t)q(t), & \forall\, t \in [t_0, t_1], \\ u_{i-1} := -\dot{u}_i(t) + Q_i(t)q(t), \forall\, i \in \{1, \ldots, p-1\}, & \forall\, t \in [T_0, T_1]. \end{cases} \tag{4.12}$$

We define

$$x(t) := \sum_{i=0}^{p-1} B_i(t)u_i(t)$$

and

$$u(t) := \dot{u}_0(t) - Q_0(t)q(t),$$

for all $t \in [t_0, t_1]$. For these constructions, the right inverse of $\mathcal{L}$ in $[t_0, t_1]$ is given by

$$\mathcal{B}(q) := (x, u). \tag{4.13}$$

Indeed, for $t \in [t_0, t_1]$ we have

$$\begin{aligned}
\mathcal{L}(x,u) &= \dot{x} - Ax - Bu \\
&= \sum_{i=0}^{p-1} \left(\dot{B}_i u_i + B_i \dot{u}_i - AB_i u_i\right) - B\left(\dot{u}_0 - Q_0 q\right) \\
&= \sum_{i=1}^{p-1} \left(\dot{B}_i u_i + B_i \dot{u}_i - AB_i u_i\right) + \dot{B}_0 u_0 - AB_0 u_0 + BQ_0 q.
\end{aligned}$$

By (4.5), we have

$$\sum_{i=1}^{p-1} \dot{B}_i u_i = \sum_{i=1}^{p-1} \left(B_{i+1} + AB_i\right) u_i.$$

Hence, by (4.12) and since $B_0 := B$, we have

$$\begin{aligned}
\mathcal{L}(x,u) &= \sum_{i=1}^{p-1} \left(B_{i+1} u_i + B_i \dot{u}_i\right) + \dot{B}_0 u_0 - AB_0 u_0 + B_0 Q_0 q \\
&= \sum_{i=1}^{p-1} \left(B_{i+1} u_i - B_i u_{i-1} + B_i Q_i q\right) + \dot{B}_0 u_0 - AB_0(t) u_0 + B_0 Q_0 q \\
&= \sum_{i=1}^{p-1} B_i Q_i q + B_p u_{p-1} - B_1 u_0 + \dot{B}_0 u_0 - AB_0 u_0 + B_0 Q_0 q.
\end{aligned}$$

Once again employing (4.5), we have

$$\mathcal{L}(x,u) = \sum_{i=1}^{p-1} B_i Q_i q + B_p u_{p-1} + B_0 Q_0 q,$$

and it follows from (4.8) that

$$\sum_{i=1}^{p-1} B_i Q_i q = q - B_p Q_p q - B_0 Q_0 q.$$

Hence, using (4.12), we arrive at

$$\mathcal{L}(x, u) = q.$$

□

Remark 4.2.3. An important consequence of the algebraic solvability of (4.7) for $\mathcal{N} = Id_{C^\infty([t_0,t_1])^m}$ is the controllability of (4.4): that is, for every $x^0, x^1 \in \mathbb{R}^m$, there exists $u \in L^\infty((T_0, T_1); \mathbb{R}^c)$ such that the solution to (4.4) with $x(T_0) = x^0$ satisfies $x(T_1) = x^1$. To show this, we extend the solution of (4.7) by zero to $[T_0, T_1]$: that is, we define

$$\begin{cases} u := 0 \text{ on } [T_0, t_0] \cup [t_1, T_1] \quad \text{and} \quad u(t) := \dot{u}_0(t) - Q_0(t)q(t) \; \forall \, t \in (t_0, t_1), \\ r := 0 \text{ on } [T_0, t_0] \cup [t_1, T_1] \quad \text{and} \quad r(t) := \displaystyle\sum_{i=0}^{p-1} B_i(t)u_i(t) \; \forall \, t \in (t_0, t_1). \end{cases} \tag{4.14}$$

Defining the solution of (4.4) as

$$x(t) := \Gamma(t) + r(t), \quad \forall \, t \in [T_0, T_1],$$

one has by (4.9) and (4.14) that $x(T_0) = x^0$ and $x(T_1) = x^1$. Furthermore, by (4.10), (4.14) and the fact that $q = 0$ on a neighbourhood of $[T_0, t_0] \cup [t_1, T_1] \in$

$[T_0, T_1]$, it follows that

$$\dot{x} = A(t)x + B(t)u \quad \text{on} \quad [T_0, t_0] \cup [t_1, T_1]. \tag{4.15}$$

Next, we verify that (4.15) is satisfied on (t_0, t_1). Since (4.7) is algebraically solvable for $\mathcal{N} = Id_{C^\infty([t_0,t_1])^m}$, then we have that for $\mathcal{B}$ defined in (4.13),

$$(\mathcal{L} \circ \mathcal{B})\, q = q;$$

hence for $t \in (t_0, t_1)$ and r, u defined in (4.14), we have that $\mathcal{L}\left((r, u)\, q\right) = q$. It follows from (4.10) that for $t \in (t_0, t_1)$,

$$\begin{aligned} \mathcal{L}(x, u) &= \mathcal{L}\left((\Gamma + r, u)\, q\right) \\ &= \dot{\Gamma} - A\Gamma + q \\ &= 0. \end{aligned}$$

Hence, we arrive at the following controllability result for linear time-variant systems.

Corollary 4.2.4. *Consider the linear time-variant control system* (4.4)*. Suppose that there exists an interval* $[t_0, t_1] \subset [T_0, T_1]$ *nonempty such that the linear partial differential operator* $\mathcal{L}$ *given in* (4.7) *has a right inverse in* $[t_0, t_1]$*. Then* (4.4) *is controllable.*

•

Remark 4.2.5. The algebraic solvability of the control system (4.4) is related to the fact that generic (in the algebraic topology sense) underdetermined linear (ordinary and partial) differential operators have right inverses [21, (B), pg. 150; Theorem, pg. 156].

•

4.3 The fictitious control method

Our goal is to prove null controllability in time T for the control system (3.1), where there are m coupled parabolic equations and less than m controls. To accomplish this for an arbitrary number of controls $c \leq m-1$, our strategy is to divide this control problem into two separate parts as was done in [13, 15].

4.3.1 Analytic control problem

We first consider following control problem: for any $\tilde{y}^0 \in L^2(\Omega)^m$, prove the existence of $(\tilde{y}, \tilde{u})$ a solution of

$$\begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla \tilde{y}) + G \cdot \nabla \tilde{y} + A\tilde{y} + \mathcal{N}\left(\mathbb{1}_\omega \tilde{u}\right), & \text{in } Q_T, \\ \quad \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0, \cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega, \end{cases} \tag{4.16}$$

such that $\tilde{y}(T, \cdot) = 0$, where $\mathcal{N}$ is a differential operator that is to be determined (cf. Section 4.4), $\tilde{u}$ *acts on all equations* in (4.16), and we denote by $\mathbb{1}_\omega$ a *smooth version* of the indicator function (this can be constructed via mollification; cf. relation (6.2) for its exact definition). Note that $(\tilde{y}, \tilde{u})$ has to be in a suitable space: in particular, depending on our choice of differential operator $\mathcal{N}$, $\tilde{u}$ has to be regular enough to withstand the derivatives applied by $\mathcal{N}$. While these restrictions make solving control system (4.16) slightly non-classical (we need to use a *weighted* observability inequality), there is hope to finding such a solution since controls appear in every equation in (4.16). We elaborate on our technique for solving this control problem in Chapter 5.

4.3.2 Algebraic control problem

We next consider a different control problem: prove the existence of a solution $(\hat{y}, \hat{u})$ of

$$\begin{cases} \partial_t \hat{y} = \operatorname{div}(D\nabla \hat{y}) + G \cdot \nabla \hat{y} + A\hat{y} + B\hat{u} + \mathcal{N}\left(\mathbb{1}_\omega \tilde{u}\right), & \text{in } Q_T, \\ \hat{y} = 0, & \text{on } \Sigma_T, \\ \hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0, & \text{in } \Omega, \end{cases} \tag{4.17}$$

where $\hat{u}$ acts only on the first c equations and $B = \begin{pmatrix} \mathrm{Id}_c & 0_{c\times(m-c)} \end{pmatrix}^T \in \mathcal{M}_{m\times c}(\mathbb{R})$. The notions of algebraic solvability, as described in Section 4.1, will be used to resolve this control problem in the next section. The analytic and algebraic control problems differ in the following ways: in the analytic problem, the controls are $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$, whereas in the algebraic problem, the controls are $\hat{u}$, and furthermore, $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$ appears but is considered to be a source term; and in the analytic problem, one has to prove that $\tilde{y}(T, \cdot) = 0$ (in our case, by means of an observability inequality), whereas in the algebraic problem, $\hat{y}(T, \cdot) = 0$ is inherited from the construction of the solution $(\hat{y}, \hat{u})$, as discussed in Remark 4.4.1.

Solving both the analytic and algebraic problems will prove the null controllability of system (3.1). Indeed, defining

$$(y, u) := (\tilde{y} - \hat{y}, -\hat{u}),$$

one notices that (y, u) is the solution to (3.1) in a suitable space with $y(T, \cdot) = 0$. We will elaborate on this construction of (y, u) in Chapter 6. Note that the controls to the analytic system, $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$, are eliminated via the subtraction $\tilde{y} - \hat{y}$; this gives meaning to the name of the method utilized in our treatment.

4.4 Algebraic solvability

In this section, we study the algebraic solvability of differential operators corresponding system (4.17) which contains m equations and c controls, for $c \in \{1, \dots, m-1\}$. To this end, we consider the linear partial differential operator defined by

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) := \partial_t \hat{y} - \operatorname{div}(D\nabla\hat{y}) - G \cdot \nabla\hat{y} - A\hat{y} - B\hat{u}, \tag{4.18}$$

which is an underdetermined operator, and we consider $\mathcal{N}(\mathbb{1}_\omega \tilde{u})$ as a source term, where $\mathcal{N}$ is to be chosen later. One can write system (4.17) as

$$\mathcal{L}\left((\hat{y} \quad \hat{u})^T\right) = \mathcal{N}(\mathbb{1}_\omega \tilde{u});\tag{4.19}$$

we study the algebraic solvability of (4.19) in q_T. Recall from Definition 4.1.2 that this is equivalent to proving the existence of a linear partial differential operator $\mathcal{B} : C^\infty(q_T)^k \to C^\infty(q_T)^m$ such that $(\hat{y}, \hat{u}) = \mathcal{B}(\mathbb{1}_\omega \tilde{u})$ for any $\mathbb{1}_\omega \tilde{u} \in C^\infty(q_T)^m$, and hence by reason of $\mathcal{B}$ being a differential operator, $(\hat{y}, \hat{u})$ will have support in q_T. With a slight abuse of notation, from now on we denote the extension by zero of $(\hat{y}, \hat{u})$ to Q_T also by $(\hat{y}, \hat{u})$, so that $\hat{y} = 0$ on Σ_T and $\hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0$ in Ω.

Remark 4.4.1. For simplicity, we formulated the notions of algebraic solvability for controls in the analytic problem $\mathbb{1}_\omega \tilde{u} \in C^\infty(q_T)$, which dictates the regularity of $(\hat{y}, \hat{u})$; however, we will need to expand the space of controls that we may access to recover null controllability results for system (4.16). For controls with weaker regularity, we must additionally show that these controls vanish at times $t = 0$ and $t = T$. This will be explained in detail in Chapter 6. For the time being, assume $(\hat{y}, \hat{u})$ are regular enough such that $\mathcal{L}\left((\hat{y}, \hat{u})^T\right)$ is well-defined. •

As we will see, for our choice of k it is easier to solve the adjoint equation of (4.19). To this end, we study the adjoint system associated to system (4.17):

$$\begin{cases} -\partial_t \hat{\psi} = \operatorname{div}(D\nabla\hat{\psi}) - G^* \cdot \nabla\hat{\psi} + A^*\hat{\psi}, & \text{in } Q_T, \\ \hat{\psi} = 0, & \text{on } \Sigma_T, \\ \hat{\psi}(T,\cdot) = \hat{\psi}^0(\cdot), & \text{in } \Omega, \end{cases} \tag{4.20}$$

for $\hat{\psi}^0 \in L^2(\Omega)^m$.

4.4.1 One underactuation

This section follows the treatment in [15, Subsection 2.1] and is presented here to contrast the existing technique to treat the null controllability of system (4.17) with one underactuation and the proposed technique in Subsection 4.4.2, which treats the case of multiple underactuations. The method presented here succeeds in algebraically solving (4.19) by utilizing the first and zero-order couplings to isolate for the unknown, and is henceforth referred to as the *isolation technique*.

Choose $k = m$; we wish to find a linear partial differential operator $\mathcal{B}$ such that

$$\mathcal{L} \circ \mathcal{B} = \mathcal{N}, \tag{4.21}$$

where $\mathcal{L}$ is given in (4.18) and $\mathcal{N}$ is to be chosen. Note that this is equivalent to solving the adjoint problem: that is, finding a linear partial differential operator $\mathcal{B}^*$ such that

$$\mathcal{B}^* \circ \mathcal{L}^* = \mathcal{N}^*. \tag{4.22}$$

We calculate the (formal) adjoint of differential operator $\mathcal{L}$: for $\hat{\psi} \in L^2(Q_T)^m$, we have

$$
\begin{aligned}
&\left(\mathcal{L}\left(\begin{pmatrix}\hat{y} & \hat{u}\end{pmatrix}^T\right), \hat{\psi}\right) \\
&= \left(\iint_{Q_T} \sum_{k=1}^{m} \left(\partial_t \hat{y}\hat{\psi}_k - \operatorname{div}(d_k \nabla \hat{y}_k)\hat{\psi}_k - \sum_{i=1}^{m} (g_{ki} \cdot \nabla \hat{y}_k + a_{ki}\hat{y}_k)\hat{\psi}_k\right)\right. \\
&\left. + \sum_{l=1}^{c} \hat{u}_l \hat{\psi}_l dxdt\right) \\
&= \iint_{Q_T} \sum_{k=1}^{m} \hat{y}_k \mathcal{L}_k^* \hat{\psi} + \sum_{l=1}^{c} \hat{u}_l \mathcal{L}_{m+l}^* \hat{\psi} \\
&= \left(\begin{pmatrix}\hat{y} & \hat{u}\end{pmatrix}^T, \mathcal{L}^* \hat{\psi}\right),
\end{aligned}
$$

and hence

$$
\mathcal{L}^*\hat{\psi} = \begin{pmatrix}
-\left(\partial_t + \operatorname{div}(d_1 \nabla)\right)\hat{\psi}_1 + \sum_{j=1}^m \left(g_{j1} \cdot \nabla - a_{j1}\right)\hat{\psi}_j \\
-\left(\partial_t + \operatorname{div}(d_2 \nabla)\right)\hat{\psi}_2 + \sum_{j=1}^m \left(g_{j2} \cdot \nabla - a_{j2}\right)\hat{\psi}_j \\
\vdots \\
-\left(\partial_t + \operatorname{div}(d_m \nabla)\right)\hat{\psi}_m + \sum_{j=1}^m \left(g_{jm} \cdot \nabla - a_{jm}\right)\hat{\psi}_j \\
\hat{\psi}_1 \\
\vdots \\
\hat{\psi}_{m-1}
\end{pmatrix}. \tag{4.23}
$$

We state the following lemma, which is a reformulation of [15, Theorem 1].

Lemma 4.4.2. *The linear partial differential equation* (4.22) *is algebraically solvable if there exists an index* $i_0 \in \{1, \dots, m-1\}$ *such that*

$$
g_{mi_0} \neq 0 \quad or \quad a_{mi_0} \neq 0. \tag{4.24}
$$

Proof. One need only look at the i_0-th entry of $\mathcal{L}^*$ to verify this assertion:

$$\begin{aligned}\mathcal{L}^*_{i_0}\hat{\psi} &= -\left(\partial_t + \operatorname{div}(d_{i_0}\nabla)\right)\hat{\psi}_{i_0} + \sum_{j=1}^{m}\left(g_{ji_0}\cdot\nabla - a_{ji_0}\right)\hat{\psi}_j \\ &= -\left(\partial_t + \operatorname{div}(d_{i_0}\nabla)\right)\mathcal{L}^*_{m+i_0}\hat{\psi} + \sum_{j=1}^{m-1}\left(g_{ji_0}\cdot\nabla - a_{ji_0}\right)\mathcal{L}^*_{m+j}\hat{\psi} \\ &+ \left(g_{mi_0}\cdot\nabla - a_{mi_0}\right)\hat{\psi}_m,\end{aligned}$$

which one can use to isolate for the unknown $\hat{\psi}_m$ and its spatial derivative:

$$\left(g_{mi_0}\cdot\nabla - a_{mi_0}\right)\hat{\psi}_m = \mathcal{L}^*_{i_0}\hat{\psi} + \left(\partial_t + \operatorname{div}(d_{i_0}\nabla)\right)\mathcal{L}^*_{m+i_0}\hat{\psi} - \sum_{j=1}^{m-1}\left(g_{ji_0}\cdot\nabla - a_{ji_0}\right)\mathcal{L}^*_{m+j}\hat{\psi}. \tag{4.25}$$

Hence, a careful choice of $\mathcal{N}^*$ yields the desired result: choosing

$$\mathcal{N}^*\hat{\psi} := \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \vdots \\ \hat{\psi}_{m-1} \\ \left(g_{mi_0}\cdot\nabla - a_{mi_0}\right)\hat{\psi}_m \end{pmatrix},$$

one can define for $\phi \in C^\infty(Q_T)^{2m-1}$

$$\mathcal{B}^*\phi := \begin{pmatrix} \phi_{m+1} \\ \phi_{m+2} \\ \vdots \\ \phi_{2m-1} \\ \phi_{i_0} + \left(\partial_t + \operatorname{div}(d_{i_0}\nabla)\right)\phi_{m+i_0} - \sum_{j=1}^{m-1}\left(g_{ji_0}\cdot\nabla - a_{ji_0}\right)\phi_{m+j} \end{pmatrix},$$

so that

$$(\mathcal{B}^* \circ \mathcal{L}^*)\,\hat{\psi} = \mathcal{N}^*\hat{\psi}$$

is verified for every $\hat{\psi} \in C^\infty(Q_T)^m$. □

4.4.2 Multiple underactuations

We specialize to the case where system (4.17) has more than one underactuations (i.e., when $c < m-1$).

Isolation technique

We begin by employing the technique presented in Subsection 4.4.1 to reveal the obstructions that limit our ability to deduce algebraic solvability of (4.17). For the moment, we focus on the simplest case, when $c = m-2$. We have

$$\mathcal{L}^*\hat{\psi} = \begin{pmatrix} -\left(\partial_t + \operatorname{div}(d_1\nabla)\right)\hat{\psi}_1 + \sum_{j=1}^m \left(g_{j1}\cdot\nabla - a_{j1}\right)\hat{\psi}_j \\ -\left(\partial_t + \operatorname{div}(d_2\nabla)\right)\hat{\psi}_2 + \sum_{j=1}^m \left(g_{j2}\cdot\nabla - a_{j2}\right)\hat{\psi}_j \\ \vdots \\ -\left(\partial_t + \operatorname{div}(d_m\nabla)\right)\hat{\psi}_m + \sum_{j=1}^m \left(g_{jm}\cdot\nabla - a_{jm}\right)\hat{\psi}_j \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{m-2} \end{pmatrix}.$$

We define a natural necessary condition for algebraic solvability of (4.17) as in Lemma (4.4.2): without loss of generality, suppose there exists indices $i_0 \in$

$\{1, \ldots, m-2\}$ and $i_1 \in \{1, \ldots, m-1\}$ such that

$$\begin{cases} g_{(m-1)i_0} \neq 0 & \text{or} \quad a_{(m-1)i_0} \neq 0, \\ g_{mi_1} \neq 0 & \text{or} \quad a_{mi_1} \neq 0. \end{cases}$$

One immediately encounters the issue that none of the entries of $\mathcal{L}^*$ can be used to isolate for the individual unknowns $\hat{\psi}_{m-1}$ and $\hat{\psi}_m$ (and their spatial derivatives). Instead, we recover the system of equations

$$\begin{cases} \left(g_{(m-1)i_0} \cdot \nabla - a_{(m-1)i_0}\right) \hat{\psi}_{m-1} + \left(g_{mi_0} \cdot \nabla - a_{mi_0}\right) \hat{\psi}_m \\ \qquad = \mathcal{L}^*_{i_0} \hat{\psi} + \left(\partial_t + \operatorname{div}(d_{i_0} \nabla)\right) \mathcal{L}^*_{m+i_0} \hat{\psi} \\ \qquad - \displaystyle\sum_{j=1}^{m-2} \left(g_{ji_0} \cdot \nabla - a_{ji_0}\right) \mathcal{L}^*_{m+j} \hat{\psi}, \\ \left(g_{(m-1)i_1} \cdot \nabla - a_{(m-1)i_1}\right) \hat{\psi}_{m-1} + \left(g_{mi_1} \cdot \nabla - a_{mi_1}\right) \hat{\psi}_m \\ \qquad = \mathcal{L}^*_{i_1} \hat{\psi} + \left(\partial_t + \operatorname{div}(d_{i_1} \nabla)\right) \mathcal{L}^*_{m+i_1} \hat{\psi} \\ \qquad - \displaystyle\sum_{j=1}^{m-2} \left(g_{ji_1} \cdot \nabla - a_{ji_1}\right) \mathcal{L}^*_{m+j} \hat{\psi}. \end{cases} \tag{4.26}$$

While one can define an appropriate $\mathcal{N}^*$ using (4.26) such that (4.17) is algebraically solvable, in general this $\mathcal{N}^*$ will have entries involving both $\hat{\psi}_{m-1}$ and $\hat{\psi}_m$ (and their spatial derivatives). Such an $\mathcal{N}^*$ introduces an unresolvable issue in Chapter 5 (see, for example, the proof of Proposition 5.1.1, where $\tilde{\psi}$ would be replaced by $\mathcal{N}^*\tilde{\psi}$ in (5.43); hence, one would need to use a Poincaré-type inequality similar to the one in Theorem 2.2.7 involving the differential operator $\mathcal{N}^*$, but with the righthand side replaced by $C||\mathcal{N}^* y||_{L^p(\Omega)}$, for $p = 2$). Alas, we are not aware of a procedure through which one can hope to recover a general sufficient condition for algebraic solvability of (4.17) using this technique.

Prolongation technique

Inspired by [13, Section 3], we present a new method to prove the algebraic solvability of (4.22) by means of *prolongation*: that is, since $\mathcal{L}^*\hat{\psi} = \mathcal{N}^*\hat{\psi}$ is an overdetermined system (i.e., there are $m+c$ equations and only m unknowns), we can expect to differentiate each equation a sufficient amount of times with respect to all of the spatial variables in order to gain more equations than "algebraic unknowns", which we make more precise in what follows. An inversion technique, which is motivated by [21, Section 2.3.8], is then used to recover the unknowns from the overdetermined system.

We consider system (4.17) for an arbitrary $c \in \{1, \dots, m-2\}$ and define the linear partial differential operator

$$\mathcal{N}\zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{pmatrix},$$

for $\zeta \in C^\infty(Q_T)^m$. With this choice of $\mathcal{N}$, it suffices to consider differential operators $\bar{\mathcal{L}} : C^\infty(Q_T)^m \to C^\infty(Q_T)^{m-c}$ and $\bar{\mathcal{N}} : C^\infty(Q_T)^{m-c} \to C^\infty(Q_T)^{m-c}$ defined by

$$\bar{\mathcal{L}}\zeta := \begin{pmatrix} \left(\partial_t - \operatorname{div}(d_{c+1}\nabla)\right)\zeta_{c+1} - \sum_{i=1}^m \left(g_{(c+1)i}\cdot\nabla + a_{(c+1)i}\right)\zeta_i \\ \left(\partial_t - \operatorname{div}(d_{c+2}\nabla)\right)\zeta_{c+2} - \sum_{i=1}^m \left(g_{(c+2)i}\cdot\nabla + a_{(c+2)i}\right)\zeta_i \\ \vdots \\ \left(\partial_t - \operatorname{div}(d_m\nabla)\right)\zeta_m - \sum_{i=1}^m \left(g_{mi}\cdot\nabla + a_{mi}\right)\zeta_i \end{pmatrix}$$

and

$$\bar{\mathcal{N}}\zeta := \begin{pmatrix} \zeta_{c+1} \\ \vdots \\ \zeta_m \end{pmatrix}$$

to prove algebraic solvability of (4.21). Indeed, with our choice of $\mathcal{N}$ we can write system (4.17) as

$$\mathcal{L}(\hat{y}, \hat{u}) = \mathbb{1}_\omega \tilde{u}, \tag{4.27}$$

where $\hat{u}$ acts on the first c equations; also, finding a linear partial differential operator $\mathcal{B}$ satisfying (4.21) is equivalent to finding $\mathcal{B}$ such that

$$\begin{cases} \hat{y}_1 = \mathcal{B}_1(\mathbb{1}_\omega \tilde{u}), \\ \quad \vdots \\ \hat{y}_m = \mathcal{B}_m(\mathbb{1}_\omega \tilde{u}), \\ \hat{u}_1 = \mathcal{B}_{m+1}(\mathbb{1}_\omega \tilde{u}), \\ \quad \vdots \\ \hat{u}_c = \mathcal{B}_{m+c}(\mathbb{1}_\omega \tilde{u}). \end{cases} \tag{4.28}$$

Hence, from our choice of B in (4.17), (4.18), (4.27) and (4.28), we have for $l \in \{1, \ldots, c\}$ that the last c entries of $\mathcal{B}$ must satisfy

$$\begin{aligned} \mathcal{B}_{m+l}(\mathbb{1}_\omega \tilde{u}) &= (\partial_t - \operatorname{div}(d_l \nabla))\, \hat{y}_l - \sum_{i=1}^m (g_{li} \cdot \nabla + a_{li})\, \hat{y}_i - \mathbb{1}_\omega \tilde{u}_l \\ &= (\partial_t - \operatorname{div}(d_l \nabla))\, \mathcal{B}_l(\mathbb{1}_\omega \tilde{u}) - \sum_{i=1}^m (g_{li} \cdot \nabla + a_{li})\, \mathcal{B}_i(\mathbb{1}_\omega \tilde{u}) - \mathbb{1}_\omega \tilde{u}_l, \end{aligned}$$

if (4.21) is to be verified. Thus, one need only to find a $\bar{\mathcal{B}} : C^\infty(Q_T)^{m-c} \to C^\infty(Q_T)^m$

to satisfy the first m lines of (4.28), as the last c lines of (4.28) are completely determined by the first m lines and the respective entry of $\tilde{u}$; consequentially, for our choice of $\mathcal{N}$, the algebraic solvability of (4.21) is equivalent to the algebraic solvability of

$$\bar{\mathcal{L}} \circ \bar{\mathcal{B}} = \bar{\mathcal{N}}. \tag{4.29}$$

We study the adjoint equation of (4.29),

$$\bar{\mathcal{B}}^* \circ \bar{\mathcal{L}}^* = \bar{\mathcal{N}}^*, \tag{4.30}$$

and we call $\bar{\mathcal{B}}^*$ the *left inverse* of $\bar{\mathcal{L}}^*$. Similar to (4.23), we have for $\hat{\psi} \in C^\infty(Q_T)^{m-c}$ that

$$\bar{\mathcal{L}}^* \hat{\psi} = \begin{pmatrix} \sum_{j=c+1}^m (g_{j1} \cdot \nabla - a_{j1}) \hat{\psi}_j \\ \vdots \\ \sum_{j=c+1}^m (g_{jc} \cdot \nabla - a_{jc}) \hat{\psi}_j \\ (-\partial_t - \operatorname{div}(d_{c+1}\nabla)) \hat{\psi}_{c+1} + \sum_{j=c+1}^m \left(g_{j(c+1)} \cdot \nabla - a_{j(c+1)}\right) \hat{\psi}_j \\ \vdots \\ (-\partial_t - \operatorname{div}(d_m\nabla)) \hat{\psi}_m + \sum_{j=c+1}^m (g_{jm} \cdot \nabla - a_{jm}) \hat{\psi}_j \end{pmatrix}$$

and

$$\bar{\mathcal{N}}^* \hat{\psi} = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}.$$

Hence, the algebraic solvability of (4.29) is equivalent to proving the existence of a differential operator $\bar{\mathcal{B}}^* : C^\infty(Q_T)^m \to C^\infty(Q_T)^{m-c}$ such that for every $\phi \in C^\infty(Q_T)^m$, if $\hat{\psi} \in C^\infty(Q_T)^{m-c}$ is a solution of

$$\begin{cases} \displaystyle\sum_{j=c+1}^{m} (g_{j1} \cdot \nabla - a_{j1})\, \hat{\psi}_j \;= \phi_1, \\ \quad\vdots \\ \displaystyle\sum_{j=c+1}^{m} (g_{jc} \cdot \nabla - a_{jc})\, \hat{\psi}_j = \phi_c, \\ (-\partial_t - \operatorname{div}(d_{c+1}\nabla))\, \hat{\psi}_{c+1} + \displaystyle\sum_{j=c+1}^{m} \left(g_{j(c+1)} \cdot \nabla - a_{j(c+1)}\right) \hat{\psi}_j = \phi_{c+1}, \\ \quad\vdots \\ (-\partial_t - \operatorname{div}(d_m\nabla))\, \hat{\psi}_m + \displaystyle\sum_{j=c+1}^{m} (g_{jm} \cdot \nabla - a_{jm})\, \hat{\psi}_j = \phi_m, \end{cases} \tag{4.31}$$

then

$$\bar{\mathcal{B}}^* \phi = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}. \tag{4.32}$$

An examination of (4.31) reveals that, in general, there are m distinct equations and only $m-c$ unknowns, them being $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$. Let us call $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ the *analytic unknowns*. If we view (4.31) as a linear algebraic system by treating every (time and spatial) derivative of $\hat{\psi}_l$ as an *independent algebraic unknown*, for $l \in \{c+1, \ldots, m\}$, then there are many more algebraic unknowns than distinct equations. Under this algebraic viewpoint, one can hope to prolong (or differentiate with respect to every spatial variable) each equation of (4.31) to introduce many new equations and a few new algebraic unknowns (owing to the symmetry property of mixed partial derivatives). Repeating this process a sufficient amount of times, one can hope that the linear algebraic system eventually becomes *overdetermined*, that is, the number of distinct equations eventually exceeds the number of algebraic unknowns. Proceeding this way, we begin by counting the number of derivatives up to the highest order contained in a prolonged version of system (4.31), which is an adaptation of the method used in [13, Subection 3.2.2].

Lemma 4.4.3. *Let $p \in \mathbb{N}$ denote the number of prolongations of* (4.31)*, and let $F(p)$ denote the distinct number of derivatives of order less than or equal to p for* smooth enough *functions having n variables. Then*

$$F(p) = \binom{p+n}{n}. \tag{4.33}$$

Furthermore, denoting by $U(p)$ and by $E(p)$ the number of algebraic unknowns and the number of equation contained in the prolonged version of system (4.31)*, respectively, we have*

$$U(p) = (m-c)\left(F(p+2) + F(p)\right), \tag{4.34}$$

and

$$E(p) = mF(p). \tag{4.35}$$

Proof. Let α be a multi-index of length n such that $|\alpha| \leq p$: that is, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $\sum_{i=1}^{n} \alpha_i \leq p$. Note that

$$(\alpha_1, \ldots, \alpha_n) \mapsto \left\{ \alpha_1 + 1, \alpha_1 + \alpha_2 + 2, \alpha_1 + \alpha_2 + \alpha_3 + 3, \ldots, \sum_{i=1}^{n} \alpha_i + n \right\}$$

defines a bijection between the set of tuples $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $|\alpha| \leq p$ and the set of subsets of $\{1, 2, \ldots, p+n\}$ having n elements. Furthermore, attributing the multi-index α to the partial derivative operator $\partial_\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_n}$ takes into account the symmetry of mixed partial derivatives, and thus only counts the *distinct* number of derivatives of order less than or equal to p. Since the cardinality of the set of subsets of $\{1, 2, \ldots, p+n\}$ having n elements is $\binom{p+n}{n}$, we have (4.33).

Since each analytic unknown contained in system (4.31) has corresponding algebraic unknowns of order up to two in space and one time derivative unknown, and there are $m - c$ analytic unknowns, (4.34) follows.

Since there are m equations, each of which is prolonged p times, and $F(p)$ can be used to represent the number of distinct equations differentiated with respect to the multi-index α, (4.35) follows. □

Concerning our system (4.31), we have the following lemma.

Lemma 4.4.4. *For all $m \in \mathbb{N}_{>1}$, $n \in \mathbb{N}^*$ and $c \in \{1, \ldots, m-2\}$ such that $c > \frac{m}{2}$, there exists $p \in \mathbb{N}^*$ such that*

$$E(p) > U(p).$$

Proof. We claim that $\exists\, p \in \mathbb{N}^*$ such that

$$c\binom{p+n}{n} > (m-c)\binom{p+n+2}{n}.$$

Indeed, we have

$$(m-c)\binom{p+n+2}{n} = (m-c)\frac{(p+n+2)(p+n+1)}{(p+2)(p+1)}\,\frac{(p+n)!}{p!n!}$$

and

$$c\binom{p+n}{n} = c\frac{(p+n)!}{p!n!}.$$

First, we show that for fixed m and n, $\exists\, p$ and c such that

$$\frac{(p+n+2)(p+n+1)}{(p+2)(p+1)} < \frac{c}{(m-c)}; \tag{4.36}$$

Indeed, since $m \in \mathbb{N}_{>1}$, we can choose $c > \frac{m}{\frac{(p+2)(p+1)}{(p+n+2)(p+n+1)}+1}$ to verify (4.36). Note that $\frac{(p+2)(p+1)}{(p+n+2)(p+n+1)} \to 1$ from below as $p \to \infty$, and thus $c > \frac{m}{2}$ is necessary for $E(p) > U(p)$. Since $m \in \mathbb{N}^*$ and $c \in \{1, \ldots, m-2\}$, $c > \frac{m}{2}$ is also sufficient since one can always choose $p \in \mathbb{N}$ large enough to verify (4.36) when $c = \lfloor \frac{m}{2} \rfloor + 1$. □

Remark 4.4.5. Lemma 4.4.4 shows that for a sufficiently regular solution $\hat{\psi}$ to system (4.20), if $c \geq \lfloor \frac{m}{2} \rfloor + 1$, then there exists $p \in \mathbb{N}$ such that we can prolong system (4.31) p times and study the resulting overdetermined linear algebraic system. One can argue the appropriate regularity of $\hat{\psi}$ as follows: without loss of generality, we can take $\hat{\psi}^0 \in H^{p+1}(\Omega)^m$ by a classical density argument; then, ones applies Theorem 2.4.2. As we will see, under certain conditions, one may hope to extract the analytic unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ from the overdetermined algebraic system. Hence, one can expect the left inverse of the differential operator associated to the prolonged version of system (4.31) to be of maximum differential order $p+2$ in space and 1 in time. Thus, by (4.28) we require the analytic system's controls, $\mathbb{1}_\omega \tilde{u}$, to accommodate $p+2$ spatial differentiations. These highly regular $\mathbb{1}_\omega \tilde{u}$ are constructed in Chapter 6 (cf. Proposition 6.2.1). •

We finish this chapter by proving the following important result.

Proposition 4.4.6. *Given m, n and c in $\mathbb{N}^*$ with $\lfloor \frac{m}{2} \rfloor + 1 \leq c \leq m$, if*

(i) $c \geq h$, where $h := (m-c)(n+1)$, and;

(ii) the matrix $C \in \mathcal{M}_h(\mathbb{R})$ given by

$$C := \begin{pmatrix} a_{(m-c)\alpha_1} & \cdots & a_{m\alpha_1} & g^1_{(m-c)\alpha_1} & \cdots & g^1_{m\alpha_1} & \cdots & g^n_{(m-c)\alpha_1} & \cdots & g^n_{m\alpha_1} \\ a_{(m-c)\alpha_2} & \cdots & a_{m\alpha_2} & g^1_{(m-c)\alpha_1} & \cdots & g^1_{m\alpha_1} & \cdots & g^n_{(m-c)\alpha_1} & \cdots & g^n_{m\alpha_1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{(m-c)\alpha_h} & \cdots & a_{m\alpha_h} & g^1_{(m-c)\alpha_h} & \cdots & g^1_{m\alpha_h} & \cdots & g^n_{(m-c)\alpha_h} & \cdots & g^n_{m\alpha_h} \end{pmatrix}$$

is non-singular for any $\{\alpha_1, \ldots, \alpha_h\} \subseteq \{1, \ldots, c\}$ with $\alpha_1 \neq \cdots \neq \alpha_h$, where g^k_{ij} is the k-th component of g_{ij}, for $k \in \{1, \ldots, n\}$ and for $i, j \in \{1, \ldots, m\}$,

then (4.21) *is algebraically solvable in q_T.*

Proof. Without loss of generality, for a given m, n and c, we fix a p large enough such that $E(p) > U(p)$. Consider the overdetermined matrix $\bar{L}^* \in \mathcal{M}_{E(p)\times U(p)}(\mathbb{R})$ with entries equal to the coefficients multiplying the algebraic unknowns generated by prolonging system (4.31) p times. We denote the vector containing the p-times prolonged unknowns by $\hat{z} \in \mathcal{M}_{U(p)\times 1}(L^2(Q_T))$, where the necessary regularity of $\hat{\psi}$ is discussed in Remark 4.4.5. Similarly, we denote the p-times prolonged version of ϕ by $\Phi \in \mathcal{M}_{E(p)\times 1}(C^\infty(Q_T))$. Hence, we can write the *algebraic version* of the prolonged system (4.31) as

$$\bar{L}^* \hat{z} = \Phi. \tag{4.37}$$

The counterpart of solving (4.31) and (4.32) simultaneously for (4.37) is to find a $P \in \mathcal{M}_{(m-c)\times E(p)}$ such that

$$P\bar{L}^* \hat{z} = \begin{pmatrix} \hat{\psi}_{c+1} \\ \vdots \\ \hat{\psi}_m \end{pmatrix}, \tag{4.38}$$

with P being the algebraic version of $\bar{\mathcal{B}}^*$. We apply Theorem 2.5.6 to $\bar{L}^*$ so that for $S_{\tilde{\sigma}}$ and S_σ the left and right permutation matrices generated by the Dulmage-Mendelsohn decomposition, respectively, we have

$$S_{\tilde{\sigma}}\bar{L}^*S_\sigma = \begin{pmatrix} \bar{L}^*_{11} & \bar{L}^*_{12} & \bar{L}^*_{13} & \bar{L}^*_{14} \\ 0 & 0 & \bar{L}^*_{23} & \bar{L}^*_{24} \\ 0 & 0 & 0 & \bar{L}^*_{34} \\ 0 & 0 & 0 & \bar{L}^*_{44} \end{pmatrix}, \tag{4.39}$$

where $\bar{L}^*_{34}$ is square and perfectly matched (i.e., it is of maximal structural rank). We must also permute $\hat{z}$ by S_σ^{-1}.

Our next steps are as follows. First, we study the structure of $\bar{L}^*$ to argue that under $S_{\tilde{\sigma}}$ and S_σ, every row of C (which appear in $\bar{L}^*$) is permuted to block $\bar{L}^*_{34}$ (possibly with some zero entries to the right), for every $\{\alpha_1, \ldots, \alpha_h\} \subseteq \{1, \ldots, c\}$ with $\alpha_1 \neq \cdots \neq \alpha_h$. Then, we argue that the unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ contained in $\hat{z}$ are being multiplied by the block $\bar{L}^*_{34}$ (and in particular, the rows of C). Immediately following the end of this proof, we supplement our explanations with Example 4.40. Furthermore, in Remark 3.1.6 we give some insight into the null controllability condition for the case where $c < h$ (which is possibly non-generic, i.e., it is the non-singularity of a sparse matrix).

By construction of $\bar{L}^*$, we have that the columns of $\bar{L}^*$ corresponding to any algebraic unknown involving a time derivative are very sparse. Indeed, each of these columns has only one nonzero entry (which is -1). This occurs since we do not prolong system (4.31) with respect to time, and hence each time derivative term appears in one (and only one) equation within the prolonged version of system (4.31). Furthermore, the row associated to any one of these nonzero column entries must correspond to the j-th equation (or its prolonged version) in system (4.31), for $j \in \{c+1, \ldots, m\}$. Hence, the coefficients corresponding to the j-th equation (or its prolonged version) in system (4.31) lie in this row, for $j \in \{c+1, \ldots, m\}$.

We claim that there exists a maximum matching M in $G(\bar{L}^*)$ that contains all of the edges (r_i, c_i) corresponding to these -1 entries. Indeed, for any matrix P, a *matching* in $G(P)$ is a subset of nonzero entries of P such that no two of which belong to the same row or column. Hence, since the columns of $\bar{L}^*$ corresponding to any algebraic unknown involving a time derivative contain only one nonzero entry, it is easy to deduce that there exists a maximum matching M in $G(\bar{L}^*)$ that be chosen to include these nonzero entries. Importantly, this choice will omit any other edges associated to coefficients corresponding to the j-th equation (or its prolonged version) in system (4.31), for $j \in \{c+1, \ldots, m\}$, from the matching, and the rows containing these coefficients will be matched (see Example 4.4.7). Furthermore, we can choose at random enough edges which make M maximal; due to the structure of $\bar{L}^*$, all of these edges will correspond to coupling coefficients of the j-th equation (or its prolonged version) in system (4.31), for $j \in \{1, \ldots, c\}$. Without loss of generality, we associate $S_{\tilde{\sigma}}$ and S_σ to this choice of maximum matching.

With our choice of M, we now study vertex sets VR and VC. Recall from Section 2.5 that

$$VR := \{\text{row vertices reachable by alternating paths from some unmatched row}\},$$

$$VC := \{\text{column vertices reachable by alternating paths from some unmatched row}\},$$

where an alternating path is a sequence of (row or column) vertices $(v_i)_{i=0}^{k}$ such that

$(v_{2i}, v_{2i+1}) \in E$ and, additionally, $(v_{2i+1}, v_{2(i+1)}) \in M$ and no vertices are repeated, for $k \in \mathbb{N}^*$. By our choice of M and since $\bar{L}^*$ is overdetermined, there exists unmatched rows, and any unmatched row must correspond to the j-th equation (or its prolonged version) in system (4.31), for $j \in \{1, \ldots, c\}$. One deduces from the structure of $\bar{L}^*$ that these unmatched rows have nonzero entries which lie in matched columns, and hence VR and VC are not empty. Furthermore, these matched columns cannot be those corresponding to algebraic unknowns involving a time derivative. By the structure of $\bar{L}^*$, all row vertices corresponding to the j-th equation in system (4.31) are reachable by an alternating path, for all $j \in \{1, \ldots, c\}$. This is a consequence of equations in system (4.31) having *first and zero-order coupling coefficients* and since $\bar{L}^*$ is generated by prolongations with respect to spatial variables. Hence, rows corresponding to the j-th equation (or its prolonged version) in system (4.31) have corresponding row vertices contained in VR, for $j \in \{1, \ldots, c\}$. It follows that columns containing coupling coefficients have corresponding column vertices contained in VC (the same search yields the column vertices in VC). Hence, the coefficients that appear in the j-th equation (or its prolonged version) in system (4.31) are permuted to the blocks $\bar{L}^*_{34}$ and $\bar{L}^*_{44}$, for $j \in \{1, \ldots, c\}$.

By examining system (4.31), one easily deduces that the unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ are being multiplied by either $\bar{L}^*_{34}$ or $\bar{L}^*_{44}$. We permute the rows contained in C (the ones from the original – and not a prolonged – system (4.31), and hence have the same number of zero entries appearing only to their right) to the top of $\bar{L}^*_{34}$; we deduce that $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$ are multiplied by $\bar{L}^*_{34}$. We denote this row permutation on $S_{\tilde{\sigma}}\bar{L}^* S_\sigma$ by $S_{\tilde{\sigma}^0}$. Finally, with a slight abuse of notation, we denote by I various identity matrices with appropriate dimensions; using the row (left) permutation

$$S_{\tilde{\sigma}^1} := \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

the column (right) permutation

$$S_{\sigma^1} := \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \end{pmatrix},$$

we permute $S_{\tilde{\sigma}^0} S_{\tilde{\sigma}} \bar{L} S_{\sigma}$ into upper-block triangular form with $\bar{L}^*_{34}$ being the top left-most block, and we define

$$P := \begin{pmatrix} \mathrm{Id}_{m-c} & 0_{(m-c)\times(h-m+c)} \end{pmatrix} \begin{pmatrix} C^{-1} & 0_{h\times(E(p)-h)} \end{pmatrix} S_{\tilde{\sigma}^1} S_{\tilde{\sigma}^0} S_{\tilde{\sigma}},$$

which verifies (4.38). Hence, by the non-singularity of C, there exists a linear combination of differentiated lines of $\bar{\mathcal{L}}^*$ that allow us to recover the analytic unknowns $\hat{\psi}_{c+1}, \ldots, \hat{\psi}_m$. We denote by $\mathcal{C}$ the differential operator associated to matrix C; it follows that $\bar{\mathcal{B}}^* := \mathcal{C}$ verifies (4.30), and hence $\bar{\mathcal{B}} = \mathcal{C}^*$ verifies (4.29). □

Example 4.4.7. *In this example, we consider the algebraic control system given by* (4.17)*, where we choose* $m = 5$*,* $c = 3$*, and for simplicity,* $n = 1$*. In solving the algebraic version of* (4.30)*, which is given by* (4.38)*, we study the linear algebraic* *operator obtained by prolonging system* (4.31) *3 times given by*

$$\bar{L}^* = \left(\begin{array}{cccccccccccccccccccc}
\enclose{circle}{-a_{41}} & -a_{51} & 0 & 0 & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{42} & \enclose{circle}{-a_{52}} & 0 & 0 & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{43} & -a_{53} & 0 & 0 & \enclose{circle}{g_{43}} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{44} & -a_{54} & 0 & \enclose{circle}{-1} & g_{44} & g_{54} & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{45} & -a_{55} & \enclose{circle}{-1} & 0 & g_{45} & g_{55} & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{41} & \enclose{circle}{-a_{51}} & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{42} & -a_{52} & \enclose{circle}{g_{42}} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & \enclose{circle}{g_{53}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{44} & -a_{54} & g_{44} & g_{54} & 0 & \enclose{circle}{-1} & d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{45} & -a_{55} & g_{45} & g_{55} & \enclose{circle}{-1} & 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & 0 & 0 & \enclose{circle}{g_{41}} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & 0 & 0 & g_{42} & \enclose{circle}{g_{52}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & 0 & 0 & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{44} & -a_{54} & 0 & 0 & g_{44} & g_{54} & 0 & \enclose{circle}{-1} & d_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{45} & -a_{55} & 0 & 0 & g_{45} & g_{55} & \enclose{circle}{-1} & 0 & 0 & d_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & 0 & 0 & \enclose{circle}{g_{41}} & g_{51} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & 0 & 0 & g_{42} & \enclose{circle}{g_{52}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & 0 & 0 & g_{43} & g_{53} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{44} & -a_{54} & 0 & 0 & g_{44} & g_{54} & 0 & \enclose{circle}{-1} & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{45} & -a_{55} & 0 & 0 & g_{45} & g_{55} & \enclose{circle}{-1} & 0 & 0 & d_2
\end{array}\right). \tag{4.40}$$

In (4.40)*, we have circled a subset of nonzero entries of* $\bar{L}^*$ *whose corresponding edges make up a maximum matching* M *in* $G(\bar{L}^*)$*. We have chosen* M *such that it contains every edge corresponding to a* -1 *entry of* $\bar{L}^*$*.*

We now populate the set of edges VR *and* VC*. Note that* r_{13} *and* r_{18} *are the only unmatched row vertices; hence, we search for row vertices that are reachable from* r_{13} *and* r_{18} *via an alternating path. A crucial observation is that there exists no alternating paths from these row vertices to the row vertices corresponding to rows containing* -1 *entries: indeed, for a walk starting from row* 13*, since* $(r_{13}, c_i) \notin M$*, for* $i \in \{7, 8, 11, 12\}$*, the next row vertex* r_k *in the walk must be such that* $(c_i, r_k) \in M$*, for* $k \in \{1, \ldots, 12, 14, \ldots, 20\}$*, hence* $r_k \neq r_j$ *for* $j \in \{4, 5, 9, 10, 14, 15, 19, 20\}$*; the* *exact same argument holds for a walk starting from row* 18. *One can easily deduce by the same reasoning that* r_j *will never be reachable by a (longer) alternating path, for* $j \in \{4, 5, 9, 10, 14, 15, 19, 20\}$. *Furthermore, every other row vertex is reachable by an alternating path from either* r_{13} *or* r_{18}*! Hence,*

$$VR = \{r_1, r_2, r_3, r_6, r_7, r_8, r_{11}, r_{12}, r_{13}, r_{16}, r_{17}, r_{18}\},$$

and it follows that

$$VC = \{c_1, c_2, c_5, c_6, c_7, c_8, c_{11}, c_{12}, c_{15}, c_{16}\}.$$

Hence, we arrive at (possibly after a row permutation)

$$\begin{pmatrix} \bar{L}^*_{34} \\ \bar{L}^*_{44} \end{pmatrix} = \begin{pmatrix}
-a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 & 0 & 0 \\
0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 & 0 & 0 \\
0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} & 0 & 0 \\
0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{41} & -a_{51} & g_{41} & g_{51} \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{42} & -a_{52} & g_{42} & g_{52} \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{43} & -a_{53} & g_{43} & g_{53}
\end{pmatrix},$$

from which we deduce the null controllability condition stated in Remark 3.1.6. •

Remark 4.4.8. For the case where $c < h$, one does not have enough equations in system (4.31) to permute the rows of $\bar{L}_{34}^*$ and $\bar{L}_{44}^*$ and construct a *square block* in the diagonal of $\bar{L}_{34}^*$ with zeros only to the right. In this case, we have to expand C to the next smallest candidate, which will contain zeros (see Example 4.4.7). One would hope for any m, n and $c > \frac{m}{2}$ that this expanded C, denoted by $\tilde{C}$, be non-singular generically. However, due to the fact that $\bar{L}_{34}^*$ can only be further decomposed into a block diagonal matrix with nonzero diagonal (see *fine decomposition* in Theorem 2.5.6), we were not able to prove that this is true in general. Nevertheless, for *small* systems with not too severe underactuation in low dimensions (e.g., for $m \leq 20$, $(m-c) \leq 4$ and $n \leq 3$), we have observed that $\tilde{C}$ be non-singular generically. Note that for $c \geq h$, we have shown that $\mathcal{B}$ will be of differential order 1 in space and 0 in time, so higher regularity of $\mathbb{1}_\omega \tilde{u}$ than was proved in [15] is no longer needed. Although the null controllability condition that $\tilde{C}$ be non-singular for $c < h$ may not be generic in some cases, it may still be useful in many engineering applications. Hence, we continue our treatment for an operator $\mathcal{B}$ that is of differential order 0 in time and at most $p + 2$ in space. •

Chapter 5

A Carleman estimate for the analytic problem

5.1 Main objective and technique summary

In this chapter, we study the analytic system:

$$\begin{cases} \partial_t \tilde{y} = \operatorname{div}(D\nabla\tilde{y}) + G\cdot\nabla\tilde{y} + A\tilde{y} + \mathbb{1}_\omega \tilde{u}, & \text{in } Q_T, \\ \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0,\cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega. \end{cases} \tag{5.1}$$

The goal of this chapter is to prove that the solution $(\tilde{y}, \tilde{u})$ to the analytic control system (5.1) satisfies the following so-called *weighted observability inequality*, which will help us deduce its null controllability. To this end, we consider the adjoint system to system (5.1) given by

$$\begin{cases} -\partial_t \tilde{\psi} = \operatorname{div}(D\nabla\tilde{\psi}) - G^*\cdot\nabla\tilde{\psi} + A^*\tilde{\psi}, & \text{in } Q_T, \\ \tilde{\psi} = 0, & \text{on } \Sigma_T, \\ \tilde{\psi}(T,\cdot) = \tilde{\psi}^0(\cdot), & \text{in } \Omega, \end{cases} \tag{5.2}$$

where $\tilde{\psi}^0 \in L^2(\Omega)^m$. We state the weighted observability inequality we aim to establish.

Proposition 5.1.1. *For every $\tilde{\psi}^0 \in L^2(\Omega)^m$, the solution $\tilde{\psi}$ of system* (5.2) *satisfies*

$$\int_\Omega \left\| \tilde{\psi}(0,x) \right\|_1^2 dx \leq C_{obs} \iint_{(0,T)\times\omega_0} e^{-2s_1\alpha}\xi^{2p+7} \left\| \tilde{\psi}(t,x) \right\|_1^2 dxdt, \tag{5.3}$$

where $C_{obs} := CT^9 e^{C(1+3T/4+1/T^5)} > 0$ and $\|\cdot\|_1$ denotes the Euclidean norm. We call (5.3) *a weighted observability inequality, with weight $\rho := e^{-2s_1\alpha}\xi^{2p+7}$, for α and ξ defined below in* (5.5) *and* (5.6)*, respectively, where $s_1 := \sigma(T^5 + T^{10})$ for $\sigma > 0$ depending on Ω and ω_0.*

We utilize the *Carleman estimate* technique to develop an estimate which will help us establish the observability inequality stated above. This chapter builds upon the technique developed in [15, Section 2.2]; in particular, it incorporates higher-order terms on the lefthand side of (5.14) which allow us to construct highly regular controls for system (5.1) (see Remarks 4.4.1 and 4.4.5 for more details). Constructing a solution $(\tilde{y}, \tilde{u})$ to system (5.1) with highly regular controls and satisfying $\tilde{y}(T, \cdot) = 0$ is treated in Chapter 6.

Carleman estimates are weighted energy estimates for solutions to PDEs with exponential weights. These types of estimates for parabolic operators are derived, for example, in [1, Section 4.7]. Carleman estimates were initially introduced in [9] to obtain uniqueness and stability results for a particular first-order initial-boundary value problem; they have since been used to derive results in many applications, including exact, approximate and null controllability results for partial differential equations with internal or boundary control.

5.2 Some notation and technical results

We begin by introducing some notation. For the multi-index β of length l consisting of multi-indices, consider the l^{th}-order tensor given by $C := (C_\beta)_\beta$, where β_i has length n_i, for $n_i \in \mathbb{N}^*$, for $i \in \{1, \ldots, l\}$. We associate to C the element-wise norm:

$$\|\cdot\|_l := \left(\sum_{i_1=1,\ldots,i_l=1}^{n_1,\ldots,n_l} C^2_{\beta_1(i_1),\ldots,\beta_l(i_l)} \right)^{1/2}.$$

This is the norm associated to the natural inner product on the tensor product of two inner product spaces. An equivalent interpretation of $\|\cdot\|_l$ is the following: given a l^{th}-order tensor C, one *vectorizes* C into a vector of length $\sum_{i=1}^{l} n_i$ and then applies the Euclidean norm to recover $\|\cdot\|_l$. Fix a sequence $(\omega_i)_{i=0}^{p+2}$ of nonempty open subsets of ω such that

$$\begin{cases} \bar{\omega}_i \subset \omega_{i-1}, \quad \text{for} \quad i \in \{1, \ldots, p+2\}, \\ \bar{\omega}_0 \subset \omega. \end{cases}$$

We have the following lemma, which is an adaptation of [19, Lemma 1.1] (see also [11, Lemma 2.68]). Its proof is included in the Appendix.

Lemma 5.2.1. *Assume that Ω is of class C^r and connected. Then, for $r \geq 2$, there exists $\eta^0 \in C^r(\bar{\Omega})$ such that*

$$\begin{cases} \left\|\nabla \eta^0\right\|_1 \geq \kappa, & \textit{in} \quad \Omega \setminus \omega_{p+2}, \\ \eta^0 > 0, & \textit{in} \quad \Omega, \\ \eta^0 = 0, & \textit{on} \quad \partial\Omega, \end{cases} \tag{5.4}$$

for some $\kappa > 0$.

For $r = p+2$, fix such an $\eta^0 \in C^{p+2}(\bar{\Omega})$, where p is chosen to be large enough such that the p-times prolonged version system (4.31) be overdetermined.

Remark 5.2.2. In (6.17), we require η^0 to be $(p+2)$–times differentiable, and hence in Theorem 3.1.5, the hypothesis on the spatial domain Ω is that it be of class C^r, where $r \geq p+2$. •

For $(t,x) \in Q_T$ we define

$$\alpha(t,x) := \frac{e^{12\lambda||\eta^0||_\infty} - e^{\lambda(10||\eta^0||_\infty + \eta^0(x))}}{t^5(T-t)^5} \tag{5.5}$$

and

$$\xi(t,x) := \frac{e^{\lambda(10||\eta^0||_\infty + \eta^0(x))}}{t^5(T-t)^5}. \tag{5.6}$$

Additionally, for $t \in (0,T)$ we define

$$\alpha^*(t) := \max_{x \in \bar{\Omega}} \alpha(t,x) \tag{5.7}$$

and

$$\xi^*(t) := \min_{x \in \bar{\Omega}} \xi(t,x). \tag{5.8}$$

For $s, \lambda > 0$ and $u \in L^2((0,T); H_0^1(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$, let us define

$$\mathcal{I}(s,\lambda;u) := s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3|u|^2 dxdt + s\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi \, \|\nabla u\|_1^2 \, dxdt. \tag{5.9}$$

In the work to follow, for $u \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$, we use a slight abuse of notation and define $\mathcal{I}(s,\lambda;u)$ as above but with $|\cdot|$ replaced by $\|\cdot\|_1$, and with $\|\cdot\|_1$ replaced by $\|\cdot\|_2$. We now state a Carleman estimate result for the heat equation; the proof is quite technical and is omitted here.

Lemma 5.2.3. *[18, Theorem 1] Assume that $d > 0$, $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$ and $f_2 \in L^2(\Sigma_T)$. Then there exists a constant $C := C(\Omega, \omega_{p+2}) > 0$ such that the solution to*

$$\begin{cases} -\partial_t u = div(d\nabla u) + f_1, & in\ Q_T, \\ \dfrac{\partial u}{\partial n} = f_2, & on\ \Sigma_T, \\ u(T,\cdot) = u^0(\cdot), & in\ \Omega, \end{cases}$$

satisfies

$$\mathcal{I}(s,\lambda;u) \leq C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3|u|^2dxdt + \iint_{Q_T} e^{-2s\alpha}|f_1|^2dxdt \right.$$
$$\left. + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^*|f_2|^2d\sigma dt \right)$$

for all $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

We can adapt the Carleman estimate in Lemma 5.2.3 to system (5.2) with Neumann boundary condition.

Lemma 5.2.4. *Assume that $\tilde{\psi}^0 \in L^2(\Omega)^m$ and $u \in L^2(\Sigma_T)^m$. Then there exists a constant $C := C(\Omega, \omega_{p+2}) > 0$ such that the solution to*

$$\begin{cases} -\partial_t\tilde{\psi} = div(D\nabla\tilde{\psi}) - G^* \cdot \nabla\tilde{\psi} + A^*\tilde{\psi}, & in\ Q_T, \\ \dfrac{\partial\tilde{\psi}}{\partial n} = u, & on\ \Sigma_T, \\ \tilde{\psi}(T,\cdot) = \tilde{\psi}^0(\cdot), & in\ \Omega, \end{cases} \tag{5.10}$$

satisfies

$$\mathcal{I}(s,\lambda;\tilde{\psi}) \leq C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\tilde{\psi}\right\|_1^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left\|u\right\|_1^2 d\sigma dt\right) \tag{5.11}$$

for all $\lambda \geq C$ *and* $s \geq C(T^5 + T^{10})$.

Proof. We denote by C various positive constants depending on Ω and ω_{p+2}. Lemma 5.2.3 can be extended to a system of parabolic PDEs resembling (5.10) by replacing $|\cdot|$ with $\|\cdot\|_1$; furthermore, if this system has first and zero-order coupling as in (5.2), then (5.11) is still verified since D is diagonal, and hence f_1 absorbs all coupling terms. Indeed, for $k \in \{1,\dots,m\}$ we let

$$f_1^k = \sum_{j=1}^{m} \left(-g_{jk}\cdot\nabla + a_{jk}\right)\tilde{\psi}_j;$$

redefining f_1 now as $f_1 := (f_1^1,\dots,f_1^m)$ yields

$$\mathcal{I}(s,\lambda;\tilde{\psi}) \leq C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\tilde{\psi}\right\|_1^2 dxdt + \iint_{Q_T} e^{-2s\alpha}\left\|f_1\right\|_1^2 dxdt \right.$$
$$\left. + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left\|u\right\|_1^2 d\sigma dt\right). \tag{5.12}$$

By (5.6), we have that

$$\min_{t\in(0,T)} \xi(t,x) = \xi\left(\frac{T}{2},x\right) = \frac{2^{10}e^{\lambda(10||\eta^0||_\infty+\eta^0(x))}}{T^{10}}, \tag{5.13}$$

and hence using (5.9),

$$\begin{aligned}\mathcal{I}(s,\lambda;\tilde{\psi}) &\geq C^7\left(\min_{x\in\bar{\Omega}}\left\{e^{\lambda(10\|\eta^0\|_\infty+\eta^0(x))}\right\}\right)^3\left(1+\frac{1}{T^5}\right)^3\iint_{Q_T} e^{-2s\alpha}\left\|\tilde{\psi}\right\|_1^2 dxdt\\ &\quad + C^3\min_{x\in\bar{\Omega}}\left\{e^{\lambda(10\|\eta^0\|_\infty+\eta^0(x))}\right\}\left(1+\frac{1}{T^5}\right)\iint_{Q_T} e^{-2s\alpha}\|\nabla\tilde{\psi}\|_2^2 dxdt\\ &\geq C\left(\iint_{Q_T} e^{-2s\alpha}\left\|\tilde{\psi}\right\|_1^2 dxdt + \iint_{Q_T} e^{-2s\alpha}\|\nabla\tilde{\psi}\|_2^2 dxdt\right),\end{aligned}$$

since $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. One can employ the triangle inequality to obtain

$$\mathcal{I}(s,\lambda;\tilde{\psi}) \geq C\iint_{Q_T} e^{-2s\alpha}\left\|f_1\right\|_1^2 dxdt,$$

and hence one can absorb the term $C\iint_{Q_T} e^{-2s\alpha}\left\|f_1\right\|_1^2 dxdt$ into the lefthand side of (5.12) to obtain the desired Carleman estimate. □

We will also use the following estimate in the ensuing treatment. Its proof is included in the Appendix.

Lemma 5.2.5. *[12, Lemma 3] Let $r \in \mathbb{R}$. There exists a $C := C(\Omega, \omega_{p+2}, r) > 0$ such that for every $T > 0$ and every $u \in L^2((0,T); H^1(\Omega))$,*

$$\begin{aligned}s^{r+2}\lambda^{r+3}\iint_{Q_T} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt \leq C\Big(s^r\lambda^{r+1}&\iint_{Q_T} e^{-2s\alpha}\xi^r\left\|\nabla u\right\|_1^2 dxdt\\ &+ s^{r+2}\lambda^{r+3}\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt\Big)\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

5.3 Carleman estimate

The goal of this section is to establish the following inequality.

Proposition 5.3.1. *There exists a constant $C := C(\Omega, \omega_0) > 0$ such that for every $\tilde{\psi}^0 \in L^2(\Omega)^m$, the solution $\tilde{\psi}$ to system (5.2) satisfies*

$$\iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \leq Cs^{2p+7}\lambda^{2p+8} \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2p+7} \left\|\tilde{\psi}\right\|_1^2 dxdt \quad (5.14)$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Remark 5.3.2. It should be surprising that (5.14) contains spatial derivatives past order one, since $\tilde{\psi}^0$ is assumed to be in $L^2(\Omega)^m$, and hence by Theorem 2.3.4, $\tilde{\psi} \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); L^2(\Omega))^m$. However, due to inequalities (5.28) and (5.29) and by the fact that the weight $e^{-2s\alpha}$ absorbs the singularity of ξ at $t = 0$, one can deduce that these integrals exist. •

Proof. We denote by C various positive constants which depend on Ω and ω_0. We define the operator

$$\mathcal{L}^* := \left(-\text{div}(D\nabla) + G^* \cdot \nabla - A^*\right). \quad (5.15)$$

By density of $H^k(\Omega)^m \cap H_0^1(\Omega)^m$ in $L^2(\Omega)^m$ for $k \in \mathbb{N}$ (this follows from the inclusion $C_c^\infty(\Omega)^m \subset H^k(\Omega)^m \cap H_0^1(\Omega)^m \subset L^2(\Omega)^m$ and since $C_c^\infty(\Omega)^m$ dense in $L^2(\Omega)^m$), we assume without loss of generality that $\tilde{\psi}^0 \in H^{2p+5}(\Omega)^m$ and $\left((\mathcal{L}^*)^k\tilde{\psi}^0\right)_{k=0}^{p+2} \subset H_0^1(\Omega)$. Hence by Theorem 2.4.2, the solution $\tilde{\psi}$ to system (5.2) is an element of

$$L^2((0,T); H^{2p+6}(\Omega))^m \cap H^{p+3}((0,T); L^2(\Omega))^m. \quad (5.16)$$

We apply the differential operator ∇^{p+2} to system (5.2) and, for β a multi-index with $|\beta| = p+2$, we denote $\partial_\beta \tilde{\psi}$ by ϕ_β so that ϕ_β satisfies

$$\begin{cases} -\partial_t \phi_\beta = \operatorname{div}(D\nabla\phi_\beta) - G^* \cdot \nabla\phi_\beta + A^*\phi_\beta, & \text{in } Q_T, \\ \dfrac{\partial \phi_\beta}{\partial n} = \nabla\phi_\beta \cdot \mathbf{n}, & \text{on } \Sigma_T, \\ \phi_\beta(T,\cdot) = \partial_\beta \tilde{\psi}^0(\cdot), & \text{in } \Omega. \end{cases} \tag{5.17}$$

Indeed, since D, G^* and A^* are constant, ∇^{p+2} commutes with all the terms in system (5.2). We define the $(p+3)$-th order tensor $\phi := (\phi_\beta)_{1 \le \beta_1, \dots, \beta_{p+2} \le n}$; applying Lemma 5.2.4 to system (5.17), we have a Carleman inequality for ϕ:

$$\begin{aligned} &\mathcal{I}(s,\lambda;\phi) \\ &\le C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \|\phi\|_{p+3}^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \|\nabla\phi \cdot n\|_{p+3}^2 d\sigma dt \right) \end{aligned} \tag{5.18}$$

for every $\lambda \ge C$ and $s \ge C(T^5 + T^{10})$. The rest of this proof follows three steps:

(i) We will estimate the boundary term on the righthand side of (5.18) with a global interior term involving $\tilde{\psi}$, which will be absorbed into the lefthand side later;

(ii) we will relate $\mathcal{I}(s,\lambda;\phi)$ with the lefthand side of (5.14);

(iii) we will estimate the local term on the righthand side of (5.18) with a local term of zero differential order (as appearing in (5.14)) and some other local terms which will be absorbed into the lefthand side.

Step (i): Consider a function $\theta \in C^2(\bar{\Omega})$ such that $\nabla\theta \cdot \mathbf{n} = \theta = 1$ in $\bar{\Omega}$, where $\mathbf{n}$ is the outward pointing normal of $\partial\Omega$. With this construction, $\nabla\theta = \mathbf{n}$. Indeed, for any

$q \in \partial\Omega$ and for any parametrized curve $\gamma : \mathbb{R} \to \Omega$ passing through point q at time 0, we have

$$\frac{d}{dt}\theta(\gamma(t))\Big|_{t=0} = \nabla\theta\Big|_q \frac{d\gamma(t)}{dt}\Bigg|_{t=0} = 0,$$

since $\theta = 1$ in $\bar{\Omega}$. Hence, since $\nabla\theta$ is orthogonal to the tangent of any curve passing through any arbitrary point $q \in \partial\Omega$ at $t = 0$, it must be equal to $\mathbf{n}$. Let β and γ be multi-indices of length n; we integrate the boundary term by parts to obtain

$$\begin{aligned} s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \|\nabla\phi \cdot \mathbf{n}\|_{p+3}^2 d\sigma dt &= s\lambda \sum_{|\beta|=p+3} \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \left(\partial_\beta\psi \cdot \nabla\theta\right)\left(\partial_\beta\psi \cdot \mathbf{n}\right) d\sigma dt \\ &= \sum_{\substack{|\beta|=p+3 \\ |\gamma|=p+4}} \Bigg(s\lambda \iint_{Q_T} e^{-2s\alpha^*}\xi^* \left(\partial_\gamma\psi\right)\left(\partial_\beta\psi \cdot \nabla\theta\right) dxdt \\ &\quad + s\lambda \iint_{Q_T} e^{-2s\alpha^*}\xi^* \nabla(\partial_\beta\psi \cdot \nabla\theta) \cdot \partial_\beta\psi dxdt \Bigg). \end{aligned}$$

Next, we employ Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned} &s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \|\nabla\phi \cdot \mathbf{n}\|_{p+3}^2 d\sigma dt \\ &\leq \lambda \int_0^T e^{-2s\alpha^*} \Bigg(\left(\int_\Omega \|(s\xi^*)^k \Delta\phi\|_{p+5}^2 dx \right)^{1/2} \left(\int_\Omega \|(s\xi^*)^{1-k} \nabla\phi \cdot \nabla\theta\|_{p+3}^2 dx \right)^{1/2} \\ &+ \left(\int_\Omega \|(s\xi^*)^k \nabla(\nabla\phi \cdot \nabla\theta)\|_{p+4}^2 dx \right)^{1/2} \left(\int_\Omega \|(s\xi^*)^{1-k} \nabla\phi\|_{p+4}^2 dx \right)^{1/2} \Bigg) dt \\ &\leq C\lambda \int_0^T e^{-2s\alpha^*} \left(||(s\xi^*)^k \tilde{\psi}||_{H^{p+4}(\Omega)^m} ||(s\xi^*)^{1-k} \tilde{\psi}||_{H^{p+3}(\Omega)^m} \right) dt \\ &\leq C\lambda \left(\int_0^T e^{-2s\alpha^*} (s\xi^*)^{2k} ||\tilde{\psi}||_{H^{p+4}(\Omega)^m}^2 dt + \int_0^T e^{-2s\alpha^*} (s\xi^*)^{2-2k} ||\tilde{\psi}||_{H^{p+3}(\Omega)^m}^2 dt \right), \end{aligned} \tag{5.19}$$

for $k \in (0,1)$ to be chosen later. We define $\hat{\tilde{\psi}} := \rho\tilde{\psi}$, with $\rho \in C^\infty([0,T])$ defined by $\rho := (s\xi^*)^a e^{-s\alpha^*}$ for some $a \in \mathbb{R}$ to be chosen later. Note that $\hat{\tilde{\psi}}(T,\cdot) = 0$ in Ω, since ρ decays exponentially to zero as $t \to T$. Similarly, $\frac{d^i}{dt^i}\rho(0) = 0$, for all $i \in \mathbb{N}$. Furthermore, $\hat{\tilde{\psi}}$ is the solution to

$$\begin{cases} -\partial_t \hat{\tilde{\psi}} = \operatorname{div}(D\nabla\hat{\tilde{\psi}}) - G^* \cdot \nabla\hat{\tilde{\psi}} + A^*\hat{\tilde{\psi}} - \dfrac{d}{dt}\rho\tilde{\psi}, & \text{in } Q_T, \\ \quad \hat{\tilde{\psi}} = 0, & \text{on } \Sigma_T, \\ \hat{\tilde{\psi}}(T,\cdot) = 0, & \text{in } \Omega. \end{cases} \tag{5.20}$$

Hence, by (5.16), one can utilize Theorem 2.4.2 to get the estimate

$$\|\hat{\tilde{\psi}}\|_{L^2((0,T);H^{2d+2}(\Omega))^m \cap H^{d+1}((0,T);L^2(\Omega))^m} \leq C \left\| \frac{d}{dt}\rho\tilde{\psi} \right\|_{L^2((0,T);H^{2d}(\Omega))^m \cap H^d((0,T);L^2(\Omega))^m} \tag{5.21}$$

for $d \in \{0, \dots, p+2\}$. Owing to (5.5) and (5.6), we have the bound

$$\left| \frac{d}{dt}\rho \right| \leq CT(s\xi^*)^{a+6/5} e^{-s\alpha^*}. \tag{5.22}$$

Indeed, for

$$\bar{c} := \min_{x\in\bar{\Omega}} \{e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}\},$$

and

$$\tilde{c} := \max_{x\in\bar{\Omega}} \{e^{12\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))}\},$$

$$\begin{aligned}\left|\frac{d}{dt}\rho\right| &= \left|as(s\xi^*)^{a-1}e^{-s\alpha^*}\frac{d}{dt}\xi^* - s(s\xi^*)^a e^{-s\alpha^*}\frac{d}{dt}\alpha^*\right| \\ &= e^{-s\alpha^*}\left|s(s\xi^*)^{a-1}\frac{5(2t-T)}{t^6(T-t)^6}\left(a\bar{c}-(s\xi^*)\tilde{c}\right)\right| \\ &= (s\xi^*)^a e^{-s\alpha^*}\left|\frac{10t-5T}{t(T-t)}\left(a-\frac{(s\xi^*)\tilde{c}}{\bar{c}}\right)\right| \\ &= (s\xi^*)^{a+6/5}e^{-s\alpha^*}\left|\frac{(10t-5T)}{\bar{c}^{6/5}}\left(\frac{at^5(T-t)^5}{s^{6/5}}-\frac{\tilde{c}}{s^{1/5}}\right)\right|,\end{aligned}$$

and since $s \geq C(T^5+T^{10})$, one can obtain (5.22). Similarly, we have

$$\left|\frac{d^r}{dt^r}\rho\right| \leq CT^r(s\xi^*)^{a+6r/5}e^{-s\alpha^*}, \tag{5.23}$$

for $r \in \mathbb{N}$. We apply (5.21) to $\hat{\tilde{\psi}}$ for $a = 1-k$ and $d = \left\lfloor \frac{p+1}{2} \right\rfloor$ to obtain

$$\begin{aligned}&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2-2k}\|\tilde{\psi}\|^2_{H^{2\left\lfloor\frac{p+3}{2}\right\rfloor}(\Omega)^m}dt \\ &\leq C\left(\int_0^T \left\|\frac{d}{dt}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right\|^2_{H^{2\left\lfloor\frac{p+1}{2}\right\rfloor}(\Omega)^m}dt\right. \\ &\left.+\sum_{r=1}^{\left\lfloor\frac{p+1}{2}\right\rfloor}\int_0^T\left\|\frac{d^r}{dt^r}\left(\frac{d}{dt}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right)\right\|^2_{L^2(\Omega)^m}dt\right).\end{aligned} \tag{5.24}$$

We now apply (5.21) to $\hat{\tilde{\psi}} = \frac{d}{dt}\rho\tilde{\psi}$ (which satisfies a system very similar to (5.20) and verifies the compatibility conditions in Theorem 2.4.2) for $a = 1-k$ and $d = \left\lfloor \frac{p+1}{2} \right\rfloor - 1$

to obtain

$$\begin{aligned}
&\int_0^T \left\| \frac{d}{dt}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right\|^2_{H^{2\lfloor \frac{p+1}{2}\rfloor}(\Omega)^m} dt \\
&+ \sum_{r=1}^{\lfloor \frac{p+1}{2}\rfloor} \int_0^T \left\| \frac{d^r}{dt^r}\left(\frac{d}{dt}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right)\right\|^2_{L^2(\Omega)^m} dt \\
&\le C\int_0^T \left\| \frac{d^2}{dt^2}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right\|^2_{H^{2\lfloor \frac{p+1}{2}\rfloor-2}(\Omega)^m} dt \\
&+ \sum_{r=1}^{\lfloor \frac{p+1}{2}\rfloor-1} \int_0^T \left\| \frac{d^r}{dt^r}\left(\frac{d^2}{dt^2}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right)\right\|^2_{L^2(\Omega)^m} dt. \qquad (5.25)
\end{aligned}$$

Repeating this way $\lfloor \frac{p+1}{2}\rfloor - 1$ more times and utilizing (5.23) yields the inequality

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2-2k}\|\tilde{\psi}\|^2_{H^{2\lfloor \frac{p+3}{2}\rfloor}(\Omega)^m} dt \\
&\le C\int_0^T \left\| \frac{d^{\lfloor \frac{p+1}{2}\rfloor+1}}{dt^{\lfloor \frac{p+1}{2}\rfloor+1}}\left(e^{-s\alpha^*}(s\xi^*)^{1-k}\right)\tilde{\psi}\right\|^2_{L^2(\Omega)^m} dt \\
&\le CT^{2\lfloor \frac{p+1}{2}\rfloor+2}\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2-2k+\frac{12}{5}\left(\lfloor \frac{p+1}{2}\rfloor+1\right)}\|\tilde{\psi}\|^2_{L^2(\Omega)^m} dt. \qquad (5.26)
\end{aligned}$$

We can get very similar estimates (5.24) and (5.25) for $a = 3k-1$, $d = \lceil \frac{p+2}{2}\rceil$, and by using (5.23), we obtain

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{6k-2}\|\tilde{\psi}\|^2_{H^{2\lceil \frac{p+4}{2}\rceil}(\Omega)^m} dt \\
&\le C\int_0^T \left\| \frac{d^{\lceil \frac{p+2}{2}\rceil+1}}{dt^{\lceil \frac{p+2}{2}\rceil+1}}\left(e^{-s\alpha^*}(s\xi^*)^{3k-1}\right)\tilde{\psi}\right\|^2_{L^2(\Omega)^m} dt \\
&\le CT^{2\lceil \frac{p+2}{2}\rceil+2}\int_0^T e^{-2s\alpha^*}(s\xi^*)^{6k-2+\frac{12}{5}\left(\lceil \frac{p+2}{2}\rceil+1\right)}\|\tilde{\psi}\|^2_{L^2(\Omega)^m} dt. \qquad (5.27)
\end{aligned}$$

Suppose for the moment that p is odd. By applying Theorem 2.2.6 to the appropriate spatial derivative of $\tilde{\psi}$ with $j = 1$, $m = q = p = r = 2$ and $\alpha = 1/2$, and then employing the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2k}\|\tilde{\psi}\|^2_{H^{p+4}(\Omega)^m}dt \\
&\leq C\int_0^T \|e^{-s\alpha^*}(s\xi^*)^{3k-1}\tilde{\psi}\|_{H^{2\lceil\frac{p+4}{2}\rceil}(\Omega)^m}\|e^{-s\alpha^*}(s\xi^*)^{1-k}\tilde{\psi}\|_{H^{2\lfloor\frac{p+3}{2}\rfloor}(\Omega)^m}dt \\
&\leq C\bigg(\int_0^T e^{-2s\alpha^*}(s\xi^*)^{6k-2}\|\tilde{\psi}\|^2_{H^{2\lceil\frac{p+4}{2}\rceil}(\Omega)^m}dt\bigg)^{\frac{1}{2}}\bigg(\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2-2k}\|\tilde{\psi}\|^2_{H^{2\lfloor\frac{p+3}{2}\rfloor}(\Omega)^m}dt\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Choosing $k = \frac{1}{2} + \frac{3}{10}\left(\left\lfloor\frac{p+1}{2}\right\rfloor - \left\lceil\frac{p+2}{2}\right\rceil\right)$ verifies

$$2 - 2k + \frac{12}{5}\left(\left\lfloor\frac{p+1}{2}\right\rfloor + 1\right) = 6k - 2 + \frac{12}{5}\left(\left\lceil\frac{p+2}{2}\right\rceil + 1\right),$$

and hence by utilizing (5.26) and (5.27), we obtain

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2k}\|\tilde{\psi}\|^2_{H^{p+4}(\Omega)^m}dt \\
&\qquad\leq CT^{\lceil\frac{p+2}{2}\rceil+\lfloor\frac{p+1}{2}\rfloor+2}\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\frac{17}{5}+\frac{9}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{3}{5}\lceil\frac{p+2}{2}\rceil}\|\tilde{\psi}\|^2_{L^2(\Omega)^m}dt.
\end{aligned} \tag{5.28}$$

Identical steps can be followed for the case when p is even to obtain

$$\begin{aligned}
&\int_0^T e^{-2s\alpha^*}(s\xi^*)^{2-2k}\|\tilde{\psi}\|^2_{H^{p+3}(\Omega)^m}dt \\
&\qquad\leq CT^{\lceil\frac{p+2}{2}\rceil+\lfloor\frac{p+1}{2}\rfloor+2}\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\frac{17}{5}+\frac{3}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{9}{5}\lceil\frac{p+2}{2}\rceil}\|\tilde{\psi}\|^2_{L^2(\Omega)^m}dt.
\end{aligned} \tag{5.29}$$

It follows from (5.19), (5.26) and (5.28) that

$$\begin{aligned} &s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \|\nabla\phi\cdot n\|_{p+3}^2 d\sigma dt \\ &\leq C\lambda \left(T^{2\lfloor\frac{p+1}{2}\rfloor+2} + T^{\lceil\frac{p+2}{2}\rceil+\lfloor\frac{p+1}{2}\rfloor+2}\right)\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\frac{17}{5}+\frac{9}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{3}{5}\lceil\frac{p+2}{2}\rceil}\|\tilde\psi\|_{L^2(\Omega)^m}^2 dt, \end{aligned}$$

for p odd, and it follows from (5.19), (5.27) and (5.29)

$$\begin{aligned} &s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi^* \|\nabla\phi\cdot n\|_{p+3}^2 d\sigma dt \\ &\leq C\lambda \left(T^{2\lceil\frac{p+2}{2}\rceil+2} + T^{\lceil\frac{p+2}{2}\rceil+\lfloor\frac{p+1}{2}\rfloor+2}\right)\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\frac{17}{5}+\frac{3}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{9}{5}\lceil\frac{p+2}{2}\rceil}\|\tilde\psi\|_{L^2(\Omega)^m}^2 dt, \end{aligned}$$

for p even. In what follows, we choose p even without loss of generality (the exact same technique can be used for p odd), and since

$$\left(T^{2\lceil\frac{p+2}{2}\rceil+2} + T^{\lceil\frac{p+2}{2}\rceil+\lfloor\frac{p+1}{2}\rfloor+2}\right) \leq Cs^{2p-\frac{3}{5}\lfloor\frac{p+1}{2}\rfloor-\frac{9}{5}\lceil\frac{p+2}{2}\rceil+\frac{17}{5}},$$

for $s \geq C(T^5+T^{10})$, we use (5.7) and (5.8) to obtain

$$\begin{aligned} s\lambda \iint_{\Sigma_T} & e^{-2s\alpha^*}\xi^* \|\nabla\phi\cdot n\|_{p+3}^2 d\sigma dt \\ &\leq Cs^{2p+34/5}\lambda\int_0^T e^{-2s\alpha^*}(\xi^*)^{\frac{17}{5}+\frac{9}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{3}{5}\lceil\frac{p+2}{2}\rceil}\|\tilde\psi\|_{L^2(\Omega)^m}^2 dt \\ &\leq Cs^{2p+34/5}\lambda\iint_{Q_T} e^{-2s\alpha}\xi^{\frac{17}{5}+\frac{9}{5}\lfloor\frac{p+1}{2}\rfloor+\frac{3}{5}\lceil\frac{p+2}{2}\rceil}\left\|\tilde\psi\right\|_1^2 dxdt. \end{aligned}$$

Denoting by $l(p)$ the exponent $\frac{17}{5}+\frac{9}{5}\left\lfloor\frac{p+1}{2}\right\rfloor+\frac{3}{5}\left\lceil\frac{p+2}{2}\right\rceil$, we arrive at the end of Step (i) to conclude that

$$\mathcal{I}(s,\lambda;\phi) \leq C\left(s^3\lambda^4\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3\|\phi\|_{p+3}^2 dxdt + s^{2p+34/5}\lambda\iint_{Q_T} e^{-2s\alpha}\xi^{l(p)}\left\|\tilde{\psi}\right\|_1^2 dxdt\right) \tag{5.30}$$

for every $\lambda \geq C$ and $s \geq C(T^5+T^{10})$.

Step (ii): In this step, we relate $\mathcal{I}(s,\lambda;\phi)$ to the lefthand side of (5.14). We apply Lemma 5.2.5 to $\tilde{\psi}$ for $r = 2p+5$ to obtain

$$s^{2p+7}\lambda^{2p+8}\iint_{Q_T} e^{-2s\alpha}\xi^{2p+7}\left\|\tilde{\psi}\right\|_1^2 dxdt \leq C\Bigg(s^{2p+5}\lambda^{2p+6}\iint_{Q_T} e^{-2s\alpha}\xi^{2p+5}\left\|\nabla\tilde{\psi}\right\|_2^2 dxdt + s^{2p+7}\lambda^{2p+8}\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^{2p+7}\left\|\tilde{\psi}\right\|_1^2 dxdt\Bigg) \tag{5.31}$$

for every $\lambda \geq C$ and $s \geq C(T^5+T^{10})$. Similarly, for $k \in \{0,\dots,p\}$, we apply Lemma 5.2.5 to $\nabla^{p+1-k}\tilde{\psi}$ for $r = 2k+3$ to obtain

$$\begin{aligned} &s^{2k+5}\lambda^{2k+6}\iint_{Q_T} e^{-2s\alpha}\xi^{2k+5}\|\nabla^{p+1-k}\tilde{\psi}\|_{p+2-k}^2 dxdt \\ &\leq C\Bigg(s^{2k+3}\lambda^{2k+4}\iint_{Q_T} e^{-2s\alpha}\xi^{2k+3}\left\|\nabla^{p+2-k}\tilde{\psi}\right\|_{p+3-k}^2 dxdt \\ &+s^{2k+5}\lambda^{2k+6}\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^{2k+5}\|\nabla^{p+1-k}\tilde{\psi}\|_{p+2-k}^2 dxdt\Bigg), \end{aligned} \tag{5.32}$$

for every $\lambda \geq C$ and $s \geq C(T^5+T^{10})$. One can upper bound the first term in the righthand side of (5.31) by (5.32) for $k = p$ and continue this way by backwards iteration on k. The global terms on the righthand side of (5.32) can be absorbed in the exact same way. Hence, a combination of (5.30), (5.31) and (5.32) gives

$$\begin{aligned}
&\iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \\
&\leq C\left(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \right.\\
&\left. + s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3\|\nabla^{p+2}\tilde{\psi}\|^2_{p+3} dxdt + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|^2_1 dxdt \right),
\end{aligned}$$

for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. By utilizing (5.30) once more, we arrive at the inequality

$$\begin{aligned}
&\iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \\
&\leq C\left(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \right.\\
&\left. + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|^2_1 dxdt \right),
\end{aligned} \tag{5.33}$$

which is verified for every $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Step (iii): In this final step, we absorb the higher-order local terms in the righthand side of (5.33). Consider the function $\theta_{p+1} \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases}
\mathrm{Supp}(\theta_{p+1}) \subseteq \omega_{p+1}, & \\
\theta_{p+1} = 1, & \text{in } \omega_{p+2}, \\
0 \leq \theta_{p+1} \leq 1 & \text{in } \Omega.
\end{cases} \tag{5.34}$$

Let β be a multi-index of length n. Since $\bar{\omega}_{p+2} \subset \omega_{p+1}$, where ω_{p+1} is an open subset of Ω, we integrate the rightmost term in (5.33) by parts and employ the the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 dxdt \\
&\leq s^3\lambda^4 \iint_{(0,T)\times\omega_{p+1}} \theta_{p+1} e^{-2s\alpha}\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 dxdt \\
&= -s^3\lambda^4 \iint_{(0,T)\times\omega_{p+1}} \sum_{\substack{i=1\\|\beta|=p+1}}^{n} \left(\partial_i(\theta_{p+1}e^{-2s\alpha}\xi^3)\partial_i\partial_\beta\tilde{\psi} + \theta_{p+1}e^{-2s\alpha}\xi^3\partial_i^2\partial_\beta\tilde{\psi}\right)\left(\partial_\beta\tilde{\psi}\right) dxdt \\
&\leq s^3\lambda^4 \iint_{(0,T)\times\omega_{p+1}} \Bigg(\left\|\nabla\left(\theta_{p+1}e^{-2s\alpha}\xi^3\right)\right\|_1 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3} \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2} \\
&+\theta_{p+1}e^{-2s\alpha}\xi^3 \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4} \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2}\Bigg) dxdt. \qquad (5.35)
\end{aligned}$$

By (5.5) and (5.6), we have that

$$\left\|\nabla\left(\theta_{p+1}e^{-2s\alpha}\xi^3\right)\right\|_1 \leq Cs\lambda e^{-2s\alpha}\xi^4. \qquad (5.36)$$

Indeed,

$$\begin{aligned}
\left\|\nabla\left(\theta_{p+1}e^{-2s\alpha}\xi^3\right)\right\|_1 &= \left\|e^{-2s\alpha}\xi^3\left(\nabla\theta_{p+1} + 2s\lambda\theta_{p+1}\xi\nabla\eta^0 + 3\lambda\theta_{p+1}\nabla\eta^0\right)\right\|_1 \\
&= s\lambda e^{-2s\alpha}\xi^4 \left\|\frac{\nabla\theta_{p+1}}{s\lambda\xi} + 2\theta_{p+1}\nabla\eta^0 + \frac{3\theta_{p+1}\nabla\eta^0}{s\xi}\right\|_1,
\end{aligned}$$

and since $s \geq C(T^5 + T^{10})$, (5.36) is verified. Hence, by (5.34), (5.36) and using

Young's inequality with $\epsilon > 0$, we have

$$
\begin{aligned}
& s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 dxdt \\
& \le Cs^3\lambda^4 \iint_{(0,T)\times\omega_{p+1}} \Big(s\lambda e^{-2s\alpha}\xi^4 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3} \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2} \\
& + e^{-2s\alpha}\xi^3 \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4} \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2} \Big) dxdt \\
& \le C \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha} \Big(\epsilon s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + \epsilon s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2 \\
& + \frac{2}{\epsilon} s^5\lambda^6\xi^5 \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2}^2 \Big) dxdt. \qquad (5.37)
\end{aligned}
$$

Observe that the first two terms in the righthand side of (5.37) can be bounded above by employing (5.33) and (5.37) recursively: indeed, by positivity of the integrand in Q_T and by (5.33), we obtain

$$
\begin{aligned}
& \epsilon \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha} \left(s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2 \right) dxdt \\
& \le C\epsilon \Bigg(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1} \|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
& + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|_1^2 dxdt \Bigg) \\
& = C\epsilon \Bigg(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1} \|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
& + s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 dxdt + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|_1^2 dxdt \Bigg), \qquad (5.38)
\end{aligned}
$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Combining (5.38) and (5.37) yields

$$\begin{aligned}
&\epsilon \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha}\left(s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2\right) dxdt \\
&\leq C\Bigg(\epsilon \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\sum_{k=3}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
&+ \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha}\epsilon^2\left(s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2\right) \\
&+ \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha}2s^5\lambda^6\xi^5 \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2}^2 dxdt \\
&+ \epsilon s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|_1^2 dxdt\Bigg),
\end{aligned} \tag{5.39}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Using the same treatment by adapting (5.37), one can bound from above the terms being multiplied by ϵ^2 in (5.39); after r of these recursions, we obtain

$$\begin{aligned}
&\epsilon \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha}\left(s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2\right) dxdt \\
&\leq C\sum_{j=1}^{r}\Bigg(\epsilon^j \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\sum_{k=3}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
&+ \epsilon^{2(r+1)} \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha}\Big(s^3\lambda^4\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 + s\lambda^2\xi \left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}^2 \\
&+2js^5\lambda^6\xi^5 \left\|\nabla^{p+1}\tilde{\psi}\right\|_{p+2}^2\Big) dxdt + \epsilon^j s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|_1^2 dxdt\Bigg),
\end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Taking ϵ sufficiently small, we obtain from (5.37) that

$$\begin{aligned}
& s^3\lambda^4 \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha}\xi^3 \left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3}^2 dxdt \\
& \leq C\Bigg(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
& + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|_1^2 dxdt \Bigg),
\end{aligned} \tag{5.40}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$, since by (5.35), if $\left\|\nabla^{p+2}\tilde{\psi}\right\|_{p+3} = 0$, then so does $\left\|\nabla^{p+3}\tilde{\psi}\right\|_{p+4}$. Hence from (5.40), we obtain

$$\begin{aligned}
& \iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=2}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt \\
& \qquad \leq C \iint_{(0,T)\times\omega_{p+1}} e^{-2s\alpha} \sum_{k=3}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|_{p+5-k}^2 dxdt,
\end{aligned} \tag{5.41}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. For $r \in \{1, \ldots, p+1\}$, consider the functions $\theta_r \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases}
\operatorname{Supp}(\theta_{p+1-r}) \subseteq \omega_{p+1-r}, & \\
\theta_{p+1-r} = 1, & \text{in } \omega_{p+2-r}, \\
0 \leq \theta_{p+1-k} \leq 1, & \text{in } \Omega.
\end{cases}$$

Using the exact same approach as was used for $r = 0$, one obtains the estimate

$$\begin{aligned} &s^{2r+3}\lambda^{2r+4} \iint_{(0,T)\times\omega_{p+2-r}} e^{-2s\alpha}\xi^{2r+3} \left\|\nabla^{p+2-r}\tilde{\psi}\right\|^2_{p+3-r} dxdt \\ &\quad \leq C\Bigg(\iint_{(0,T)\times\omega_{p+2}} e^{-2s\alpha} \sum_{k=3+r}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \\ &\quad + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|^2_1 dxdt \Bigg), \end{aligned}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Hence, it follows that

$$\begin{aligned} &\iint_{Q_T} e^{-2s\alpha} \sum_{k=1}^{p+4} s^{2k-1}\lambda^{2k}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}\|^2_{p+5-k} dxdt \\ &\quad \leq C\Bigg(s^{2p+7}\lambda^{2p+8} \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2p+7} \left\|\tilde{\psi}\right\|^2_1 dxdt \\ &\quad + s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|^2_1 dxdt \Bigg), \end{aligned} \tag{5.42}$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. Finally, by (5.6) we have the estimate

$$s^{2p+34/5}\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{l(p)} \left\|\tilde{\psi}\right\|^2_1 dxdt \leq C s^{2p+7}\lambda^{2p+8} \iint_{Q_T} e^{-2s\alpha}\xi^{2p+7} \left\|\tilde{\psi}\right\|^2_1 dxdt,$$

for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$ large enough; from now on, we denote this choice of s by s_0. Hence, one can absorb the global term in the righthand side of (5.42) into its lefthand side, and thus (5.14) is verified. □

5.4 Observability inequality

In this section, we prove Proposition 5.1.1.

Proof of Proposition 5.1.1. We denote by C various positive constant depending on Ω and ω_0. From (5.14), we deduce

$$\iint_{Q_T} e^{-2s\alpha}\xi^{2p+7}\left\|\tilde{\psi}\right\|_1^2 dxdt \le C\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2p+7}\left\|\tilde{\psi}\right\|_1^2 dxdt, \tag{5.43}$$

for $\lambda \ge C$ and $s \ge s_0$. Note that for $t \in \left[\frac{T}{4}, \frac{3T}{4}\right]$, we have

$$\begin{aligned}
&\min_{t\in\left[\frac{T}{4},\frac{3T}{4}\right]}\{e^{-2s\alpha}\xi^{2p+7}\}\\
&= \left(e^{-2s\alpha}\xi^{2p+7}\right)\left(\frac{T}{4},\cdot\right) = \left(e^{-2s\alpha}\xi^{2p+7}\right)\left(\frac{3T}{4},\cdot\right)\\
&= \left(e^{-2s\frac{4^{10}}{3^5}\left(\frac{e^{12\lambda\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty+\eta^0(x))}}{T^{10}}\right)}\right)\left(\frac{4^{10}e^{(2p+7)\lambda(10\|\eta^0\|_\infty+\eta^0(x))}}{3^5T^{10}}\right).
\end{aligned} \tag{5.44}$$

We can choose s sufficiently large such that

$$\frac{4^{10}}{3^5T^{10}}e^{-\frac{s}{T^{10}}} \le e^{-2s\alpha}\xi^{2p+7}, \tag{5.45}$$

for all $t \in \left[\frac{T}{4}, \frac{3T}{4}\right]$. Indeed, choosing

$$s \ge s_1 := \max\left\{s_0, \left(\frac{3^5(2p+7)\lambda}{4^{10}}\right)\max_{x\in\bar{\Omega}}\left\{\frac{10\|\eta^0\|_\infty+\eta^0(x)}{e^{12\lambda\|\eta^0\|_\infty} - e^{\lambda(10\|\eta^0\|_\infty+\eta^0(x))}}\right\}\right\}$$

in (5.44) will ensure that (5.45) is verified. Note that we can write s_1 as $s_1 = \sigma\left(T^5+T^{10}\right)$, where $\sigma>0$ depends only on Ω and ω_0. Fixing $s=s_1$ from now on, we deduce from (5.43) and (5.45) that

$$\iint_{\left(\frac{T}{4},\frac{3T}{4}\right)\times\Omega}\left\|\tilde{\psi}\right\|_1^2 dxdt \le CT^{10}e^{C\left(1+1/T^5\right)}\iint_{(0,T)\times\omega_0} e^{-2s_1\alpha}\xi^7\left\|\tilde{\psi}\right\|_1^2 dxdt$$

for every $\lambda \geq C$ and $s \geq s_1$. We claim that

$$\int_{\Omega} \left\| \tilde{\psi}(\cdot, T/4) \right\|_1^2 dx \leq \frac{C}{T} e^{CT/2} \iint_{\left(\frac{T}{4}, \frac{3T}{4}\right) \times \Omega} \left\| \tilde{\psi} \right\|_1^2 dxdt \tag{5.46}$$

and

$$\int_{\Omega} \left\| \tilde{\psi}(\cdot, 0) \right\|_1^2 dx \leq e^{CT/4} \int_{\Omega} \left\| \tilde{\psi}(\cdot, T/4) \right\|_1 dx, \tag{5.47}$$

from which we can deduce (5.3). Indeed, we can multiply system (5.2) by $\tilde{\psi}$, integrate the resulting equation by parts over Ω and use the Cauchy-Schwarz and Young's inequalities to obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\Omega} \left\| \tilde{\psi} \right\|_1^2 dx + D \int_{\Omega} \|\nabla \tilde{\psi}\|_2^2 dx \\
&= -\int_{\Omega} \left(\partial_t \tilde{\psi}\right) \tilde{\psi} dx + \int_{\Omega} \operatorname{div}(D \nabla \tilde{\psi}) \tilde{\psi} dx \\
&= -\int_{\Omega} \left(G^* \cdot \nabla \tilde{\psi}\right) \tilde{\psi} dx + \int_{\Omega} \left(A^* \tilde{\psi}\right) \tilde{\psi} dx \\
&\leq \frac{1}{2} \int_{\Omega} \left\| G^* \cdot \nabla \tilde{\psi} \right\|_1^2 dx + \left(1 + \frac{\|A^*\|_\infty}{2}\right) \int_{\Omega} \left\| \tilde{\psi} \right\|_1^2 dx.
\end{aligned}$$

Hence, since (2.10) satisfies the uniform ellipticity condition (see (2.7)), we obtain

$$-\frac{d}{dt}\int_{\Omega} \left\| \tilde{\psi} \right\|_1^2 dx + \int_{\Omega} \|\nabla \tilde{\psi}\|_2^2 dx \leq C \int_{\Omega} \left\| \tilde{\psi} \right\|_1^2 dx,$$

from which we deduce

$$\begin{aligned}\frac{d}{dt}\left(e^{Ct}\int_\Omega \left\|\tilde{\psi}\right\|_1^2 dx\right) &= e^{Ct}\left(C\int_\Omega \left\|\tilde{\psi}\right\|_1^2 dx + \frac{d}{dt}\int_\Omega \left\|\tilde{\psi}\right\|_1^2 dx\right)\\ &\geq e^{Ct}\int_\Omega \|\nabla\tilde{\psi}\|_2^2 dx\\ &\geq 0, \qquad (5.48)\end{aligned}$$

for all $t > 0$. We integrate (5.48) over $\left[\frac{T}{4}, t\right]$ to obtain

$$\begin{aligned}\int_\Omega \left\|\tilde{\psi}\right\|_1^2 dx &\geq e^{C(T/4-t)}\int_\Omega \left\|\tilde{\psi}\left(T/4,\cdot\right)\right\|_1^2 dx\\ &\geq e^{-CT/2}\int_\Omega \left\|\tilde{\psi}\left(T/4,\cdot\right)\right\|_1 dx, \qquad (5.49)\end{aligned}$$

for every $t \in \left[\frac{T}{4}, \frac{3T}{4}\right]$. We integrate (5.49) once more over $\left[\frac{T}{4}, \frac{3T}{4}\right]$ to obtain

$$\frac{T}{2}\int_\Omega \left\|\tilde{\psi}\left(T/4,\cdot\right)\right\|_1^2 dx \leq e^{CT/2}\iint_{\left(\frac{T}{4},\frac{3T}{4}\right)\times\Omega} \left\|\tilde{\psi}\right\|_1^2 dx,$$

which verifies (5.46). To show that (5.47) is verified, we integrate (5.48) over $t \in \left[0, \frac{T}{4}\right]$ and conclude at once that

$$\int_\Omega \left\|\tilde{\psi}(0,\cdot)\right\|_1^2 dx \leq e^{CT/4}\int_\Omega \left\|\tilde{\psi}\left(T/4,\cdot\right)\right\|_1^2 dx.$$

□

Chapter 6

Null controllability of the analytic problem and proof of main theorem

Recall from Chapter 3 that our principal goal was to prove null controllability of system (3.1) with multiple underactuations. To this end, we studied an algebraic system and an analytic system both related to system (3.1). In Chapter 4, we developed an algebraic method which allowed us to solve the algebraic control problem posed in (4.17) under the assumption that the source term $\mathbb{1}_\omega \tilde{u}$ be regular enough so that our algebraic solution $\mathcal{B}(\mathbb{1}_\omega \tilde{u})$ be well-defined, where $\mathcal{B}$ is a differential operator of differential order zero in time and at most $p+2$ in space. In Chapter 5, we established the weighted observability inequality (5.3) for the analytic system (5.1).

The goal of this chapter is solve the analytic control problem (4.16) with *regular enough* controls $\mathbb{1}_\omega \tilde{u}$ so that the algebraic control problem (4.17) also be solved. The treatment presented in this chapter is an extension of that used in [15, Section 2.3]. In particular, since the right inverse $\mathcal{B}$ of $\mathcal{L}$ derived implicitly in Chapter 4 is in general of order at most $p+2$ in space, we require much more regular controls for the analytic problem than in [15].

6.1 An optimal control result

We do not use the weighted observability inequality to directly deduce null controllability of system (5.1). Instead, we use a method developed in [19] to construct controls with high regularity which will help us deduce controllability results; to do this, we rely on the following unconstrained optimal control result.

Theorem 6.1.1. *[24, Chapter 3, Theorem 2.2] Let $y^0 \in L^2(\Omega)^m$, $u \in L^2(Q_T)^m$, $B \in \mathscr{L}(L^2(Q_T)^m; L^2(Q_T)^m)$, and suppose $\mathcal{L}$ given in (2.10) satisfies the uniform ellipticity condition (2.7). Let $N \in \mathscr{L}(L^2(Q_T)^m; L^2(Q_T)^m)$ such that $(Nu, u)_{L^2(Q_T)^m} \geq \nu \|u\|^2_{L^2(Q_T)^m}$ for $\nu > 0$ and for all $u \in L^2(Q_T)^m$, and let $D \in \mathscr{L}(H^1_0(\Omega))^m; H^1_0(\Omega))^m)$. Consider the optimal control problem associated to system (2.11) with cost functional $J(u) : L^2(Q_T)^m \to \mathbb{R}^+$ given by*

$$J(u) := (Nu, u)_{L^2(Q_T)^m} + (Dy_u(T, \cdot) - z_d)^2_{L^2(\Omega)^m}\,, \tag{6.1}$$

for some $z_d \in H^1_0(\Omega)^m$. This problem has a unique solution, and the optimal control is characterized by the following relations:

$$\begin{cases} \partial_t y_u = div(D\nabla y_u) + G \cdot \nabla y_u + Ay_y + Bu, & in\ Q_T, \\ y_u = 0, & on\ \Sigma_T, \\ y_u(0, \cdot) = y^0(\cdot), & in\ \Omega, \end{cases}$$

$$\begin{cases} -\partial_t \psi_u = div(D\nabla \psi_u) - G^* \cdot \nabla \psi_u + A^* \psi_u, & in\ Q_T, \\ \psi_u = 0, & on\ \Sigma_T, \\ \psi_u(T, \cdot) = D^*\left(Dy_u(T, \cdot) - z_d\right), & in\ \Omega, \end{cases}$$

and

$$B^*\psi_u + Nu = 0.$$

Hence, for this unconstrained optimal control problem, the second term in (6.1) *has no dependence on u (nor do the primal/adjoint systems).*

6.2 Null controllability of the analytic problem

Recall that we fixed a p large enough such that the prolonged version of system (4.31) be overdetermined. In this section, we establish the following proposition.

Proposition 6.2.1. *Consider $\theta \in C^{p+2}(\bar{\Omega})$ such that*

$$\begin{cases} Supp(\theta) \subseteq \omega, & \\ \theta = 1, & in\ \omega_0, \\ 0 \leq \theta \leq 1, & in\ \Omega. \end{cases} \tag{6.2}$$

Then there exists $v \in L^2(Q_T)^m$ such that

$$(\tilde{y}, \theta v) \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m \times L^2(Q_T)^m$$

is a solution to the analytic control problem (4.16) *satisfying $\tilde{y}(T,\cdot) = 0$ in Ω. Moreover, for every $K \in (0,1)$, we have*

$$e^{Ks_1\alpha^*} v \in L^2((0,T); H^{p+2}(\Omega) \cap H_0^1(\Omega))^m \cap H^1((0,T); L^2(\Omega))^m,$$

and

$$\|e^{Ks_1\alpha^*} v\|_{L^2((0,T);H^{p+2}(\Omega)\cap H_0^1(\Omega))^m \cap H^1((0,T);L^2(\Omega))^m} \leq C\|\tilde{y}^0\|_{L^2(\Omega)^m}. \tag{6.3}$$

Proof. Let $\tilde{y}^0 \in L^2(\Omega)^m$, $\rho := e^{-2s_1\alpha}\xi^{2p+7}$ and $C := C(\Omega, \omega_0, T) > 0$. Let $k \in \mathbb{N}^*$ and denote by $L^2(Q_T, \rho^{-1/2})^m$ the space of functions which, when multiplied by $\rho^{-1/2}$, are L^2-integrable (i.e., for $u \in L^2(Q_T, \rho^{-1/2})^m$, we require $\iint_{Q_T} \rho^{-1} \|u\|_1^2 \, dxdt < \infty$). Consider the following optimal control problem:

$$\begin{cases} \text{minimize} \quad J_k(v) := \frac{1}{2} \iint_{Q_T} \rho^{-1} \|v\|_1^2 \, dxdt + \frac{k}{2} \int_\Omega \|\tilde{y}(T, \cdot)\|_1^2 \, dx, \\ \text{subject to} \quad v \in L^2(Q_T, \rho^{-1/2})^m, \end{cases} \tag{6.4}$$

where $\tilde{y} \in L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m$. The functional J_k is differentiable, coercive and strictly convex on $L^2(Q_T, \rho^{-1/2})^m$. By Theorem 6.1.1 (for $D = \sqrt{k}$, $N = \rho^{-1}$ and $z_d = 0$ in Q_T), there exists a unique solution to this optimal control problem, and the optimal control is characterized by the solution $\tilde{y}_k$ to the analytic system

$$\begin{cases} \partial_t \tilde{y}_k = \operatorname{div}(D\nabla \tilde{y}_k) + G \cdot \nabla \tilde{y}_k + A\tilde{y}_k + \theta v_k, & \text{in } Q_T, \\ \tilde{y}_k = 0, & \text{on } \Sigma_T, \\ \tilde{y}_k(0, \cdot) = \tilde{y}^0(\cdot), & \text{in } \Omega, \end{cases} \tag{6.5}$$

the solution $\tilde{\psi}_k$ to its adjoint system

$$\begin{cases} -\partial_t \tilde{\psi}_k = \operatorname{div}(D\nabla \tilde{\psi}_k) - G^* \cdot \nabla \tilde{\psi}_k + A^* \tilde{\psi}_k, & \text{in } Q_T, \\ \tilde{\psi}_k = 0, & \text{on } \Sigma_T, \\ \tilde{\psi}_k(T, \cdot) = k\tilde{y}(T, \cdot), & \text{in } \Omega, \end{cases} \tag{6.6}$$

and the relation

$$\begin{cases} v_k = -\rho\theta\tilde{\psi}_k, \quad \text{in } Q_T, \\ v_k \in L^2(Q_T, \rho^{-1/2})^m. \end{cases} \tag{6.7}$$

From (6.5) and (6.6), we calculate

$$\begin{aligned} &\int_0^T \Big((\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \Big)\, dt \\ &= \frac{d}{dt} \int_0^T (\tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} dt \\ &= (\tilde{y}_k(T, \cdot), k\tilde{y}_k(T, \cdot))_{L^2(\Omega)^m} - (\tilde{y}^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m}, \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} &(\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \\ &= (\tilde{y}_k, -\mathrm{div}(D\nabla\tilde{\psi}_k) + G^* \cdot \nabla\tilde{\psi}_k - A^*\tilde{\psi}_k)_{L^2(\Omega)^m} \\ &+ (\mathrm{div}(D\nabla\tilde{y}_k) + G \cdot \nabla\tilde{y}_k + A\tilde{y}_k + \theta v_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \\ &= (\theta v_k, \tilde{\psi}_k)_{L^2(\Omega)^m}. \end{aligned} \tag{6.9}$$

It follows from (6.7), (6.8) and (6.9) that

$$\begin{aligned} J_k(v_k) &= -\frac{1}{2}\int_0^T (\theta\tilde{\psi}_k, v_k)_{L^2(\Omega)^m} dt + \frac{1}{2}(\tilde{y}_k(T, \cdot), \tilde{\psi}_k(T, \cdot))_{L^2(\Omega)^m} \\ &= -\frac{1}{2}\int_0^T (\tilde{\psi}_k, \theta v_k)_{L^2(\Omega)^m} dt + \frac{1}{2}\int_0^T \Big((\tilde{y}_k, \partial_t \tilde{\psi}_k)_{L^2(\Omega)^m} + (\partial_t \tilde{y}_k, \tilde{\psi}_k)_{L^2(\Omega)^m} \Big)\, dt \\ &+ \frac{1}{2}(y^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m} \\ &= \frac{1}{2}(y^0, \tilde{\psi}_k(0, \cdot))_{L^2(\Omega)^m}. \end{aligned} \tag{6.10}$$

Moreover, employing the weighted observability inequality (5.3) along with (6.2), (6.7), (6.4), (6.10) and the Cauchy-Schwarz inequality successively, we have

$$\begin{aligned}
\|\tilde{\psi}_k(0,\cdot)\|^2_{L^2(\Omega)^m} &\leq C_{obs} \iint_{(0,T)\times\omega_0} \rho\theta^2 \left\|\tilde{\psi}_k\right\|_1^2 dxdt \\
&\leq C_{obs} \iint_{Q_T} \rho\theta^2 \left\|\tilde{\psi}_k\right\|_1^2 dxdt \\
&= C_{obs} \iint_{Q_T} \rho^{-1} \left\|v_k\right\|_1^2 dxdt \\
&\leq 2C_{obs} J_k(v_k) \\
&\leq 2C_{obs} \|\tilde{\psi}_k(0,\cdot)\|_{L^2(\Omega)^m} \|y^0\|_{L^2(\Omega)^m},
\end{aligned}$$

from which we deduce

$$\|\tilde{\psi}_k(0,\cdot)\|_{L^2(\Omega)^m} \leq 2C_{obs}\|y^0\|_{L^2(\Omega)^m}. \tag{6.11}$$

Furthermore, by (6.10), (6.11) and the Cauchy-Schwarz inequality, we obtain

$$J_k(v_k) \leq C_{obs}\|y^0\|^2_{L^2(\Omega)^m}. \tag{6.12}$$

One can deduce from Theorem 2.3.3, (6.2) and (6.12) that

$$\begin{aligned}
\|\tilde{y}_k\|_{L^2((0,T);H_0^1(\Omega))^m \cap H^1((0,T);H^{-1}(\Omega))^m} &\leq C\left(\|\theta v_k\|_{L^2(Q_T)^m} + \|\tilde{y}^0\|_{L^2(\Omega)^m}\right) \\
&\leq C\left(\|v_k\|_{L^2(Q_T)^m} + \|\tilde{y}^0\|_{L^2(\Omega)^m}\right) \\
&\leq C(1+\sqrt{2C_{obs}})\|\tilde{y}^0\|_{L^2(\Omega)^m}, \qquad (6.13)
\end{aligned}$$

since for our choice of s_1 (which depends on p) and by (5.5) and (5.6), $\rho \leq 1$ in Q_T. Owing to the well-known result that in Hilbert spaces, bounded sequences have weakly convergent subsequences (see, for example, [10]), along with (6.4) (6.12), and (6.13), one can extract subsequences of $(v_k)_k$ and $(\tilde{y}_k)_k$ (which we still denote by v_k and $\tilde{y}_k$) such that

$$\begin{cases} v_k \rightharpoonup v & \text{in } L^2(Q_T, \rho^{-1/2})^m, \\ \tilde{y}_k \rightharpoonup \tilde{y} & \text{in } L^2((0,T); H_0^1(\Omega))^m \cap H^1((0,T); H^{-1}(\Omega))^m, \\ \tilde{y}_k(T,\cdot) \rightharpoonup 0 & \text{in } L^2(\Omega)^m. \end{cases}$$

Hence, $(\tilde{y}, \theta v)$ is the solution to the analytic control problem (4.16) with $\theta v \in L^2(Q_T, \rho^{-1/2})$. Furthermore, we deduce from (6.4) by taking $k \to \infty$ that $\tilde{y}(T,\cdot) = 0$ (in the sense of Definition 2.3.1). In addition, by (6.12) and since $\rho \leq 1$ in Q_T for our choice of s_1, it follows that

$$\|v\|^2_{L^2(Q_T)} \leq \sqrt{2C_{obs}} \|y^0\|^2_{L^2(\Omega)^m},$$

as claimed. It is left to show that (6.3) is verified. Note that for every $K \in (0,1)$, there exists a $C_K := C_K(\Omega)$ such that

$$e^{2Ks_1\alpha^*} \leq C_K \xi^{-2p-7} e^{2s_1\alpha}, \tag{6.14}$$

for all $(t,x) \in Q_T$. Hence, utilizing (6.14), (6.4) and then (6.12), we obtain

$$\begin{aligned} \|e^{2Ks_1\alpha^*} v_k\|^2_{L^2(Q_T)^m} &\le C_K \iint_{Q_T} \rho^{-1} \|v_k\|_1^2 \, dxdt \\ &\le C_K \|\tilde{y}^0\|^2_{L^2(\Omega)^m}. \end{aligned} \tag{6.15}$$

Similar to (5.22), for $a > 0$, one has

$$|\partial_t(\xi^a e^{-2s_1\alpha})| \le CT\xi^{a+6/5} e^{-2s_1\alpha}. \tag{6.16}$$

Furthermore, for $r \in \mathbb{N}$ one has

$$\left\|\nabla^r(\xi^a e^{-2s_1\alpha})\right\|_r \le C\xi^{a+r} e^{-2s_1\alpha}. \tag{6.17}$$

Indeed,

$$\begin{aligned} \nabla(\xi^a e^{-2s_1\alpha}) &= a\xi^{a-1}\lambda\nabla\eta^0\xi e^{-2s_1\alpha} - 2s_1\xi^a e^{-2s_1\alpha}\left(-\lambda\nabla\eta^0\xi\right) \\ &= \lambda\nabla\eta^0\left(\frac{a}{\xi} + 2s_1\right)\xi^{a+1}e^{-2s_1\alpha}, \end{aligned}$$

and since $C := C(\Omega, \omega_0, T)$, (6.17) is verified for $r = 1$. The same reasoning can be used for the r-th derivative, where we have fixed $\eta^0 \in C^r(\bar{\Omega})$ (the existence of such an η^0 is verified in Section 8.1) for $r = p + 2$. Hence, by (6.7), the triangle inequality and then (6.17) for $a = 2p+7$, we obtain

$$\begin{aligned}
&\|e^{Ks_1\alpha^*}\nabla v_k\|^2_{L^2(Q_T)^m} \\
&= \iint_{Q_T} e^{2Ks_1\alpha^*}\|\nabla v_k\|_2^2 dxdt \\
&= \iint_{Q_T} e^{2Ks_1\alpha^*}\|\nabla(-\xi^{2p+7}e^{-2s_1\alpha}\theta\tilde{\psi}_k)\|_2^2 dxdt \\
&\leq C\iint_{Q_T} e^{2Ks_1\alpha^*}\left(\|\nabla(\xi^{2p+7}e^{-2s_1\alpha})\|_1^2\left\|\tilde{\psi}_k\right\|_1^2 + \left\|\xi^{2p+7}e^{-2s_1\alpha}\nabla\tilde{\psi}_k\right\|_2^2\right)dxdt \\
&\leq C\iint_{Q_T} e^{2Ks_1\alpha^*-4s_1\alpha}\left(\xi^{4p+16}\left\|\tilde{\psi}_k\right\|_1^2 + \xi^{4p+14}\|\nabla\tilde{\psi}_k\|_2^2\right)dxdt, \qquad (6.18)
\end{aligned}$$

and similarly, for $r \in \{1,\ldots,p+2\}$, we obtain

$$\|e^{Ks_1\alpha^*}\nabla^r v_k\|^2_{L^2(Q_T)^m} \leq C\iint_{Q_T} e^{2Ks_1\alpha^*-4s_1\alpha}\left(\sum_{l=0}^{r}\xi^{4p+14+2l}\|\nabla^{r-l}\tilde{\psi}_k\|^2_{r-l+1}\right)dxdt. \qquad (6.19)$$

By (6.16) and since $\tilde{\psi}_k$ satisfies system (6.6), we obtain

$$\begin{aligned}
&\|\partial_t(e^{Ks_1\alpha^*}v_k)\|^2_{L^2(Q_T)^m} \qquad (6.20)\\
&\leq C\iint_{Q_T} e^{2Ks_1\alpha^*-4s_1\alpha}\left(\xi^{(20p+82)/5}\left\|\tilde{\psi}_k\right\|_1^2 + \xi^{2p+14}\left\|\partial_t\tilde{\psi}_k\right\|_1^2\right)dxdt \\
&\leq C\iint_{Q_T} e^{2Ks_1\alpha^*-4s_1\alpha}\Bigg(\xi^{(20p+82)/5}\left\|\tilde{\psi}_k\right\|_1^2 \\
&+ \xi^{2p+14}\left(\|\nabla\nabla\tilde{\psi}_k\|_3^2 + \|\nabla\tilde{\psi}_k\|_2^2 + \left\|\tilde{\psi}_k\right\|_1^2\right)\Bigg)dxdt. \qquad (6.21)
\end{aligned}$$

Note that for every $a, b > 0$ and $K \in (0,1)$, there exists $C_{a,b,K} := C_{a,b,K}(\Omega) > 0$ such that

$$\left|\xi^a e^{2Ks_1\alpha^*-4s_1\alpha}\right| \leq C_{a,b,K}\xi^b e^{2s_1\alpha}. \qquad (6.22)$$

From (6.15), (6.18), (6.19), (6.20) and utilizing (6.22) for appropriate a and b, we obtain

$$\|e^{Ks_1\alpha^*}v_k\|_{L^2((0,T);H^{p+2}(\Omega)\cap H_0^1(\Omega))^m\cap H^1((0,T);L^2(\Omega))^m} \\ \leq C_{max,K}\iint_{Q_T} e^{-2s_1\alpha}\sum_{k=2}^{p+4}\xi^{2k-1}\|\nabla^{p+4-k}\tilde{\psi}_k\|_{p+5-k}^2 dxdt,$$

where $C_{max,K} := \max\{\max_{a,b}\{C_{a,b,K}\}, C_K\}$. Owing to (6.2), Proposition 5.3.1 and (6.7), we deduce

$$\|e^{Ks_1\alpha^*}v_k\|_{L^2((0,T);H^{p+2}(\Omega)\cap H_0^1(\Omega))^m\cap H^1((0,T);L^2(\Omega))^m} \\ \leq C_{max,K}C_{obs}\iint_{Q_T} e^{-2s_1\alpha}\xi^{2p+7}\left\|\theta\tilde{\psi}_k\right\|_1^2 dxdt \\ = C_{max,K}C_{obs}\|v_k\|_{L^2(Q_T)^m}^2.$$

Lastly, for $\bar{C}_K := \bar{C}_K(\Omega,\omega_0,T)$, (6.12) yields the inequality

$$\|e^{Ks_1\alpha^*}v_k\|_{L^2((0,T);H^{p+2}(\Omega)\cap H_0^1(\Omega))^m\cap H^1((0,T);L^2(\Omega))^m} \leq \bar{C}_K\|\tilde{y}^0\|_{L^2(\Omega)^m},$$

from which (6.3) is verified by extracting a convergent subsequence and letting $k \to \infty$. □

With algebraic solvability of the algebraic control problem (4.17) and null controllability of the analytic control problem (4.16) both established for highly regular controls, we can now prove null controllability of the system (3.1) with internal controls $\hat{u} \in L^2(q_T)^c$, where $c < m-1$.

In Proposition 6.2.1, we showed the existence of $(\tilde{y},\theta v) \in L^2((0,T);H_0^1(\Omega))^m \cap$

$H^1((0,T);H^{-1}(\Omega))^m \times L^2(Q_T)^m$ satisfying

$$\begin{cases} \partial_t\tilde{y} = \operatorname{div}(D\nabla\tilde{y}) + G\cdot\nabla\tilde{y} + A\tilde{y} + \theta v, & \text{in } Q_T, \\ \tilde{y} = 0, & \text{on } \Sigma_T, \\ \tilde{y}(0,\cdot) = y^0(\cdot), & \text{in } \Omega, \end{cases} \tag{6.23}$$

such that $\tilde{y}(T,\cdot) = 0$ in Ω. Furthermore, we established the following higher regularity for v:

$$e^{Ks_1\alpha^*}v \in L^2((0,T);H^{p+2}(\Omega)\cap H_0^1(\Omega))^m \cap H^1((0,T);L^2(\Omega))^m, \tag{6.24}$$

for all $k\in(0,1)$. Notice that (6.24) implies that v is exponentially decaying as $t\to 0$ and $t\to T$. For the linear partial differential operator $\mathcal{B}$ (of order zero in time and at most $p+2$ in space) constructed implicitly in Proposition 4.4.6, let us define

$$\begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix} := \mathcal{B}\left(\theta v\right),$$

which is well-defined by (6.24). By virtue of $\mathcal{B}$ being a linear partial differential operator of the stated orders with constant coefficients, we conclude that

$$(\hat{y},\hat{u}) \in L^2(q_T)\times L^2(q_T)^c; \tag{6.25}$$

we then extend $(\hat{y},\hat{u})$ by zero to Q_T. Since v decays exponentially as $t\to 0$ and $t\to T$, $\hat{y}(0,\cdot) = \hat{y}(T,\cdot) = 0$ in Ω. Furthermore, it follows from the discussions in

Chapter 4 that $(\hat{y}, \hat{u})$ is the solution to

$$\begin{cases} \partial_t \hat{y} = \operatorname{div}(D\nabla \hat{y}) + G \cdot \nabla \hat{y} + A\hat{y} + B\hat{u} + \theta v, & \text{in } Q_T, \\ \hat{y} = 0, & \text{on } \Sigma_T, \\ \hat{y}(0, \cdot) = \hat{y}(T, \cdot) = 0, & \text{in } \Omega, \end{cases} \tag{6.26}$$

where, by (6.25) and by the parabolic regularity discussed in Section 2.3, $(\hat{y}, \hat{u})$ satisfies Definition 2.3.1. Defining $(y, u) := (\tilde{y} - \hat{y}, -\hat{u})$, it is immediate that (y, u) is the solution to (3.1) with $y(T, \cdot) = 0$ in Ω. This finishes the proof of Theorem 3.1.5.

Chapter 7

Conclusions and future work

7.1 Summary

In this work, we derived a sufficient condition for the null controllability of a system of coupled parabolic PDEs, where the couplings were constant in space and time and of first and zero-order, when more than half of the equations in the system were actuated. This controllability condition is generic for the case of $c \geq h$, where c denotes the number of controls and for h defined in Theorem 3.1.5. Furthermore, we demonstrated that for $c < h$, the possibly non-generic nature of this controllability condition is purely technical and is an artifact of our treatment in Section 4.4. In the process of deriving our main result, we used a powerful technique, the so-called fictitious control method, which allowed us to pose our controllability problem as two interconnected problems.

7.2 Future work

Here are two directions of research which build upon the work in this thesis:

(i) in light of the shortcomings of Theorem 3.1.5 for $c < h$, one may wish to explore other decomposition methods than the one we relied on in Theorem 2.5.6 to construct a right inverse to the differential operator $\mathcal{L}$ defined in (4.18). Furthermore, since the controllability condition stated in Theorem 3.1.5 are only sufficient, a different sparse matrix decomposition and a different choice of differential operator $\mathcal{N}$ may yield a milder controllability condition that closes the gap between sufficiency and necessity (as in Lemma 4.4.2 for one underactuation), and;

(ii) to our knowledge, the fictitious control method has yet to be applied to problems of stabilizability of control systems at the time of this writing. One could possibly extend this treatment to stabilizability of control systems by means of internal controls.

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Chapter 8

Appendix

8.1 Omitted proofs

Proof of Lemma 5.2.1. Consider a function $\theta \in C^r(\mathbb{R}^n)$ such that $\Omega = \{x \in \mathbb{R}^n : \theta(x) > 0\}$ and $\|\nabla\theta(x)\|_1 > 0$ for all $x \in \partial\Omega$. In can be deduced from [5, Theorem 5.31] that there exists a sequence of Morse functions $(\theta_k)_{k\in\mathbb{N}^*}$ such that

$$\theta_k \to \theta \quad \text{in } C^r(\bar{\Omega}) \text{ as } k \to \infty. \tag{8.1}$$

We define the set of critical points of θ by $\mathcal{C} := \{x \in \mathbb{R}^n : \nabla\theta(x) = 0\}$. Since $\|\nabla\theta\|_1 > 0$ on $\partial\Omega$, there exists an open set $\Theta \subset \mathbb{R}^n$ such that

$$\bar{\Theta} \cap \mathcal{C} = \emptyset, \quad \text{where} \quad \partial\Omega \subset \Theta, \tag{8.2}$$

that is, the critical points of θ are not limit points in Θ. Let $e \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(e) = \Theta$ and $e = 1$ on $\partial\Omega$. We define a function $g_k(x) := \theta_k + e(\theta - \theta_k)$. By the construction of θ and e, it follows that

$$g_k = 0 \quad \text{on} \quad \partial\Omega. \tag{8.3}$$

Indeed, by construction of θ, $\theta = 0$ on $\partial\Omega$. Furthermore, by construction on e,

$$\nabla g_k = \begin{cases} \nabla\theta_k & \text{in} \quad \overline{\Omega \setminus \Theta}, \\ \nabla\theta_k + e(\nabla\theta - \nabla\theta_k) + \nabla e(\theta - \theta_k) & \text{in} \quad \Omega \cap \Theta. \end{cases} \tag{8.4}$$

By (8.4) and then (8.1), for all $\epsilon > 0$, there exists an integer $k_0 := k_0(\epsilon)$ such that

$$\begin{aligned} |\nabla g_k\|_1 &\geq \|\nabla\theta_k\|_1 - \|-e(\nabla\theta - \nabla\theta_k) - \nabla e(\theta - \theta_k)\|_1 \\ &\geq \|\nabla\theta_k\|_1 - \|e(\nabla\theta - \nabla\theta_k)\|_1 - \|\nabla e(\theta - \theta_k)\|_1 \\ &\geq \|\nabla\theta_k\|_1 - \sup_{x\in\Theta}\{e\} \|\nabla\theta - \nabla\theta_k\|_1 - \sup_{x\in\Theta}\{\nabla e\} |\theta - \theta_k| \\ &\geq \|\nabla\theta_k\|_1 - \|e\|_{C^1(\Theta)} (\|\nabla\theta - \nabla\theta_k\|_1 + |\theta - \theta_k|) \\ &\geq \|\nabla\theta_k\|_1 - \epsilon, \end{aligned}$$

for $k > k_0$ and for $x \in \Omega \cap \Theta$. It follows from (8.1), (8.2) and the above inequality that there exists $\epsilon > 0$ and a $\hat{k}$ sufficiently large such that

$$\|\nabla g_{\hat{k}}\|_1 > 0 \quad \text{in} \quad \Omega \cap \Theta.$$

Hence, defining $g := g_{\hat{k}}$, we have shown the existence of a Morse function $g \in C^r(\bar{\Omega})$ such that $g > 0$ in Ω, $g = 0$ on $\partial\Omega$, $\|\nabla g\|_1 > 0$ in $\partial\Omega$. Furthermore, since $\bar{\Omega}$ is compact, the set of critical points of g is finite (indeed, non-degenerate critical points are isolated points, so must be finite in a compact set; see [5, Corollary 5.25] for the details).

For $k \in \mathbb{N}$, let a_i denote the i^{th} critical point of g, for $i \in \{1, \dots, k\}$. Let $\gamma_i \in C^\infty([0,1]; \Omega)$ be such that

$$\begin{gathered} \gamma_i \text{ is one to one for every } i \in \{1, \dots, k\}; \\ \gamma_i([0,1]) \cap \gamma_j([0,1]) = \emptyset \; \forall \; (i,j) \in \{1, \dots, k\}^2 \text{ such that } i \neq j; \\ \gamma_i(0) = a_i, \; \forall \, i \in \{1, \dots, k\}, \text{ and;} \\ \gamma_i(1) \in \omega_2, \; \forall \, i \in \{1, \dots, k\}. \end{gathered} \tag{8.5}$$

The existence of such γ_i's follows from the connectedness of Ω: for $n > 2$, one uses a transversality argument (two intersecting embedded curves can be perturbed so that they no longer intersect); for $n = 2$, one proceeds by induction on k and by noticing that for Γ a set of k of disjoint embedded paths in Ω, $\Omega \setminus \Gamma$ is still connected.

Consider a C^∞ vector field $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\overline{\{x \in \mathbb{R}^n : X(x) \neq 0\}} \subset \Omega, \text{ and;} \tag{8.6}$$

$$X(\gamma_i(t)) = \frac{d\gamma_i(t)}{dt}, \; \forall \, i \in \{1, \dots, k\}. \tag{8.7}$$

Let Φ be the flow associated to X, i.e., $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\frac{d\Phi}{dt} = X(\Phi), \quad \Phi(0,x) = x, \; \forall \, x \in \mathbb{R}^n. \tag{8.8}$$

From (8.7) and (8.8), we have

$$\Phi(t, a_i) = \gamma_i(t),\ \forall\, t \in [0, 1],$$

which follows from uniqueness of solutions to equations of type (8.8). Hence, by (8.5), it follows that

$$\Phi(1, a_i) \in \omega_2. \tag{8.9}$$

Note that for every $\tau \in \mathbb{R}$, $\Phi(\tau, \cdot)$ is a diffeomorphism (with the inverse $\Phi(\tau, \cdot)^{-1} = \Phi(-\tau, \cdot)$). Hence, by (8.6), for every $\tau \in \mathbb{R}$, $\Phi(\tau, \Omega) = \Omega$. Indeed, since Φ^{-1} is continuous, we have $\Phi(\tau, \Omega) = \omega$ for some $\omega \subset \Omega$ open. Suppose $\omega \neq \Omega$; then, $\Phi(-\tau, \omega) = \Omega$, which yields a contradiction since Φ is a diffeomorphism. Furthermore, by (8.6) and (8.8), Φ is the identity map in a neighbourhood of $\partial\Omega$. We define

$$\eta^0(x) := g(\Phi(-1, x)),\ \forall\, x \in \bar{\Omega},$$

which satisfies the desired properties. Indeed, $\eta^0 \in C^r(\bar{\Omega})$ and η^0 inherits from g the properties that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$ and $\|\nabla\eta^0\|_1 > 0$ on $\partial\Omega$. Lastly, the critical points of η^0 in Ω are located where $\Phi(-1, x) = a_i =$ for $i \in \{1, \ldots, k\}$, and hence by (8.9), they occur for

$$\begin{aligned} x &= \Phi(0, x) \\ &= \Phi(1, \Phi(-1, x)) \\ &= \Phi(1, a_i) \in \omega_2, \end{aligned}$$

thus $\|\nabla\eta^0(x)\|_1 > 0$ for $x \in \Omega \setminus \omega_2$. $\square$

Proof of Lemma 5.2.5. We denote by C various positive constants depending on Ω, ω_{p+2} and r. From now on, let $\mathbf{n}$ denote the outward-pointing normal associated to $\partial\Omega$. Since $C^m(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$ (see, for example, [17, Theorem 3, Subection 5.3.3]), we can take $u \in C^0([0,T];C^1(\bar{\Omega})$. Consider the integral

$$\int_\Omega e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\nabla u\right)udxdt. \tag{8.10}$$

Using (5.5), we calculate

$$\begin{aligned}
&\nabla\cdot\left(e^{-2s\alpha}\xi^{r+1}\nabla\eta^0 u\right)\\
&= 2s\lambda e^{-2s\alpha}\xi^{r+2}\left\|\nabla\eta^0\right\|_1^2 u + e^{-2s\alpha}\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)u + e^{-2s\alpha}\xi^{r+1}\nabla u\cdot\nabla\eta^0.
\end{aligned}$$

Hence, integrating (8.10) by parts over Ω yields

$$\begin{aligned}
\int_\Omega e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\nabla u\right)udxdt = &\int_{\partial\Omega} e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\mathbf{n}\right)|u|^2 d\sigma dt\\
&- 2s\lambda\int_\Omega e^{-2s\alpha}\xi^{r+2}\left\|\nabla\eta^0\right\|_1^2|u|^2dxdt\\
&- \int_\Omega e^{-2s\alpha}\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)|u|^2dxdt\\
&- \int_\Omega e^{-2s\alpha}\xi^{r+1}\left(\nabla u\cdot\nabla\eta^0\right)udxdt.
\end{aligned}$$

Multiplying by $s\lambda$ and integrating over $t \in (0,T)$, one gets

$$\begin{aligned} s\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\nabla u\right) u dxdt &= \frac{s\lambda}{2}\iint_{\Sigma_T} e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\mathbf{n}\right)|u|^2 d\sigma dt \\ &- s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}\left\|\nabla\eta^0\right\|_1^2|u|^2 dxdt \\ &- \frac{s\lambda}{2}\iint_{Q_T} e^{-2s\alpha}\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)|u|^2 dxdt. \end{aligned}$$

By (5.4), we have that $(\nabla\eta^0\cdot\mathbf{n}) \leq 0$ which gives

$$\begin{aligned} s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}\left\|\nabla\eta^0\right\|_1^2|u|^2 dxdt &\leq -s\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{r+1}\left(\nabla\eta^0\cdot\nabla u\right) u dxdt \\ &- \frac{s\lambda}{2}\iint_{Q_T} e^{-2s\alpha}\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)|u|^2 dxdt. \end{aligned}$$

One can make an even coarser estimate:

$$\begin{aligned} s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}\left\|\nabla\eta^0\right\|_1^2|u|^2 dxdt &\leq s\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{r+1}\left\|\nabla\eta^0\right\|_1\left\|\nabla u\right\|_1|u| dxdt \\ &+ \frac{s\lambda}{2}\iint_{Q_T} e^{-2s\alpha}|\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)||u|^2 dxdt. \end{aligned}$$

Note that for $\lambda \geq C$,

$$|\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)| \leq C\lambda\xi^{r+1}.$$

Indeed, from (5.6) we compute $\nabla\xi^{r+1} = \lambda(r+1)\xi^{r+1}\nabla\eta^0$; hence

$$\begin{aligned} |\nabla\cdot\left(\xi^{r+1}\nabla\eta^0\right)| &= \xi^{r+1}\lambda(r+1)\left|\left\|\nabla\eta^0\right\|_1^2 + \frac{\Delta\eta^0}{\lambda(r+1)}\right| \\ &\leq \xi^{r+1}\lambda(r+1)\left|\left\|\nabla\eta^0\right\|_1^2 + \frac{\Delta\eta^0}{C(r+1)}\right| \\ &\leq C\lambda\xi^{r+1}. \end{aligned}$$

It follows that that

$$s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2} \left\|\nabla\eta^0\right\|_1^2 |u|^2 dxdt \leq s\lambda \iint_{Q_T} e^{-2s\alpha}\xi^{r+1} \left\|\nabla\eta^0\right\|_1 \|\nabla u\|_1 |u| dxdt + \frac{s\lambda^2}{2} \iint_{Q_T} Ce^{-2s\alpha}\xi^{r+1}|u|^2 dxdt. \tag{8.11}$$

We use Cauchy-Schwarz and Young's inequalities with $\epsilon > 0$ on the first term in the righthand of (8.11) along with the fact that η^0 is chosen and C depends on Ω:

$$\begin{aligned} s\lambda &\iint_{Q_T} e^{-2s\alpha}\xi^{r+1} \left\|\nabla\eta^0\right\|_1 \|\nabla u\|_1 |u| dxdt \\ &= \iint_{Q_T} \left(\epsilon^{-1/2} e^{-s\alpha}\xi^{r/2} \|\nabla u\|_1 |\eta^0|\right) \left(\epsilon^{1/2} s\lambda e^{-s\alpha}\xi^{(r+2)/2}|u|\right) dxdt \\ &\leq \int_{(0,T)} \left(\int_\Omega \frac{1}{\epsilon} e^{-2s\alpha}\xi^r \|\nabla u\|_1^2 |\eta^0|^2 dx\right)^{1/2} \left(\epsilon s^2\lambda^2 \int_\Omega e^{-2s\alpha}\xi^{r+2}|u|^2 dx\right)^{1/2} dt \\ &\leq \frac{C}{\epsilon} \iint_{Q_T} e^{-2s\alpha}\xi^r \|\nabla u\|_1^2 dxdt + \frac{\epsilon s^2\lambda^2}{2} \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt. \end{aligned}$$

For $s \geq \frac{C(T^5+T^{10})}{\epsilon}$, the second term in the righthand side of (8.11) can be upper bounded: by (5.6), (5.13) and since $e^{\lambda(10\|\eta^0\|_\infty + \eta^0(x))} > 1$ for all $x \in \Omega$,

$$\begin{aligned} \frac{s\lambda^2}{2} \iint_{Q_T} Ce^{-2s\alpha}\xi^{r+1}|u|^2 dxdt &\leq \frac{\epsilon s^2\lambda^2}{2} \iint_{Q_T} \frac{1}{(T^5+T^{10})} e^{-2s\alpha}\xi^{r+1}|u|^2 dxdt \\ &\leq \frac{\epsilon s^2\lambda^2}{2} \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt. \end{aligned} \tag{8.12}$$

Hence, a coarser bound for (8.11) is

$$s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2} \left\|\nabla\eta^0\right\|_1^2 |u|^2 dxdt \leq \frac{C}{\epsilon} \iint_{Q_T} e^{-2s\alpha}\xi^r \left\|\nabla u\right\|_1^2 dxdt + \epsilon s^2\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt. \tag{8.13}$$

We multiply (8.13) by $s^r\lambda^{r+1}$ and take ϵ small enough such that the rightmost term becomes dominated by $Cs^2\lambda^2 \iint_{\omega_{p+2}} e^{-2s\alpha}\xi^{r+2}|u|^2 dxdt$ (recall that C depends on Ω and ω_{p+2}). This proves the desired inequality for $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$. □