Dissipative Dynamics of Atomic Bose-Einstein Condensates at Zero Temperature

by

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Abstract

In this thesis we study various dissipative processes that are associated with the flow of an atomic Bose-Einstein condensate at zero temperature. In particular, we investigate the effect of a weak correlated disorder potential on the collective dipole motion of a harmonically-confined elongated condensate. By using an extension of the Harmonic Potential Theorem, we demonstrate that the dynamics of the system can be described equivalently in terms of a disorder potential oscillating relative to a stationary condensate. This latter point of view allows the application of linear response theory to determine the drag force experienced by the condensate and to evaluate the damping rate of the centre of mass oscillation. The density response function for the elongated condensate is determined with a new local density approximation that takes into account the tight radial confinement of the atomic cloud. Our linear response theory reveals the detailed dependence of the damping rate on various system parameters. A comparison with available experimental data is only partially successful and points to the need for additional experiments. In addition to disorder induced dissipation, we also consider a variety of other problems that can be addressed by means of linear response theory. For example, we study momentum transferred to a condensate by a Bragg pulse and the energy absorption of a gas in an
optical lattice that is parametrically modulated in different ways. All of these applications demonstrate the utility of linear response theory in describing the dynamics of Bose-condensed systems which are subjected to weak perturbations.
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Statement of Originality

I hereby certify that all of the work described within this thesis is the original work of the author. Any published (or unpublished) ideas and/or techniques from the work of others are fully acknowledged in accordance with the standard referencing practices.
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Chapter 1

Introduction

1.1 Historical overview

This thesis is devoted to the study of certain aspects of dilute atomic Bose-Einstein condensates in various trapping geometries. These physical systems were first realized in the laboratory in 1995 [1, 2] and have subsequently become active areas of experimental and theoretical research. This research is now multi-faceted and has evolved in directions which were not originally anticipated when the seminal discovery of Bose-Einstein condensation [1, 2] was made.

The phenomenon of Bose-Einstein condensation (BEC) was first predicted by Einstein in 1924 for an ideal gas of Bosons in thermal equilibrium. By extending the analysis of Bose for photons, Einstein argued that a gas of material particles would undergo a phase transition at a critical temperature below which a macroscopic fraction of the particles “condense” into the lowest quantum state. This prediction is based on the properties of the Bose-Einstein distribution which governs the occupancy of the available single particle quantum states. As the temperature is lowered, the occupancy of each quantum state decreases and eventually a temperature is reached...
below which the excited states can no longer accommodate all of the particles. At this point, a finite fraction of the particles is “forced” to occupy the single-particle ground state. Qualitatively, condensation occurs when the thermal de Broglie wavelength of the particles is comparable to the mean inter-particle separation. At this point, the quantum nature of the particles becomes manifest.

Einstein’s work did not receive much attention for over a decade after its publication for several reasons. At that time, the notion of quantum statistics was not well-understood, and even the distinction between bosons and fermions as fundamental particles was not appreciated. Furthermore, the concept of an ideal gas was far removed from reality. Almost all substances in Nature solidify due to the interactions between the constituent particles when cooled down to very low temperatures. The one exception to this rule is helium (including both its $^3$He and $^4$He isotopes) which remains a liquid down to absolute zero under its own vapour pressure. But even here, the liquid is relatively dense and the interaction between the atoms is important.

Following the discovery that liquid $^4$He exhibits a superfluid phase when cooled below 2.2 K, F. London [3] suggested that superfluidity might be a manifestation of BEC. In some sense, this was a bold idea since, as already stated, liquid $^4$He is far from an ideal gas. His suggestion was no doubt based on the observation that the BEC critical temperature for an ideal gas of $^4$He atoms at the density of the liquid is close to the normal fluid-superfluid transition temperature for liquid $^4$He. But perhaps more importantly, London saw that a macroscopic occupation of the zero-momentum state might account for a coherent component of the liquid that could flow without dissipation.
This idea of a superfluid component was taken further by Tisza [4] and, in particular, Landau [35], who developed a successful two-fluid model that could account for many of the properties of superfluid $^4$He. Interestingly, in doing so, Landau did not actually invoke the phenomenon of Bose-Einstein condensation. This connection was only established later when the momentum distribution of the helium atoms in the superfluid phase was measured [5]. These experiments convincingly showed that BEC does accompany the normal fluid to superfluid transition, although only about 10% of the atoms (in contrast to 100% in the case of an ideal gas) “condense” into the zero-momentum state at $T = 0$. The significant reduction of the zero-momentum state occupancy is due to the strong interactions between the helium atoms.

For a long time, superfluid $^4$He was the only physical system known to exhibit BEC and it was natural to ask whether other examples might exist. One possibility was spin-polarized hydrogen [6] which was studied intensively in several laboratories in the 1970s [7, 8]. However, atomic hydrogen is an extremely volatile substance and there were many technical challenges that had to be overcome in order to achieve stable trapped gases of sufficiently low density and temperature. These challenges were not in fact overcome until 1998 [9]. Prior to this, however, was a parallel and independent effort devoted to the study of trapped alkali atoms. Here too there were numerous experimental challenges [10] but these systems ultimately provided the first successful route to BEC in a dilute atomic gas.

One of the challenges is that dilute atomic gases are in fact metastable. The lowest energy configuration of atoms at low temperatures is a solid. Reaching this state from a dilute gas, however, must first be preceded by the formation of molecules
which require three-body collisions. If the gas is extremely dilute, the rate of three-body collisions can be reduced to the point that the gas is sufficiently long-lived. This in turn leads to the next challenge: BEC occurs at such low temperatures that conventional cryogenic techniques cannot be used. Fortunately, the advent of laser cooling and trapping of atoms in the 1980s opened the door to the experimental realization of BEC in atomic gases. A decade of effort and a combination of ingenious cooling techniques culminated in the realization of the first atomic condensates with $^{87}\text{Rb}$ [1] and $^{23}\text{Na}$ [2] atoms. The required temperatures of the atomic condensates is of the order of 100 nK; the mere fact that one can cool a system down to such temperatures is rather astounding.

The atomic Bose-condensates are fascinating physical systems for many reasons. Like superfluid $^4\text{He}$ and superconductors, atomic Bose-condensates are systems that exhibit quantum effects at a macroscopic level. In fact, they have become ideal laboratories in which to explore superfluid collective modes [11], quantized vortices and vortex lattices [12] and matter wave interference [13], to name just a few. Furthermore, because of their dilute and weakly interacting nature, atomic condensates provide a very good platform for the precise test of various microscopic theories and concepts, many of which were originally developed to provide some understanding of superfluid $^4\text{He}$. Finally, experimentalists have many ways of controlling the properties of cold atomic gases and their environment. For example, Feshbach resonances [14] can be used to tune the interactions between the atoms continuously from attractive to repulsive. On the other other hand, the physics of strongly-correlated systems can be investigated by placing cold atoms in optical lattices [15]. Although many novel physical phenomena have already been studied using cold atom systems, much
remains to be explored.

1.2 Motivation of Thesis

Understanding the role that disorder plays in shaping the equilibrium and transport properties of superfluid systems (charged or neutral) has always been an important topic in condensed matter research [19, 20, 21]. A prime example of a disordered superfluid system is superfluid helium in porous media. Of interest here is the effect that confinement within a porous structure has on various properties of the helium superfluid such as the superfluid phase transition, thermodynamic quantities and the dissipation of superfluid flow [22].

Similar studies of the interplay between disorder and superfluidity can now be carried out using atomic Bose-Einstein condensates. The advantage of these systems, however, is that the disorder environment is highly controllable. Typically, disorder is introduced by means of an optical speckle pattern which consists of a random spatial distribution of light intensity. The dipolar atom-light interaction then leads to a disorder potential acting on the cold atoms [23, 24]. Various disorder-related aspects have been studied in these systems such as the transport of a BEC in a disordered medium [16, 17, 18], fragmentation of a BEC [17], diffusion of an atomic cloud [25] and observation of Anderson localization [26, 27].

One of the first experiments on this topic [16] observed the effect of a weak disorder potential on both the diffusive expansion of an atomic cloud and the collective modes of the condensate. In particular, it was found that even a very weak disorder potential led to an appreciable damping of the centre of mass, or dipole, mode of oscillation. This same damping was investigated in much more detail in later experiments [17, 18].
and the dependence on various system parameters was explored. These experiments and the data they have provided are the motivation for much of the theoretical work carried out in this thesis. Our ultimate objective is to provide a detailed interpretation of these experimental results.

Most of the existing theoretical studies of this problem [28, 29, 30] are based on numerical simulations of the time-dependent Gross-Pitaevskii equation. This approach, however, is not ideal since it is difficult to ensure that a specific realization of the disorder is representative of the statistical properties of the disorder potential. To check this, one would have to at least repeat the simulation several times. As a result, it is difficult and time-consuming to investigate in detail the dependences on various system parameters. Furthermore, numerical simulations might not provide much insight into the underlying physical mechanisms responsible for the numerical data obtained.

In this thesis we propose a semi-analytic approach to study the zero temperature dissipative dynamics of the centre-of-mass motion of a trapped condensate in the presence of a correlated disorder potential. Our general framework is linear response theory, which is valid in the limit of a weak perturbation. As usually formulated, one considers the response of a system which is initially in a state of equilibrium. However, in the experiments of interest, the centre of mass degree of freedom of the condensate is in a highly non-equilibrium state and it is not obvious that a linear response approach is applicable. One of the main contributions of this thesis is to show that the dynamics of the system can be reformulated so that linear response theory can indeed be used.
1.3 Overview of Thesis

We now give an overview of the content of this thesis. In Chapter 2, we provide the theoretical background required to describe trapped atomic gases. In particular, a number-conserving Bogoliubov theory of the excitations of an arbitrary non-homogeneous condensate is developed. The connection of this theory to hydrodynamics is also discussed. Chapter 3 is devoted to the calculation of density response functions of atomic Bose-condensates within the Bogoliubov approximation. Response functions for the homogeneous gas and the uniform cylindrical condensate are determined and their properties are discussed in detail. We also develop a new local density approximation, the cylindrical local density approximation (CLDA), which can be used to determine the density response function of a highly elongated (cigar-shaped) condensate. All of these ingredients are used in subsequent chapters.

Our first application of linear response theory appears in Chapter 4 where we analyze the energy dissipation associated with the flow of a condensate past some external potential. By Galilean invariance, this can be viewed equivalently as the motion of the perturbation (i.e. impurity) through a stationary condensate. The rate of energy dissipation is calculated as a function of the impurity velocity. In this chapter, we also provide the first application of the CLDA to a trapped condensate in the context of Bragg spectroscopy. In these experiments, the condensate is probed by means of a Bragg pulse which imparts momentum to the condensate. This momentum gain is calculated using linear response theory and the results are found to compare favourably with experiment. The main conclusion of these calculations is that the CLDA provides a reliable approximation for the determination of the response properties of an elongated condensate.
1.3. OVERVIEW OF THESIS

The linear response approach developed in Chapter 4 is extended to more general dynamical situations in Chapter 5. The main topic studied there is the centre of mass dynamics of a trapped condensate in the presence of a disorder potential. We show that this dynamics can be analyzed from an alternative point of view in which the disorder potential itself oscillates harmonically with respect to the trapping potential confining the condensate. The advantage of this picture is that conventional linear response theory can again be used. Our approach allows us to conveniently determine the dependence of the damping of the centre of mass motion on various system parameters.

Finally, in Chapter 6 we present some preliminary results on the amplitude and phase modulation of a gas in an optical lattice. We derive general expressions for the energy absorption for these two kinds of modulation using linear response theory. We also critique some previous theoretical work.

All of our results are summarized in a concluding chapter.
Chapter 2

Theoretical background

The main purpose of this chapter is to provide a theoretical background on trapped atomic Bose-Einstein condensates. Our focus will be on a derivation of the essential equations that are used to describe the ground state and low energy excited states of the condensate, starting from certain assumptions. We will also discuss the effect of far-detuned laser radiation on alkali atoms, i.e., the optical dipole potential, and some of its experimental applications. Among these, we discuss the formation of periodic lattice potentials, the realization of disorder potentials and Bragg spectroscopy. There are many other aspects, such as the physics of trapping and cooling, which are needed to understand the creation of a condensate in a cloud of atoms. However, this will not be discussed here as this information is available elsewhere [31]. We only include here a discussion of those topics which are particularly germane to the present thesis.
2.1 Condensate wave function

2.1.1 General description of the atomic system

In many experiments, the atomic cloud is confined within a harmonic potential and consists of bosonic alkali atoms, such as rubidium ($^{87}\text{Rb}$, $^{85}\text{Rb}$), sodium ($^{23}\text{Na}$) or lithium ($^7\text{Li}$). The typical number of atoms in the cloud is of the order of $10^4 - 10^6$, and once Bose-condensed the density at the centre of the condensate is typically $10^{13} - 10^{15}$ cm$^{-3}$. Comparing this to the density of air molecules at room temperature and atmospheric pressure, which is about $10^{19}$ cm$^{-3}$, we can see that the atomic condensate is extremely dilute.

The atoms are prepared in their electronic ground state but can reside in different hyperfine levels. We will restrict ourselves to those cases where all the atoms are in the same hyperfine level of their electronic ground state and thus can be treated as “spinless”. In addition, we will usually be considering three-dimensional systems unless stated otherwise. The Hamiltonian of such a system can be written as

$$\hat{H} = \sum_i \left( \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{r}_i) \right) + \sum_{i<j} v(\hat{r}_i - \hat{r}_j), \quad (2.1)$$

where $m$ denotes the atomic mass and

$$V_{\text{trap}}(\mathbf{r}) = \frac{1}{2} m \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) \quad (2.2)$$

is the harmonic trapping potential. The inter-atomic potential is denoted by $v(\mathbf{r})$. This potential has the conventional form of a strongly repulsive core and an attractive (van der Waals) long-range part. In principle it can be determined theoretically but
is usually determined experimentally by means of scattering experiments. However, for cold atomic gases, a detailed knowledge of the form of the inter-atomic potential is not necessary. This, in fact, is a fortunate circumstance since dealing directly with the potential in a many-body context is extremely difficult. We discuss next how this potential is represented in practice.

### 2.1.2 Atomic pseudopotential

The simplifying characteristic of atomic gases is that they are extremely dilute. For typical condensates, the average density $n$ is so small that the mean atomic separation $n^{-1/3}$ is much larger than the range of the atomic potential. In this dilute limit, the effect of the atomic interactions comes in mainly through binary collisions, which at low energies are characterized by their scattering lengths (for bosons the s-wave scattering dominates at low energies). The equations employed to describe the properties of the ground and low energy excited states of the condensate can be derived in relatively simple terms, if the true atomic potential $v(r - r')$ is replaced by an effective potential, or pseudopotential, of the contact form

$$v_{\text{eff}}(r - r') = g\delta(r - r'),$$

(2.3)

where $g = 4\pi\hbar^2a_s/m$, with $a_s$ being the s-wave scattering length. In fact, the use of this effective potential is the starting point for almost all the theoretical work that has been done in cold atom physics. Its validity has been firmly established by the extraordinary agreement between numerous theoretical predictions and experimental observations. Simplistic derivations of this effective potential are possible based on
the assumption that the original potential possesses a Fourier transform. More importantly, the potential is assumed to be sufficiently weak that the low-energy two-body scattering events can be described in terms of the Born approximation. Neither of these conditions is satisfied by realistic inter-atomic potentials. A more rigorous justification of the use of the pseudopotential can be provided by means of many-body diagrammatic perturbation theory [32]. We shall thus accept Eq. (2.3) as a suitable way of treating the atomic interactions.

2.1.3 Condensate wave function

Central to our discussion is the concept of a condensate wave function, which can be defined most rigorously via the criterion for Bose-Einstein condensation first presented by Penrose and Onsager [33]. This criterion is applicable to interacting systems and is formulated in terms of the one-body density matrix

\[ \rho(r, r') = \langle \hat{\psi}^\dagger(r') \hat{\psi}(r) \rangle. \]  

Here \( \hat{\psi}(r) \) and \( \hat{\psi}^\dagger(r') \) are the standard field operators which satisfy the Bose commutation relations

\[ [\hat{\psi}(r), \hat{\psi}^\dagger(r')] = \delta(r - r'); \quad [\hat{\psi}(r), \hat{\psi}(r')] = 0. \]  

The angular brackets in Eq. (2.4) in general stand for an average with respect to a statistical (thermal) density matrix. In the limit of zero temperature, the average reduces to a ground state expectation value.
From its definition, it is clear that
\[ \rho^*(r, r') = \rho(r', r). \]  
(2.6)

As such, it can be viewed as a Hermitian operator. Its eigenvalues and eigenfunctions are defined according to
\[ \int d'r' \rho(r, r') \chi_i(r') = N_i \chi_i(r). \]  
(2.7)

Since \( \rho \) is Hermitian, the eigenvalues \( N_i \) are real and the eigenfunctions \( \chi_i(r) \) can be taken to be a complete orthonormal set. This completeness property implies that \( \rho(r, r') \) can be written as
\[ \rho(r, r') = \sum_i N_i \chi_i^*(r') \chi_i(r). \]  
(2.8)

The diagonal element of the density matrix gives the particle density
\[ n(r) = \rho(r, r) = \sum_i N_i |\chi_i(r)|^2, \]  
(2.9)

and the total number of particles is then given by
\[ N = \int d'r n(r) = \sum_i N_i. \]  
(2.10)

This suggests that \( N_i \) can be interpreted as the average occupation number of the single particle state \( \chi_i(r) \).
Now, according to Penrose and Onsarge, Bose-Einstein condensation occurs if the largest eigenvalue of $\rho$, denoted by $N_0$, is of order of $N$. In this situation, the ratio $N_0/N$ takes a finite limiting value in the thermodynamic limit.\(^1\) If $N_i/N \to 0$ for all other eigenvalues in this limit, $\chi_0(r)$ plays a special role and is used to define the condensate wave function

$$\Phi_0(r) \equiv \sqrt{N_0} \chi_0(r).$$

(2.11)

The single-particle state $\chi_0(r)$ is often referred to as the condensate mode. Since the average occupation number of the condensate mode is macroscopic (i.e., of order $N$), it dominates the physical properties of the system.

The fraction of particles in the condensate mode $N_0/N$ depends on the temperature, as well as the density of the system and the particle interactions. The atomic condensates we study in this thesis are dilute and weakly interacting systems, and as such, almost all particles are in the condensate mode at zero temperature. This in fact is the most important reason why a quantitative theoretical description of atomic condensates is possible.

### 2.2 Ground state: Gross-Pitaevskii equation

In this section we present a mean-field theory description of the ground state of the trapped atomic condensate. This theory is based on a variational (Hartree-Fock) ansatz for the ground state wave function of the form

$$\Psi_0(r_1, \cdots, r_N) = \prod_{i=1}^{N} \varphi(r_i),$$

(2.12)

\(^1\)For a finite system, one cannot actually take the thermodynamic limit. In this case, we must have $N_0/N \gg N_i/N$ for all $i \neq 0$.\]
2.2. GROUND STATE: GROSS-PITAEVSKII EQUATION

where $\varphi(r)$, to be determined, is a normalized single-particle wave function. This choice of many-body wave function is based on the idea that all the particles in a Bose-condensed system reside in “the lowest quantum state”. This of course cannot be the true ground state for an interacting system. Nevertheless, it provides a remarkably accurate description of the ground state properties of trapped dilute Bose gases.

Using this ansatz and the pseudopotential given in Eq. (2.3), the average energy takes the form

$$ E[\varphi, \varphi^*] \simeq N \int dr \varphi^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \varphi(r) + \frac{1}{2} g N^2 \int dr |\varphi(r)|^4. \quad (2.13) $$

Here and in the following, we make the approximation $\frac{1}{2} g N(N - 1) \approx \frac{1}{2} g N^2$, since the resulting relative error is of order $1/N$. The wave function $\varphi(r)$ is now determined by requiring it to minimize the above energy functional. The variation of (2.13) with respect to $\varphi^*(r)$, subject to the normalization constraint $\int dr |\varphi(r)|^2 = 1$, leads to the equation

$$ \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \varphi(r) + gN|\varphi(r)|^2 \varphi(r) = \mu \varphi(r). \quad (2.14) $$

Here, $\mu$ is a Lagrange multiplier introduced to fix the normalization of $\varphi(r)$. The self-consistent solution of Eq. (2.14) with the smallest value of $\mu$ will be denoted by $\varphi_0(r)$.

Within this variational ansatz, the density of the condensate is given by $n(r) = N|\varphi_0(r)|^2$. Comparing this to Eq. (2.9) and assuming that $N_0 \approx N$, we see that within mean-field theory,

$$ \chi_0(r) \simeq \varphi_0(r). \quad (2.15) $$

In other words, the self-consistent solution of Eq. (2.14) provides an approximation
for the wave function of the condensate mode. It follows that the condensate wave function is then given by

$$\Phi_0(r) = \sqrt{N_0} \chi_0(r) \simeq \sqrt{N} \varphi_0(r). \quad (2.16)$$

In terms of the condensate wave function, Eq. (2.14) can be rewritten as

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r)\right) \Phi_0(r) + g|\Phi_0(r)|^2 \Phi_0(r) = \mu \Phi_0(r), \quad (2.17)$$

where \(\Phi_0(r)\) has the normalization

$$\int dr |\Phi_0(r)|^2 = N. \quad (2.18)$$

Eq. (2.17) is the time-independent Gross-Pitaevskii (GP) equation [34]. Once the condensate wave function is determined from the GP equation, the ground state energy is given by

$$E_{\text{GP}}^0 = \int dr \Phi_0^*(r) \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r)\right) \Phi_0(r) + \frac{1}{2} g \int dr |\Phi_0(r)|^4, \quad (2.19)$$

and the density of the atomic cloud is

$$n(r) = |\Phi_0(r)|^2. \quad (2.20)$$

One can show that \(\mu = \partial E_{\text{GP}}^0 / \partial N\), thus allowing \(\mu\) to be interpreted as the chemical potential of the system.

Due to the non-linearity, the time-independent GP equation cannot be solved
2.2. GROUND STATE: GROSS-PITAEVSKII EQUATION

analytically and in general, numerical methods are required. However, for a large condensate, a very good approximate solution can be obtained by neglecting the kinetic energy term in Eq. (2.17). This so-called Thomas-Fermi approximation [31] yields the density

$$n_{TF}(r) = \frac{1}{g} (\mu_{TF} - V_{\text{trap}}(r)).$$  \hspace{1cm} (2.21)

This expression is valid within a surface defined by $V_{\text{trap}}(r) = \mu_{TF}$; outside this region the density is zero. This behaviour is of course unphysical since the density must go to zero smoothly as the wave function penetrates the classically forbidden region. The error, however, occurs only in a small region near the boundary of the condensate, and does not significantly affect the quantitative predictions regarding the most important properties of the condensate.

The size of the cloud can be measured by the extent of the TF wave function along the three semi-axes. These are given by

$$R_i = \sqrt{\frac{2\mu_{TF}}{m\omega_i^2}}, \quad i = x, y, z$$  \hspace{1cm} (2.22)

which are referred to as the Thomas-Fermi radii of the cloud. The chemical potential $\mu_{TF}$ is determined by substituting the solution $\Phi_{TF}(r) = \sqrt{n_{TF}(r)}$ into the normalization condition Eq. (2.18). One finds that the TF chemical potential is

$$\mu_{TF} = \frac{15^{2/5}}{2} \left( \frac{N a_s}{\bar{a}} \right)^{2/5} \bar{\hbar}\bar{\omega},$$  \hspace{1cm} (2.23)

where $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$ and $\bar{a} = \sqrt{\hbar/m\bar{\omega}}$. This expression for $\mu_{TF}$ shows that $R_i$ scales as $(a_s/\bar{a})^{1/5}$.

It is clear from the expression in Eq. (2.21) that the trap potential plays an
important role in determining the density profile of the condensate. One class of trapping potentials commonly used are those possessing axial symmetry for which \( \omega_x = \omega_y = \omega_\perp \). The ratio between the axial and radial trap frequencies, \( \lambda = \omega_z/\omega_\perp \), is called the aspect ratio; this parameter specifies the anisotropy of the density profile of the condensate. The condensates under consideration in this thesis will have small aspect ratios; the condensate density profile in such cases is elongated along the \( z \) axis and has a cigar-like shape. In the \( \lambda = 0 \) limit, one has a condensate that is trapped only in the radial direction and is uniform along the \( z \) axis. The condensate in this limit will be referred to as a uniform cylindrical condensate.

### 2.3 Collective excitations: Bogoliubov-de Gennes equations

The concept of elementary excitations, first introduced by Landau [35], is the cornerstone for understanding various interacting many-body systems. These are the low-lying excited states whose energies (relative to the ground state) are single-particle like; namely they can be well characterized by a few quantum numbers such as momentum and spin. If these excitations are weakly interacting, they effectively behave as an ideal gas. As a result, these excitations can be superimposed to construct all the low-lying, excited states that contribute to the low temperature properties of the system. Strictly speaking, there are two kinds of elementary excitations, quasi-particles and collective excitations. The former are analogous to the single-particle excitations in a non-interacting system, while the latter are related to the density fluctuations of the system. For a Bose-Einstein condensate, it turns out that there is an intimate connection between these excitations [36] which allows one to use these two terms interchangeably.
2.3. COLLECTIVE EXCITATIONS: BOGOLIUBOV-DE GENNES EQUATIONS

2.3.1 Heisenberg equations of motion and the Bogoliubov transformation

Although Landau postulated the existence of elementary excitations in the form of phonons and rotons in superfluid $^4$He, it was Bogoliubov who first demonstrated that phonon-like excitations naturally arise in a weakly interacting Bose-Einstein condensate [37]. A Bose-condensed ideal gas does not possess phonon-like excitations and as a result, it does not exhibit superfluidity [32]. In fact, this was the basis of objections against London’s conjecture that Bose-Einstein condensation could provide an understanding of superfluid $^4$He [37]. Thus Bogoliubov’s work was a significant step towards the vindication of this important idea.

Bogoliubov considered a uniform weakly interacting Bose gas, but his theory can also be extended to treat non-uniform systems such as trapped gases [31]. In the way that the Bogoliubov theory is usually formulated [31], it does not conserve the total number of particles in the system. Several number-conserving versions of the theory have appeared in the literature [38, 39, 40, 41] to circumvent this apparent shortcoming. Here we present an alternative method of enforcing particle number conservation. In the end, however, these number-conserving approaches lead to results which are in fact consistent with Bogoliubov’s original approach.

Let us start from the second-quantized Hamiltonian for the trapped gas

$$
\hat{H} = \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{r}) \right) \hat{\psi}(\mathbf{r}) + \frac{1}{2} g \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}).
$$

We express the field operator in the following form

$$
\hat{\psi}(\mathbf{r}) = \chi_0(\mathbf{r}) \hat{a}_0 + \delta \hat{\psi}(\mathbf{r}),
$$
where $\chi_0(\mathbf{r})$ will be approximated by $\varphi_0(\mathbf{r})$ and $\delta\hat{\psi}(\mathbf{r}) = \sum_{i>0} \chi_i(\mathbf{r}) \hat{a}_i$. We will refer to $\delta\hat{\psi}(\mathbf{r})$ and $\delta\hat{\psi}^\dagger(\mathbf{r})$ as field fluctuation operators. In view of the orthonormality relations of the wave functions $\chi_i(\mathbf{r})$, Eq. (2.25) leads to the following expression for the total particle number operator $\hat{N}$

$$\hat{N} = \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$$
$$= \hat{N}_0 + \int d\mathbf{r} \delta\hat{\psi}^\dagger(\mathbf{r})\delta\hat{\psi}(\mathbf{r}), \quad (2.26)$$

where $\hat{N}_0 \equiv \hat{a}_0^\dagger\hat{a}_0$ is the number operator for the particles in the condensate mode.

The main premise of the Bogoliubov theory is that, in the ground state $|\Psi_0\rangle$ as well as the low-lying excited states, the number of particles in the condensate mode $\langle \hat{N}_0 \rangle$ is only slightly different from the total number of particles, i.e., $\langle \hat{N}_0 \rangle \simeq N$. Therefore we can use Eq. (2.25) to expand physical operators such as the Hamiltonian and the density operator in terms of the field fluctuation operators whose effects can be treated perturbatively.

Following Bogoliubov [37], we begin by investigating the elementary excitations of the system by means of the Heisenberg equations of motion for the operators $\hat{a}_{0,H}(t)$ and $\delta\hat{\psi}_H(\mathbf{r},t)$. Here and in the following, the subscript ‘$H$’ is used to denote Heisenberg operators defined with respect to the Hamiltonian $\hat{H}$, that is, $\hat{O}_H(t) = e^{i\hat{H}t/\hbar}\hat{O}e^{-i\hat{H}t/\hbar}$. These operators satisfy the Heisenberg equation of motion

$$i\hbar \frac{\partial \hat{O}_H}{\partial t} = [\hat{O}_H, \hat{H}]. \quad (2.27)$$
Substituting Eq. (2.25) into Eq. (2.24), we have

\[ \hat{H} = \hat{a}_0^{\dagger} \hat{a}_0 \int dr \chi_0^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + \frac{1}{2} g \hat{a}_0^{\dagger} \hat{a}_0^{\dagger} \hat{a}_0 \hat{a}_0 \int dr |\chi_0(r)|^4 + f(\hat{\psi}, \hat{\psi}^\dagger), \] (2.28)

where \( f(\hat{\psi}, \hat{\psi}^\dagger) \) contains all the terms involving the field fluctuation operators. Using this Hamiltonian in the Heisenberg equation of motion for \( \hat{a}_{0,H} \), we find that to lowest order in field fluctuation operators

\[ i\hbar \frac{\partial \hat{a}_{0,H}}{\partial t} \simeq \left[ \int dr \chi_0^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + g \hat{N}_{0,H} \int dr |\chi_0(r)|^4 \right] \hat{a}_{0,H} \]

\[ \simeq \left[ \int dr \chi_0^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + g N \int dr |\chi_0(r)|^4 \right] \hat{a}_{0,H} \]

\[ = \mu \hat{a}_{0,H}, \] (2.29)

where Eq. (2.26) is used to eliminate \( \hat{N}_{0,H} \) in the second line and Eq. (2.14) is used to arrive at the final result. The solution of the above equation is simply

\[ \hat{a}_{0,H}(t) \simeq \hat{a}_0 e^{-i\mu t/\hbar}. \] (2.30)

This result implies that the state \( \hat{a}_0 |\Psi_0\rangle \) in which one particle is removed from the condensate is physically indistinguishable from the ground state of the system with \( N-1 \) particles. To see this, we act with \( \hat{a}_{0,H}(t) \) on the ground state \( |\Psi_0\rangle \) of \( \hat{H} \). Using Eq. (2.30), we have

\[ \hat{a}_{0,H}(t) |\Psi_0\rangle = e^{-i\mu t/\hbar} \hat{a}_0 |\Psi_0\rangle. \] (2.31)
Alternatively, using $\hat{a}_0(t) \equiv e^{\hat{H}t/\hbar}\hat{a}_0 e^{-i\hat{H}t/\hbar}$ in Eq. (2.31), we obtain

$$e^{i\hat{H}t/\hbar}\hat{a}_0|\Psi_0\rangle = e^{i(E_0-\mu)t/\hbar}\hat{a}_0|\Psi_0\rangle,$$  \hspace{1cm} (2.32)$$

where $E_0$ is the ground state energy of $|\Psi_0\rangle$. Thus we see that the state $\hat{a}_0|\Psi_0\rangle$ is an eigenstate of Hamiltonian $\hat{H}$ with energy $E_0 - \mu$, implying that it is the ground state of the system with $N-1$ particles.

To find an approximate equation of motion for $\delta\hat{\psi}(r, t)$, we start with the Heisenberg equation of motion for the field operator $\hat{\psi}(r, t)$

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(r, t) = \left( -\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}}(r) \right) \hat{\psi}(r, t) + g\hat{\psi}^\dagger_H(r, t)\hat{\psi}(r, t)\hat{\psi}(r, t).$$  \hspace{1cm} (2.33)$$

Using the decomposition in Eq. (2.25), we find to first order in field fluctuation operators

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(r, t) = \left( -\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}}(r) + gN|\chi_0|^2 \right) \hat{\psi}(r, t) + g\hat{\psi}^\dagger_H(r, t)\hat{\psi}(r, t)\hat{\psi}(r, t).$$  \hspace{1cm} (2.34)$$

Substituting Eq. (2.29) into Eq. (2.34) and using Eq. (2.14) we get

$$i\hbar\frac{\partial}{\partial t}\delta\hat{\psi}(r, t) = \left( -\frac{\hbar^2\nabla^2}{2m} + V_{\text{trap}}(r) + 2gN|\chi_0|^2 \right) \delta\hat{\psi}(r, t) + g\hat{\psi}^\dagger_H(r, t)\hat{\psi}(r, t)\hat{\psi}(r, t).$$  \hspace{1cm} (2.35)$$

where we recall that $\Phi_0(r) = \sqrt{N}\chi_0(r)$. Using Eq. (2.30), the above equation can be
2.3. COLLECTIVE EXCITATIONS: BOGOLIUBOV-DE GENNES EQUATIONS

written as

\[ i\hbar \frac{\partial}{\partial t} \delta \hat{\psi}_K = \mathcal{L} \delta \hat{\psi}_K + g \hat{b}_0 \hat{b}_0 \Phi_0^2 \delta \hat{\psi}_K^\dagger, \]  

(2.36)

where \( \delta \hat{\psi}_K \equiv e^{i\mu t/\hbar} \delta \hat{\psi}_H \), \( \hat{b}_0 \equiv \hat{a}_0 / \sqrt{N} \) and

\[ \mathcal{L} = -(h^2/2m) \nabla^2 + V_{\text{trap}}(r) + 2g|\Phi_0|^2 - \mu. \]  

(2.37)

Since Eq. (2.36) for \( \delta \hat{\psi}_K \) involves \( \delta \hat{\psi}_K^\dagger \), we also need an equation for \( \delta \hat{\psi}_K^\dagger \). This is obtained by simply taking the Hermitian conjugate of Eq. (2.36), and we find

\[ -i\hbar \frac{\partial}{\partial t} \delta \hat{\psi}_K^\dagger = \mathcal{L} \delta \hat{\psi}_K^\dagger + g \hat{b}_0^\dagger \hat{b}_0^\dagger \Phi_0^2 \delta \hat{\psi}_K. \]  

(2.38)

Eqs. (2.36) and (2.38) are the fundamental equations of motion that we will eventually solve.

We point out that \( \delta \hat{\psi}_K = e^{i\mu t/\hbar} \delta \hat{\psi}_H \) is in fact the Heisenberg operator defined with respect to the grand canonical Hamiltonian \( \hat{K} = \hat{H} - \mu \hat{N} \). Such an operator would be defined as

\[ e^{i\hat{K}t/\hbar} \delta \hat{\psi} e^{-i\hat{K}t/\hbar} = e^{i\hat{H}t/\hbar} e^{-i\mu \hat{N}t/\hbar} \delta \hat{\psi} e^{i\mu \hat{N}t/\hbar} e^{-i\hat{H}t/\hbar}. \]  

(2.39)

We now observe that, for any operator \( \hat{A} \) which annihilates a single particle,

\[ e^{-i\mu \hat{N}t/\hbar} \hat{A} e^{i\mu \hat{N}t/\hbar} = \hat{A} e^{i\mu t/\hbar}. \]  

(2.40)
Using this result on Eq. (2.39), we obtain

\[ e^{i\hat{K}t/\hbar} \delta \hat{\psi} e^{-i\hat{K}t/\hbar} = e^{i\mu t/\hbar} \delta \hat{\psi}_H = \delta \hat{\psi}_K. \]

Furthermore, defining \( \hat{a}_{0,K}(t) \equiv e^{i\hat{K}t/\hbar} \hat{a}_0 e^{-i\hat{K}t/\hbar} \), we have

\[ \hat{a}_{0,K}(t) = \hat{a}_{0,H} e^{i\mu t/\hbar} \simeq \hat{a}_0, \tag{2.41} \]

where Eq. (2.30) is used in getting the final result. The operator \( \hat{a}_{0,K}(t) \) is thus strictly time-independent.

Since the number of particles in the condensate mode is macroscopically large in the ground and the low-lying excited states, that is \( \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \simeq N \), one has \( \langle \hat{b}_0^\dagger \hat{b}_0 \rangle \simeq 1 \). In view of the fact that \( [\hat{b}_0, \hat{b}_0^\dagger] = 1/N \), we see that \( \langle \hat{b}_0 \hat{b}_0^\dagger \rangle = \langle \hat{b}_0^\dagger \hat{b}_0 \rangle + 1/N \simeq 1 \). Thus, \( \hat{b}_0 \) and \( \hat{b}_0^\dagger \), and hence \( \hat{a}_0 \) and \( \hat{a}_0^\dagger \), effectively commute. As a result, the field fluctuation operators satisfy the following commutation relations

\[ [\delta \hat{\psi}(r), \delta \hat{\psi}^\dagger(r')] \simeq [\hat{\psi}(r), \hat{\psi}^\dagger(r')] = \delta(r - r'), \quad [\delta \hat{\psi}(r), \delta \hat{\psi}(r')] = 0. \tag{2.42} \]

Our goal now is to solve Eq. (2.36) and Eq. (2.38) with the approximation

\[ \hat{b}_0 \hat{b}_0^\dagger \simeq \hat{b}_0^\dagger \hat{b}_0 = 1. \tag{2.43} \]

We note that this is a weaker approximation than \( \hat{b}_0 = \hat{b}_0^\dagger = 1 \), which is commonly employed [31]. The latter results in a theory in which particle number is not conserved.

As a preliminary to the full solution of Eqs. (2.36) and (2.38), we first solve these equations using the stronger approximation referred to above, namely treating \( \hat{b}_0^\dagger \) and \( \hat{b}_0 \) as c-numbers and replacing them by unity. Although this approximation leads to (2.43), the converse is not true. With this stronger approximation, Eqs. (2.36) and
(2.38) become

\[
\begin{align*}
  i\hbar \frac{\partial}{\partial t} \delta \hat{\psi}_K &= \mathcal{L} \delta \hat{\psi}_K + g \Phi_0^* \delta \hat{\psi}^\dagger_K; \\
- i\hbar \frac{\partial}{\partial t} \delta \hat{\psi}^\dagger_K &= \mathcal{L} \delta \hat{\psi}^\dagger_K + g \Phi_0^2 \delta \hat{\psi}_K.
\end{align*}
\]

(2.44) and (2.45)

Although treating \( \hat{b}^\dagger_0 \) and \( \hat{b}_0 \) as c-numbers leads to a number non-conserving theory, we shall see that a slight modification of the solution to Eqs. (2.44) and (2.45) can provide a solution to Eqs. (2.36) and (2.38), thereby restoring particle number conservation.

The pair of coupled equations (2.44) and (2.45) can be brought into a compact matrix form by introducing the two-component column vector

\[
\hat{\Psi}_K(r, t) = \begin{pmatrix} \delta \hat{\psi}_K(r, t) \\ \delta \hat{\psi}^\dagger_K(r, t) \end{pmatrix}.
\]

(2.46)

Using this notation, Eqs. (2.44) and (2.45) take the form of the following single matrix equation

\[
\begin{align*}
  i\hbar \frac{\partial}{\partial t} \sigma_z \hat{\Psi}_K &= \mathcal{M} \hat{\Psi}_K,
\end{align*}
\]

(2.47)

where

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(2.48)

and

\[
\mathcal{M} = \begin{pmatrix} \mathcal{L} & g \Phi_0^2(r) \\ g \Phi_0^*(r)^2 & \mathcal{L} \end{pmatrix}.
\]

(2.49)
Multiplying both sides of Eq. (2.47) by $\sigma_z$, we get

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}_K = \sigma_z \mathcal{M} \hat{\Psi}_K. \quad (2.50)$$

This operator equation can be solved by first determining the eigenvectors of the matrix differential operator $\sigma_z \mathcal{M}$, namely

$$\sigma_z \mathcal{M} w_i = \epsilon_i w_i. \quad (2.51)$$

The eigenvector $w_i$ is written as

$$w_i = \begin{pmatrix} u_i(r) \\ -v_i(r) \end{pmatrix}, \quad (2.52)$$

where the minus sign is simply a convention. Equation (2.51) is in fact a matrix version of the Bogoliubov-de Gennes equations. More explicitly, in component form it gives

$$\mathcal{L} u_i(r) - g \Phi_0(r)^2 v_i(r) = \epsilon_i u_i(r)$$
$$\mathcal{L} v_i(r) - g \Phi_0^*(r)^2 u_i(r) = -\epsilon_i v_i(r), \quad (2.53)$$

where $u_i(r)$ and $v_i(r)$ are referred to as the Bogoliubov amplitudes. The importance of these equations will become clear in the following discussions.

To facilitate the investigation of the matrix equation (2.51), we define the inner
product of two vectors $a$ and $b$ as

$$a^\dagger b \equiv \sum_{\alpha=1,2} \int dr a^\dagger_\alpha(r) b_\alpha(r),$$

(2.54)

where $\alpha$ labels the components of the vectors. From Eq. (2.51), we have $\epsilon_i w_i^\dagger \sigma_z w_i = w_i^\dagger M w_i$. Since $M$ is a Hermitian matrix, we observe that all the eigenvalues $\epsilon_i$ are real, provided that $w_i^\dagger \sigma_z w_i \neq 0$. For more details, see Ref. [31].

However, since the matrix $\sigma_z M$ appearing in Eq. (2.51) is not Hermitian, its eigenvectors have properties that differ from those of Hermitian matrices. First, the orthogonality condition for its eigenvectors is not $w_i^\dagger w_j = 0$ for $\epsilon_i \neq \epsilon_j$. Using Eq. (2.51) and the fact that $M$ is Hermitian, we can show that

$$(\epsilon_i - \epsilon_j) w_i^\dagger \sigma_z w_j = \left( w_i^\dagger M w_j \right) - \left( w_j^\dagger M w_i \right)^* = 0,$$

(2.55)

where we used the fact that $\epsilon_i$ are real. The orthogonality condition is thus given by

$$w_i^\dagger \sigma_z w_j = 0, \quad \text{if} \quad \epsilon_i \neq \epsilon_j.$$

(2.56)

We next observe that the eigenvalues can be positive, negative or zero. In fact, one can easily check that $w_0 = (\varphi_0, -\varphi_0)^T$ is a solution with eigenvalue $\epsilon_0 = 0$. Furthermore, we observe that if $(u_j, -v_j)^T$ is an eigenvector of Eq. (2.51) with eigenvalue $\epsilon_j$, then $(v_j^*, -u_j^*)^T$ is also an eigenvector with eigenvalue $-\epsilon_j$. Thus for each positive eigenvalue there is a corresponding negative eigenvalue. The eigenvectors corresponding to the positive and negative eigenvalues will be denoted by $w_{j+} = (u_j, -v_j)^T$ and $w_{j-} = (v_j^*, -u_j^*)^T$ respectively. If we choose the normalization of the eigenvector $w_{i+}$
as $w_{i+}^\dagger \sigma_z w_{i+} = 1$, we then have the orthonormality condition

$$w_{i+}^\dagger \sigma_z w_{j+} = \delta_{ij}. \quad (2.57)$$

Similarly, the orthonormality condition for $w_{j-}$ is given by

$$w_{i-}^\dagger \sigma_z w_{j-} = -\delta_{ij}. \quad (2.58)$$

In terms of the $w_{j\pm}$ vectors, Eq. (2.56) of course implies that

$$w_{j+}^\dagger \sigma_z w_{i-} = w_{j-}^\dagger \sigma_z w_{i+} = 0. \quad (2.59)$$

When written out explicitly, either of the orthonormality conditions in Eq. (2.57) or Eq. (2.58) is equivalent to the single equation

$$\int dr [u_i(r)u_j^*(r) - v_i(r)v_j^*(r)] = \delta_{ij}. \quad (2.60)$$

Therefore the BdeG equations (2.53) should be solved with the amplitudes satisfying the normalization condition

$$\int dr (|u_i(r)|^2 - |v_i(r)|^2) = 1. \quad (2.61)$$

To determine the closure relation for the eigenvectors of $\sigma_z M$, we first observe that $(w_1 w_2^\dagger \sigma_z) w_3 = w_1 (w_2^\dagger \sigma_z w_3)$ for any three column vectors $w_1$, $w_2$ and $w_3$. Choosing $w_1 = w_2 = w_{j+}$ and $w_3 = w_{i+}$ and summing over $j$ (with the exclusion of the zero
mode), we find

\[
\left( \sum_j w_j + w_j^\dagger \sigma_z \right) w_{i+} = \sum_j w_j^\dagger \left( w_j^\dagger \sigma_z w_{i+} \right). \tag{2.62}
\]

Using Eq. (2.57), we thus have

\[
\left( \sum_j w_j + w_j^\dagger \sigma_z \right) w_{i+} = w_{i+}. \tag{2.63}
\]

Similarly, using Eq. (2.59) and Eq. (2.58) we have

\[
\left( \sum_j w_j - w_j^\dagger \sigma_z \right) w_{i-} = \sum_j w_j \left( w_j^{\dagger} \sigma_z w_{i-} \right) = 0, \tag{2.64}
\]

\[
\left( \sum_j w_j - w_j^{\dagger} \sigma_z \right) w_{i-} = \sum_j w_j \left( w_j^{\dagger} \sigma_z w_{i-} \right) = -w_{i-}, \tag{2.65}
\]

and

\[
\left( \sum_j w_j - w_j^{\dagger} \sigma_z \right) w_{i+} = \sum_j w_j \left( w_j^{\dagger} \sigma_z w_{i+} \right) = 0. \tag{2.66}
\]

From Eqs. (2.63-2.66) we find that for any vector of the form \( w = \sum_i (c_i w_{i+} + d_i w_{i-}) \)

\[
\left( \sum_j w_j + w_j^\dagger \sigma_z - \sum_j w_j - w_j^\dagger \sigma_z \right) w = w. \tag{2.67}
\]

We thus see that, in the space spanned with the \( w_{j, \pm} \) vectors, the bracketed quantity in Eq. (2.67) acts as a unit operator, that is

\[
\sum_j w_j + w_j^\dagger \sigma_z - \sum_j w_j - w_j^\dagger \sigma_z = \hat{I}. \tag{2.68}
\]
We now have all the ingredients to solve Eq. (2.50). Inserting the unit operator $\hat{I}$ given by Eq. (2.68) into Eq. (2.50) and using Eq. (2.51), we find

\[
\begin{align*}
\frac{i\hbar}{\partial t} \hat{\Psi}_K &= \sigma_z M \hat{I} \hat{\Psi}_K \\
&= \sum_j \epsilon_j \left( w_{j+} w^\dagger_{j+} \sigma_z + w_{j-} w^\dagger_{j-} \sigma_z \right) \hat{\Psi}_K.
\end{align*}
\] (2.69)

Defining

\[
\hat{\alpha}_{i,K} \equiv w^\dagger_{i+} \sigma_z \hat{\Psi}_K
\] (2.70)

and using the orthonormality conditions Eq. (2.57) and Eq. (2.59), we find from Eq. (2.69) that

\[
\frac{i\hbar}{\partial t} \hat{\alpha}_{i,K} = \epsilon_i \hat{\alpha}_{i,K}.
\] (2.71)

Similarly we find

\[
-\frac{i\hbar}{\partial t} \hat{\alpha}^\dagger_{i,K} = \epsilon_i \hat{\alpha}^\dagger_{i,K},
\] (2.72)

where

\[
\hat{\alpha}^\dagger_{i,K} \equiv w^\dagger_{i-} \sigma_z \hat{\Psi}_K = \left( w^\dagger_{i+} \sigma_z \hat{\Psi}_K \right)^\dagger.
\] (2.73)

Thus the coupled equations (2.44) and (2.45) are transformed into a pair of independent equations (2.71) and (2.72), which can be solved immediately. One finds

\[
\hat{\alpha}_{i,K} = \hat{\alpha}_i e^{-i\epsilon_i t/\hbar}, \quad \hat{\alpha}^\dagger_{i,K} = \hat{\alpha}^\dagger_i e^{i\epsilon_i t/\hbar}.
\] (2.74)
By setting $t = 0$ in the above equations and using Eq. (2.70) and Eq. (2.73) we obtain

$$
\hat{\alpha}_i = w_i^\dagger \sigma_z \hat{\Psi}; \quad \hat{\alpha}^\dagger_i = w_i^\dagger - \sigma_z \hat{\Psi},
$$
(2.75)

where

$$
\hat{\Psi} \equiv \hat{\Psi}_K(t = 0) = \begin{pmatrix} \delta \hat{\psi}(r) \\ \delta \hat{\psi}^\dagger(r) \end{pmatrix}.
$$
(2.76)

Evaluating Eqs. (2.75) explicitly, we have

$$
\hat{\alpha}_i = \int dr [u_i^\ast(r) \delta \hat{\psi}(r) + v_i^\ast(r) \delta \hat{\psi}^\dagger(r)] = \int dr [u_i(r) \delta \hat{\psi}(r) + v_i(r) \delta \hat{\psi}^\dagger(r)],
$$

$$
\hat{\alpha}^\dagger_i = \int dr [u_i^\ast(r) \delta \hat{\psi}^\dagger(r) - v_i^\ast(r) \delta \hat{\psi}(r)].
$$
(2.77)

Conversely we can also express the fluctuation operators in terms of $\hat{\alpha}_i$ and $\hat{\alpha}^\dagger_i$.

Using Eqs. (2.75), we find

$$
\sum_i \left( w_{i+} \hat{\alpha}_i - w_{i-} \hat{\alpha}^\dagger_i \right) = \left( \sum_i w_{i+} w_{i+}^\dagger \sigma_z - \sum_i w_{i-} w_{i-}^\dagger \sigma_z \right) \hat{\Psi} = \hat{\Psi},
$$
(2.78)

where we used the completeness relation in Eq. (2.68). More explicitly we have

$$
\delta \hat{\psi}(r) = \sum_i [u_i(r) \hat{\alpha}_i - v_i^\ast(r) \hat{\alpha}^\dagger_i],
$$

$$
\delta \hat{\psi}^\dagger(r) = \sum_i [u_i^\ast(r) \hat{\alpha}^\dagger_i - v_i(r) \hat{\alpha}_i].
$$
(2.79)
Similarly, we can solve Eq. (2.70) and Eq. (2.73) for $\delta \hat{\psi}_K$ and $\delta \hat{\psi}_K^\dagger$ using the completeness relation (2.68). We find

$$\delta \hat{\psi}_K(r, t) = \sum_i [u_i(r) \hat{\alpha}_{i,K} - v_i^*(r) \hat{\alpha}_{i,K}^\dagger],$$

$$\delta \hat{\psi}_K^\dagger(r, t) = \sum_i [u_i^*(r) \hat{\alpha}_{i,K}^\dagger - v_i(r) \hat{\alpha}_{i,K}].$$

(2.80)

Substituting Eqs. (2.74) into Eqs. (2.80), we obtain

$$\delta \hat{\psi}_K(r, t) = \sum_i [u_i(r) \hat{\alpha}_{i} e^{-i\epsilon_i t/\hbar} - v_i^*(r) \hat{\alpha}_{i}^\dagger e^{i\epsilon_i t/\hbar}],$$

(2.81)

$$\delta \hat{\psi}_K^\dagger(r, t) = \sum_i [u_i^*(r) \hat{\alpha}_{i}^\dagger e^{i\epsilon_i t/\hbar} - v_i(r) \hat{\alpha}_{i} e^{-i\epsilon_i t/\hbar}].$$

(2.82)

These are solutions of the coupled equations (2.44) and (2.45). Using the BdeG equations (2.53), it is straightforward to verify that Eqs. (2.81) and (2.82) indeed satisfy Eqs. (2.44) and (2.45).

We now come back to the problem of solving Eqs. (2.36) and (2.38) under the approximation given in Eq. (2.43). The close similarity between Eqs. (2.36) and (2.38) and Eqs. (2.44-2.45) leads us to consider the following trial solution for the former equations

$$\delta \hat{\psi}_K(r, t) = \sum_i [u_i(r) \hat{\beta}_{i,K} - v_i^*(r) \hat{b}_0 \hat{b}_0^\dagger \hat{\beta}_{i,K}];$$

(2.83)

$$\delta \hat{\psi}_K^\dagger(r, t) = \sum_i [u_i^*(r) \hat{\beta}_{i,K}^\dagger - v_i(r) \hat{b}_0^\dagger \hat{b}_0 \hat{\beta}_{i,K}].$$

(2.84)
Inserting the above trial solutions into Eqs. (2.36) and (2.38) and using the approximation given in Eq. (2.43) and the BdeG equations (2.53), we find

\[ i\hbar \frac{\partial}{\partial t} \hat{\beta}_{i,K} = \epsilon_i \hat{\beta}_{i,K}; \quad -i\hbar \frac{\partial}{\partial t} \hat{\beta}_{i,K}^\dagger = \epsilon_i \hat{\beta}_{i,K}^\dagger, \]  

(2.85)

which leads to

\[ \hat{\beta}_{i,K} = \hat{\beta}_i e^{-i\epsilon_i t/\hbar}; \quad \hat{\beta}_{i,K}^\dagger = \hat{\beta}_i^\dagger e^{i\epsilon_i t/\hbar}. \]  

(2.86)

Substituting Eq. (2.86) into Eq. (2.83) and (2.84), we find that the solutions to Eqs. (2.36) and (2.38) are given by

\[ \delta \hat{\psi}_K (r,t) = \sum_i [u_i(r) \hat{\beta}_i e^{-i\epsilon_i t/\hbar} - v_i^*(r) \hat{b}_0 \hat{b}_0^\dagger \hat{\beta}_i^\dagger e^{i\epsilon_i t/\hbar}], \]  

(2.87)

\[ \delta \hat{\psi}_K^\dagger (r,t) = \sum_i [u_i^*(r) \hat{\beta}_i^\dagger e^{i\epsilon_i t/\hbar} - v_i(r) \hat{b}_0 \hat{b}_0^\dagger \hat{\beta}_i e^{-i\epsilon_i t/\hbar}], \]  

(2.88)

Setting \( t = 0 \) in the above equations we have

\[ \delta \hat{\psi}(r) = \sum_i [u_i(r) \hat{\beta}_i - v_i^*(r) \hat{b}_0 \hat{b}_0^\dagger \hat{\beta}_i^\dagger]; \]

\[ \delta \hat{\psi}^\dagger(r) = \sum_i [u_i^*(r) \hat{\beta}_i^\dagger - v_i(r) \hat{b}_0 \hat{b}_0^\dagger \hat{\beta}_i], \]  

(2.89)

which will be referred to as the number-conserving Bogoliubov transformation. Using the orthonormality conditions for the Bogoliubov amplitudes Eq. (2.60) and the
approximation given in Eq. (2.43), we can invert Eq. (2.89) to find

$$\hat{\beta}_i = \int dr [u^*_i(r)\delta\hat{\psi}(r) + v^*_i(r)\hat{b}_0\delta\hat{\psi}^\dagger(r)],$$

$$\hat{\beta}_i^\dagger = \int dr [u_i(r)\delta\hat{\psi}^\dagger(r) + v_i(r)\hat{b}_0\delta\hat{\psi}(r)].$$

(2.90)

These new operators will be referred to as the number-conserving quasi-particle operators. Making use of the commutation relations in Eq. (2.42) and the orthonormal conditions in Eq. (2.60), one can verify that these operators satisfy the usual Bose commutation rules

$$[\hat{\beta}_i, \hat{\beta}_j^\dagger] = \delta_{i,j}; \quad [\hat{\beta}_i, \hat{\beta}_j] = 0.$$  

(2.91)

These commutation relations imply that the new operators can be interpreted as the annihilation and creation operators for certain quasi-particles. In addition, the equations in (2.85) indicate that the quasi-particles carry energy $\epsilon_i$, which are determined by the BdeG equations. The physical meaning of the quasi-particles will become clear when we discuss the system Hamiltonian in the next section.

### 2.3.2 Bogoliubov Hamiltonian and its diagonalization

In order to obtain an approximate Hamiltonian that is consistent with the Heisenberg equations of motion (2.36) and (2.38), we substitute Eq. (2.25) into Eq. (2.24) and retain terms up to second order in field fluctuation operators. We find that the
approximate Hamiltonian takes the form (see Appendix A for more details)

\[
\hat{H}_B = E^{GP}_0 + \int dr \delta \hat{\psi}^\dagger(r) \mathcal{L} \delta \hat{\psi}(r) \\
+ \frac{1}{2} g \hat{b}_0^\dagger \hat{b}_0 \int dr \Phi_0(r)^2 \delta \hat{\psi}^\dagger(r) \delta \hat{\psi}(r) + \frac{1}{2} g \hat{b}_0^\dagger \hat{b}_0 \int dr \Phi_0^*(r)^2 \delta \hat{\psi}(r) \delta \hat{\psi}(r),
\]

(2.92)

where \( E^{GP}_0 \) is the GP ground energy given in Eq. (2.19). This Hamiltonian will be referred to as the number-conserving Bogoliubov Hamiltonian. We now demonstrate that the Bogoliubov transformation in Eq. (2.89) diagonalizes this Hamiltonian. Inserting Eq. (2.89) into Eq. (2.92) and using the approximation given in Eq. (2.43), we find

\[
\hat{H}_B = E^{GP}_0 + \sum_{i,j} \int dr \left[ u^*_i \mathcal{L} u_j \hat{\beta}^\dagger_i \hat{\beta}_j + v_i \mathcal{L} v_j^* \hat{\beta}^\dagger_i \hat{\beta}_j - u^*_i \mathcal{L} v_j^* \hat{b}_0^\dagger \hat{\beta}^\dagger_i \hat{\beta}_j - u^*_i v_j \hat{b}_0^\dagger \hat{\beta}_j - u^*_i \hat{b}_0 \hat{\beta}^\dagger_i \hat{\beta}_j \right] \\
+ \frac{1}{2} g \sum_{i,j} \int dr \Phi_0^2 \left[ u^*_i u_j \hat{b}_0^\dagger \hat{\beta}_j + v_i \mathcal{L} v_j^* \hat{\beta}^\dagger_i \hat{\beta}_j - v_i u_j \hat{b}_0^\dagger \hat{\beta}_j - u^*_i v_j \hat{b}_0 \hat{\beta}^\dagger_i \hat{\beta}_j \right] \\
+ \frac{1}{2} g \sum_{i,j} \int dr \Phi_0^2 \left[ u^*_i u_j \hat{b}_0 \hat{\beta}_j + v_i \mathcal{L} v_j^* \hat{\beta}^\dagger_i \hat{\beta}_j - v_i u_j \hat{b}_0 \hat{\beta}_j - u^*_i v_j \hat{b}_0 \hat{\beta}_j \right] \\
+ \frac{1}{2} g \sum_{i,j} \int dr \Phi_0^2 \left[ u^*_i u_j \hat{b}_0 \hat{\beta}_j + v_i \mathcal{L} v_j^* \hat{\beta}^\dagger_i \hat{\beta}_j - v_i u_j \hat{b}_0 \hat{\beta}_j - u^*_i v_j \hat{b}_0 \hat{\beta}_j \right]
\]

(2.93)
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Reorganizing the various terms, the above equation can be written as

\[ \hat{H}_B = E_{0}^{GP} + \sum_{i,j} \int dr \left[ \left( \begin{array}{c} u_i \mathcal{L} u_j - \frac{1}{2} g \Phi_0(r)^2 u_i^* v_j - \frac{1}{2} g \Phi_0^*(r)^2 v_i^* u_j \end{array} \right) \hat{\beta}_i \hat{\beta}_j \right. \]

\[ + \sum_{i,j} \int dr \left[ \left( \begin{array}{c} v_i \mathcal{L} v_j^* - \frac{1}{2} g \Phi_0(r)^2 v_i u_j^* - \frac{1}{2} g \Phi_0^*(r)^2 u_i v_j^* \end{array} \right) \hat{\beta}_i \hat{\beta}_j \right. \]

\[ + \hat{b}_0 \hat{b}_0 \sum_{i,j} \int dr \left[ \left( \begin{array}{c} -v_i \mathcal{L} u_j + \frac{1}{2} g \Phi_0^*(r)^2 v_i v_j \end{array} \right) \hat{\beta}_i \hat{\beta}_j \right. \]

\[ + \hat{b}_0 \hat{b}_0 \sum_{i,j} \int dr \left[ \left( \begin{array}{c} -u_i \mathcal{L} v_j^* + \frac{1}{2} g \Phi_0^*(r)^2 u_i^* u_j \end{array} \right) \hat{\beta}_i \hat{\beta}_j \right]. \] (2.94)

Using the BdeG equations (2.53), the non-diagonal terms in Eq. (2.94) can be shown to vanish and the above expression is simplified to

\[ \hat{H}_B = E_{0}^{GP} + \frac{1}{2} \sum_{i,j} (\epsilon_i + \epsilon_j) \int dr u_i^* u_j \hat{\beta}_i^\dagger \hat{\beta}_j \]

\[ + \frac{1}{2} \sum_{i,j} (\epsilon_i + \epsilon_j) \int dr v_i^* v_j \hat{\beta}_i^\dagger \hat{\beta}_j \]

\[ = E_0 + \frac{1}{2} \sum_{i,j} (\epsilon_i + \epsilon_j) \int dr (u_i^* u_j - v_i^* v_j) \hat{\beta}_i^\dagger \hat{\beta}_j \]

\[ = E_0 + \sum_i \epsilon_i \hat{\beta}_i^\dagger \hat{\beta}_i, \] (2.95)

where \( E_0 = E_{0}^{GP} - \sum_i \epsilon_i \int dr |v_i(r)|^2 \) and Eq. (2.60) is used in arriving at the final result.

The above diagonalized Hamiltonian implies that within the Bogoliubov theory, the ground state energy of the system is given by \( E_0 \). To see this, we observe that the expectation value of \( \hat{H}_B \) for any state is given by \( \langle \hat{H}_B \rangle = E_0 + \sum_i \epsilon_i \langle \hat{\beta}_i^\dagger \hat{\beta}_i \rangle \). Since both \( \epsilon_i \) and \( \langle \hat{\beta}_i^\dagger \hat{\beta}_i \rangle \) are non-negative, the lowest energy of the system is achieved in a state for which \( \langle \Psi_0 | \hat{\beta}_i^\dagger \hat{\beta}_i | \Psi_0 \rangle = 0 \). This relation is satisfied if \( |\Psi_0 \rangle \) is taken to be the quasi-particle vacuum, that is, \( \hat{\beta}_i |\Psi_0 \rangle = 0 \). Furthermore, we can use the quasi-particle
operators to construct all the low-lying eigenstates of the system. For example, let us consider a many-body state such as $|\Psi_i\rangle \equiv \hat{b}_0\hat{\beta}_i^\dagger |\Psi_0\rangle$. Since $\hat{b}_0^\dagger \hat{b}_0 = 1$ and $\hat{\beta}_j |\Psi_0\rangle = 0$, the state $|\Psi_i\rangle$ is found to be normalized. In fact it is an eigenstate of the system with energy $E_i = E_0 + \epsilon_i$. This can be seen directly from

$$
\hat{H}_B |\Psi_i\rangle = \left( E_0 + \sum_j \epsilon_j \hat{\beta}_j^\dagger \hat{\beta}_j \right) |\Psi_i\rangle = E_i |\Psi_i\rangle,
$$

where we used the commutation relation of the quasi-particle operators and the fact that $\hat{\beta}_j |\Psi_0\rangle = 0$. By its definition, $|\Psi_i\rangle$ only contains one quasi-particle. The eigenstates that contain more than one quasi-particle can be constructed by the repeated application of the creation operator $\hat{\beta}_j^\dagger$.

### 2.3.3 Bogoliubov excitations and density fluctuations

In this section we will demonstrate that the low-lying elementary excitations of the Bose-Einstein condensate are related to the density fluctuations of the atomic cloud. This important relationship underlies the fact that the Bogoliubov frequencies and amplitudes of the low-lying modes can be probed experimentally by measuring the density fluctuations of the atomic cloud.

To establish this relationship, we first express the density operator in terms of the Bogoliubov quasi-particle operators. This will allow us to find the time-dependent density for the system in a non-equilibrium state. The expansion in Eq. (2.25) can be rewritten $\hat{\psi}(\mathbf{r}) = \Phi_0(\mathbf{r}) \hat{b}_0 + \delta \hat{\psi}(\mathbf{r})$. Substituting this expression into $\hat{n}(\mathbf{r}) = \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r})$,}

```latex
\begin{align*}
2.3. \text{ COLLECTIVE EXCITATIONS: BOGOLIUBOV-DE GENNES EQUATIONS} & 37 \\
\end{align*}
```
we find that to the first order in fluctuation operators

\[ \hat{n}(r) = n_0(r) + \Phi_0(r)\hat{b}_0^\dagger\hat{\psi}(r) + \Phi_0(r)\hat{b}_0\hat{\psi}^\dagger(r), \]  

(2.97)

where \( \Phi_0(r) \) is assumed to be real and \( n_0(r) = \Phi_0^2(r) \) is the density of the condensate in the ground state. The Heisenberg density operator is then given by

\[ \hat{n}_H(r,t) = n_0(r) + \Phi_0(r)\hat{b}_0^\dagger_H\hat{\psi}_H(r,t) + \Phi_0(r)\hat{b}_0\hat{\psi}_H^\dagger(r,t) \]

\[ = n_0(r) + \Phi_0(r)\hat{b}_0^\dagger\hat{\psi}_K(r,t) + \Phi_0(r)\hat{b}_0\hat{\psi}_K^\dagger(r,t), \]  

(2.98)

where Eq. (2.39) and Eq. (2.41) are used. Inserting Eq. (2.87) and Eq. (2.88) into the above equation, we find

\[ \hat{n}_H(r,t) = n_0(r) + \sum_i [\Phi_0(r)\psi_i^{-}(r)e^{-i\epsilon_i t/\hbar}\hat{b}_0^\dagger\hat{\beta}_i + \text{h.c.}], \]  

(2.99)

where \( \psi_i^{-}(r) \equiv u_i(r) - v_i(r) \).

Now we suppose that the condensate is in a normalized initial state of the form

\[ |\Psi\rangle = c_0|\Psi_0\rangle + c_j|\Psi_j\rangle, \]  

(2.100)

where \( |\Psi_j\rangle = \hat{b}_0\hat{\beta}_j^\dagger|\Psi_0\rangle \) is an excited state containing one Bogoliubov quasi-particle. Clearly this initial state is a non-equilibrium state in which the density deviates from its value in the ground state. Using Eqs. (2.99) and (2.100) we find that the subsequent temporal evolution of the density is given by

\[ \langle \Psi|\hat{n}_H(r,t)|\Psi\rangle = n_0(r) + \delta n_j(r,t), \]  

(2.101)
where
\[
\delta n_j(r, t) = c_0^* c_j \delta n_j(r) e^{-i\epsilon_j t/\hbar} + \text{c.c.} \quad (2.102)
\]
with
\[
\delta n_j(r) = \Phi_0(r) \psi_j^-(r). \quad (2.103)
\]
This shows that the density fluctuation of the condensate in this particular dynamic state oscillates with a frequency \( \omega_j = \epsilon_j/\hbar \) and with an amplitude that is proportional to \( \delta n_j(r) \). Using Eq. (2.99) and the fact that \( \hat{n}(r) = \hat{n}_H(r, t = 0) \), it is easy to show that
\[
\delta n_j(r) = \langle \Psi_0 | \hat{n}(r) | \Psi_j \rangle. \quad (2.104)
\]

The foregoing analysis implies that the energies of the excitations and the Bogoliubov amplitudes can be measured experimentally by exciting the condensate in definite ways and recording its density oscillations. Indeed many low-lying excitations have been measured in this way and they are found to be in good agreement with theoretical predictions based on Bogoliubov theory [11].

### 2.4 Collective excitations: Hydrodynamic theory

We have seen in the previous section that there is an intimate connection between the Bogoliubov excitations and the density fluctuations in non-equilibrium states of the system. Indeed, we shall see that the energies of the Bogoliubov quasi-particles and the associated Bogoliubov amplitudes can be found alternatively by means of the hydrodynamic equations, which describe the temporal evolution of the density fluctuation \( \delta n(r, t) \) and the velocity field \( \mathbf{v}(r, t) \) of the condensate. Quite generally,
the density fluctuation \( \delta n(\mathbf{r}, t) \) and the velocity field \( \mathbf{v}(\mathbf{r}, t) \) can be expressed in terms of the expectation values of the Heisenberg density operator \( \hat{n}_H(\mathbf{r}, t) \) and the current density operator \( \hat{j}_H(\mathbf{r}, t) \), namely\(^2\)

\[
\delta n(\mathbf{r}, t) = \langle \hat{n}_H(\mathbf{r}, t) \rangle - n_0(\mathbf{r}); \tag{2.105}
\]

\[
\mathbf{v}(\mathbf{r}, t) = \frac{1}{n_0(\mathbf{r})} \langle \hat{j}_H(\mathbf{r}, t) \rangle, \tag{2.106}
\]

where the average is taken with respect to the initial many-body state of the system \( |\Psi\rangle \). As shown earlier, within Bogoliubov theory \( \hat{n}_H(\mathbf{r}, t) \) is given by Eq. (2.98). To find the current density operator within the same approximation, we write \( \hat{j}(\mathbf{r}) \) in the second-quantization form

\[
\hat{j}(\mathbf{r}) = \frac{\hbar}{2m_i} \left[ \hat{\psi}^\dagger(\mathbf{r}) \nabla \hat{\psi}(\mathbf{r}) - \nabla \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \right]. \tag{2.107}
\]

To first order in field fluctuation operators, we find

\[
\hat{j}(\mathbf{r}) = \frac{\hbar}{2m_i} \left[ \Phi_0 \delta \hat{b}_0 \nabla \hat{\psi} + \nabla \Phi_0 \delta \hat{b}_0 \hat{\psi}^\dagger + \nabla \Phi_0 \delta \hat{b}_0 \hat{\psi} - \Phi_0 \delta \nabla \hat{\psi}^\dagger \right] \\
= \frac{\hbar}{2m_i} n_0(\mathbf{r}) \nabla \left( \frac{1}{\Phi_0(\mathbf{r})} \left[ \delta \hat{b}_0 \hat{\psi}(\mathbf{r}) - \hat{b}_0 \delta \hat{\psi}^\dagger(\mathbf{r}) \right] \right), \tag{2.108}
\]

where \( \Phi_0(\mathbf{r}) \) is again taken to be real. The Heisenberg current density operator is then given by

\[
\hat{j}_H(\mathbf{r}, t) = \frac{\hbar}{2m_i} n_0(\mathbf{r}) \nabla \left( \frac{1}{\Phi_0(\mathbf{r})} \left[ \delta \hat{b}_0 \hat{\psi}_K(\mathbf{r}, t) - \hat{b}_0 \delta \hat{\psi}^\dagger_K(\mathbf{r}, t) \right] \right). \tag{2.109}
\]

\(^2\)Strictly speaking, the velocity field \( \mathbf{v}(\mathbf{r}, t) \) is defined by the relation \( \langle \hat{j}_H(\mathbf{r}, t) \rangle = \langle \hat{n}_H(\mathbf{r}, t) \rangle \mathbf{v}(\mathbf{r}, t) \). However, the definition in Eq. (2.106) amounts to a linearization of this relation and is consistent with the Bogoliubov theory.
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Eq. (2.98) and Eq. (2.109) imply that within Bogoliubov theory, the time-evolution of \( \delta n(\mathbf{r},t) \) and \( \mathbf{v}(\mathbf{r},t) \) are essentially determined by the time-evolution of the field fluctuation operators. In other words, the hydrodynamic equations satisfied by \( \delta n(\mathbf{r},t) \) and \( \mathbf{v}(\mathbf{r},t) \) can be derived by means of the Heisenberg equations of motion for these fluctuation operators given in Eq. (2.36) and Eq. (2.38).

Using Eq. (2.98) and Eq. (2.105) we find

\[
\frac{\partial}{\partial t} \delta n(\mathbf{r},t) = \Phi_0(\mathbf{r}) \left\langle \hat{b}_0^\dagger \frac{\partial}{\partial t} \delta \hat{\psi}_K(\mathbf{r},t) + \hat{b}_0 \frac{\partial}{\partial t} \delta \hat{\psi}_K^\dagger(\mathbf{r},t) \right\rangle.
\]  

(2.110)

Substituting Eqs. (2.36) and (2.38) into the above equation and using the GP equation, we find

\[
\frac{\partial}{\partial t} \delta n(\mathbf{r},t) = -\nabla \cdot \left( n_0 \mathbf{v} \right).
\]  

(2.111)

We recognize that this is simply the continuity equation for the flow of the condensate.

A similar but slightly more involved procedure leads to the second hydrodynamic equation

\[
m \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left[ g \delta n - \frac{\hbar^2}{4m} \frac{\nabla^2 \left( \delta n/\sqrt{n_0} \right)}{\sqrt{n_0}} + \frac{\hbar^2}{4m} \frac{\nabla^2 \sqrt{n_0}}{n_0^{3/2}} \delta n \right].
\]  

(2.112)

Particular solutions of these hydrodynamic equations are of course specified by the initial values of the density fluctuation and the velocity field, which, according to the definitions in Eq. (2.105) and Eq. (2.106), are determined by the initial many-body state of the system. In other words, every solution to the hydrodynamic equations (2.111) and (2.112) is associated with a certain many-body state of the system. In fact, as previously shown, if the initial state of the system is that given by Eq. (2.100), i.e., \( |\Psi\rangle = c_0 |\Psi_0\rangle + c_j |\Psi_j\rangle \), the time-dependent density fluctuation is
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given by Eq. (2.102), namely,

\[ \delta n_j(\mathbf{r}, t) = c_0^* c_j \delta n_j(\mathbf{r}) e^{-i\epsilon_j t/\hbar} + \text{c.c.}, \]  

(2.113)

where we recall that

\[ \delta n_j(\mathbf{r}) = \Phi_0(\mathbf{r}) \psi_j^- (\mathbf{r}). \]  

(2.114)

Likewise, the velocity field can be obtained from Eq. (2.106) and Eq. (2.109). Substituting Eqs. (2.87) and (2.88) into Eq. (2.109), we find

\[ \hat{j}_H(\mathbf{r}, t) = \frac{\hbar}{2m_i} n_0(\mathbf{r}) \nabla \left( \frac{\psi^+(\mathbf{r})}{\Phi_0(\mathbf{r})} \right) \left[ \hat{b}_0 \hat{\delta}_j e^{-i\epsilon_j t/\hbar} - \text{c.c.} \right], \]  

(2.115)

where \( \psi_j^+(\mathbf{r}) \equiv u_j(\mathbf{r}) + v_j(\mathbf{r}) \). Evaluating the expectation value of the above expression using the state given in Eq. (2.100), we find from Eq. (2.106) that the velocity field is given by

\[ v_j(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \theta_j(\mathbf{r}, t), \]  

(2.116)

where

\[ \theta_j(\mathbf{r}, t) = c_0^* c_j \theta_j(\mathbf{r}) e^{-i\epsilon_j t/\hbar} + \text{c.c.}, \]  

(2.117)

with

\[ \theta_j(\mathbf{r}) = \frac{\psi_j^+(\mathbf{r})}{2i\Phi_0(\mathbf{r})}. \]  

(2.118)

We see from Eq. (2.114) and Eq. (2.118) that \( \delta n_j(\mathbf{r}) \) and \( \theta_j(\mathbf{r}) \) are determined by the Bogoliubov amplitudes. The normalization condition for these Bogoliubov amplitudes
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in Eq. (2.61) can be expressed alternatively as

\[ \frac{1}{2} \int dr \left[ \psi_j^-(r) \psi_j^+(r) + \text{c.c.} \right] = 1. \quad (2.119) \]

Using Eq. (2.114), Eq. (2.118) and Eq. (2.119), we find that \( \delta n_j(r) \) and \( \theta_j(r) \) satisfy the following relation

\[ i \int dr \left[ \delta n_j^*(r) \theta_j(r) - \text{c.c.} \right] = 1. \quad (2.120) \]

Eq. (2.113) and Eq. (2.116) are necessarily solutions to Eq. (2.111) and Eq. (2.112), since they are obtained directly from the definitions given in Eq. (2.105) and Eq. (2.106). The foregoing analysis then implies that the Bogoliubov amplitudes and frequencies can also be found by solving the hydrodynamic equations. More specifically, if we look for solutions to Eq. (2.111) and (2.112) of the form

\[ \delta n(r, t) = \delta n(r) e^{-i\omega t} + \text{c.c.} \quad (2.121) \]
\[ \mathbf{v}(r, t) = \frac{\hbar}{m} \nabla \theta(r) e^{-i\omega t} + \text{c.c.} \quad (2.122) \]

and impose the following normalization condition for the amplitudes

\[ i \int dr \left[ \delta n^*(r) \theta(r) - \text{c.c.} \right] = 1, \quad (2.123) \]

the various solutions we find for \( \delta n(r) \) and \( \theta(r) \) will correspond to the Bogoliubov modes given by Eq. (2.114) and Eq. (2.118) respectively.
2.4.1 Hydrodynamic theory in the Thomas-Fermi limit

In the limit that the condensate contains a large number of atoms, the Thomas-Fermi approximation can be used. Within this approximation, the terms involving the Laplacian within the square bracket of Eq. (2.112) are neglected. The resulting hydrodynamic equations are

\[
\frac{\partial \delta n}{\partial t} = -\nabla \cdot (n_0 \mathbf{v}); \tag{2.124}
\]

\[
m \frac{\partial \mathbf{v}}{\partial t} = -g \nabla \delta n. \tag{2.125}
\]

These equations are easier to solve than the full hydrodynamic equations or the BdeG equations. To be consistent, the equilibrium condensate density appearing in Eq. (2.124) is that given by the TF solution to the GP equation.

Using Eq. (2.125) to eliminate the velocity variable from Eq. (2.124), we obtain the equation

\[
\frac{\partial^2 \delta n}{\partial t^2} = \frac{g}{m} \nabla \cdot (n_0 \nabla \delta n), \tag{2.126}
\]

which involves only the density fluctuation. For harmonic solutions of the form given in Eq. (2.121), Eq. (2.126) reduces to

\[
\omega^2 \delta n(\mathbf{r}) = -\frac{g}{m} \nabla \cdot (n_0 \nabla \delta n(\mathbf{r})). \tag{2.127}
\]

Either of Eq. (2.126) or Eq. (2.127) is referred to as the Stringari equation [42]. Eq. (2.127) is an eigenvalue problem for the mode frequencies \(\omega\) and the corresponding mode density fluctuations \(\delta n(\mathbf{r})\).

Once \(\omega\) and \(\delta n(\mathbf{r})\) have been determined, \(\delta n(\mathbf{r})\) must be properly normalized
according to Eq. (2.123). The phase fluctuation in this equation is eliminated by means of Eq. (2.125) and Eq. (2.122). Eq. (2.123) then becomes

\[
\int dr |\delta n(r)|^2 = \frac{\hbar \omega}{2g},
\]

which is the required normalization condition for the density fluctuation.

This completes our background discussion regarding the dynamics of the condensate. Much of this material will be used in later chapters. Before delving into these matters, we will first briefly discuss some of the ways that condensates are manipulated experimentally.

2.5 Optical dipole potential

The interaction between atoms and laser radiation is of great importance in cold atom physics. Lasers are used extensively not only to trap and cool atomic clouds, but also to probe them in various experiments [43]. From a quantum mechanical perspective, an atom interacts with the laser radiation through both real and virtual processes of absorption and emission of photons. The real absorption of a photon followed by emission is a scattering process and is responsible for imparting a random momentum to the atom. When repeated many times, these processes give rise to a radiation pressure force. Virtual processes result in an energy shift (AC Stark shift) of the atom that is proportional to the intensity of the laser radiation. If the intensity varies with position, an atom effectively experiences a potential which can also give rise to forces acting on the atom. This potential can be understood semiclassically in terms of the interaction between the induced dipole moment of the atom and the time-dependent radiation field. For this reason, it is referred to as an optical dipole
potential. Although all of these effects are important in cold atom physics, we will be mainly concerned with optical dipole forces in this thesis. These in fact are the dominant effects if the laser frequency is detuned far from the resonant frequency of the relevant atomic transition.

To make the above discussion a bit more quantitative, let us consider an atom in its ground state, subjected to an oscillating electric field $\mathbf{E}(\mathbf{r}, t) = \frac{1}{2}(\mathbf{E}(\mathbf{r})e^{i\omega t} + \text{c.c.})$. The atom couples to the electric field via the dipole moment operator $\hat{d}$. Treating this coupling perturbatively, the atom acquires an induced dipole moment

$$d(t) \equiv \langle \hat{d} \rangle = \alpha(\omega)\mathbf{E}(\mathbf{r}, t)$$

(2.129)

that oscillates at the same frequency as the applied electric field. The proportionality constant $\alpha(\omega)$ is (the real part of) the frequency-dependent polarizability.

The dipole potential is defined in terms of the energy shift due to the interaction between the dipole and the electric field which is given by

$$\Delta \mathbf{E}(\mathbf{r}) = -\frac{1}{2}\langle d(t) \cdot \mathbf{E}(\mathbf{r}, t) \rangle_t.$$  

(2.130)

The factor $1/2$ arises from the fact that the dipole moment is induced. The brackets $\langle \cdots \rangle_t$ indicate a time-average which is justified because the radiation period is much shorter than typical time scales of atomic motion. Performing the time average, one finds

$$V_{\text{dip}}(\mathbf{r}) = -\frac{\alpha(\omega)}{2\epsilon_0c}I(\mathbf{r}),$$  

(2.131)

where $I(\mathbf{r}) = \epsilon_0c|\mathbf{E}(\mathbf{r})|^2/2$ is the intensity of the light.

The polarizability $\alpha(\omega)$ depends on the detuning $\delta$ of the laser frequency from
the frequency of the relevant atomic transition. In particular, it is negative for $\delta > 0$ (blue detuning) and positive for $\delta < 0$ (red detuning). As a result, a blue-detuned laser produces a potential that repels atoms from high intensity regions; the opposite is true for a red-detuned laser. In the following, we discuss some of the experimental applications of the optical dipole potential that are relevant to this thesis.

### 2.5.1 Periodic lattice potential

The ability to introduce a periodic lattice potential into cold atomic systems has made them valuable tools for studying some important models in condensed matter physics. A one-dimensional periodic lattice potential is formed by superimposing two counter-propagating laser beams with the same frequency. The interference of the laser beams results in an optical standing wave and its electric field is

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) + E_0 \cos(\omega t + \mathbf{k} \cdot \mathbf{r})$$

$$= 2E_0 \cos(\omega t) \cos(\mathbf{k} \cdot \mathbf{r}).$$

(2.132)

According to Eq. (2.131), the corresponding dipole potential has the following form

$$V_{\text{lattice}}(\mathbf{r}) = V_0 \cos^2(\mathbf{k} \cdot \mathbf{r}) = \frac{1}{2}V_0[1 + \cos(2\mathbf{k} \cdot \mathbf{r})],$$

(2.133)

where the absolute value of $V_0$ describes the depth of the potential. The lattice spacing, defined as the distance between the neighbouring minima of the potential, is $\lambda_{\text{las}}/2 = \pi/k$, where $\lambda_{\text{las}}$ is the wavelength of the laser. Two and three dimensional periodic potentials can be formed by an appropriate arrangement of different laser beams.
2.5. OPTICAL DIPOLE POTENTIAL

2.5.2 Bragg spectroscopy

Another important application of optical dipole potentials in cold atom experiments is in Bragg spectroscopy. Just like the Bragg diffraction of X-rays by a crystal structure, a coherent atomic beam can be Bragg diffracted by a stationary periodic lattice potential. Experimentally Bragg diffraction is performed by imposing a moving periodic lattice potential on a stationary condensate [44]. Because these experiments can be used to probe the excitations of the condensate, they are referred to as the Bragg spectroscopy [45].

A moving optical lattice potential can be formed by superimposing two lasers beams with slightly different frequencies $\omega_1$ and $\omega_2$. The resulting electric field is given by

$$E(r, t) = E_0 \cos(\omega_1 t - k_1 \cdot r) + E_0 \cos(\omega_2 t - k_2 \cdot r)$$

$$= 2E_0 \cos \left( \frac{\omega_1 + \omega_2}{2} t - \frac{k_1 + k_2}{2} \cdot r \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t - \frac{k_1 - k_2}{2} \cdot r \right).$$

(2.134)

To obtain the dipole potential, we substitute the above expression into Eq. (2.131) and perform the average over a period of time $\pi/(\omega_1 + \omega_2)$. Since $(\omega_1 - \omega_2)/2 \ll (\omega_1 + \omega_2)/2$, the effect of time-average on

$$\cos^2 \left( \frac{\omega_1 - \omega_2}{2} t - \frac{k_1 - k_2}{2} \cdot r \right)$$

can be neglected. With an appropriate energy reference point, we obtain the following
dipole potential

\[ V_{\text{Bragg}}(\mathbf{r}, t) = V_B \cos (\omega_B t - \mathbf{k}_B \cdot \mathbf{r}), \]  

(2.135)

where \( \omega_B = \omega_1 - \omega_2 \) and \( \mathbf{k}_B = \mathbf{k}_1 - \mathbf{k}_2 \). This potential moves in the direction of \( \mathbf{k}_B \) with a velocity \( v_B = \omega_B/k_B \) which can be controlled by adjusting the angle between the directions of propagation of the two laser beams. A slow modulation of the amplitude \( V_B \) can be used to generate a Bragg pulse.

### 2.5.3 Disorder potential for atomic clouds: optical speckle pattern

Unlike solid state systems, cold atomic systems are intrinsically clean. Thus in order to study the interplay between disorder and particle interactions in cold atom experiments, some form of disorder potential has to be introduced into the system. Because of the fragile nature of these atomic systems, the disorder potential needs to be well controlled and calibrated. A random light intensity pattern formed after a laser passes through a diffusive medium, provides one ideal implementation of such a disorder potential. A typical example of a diffusive medium is a ground glass plate with an optically rough surface. When a coherent light beam is passed through such a plate, each point on the surface can be viewed as a secondary source of radiation (Huygens Principle). The surface roughness imparts a random phase to each secondary source. Thus, when the radiation from all the sources is superposed on an observation plane, the interference of all the waves produces a randomly distributed pattern of light. This pattern is referred to as optical speckle.

Shown in Fig. 2.1 is a typical experimental setup for producing a disorder potential on a Bose condensate. A laser beam is shone through a diffusive plate and produces an optical speckle pattern, which is imaged onto a cold atomic system. The speckle
2.5. OPTICAL DIPOLE POTENTIAL

50
gives rise to a static disorder dipole potential $V_{\text{dis}}(\mathbf{r})$. The effect of this disorder potential on the atomic cloud is then observed by means of an imaging beam.

Figure 2.1: (a) Experimental setup for the speckle potential and the imaging system for a BEC. (b) The intensity distribution of the speckle potential in a 2D plane (left) and its Fourier transform (right). Figures taken from [16].

Two important statistical properties of a speckle pattern are the standard deviation of the light intensity in space and the two-point light intensity auto-correlation function. As explained in [46], the statistical distribution of the light intensity $I(\mathbf{r})$ in space follows an exponential distribution

$$
P(I) = \frac{1}{\bar{I}} e^{-I/\bar{I}},
$$

(2.136)

where $\bar{I}$ is the mean value of the intensity. From this distribution the standard
deviation of the light intensity $\sigma_I \equiv \sqrt{\overline{I^2} - \overline{I}^2}$ is found to be the same as $\bar{I}$. The intensity auto-correlation function is defined as

$$\gamma_I(r) = \overline{I(r')I(r' + r)},$$

(2.137)

where the overbar denotes a spatial average with respect to the variable $r'$.

Since $\sigma_I = \bar{I}$, Eq. (2.137) can be rewritten as

$$\gamma_I(r) = \bar{I}^2[1 + C(r)],$$

(2.138)

where the function $C(r)$ has the property that $C(r = 0) = 1$.

Since the dipole potential is proportional to the light intensity, the statistical properties of the disorder potential follow from those of the speckle light pattern. The standard deviation of the disorder potential

$$V_D \equiv \sqrt{\overline{V_{\text{dis}}^2(r)} - \overline{V_{\text{dis}}(r)}^2} = \overline{V_{\text{dis}}(r)}$$

(2.139)

is often referred to as the strength of the disorder potential. From Eq. (2.138) we see that the auto-correlation function of the disorder potential is given by

$$\gamma_V(r) = V_D^2[1 + C(r)],$$

(2.140)

The correlation function $C(r)$ typically decays as a function of $r$ and its widths in the $x$, $y$ and $z$ directions characterize the typical size of a speckle grain. These

---

3The spatial average can equivalently be viewed as an ensemble average. For an ensemble representing a translationally invariant system, the resulting average is independent of $r'$. 

---
widths can be adjusted experimentally to produce an effectively one-dimensional disorder potential. For instance, let us consider a condensate that is elongated in the $z$ direction. If the size of the speckle grains in the transverse direction is much larger than the transverse TF radius $R_\perp$ of the cloud, the disorder potential produced by the speckle pattern can be viewed as effectively one-dimensional (along the $z$ direction).

In this situation $C(r) = C(z)$. To a good approximation, $C(z)$ is given by [46]

$$C(z) = e^{-(z/\sigma)^2}, \quad (2.141)$$

where $\sigma$ is sometimes referred to as the disorder correlation length.

As we shall see, it is useful to express the statistical properties of the one-dimensional disorder potential in terms of the Fourier components of the potential $V_{\text{dis}}(r)$,

$$\tilde{V}_{\text{dis}}(q) = \int dr e^{-iq \cdot r} V_{\text{dis}}(r). \quad (2.142)$$

The ensemble average of this quantity is

$$\overline{V_{\text{dis}}(q)} = \int dr e^{-iq \cdot r} V_{\text{dis}}(r) = (2\pi)^3 \delta(q) V_D. \quad (2.143)$$

Similarly, we have

$$\overline{V_{\text{dis}}^*(q)V_{\text{dis}}(q')} = \int dr \int dr' e^{iq \cdot r} e^{-iq' \cdot r'} V_{\text{dis}}(r) V_{\text{dis}}(r')$$

$$= \int dr e^{i(q-q) \cdot r} \int ds e^{-iq' \cdot s} \gamma_V(s)$$

$$= (2\pi)^3 \delta(q - q') V_D^2[(2\pi)^3 \delta(q) + \tilde{C}(q)], \quad (2.144)$$
2.5. OPTICAL DIPOLE POTENTIAL

where $\tilde{C}(q)$ is the Fourier transform of $C(r)$. For the one-dimensional correlation function we have

$$\tilde{C}(q) = \int dr e^{-i q \cdot r} C(r)$$

$$= (2\pi)^2 \delta(q_x) \delta(q_y) \sigma \sqrt{\pi} e^{-\frac{1}{4}\sigma^2 q^2_z}. \hspace{1cm} (2.145)$$

These results will be used in Chapter 5.

Finally, we point out that both the disorder strength and the auto-correlation function can be measured accurately in experiments. Fig. 2.2 (a) shows an example of the spatial variation of a one-dimensional disorder potential. The corresponding auto-correlation function is shown in Fig. 2.2 (b). The correlation length is about $10 \mu m$ which is comparable to the average distance between the peaks in Fig. 2.2 (a).
Chapter 3

Density response functions

In this chapter we present detailed calculations of zero-temperature density response functions of Bose-Einstein condensates within the Bogoliubov theory. We will consider condensates having a uniform density as well as those in trapped geometries. These density response functions will be used extensively in later chapters when we investigate the dissipative dynamics of condensates using linear response theory. Within the Bogoliubov approximation, the zero-temperature density response function of a Bose condensate can be expressed in terms of the Bogoliubov amplitudes and hence its determination becomes a matter of solving the Bogoliubov-de Gennes equations. It is relatively straightforward to carry out such calculations for both the uniform condensate and the uniform cylindrical condensate. However, the numerical calculations are much more difficult for an arbitrary trapped gas. Nevertheless, for an elongated condensate we can invoke a local density approximation which makes use of the response properties of a uniform cylindrical condensate. The formulation of this approximation and its justification will be explained.
3.1 Definition and general properties of the density response function

The zero-temperature density response function for the atomic Bose-Einstein condensate, described by the Hamiltonian $\hat{H}$ given in Eq. (2.24), is defined as

$$\chi(\mathbf{r}, \mathbf{r}', t - t') = \frac{i}{\hbar} \Theta(t - t') \langle \Psi_0 | [\hat{n}_H(\mathbf{r}, t), \hat{n}_H(\mathbf{r}', t')] | \Psi_0 \rangle,$$  

(3.1)

where $|\Psi_0\rangle$ denotes the ground state of $\hat{H}$ and $\hat{n}(\mathbf{r}, t)$ is the density operator in the Heisenberg picture $\hat{n}_H(\mathbf{r}, t) = e^{i\hat{H}t/\hbar} \hat{n}(\mathbf{r}) e^{-i\hat{H}t/\hbar}$. One can obtain a more explicit expression by making use of the closure relation $\sum_m |\Psi_m\rangle \langle \Psi_m| = 1$, where $|\Psi_m\rangle$ is an eigenstate of $\hat{H}$ with energy $E_m$. The insertion of this relation in Eq. (3.1) leads to the spectral representation

$$\chi(\mathbf{r}, \mathbf{r}', t - t') = \frac{i}{\hbar} \Theta(t - t') \sum_{m \neq 0} \left\{ e^{-i\omega_m (t - t')} \langle \Psi_0 | \hat{n}(\mathbf{r}) | \Psi_m \rangle \langle \Psi_m | \hat{n}(\mathbf{r}') | \Psi_0 \rangle - \text{c.c.} \right\},$$  

(3.2)

where $\omega_m \equiv (E_m - E_0)/\hbar$. The term with $m = 0$ can be excluded from the summation since it gives a zero contribution. We emphasize that this is a general expression which is applicable to any system in its ground state. It can be extended straightforwardly to finite temperatures by making use of an appropriate statistical density matrix.

In calculations, Fourier transforms of the density response function are often more useful. Since $\chi(\mathbf{r}, \mathbf{r}', \tau)$ for an inhomogeneous system depends on the two independent variables $\mathbf{r}$ and $\mathbf{r}'$, we define the double spatial Fourier transform

$$\chi(\mathbf{q}, \mathbf{q}', \tau) \equiv \int d\mathbf{r} \int d\mathbf{r}' e^{-i(\mathbf{q} \cdot \mathbf{r} - \mathbf{q}' \cdot \mathbf{r}')} \chi(\mathbf{r}, \mathbf{r}', \tau).$$  

(3.3)
3.1. DEFINITION AND GENERAL PROPERTIES OF THE DENSITY RESPONSE FUNCTION

Substituting Eq. (3.2) into the above definition one finds

\[ \chi(q, q', \tau) = \frac{i}{\hbar} \theta(\tau) \sum_{m \neq 0} \left\{ e^{-i\omega_m \tau} n_{0m}(q) n_{0m}^*(q') - e^{i\omega_m \tau} n_{m0}(q) n_{m0}^*(q') \right\}, \]

\[ = \frac{i}{\hbar} \theta(\tau) \sum_{m \neq 0} n_{0m}(q) n_{0m}^*(q') \left( e^{-i\omega_m \tau} - e^{i\omega_m \tau} \right), \]

(3.4)

where

\[ n_{mn}(q) = \langle \Psi_m | \hat{n}(q) | \Psi_n \rangle \]

(3.5)

with

\[ \hat{n}(q) \equiv \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \hat{n}(\mathbf{r}). \]

(3.6)

In arriving at the second line of Eq. (3.4), we have used the fact that the system possesses time-reversal symmetry. This symmetry implies that for each excited state \( |\Psi_m\rangle \) there exists a degenerate time-reversed state \( |\Psi_{\bar{m}}\rangle \) with the property that \( \langle \Psi_0 | \hat{n}(\mathbf{r}) | \Psi_m \rangle = \langle \Psi_0 | \hat{n}(\mathbf{r}) | \Psi_{\bar{m}} \rangle^* \). As a result, \( n_{m0}(q) = n_{0m}(q) \) and, since a summation over \( m \) is equivalent to a summation over \( \bar{m} \), the second line of Eq. (3.4) follows.

Using the same reasoning, we can show that \( \chi(q, q', \tau) \) has the property

\[ \chi(-q, -q', \tau) = \chi(q, q', \tau). \]

(3.7)

We now introduce the spectral function

\[ S(q, q', \omega) \equiv \sum_{m \neq 0} n_{0m}(q) n_{0m}^*(q') \delta(h\omega - \hbar \omega_{m0}). \]

(3.8)

Since \( \omega_{m0} > 0 \), we observe that the spectral function is zero for \( \omega < 0 \). With this
3.1. DEFINITION AND GENERAL PROPERTIES OF THE DENSITY RESPONSE FUNCTION

Definition, \( \chi(q, q'; \tau) \) can be written as

\[
\chi(q, q', \tau) = i\theta(\tau) \int d\omega S(q, q', \omega) (e^{-i\omega \tau} - e^{i\omega \tau}).
\] (3.9)

Using Eq. (3.9) and the following Fourier representation of the step function

\[
\theta(\tau) = i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega \tau} \omega + i\epsilon,
\] (3.10)

we obtain the Fourier transform

\[
\chi(q, q', \omega) \equiv \int d\tau e^{i\omega \tau} \chi(q, q', \tau)
= \int d\omega' S(q, q', \omega') \left( \frac{1}{\omega + \omega' + i\epsilon} - \frac{1}{\omega - \omega' + i\epsilon} \right),
\] (3.11)

where the positive, real infinitesimal \( \epsilon \) ensures the causality of the response. The above expression indicates that knowledge of the spectral function \( S(q, q', \omega) \) completely determines the density response function.

Making use of the identity

\[
\frac{1}{x + i\epsilon} = P\frac{1}{x} - i\pi \delta(x),
\] (3.12)

Eq. (3.11) can be written

\[
\chi(q, q', \omega) = \chi'(q, q', \omega) + i\chi''(q, q', \omega),
\] (3.13)
where

\[
\chi'(q, q', \omega) = P \int d\omega' S(q, q', \omega') \left( \frac{1}{\omega + \omega'} - \frac{1}{\omega - \omega'} \right), \quad (3.14)
\]

\[
\chi''(q, q', \omega) = \pi [S(q, q', \omega) - S(q, q', -\omega)]. \quad (3.15)
\]

Since the spectral function \( S(q, q', \omega) \) is not in general real, \( \chi' \) and \( \chi'' \) can be complex and therefore are not necessarily the real and imaginary parts of \( \chi(q, q', \omega) \). We observe from the above expressions that \( \chi'(q, q', \omega) \) is quite generally an even function of \( \omega \) while \( \chi''(q, q', \omega) \) is an odd function of \( \omega \). In addition, they are related through the Kramers-Kronig relation

\[
\chi'(q, q', \omega) = \frac{1}{\pi} P \int d\omega' \frac{\chi''(q, q', \omega')}{\omega' - \omega}, \quad (3.16)
\]

which is a direct consequence of the causal nature of the response function.

The density response function with equal momenta \( \chi(q, q, \omega) \) will be of particular importance to us. The spectral function for \( q = q' \) is given by

\[
S(q, \omega) \equiv S(q, q, \omega) = \sum_{m \neq 0} |n_{0m}(q)|^2 \delta(\hbar \omega - \hbar \omega_{m0}) \quad (3.17)
\]

and is real. In view of this fact, Eq. (3.14) and Eq. (3.15) imply that \( \chi'(q, q, \omega) \) and \( \chi''(q, q, \omega) \) are real and hence,

\[
\chi'(q, q, \omega) = \text{Re} \chi(q, q, \omega); \quad \chi''(q, q, \omega) = \text{Im} \chi(q, q, \omega). \quad (3.18)
\]
3.1. DEFINITION AND GENERAL PROPERTIES OF THE DENSITY RESPONSE FUNCTION

$S(q, \omega)$, as given by Eq. (3.17), is called the dynamic structure factor and is an important physical quantity. In solid state systems as well as in liquid helium, it can be directly measured in neutron scattering experiments. In cold atom systems, however, scattering probes of this kind cannot be used. For these systems it is more convenient to make use of the optical coupling between the atoms and laser beams. One such method is Bragg diffraction [47] which has been used experimentally to obtain information about the dynamic structure factor. We will give a detailed example of this in the next chapter.

On the theoretical side, it is well-known that the dynamic structure factor $S(q, \omega)$ satisfies an important relation, the f-sum rule, which states that [48]

$$\int_0^\infty d\omega \omega S(q, \omega) = \frac{Nq^2}{2m}. \tag{3.19}$$

This f-sum rule is an exact relation and can serve as a useful check on the consistency of an approximate theory. In addition, from the fact that

$$n_{0m}(q = 0) = \langle \Psi_0 | \hat{N} | \Psi_m \rangle = N\delta_{0,m}, \tag{3.20}$$

where $\hat{N} = \int d\mathbf{r} \hat{n}(\mathbf{r})$ is the operator for the total number of particles, we see that the dynamic structure factor vanishes in the long wave length limit, namely

$$S(q = 0, \omega) = 0. \tag{3.21}$$

In the following we shall find that the dynamic structure factor of Bose condensates obtained within the Bogoliubov theory satisfies both of these properties.
3.2 Density response function within Bogoliubov theory

We now present the derivation of the density response function using the Bogoliubov theory. In view of Eq. (3.11), it is sufficient to evaluate the spectral function $S(q, q', \omega)$. In the Bogoliubov theory, the Hamiltonian $\hat{H}$ is approximated by the Bogoliubov Hamiltonian $\hat{H}_B$ given in Eq. (2.92). Furthermore, from Eq. (2.99) we see that the density operator $\hat{n}(r)$ is linear in quasi-particle operators. This implies that the excited states involved in Eq. (3.2) are restricted to those containing only a single quasi-particle. These states and their corresponding energies are given by $|\Psi_i\rangle = \hat{b}_0 \hat{\alpha}_i^\dagger |\Psi_0\rangle$ and $E_i = E_0 + \epsilon_i$ respectively. The relevant matrix elements $\langle \Psi_0 | \hat{n}(r) | \Psi_i \rangle$ were obtained earlier and are given by Eq. (2.103), namely

$$\delta n_i(r) \equiv \langle \Psi_0 | \hat{n}(r) | \Psi_i \rangle = \Phi_0(r) \psi_i^-(r), \quad (3.22)$$

where we recall that $\psi_i^-(r) = u_i(r) - v_i(r)$. Using these results in Eq. (3.8) we immediately arrive at [49]

$$S(q, q', \omega) = \sum_i \delta n_i^*(q) \delta n_i(q') \delta(\hbar \omega - \hbar \omega_i), \quad (3.23)$$

where $\omega_i = \epsilon_i/\hbar$ and

$$\delta n_i(q) = \int dr \delta n_i(r) e^{-iq \cdot r}$$

$$= \int dr \Phi_0(r) [u_i(r) - v_i(r)] e^{-iq \cdot r}. \quad (3.24)$$

It is clear from the orthogonality properties of the solutions to the BdeG equations...
that
\[ \delta n_i(q = 0) = \int dr \Phi_0(r) [u_i(r) - v_i(r)] = 0, \tag{3.25} \]
which leads to \( S(q = 0, \omega) = 0 \). We further demonstrate that the dynamic structure factor obtained in Bogoliubov theory also satisfies the f-sum rule [48]. Following Ref. [48], we consider the commutator expression \( \langle \Psi_0 | [[\hat{n}(q), \hat{H}], \hat{n}(-q)] | \Psi_0 \rangle \). By inserting a complete set of states in the appropriate places, one finds
\[
\langle \Psi_0 | [[\hat{n}(q), \hat{H}], \hat{n}(-q)] | \Psi_0 \rangle = \sum_{m \neq 0} \hbar \omega_i (|n_{0m}(q)|^2 + |n_{0m}(-q)|^2)
= 2 \sum_{m \neq 0} \hbar \omega_{m0} |n_{0m}(q)|^2, \tag{3.26} \]
where we have again made use of time-reversal symmetry. From Eq. (3.17), we observe that this final result is equal to \( 2\hbar^2 \int_0^\infty d\omega \omega S(q, \omega) \). This relation is still valid in the Bogoliubov theory, but the Hamiltonian \( \hat{H} \) must be replaced by \( \hat{H}_B \). We thus have
\[
2\hbar^2 \int_0^\infty d\omega \omega S(q, \omega) = \langle \Psi_0 | [[\hat{n}(q), \hat{H}_B], \hat{n}(-q)] | \Psi_0 \rangle. \tag{3.27} \]
On the other hand, the commutator \([ [\hat{n}(q), \hat{H}_B], \hat{n}(-q)] \) can be evaluated explicitly. To do so we first evaluate
\[
[\hat{n}(q), \hat{H}_B] = \int dr e^{-iq \cdot r} [\hat{n}(r), \hat{H}_B]. \tag{3.28} \]
Using Eq. (2.92) and Eq. (2.97), we find
\[
[\hat{n}(r), \hat{H}_B] = \hat{b}_0^\dagger \Phi_0^* \mathcal{L} \hat{\psi} + g\hat{b}_0 \Phi_0 |\Phi_0|^2 \hat{\psi}^\dagger - \text{h.c.}, \tag{3.29} \]
where we recall that the operator $\mathcal{L}$ is given in Eq. (2.37). Substituting Eq. (3.29) into Eq. (3.28) we obtain

$$[\hat{n}(\mathbf{q}), \hat{H}_B] = \hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \mathcal{L} \delta \hat{\psi} + g \hat{b}_0 \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} |\Phi_0|^2 \delta \hat{\psi}^\dagger$$

$$- \hat{b}_0 \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \mathcal{L} \delta \hat{\psi}^\dagger - g \hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} |\Phi_0|^2 \delta \hat{\psi}^\dagger.$$  

(3.30)

Since $\mathcal{L}$ is Hermitian we have

$$\hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \mathcal{L} \delta \hat{\psi} = \hat{b}_0^\dagger \int d\mathbf{r} \left[ \mathcal{L} \left( e^{i\mathbf{q} \cdot \mathbf{r}} \Phi_0 \right) \right]^* \delta \hat{\psi}$$

$$= \hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \left[ \frac{\hbar^2 \mathbf{q}^2}{2m} \Phi_0^* + \frac{i\hbar^2}{m} \mathbf{q} \cdot \nabla \Phi_0^* + \mathcal{L} \Phi_0 \right] \delta \hat{\psi}$$

$$= \hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \left[ \frac{\hbar^2 \mathbf{q}^2}{2m} \Phi_0^* + \frac{i\hbar^2}{m} \mathbf{q} \cdot \nabla \Phi_0^* + g |\Phi_0|^2 \Phi_0^* \right] \delta \hat{\psi}^\dagger.$$  

(3.31)

where we used the GP equation (2.17) in the last line. Similarly we find

$$\hat{b}_0 \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \mathcal{L} \delta \hat{\psi}^\dagger = \hat{b}_0 \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \left[ \frac{\hbar^2 \mathbf{q}^2}{2m} \Phi_0 + \frac{i\hbar^2}{m} \mathbf{q} \cdot \nabla \Phi_0 + g |\Phi_0|^2 \Phi_0 \right] \delta \hat{\psi}^\dagger.$$  

(3.32)

Substituting Eq. (3.32) and Eq. (3.31) into Eq. (3.30), we find

$$[\hat{n}(\mathbf{q}), \hat{H}_B] = \hat{b}_0^\dagger \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \left[ \frac{\hbar^2 \mathbf{q}^2}{2m} \Phi_0^*(\mathbf{r}) + \frac{i\hbar^2}{m} \mathbf{q} \cdot \nabla \Phi_0^*(\mathbf{r}) \right] \delta \hat{\psi}(\mathbf{r})$$

$$- \hat{b}_0 \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \left[ \frac{\hbar^2 \mathbf{q}^2}{2m} \Phi_0(\mathbf{r}) + \frac{i\hbar^2}{m} \mathbf{q} \cdot \nabla \Phi_0(\mathbf{r}) \right] \delta \hat{\psi}^\dagger(\mathbf{r})$$

(3.33)
Finally, using the above result to evaluate \([\hat{n}(q), \hat{H}_B], \hat{n}(-q)\], we find that

\[
[\hat{n}(q), \hat{H}_B], \hat{n}(-q) = N\frac{\hbar^2 q^2}{m},
\]

(3.34)

where we have used the fact that \(\int d\mathbf{r} |\Phi_0(\mathbf{r})|^2 = N\). Comparing the above equation to Eq. (3.27), we immediately arrive at the f-sum rule Eq. (3.19).

### 3.3 Examples of density response functions

#### 3.3.1 Uniform condensate

For a uniform system, the condensate wave function within the GP theory is simply

\[
\Phi_0 = \sqrt{\frac{N}{\Omega}},
\]

(3.35)

where \(\Omega\) is the volume of the system. In this case, the GP equation yields the chemical potential

\[
\mu = g n_0,
\]

(3.36)

where \(n_0 = N/\Omega\) is the condensate density.

Because of the translational symmetry of the uniform condensate, the solutions to Bogoliubov-de Gennes (BdeG) equations (2.53) take the form

\[
\begin{align*}
    u_k(\mathbf{r}) &= u_k \frac{e^{i k \cdot \mathbf{r}}}{\sqrt{\Omega}}, \\
    v_k(\mathbf{r}) &= v_k \frac{e^{i k \cdot \mathbf{r}}}{\sqrt{\Omega}},
\end{align*}
\]

(3.37)

where \(k\) is the wave vector used to label the excitations. The normalization condition
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Eq. (2.61) for the amplitudes becomes

\[ u_k^2 - v_k^2 = 1. \]  \hfill (3.38)

Substituting the above expressions for the Bogoliubov amplitudes and the chemical potential given by Eq (3.36) into Eq. (2.53), one finds

\[
\left( \frac{\hbar k^2}{2m} + gn_0 \right) u_k - gn_0 v_k = \hbar \omega_k u_k \\
- \left( \frac{\hbar k^2}{2m} + gn_0 \right) v_k + gn_0 v_k = \hbar \omega_k v_k. \]  \hfill (3.39)

These equations together with the normalization condition in Eq. (3.38) can easily be solved for the Bogoliubov amplitudes with the result

\[
u_k = \sqrt{\frac{1}{2} \left( \frac{\xi_k}{E_k} + 1 \right)}; \quad v_k = \sqrt{\frac{1}{2} \left( \frac{\xi_k}{E_k} - 1 \right)}, \]  \hfill (3.40)

where \( E_k \equiv \hbar \omega_k = \sqrt{\varepsilon_k^2 + 2gn_0 \varepsilon_k} \), \( \varepsilon_k = \hbar^2 k^2 / 2m \) and \( \xi_k = \varepsilon_k + gn_0 \). In the long wavelength limit the excitation spectrum exhibits a phonon-like behaviour, namely

\[
\omega_k = ck + O(k^3), \]  \hfill (3.41)

where \( c = \sqrt{gn_0 / m} \) is the speed of sound for the uniform condensate.

Using these results for the condensate wave function and the Bogoliubov amplitudes in Eq. (3.24), the Fourier transform of the density fluctuation is found to be

\[
\delta n_k(q) = \delta_{k,q} \sqrt{N} (u_k - v_k). \]  \hfill (3.42)
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Substituting this expression into Eq. (3.23) yields the spectral function

\[ S(q, q', \omega) = \sum_k \delta n_k^*(q) \delta n_k(q') \delta(\hbar \omega - \hbar \omega_k) \]
\[ = N \delta_{q, q'} \frac{\varepsilon_q}{E_q} \delta(\hbar \omega - \hbar \omega_q). \]  

(3.43)

One can easily check that the dynamic structure factor \( S(q, \omega) \) obtained from Eq. (3.43) indeed satisfies the f-sum rule.

3.3.2 Uniform cylindrical condensate

Our next example concerns a uniform cylindrical condensate with an axial length \( L \) along the \( z \) axis and a linear density \( \nu \equiv N/L \). For the applications we have in mind, it will only be necessary to determine the density response function \( \chi(q, q', \omega) \) for wave vectors restricted to \( q = q_z \hat{z} \) and \( q' = q'_z \hat{z} \). For convenience, these functions will be denoted simply by \( \chi(q_z, q'_z, \omega) \). A similar notation will be used for the spectral function \( S(q_z, q'_z, \omega) \) and for the dynamic structure factor \( S(q_z, \omega) \).

The solutions of the GP equation and the BdeG equations are most conveniently discussed in the cylindrical coordinate system \((\rho, \phi, z)\), since the system has translational symmetry along the \( z \) axis and rotational symmetry about the \( z \) axis. These symmetries imply that the condensate wave function does not depend on the axial and azimuthal coordinates that is, \( \Phi_0(r) = \Phi_0(\rho) \). In this case, the BdeG equations admit solutions having the general form

\[ u_{jkm}(r) = \frac{1}{\sqrt{L}} e^{ikz} \frac{1}{\sqrt{2\pi}} e^{im\phi} u_{jkm}(\rho); \quad v_{jkm}(r) = \frac{1}{\sqrt{L}} e^{ikz} \frac{1}{\sqrt{2\pi}} e^{im\phi} v_{jkm}(\rho), \]  

(3.44)

where \( j, k \) and \( m \) are referred to as the radial, axial and azimuthal quantum numbers,
respectively. The $z$ and $\phi$ dependence reflect the fact that the $z$-component of the total momentum and total angular momentum are conserved quantities. The Bogoliubov frequencies corresponding to these solutions will be denoted by $\omega_{jm}(k)$.

From Eq. (3.22) we see that the density fluctuations of the modes can be written as

$$
\delta n_{jkm}(r) = \frac{1}{\sqrt{L}} e^{ikz} \frac{1}{\sqrt{2\pi}} e^{im\phi} \delta n_{jkm}(\rho),
$$

(3.45)

where

$$
\delta n_{jkm}(\rho) = \Phi_0(\rho)[u_{jkm}(\rho) - v_{jkm}(\rho)].
$$

(3.46)

The relevant Fourier components of the density fluctuations are

$$
\delta n_{jkm}(q_x = q_y = 0, q_z) = \int \! d\mathbf{r} e^{-iq_z z} \delta n_{jkm}(\mathbf{r})
$$

$$
= \sqrt{L} \delta_{q_z k} \delta_{m,0} \delta n_j(q_z)
$$

(3.47)

with

$$
\delta n_j(k) \equiv \sqrt{2\pi} \int_0^{\infty} \! d\rho \rho \delta n_{j,k0}(\rho)
$$

$$
= \sqrt{2\pi} \int_0^{\infty} \! d\rho \Phi_0(\rho)[u_{j,k0}(\rho) - v_{j,k0}(\rho)].
$$

(3.48)

From Eq. (3.23) one finds that the spectral function of interest is given by

$$
S(q_z, q'_z, \omega) = L \delta_{q_z q'_z} \sum_j |\delta n_j(q_z)|^2 \delta(\hbar \omega - \hbar \omega_j(q_z))
$$

$$
= N \delta_{q_z q'_z} \sum_j W_j(q_z) \delta(\hbar \omega - \hbar \omega_j(q_z)),
$$

(3.49)
where for simplicity we now denote $\omega_{j0}(q_z)$ by $\omega_j(q_z)$. The quantity

$$W_j(q_z) = |\delta n_j(q_z)|^2/\nu$$

appearing in Eq. (3.49) is a dimensionless weight factor. We recall that $\nu = N/L$.

Unlike the situation for the uniform condensate, the Bogoliubov amplitudes and mode frequencies cannot be obtained analytically for the cylindrical condensate. To determine the relevant mode frequencies and amplitudes, we will first solve the hydrodynamic equations in the Thomas-Fermi (TF) limit [50] and then solve the BdeG equations numerically. The solutions obtained from the TF approach are in fact the long wave-length limit of the BdeG solutions. Although the TF solutions are in general approximate, they have the advantage of being semi-analytic and can therefore be more informative than the full numerical approach. In any case, a comparison of the results obtained from the two approaches will allow us to establish more precisely the region of validity for the TF hydrodynamic theory.

**Hydrodynamic theory in the TF limit**

As we have seen in the previous chapter, the Bogoliubov equations and the hydrodynamic equations are complementary ways of determining the elementary excitations of a condensate. In the limit that the condensate contains a large number of atoms, the Thomas-Fermi approximation can be used. As we have shown, in this limit the coupled hydrodynamic equations can be manipulated to yield the Stringari equation (2.127).

Now we solve Eq. (2.127) for the cylindrical condensate. The equilibrium TF
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Density in this case is given by

\[ n_0(\rho) = \frac{m\omega^2}{2g}(R^2_{\perp} - \rho^2), \]  

where \( \omega_{\perp} \) is the radial trapping frequency. \( R_{\perp} \) is the TF radius of the condensate and is determined by the total number of particles

\[ N = \int_0^L dz \int_0^{2\pi} d\phi \int_0^{R_{\perp}} d\rho n_0(\rho) = \frac{\pi m\omega^2_{\perp} LR^4_{\perp}}{4g}, \]

that is,

\[ \nu = \frac{N}{L} = \frac{\pi m\omega^2_{\perp} R^4_{\perp}}{4g}. \]

Assuming a density fluctuation of the form given in Eq. (3.45), the normalization condition in Eq. (2.128) can be expressed as

\[ \int_0^{R_{\perp}} d\rho \rho^2 n^2_{jkm}(\rho) = \frac{\hbar\omega_{jm}(k)}{2g}. \]

Since only the \( m = 0 \) modes are of interest according to Eq. (3.47), we will only look for solutions of Eq. (2.127) corresponding to these particular modes.

Substituting Eq. (3.51) and \( \delta n(\mathbf{r}) = \frac{1}{\sqrt{L}} e^{ikz} \frac{1}{\sqrt{2\pi}} \delta n(\rho) \) into Eq. (2.127), we obtain

\[ \omega^2 \delta n = \frac{1}{2} \omega_{\perp}^2 \left\{ k^2 (R_{\perp}^2 - \rho^2) \delta n - \frac{1}{\rho} (R_{\perp}^2 - 3\rho^2) \frac{\partial \delta n}{\partial \rho} - (R_{\perp}^2 - \rho^2) \frac{\partial^2 \delta n}{\partial \rho^2} \right\}, \]

where \( \delta n \) now denotes \( \delta n(\rho) \) and we have suppressed the \( j, k \) and \( m = 0 \) subscripts.
for simplicity. This equation can be cast into a standard Sturm-Liouville form. Introducing the new variable $x = 2\rho^2/R^2_\perp - 1$ and writing $\delta n(\rho) = y(x)$, we obtain the equation
\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dy}{dx} \right] + \frac{1}{2} \bar{\omega}^2 y - \frac{1}{8} \bar{k}^2 (1 - x)y = 0,
\]
where $\bar{\omega} = \omega/\omega_\perp$ and $\bar{k} = kR_\perp$ are now dimensionless frequency and wave number variables. Eq. (3.55) can be solved by expanding $y(x)$ as
\[
y(x) = \sum_l a_l P_l(x),
\]
where $P_l(x)$ is a Legendre polynomial of order $l$, but normalized according to
\[
\int_{-1}^1 dx P_l(x)P_l'(x) = \delta_{ll}'.
\]
Substituting Eq. (3.56) into Eq. (3.55), we obtain the system of linear equations
\[
\left( \frac{1}{2} \bar{\omega}^2 - l(l+1) - \frac{1}{8} \bar{k}^2 \right) a_l + \frac{1}{8} \bar{k}^2 \sum_{l'} M_{ll'} a_{l'} = 0,
\]
where the symmetric matrix $M_{ll'}$ is given by
\[
M_{ll'} = \int_{-1}^1 dx P_l(x)P_{l'}(x),
= \delta_{l,l'+1} \frac{l}{\sqrt{(2l-1)(2l+1)}} + \delta_{l,l'-1} \frac{l+1}{\sqrt{(2l+1)(2l+3)}}.
\]
Eq. (3.58) defines an eigenvalue problem for the scaled mode frequencies $\bar{\omega}_j(\bar{k}) = \omega_j(k)/\omega_\perp$ and the expansion coefficients $a_l(j, \bar{k})$ for the various radial modes labelled by the index $j$. Once these coefficients have been determined, the corresponding mode
density fluctuations are obtained from Eq. (3.56) as

$$\delta n_{j,0}(\rho) = \sum_l a_l(j, \bar{k}) P_l \left( 2\rho^2/R_\perp^2 - 1 \right).$$

Substituting Eq. (3.60) into Eq. (3.54), one finds that the expansion coefficients $a_l(j, \bar{k})$ must be normalized according to

$$\sum_l a_l^2(j, \bar{k}) = \frac{2\hbar \omega_\perp \bar{\omega}_j(\bar{k})}{g R_\perp^2}.$$  

(3.61)

In numerical calculations, however, it is more convenient to define expansion coefficients having a unit norm, namely

$$\sum_l \bar{a}_l^2(j, \bar{k}) = 1.$$  

(3.62)

In view of Eq. (3.61) these coefficients are given by

$$\bar{a}_l^2(j, \bar{k}) = \frac{g R_\perp^2}{2\hbar \omega_\perp \bar{\omega}_j(\bar{k})} a_l^2(j, \bar{k}).$$

(3.63)

The behaviour of the mode frequencies can be determined easily in the long wavelength limit from Eq. (3.58) [50]. Since the diagonal elements of the matrix $M_{ll'}$ are zero, the effect of the $M_{ll'}$ term in Eq. (3.58) yields corrections to the mode frequencies which are at least of order $k^4$. A perturbative analysis of Eq. (3.58) then leads to

$$\omega_j^2(k) = 2j(j+1)\omega_\perp^2 + \frac{1}{4}(kR_\perp)^2\omega_\perp^2 + O(k^4),$$

(3.64)

where $j = 0, 1, 2, \cdots$. In the $k \to 0$ limit, the mode frequencies are given by
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\[ \sqrt{2j(j+1)} \omega_\perp. \] The corresponding expansion coefficients, if normalized to unity, are

\[ \bar{a}_i(j, \bar{k}) = \delta_{lj} + O(k^2). \quad (3.65) \]

The \( j = 0 \) mode is of particular interest. A perturbative analysis leads to the following dispersion relation

\[ \omega_{j=0}(k) = c_0 k - \frac{1}{96} c_0 R_\perp^2 k^3 + O(k^5). \quad (3.66) \]

We see that this is an acoustic mode in the long wavelength limit with a sound speed \( c_0 \) given by

\[ c_0 = \frac{1}{2} R_\perp \omega_\perp = \sqrt{gn_0(0)/2m}. \quad (3.67) \]

Here \( n_0(0) \) is the maximum density of the condensate on the cylindrical axis. Eq. (3.67) is similar to the expression obtained for the uniform system with the mean density \( n_0 \) replaced by \( n_0(0)/2 \). This latter quantity is in fact the cross-sectional average of the density of the uniform cylindrical condensate. We also observe that the acoustic mode has a negative curvature in the TF approximation.

With the above ingredients we can now evaluate \( W_j(q_z) \) in Eq. (3.50). This quantity along with \( \omega_j(q_z) \) determines the spectral function \( S(q_z, q_z', \omega) \) by means of Eq. (3.49). Substituting Eq. (3.60) into Eq. (3.48) and using Eq. (3.63), we find the following simple result

\[ \delta n_j(q_z) = \bar{a}_0(j, \bar{q}_z) \sqrt{\nu \bar{\omega}_j(\bar{q}_z) \frac{\hbar \omega_\perp}{\mu_{TF}}}, \quad (3.68) \]

where \( \nu = N/L \) is the linear density of the cylindrical condensate and the TF chemical
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potential is given in terms of the TF radius as

$$\mu_{TF} = \frac{1}{2}m\omega_\perp^2 R_\perp^2.$$  \hfill (3.69)

We observe that of all the expansion coefficients determining the $j$-th mode, only $\bar{a}_0(j, \bar{q}_z)$ contributes to $\delta n_j(q_z)$. From Eq. (3.50) we immediately have

$$W_j(q_z) = \bar{\omega}_j(\bar{q}_z)\bar{a}_0^2(j, \bar{q}_z) \frac{\hbar\omega_\perp}{\mu_{TF}}.$$  \hfill (3.70)

It is worth pointing out that the dynamic structure factor obtained using this hydrodynamic approach is also consistent with the f-sum rule. Using Eq. (3.49) with $\delta n_j(q_z)$ given by Eq. (3.68), and recalling that $S(q_z,\omega) = S(q_z, q_z, \omega)$, we have

$$\int_0^{\infty} d\omega \omega S(q_z, \omega) = \frac{N\omega_\perp^2}{\mu_{TF}} \sum_j \bar{\omega}_j^2(q_z)\bar{a}_0^2(j, \bar{q}_z).$$  \hfill (3.71)

To evaluate the summation on the right hand side of this equation, we multiply both sides of Eq. (3.58) for $l = 0$ by $\bar{a}_0(j, \bar{k})$ and take the summation with respect to $j$. This yields

$$\frac{1}{2} \sum_j \bar{\omega}_j^2(\bar{k})\bar{a}_0^2(j, \bar{k}) - \frac{1}{8} \bar{k}^2 \sum_j \bar{a}_0^2(j, \bar{k}) + \frac{1}{8} \sum_\nu M_{0\nu} \sum_j \bar{a}_\nu(j, \bar{k})\bar{a}_0(j, \bar{k}) = 0.$$  \hfill (3.72)

From the orthonormal condition $\sum_j \bar{a}_\ell(j, \bar{k})\bar{a}_\nu(j, \bar{k}) = \delta_{\ell\nu}$ and the fact that $M_{ll} = 0$, we immediately arrive at

$$\sum_j \bar{\omega}_j^2(\bar{k})\bar{a}_0^2(j, \bar{k}) = \frac{1}{4} \bar{k}^2.$$  \hfill (3.73)

Making use of the above identity, we obtain from Eq. (3.71) the f-sum rule Eq. (3.19)
for the special case of \( \mathbf{q} = \hat{z}q_z \).

In addition, a direct integration yields

\[
\int_0^\infty d\omega \frac{S(q_z, \omega)}{\omega} = \frac{N}{2mc_0^2} \sum_j \bar{a}_0^2(j, \bar{q}_z).
\]

(3.74)

This, combined with the fact that \( \lim_{\bar{q}_z \to 0} \bar{a}_0(j, \bar{q}_z) = \delta_{j0} \), leads to

\[
\lim_{q_z \to 0} \int_0^\infty d\omega S(q_z, \omega) \frac{\omega}{\omega} = \frac{N}{2mc_0^2}.
\]

(3.75)

The above expression is similar to the so-called compressibility sum rule [48] for uniform systems. We emphasize, however, that the system we are dealing with is inhomogeneous in the transverse direction, and that the result is obtained for the special case of \( \mathbf{q} = \hat{z}q_z \).

**Numerical solutions of BdeG equations**

We outline here the numerical procedure that is employed to solve the BdeG equations for the uniform cylindrical condensate [51]. Substituting the expressions for the \( m = 0 \) Bogoliubov amplitudes Eq. (3.44) into the BdeG equations Eq. (2.53) we obtain

\[
\left( \mathcal{L}_\perp + \frac{\hbar^2 k^2}{2m} \right) u_{jk0}(\rho) - g \Phi_0^2(\rho) v_{jk0}(\rho) = \hbar \omega_j(k) u_{jk0}(\rho)
\]

\[
- \left( \mathcal{L}_\perp + \frac{\hbar^2 k^2}{2m} \right) v_{jk0}(\rho) + g \Phi_0^2(\rho) u_{jk0}(\rho) = \hbar \omega_j(k) v_{jk0}(\rho),
\]

(3.76)

where

\[
\mathcal{L}_\perp \equiv -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{1}{2} m \omega_\perp^2 \rho^2 - \mu + 2g \Phi_0^2(\rho).
\]

(3.77)
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The condensate wave function $\Phi_0(\rho)$ and the chemical potential $\mu$ are obtained by numerically solving the following GP equation

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{1}{2} m \omega_\perp^2 \rho^2 + g \Phi_0^2(\rho) \right] \Phi_0(\rho) = \mu \Phi_0(\rho) \quad (3.78)$$

with the normalization

$$2\pi \int_0^\infty d\rho \rho \Phi_0^2(\rho) = \nu. \quad (3.79)$$

In numerical solutions of the GP equation and the BdeG equations, it is convenient to express all lengths in units of the oscillator length $a_\perp = (\hbar/m \omega_\perp)^{1/2}$ and all frequencies in units of the radial trapping frequency $\omega_\perp$. In doing so, one finds that the solutions depend on a single characteristic system parameter [51], namely

$$\eta \equiv \frac{\mu_{\text{TF}}}{\hbar \omega_\perp} = \sqrt{4a_s \nu}. \quad (3.80)$$

The self-consistent solution of Eq. (3.78) is conveniently achieved using the method of imaginary time propagation [51]. In Fig. 3.1 the condensate wave functions determined numerically are compared to the corresponding TF solutions for condensates with $\eta = 10$ and $\eta = 70$. As we can see, for both condensates the TF solution agrees rather well with the exact solution except at the edge of the condensate. In general, the accuracy of the TF solution improves as the parameter $\eta$ increases. This can be seen from the fact the non-linear term in Eq. (3.78) is proportional to $g \Phi_0^2(\rho) \propto a_s \nu \propto \eta^2$. Thus the relative importance of the kinetic energy term diminishes with increasing $\eta$. 


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Figure 3.1: Condensate wave functions plotted as a function of the radial coordinate for $\eta = 10$ (left figure) and $\eta = 70$ (right figure). The wave functions are in units of $\sqrt{\nu/a_\perp}$ and the radial coordinate is in units of $a_\perp$. We point out that for a given scattering length, the value of $\sqrt{\nu/a_\perp}$ scales with $\eta$. The solid lines represent the numerical solutions to the GP equation and the dashed lines represent the TF approximation.

To solve Eq. (3.76), we represent $u_{jk0}(\rho)$ and $v_{jk0}(\rho)$ by the following Fourier-Bessel series

$$u_{jk0}(\rho) = \sum_i \frac{\sqrt{2}}{\rho_c J_1(\beta_i)} c_i(k) J_0(\beta_i \rho / \rho_c)$$
$$v_{jk0}(\rho) = \sum_i \frac{\sqrt{2}}{\rho_c J_1(\beta_i)} d_i(k) J_0(\beta_i \rho / \rho_c),$$

where $\beta_i$ is the $i$-th zero of the Bessel function of the first kind $J_0(x)$, and $\rho_c$ specifies a cut-off radius which is chosen to be sufficiently large to ensure that all localized quantities are unaffected by the choice. The set of Bessel functions satisfy the following orthonormal relations

$$\int_0^{\rho_c} d\rho \rho J_0(\beta_i \rho / \rho_c) J_0(\beta_j \rho / \rho_c) = \delta_{ij} \frac{\rho_c^2 J_1^2(\beta_i)}{2}.$$
By means of this relation, the normalization of the expansion coefficients can be obtained from that of the Bogoliubov amplitudes and is given by

\[
\sum_i \left[ c_i^2(k) - d_i^2(k) \right] = 1.
\] (3.83)

With the series expansions in Eq. (3.81), Eqs. (3.76) are transformed into a matrix eigenvalue problem, which is then solved numerically for the frequencies \( \omega_i(k) \) and the expansion coefficients \( c_i(k) \) and \( d_i(k) \).

Figure 3.2: Bogoliubov frequencies \( \omega_j(k) \) plotted as a function of axial wave vector \( k \) for \( \eta = 10 \) (left figure) and \( \eta = 70 \) (right figure). The solid lines represent results from the full BdeG solutions and the dashed lines represent results from the TF hydrodynamic theory. The wave vector is in units of \( a_{\perp}^{-1} \) and the frequency is in units of \( \omega_{\perp} \).

In the following we present the results obtained from both the TF hydrodynamic approach and the full Bogoliubov theory for the Bogoliubov mode frequencies \( \omega_j(k) \) and the dimensionless spectral weights \( W_j(k) \) defined in Eq. (3.50), which are used to determine the spectral function. In Fig. 3.2 the Bogoliubov mode dispersions are plotted for condensates with \( \eta = 10 \) and \( \eta = 70 \); the spectral weights of the first four
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Figure 3.3: Dimensionless spectral weight $W_j(k)$ plotted as a function of axial wave vector $k$ for $\eta = 10$ (left figure) and $\eta = 70$ (right figure); the numbers indicate the branch index $j$. The solid lines represent results from the full BdeG solutions and the dashed lines represent results from the TF hydrodynamic theory.

radial modes are shown in Fig. 3.3. For the condensates considered, there is generally good agreement between the two approaches for the lowest few modes for $ka_\perp \leq 1$. In particular the speed of sound obtained from the TF approach, namely Eq. (3.67), agrees very well with that found by solving the Bogoliubov equations. To facilitate the comparison, we rewrite Eq. (3.67) as

$$c_0 = \sqrt{\frac{\eta}{2}a_\perp \omega_\perp}.$$  \hfill (3.84)

The agreement in the speed of sound obtained from both approaches is illustrated in Fig. 3.4. In the short wavelength limit (large $k$), the numerical solutions show that the Bogoliubov modes exhibit a particle-like dispersion, namely $\omega_j(k) \to \hbar^2 k^2 / 2m$. This behaviour is not captured in the TF approach since it arises from the gradient terms in Eq. (2.112) which are neglected in the TF approximations.
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Figure 3.4: Speed of sound plotted as a function of the dimensionless parameter $\eta$. The solid line represents the results given by Eq. (3.84); the circles represent the results obtained from the numerical solutions of BdeG equations.

Figure 3.5: The mode density fluctuations $\delta n_{j\ell 0}(\rho)$ for $j = 0, 1$ and $ka_\perp = 0.1$ plotted as a function of the radial coordinate for $\eta = 10$ (left figure) and $\eta = 70$ (right figure). The solid lines represent results from the full Bogoliubov solutions and the dashed lines represent results from the TF hydrodynamic theory, given according to Eq. (3.60). The numbers indicate the branch index $j$. 
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Finally we show in Fig. 3.5 and Fig. 3.6 a few examples of the mode density fluctuations $\delta n_{jko}(\rho)$ obtained by both approaches. The results are qualitatively similar except in the vicinity of $R_{\text{TF}}$. The BdeG density fluctuations vary continuously with $\rho$ and decay to zero smoothly at the edge of the condensate. In the TF approximation on the other hand, the density fluctuations drop to zero discontinuously at $\rho = R_{\text{TF}}$. This unphysical behaviour is associated with the fact that the equilibrium TF density goes to zero abruptly at $\rho = R_{\text{TF}}$. In spite of this, we see that most of the important physical properties (see Figs. 3.2-3.4) are well-described by the TF approach in the long wave length limit.

3.4 Elongated condensate: local density approximations

In this section we make use of a local density approximation (LDA) to determine the density response function of interest for an elongated condensate. To motivate this approximation, we first discuss the more commonly used bulk LDA [52], which can be formulated in terms of an approximation to the density response function of
3.4. ELONGATED CONDENSATE: LOCAL DENSITY APPROXIMATIONS

a general inhomogeneous system having a density \( n_0(r) \).

Introducing the variable transformations \( \bar{r} = (r + r')/2 \) and \( s = r - r' \), we have

\[
\chi(r, r', \omega) = \chi \left( \bar{r} + \frac{1}{2}s, \bar{r} - \frac{1}{2}s, \omega \right),
\]

(3.85)

where

\[
\chi(r, r', \omega) \equiv \int d\tau e^{i\omega\tau} \chi(r, r', \tau).
\]

(3.86)

The bulk LDA is then based on the assumption that the range of \( \chi(\bar{r} + s/2, \bar{r} - s/2, \omega) \) as a function of \( s \) is small on the scale of the spatial variations of the system. In this situation, we can approximate \( \chi \) as

\[
\chi \left( \bar{r} + \frac{1}{2}s, \bar{r} - \frac{1}{2}s, \omega \right) \approx \chi_{\text{bulk}}(s, \omega; n_0(\bar{r})),
\]

(3.87)

where \( \chi_{\text{bulk}}(s, \omega; n_0(\bar{r})) \) is the density response function for a uniform system having a density equal to the local density \( n_0(\bar{r}) \) at the position \( \bar{r} \) of the inhomogeneous system. (Note that as a result of translational invariance in the homogeneous system, \( \chi_{\text{bulk}}(r, r', \omega) = \chi_{\text{bulk}}(r - r', 0, \omega) \equiv \chi_{\text{bulk}}(s, \omega) \)). Since the Fourier transform \( \chi(q, q', \omega) \) is given by

\[
\chi(q, q', \omega) = \int d\bar{r} \int d\bar{r}' e^{-i(q - q') \cdot \bar{r}} \chi(\bar{r}, \bar{r}', \omega)
\]

\[
= \int d\bar{r} e^{-i(q - q') \cdot \bar{r}} \int ds e^{-i(q + q') \cdot s/2} \chi \left( \bar{r} + \frac{1}{2}s, \bar{r} - \frac{1}{2}s, \omega \right),
\]

(3.88)

the bulk LDA leads to

\[
\chi(q, q', \omega) \simeq \int d\bar{r} e^{-i(q - q') \cdot \bar{r}} \chi_{\text{bulk}} \left( \frac{q + q'}{2}, \omega; n_0(\bar{r}) \right),
\]

(3.89)
where
\[ \chi_{\text{bulk}}(\mathbf{q}, \omega; n_0(\bar{r})) \equiv \int d\mathbf{s} e^{-i\mathbf{q} \cdot \mathbf{s}} \chi_{\text{bulk}}(\mathbf{s}, \omega; n_0(\bar{r})). \] (3.90)

It is straightforward to show that the bulk LDA can be expressed alternatively in terms of the spectral function as
\[ S_{\text{LDA bulk}}(\mathbf{q}, \mathbf{q}', \omega) = \int d\bar{r} e^{-i(\mathbf{q} - \mathbf{q}') \cdot \bar{r}} \bar{S}_{\text{bulk}} \left( \frac{\mathbf{q} + \mathbf{q}'}{2}, \omega; n_0(\bar{r}) \right), \] (3.91)
where \( S_{\text{bulk}} \equiv S_{\text{bulk}}/\Omega \) and \( S_{\text{bulk}} \) is the structure factor of a homogeneous system. In the Bogoliubov approximation, one finds from Eq. (3.43) that
\[ \bar{S}_{\text{bulk}}(\mathbf{q}, \omega; n_0(\bar{r})) = n_0(\bar{r}) \frac{\varepsilon_{\mathbf{q}}}{E_{\mathbf{q}}(n_0)} \delta \left( \hbar \omega - \hbar \omega_{\mathbf{q}}(n_0) \right), \] (3.92)
where \( E_{\mathbf{q}}(n_0) = \hbar \omega_{\mathbf{q}}(n_0) \) is the local excitation energy of the inhomogeneous system, namely the excitation energy of a uniform condensate with a density equal to the local density \( n_0(\bar{r}) \) of the inhomogeneous system.

Let us consider the spectral function calculated using the bulk LDA in two special cases, i.e., \( \mathbf{q}' = \pm \mathbf{q} \). Since \( S_{\text{bulk}}(\mathbf{q} = 0, \omega) = 0 \), we immediately have
\[ S_{\text{LDA bulk}}^{\text{LDA}}(\mathbf{q}, -\mathbf{q}, \omega) = 0. \] (3.93)

Setting \( \mathbf{q}' = \mathbf{q} \) and substituting Eq. (3.92) into Eq. (3.91), we obtain the dynamic structure factor
\[ S_{\text{bulk}}^{\text{LDA}}(\mathbf{q}, \omega) = \int d\mathbf{r} n_0(\mathbf{r}) \frac{\varepsilon_{\mathbf{q}}}{\hbar \omega} \delta \left( \hbar \omega - \hbar \omega_{\mathbf{q}}(n_0) \right). \] (3.94)
A closed form expression can be obtained for this quantity if the condensate density is approximated by the TF result in Eq. (2.21). One finds [52]

\[
S_{\text{LDA}}^\text{bulk}(q, \omega) = \frac{15N}{4\mu_{\text{TF}}} \left( \frac{(\hbar\omega)^2 - \varepsilon_q^2}{2\varepsilon_q\mu_{\text{TF}}} \right) \sqrt{1 - \frac{(\hbar\omega)^2 - \varepsilon_q^2}{2\varepsilon_q\mu_{\text{TF}}}},
\]

for \( \varepsilon_q < |\hbar\omega| < \sqrt{\varepsilon_q^2 + 2\varepsilon_q\mu_{\text{TF}}} \). Outside the range of frequencies specified by this inequality, the bulk LDA dynamic structure factor is zero. It is worth pointing out that \( S_{\text{LDA}}^\text{bulk}(q, \omega) \) given in Eq. (3.95) is an isotropic function of \( q \), even if the condensate is anisotropic. This is one limitation of the bulk LDA which makes it inaccurate in the case of a highly anisotropic condensate.

We now consider a condensate that is tightly confined along the radial direction and highly elongated along the \( z \) (axial) direction. We will see in later chapters that the physical quantity required is the integrated response function

\[
\bar{\chi}(z, z', \omega) \equiv \int dr_\perp \int dr'_\perp \chi(r, r', \omega)
\]

and its spatial Fourier transform

\[
\bar{\chi}(q_z, q'_z, \omega) = \int dz \int dz' e^{-i q_z z + i q'_z z'} \bar{\chi}(z, z', \omega).
\]

Due to the tight confinement in the radial direction, the density varies rapidly in this direction and the bulk LDA as given by Eq. (3.87) ceases to be a good approximation. However, an alternative approximation is available which makes use of the fact that the linear density varies relatively slowly along the axial direction. If the elongated condensate is divided along the axial direction into small segments, the linear density
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within each is relatively constant. The essential idea is to think of each segment as a uniform cylindrical condensate having a linear density corresponding to this part of the inhomogeneous condensate. The response properties of the segment are then approximated by those of a uniform cylindrical condensate. These physical ideas constitute what we refer to as the cylindrical LDA. To quantify the approximation, we make the variable transformations \( \bar{z} = (z + z')/2 \) and \( s = z - z' \) in \( \tilde{\chi}(z, z', \omega) \) and write

\[
\tilde{\chi}(z, z', \omega) = \tilde{\chi}\left(\bar{z} + \frac{1}{2}s, \bar{z} - \frac{1}{2}s, \omega\right). 
\] (3.98)

Assuming that the range of \( \tilde{\chi}(\bar{z} + s/2, \bar{z} - s/2, \omega) \) as a function of \( s \) is small compared to the axial distance over which the linear density has an appreciable variation, we then make the approximation

\[
\tilde{\chi}\left(\bar{z} + \frac{1}{2}s, \bar{z} - \frac{1}{2}s, \omega\right) \simeq \tilde{\chi}_{cyl}(s, \omega; \nu(\bar{z})), 
\] (3.99)

where \( \tilde{\chi}_{cyl}(s, \omega; \nu(\bar{z})) \equiv \tilde{\chi}_{cyl}(z, z', \omega) = \tilde{\chi}(z - z', 0, \omega) \) is the response function of a uniform cylindrical condensate having a linear density equal to the local linear density \( \nu(\bar{z}) \) at the axial position \( \bar{z} \) of the elongated condensate.

The Fourier transform of Eq. (3.98) is

\[
\chi(q_z, q'_z, \omega) = \int d\bar{z}e^{-i(q_zq'_z)\bar{z}} \int ds e^{-i(q_zq'_z)s/2} \tilde{\chi}\left(\bar{z} + \frac{1}{2}s, \bar{z} - \frac{1}{2}s, \omega\right). 
\] (3.100)

Using the approximation in Eq. (3.99), \( \chi(q_z, q'_z, \omega) \) in the cylindrical LDA is then given by

\[
\chi(q_z, q'_z, \omega) \simeq \int d\bar{z}e^{-i(q_zq'_z)\bar{z}} \tilde{\chi}cyl\left(\frac{q_z + q'_z}{2}, \omega; \nu(\bar{z})\right). 
\] (3.101)
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where

\[ \bar{\chi}_{cyl}(q_z, \omega; \nu(\bar{z})) \equiv \int ds e^{-iq_zs} \bar{\chi}_{cyl}(s, \omega; \nu(\bar{z})). \]  

(3.102)

The cylindrical LDA also yields an approximation to the spectral function. It is given by

\[ S_{cyl}^{LDA}(q_z, q'_z, \omega) = \int d\bar{z} e^{-i(q_z-q'_z)\bar{z}} \bar{S}_{cyl}\left(\frac{q_z + q'_z}{2}, \omega; \nu(\bar{z})\right), \]  

(3.103)

where \( \bar{S}_{cyl}(q_z, \omega) \equiv S_{cyl}(q_z, \omega)/L. \) From Eq. (3.49) one finds

\[ \bar{S}_{cyl}(q_z, \omega; \nu(\bar{z})) = \nu(\bar{z}) \sum_j W_j(q_z; \nu) \delta(h\omega - h\omega_j(q_z; \nu)), \]  

(3.104)

where \( W_j(q_z; \nu) \) and \( \omega_j(q_z; \nu) \) are determined for a uniform cylindrical condensate with a linear density equal to the local linear density \( \nu(\bar{z}) \) of the elongated condensate.

As found in the bulk LDA, we have

\[ S_{cyl}^{LDA}(q_z, -q_z, \omega) = 0 \]  

(3.105)

due to the fact that \( S_{cyl}(q_z = 0, \omega) = 0. \) However, unlike the bulk LDA, we do not have a closed analytic expression for \( S_{cyl}^{LDA}(q_z, \omega). \) Using Eq. (3.104) in Eq. (3.103) for \( q_z = q'_z, \) one finds

\[ S_{cyl}^{LDA}(q_z, \omega) = \int d\bar{z} \nu(\bar{z}) \sum_j W_j(q_z; \nu(\bar{z})) \delta(h\omega - h\omega_j(q_z; \nu(\bar{z}))) \]

\[ = \frac{1}{\hbar} \sum_j \frac{\nu(\bar{z})W_j(q_z; \nu)}{|\partial\omega_j(q_z; \nu(\bar{z}))/\partial\bar{z}|} \bigg|_{\bar{z} = \bar{z}_j}, \]  

(3.106)
where $\bar{z}_j$ is a function of $q_z$ and $\omega$ and is defined as the solution to the equation

$$\omega = \omega_j(q_z; \nu(\bar{z}_j)).$$

From Eq. (3.106) we see that $S_{cyl}^{\text{LDA}}(q_z, \omega)$ may exhibit singularities at frequencies given by

$$\omega = \omega_j(q_z; \nu(\bar{z}_s)), \quad \text{(3.108)}$$

where $\bar{z}_s$ are determined by

$$\frac{\partial}{\partial \bar{z}_s} \omega_j(q_z; \nu(\bar{z}_s)) = 0. \quad \text{(3.109)}$$

It should be noted, however, that these singularities are integrable and do not lead to any difficulties in calculations. This can be most easily seen by integrating the first line of Eq. (3.106) with respect to $\omega$. One finds a well-defined static structure factor given by

$$S_{cyl}^{\text{LDA}}(q_z) \equiv \int d\omega S_{cyl}^{\text{LDA}}(q_z, \omega) = \frac{1}{\hbar} \int d\bar{z} \nu(\bar{z}) \sum_j W_j(q_z; \nu(\bar{z})). \quad \text{(3.110)}$$

In fact, when the dynamic structure factor is used later in determining physical quantities of interest, Eq. (3.106) does not need to be evaluated explicitly. These calculations involve integrations with respect to the frequency variable $\omega$ of the kind that led to Eq. (3.110).

To illustrate the difference between the bulk and cylindrical LDAs, we consider the dynamic structure factor $S(q_z, \omega)$ for an elongated condensate of the kind studied
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experimentally [53]. Specifically we consider a condensate consisting of $N = 10^5$ $^{87}\text{Rb}$ atoms in a trap with $\omega_\perp = 2\pi \times 220\text{Hz}$ and $\omega_z = 2\pi \times 25\text{Hz}$. The linear density appearing in Eq. (3.106) is determined within the TF approximation, which is sufficiently accurate for a large condensate. From Eq. (2.21) we find that the radial extent of the condensate at axial position $z$ is given by

$$R_\perp(z) = \lambda R_z \sqrt{1 - \frac{z^2}{R_z^2}},$$

(3.111)

where $R_z$ is the extent of the condensate in the axial direction and $\lambda = \omega_z/\omega_\perp$ is the aspect ratio of the trap. The density per unit length $\nu(z)$ is thus given by

$$\nu(z) = \int_0^{2\pi} d\phi \int_0^{R_\perp(z)} d\rho\, n(r)$$

$$= \frac{\pi m \omega_\perp^2}{4g} R_\perp^4(z)$$

$$= \frac{\pi m \omega_\perp^2 \lambda^4 R_z^4}{4g} \left(1 - \frac{z^2}{R_z^2}\right)^2.$$

(3.112)

As we can see from the plot of the linear density in Fig. 3.7, the centre of the condensate gives the largest contribution to the dynamic structure factor given in Eq. (3.106).

In Fig. 3.8 we plot $S^{\text{LDA}}_{\text{cyl}}(q_z, \omega)$ for this condensate as a function of frequency for $q_z a_\perp = 8$. The results in this figure are obtained by replacing the delta function in Eq. (3.106) by a Lorentzian

$$\frac{1}{\pi \hbar (\omega - \omega_j)^2 + \Gamma^2},$$

(3.113)

and then performing the integration with respect to $\tilde{z}$ numerically. For comparison, the result from the bulk LDA is also shown. We observe that the dynamic structure factor obtained in the cylindrical LDA exhibits multiple peaks as a function of the
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Figure 3.7: The (normalized) linear density $\nu(z)/\nu(0)$ plotted as a function of $z/R_z$.

Figure 3.8: Dynamic structure factor $S(q_z, \omega)$ plotted as a function of frequency for $qa_\perp = 8$. The solid line represents the result from the cylindrical LDA. The singularities are broadened as a result of the Lorenzian approximation to the delta function in Eq. (3.106). The width of the Lorenzian in Eq. (3.113) was taken to be $\Gamma = 0.05\omega_\perp$. The dashed line represents the result from the bulk LDA, given in Eq. (3.95).
frequency, which reflects the fact that the highly elongated condensate still preserves a multi-branch structure in the Bogoliubov excitation spectrum. It is also important to point out that in the limit that the aspect ratio $\lambda = \omega_z/\omega_\perp$ vanishes, the system returns to a uniform cylindrical condensate and the cylindrical LDA becomes exact; the bulk LDA remains an approximation in this limit. For these reasons, the cylindrical LDA is expected to be a more accurate approximation than the bulk LDA for highly elongated condensates.

Lastly, we point out that the dynamic structure factors obtained from both LDAs satisfy the f-sum rule. For the bulk LDA, a direct integration of Eq. (3.95) yields

$$\int_0^\infty d\omega \omega S_{\text{LDA bulk}}(q,\omega) = \frac{Nq^2}{2m}. \quad (3.114)$$

For the cylindrical LDA, Eq. (3.103) gives

$$S_{\text{LDA cyl}}(q_z,\omega) = \int d\bar{z} S_{\text{cyl}}(q_z,\omega; \nu(\bar{z})). \quad (3.115)$$

As shown earlier, $\tilde{S}_{\text{cyl}}(q_z,\omega)$ satisfies the sum rule

$$\int_0^\infty d\omega \omega \tilde{S}_{\text{cyl}}(q_z,\omega; \nu) = \frac{\nu q_z^2}{2m}. \quad (3.116)$$

As a result, we find from Eq. (3.115) that

$$\int_0^\infty d\omega \omega S_{\text{cyl}}^{\text{LDA}}(q_z,\omega) = \int d\bar{z} \nu(\bar{z}) q_z^2 = \frac{Nq_z^2}{2m}. \quad (3.117)$$

Equation (3.117) also serves as a useful check of the accuracy of the numerical
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Figure 3.9: The first moment of the dynamic structure factor (in units of $N\omega_\perp/h$) plotted as a function of wave vector $q_z$. The circles represent the first moment of the dynamic structure factor calculated using the cylindrical LDA and the solid line represents the function $Nq_z^2/2m$ (in units of $N\omega_\perp/h$).

The above expression is then numerically evaluated and compared to the right hand side of Eq. (3.117). The agreement shown in Fig. 3.9, is an indication that our numerical procedure is accurate.

In summary, we have determined density response functions within the Bogoliubov theory for both a uniform condensate and a uniform cylindrical condensate. These results were then used to define two different local density approximations, the bulk LDA and the cylindrical LDA. Both of these can be applied to obtain an approximate density response function for an elongated condensate. As argued above, we expect
the cylindrical LDA to be a superior approximation since it incorporates the strong transverse confinement of the elongated condensate.
Chapter 4

Dissipation of the uniform flow of Bose condensates

In this chapter we address the question of dissipation in the flow of a Bose-Einstein condensate. This question is intimately associated with the possibility of superfluidity as first exhibited by liquid helium in various physical situations. A classic example concerns the flow of the liquid through a narrow capillary [54]. Below the lambda-point it is found that the liquid can flow through the capillary without dissipation if the flow velocity is sufficiently small. In other words, the liquid exhibits no viscosity. It is for this reason that it is referred to as a superfluid. Similar behaviour is also exhibited by Bose-condensed atomic gases although manifestations of superfluidity are not as directly apparent due to the spatial confinement of the atomic cloud. Nevertheless, confirmation of superfluidity in these systems is provided by various experiments, most notably by the observation of quantized vortices [12].

The breakdown of superfluidity and the onset of dissipation is associated with the interaction of a flowing Bose condensate with some external perturbation. The
perturbation leads to the creation of excitations which convert the kinetic energy associated with the macroscopic flow into heat. As a result, the condensate experiences a frictional force and the flow decays as a function of time. These effects can be illustrated by considering model perturbations which correspond to physical scenarios that have been investigated experimentally. In particular, we consider in this chapter perturbations which specifically couple to the particle density. If the perturbation is sufficiently weak, the response of the system can be described quite generally in terms of density response functions. The response functions obtained in the previous chapter can be used to describe the dissipative processes of interest.

4.1 Linear response theory of energy dissipation

In this section we present a general framework which allows us to calculate the energy dissipation of a flowing Bose condensate. The formulation of the problem is motivated by considering a uniform condensate flowing with velocity \(-\mathbf{v}\) past a fixed potential perturbation \(V_{\text{ext}}(\mathbf{r})\). In this context, energy dissipation is understood to be the process by which the kinetic energy of the moving system is converted into thermal energy by means of the interactions between the atoms and the external potential. The thermal energy produced in this process is referred to as the dissipated energy. By making a Galilean transformation to the rest frame of the condensate, we arrive at an alternative but physically equivalent situation in which the external potential moves at a velocity \(\mathbf{v}\) with respect to a stationary condensate. In this case, the moving potential perturbs the system from its equilibrium state and imparts energy to the system. Galilean invariance implies that the increase in energy of the stationary condensate resulting from internal excitations is the same as the energy dissipated by the moving
condensate. This second point of view can be extended to inhomogeneous condensates such as trapped atomic gases. Here the condensate is necessarily localized and it is then more natural to consider the external potential as a dynamic perturbation. A possible physical realization would be the passage of a particle through a condensate which is confined within a trap.

In the absence of the moving external potential, the Hamiltonian of the stationary condensate is \( \hat{H} \), as given in Eq. (2.24). The interaction between the moving potential and the condensate takes the form

\[
\hat{V}_{\text{ext}}(t) = \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i - vt) = \int d\mathbf{r} V_{\text{ext}}(\mathbf{r} - vt) \hat{n}(\mathbf{r}),
\]

where \( V_{\text{ext}}(\mathbf{r}) \) is the external potential and \( \hat{n}(\mathbf{r}) \) is the density operator. For the moment, we do not need to specify the form of the external potential, but in later sections we will consider, as examples, a spatially localized impurity potential and a periodical lattice potential, both of which can be realized in cold atom experiments.

We assume that the perturbation is absent for \( t < t_0 \), and that the system is in its ground state \( |\Psi_0\rangle \). For \( t > t_0 \), the dynamic state of the condensate \( |\Psi(t)\rangle \) evolves according to

\[
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{\mathcal{H}}(t)|\Psi(t)\rangle,
\]

where \( \hat{\mathcal{H}}(t) = \hat{H} + \hat{V}_{\text{ext}}(t) \) is the full Hamiltonian of the system. The rate of change of the total energy \( E(t) = \langle \Psi(t)|\hat{\mathcal{H}}|\Psi(t)\rangle \) is given by

\[
\frac{dE}{dt} = \left< \Psi(t) \left| \frac{\partial \hat{V}_{\text{ext}}(t)}{\partial t} \right| \Psi(t) \right> = -\mathbf{v} \cdot \int d\mathbf{r} \nabla V_{\text{ext}}(\mathbf{r} - vt) n(\mathbf{r}, t),
\]

where \( n(\mathbf{r}, t) = \langle \Psi(t)|\hat{n}(\mathbf{r})|\Psi(t)\rangle \) is the expectation value of the density. We observe
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that the integral in Eq. (4.3) is just the total force

\[ F(t) = -\int dr \nabla V_{\text{ext}}(r - vt)n(r, t) \] (4.4)

exerted by the potential on the system. We then have

\[ \frac{dE}{dt} = F(t) \cdot v. \] (4.5)

We refer to \( F \) as the drag force on the condensate.

Writing \( n(r, t) = n_{eq}(r) + \delta n(r, t) \), where \( n_{eq}(r) = \langle \Psi_0 | \hat{n}(r) | \Psi_0 \rangle \) is the initial equilibrium density of the condensate and \( \delta n(r, t) \) is the density fluctuation induced by the moving potential, Eq. (4.3) can be written as

\[ \frac{dE}{dt} = -v \cdot \int dr \nabla V_{\text{ext}}(r - vt)n_{eq}(r) - v \cdot \int dr \nabla V_{\text{ext}}(r - vt)\delta n(r, t). \] (4.6)

The first term on the right hand side of Eq. (4.6) is due to the force the external potential exerts on the system in its equilibrium state. For a uniform system, \( n_{eq}(r) = \bar{n}, \) this term vanishes identically. On the other hand, if \( n_{eq}(r) \) is localized, this term is equal to

\[ \frac{\partial}{\partial t} \int dr V_{\text{ext}}(r - vt)n_{eq}(r). \] (4.7)

This is just the time variation of the potential energy of the equilibrium density in the moving potential. Integrating this over all times gives a vanishing result if \( V_{\text{ext}}(r) \) is a localized potential. Thus this term has no net effect and can be ignored. In any case it is of no interest since it has nothing to do with the internal excitations of the system which are accounted for in the second term. The energy dissipation rate of
interest is then given by the second term, i.e.,

\[
\frac{dE}{dt} = -\mathbf{v} \cdot \int d\mathbf{r} \nabla V_{\text{ext}}(\mathbf{r} - \mathbf{vt}) \delta n(\mathbf{r}, t). \tag{4.8}
\]

We now assume that the external potential is weak, in which case the density fluctuation can be determined by linear response theory as

\[
\delta n(\mathbf{r}, t) = -\int d\mathbf{r}' \int_{t_0}^{t} dt' \chi(\mathbf{r}, \mathbf{r}', t - t') V_{\text{ext}}(\mathbf{r}' - \mathbf{v}t'), \tag{4.9}
\]

where the ground state density response function \( \chi(\mathbf{r}, \mathbf{r}', t - t') \) is defined in Eq. (3.1). Substituting Eq. (4.9) into Eq. (4.8) and taking Fourier transforms, we obtain the energy dissipation rate in the form

\[
\frac{dE}{dt} = -\frac{1}{\Omega^2} \sum_{\mathbf{q}, \mathbf{q}'} i\mathbf{q} \cdot \mathbf{v} \tilde{V}_{\text{ext}}^{*}(\mathbf{q}) \tilde{V}_{\text{ext}}(\mathbf{q}') \int_{t_0}^{t} dt' e^{i(q \cdot \mathbf{v} t - q' \cdot \mathbf{v} t')} \chi(\mathbf{q}, \mathbf{q}', t - t'), \tag{4.10}
\]

where \( \Omega \) is the volume of the system, \( \chi(\mathbf{q}, \mathbf{q}', t - t') \) is defined according to Eq. (3.3) and \( \tilde{V}_{\text{ext}}(\mathbf{q}) \) is the Fourier component of the external potential

\[
\tilde{V}_{\text{ext}}(\mathbf{q}) = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} V_{\text{ext}}(\mathbf{r}). \tag{4.11}
\]

Using the expression in Eq. (3.9) for \( \chi(\mathbf{q}, \mathbf{q}', \tau) \) and performing the integral in Eq. (4.10), we find

\[
\frac{dE}{dt} = \frac{1}{\Omega^2} \sum_{\mathbf{q}, \mathbf{q}'} i\mathbf{q} \cdot \mathbf{v} \tilde{V}_{\text{ext}}^{*}(\mathbf{q}) \tilde{V}_{\text{ext}}(\mathbf{q}') e^{i(q - q') \cdot \mathbf{v} t} \times \int d\omega S(\mathbf{q}, \mathbf{q}' \cdot \omega) \left\{ \frac{1 - e^{i(q' \cdot \mathbf{v} - \omega)(t - t_0)}}{\mathbf{q}' \cdot \mathbf{v} - \omega} - \frac{1 - e^{i(q' \cdot \mathbf{v} + \omega)(t - t_0)}}{\mathbf{q}' \cdot \mathbf{v} + \omega} \right\}. \tag{4.12}
\]
In view of the relation between $S(q, q', \omega)$ and $\chi''(q, q', \omega)$, as given in Eq. (3.15), we can write Eq. (4.12) alternatively as

$$\frac{dE}{dt} = \frac{1}{\pi \Omega^2} \sum_{q, q'} iq \cdot v \tilde{V}_{\text{ext}}^*(q) \tilde{V}_{\text{ext}}(q') \chi''(q, q') \frac{1 - e^{i(q' \cdot v - \omega)(t - t_0)}}{q' \cdot v - \omega} \int d\omega \chi''(q, q', \omega).$$

(4.13)

We emphasize that this is the energy dissipation rate for an arbitrary time $t > t_0$. As such, this formula, or equivalently Eq. (4.12), can be used to investigate the transients in the energy dissipation rate. Of more interest, however, is the energy dissipation rate that arises in the $t - t_0 \to \infty$ limit. The simplest way to arrive at this quantity is to take the $t_0 \to -\infty$ limit in Eq. (4.10). The upper limit of the time integral can also be extended to $\infty$ due to the step function $\theta(t - t')$ contained in the response function. Performing the time integral, we obtain

$$\frac{dE}{dt} \bigg|_{t_0 = -\infty} = \frac{1}{\Omega^2} \sum_{q, q'} iq \cdot v \tilde{V}_{\text{ext}}^*(q) \tilde{V}_{\text{ext}}(q') \chi(q, q', \omega) = q' \cdot v e^{i(q' \cdot v - \omega)(t - t_0)}.$$

(4.14)

From this result we see that the energy dissipation rate has an explicit time-dependence for a non-uniform system even in the $t - t_0 \to \infty$ limit. However, for a uniform system the energy dissipation rate eventually reaches a steady-state limit. This can be seen from the fact that $\chi(q, q', \omega) = \delta_{qq'} \chi(q, q, \omega)$ for a uniform system. Using this result in Eq. (4.14), the energy dissipation rate becomes

$$\frac{dE}{dt} \bigg|_{t_0 = -\infty} = -\frac{1}{\Omega^2} \sum_q iq \cdot v |\tilde{V}_{\text{ext}}(q)|^2 \chi(q, q, \omega) = q \cdot v,$$

(4.15)

which is time independent. In the next two sections we apply Eq. (4.12) and Eq. (4.14)
4.1. LINEAR RESPONSE THEORY OF ENERGY DISSIPATION

...to discuss a variety of physical situations.

So far we have only considered the energy imparted to a condensate by a moving potential. In the rest of this section, we show that the total momentum imparted to a trapped condensate by a moving external potential is determined by the drag force acting on the condensate. This result can be obtained by noting that the Heisenberg equation of motion for the centre of mass coordinate $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} \hat{r}_i$ leads to

$$\frac{d^2 R_\mu(t)}{dt^2} + \omega_\mu^2 R_\mu(t) = \frac{F_\mu(t)}{M},$$

(4.16)

where $R_\mu(t) = \langle \Psi(t)|\hat{R}_\mu|\Psi(t)\rangle$, $M$ is the total mass of the system and $F_\mu(t)$ is the $\mu$-component of the drag force defined in Eq. (4.4). The total momentum of the condensate is then given by

$$P_\mu(t) = M \frac{dR_\mu(t)}{dt}.$$  

(4.17)

We see that equation (4.16) describes the motion of a driven harmonic oscillator. Assuming that the condensate is stationary at time $t_0$, the solution to Eq. (4.16) is

$$R_\mu(t) = \frac{1}{M \omega_\mu} \int_{t_0}^{t} dt' \sin \omega_\mu(t - t') F_\mu(t').$$

(4.18)

Substituting Eq. (4.18) into Eq. (4.17), we find

$$P_\mu(t) = \int_{t_0}^{t} dt' \cos \omega_\mu(t - t') F_\mu(t').$$

(4.19)

This result will be used in the last section of this chapter when we discuss the Bragg spectroscopy of Bose condensates.

---

1A more detailed discussion of the centre of mass equations of motion will be given in Chapter 5.
4.2  Flow of uniform condensate

As our first example, we consider the flow of a uniform condensate. Substituting the spectral function for the uniform condensate Eq. (3.43) into Eq. (4.12) and converting the sum to an integral we obtain

\[
\frac{dE}{dt} = i \frac{n_0}{\hbar} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \mathbf{v} |\tilde{V}_{\text{ext}}(\mathbf{q})|^2 \frac{\epsilon_\mathbf{q}}{\omega_\mathbf{q}} \left\{ \frac{1 - e^{i(\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q})t}}{\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q}} - \frac{1 - e^{i(\mathbf{q} \cdot \mathbf{v} + \omega_\mathbf{q})t}}{\mathbf{q} \cdot \mathbf{v} + \omega_\mathbf{q}} \right\},
\]  

(4.20)

where the initial time \( t_0 \) of the perturbation is set to zero for convenience. Because the potential \( V_{\text{ext}}(\mathbf{r}) \) is real, the Fourier components have the property that \( \tilde{V}_{\text{ext}}^*(\mathbf{q}) = \tilde{V}_{\text{ext}}(-\mathbf{q}) \), which implies that \( |\tilde{V}_{\text{ext}}(\mathbf{q})|^2 \) is an even function of \( \mathbf{q} \). We thus see that the real part of the integrand in the above integral is odd in \( \mathbf{q} \) and the imaginary part is even. Only the latter contribution survives and we have

\[
\frac{dE}{dt} = \frac{n_0}{\hbar^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \mathbf{v} |\tilde{V}_{\text{ext}}(\mathbf{q})|^2 \frac{\epsilon_\mathbf{q}}{\omega_\mathbf{q}} \sin (\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q})t
\]

(4.21)

To obtain the steady-state energy dissipation rate, we take the \( t \to \infty \) limit in Eq. (4.21) and use the identity

\[
\lim_{t \to \infty} \frac{\sin (\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q})t}{\pi (\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q})} = \delta(\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q}).
\]  

(4.22)

This yields

\[
\left. \frac{dE}{dt} \right|_{t \to \infty} = \frac{2\pi n_0}{\hbar^2} \int \frac{d\mathbf{q}}{(2\pi)^3} |\tilde{V}_{\text{ext}}(\mathbf{q})|^2 \epsilon_\mathbf{q} \delta(\mathbf{q} \cdot \mathbf{v} - \omega_\mathbf{q}).
\]  

(4.23)
This result can be derived alternatively from Eq. (4.15). Making use of the fact that $\text{Re}\chi(q, q, \omega = q \cdot v)$ is even in $q$, Eq. (4.15) can be written as

$$
\frac{dE}{dt} \bigg|_{t_0=-\infty} = \frac{1}{\Omega^2} \sum_q q \cdot v |\tilde{V}_{\text{ext}}(q)|^2 \text{Im}\chi(q, q, \omega = q \cdot v)
$$

$$
= \frac{2\pi}{\Omega^2} \sum_q q \cdot v |\tilde{V}_{\text{ext}}(q)|^2 S(q, \omega = q \cdot v).
$$

(4.24)

Substituting Eq. (3.43) into Eq. (4.24) and converting the sum into an integral, we find again the result in Eq. (4.23).

We now assume that the external potential is isotropic. The integral in Eq. (4.23) can then be evaluated using polar coordinates. We find

$$
\frac{dE}{dt} \bigg|_{t \to \infty} = \frac{n_0}{4\pi mv} \int_0^{q_c} dq q^3 |\tilde{V}_{\text{ext}}(q)|^2 \theta(1 - \omega q / qv)
$$

$$
= \frac{n_0}{4\pi mv} \int_0^{q_c} dq q^3 |\tilde{V}_{\text{ext}}(q)|^2,
$$

(4.25)

where $q_c = \frac{2m}{\hbar} \sqrt{v^2 - c^2}$ is the solution to the equation $qv - \omega q = 0$ and $c = \sqrt{gn_0/m}$ is the speed of sound in the uniform condensate. The final result in Eq. (4.25) is valid when $v$ is greater than $c$; when the potential moves at a speed less than $c$, the steady-state energy dissipation rate vanishes. This is in agreement with the general result given by the Landau criterion [32], which states that superfluid flow in a uniform system is stable if the velocity is less than some critical value. For the dilute Bose gas, the critical velocity entering the Landau criterion is the speed of sound $c$.

The more general result in Eq. (4.21), however, indicates that the energy dissipation is in fact finite during the transient period following the introduction of the external potential. It can be seen from Eq. (4.21) that the energy dissipation rate
starts from zero at \( t = 0 \), and then becomes finite as the density fluctuation builds up. In view of the long time limit given by Eq. (4.25) for \( v > c \), we expect the energy dissipation rate to grow to the limiting value given by this equation. On the other hand, the energy dissipation rate for \( v < c \) is expected to grow to a maximum value and then eventually to diminish to zero. Detailed calculations presented in the following section confirm this expected behaviour.

### 4.2.1 Gaussian impurity

To make the above analysis more quantitative, we must specify the form of the external potential. Here we consider a Gaussian impurity potential \( V_{\text{imp}}(r) = V_0 e^{-r^2/w^2} \), where \( w \) is the width of the potential. Substituting the Fourier transform of the potential \( \tilde{V}_{\text{imp}}(q) = \frac{\pi^{3/2} V_0 w^3 e^{-w^2 q^2/4}}{2} \) into Eq. (4.25) we obtain for \( v > c \)

\[
\frac{dE}{dt}\bigg|_{t \to \infty} = \frac{V_0^2 n_0 \xi^3}{\hbar} \times \frac{\pi^2 \bar{w}^2}{\sqrt{2} \bar{v}} \left[ 1 - \left( \frac{1}{2} \bar{w}^2 \bar{q}_c^2 + 1 \right) e^{-\frac{1}{2} \bar{q}_c^2} \right],
\]

where \( \bar{q}_c = q_c \xi \), \( \bar{w} = w / \xi \) and \( \bar{v} = v / c \). Here \( \xi = \frac{\hbar}{\sqrt{2} mc} \) is the healing length of the uniform condensate.

The finite time energy dissipation rate given in Eq. (4.21) is evaluated numerically for a Gaussian impurity potential having a width of \( w = 10 \xi \). In Fig. 4.1, we plot the energy dissipation rate in units of \( V_0^2 n_0 \xi^3 / \hbar \) as a function of time for velocities above and below the speed of sound. We see that the plots confirm the general arguments given earlier. There are several details worth pointing out. For \( v > c \), the energy dissipation rate reaches its asymptotic limit in a non-monotonic fashion. For \( v < c \), the energy dissipation rate passes through a maximum and then exhibits a negative excursion before eventually going to zero. The negative values should not be cause
4.2. FLOW OF UNIFORM CONDENSATE

Figure 4.1: Energy dissipation rates (in units of $V_0^2 n_0 \xi^3 / \hbar$) as a function of time (solid lines) for the velocities $v/c = 1.2$ (left figure) and $v/c = 0.8$ (right figure). The time is in units of $\xi/c$. The dashed lines indicate the long time limits of the energy dissipation rates at the corresponding velocities; for $v > c$ it is given by Eq. (4.25) and for $v < c$ it is zero. The width of the Gaussian potential in the plots is $w = 10 \xi$. The time is in units of $\xi/c$. The dashed lines indicate the long time limits of the energy dissipation rates at the corresponding velocities; for $v > c$ it is given by Eq. (4.25) and for $v < c$ it is zero. The width of the Gaussian potential in the plots is $w = 10 \xi$.

for alarm. At these times, the transient density fluctuation is such that the net force acting on the condensate is in a direction opposite to that in which the impurity is moving. Once the steady state has been reached, the net force on the condensate vanishes. Finally, we observe that the initial transient response occurs on a time scale of $\tau \simeq 10 \xi/c \simeq w/c$. That is the time it takes a sound wave to traverse the width of the external potential. The relaxation to the steady state occurs on a time scale about ten times longer.

4.2.2 Point-like impurity

We can also consider other impurity potentials such as that produced by a massive atomic projectile of a different species. If the atoms in the condensate scatter from this massive impurity atom with an $s$-wave scattering length $b$, the impurity potential
4.2. FLOW OF UNIFORM CONDENSATE

can be approximated by the pseudopotential \( V_{\text{imp}}(r) = \frac{2\pi \hbar^2 b}{m} \delta(r) \). The drag force that such a moving point-like impurity exerts on a uniform condensate was determined in [55] and here we rederive their result using the linear response approach we have developed.

The magnitude of the drag force can be calculated easily from Eq. (4.25) and Eq. (4.5) and the result for \( v > c \) is [55]

\[
F = \frac{4\pi n_0 b^2 m c^4}{v^2} \left( v^2/c^2 - 1 \right)^2.
\] (4.27)

For \( v \gg c \), we see that \( F \propto v^2 \). However, this behaviour must eventually break down when the velocity is so high that the \( s \)-wave scattering length \( b \) no longer provides a good description of the scattering.

![Graph](image)

**Figure 4.2:** Forces (in units of \( 4\pi n_0 b^2 m c^2 \)) due to the point impurity (solid line) and Gaussian impurities (other lines) plotted as a function of \( v/c \).

Let us compare the drag force exerted on the condensate by a Gaussian impurity to that of a point-like impurity. The drag force from a moving Gaussian impurity is
determined by Eq. (4.26) and Eq. (4.5) as

\[
F = \frac{\pi^2 \hbar^2 V_0^2 n_0 \bar{w}^2}{4 m^3 c^2 v^2} \left[ 1 - \left( \frac{1}{2} \bar{w}^2 q_c^2 + 1 \right) e^{-\frac{1}{2} \bar{w}^2 q_c^2} \right]
\] (4.28)

for \( v > c \). As the speed \( v \) approaches \( c \), we observe that \( F \propto (v^2/c^2 - 1)^2 \) for both kinds of impurity potentials. However, in the large velocity limit, \( F \propto v^{-2} \) for the Gaussian impurity which differs markedly from the \( F \sim v^2 \) behaviour for the point impurity. To further facilitate the comparison between these two cases it is convenient to fix the strength of the Gaussian potential according to

\[
V_0 \int dr e^{-r^2/\bar{w}^2} = \frac{2 \pi \hbar^2 b}{m}.
\] (4.29)

In the limit that \( \bar{w} q_c \ll 1 \), we then obtain for the Gaussian impurity

\[
F = \frac{4 \pi n_0 b^2 m c^4}{v^2} (v^2/c^2 - 1)^2 \left[ 1 - \frac{1}{3} \bar{w}^2 q_c^2 + O(\bar{w}^4 q_c^4) \right]
\] (4.30)

Thus, the Gaussian impurity can be viewed as a point impurity as long as \( \bar{w} q_c \ll 1 \), that is when \( w/\xi << 1/\sqrt{v^2/c^2 - 1} \). This inequality will break down when \( v/c \) becomes sufficiently large. Detailed comparisons between these two forces are shown in Fig. 4.2 as a function of \( v \) for a range of widths of the Gaussian potential.

### 4.3 Dissipation in a cylindrical condensate

We next consider the flow of a uniform cylindrical condensate along the axial direction, where the perturbing external potential is assumed, for simplicity, to be one-dimensional, i.e., \( V_{\text{ext}}(r) = V_{\text{ext}}(z) \). In this situation, Eq. (4.12) becomes (we
4.3. DISSIPATION IN A CYLINDRICAL CONDENSATE

again take $t_0 = 0$)

$$\frac{dE}{dt} = \frac{1}{L^2} \sum_{q_z} i q_z v |\tilde{V}_{\text{ext}}(q_z)|^2 \int d\omega S(q_z, \omega) \left\{ \frac{1 - e^{i(q_z v - \omega)t}}{q_z v - \omega} - \frac{1 - e^{i(q_z v + \omega)t}}{q_z v + \omega} \right\} , \quad (4.31)$$

where

$$\tilde{V}_{\text{ext}}(q_z) = \int dz V_{\text{ext}}(z) e^{-iq_z z}. \quad (4.32)$$

Substituting $S(q_z, \omega)$ obtained from Eq. (3.49) into Eq. (4.31), we find

$$\frac{dE}{dt} = \nu \frac{v}{\pi \hbar} \sum_j \int_{-\infty}^{\infty} dq_z |\tilde{V}_{\text{ext}}(q_z)|^2 W_j(q_z) \frac{\sin \left( q_z v - \omega_j(q_z) \right) t}{q_z v - \omega_j(q_z)}. \quad (4.33)$$

The energy dissipation rate in the long time limit can be found from Eq. (4.14) as

$$\left. \frac{dE}{dt} \right|_{t_0 \to -\infty} = \frac{\nu v}{L} \int_{-\infty}^{\infty} dq_z |\tilde{V}_{\text{ext}}(q_z)|^2 \text{Im} \chi(q_z, q_z, \omega = q_z v)$$

$$= \frac{\nu}{L} \int_{0}^{\infty} dq_z |\tilde{V}_{\text{ext}}(q_z)|^2 S(q_z, \omega = q_z v). \quad (4.34)$$

As a consequence of the f-sum rule for the dynamic structure factor, we find from Eq. (4.34) the simple relation

$$\int_{0}^{\infty} dv \left. \frac{dE}{dt} \right|_{t_0 \to -\infty} = \frac{\nu}{2m} \int_{0}^{\infty} dq_z |\tilde{V}_{\text{ext}}(q_z)|^2. \quad (4.35)$$

This expression can be thought of as a velocity sum rule for the energy dissipation rate.

Substituting the expression for the dynamic structure factor $S(q_z, \omega)$, given in
Eq. (3.49), into Eq. (4.34), we find

\[
\left. \frac{dE}{dt} \right|_{t_0 \to -\infty} = \frac{\nu \nu}{\hbar} \sum_j \int_0^\infty dq_z q_z |\tilde{V}_{\text{ext}}(q_z)|^2 W_j(q_z) \delta(q_zv - \omega_j(q_z)).
\]

(4.36)

Alternatively, this result can also be obtained from Eq. (4.33) by taking the limit \( t \to \infty \).

In the long time limit, there is a well-defined critical velocity for the onset of energy dissipation for the cylindrical condensate, just as in the case of uniform condensate. However, it is important to note that, unlike the uniform condensate, the critical velocity here is not the speed of sound of the system. This is due to the special nature of the dispersion of the excitations in a cylindrical condensate.

To be specific, we consider a uniform cylindrical condensate with \( \eta = 40 \). It is clear from Eq. (4.36) that dissipation of energy occurs if there are solutions to at least one of the following equations

\[
v - v_j(q_z) = 0,
\]

(4.37)

where \( v_j(q_z) \) is the phase velocity of the \( j \)-th mode

\[
v_j(q_z) = \frac{\omega_j(q_z)}{q_z}.
\]

(4.38)

In view of the condensate modes plotted in Fig. 4.3 we see that dissipation sets in when \( v \) exceeds the minimum phase velocity of the lowest \( j = 0 \) mode, namely

\[
v \geq \min \{v_0(q_z)\}.
\]

(4.39)
4.3. DISSIPATION IN A CYLINDRICAL CONDENSATE

A unique critical velocity is then given by

\[ v_{cr} = v_0(q_z = q_{0,cr}), \]  

(4.40)

where \( q_{0,cr} \) is the wave vector at which the phase velocity \( v_0(q_z) \) assumes its minimum value. This is illustrated in Fig. 4.3, where for this specific condensate with \( \eta = 40 \), the critical velocity is found to be \( v_{cr} = 0.57 c_0 \) and \( q_{0,cr} \) is found to be \( 2.2 a_\perp^{-1} \).

If we consider the contribution to the energy dissipation arising from each of the radial modes individually, we can introduce a critical velocity for each. These critical velocities are defined by

\[ v_{j,cr} \equiv v_j(q_z = q_{j,cr}), \]  

(4.41)

where \( q_{j,cr} \) is the wave vector at which the phase velocity \( v_j(q_z) \) has a minimum. Of course the actual critical velocity is the smallest of all the mode-dependent critical velocities, namely \( v_{cr} = v_{0,cr} \). For \( v < v_{j,cr} \) the dissipation rate from the \( j \)-th mode is zero; it turns on for \( v > v_{j,cr} \) and is given by

\[
\left. \frac{dE_j}{dt} \right|_{t_0 \to -\infty} = \frac{\nu v}{\hbar} \sum_\alpha q_{j,\alpha} |\tilde{V}_{ext}(q_{j,\alpha})|^2 \frac{W_j(q_{j,\alpha})}{|\omega'_j(q_{j,\alpha}) - v|} \\
= \frac{\nu v}{\hbar} \sum_\alpha |\tilde{V}_{ext}(q_{j,\alpha})|^2 \frac{W_j(q_{j,\alpha})}{|v'_j(q_{j,\alpha})|},
\]

(4.42)

where the wavevectors \( q_{j,\alpha} \), as functions of velocity \( v \), are the solutions of Eq. (4.37), \( \omega'_j(q_z) \equiv d\omega_j(q_z)/dq_z \) and \( v'_j(q_z) \equiv dv_j(q_z)/dq_z \). Since Eq. (4.37) may admit more than one solution (see Fig. 4.3), an additional index \( \alpha \) is used to distinguish the different solutions.

To investigate the way in which dissipation sets in for each mode, we consider
### 4.3. DISSIPATION IN A CYLINDRICAL CONDENSATE

Figure 4.3: The excitation mode dispersion (solid lines) plotted on the left and phase velocity of the lowest mode (solid line) plotted on the right for the cylindrical condensate with $\eta = 40$. In the left figure, the slope of the blue dashed line corresponds to the critical velocity and the slope of the red dashed line corresponds to a velocity greater than the critical velocity. The values of these two velocities are indicated by the horizontal lines in the right figure.

Velocities just above $v_{j,cr}$. In this case $q_{j,\alpha}$ will be close to $q_{j,cr}$ and we can make the expansion

$$v = v_j(q_{j,\alpha}) \simeq v_{j,cr} + \frac{1}{2} v''(q_{j,cr})(q_{j,\alpha} - q_{j,cr})^2,$$

where we have used the fact that $v'(q_{j,cr}) = 0$. Similarly, the denominator in Eq. (4.42) has the expansion

$$v_j'(q_{j,\alpha}) \simeq v''(q_{j,cr})(q_{j,\alpha} - q_{j,cr}).$$

Using Eq. (4.43) to eliminate $q_{j,\alpha} - q_{j,cr}$ from Eq. (4.44), we find

$$v_j'(q_{j,\alpha}) = \pm \sqrt{2v''(q_{j,cr})(v - v_{j,cr})}.$$

We thus see that the denominator in Eq. (4.42) is proportional to $\sqrt{v - v_{j,cr}}$, that is,
the dissipation rate exhibits a square-root singularity at each of the critical velocities. We emphasize that the singularity appears only in the $t \to \infty$ (or $t_0 \to -\infty$) limit; there is no divergence in the energy dissipation at finite times.

Another peculiar feature of the velocity-dependent dissipation rate is that the contribution from the lowest mode exhibits a discontinuity at $v = c_0$. As we shall see, this feature reflects the underlying structure of the lowest mode dispersion. From the right hand side figure in Fig 4.3 we observe that there are two solutions to Eq. (4.37) with $j = 0$ when the velocity approaches the speed of sound ($v_0(q_z)/c_0 \to 1$) from below; we denote the one that is smaller than $q_{0,cr}$ by $q_<$ and the other by $q_>$. Only one of these solutions remains for $v > c_0$. As $v \to c_0$ from below, $q_< \to 0$ and the discontinuity comes from the fact that the contribution to Eq. (4.42) with $j = 0$ from $q_<$ has a finite limiting value.

To illustrate this behavior, we use the TF hydrodynamic theory, which in the long wave length limit reproduces the results of the full Bogoliubov theory. Using Eq. (3.64), we find to lowest order in $q_z$ that

$$v'_0(q_z) = -\frac{\eta c_0 a^2}{24} q_z + O(q_z^3). \quad (4.46)$$

Similarly using Eq. (3.70), Eq. (3.66) and Eq. (3.65) we have

$$W_0(q_z) = \frac{c_0}{\eta \omega_\perp} q_z + O(q_z^3). \quad (4.47)$$

Substituting Eqs. (4.46) and (4.47) into Eq. (4.42) we find that the dissipation rate
from the lowest mode drops at $v = c_0$ by 

$$\Delta = \frac{12\sqrt{2}\nu|\tilde{V}_{\text{ext}}(0)|^2}{\eta^{3/2}h a_\perp}.$$  \hspace{1cm} (4.48)$$

We emphasize that this kind of discontinuity only arises for the lowest mode.

![Figure 4.4: The dimensionless weight $W_j(q_z)$ (left figure) and the first derivative of the phase velocity $v'_j(q_z)$ (right figure) plotted as functions of $q_z$. The blue lines (0) represent the lowest radial mode and the red lines (1) represent the first excited mode. As $q_z \to 0$, $v'_j(q_z)$ approaches zero for the lowest radial mode and diverges for the first excited mode.](image)

As a concrete example of the foregoing analysis, we now assume a one-dimensional Gaussian impurity potential $V_{\text{imp}}(z) = V_0 e^{-z^2/w_z^2}$, where $w_z$ is the width of the potential. The Fourier transform of the potential is $\tilde{V}_{\text{imp}}(q_z) = \sqrt{\pi}V_0 w_z e^{-w_z^2 q_z^2/4}$. The dimensionless weight $W_j(q_z)$ and the quantity $v'_j(q_z)$ are plotted as functions of $q_z$ for the lowest two modes in Fig. 4.4. With these ingredients we plot in Fig. 4.5 the energy dissipation rates for the lowest two modes, in units of $V_0^2 \nu a_\perp/h$, as functions of $v$. We see that the overall behaviour of the energy dissipation rate as a function of velocity depends significantly on the width of the impurity potential. In particular,
the singularities at the critical velocities are strongly suppressed for potentials with large widths. This is because the Gaussian potential effectively restricts the available modes to those with wave vector \( q < w_z^{-1} \); if \( w_z \) is so large that \( q_{j, cr} > w_z^{-1} \), the contribution from the vicinity of the singularities is strongly suppressed. Although the singularities are clearly apparent for \( w_z = 0.1a_\perp \), they are barely visible in the left-hand panel of Fig. 4.5 for \( w_z = 1.5a_\perp \).

![Figure 4.5: The mode-dependent energy dissipation rates (in units of \( V_0^2 \nu a_\perp /\hbar \)) for the lowest two modes plotted as a function of the velocity for the impurity potentials with \( w_z = 1.5a_\perp \) (left figure) and \( w_z = 0.1a_\perp \) (right figure). The blue solid line represents the \( j = 0 \) mode and the red solid line represents the \( j = 1 \) mode. The dashed lines indicate the positions of the corresponding singularities.](image)

It is interesting to observe that for the case of the Gaussian impurity potential, Eq. (4.35) gives

\[
\int_0^\infty dv \left. \frac{dE}{dt} \right|_{t_0 \to -\infty} = \frac{\pi V_0^2 \nu}{2m},
\]

which is independent of the width \( w_z \) of the Gaussian. This property applies to any impurity potential whose spatial dependence has the scaling form \( V_{\text{imp}}(z/w_z) \). Although \( dE/dt \) depends strongly on \( w_z \), the area under the \( dE/dt \) vs \( v \) curve has a
fixed value given by Eq. (4.49). Changes in \( w_z \) simply redistribute the weight under the curve. The velocity sum rule in Eq. (4.49) is all the more remarkable given the dramatic redistribution of the mode-dependent energy dissipation rates see in Fig. 4.5.

4.4 Bragg spectroscopy of Bose-condensates

Our last example concerns the flow of an elongated condensate through an optical lattice potential. The physically equivalent scenario, namely a optical lattice potential moving through a stationary condensate, has been studied in several experiments [44, 45, 47, 53]. The moving optical lattice potential (Bragg pulse) excites the condensate and imparts energy and momentum to it. Since the measurement of these physical quantities provides information on the dynamic structure factor of the condensate, it is referred to as Bragg spectroscopy.

Here we restrict ourselves to the experiments carried out in [53], where a Bragg pulse is applied along the axial direction of an elongated condensate for a certain duration of time and the momentum imparted to the condensate is measured. In these experiments, the trapping potential is switched off after the duration of the Bragg pulse and the atomic cloud is allowed to expand freely. After some period of free expansion, absorption images of the cloud are taken, from which the total momentum of the condensate can be determined. In the experiments, the condensate consisted of \( N = 10^5 \) number of \(^{87}\)Rb atoms, with a thermal fraction of 5% or less. The radial and axial trapping frequencies were \( \omega_\perp = 2\pi \times 220\text{Hz} \) and \( \omega_z = 2\pi \times 25\text{Hz} \). For these parameters, the transverse TF radius is \( R_\perp = 3.1\mu\text{m} \) [53].

In order to compare theory with experiment, we have used linear response theory to calculate the momentum imparted to a condensate exposed to Bragg pulses of
varying durations. In our calculations, the Bragg pulse is represented by the external potential

\[ V_{\text{Bragg}}(t) = V_B \cos(q_B z - \omega_B t), \tag{4.50} \]

which can be viewed as an optical lattice potential \( V_{\text{ext}}(z) = V_B \cos(q_B z) \) moving at a velocity \( v = \omega_B / q_B \). We assume the condensate is initially in its ground state \(|\Psi_0\rangle\) when the Bragg pulse is switched on \((t_0 = 0)\). The drag force exerted on the condensate at time \( t \) is found from Eq. (4.5) and Eq. (4.12) and is given by

\[
F_z(t) = i \int_{-\infty}^{\infty} dq_z \int_{-\infty}^{\infty} dq'_z q_z \tilde{V}_{\text{ext}}^*(q_z) \tilde{V}_{\text{ext}}(q'_z) \times \int_{-\infty}^{\infty} d\omega S(q_z, q'_z; \omega) \left\{ \frac{1 - e^{i(q_z v - \omega)t}}{q_z v - \omega} - \frac{1 - e^{i(q'_z v + \omega)t}}{q'_z v + \omega} \right\}, \tag{4.51} \]

where \( \tilde{V}_{\text{ext}}(q_z) = \pi V_B [\delta(q_z - q_B) + \delta(q_z + q_B)] \). To evaluate the above expression, we note that in general \( S(-q_z, -q_z; \omega) = S(q_z, q_z; \omega) \) as a result of Eq. (3.7). Furthermore, the spectral function \( S(q_z, q'_z; \omega) \) for this elongated condensate will be determined using the cylindrical LDA. Within this approximation we recall from Eq. (3.105) that \( S_{\text{cyl}}^{\text{LDA}}(q_z, -q_z; \omega) = 0 \). Using these properties, we obtain

\[
F_z(t) = \frac{V_B^2 q_B}{2} \int_{0}^{\infty} d\omega S_{\text{cyl}}^{\text{LDA}}(q_B, \omega) \left\{ \frac{\sin(\omega_B - \omega)t}{\omega_B - \omega} - \frac{\sin(\omega_B + \omega)t}{\omega_B + \omega} \right\}. \tag{4.52} \]

Substituting Eq. (4.52) into Eq. (4.19), we find that the momentum imparted to the
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condensate after a time interval \( t_B \) is

\[
P_z(t_B) = \frac{V_B^2 q_B}{2} \int_0^{\infty} d\omega S_{\text{cyl}}^{\text{LDA}}(q_B, \omega) \times \left\{ \frac{\cos(\omega + \omega_B) t_B - \cos \omega_z t_B}{(\omega + \omega_B)^2 - \omega_z^2} - \frac{\cos(\omega - \omega_B) t_B - \cos \omega_z t_B}{(\omega - \omega_B)^2 - \omega_z^2} \right\}. \tag{4.53}
\]

We emphasize that this result applies to the nonhomogeneous elongated condensate. One can recover the corresponding formula for a uniform cylindrical condensate simply by taking the \( \omega_z \to 0 \) limit. The result obtained is

\[
P_z(t_B) = NV_B^2 q_B t_B^2 \sum_j W_j(q_B) \left\{ \left[ \frac{\sin(\omega_j - \omega_B) t_B / 2}{(\omega_j - \omega_B) t_B / 2} \right]^2 - \left[ \frac{\sin(\omega_j + \omega_B) t_B / 2}{(\omega_j + \omega_B) t_B / 2} \right]^2 \right\}, \tag{4.54}
\]

which was obtained previously [51] using a slightly different approach. We see here that it follows from the more general result derived for an elongated condensate.

We now present results for the momentum of the condensate calculated from Eq. (4.53) and compare them with the experimental data from [53]. In the experiments carried out in [53], the strengths of the Bragg pulses \( V_B \) were not specified. Therefore in our theoretical calculations the strength of the Bragg pulses is a fitting parameter and is chosen to provide the best fit to the experimental data\(^2\). In the left hand side figure of Fig. 4.6 the momentum imparted is plotted as a function of the Bragg frequency for Bragg pulses with a relatively short duration \( t_B = 1 \text{ms} = 1.38 \omega_\perp^{-1} \). In the right hand side figure of Fig. 4.6 the result for a Bragg pulse with a longer duration \( t_B = 6 \text{ms} = 8.29 \omega_\perp^{-1} \) is shown.

As we can see, our theoretical results are in reasonably good agreement with the \(^2\)We observe that the momentum gain scales as \( V_B^2 \) in linear response theory
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Figure 4.6: The momentum imparted (solid lines) by Bragg pulses plotted as a function of the frequency of the pulse $\omega_B$. The experimental data (circles with error bars) are taken from [53]. In the left figure the duration of the pulse is $t_B = 1$ms and in the right figure the durations is $t_B = 6$ms. For the shorter duration pulse, the results for Bragg pulses with three different wave vectors are shown. The strength of the Bragg pulse for the right figure is chosen to be $V_B = 0.15\hbar\omega_\perp$. For comparison, the result obtained using the bulk LDA (dot-dashed line) is also shown in this figure. The dashed line connecting the experimental data points in the right figure is a guide to the eye.

Figure 4.7: The momentum imparted by a Bragg pulse of wavevector $q_B = 3.1\mu m^{-1}$ for pulses of increasing duration.
experimental data. A prominent feature of the experimental results is that a multi-peak structure of the momentum-frequency curve develops for Bragg pulses with durations long compared to \( \omega^{-1} \). This gradual development is shown in Fig. 4.7 for Bragg pulses of increasing duration. We observe from Fig. 3.2 that the cylindrical modes are spaced in frequency by roughly \( \omega_{\perp} \). To resolve these modes in the Bragg pulse experiment, we expect that the duration of the Bragg pulse must be sufficiently long, that is \( t_{B} \omega_{\perp} > 1 \). This dependence on \( t_{B} \) is well accounted for by our results which are obtained with the dynamic structure factor determined within the cylindrical LDA. As shown in Fig. 4.6, calculations based on the bulk LDA fail to capture this experimental observation. We should also point out that the results obtained by means of a time-dependent GP simulation [53] were also in good agreement with the experimental data, although the degree of agreement is no better than that achieved by means of linear response theory.
Chapter 5

Dissipative dynamics of trapped Bose condensates

In many cold atom experiments, the condensates are confined in harmonic traps. An important consequence of this kind of confinement is that, in the absence of any external perturbation, the centre of mass of the system oscillates about the centre of the trap in simple harmonic motion without dissipation. This particular collective oscillation is referred to as the centre of mass or dipole mode. Its existence is a generic property of a harmonically confined system in which the interaction between particles depends only on their relative separation. The undamped dipole oscillation was in fact used early on to accurately determine the trapping frequencies for the condensates [56].

However, when either the harmonicity of the confining potential or the pairwise-necessity of the particle interaction is compromised, the centre of mass degree of freedom is coupled to the internal ones and its motion becomes sensitive to the intrinsic properties of the system, such as quantum statistics, dimensionality of the system and particle interactions. For this reason the dipole oscillation has been used experimentally as a diagnostic of various perturbations to the trapped condensate. For example, several experiments [16, 18, 17] have used dipole oscillations to study the transport of
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A condensate through a disordered medium. Although the motion of the condensate in these experiments does not lose its collectivity, dissipation does occur and leads to the damping of the centre of mass motion. The main goal of this chapter is to provide a theoretical account of the energy dissipation and the damping of dipole oscillations, specifically for perturbations corresponding to a weak disorder potential. Through a transformation based on the Harmonic Potential Theorem (HPT) [57], we formulate the dynamics from the perspective of the response of a stationary superfluid to a dynamic external potential, thus allowing us to extend the linear response theory of energy dissipation developed in the previous chapter. Based on this formulation, we are able to investigate the dependence of the damping of dipole oscillations on various parameters characterizing the disorder potential and the condensate itself.

5.1 Dipole modes of trapped Bose condensates

As a preliminary to the development of the HPT and its extensions, we discuss in this section the dipole modes of a trapped Bose condensate and the underlying physics for the existence of such modes. The dipole modes are the low-lying collective excitations that have frequencies equal to the frequencies of the trap. Unlike other low-lying excitations, the frequencies of these modes are independent of the total number of trapped atoms and the atomic interactions.

Some insight into dipole modes is provided by considering the special case of an isotropic trap ($\lambda = 1$) in the TF limit [31]. As we have seen, the collective modes for this situation are determined by the Stringari equation (2.127). Written in polar
coordinates \((r, \theta, \varphi)\), this equation takes the form

\[
\omega^2 \delta n = \omega_0^2 r \frac{\partial}{\partial r} \delta n - \frac{\omega_0^2}{2} (R^2 - r^2) \nabla^2 \delta n,
\]

where \(\omega_0\) is the trap frequency and \(R\) is the TF radius of the cloud. Due to the spherical symmetry, the eigenmodes have a density proportional to a spherical harmonic \(Y_{lm}(\theta, \varphi)\), where \(l\) and \(m\) are angular momentum quantum numbers. Of all the possible modes, one subset has a radial dependence given by [31]

\[
\delta n = Cr^l Y_{lm}(\theta, \varphi),
\]

where \(C\) is an arbitrary constant. The corresponding mode frequencies are \(\omega = \sqrt{l} \omega_0\).

There are three \(l = 1\) modes with frequency \(\omega = \omega_0\) corresponding to \(m = 0, \pm 1\). These are the dipole modes for the isotropic trap. We observe that the density fluctuations of these modes are in fact proportional to the gradient of the equilibrium density. For example, for the \(l = 1, m = 0\) mode, the density fluctuation is proportional to \(rY_{10} \propto z\). Recalling that the TF density is \(n_{\text{TF}}(r) \propto (1 - r^2/R^2)\), we see that \(\delta n \propto -\partial n_{\text{TF}} / \partial z\). This kind of density fluctuation is associated with a rigid displacement of the equilibrium density. Writing \(n(r) = n_{\text{TF}}(r - \eta)\) and expanding in \(\eta\), we have \(\delta n(r) = n(r) - n_{\text{TF}}(r) \simeq -\eta \cdot \nabla n_{\text{TF}}(r)\). With \(\eta = \eta_0 \hat{z}\), we see that \(\delta n(r) \propto -\partial n_{\text{TF}} / \partial z\) as found directly by solving Eq. (5.1).

By introducing the coordinate scaling \(\bar{r} = r/R\) in Eq. (5.1), we observe that all the modes in the TF limit depend only on the single parameter \(\omega_0\). This is not true in general, as such a scaling does not exist for the BdeG equations or for the full hydrodynamic equations. However, it can be directly checked that \(\omega = \omega_\mu(\mu = x, y, z)\)
is still a solution to the full hydrodynamic equations (2.111) and (2.112) with the density variation \( \delta n \propto -\partial n_0/\partial r_\mu \), provided that the equilibrium density \( n_0(r) \) is determined by solution of the GP equation. This indicates that the existence of the dipole modes is not a consequence of the TF approximation.

In fact it can be shown rigorously that such dipole modes exist for any harmonically confined (bosonic or fermionic) system in which interactions depend only on the relative coordinates of the particles. The fundamental reason behind this is that the centre of mass degree of freedom is separable from all the internal degrees of freedom, which implies that there are excitations associated solely with motion of the centre of mass. We shall see that the dipole modes belong to this category of excitations.

To demonstrate this, we consider a harmonically confined many-body system described by the generic Hamiltonian

\[
\hat{H} = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{r}_i) \right) + \sum_{i<j} v(\hat{r}_i - \hat{r}_j).
\] (5.3)

The centre of mass degree of freedom of this system is defined as \( \hat{R} = \frac{1}{N} \sum_{i=1}^{N} \hat{r}_i \). From the Heisenberg equation of motion one can show that its conjugate momentum variable is simply the total momentum operator of the system \( \hat{P} = \sum_i \hat{p}_i \). These two variables satisfy the commutation relation \([\hat{R}_\mu, \hat{P}_\nu] = i\hbar \delta_{\mu\nu}\). We observe that the Hamiltonian \( \hat{H} \) can be written as

\[
\hat{H} = \hat{H}_{\text{cm}} + \hat{H}_{\text{int}}.
\] (5.4)
Here
\[ \hat{H}_{\text{cm}} = \frac{\hat{P}^2}{2M} + \frac{1}{2} M \sum_{\mu=x,y,z} \omega_{\mu}^2 \hat{R}_{\mu} \] (5.5)
and
\[ \hat{H}_{\text{int}} = \sum_{i=1}^{N} \left( \frac{\hat{P'}_i}{2m} + V_{\text{trap}}(\hat{\mathbf{r}}'_i) \right) + \sum_{i<j} v(\hat{\mathbf{r}}'_i - \hat{\mathbf{r}}'_j), \] (5.6)
where \( \hat{\mathbf{r}}'_i = \hat{\mathbf{r}}_i - \hat{\mathbf{R}} \) and \( \hat{\mathbf{p}}'_i = \hat{\mathbf{p}}_i - \frac{\hat{\mathbf{P}}}{N} \). \( \hat{H}_{\text{cm}} \) is clearly the Hamiltonian for the centre of mass degree of freedom and \( \hat{H}_{\text{int}} \) is the Hamiltonian determining the internal dynamics of the system. It can be checked that the centre of mass variables commute with \( \hat{H}_{\text{int}} \), namely
\[ [\hat{R}_{\mu}, \hat{H}_{\text{int}}] = 0; \quad [\hat{P}_{\mu}, \hat{H}_{\text{int}}] = 0. \] (5.7)

One simple implication of Eqs. (5.7) is that the centre of mass always undergoes simple harmonic motion at the frequencies of the trap. Using Eq. (5.5) and (5.7), one easily obtains the Heisenberg equations of motion for the centre of mass coordinate and the total momentum, namely
\[ \frac{d\hat{R}_{\mu}(t)}{dt} = \frac{1}{i\hbar} [\hat{R}_{\mu}(t), \hat{H}] = \frac{\hat{P}_{\mu}(t)}{M}, \] (5.8)
\[ \frac{d\hat{P}_{\mu}(t)}{dt} = \frac{1}{i\hbar} [\hat{P}_{\mu}(t), \hat{H}] = -M \omega_{\mu}^2 \hat{R}_{\mu}(t). \] (5.9)
where \( \hat{R}_{\mu}(t) \equiv e^{i\hat{H}t/\hbar} \hat{R}_{\mu} e^{-i\hat{H}t/\hbar} \) and \( \hat{P}_{\mu}(t) \equiv e^{i\hat{H}t/\hbar} \hat{P}_{\mu} e^{-i\hat{H}t/\hbar} \). (To simplify the notations, we suppress the subscript ‘H’ for Heisenberg operators.) Equations (5.8) and (5.9) lead to the simple harmonic motion equation
\[ \frac{d^2 \hat{R}_{\mu}(t)}{dt^2} + \omega_{\mu}^2 \hat{R}_{\mu}(t) = 0. \] (5.10)
The formal solution of this equation is

\[ \hat{R}_\mu(t) = \hat{R}_\mu \cos \omega_\mu t + \frac{\hat{P}_\mu}{M \omega_\mu} \sin \omega_\mu t, \quad (5.11) \]

\[ \hat{P}_\mu(t) = -M \omega_\mu \hat{R}_\mu \sin \omega_\mu t + \hat{P}_\mu \cos \omega_\mu t. \quad (5.12) \]

From this we see that the expectation value \( R_\mu(t) = \langle \Psi(t) | \hat{R}_\mu | \Psi(t) \rangle \) for an arbitrary state |\( \Psi(t) \rangle \) evolves in time according to the equation

\[ R_\mu(t) = R_\mu(0) \cos \omega_\mu t + \frac{P_\mu(0)}{M \omega_\mu} \sin \omega_\mu t = A \cos(\omega_\mu t + \phi_0), \quad (5.13) \]

where the amplitude \( A \) and phase angle \( \phi_0 \) are determined by the initial conditions \( R_\mu(0) \) and \( P_\mu(0) \). This undamped harmonic oscillation of the centre of mass coordinate is the dipole oscillation we have been referring to.

Furthermore, it follows from Eq. (5.7) that the centre of mass Hamiltonian \( \hat{H}_{\text{cm}} \) commutes with \( \hat{H}_{\text{int}} \). This means that the motion of the centre of mass is decoupled from all the internal dynamics of the system and, as a consequence, the system has excitations that are associated purely with the motion of the centre of mass. To see this in more detail, we define the centre of mass annihilation and creation operators

\[ \hat{a}_\mu = \sqrt{\frac{m \omega_\mu}{2 \hbar}} \left( \hat{R}_\mu + \frac{i}{m \omega_\mu} \hat{P}_\mu \right) ; \quad \hat{a}^\dagger_\mu = \sqrt{\frac{m \omega_\mu}{2 \hbar}} \left( \hat{R}_\mu - \frac{i}{m \omega_\mu} \hat{P}_\mu \right). \quad (5.14) \]

Using these definitions, the Hamiltonian \( \hat{H} \) can be written as

\[ \hat{H} = \sum_\mu \hbar \omega_\mu \left( \frac{1}{2} + \hat{a}^\dagger_\mu \hat{a}_\mu \right) + \hat{H}_{\text{int}}. \quad (5.15) \]
From Eq. (5.7) and the definition in Eq. (5.14), we observe that \( \hat{a}_\mu \) and \( \hat{a}_\mu^\dagger \) commute with \( \hat{H}_{\text{int}} \). Taking \( |\Psi_\alpha\rangle \) to be an eigenstate of \( \hat{H} \) with energy \( E_\alpha \), we find that the state \( |\Psi\rangle = \frac{1}{\sqrt{n!}} (a_\mu^\dagger)^n |\Psi_\alpha\rangle \) satisfies

\[
\hat{H}|\Psi\rangle = (E_\alpha + n\hbar\omega_\mu) |\Psi\rangle,
\]

that is, \( |\Psi\rangle \) remains an eigenstate of \( \hat{H} \) with energy \( E = E_\alpha + n\hbar\omega_\mu \); the application of \( \hat{a}_\mu^\dagger \) creates a quantum of excitation of the centre of mass oscillation with energy \( \hbar\omega_\mu \).

Finally, we mention the well known fact [58] that a simple harmonic oscillator has wave packet quantum states that move harmonically without any change in shape. These states are the so-called coherent states. Since the centre of mass degree of freedom is effectively a harmonic oscillator, analogous states also exist for the many-body system described by the Hamiltonian \( \hat{H} \). This property is encapsulated by the HPT discussed in the next section.

### 5.2 Harmonic Potential Theorem

We have shown that, for the many-body system described by the Hamiltonian \( \hat{H} \), the centre of mass variables are separable from the internal degrees of freedom. This property allows us to determine excitations of the many-body system that are associated solely with the motion of the centre of mass. In this section we discuss a further implication of this property, namely the existence of a class of dynamical many-body states for which the probability density oscillates without change in shape. This statement is essentially the content of the Harmonic Potential Theorem [57] which will form the basis of our study in the following section of the dissipative dynamics.
of a trapped condensate.\footnote{As discussed in Appendix B, the original statement of the HPT includes the effect of an external force acting on the centre of mass degree of freedom. This generalization is not needed for our present purposes.}

To begin, we consider the dynamical evolution of the system when prepared at \( t = 0 \) in the state \( |\Psi(0)\rangle \)
\[
|\Psi(0)\rangle = e^{-i\hat{P} \cdot x / \hbar}|\Psi_\alpha\rangle,
\]
(5.17)
where \( |\Psi_\alpha\rangle \) is an eigenstate of \( \hat{H} \) with eigenenergy \( E_\alpha \) and \( x \) is a position vector. We recognize \( e^{-i\hat{P} \cdot x / \hbar} \) as a translation operator in position space having the property
\[
\langle r_1, \cdots, r_N | e^{-i\hat{P} \cdot x / \hbar} = \langle r_1 - x, \cdots, r_N - x |. \]
This implies that \( \langle r_1, \cdots, r_N | \Psi(0)\rangle = \Psi(r_1, \cdots, r_N; t = 0) = \Psi_\alpha(r_1 - x, \cdots, r_N - x) \). Thus the initial state in Eq. (5.17) corresponds to the state \( |\Psi_\alpha\rangle \) being translated rigidly in position space through the vector \( x \). In cold atom experiments, such an initial state can be realized by a sudden displacement of the trap relative to the condensate.

The dynamical state of the system at time \( t \) is given by
\[
|\Psi(t)\rangle = e^{-i\hat{H}t / \hbar}e^{-i\hat{P} \cdot x / \hbar}|\Psi_\alpha\rangle \]
\[
= e^{-iE_\alpha t / \hbar}e^{-i\hat{P} \cdot x / \hbar}e^{i\hat{H}t / \hbar}|\Psi_\alpha\rangle. \quad (5.18)
\]
We observe that the operator \( e^{-i\hat{P} \cdot x / \hbar}e^{i\hat{H}t / \hbar} \) is simply \( e^{-i\hat{P}(t) \cdot x / \hbar} \). From Eq. (5.12), we have
\[
\hat{P}_\mu(-t) = M \omega_\mu \hat{R}_\mu \sin \omega_\mu t + \hat{P}_\mu \cos \omega_\mu t. \quad (5.19)
\]
We thus find
\[
e^{-i\hat{H}t / \hbar}e^{-i\hat{P} \cdot x / \hbar}e^{i\hat{H}t / \hbar} = \exp \left\{ \frac{i}{\hbar} \left( \hat{p}(t) \cdot \hat{R} - \hat{x}(t) \cdot \hat{P} \right) \right\}, \quad (5.20)
where

\[ x_\mu(t) = x_\mu \cos \omega_\mu t \]

\[ p_\mu(t) = -M \omega_\mu x_\mu \sin \omega_\mu t. \]  \hspace{1cm} (5.21)

These in fact are the coordinates of an oscillator with initial conditions \( x_\mu(0) = x_\mu \) and \( p_\mu(0) = 0 \).

Equation (5.20) leads us to define the following important unitary operator

\[ \hat{T}(x, p) \equiv \exp \left\{ -i \frac{\bar{\hbar}}{2} \left( p \cdot \hat{R} - x \cdot \hat{P} \right) \right\}, \]  \hspace{1cm} (5.22)

which is a generalization of the operator \( e^{-i \hat{p} \cdot x / \hbar} \) introduced in Eq. (5.17). Acting on any state, the operator \( \hat{T}(x, p) \) shifts the state by \( x \) in position space and by \( p/N \) in momentum space. To see this, we make use of the Baker-Hausdorff formula\(^2\) and write

\[ \hat{T}(x, p) = \exp \left\{ -i \frac{\bar{\hbar}}{2} x \cdot p \right\} \exp \left\{ \frac{i}{\hbar} p \cdot \hat{R} \right\} \exp \left\{ -i \frac{\bar{\hbar}}{2} x \cdot \hat{P} \right\}. \]  \hspace{1cm} (5.23)

Defining the state \( |\Psi'\rangle = \hat{T}(x, p)|\Psi\rangle \) and using Eq. (5.23), we find

\[ \Psi'(r_1, \ldots, r_N) = \exp \left[ \frac{i}{\hbar} p \cdot (R - x/2) \right] \Psi(r_1 - x, \ldots, r_N - x), \]  \hspace{1cm} (5.24)

where \( R = \sum_{i=1}^N r_i/N \). We thus have \( |\Psi'(r_1, ..., r_N)|^2 = |\Psi(r_1 - x, ..., r_N - x)|^2 \).

\(^2\)The Baker-Hausdorff formula states that \( e^{A+B} = e^A e^B e^{-C/2} \) if \( \hat{A} \) and \( \hat{B} \) commute with \( \hat{C} = [\hat{A}, \hat{B}] \).
Similarly we can write

$$\hat{T}(x, p) = \exp \left\{ \frac{i}{2\hbar} x \cdot p \right\} \exp \left\{ -\frac{i}{\hbar} x \cdot \hat{P} \right\} \exp \left\{ \frac{i}{\hbar} p \cdot \hat{R} \right\}. \quad (5.25)$$

In this case we find that

$$\tilde{\Psi}'(p_1, \ldots, p_N) = \langle p_1, \ldots, p_N | \Psi' \rangle$$

$$= \exp \left[ -\frac{i}{\hbar} x \cdot (P - p/2) \right] \tilde{\Psi}(p_1 - p/N, \ldots, p_N - p/N), \quad (5.26)$$

where $P = \sum_{i=1}^{N} p_i$. Thus $|\tilde{\Psi}'(p_1, \ldots, p_N)|^2 = |\tilde{\Psi}(p_1 - p/N, \ldots, p_N - p/N)|^2$, which implies that the total momentum of the state is boosted by $p$. Equations (5.23) and (5.25) also imply that the operator $\hat{T}(x, p)$ has the following properties:

$$\hat{T}^\dagger(x, p) \sum_{i=1}^{N} f(\hat{r}_i) \hat{T}(x, p) = \sum_{i=1}^{N} f(\hat{r}_i + x), \quad (5.27)$$

$$\hat{T}^\dagger(x, p) \sum_{i=1}^{N} f(\hat{p}_i) \hat{T}(x, p) = \sum_{i=1}^{N} f(\hat{p}_i + p/N). \quad (5.28)$$

We will make use of these transformation properties in the following.

Finally, we observe that the result in Eq. (5.20) has the generalization

$$e^{-i\hat{H}t/\hbar} \hat{T}(x, p)e^{i\hat{H}t/\hbar} = \hat{T}(x(t), p(t)), \quad (5.29)$$
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where

\[ x_\mu(t) = x_\mu \cos \omega_\mu t + \frac{p_\mu}{M \omega_\mu} \sin \omega_\mu t, \quad (5.30) \]

\[ p_\mu(t) = -M \omega_\mu x_\mu \sin \omega_\mu t + p_\mu \cos \omega_\mu t. \quad (5.31) \]

These are the solutions of the harmonic oscillator equation with initial conditions \( x_\mu(0) = x_\mu \) and \( p_\mu(0) = p_\mu \). The result given in Eq. (5.21) is the special case \( p = 0 \).

Returning to Eq. (5.18), we see that

\[ |\Psi(t)\rangle = e^{-iE_\alpha t/\hbar} \hat{T}(x(t), p(t)) |\Psi_\alpha\rangle, \quad (5.32) \]

where in this case \( x(t) \) and \( p(t) \) are given by Eq. (5.21). Since the operator \( \hat{T}(x(t), p(t)) \) shifts the state in position space by \( x(t) \), the wave function corresponding to the state \( |\Psi(t)\rangle \) is 

\[ \Psi(r_1, \ldots, r_N; t) = e^{i\theta} e^{-iE_\alpha t/\hbar} \Psi_\alpha(r_1 - x(t), \ldots, r_N - x(t)), \]

where \( \theta \) is some phase angle. In other words, the probability density simply oscillates rigidly, following a trajectory given by \( x(t) \). This is essentially the content of the Harmonic Potential Theorem, although its original formulation \([57]\) was rather different.

We can also understand the result in Eq. (5.32) from the perspective of coherent states. For concreteness we take \( x = \hat{z}z_0 \) in Eq. (5.17). Using Eq. (5.14), this initial state can be written as

\[ |\Psi(0)\rangle = e^{-\gamma \hat{a}_z + \gamma^* \hat{a}_z^\dagger} |\Psi_\alpha\rangle, \quad (5.33) \]

where \( \gamma = z_0 \sqrt{m\hbar \omega_z / 2\hbar} \). We see that this state is analogous to a coherent state of a
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simple harmonic oscillator [58]. The dynamic state is then given by

\[ |\Psi(t)\rangle = e^{-i\hat{H}_t/\hbar} e^{-\gamma \hat{a}_z + \gamma^* \hat{a}_z^\dagger} |\Psi_\alpha\rangle \]

\[ = e^{-iE_\alpha t/\hbar} e^{-\gamma \hat{a}_z(-t) + \gamma^* \hat{a}_z^\dagger(-t)} |\Psi_\alpha\rangle, \quad (5.34) \]

where

\[ \hat{a}_z(t) \equiv e^{i\hat{H}_t/\hbar} \hat{a}_z e^{-i\hat{H}_t/\hbar} = e^{-i\omega_z t} \hat{a}_z. \quad (5.35) \]

Substituting Eq. (5.35) into Eq. (5.34) and using Eqs. (5.14), we find

\[ |\Psi(t)\rangle = e^{-iE_\alpha t/\hbar} \exp \left\{ \frac{i}{\hbar} \left( -M \omega_z z_0 \sin \omega_z t \hat{R}_z + z_0 \cos \omega_z t \hat{P}_z \right) \right\} |\Psi_\alpha\rangle, \quad (5.36) \]

which is a special case of Eq. (5.32).

So far we have only discussed the evolution of many-body states. More generally, we can consider a system described by a statistical density matrix of the form

\[ \hat{\rho}_0 = \sum_\alpha \lambda_\alpha |\Psi_\alpha\rangle \langle \Psi_\alpha|. \quad (5.37) \]

A density matrix of this form encompasses the case of a system in thermal equilibrium. We now imagine that all of the states are displaced by \( \hat{T}(x, p) \) at time \( t = 0 \). The resulting density matrix is then given by

\[ \hat{\rho}(0) = \hat{T}(x, p) \hat{\rho}_0 \hat{T}^\dagger(x, p). \quad (5.38) \]
The time evolution of this density matrix is given by

\[
\hat{\rho}(t) = e^{-i\hat{H}t/\hbar} \hat{\rho}(0) e^{i\hat{H}t/\hbar} = \hat{T}(x(t), p(t)) e^{-i\hat{H}t/\hbar} \hat{\rho}_0 e^{i\hat{H}t/\hbar} \hat{T}^\dagger(x(t), p(t)),
\]

(5.39)

where Eq. (5.30) and Eq. (5.31) define the time evolution of the displacement operator.

If the states \(|\Psi_\alpha\rangle\) in \(\hat{\rho}_0\) are in fact eigenstates of \(\hat{H}\), we have the simpler result

\[
\hat{\rho}(t) = \hat{T}(x(t), p(t)) \hat{\rho}_0 \hat{T}^\dagger(x(t), p(t)).
\]

(5.40)

The expectation value of any operator \(\hat{O}\) is given by \(\text{Tr}[\hat{\rho}(t)\hat{O}]\). In particular, the time-dependent density of the system is

\[
n(r, t) = \text{Tr}[\hat{\rho}(t)\hat{n}(r)]
\]

\[
= \text{Tr}[\hat{T}(x(t), p(t))\hat{\rho}_0 \hat{T}^\dagger(x(t), p(t))\hat{n}(r)]
\]

\[
= \text{Tr}[\hat{\rho}_0 \hat{T}^\dagger(x(t), p(t))\hat{n}(r) \hat{T}(x(t), p(t))].
\]

(5.41)

Recalling that \(\hat{n}(r) = \sum_{i=1}^N \delta(\hat{r}_i - r)\) and using Eq. (5.27), we find

\[
n(r, t) = \text{Tr}[\hat{\rho}(t)\hat{n}(r - x(t))]
\]

\[
= n_0(r - x(t)),
\]

(5.42)

where \(n_0(r) = \text{Tr}[\hat{\rho}_0\hat{n}(r)]\) is the density of the system before its displacement. We thus see that the density of the system experiences the same kind of rigid oscillation in the density matrix description as it does for a pure state.
5.3 Dissipative dynamics of a trapped condensate in the presence of a disorder potential

The theoretical development in this section is motivated by several experiments [16, 17, 18], which have studied the centre of mass dynamics of trapped Bose condensates in the presence of a disorder potential. In these experiments, the motion of the condensate is often initiated by an abrupt displacement of the trap potential in a certain direction. As explained in the previous section, the HPT dictates that the condensate will undergo rigid, undamped oscillations in the absence of any additional perturbation. A disorder potential, however, couples the centre of mass and the internal degrees of freedom. As a result, the energy associated with the centre of mass motion is gradually converted into heat through internal excitations of the condensate. The disorder potential effectively exerts a drag force on the condensate and the centre of mass oscillation is damped.

We thus encounter a physical situation where it is of interest to determine the force on a harmonically confined condensate that moves through a static disorder potential with a non-uniform velocity. In the previous chapter, we considered the problem of a homogeneous condensate moving past an impurity potential with a constant velocity. In this case, Galilean invariance allowed us to consider the physically equivalent scenario where the impurity moved through a stationary condensate. Linear response theory could then be used to calculate the drag force on the condensate. For a harmonically trapped condensate, however, a Galilean transformation does not apply and it is therefore not obvious that an analogous description can be achieved. Thus the main goal of this section is to demonstrate that such a reformulation is indeed possible through an extension of the HPT. For a harmonically trapped condensate, we
find an equivalence between the motion of the condensate through a disorder potential that is at rest relative to the confining potential, and the harmonic motion of the disorder potential itself relative to the trapping potential and the condensate. Within this latter point of view, linear response theory can again be used to determine the drag force acting on the condensate.

In the experiments we consider [18, 17], two slightly different protocols are used to introduce the disorder potential: the disorder potential is either (i) suddenly switched on sometime after the commencement of the free oscillation [18] or (ii) it is slowly turned on to its full strength before the displacement of the trap [17]. Although the physics involved in these two cases are very similar, they nevertheless require separate treatments which will be discussed in turn.

5.3.1 Case (i)

As described in experiment [18], the condensate is set into motion by abruptly displacing the centre of the trap along the $z$ axis by a distance $z_0$ at $t = 0$. After a period $t_0 = T/4$ of free oscillation, where $T = 2\pi/\omega_z$, a disorder potential $\hat{V}_{\text{dis}} = \sum_{i=1}^{N} V_{\text{dis}}(\hat{r}_i)$ is suddenly switched on. To describe this situation, we define the Hamiltonian for $t < 0$ to be

$$\hat{H}' = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{r}_i - \hat{x}) \right] + \sum_{i<j} v(\hat{r}_i - \hat{r}_j), \tag{5.43}$$

where the trapping potential $V_{\text{trap}}(\hat{r} - \hat{x})$ is centred at $\hat{x} = z z_0$. This initial potential is depicted by the dashed curve in Fig. 5.1 (a). Also indicated by the shaded region is the initial condensate which is taken to be in its ground state with respect to $\hat{H}'$. At $t = 0$, the trap is suddenly shifted to the origin of coordinates and the Hamiltonian is then given by $\hat{H}$ in Eq. (5.3). The trapping potential following the shift is depicted
Figure 5.1: (a) At time $t < 0$, the condensate is in its ground state with respect to the trapping potential indicated by the dashed curve. At $t = 0$, the trapping potential is suddenly shifted to the origin, initiating the free oscillation of the condensate. (b) At $t = t_0$, the condensate arrives at the origin with the velocity $v_0 = -z_0 \omega \hat{z}$ and the disorder potential is suddenly switched on. For $t > t_0$, the condensate evolves according to the Hamiltonian in Eq. (5.48). (c) The disorder potential has a velocity $-v_0$ at $t = t_0$ and subsequently oscillates according to the free centre of mass motion. The condensate, initially stationary, begins to respond to the perturbation of the dynamic disorder potential.
by the solid curve in Fig. 5.1 (a). The condensate, of course, remains in its original position.

The initial state of the system, denoted by $|\Psi'_0\rangle$, is the ground state of $\hat{H}'$ and is displaced from the origin by $x$. This state is related to $|\Psi_0\rangle$, the ground state of $\hat{H}$, according to the expression

$$|\Psi'_0\rangle = e^{-i\hat{p} \cdot x/\hbar} |\Psi_0\rangle,$$

(5.44)

This state is taken to be the initial state $|\Psi(0)\rangle$ for the subsequent evolution according to the Hamiltonian $\hat{H}$ in the time interval $0 < t < t_0$. After this period of free oscillation, the state is

$$|\Psi(t_0)\rangle = \hat{T}(x(t_0), p(t_0)) |\Psi_0\rangle,$$

(5.45)

where

$$x(t) = \hat{z}z_0 \cos \omega_z t,$$

(5.46)

$$p(t) = Mv(t) = -\hat{z}M\omega_z z_0 \sin \omega_z t.$$

(5.47)

At the instant of time $t = t_0$, the condensate is at the centre of the trap and is moving with a velocity $v_0 \equiv v(t_0) = -z_0\omega_z \hat{z}$. This situation is schematically depicted by the shaded region in Fig. (5.1) (b). The disorder potential is then turned on at $t = t_0$ and the system evolves according to the total Hamiltonian

$$\hat{H} = \hat{H} + \hat{V}_{\text{dis}}.$$

(5.48)

---

We have dropped an inconsequential phase factor of $e^{-iE_0t_0/\hbar}$ from Eq. (5.45) for convenience (see Eq. (5.32)).
The state of the system for \( t > t_0 \) is given by \(|\Psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}|\Psi(t_0)\rangle\).

To investigate the dynamics of the centre of mass for \( t > t_0 \), we introduce the Heisenberg operators

\[
\hat{R}_\mu(t) = e^{i\hat{R}(t-t_0)/\hbar}\hat{R}_\mu e^{-i\hat{R}(t-t_0)/\hbar}, \quad \hat{P}_\mu(t) = e^{i\hat{H}(t-t_0)/\hbar}\hat{P}_\mu e^{-i\hat{H}(t-t_0)/\hbar}.
\] (5.49)

These operators satisfy the Heisenberg equations of motion

\[
\frac{d\hat{R}_\mu(t)}{dt} = \frac{1}{i\hbar}[\hat{R}_\mu(t), \hat{H}] = \frac{\hat{P}_\mu(t)}{M}, \quad (5.50)
\]

\[
\frac{d\hat{P}_\mu(t)}{dt} = \frac{1}{i\hbar}[\hat{P}_\mu(t), \hat{H}] = -M\omega^2_\mu \hat{R}_\mu(t) + \hat{F}_\mu(t), \quad (5.51)
\]

where

\[
\hat{F}_\mu = -\sum_{i=1}^{N} \partial V_{\text{dis}}(\hat{r}_i) / \partial \hat{r}_{i,\mu}
\] (5.52)

is \( \mu \)-component of the disorder force operator. Equations (5.50) and (5.51) then lead to

\[
\frac{d^2\hat{R}_\mu(t)}{dt^2} + \omega^2_\mu \hat{R}_\mu(t) = \frac{\hat{F}_\mu(t)}{M}. \quad (5.53)
\]

Taking the expectation value of both sides of Eq. (5.53) with respect to the initial state \(|\Psi(t_0)\rangle\), we find that the \( z \)-component of the centre of mass position satisfies the equation

\[
\frac{d^2Z(t)}{dt^2} + \omega^2 z Z(t) = \frac{F(t)}{M}, \quad (5.54)
\]

where \( Z(t) = \langle \Psi(t_0)|\hat{R}_z(t)|\Psi(t_0)\rangle = \langle \Psi(t)|\hat{R}_z|\Psi(t)\rangle \) and

\[
F(t) = \langle \Psi(t_0)|\hat{F}_z(t)|\Psi(t_0)\rangle = \langle \Psi(t)|\hat{F}_z|\Psi(t)\rangle. \quad (5.55)
\]
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Equation (5.54) is an exact statement of the centre of mass dynamics. From
Eq. (5.55), we see that the evaluation of the force appearing in Eq. (5.54) requires
knowledge of the dynamic state $|\Psi(t)\rangle$, which satisfies the time-dependent Schrödinger
equation

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle = \left(\hat{H} + V_{\text{dis}}\right)|\Psi(t)\rangle. \quad (5.56)$$

This equation is formally solved by going to the interaction picture. The interaction
picture state vector

$$|\Psi_1(t)\rangle \equiv e^{i\hat{H}(t-t_0)/\hbar}|\Psi(t)\rangle \quad (5.57)$$

satisfies the equation

$$|\Psi_1(t)\rangle = |\Psi(t_0)\rangle - \frac{i}{\hbar} \int_{t_0}^{t} dt' \hat{V}_{\text{dis},1}(t')|\Psi_1(t')\rangle, \quad (5.58)$$

with $\hat{V}_{\text{dis},1}(t) = e^{i\hat{H}(t-t_0)/\hbar}\hat{V}_{\text{dis}}e^{-i\hat{H}(t-t_0)/\hbar}$. Since $|\Psi(t_0)\rangle$ is given by Eq. (5.45), we see
that the application of the operator $\hat{T}^\dagger(x(t_0), p(t_0))$ to both sides of Eq. (5.58) leads to

$$|\tilde{\Psi}_1(t)\rangle = |\Psi_0\rangle - \frac{i}{\hbar} \int_{t_0}^{t} dt' \hat{T}^\dagger(x(t_0), p(t_0))\hat{V}_{\text{dis},1}(t')|\Psi_1(t')\rangle, \quad (5.59)$$

where we have defined

$$|\tilde{\Psi}_1(t)\rangle \equiv \hat{T}^\dagger(x(t_0), p(t_0))|\Psi_1(t)\rangle. \quad (5.60)$$
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Using Eq. (5.60) to eliminate $|\Psi_1(t')\rangle$ from Eq. (5.59), we obtain

$$|	ilde{\Psi}_1(t)\rangle = |\Psi_0\rangle - \frac{i}{\hbar} \int_{t_0}^{t} dt' \hat{T}^\dagger(x(t_0), p(t_0)) \hat{V}_{\text{dis}}(t') \hat{T}(x(t_0), p(t_0)) |\tilde{\Psi}_1(t')\rangle. \quad (5.61)$$

The operator acting on $|\tilde{\Psi}_1(t')\rangle$ in the above equation is the effective time-dependent 
perturbation and is given explicitly as

$$\hat{T}^\dagger(x(t_0), p(t_0)) e^{i\hat{H}(t-t_0)/\hbar} \hat{V}_{\text{dis}} e^{-i\hat{H}(t-t_0)/\hbar} \hat{T}(x(t_0), p(t_0)). \quad (5.62)$$

Since $\hat{T}$ satisfies the time translation property (see Eq. (5.29))

$$\hat{T}(x(t), p(t)) = e^{-i\hat{H}(t-t_0)/\hbar} \hat{T}(x(t_0), p(t_0)) e^{i\hat{H}(t-t_0)/\hbar}, \quad (5.63)$$

we see that

$$e^{-i\hat{H}(t-t_0)/\hbar} \hat{T}(x(t_0), p(t_0)) = \hat{T}(x(t), p(t)) e^{-i\hat{H}(t-t_0)/\hbar} \quad (5.64)$$

and Eq. (5.62) becomes

$$e^{i\hat{H}(t-t_0)/\hbar} \hat{T}^\dagger(x(t), p(t)) \hat{V}_{\text{dis}} \hat{T}(x(t), p(t)) e^{-i\hat{H}(t-t_0)/\hbar}. \quad (5.65)$$

Using Eq. (5.27), we have

$$\hat{T}^\dagger(x(t), p(t)) \hat{V}_{\text{dis}} \hat{T}(x(t), p(t)) = \sum_{i}^{N} V_{\text{dis}}(\hat{r}_i + x(t)) \equiv \hat{V}_{\text{dis}}(x(t)). \quad (5.66)$$

This expression represents the disorder potential displaced by the time-dependent 
vector $-x(t)$. Since $x(t)$ is given by Eq. (5.46), we see that this time-dependent 
perturbation corresponds to the disorder potential oscillating at the frequency $\omega_z$. 
We thus see that Eq. (5.61) becomes

$$|\tilde{\Psi}_I(t)\rangle = |\Psi_0\rangle - \frac{i}{\hbar} \int_{t_0}^{t} dt \tilde{V}_{\text{dis},1}(x(t'), t') |\tilde{\Psi}_I(t')\rangle,$$

(5.67)

where

$$\tilde{V}_{\text{dis},1}(x(t), t) = e^{i\hat{H}(t-t_0)/\hbar} \tilde{V}_{\text{dis}}(x(t)) e^{-i\hat{H}(t-t_0)/\hbar}. \quad (5.68)$$

From Eq. (5.67), we see that $|\tilde{\Psi}_I(t)\rangle$ can be interpreted as the state (in the interaction picture) that evolves from $|\Psi_0\rangle$ at $t = t_0$ according to the Hamiltonian

$$\hat{H}_{\text{osc}}(t) = \hat{H} + \tilde{V}_{\text{dis}}(x(t)),$$

(5.69)

in which the disorder potential is oscillating harmonically. In other words, the corresponding state in the Schrödinger picture is

$$|\tilde{\Psi}(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\tilde{\Psi}_I(t)\rangle.$$

(5.70)

At $t = t_0$, $|\tilde{\Psi}(t_0)\rangle = |\Psi_0\rangle$ which is the state depicted by the shaded region in Fig. 5.1 (c). The evolution of this state for $t > t_0$ is governed by the Hamiltonian in Eq. (5.69).

In Fig. 5.1 (c) we use an arrow to indicate that the disorder potential is oscillating.

From the foregoing analysis we can make a connection between the $|\Psi(t)\rangle$ and
We thus see that the state of interest $|\Psi(t)\rangle$, which evolves from $|\Psi(t_0)\rangle$ given by Eq. (5.45) according to the stationary Hamiltonian $\hat{H} = \hat{H} + \hat{V}_{\text{dis}}$, is obtained through a displacement of the state $|\tilde{\Psi}(t)\rangle$. This relationship implies that the force appearing in (5.54) can be expressed as

$$F(t) = \langle \Psi(t) | \hat{F}_{\text{z}} | \Psi(t) \rangle$$

$$= \langle \tilde{\Psi}(t) | \hat{T}^\dagger(\mathbf{x}(t), \mathbf{p}(t)) \hat{F}_{\text{z}} \hat{T}(\mathbf{x}(t), \mathbf{p}(t)) | \tilde{\Psi}(t) \rangle$$

$$= \langle \tilde{\Psi}(t) | \hat{F}_{\text{z}}(t) | \tilde{\Psi}(t) \rangle$$

$$\equiv \tilde{F}(t), \quad (5.72)$$

where we have used the property in Eq. (5.27) to obtain

$$\hat{F}_{\text{z}}(t) = - \sum_{i=1}^{N} \frac{\partial V_{\text{dis}}(\hat{r}_i + \mathbf{x}(t))}{\partial \hat{z}_i}. \quad (5.73)$$

Equation (5.72) is our key result and shows that there is an intimate connection between the two very distinct physical situations depicted in Fig. 5.1 (b) and (c). In the first, one starts with an excited state corresponding to a moving condensate. This state then evolves according to Eq. (5.58) in the presence of a static disorder

$$|\tilde{\Psi}(t)\rangle$$ states. Using Eqs. (5.56), (5.60) and (5.63), we find

$$|\Psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\Psi_I(t)\rangle$$

$$= e^{-i\hat{H}(t-t_0)/\hbar} \hat{T}(\mathbf{x}(t_0), \mathbf{p}(t_0)) |\tilde{\Psi}_I(t)\rangle$$

$$= e^{-i\hat{H}(t-t_0)/\hbar} \hat{T}(\mathbf{x}(t_0), \mathbf{p}(t_0)) e^{i\hat{H}(t-t_0)/\hbar} |\tilde{\Psi}(t)\rangle$$

$$= \hat{T}(\mathbf{x}(t), \mathbf{p}(t)) |\tilde{\Psi}(t)\rangle.$$

(5.71)
potential. Even though the condensate follows a damped trajectory that eventually ends with the cloud being in quasi-equilibrium\textsuperscript{4} with the static disorder, the total energy of the system is conserved during this evolution. In the alternative situation (Fig. 5.1 (c)) described by Eq. (5.67), the condensate starts in its ground state and is driven by a \textit{dynamic} disorder potential moving according to the unperturbed centre of mass motion. In this case, the dynamic perturbation continually excites the condensate and the total energy increases as a function of time. From Eq. (5.71) we see that $|\tilde{\Psi}(t)\rangle = \tilde{T}^\dagger(x(t), p(t))|\Psi(t)\rangle$ and at long times this state is given by $|\tilde{\Psi}(t)\rangle = \tilde{T}^\dagger(x(t), p(t))|\Psi_\infty\rangle$, where $|\Psi_\infty\rangle \equiv \lim_{t \to \infty} |\Psi(t)\rangle$ is the quasi-equilibrium state referred to above. In other words, the atomic cloud eventually reaches a steady state in which it moves with the disorder potential with no further increase in energy. That the cloud experiences the same force due to the disorder in these two situations is by no means obvious and is a consequence of the fact that the system is harmonically confined.

The difference between these two physical scenarios can also be seen by looking at the centre of mass dynamics. The solution of Eq. (5.54) with the initial conditions $Z(t_0) = 0$ and $\dot{Z}(t_0) = -v_0$ corresponding to the motion depicted in Fig. 5.1 (b) is

$$Z(t) = -z_0 \sin \omega_z(t - t_0) - \frac{1}{M\omega_z} \int_{t_0}^{t} dt' \sin \omega_z(t - t') F(t'), \quad (5.74)$$

where $z_0 = v_0/\omega_z$. In the alternative situation depicted by Fig. 5.1 (c), the centre of

\textsuperscript{4}At long times the quantum state $|\Psi(t)\rangle$ is a highly excited state which is a complex superposition of all eigenstates of $\hat{H}$. However, physically we expect that this state is essentially equivalent to a thermal equilibrium state in the sense that the expectation values of various macroscopic physical quantities obtained using the dynamical state will be very similar to the averages calculated using a thermal equilibrium ensemble.
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mass coordinate $Z(t)$ satisfies the equation

$$\frac{d^2Z(t)}{dt^2} + \omega_z^2 Z(t) = \frac{\tilde{F}(t)}{M}$$

(5.75)

with the initial conditions $Z(t_0) = 0$ and $\dot{Z}(t_0) = 0$. The solution of this problem is

$$\tilde{Z}(t) = \frac{1}{M \omega_z} \int_{t_0}^{t} dt' \sin \omega_z(t-t') \tilde{F}(t').$$

(5.76)

In view of the fact that $\tilde{F}(t) = F(t)$, a comparison of Eqs. (5.74) and (5.76) shows that

$$\tilde{Z}(t) = Z(t) + z_0 \sin \omega_z(t-t_0).$$

(5.77)

Since the displacement of the disorder potential in Fig. 5.1 (c) is given by $Z_{\text{dis}}(t) = z_0 \sin \omega_z(t-t_0)$, we see that $Z(t) = \tilde{Z}(t) - Z_{\text{dis}}(t)$, that is, the displacement $\tilde{Z}(t)$ relative to the rest frame of the disorder potential is equal to $Z(t)$. Since $\lim_{t \to \infty} Z(t) = 0$, we thus conclude that the condensate in Fig. 5.1 (c) moves synchronously with the disorder potential at long times.

### 5.3.2 Case (ii)

This case corresponds to the situation described in [17] in which the condensate is initially in equilibrium with the disorder potential and the trapping potential. At time $t = 0$ the trap is suddenly displaced along the $z$ axis by a distance $z_0$. To
describe this situation, we define the Hamiltonian of the system for \( t < 0 \) to be

\[
\hat{\mathcal{H}}' = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{\mathbf{r}}_i - \mathbf{x}) \right] + \sum_{i<j} v(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) + \sum_{i=1}^{N} V_{\text{dis}}(\hat{\mathbf{r}}_i),
\]

(5.78)

where \( \mathbf{x} = \hat{\mathbf{z}}_{z_0} \), while for \( t \geq 0 \), the system evolves according to the Hamiltonian \( \hat{\mathcal{H}} \) given in Eq. (5.48). In other words, the trap potential for \( t < 0 \) is centred at \( \mathbf{x} \) and is suddenly displaced to the origin at \( t = 0 \). These trapping potentials are represented in Fig. 5.2 (a) by the dashed and solid curves, respectively. The initial state of the system at \( t = 0 \) is \( |\Psi(0)\rangle = |\Phi'_0\rangle \), the ground state of \( \hat{\mathcal{H}}' \). We emphasize that this state is different from the \( t = 0 \) state \( |\Psi'_0\rangle \) in case (i), which is the ground state in the absence of the disorder potential, that is the ground state of \( \hat{\mathcal{H}}' \). From Eq. (5.27), we see that the Hamiltonian in Eq. (5.78) can be expressed as \( \hat{\mathcal{H}}' = \hat{T}(\mathbf{x}, \mathbf{p})\tilde{\mathcal{H}}\hat{T}^\dagger(\mathbf{x}, \mathbf{p}) \) with \( \mathbf{p} = 0 \), where \( \tilde{\mathcal{H}} \) is

\[
\tilde{\mathcal{H}} = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{\mathbf{r}}_i) \right] + \sum_{i<j} v(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) + \sum_{i=1}^{N} V_{\text{dis}}(\hat{\mathbf{r}}_i + \mathbf{x}).
\]

(5.79)

The trapping potential in this Hamiltonian is centred at the origin but the disorder potential is shifted by \( -\mathbf{x} \) relative to the disorder potential in Fig. 5.2 (a). The potentials corresponding to \( \tilde{\mathcal{H}} \) are illustrated in Fig. 5.1 (b). The relation between \( \hat{\mathcal{H}}' \) and \( \tilde{\mathcal{H}} \) indicates that the ground state of \( \tilde{\mathcal{H}} \) is \( |\tilde{\Phi}_0\rangle = \hat{T}^\dagger(\mathbf{x}, \mathbf{p})|\Phi'_0\rangle \), namely the state \( |\Phi'_0\rangle \) translated by \( -\mathbf{x} \) in position space.

Since the evolution of the condensate for \( t > 0 \) is governed by the Hamiltonian \( \hat{\mathcal{H}} \) in Eq. (5.48), the equation of motion for the centre of mass is given by Eq. (5.54). We note that in this case the disorder force acts on the condensate at all times and that there is no period of free oscillation following the displacement of the trap as in
5.3. DISSIPATIVE DYNAMICS OF A TRAPPED CONDENSATE IN THE PRESENCE OF A DISORDER POTENTIAL

Figure 5.2: (a) The condensate, originally in equilibrium with the unshifted trap (dashed) and the disorder potential, begins to oscillate about the centre of the shifted trap (solid). (b) The condensate, originally in equilibrium with the trap and disorder potential, is driven by an oscillating disorder potential.
5.3. DISSIPATIVE DYNAMICS OF A TRAPPED CONDENSATE IN THE PRESENCE OF A DISORDER POTENTIAL

To determine the dynamic state $|\Psi(t)\rangle$, which evolves from $|\Psi(0)\rangle = |\Phi'_0\rangle$ according to the Hamiltonian $\hat{H}$, we will follow a procedure similar to that used in case (i). However, since the perturbation starts at time $t = 0$ in this case, it is more appropriate to define the interaction picture according to

$$|\Psi_I(t)\rangle = e^{i\hat{H}t/\hbar} |\Psi(t)\rangle.$$  

This implies that any operator $\hat{O}$ in the interaction picture is defined as $\hat{O}_I(t) = e^{i\hat{H}t/\hbar} \hat{O} e^{-i\hat{H}t/\hbar}$. The dynamic state $|\Psi_1(t)\rangle$ satisfies the equation

$$|\Psi_1(t)\rangle = |\Psi(0)\rangle - \frac{i}{\hbar} \int_0^t dt' \hat{V}_{\text{dis},1}(t')|\Psi_1(t')\rangle.$$  

Since the initial state in (5.81) can be written as $|\Psi(0)\rangle = |\Phi'_0\rangle = T(x,p)|\tilde{\Phi}_0\rangle$, we find that the state

$$|\tilde{\Psi}_1(t)\rangle \equiv \hat{T}^\dagger(x,p)|\Psi_1(t)\rangle$$  

satisfies the equation

$$|\tilde{\Psi}_1(t)\rangle = |\tilde{\Phi}_0\rangle - \frac{i}{\hbar} \int_0^t dt' \hat{T}^\dagger(x,p)\hat{V}_{\text{dis},1}(t')\hat{T}(x,p)|\tilde{\Psi}_1(t')\rangle.$$  

Using the time translation property of the operator $\hat{T}$ in Eq. (5.29), we find that the
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THE PRESENCE OF A DISORDER POTENTIAL

Effective time-dependent perturbation acting on $|\tilde{\Psi}_I(t')\rangle$ is given by

$$\tilde{T}^\dagger(x, p)\tilde{V}_{\text{dis,1}}(x(t'), t')\tilde{T}(x, p) = \tilde{T}^\dagger(x, p)e^{i\tilde{H}_I'\hbar}\tilde{V}_{\text{dis}}e^{-i\tilde{H}_I'\hbar}\tilde{T}(x, p) = e^{i\tilde{H}_I'/\hbar}\tilde{V}_{\text{dis}}(x(t'))\tilde{T}(x(t'), p(t'))e^{-i\tilde{H}_I'/\hbar}$$

where $x(t')$ and $p(t')$ are the same as those given in Eqs. (5.46) and (5.47). $\tilde{V}_{\text{dis}}(x(t))$ is the oscillating disorder potential defined in Eq. (5.66), but $\tilde{V}_{\text{dis,1}}(x(t'), t)$ is here defined as $\tilde{V}_{\text{dis,1}}(x(t), t) = e^{i\tilde{H}_I'/\hbar}\tilde{V}_{\text{dis}}(x(t))e^{-i\tilde{H}_I'/\hbar}$.

We thus see from Eq. (5.83) that $|\tilde{\Psi}_I(t)\rangle$ is the state (in the interaction picture) that evolves from $|\tilde{\Phi}_0\rangle$ at $t = 0$ according to the time-dependent Hamiltonian $\tilde{\mathcal{H}}_{\text{osc}}(t) = \tilde{\mathcal{H}} + \tilde{V}_{\text{dis}}(x(t))$; this evolution is depicted in Fig. 5.2 (b). It is worth pointing out that at $t = 0$, the potential $\tilde{V}_{\text{dis}}(x(t))$ is in fact the static potential in Fig. 5.2 (a) displaced by $-x(0) = -x$. Thus, at $t = 0$ the Hamiltonian $\tilde{\mathcal{H}}_{\text{osc}}(t = 0)$ coincides with the Hamiltonian $\tilde{\mathcal{H}}$ defined in Eq. (5.79).

As for case (i), we can establish a useful relationship between the state $|\Psi(t)\rangle$ and the state $|\tilde{\Psi}(t)\rangle = e^{-i\tilde{H}_I'/\hbar}|\tilde{\Psi}_I(t)\rangle$. Using Eqs. (5.80) and (5.82) and following the steps used to derive Eq. (5.71), we find

$$|\Psi(t)\rangle = \tilde{T}(x(t), p(t))|\tilde{\Psi}(t)\rangle,$$

which has exactly the same form as Eq. (5.71). This again implies that the force
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given in Eq. (5.55) for the physical situation in this case can be expressed as

\[ F(t) = \langle \Psi(t) | \tilde{F}_z | \Psi(t) \rangle = \langle \tilde{\Psi}(t) | \tilde{F}_z(t) | \tilde{\Psi}(t) \rangle = \tilde{F}(t), \]

(5.86)

where \( \tilde{F}_z(t) \) is the force operator defined in Eq. (5.73).

The calculation of the force \( \tilde{F}(t) \) is essentially the same in cases (i) and (ii). The similarity is emphasized by the schematic representations in Fig. 5.1 (c) and Fig. 5.2 (b), in which the disorder potential is executing free oscillations. The difference arises in the initial state. For case (i) the initial state is the ground state for the harmonic potential in the absence of the disorder potential, whereas for case (ii) the initial state is the ground state in the presence of the disorder potential. In either case, the subsequent evolution is governed by the same Hamiltonian in Eq. (5.69). Analogous to the physical situations we have discussed in the previous chapter, the force \( \tilde{F}(t) \) can be determined from the response of a stationary condensate to a moving disorder potential; the only difference is that here the external (disorder) potential moves with a time-dependent velocity instead of a constant velocity. As before, the force \( \tilde{F}(t) \) can be shown to enter the expression for the rate of energy absorption. Specifically, we have (see also Eq. (4.3))

\[
\frac{d\tilde{E}}{dt} = \left\langle \tilde{\Psi}(t) \left| \frac{\partial \hat{V}_{\text{dis}}(\mathbf{x}(t))}{\partial t} \right| \tilde{\Psi}(t) \right\rangle
= -\mathbf{v}_{\text{dis}}(t) \cdot \int d\mathbf{r} \nabla \hat{V}_{\text{dis}}(\mathbf{r} + \mathbf{x}(t)) n(\mathbf{r}, t),
\]

(5.87)

where \( n(\mathbf{r}, t) = \langle \tilde{\Psi}(t) | \hat{n}(\mathbf{r}) | \tilde{\Psi}(t) \rangle \) is the expectation value of the density and \( \mathbf{v}_{\text{dis}}(t) = \)
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\(-d\mathbf{x}(t)/dt\) is the velocity of the disorder potential. Since \(\mathbf{v}_{\text{dis}}(t) = \mathbf{v}_{\text{dis}}(t) \hat{z}\), we have

\[
\frac{d\tilde{E}}{dt} = \mathbf{v}_{\text{dis}}(t) \tilde{F}(t),
\]

(5.88)

where \(\tilde{F}(t)\) is defined in Eqs. (5.72) or (5.86). This result is a simple generalization of Eq. (4.5) to a time-dependent velocity. The present discussion makes it clear why such a generalization is of interest.

5.4 Linear response calculation of damping rates

In the previous section, we have seen that the centre of mass trajectory of the moving condensate \(Z(t)\) is completely determined by the force \(F(t)\) exerted by the static disorder potential. In general, \(F(t)\), and hence \(Z(t)\), will be very complex functions of time and vary from one configuration of the disorder potential to another. However, we are only interested in the dynamics of the condensate averaged over all possible realizations of the disorder potential. Performing such an average of Eq. (5.54), we obtain

\[
\frac{d^2 \bar{Z}(t)}{dt^2} + \omega^2 \bar{Z}(t) = \frac{\bar{F}(t)}{M},
\]

(5.89)

where the bar denotes a disorder average. To determine the disorder-averaged force, we will restrict ourselves to the case of a weak disorder potential, where the motion of the centre of mass is expected to be weakly damped.

5.4.1 Case (i)

To motivate the method by which the theoretical damping rate is determined, we show in Fig. 5.3 (left) an experimental trace of the centre of mass position vs time.
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It is clear from the time-dependence of the envelope of the oscillations in this figure that the damping rate is in fact time-dependent. Much of this time-dependence is due to the fact that the properties of the condensate change significantly over the duration of the experiment. In particular, as the centre of mass motion decays, the internal temperature of the system continuously increases. This can be seen in Fig. 5.3 (right) which shows that the size of the condensate is much smaller by the time the oscillations have damped out.

Figure 5.3: On the left, centre of mass damped motion of a condensate due to disorder potential. On the right, in situ phase-contrast images of the condensate at various times corresponding to the data in the left figure. Both figures are taken from [18].

To extract a damping rate experimentally [17, 18], one can fit the oscillations in some time interval to a damped oscillation, that is

\[ \ddot{Z}(t) = A e^{-bt} \cos(\Omega t + \phi), \]  

(5.90)

where \( A, b, \Omega \) and \( \phi \) are parameters that depend on the specific interval chosen. In the experimental analysis, the frequency \( \Omega \) is fixed to be \( \sqrt{\omega_z^2 - b^2} \), which for weak
damping is very close to $\omega_z$. In this analysis, the damping is characterized by the parameter $b$.

Our theoretical analysis, based on linear response theory, cannot of course be used to describe the experimental behaviour over the full range of times since the theory assumes that the system is weakly perturbed (internally) from its ground state. This limits the theoretical calculations of the damping to early times following the initiation of the centre of mass oscillation. In other words, we are only able to extract a damping rate at early times. To do so, we assume that the displacement given in Eq. (5.74) for case (i) corresponds to that of a damped oscillator

$$\ddot{Z}(t) = -\frac{v_0}{\Omega} e^{-bt} \sin \Omega t. \tag{5.91}$$

It will turn out to be more convenient to consider the velocity which for the damped oscillator is given by

$$\dot{Z}(t) = -v_0 e^{-bt} \left( \cos \Omega t - \frac{b}{\Omega} \sin \Omega t \right). \tag{5.92}$$

If $b \ll \omega_z$, one finds that

$$\dot{Z}(t + T) = e^{-bT} \dot{Z}(t) + O \left( \frac{b^2}{\omega_z^2} \right). \tag{5.93}$$

Thus we have the approximate formula

$$b \simeq \frac{1}{T} \left| \frac{\Delta \dot{Z}(t)}{\dot{Z}(t)} \right|, \tag{5.94}$$

---

We now take $t_0 = 0$ for convenience.
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where \( \Delta \dot{Z}(t) = \dot{Z}(t + T) - \dot{Z}(t) \). This shows that the velocity change over one period of free oscillation is a useful measure of the damping in the \( b \ll \omega_z \) limit.

From Eq. (5.74), we find that the centre of mass velocity is given by

\[
\dot{Z}(t) = -v_0 \cos \omega_z t + \frac{1}{M} \int_0^t dt' \cos \omega_z (t - t') \tilde{F}(t').
\]  

(5.95)

Defining \( \Delta \dot{Z}_l \equiv \dot{Z}(T_l) - \dot{Z}(T_{l-1}) \) where \( T_l = lT \), we have

\[
\Delta \dot{Z}_l = \frac{1}{M} \int_{T_{l-1}}^{T_l} dt' \cos \omega_z t' \tilde{F}(t').
\]  

(5.96)

If the damping is indeed weak, we have \( \dot{Z}(T_{l-1}) \simeq -v_0 \) in Eq. (5.94). Thus we can approximate the damping rate in the \( l \)-th interval as

\[
b_l \simeq \frac{1}{T} \left| \frac{\Delta \dot{Z}_l}{v_0} \right|.
\]  

(5.97)

Using Eq. (5.96) and Eq. (5.97), we find that the relative damping rate is given by

\[
b_l \omega_z = \frac{1}{2 \pi M v_0^2} \int_{T_{l-1}}^{T_l} dt' v_{\text{dis}}(t') \tilde{F}(t'),
\]  

(5.98)

where we recall that \( v_{\text{dis}}(t) = v_0 \cos \omega_z t \). From Eq. (5.72), we recall that \( F(t) = \tilde{F}(t) \), where \( \tilde{F}(t) \) is the force experienced by the condensate for the situation in Fig. 5.1 (c). We thus observe that the integrand in Eq. (5.98) is determined by the energy absorption rate in Eq. (5.88). The relative damping rate in Eq. (5.98) can then be
expressed conveniently as

\[
\frac{b_I}{\omega_z} = \frac{1}{2\pi M v_0^2} \int_{T_{l-1}}^{T_l} \frac{d\tilde{E}}{dt} dt.
\] (5.99)

We shall use this formula to analyze the experimental damping rate at early times.

We now determine \(b_I/\omega_z\) appearing in Eq. (5.99) using linear response theory. The density in Eq. (5.87) can be written as

\[
n(r, t) = n_{eq}(r) + \delta n(r, t),
\] (5.100)

where \(n_{eq}(r) = \langle \Psi_0 | \hat{n}(r) | \Psi_0 \rangle\) is the initial equilibrium density of the condensate and \(\delta n(r, t)\) is the density fluctuation induced by the moving potential. Assuming that the disorder potential is weak, the density fluctuation is given by the linear response theory expression

\[
\delta n(r, t) = -\int dr' \int_0^t dt' \chi(r, r'; t - t') V_{\text{dis}}(r' + x(t')), \tag{5.101}
\]

where the ground state density response function \(\chi(r, r', t - t')\) is defined in Eq. (3.1) and \(x(t) = z_0 \sin \omega_z t\). Substituting Eq. (5.100) with \(\delta n(r, t)\) given by Eq. (5.101) into Eq. (5.87), we get

\[
\frac{d\tilde{E}}{dt} = v(t) \cdot \int dr \nabla V_{\text{dis}}(r + x(t)) n_{eq}(r) \\
- v(t) \cdot \int dr \int dr' \int_0^t dt' \nabla V_{\text{dis}}(r + x(t)) V_{\text{dis}}(r' + x(t')) \chi(r, r', t - t').
\] (5.102)
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The disorder potential in Eq. (5.102) is represented as

\[ V_{\text{dis}}(r) = \int \frac{dq}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{r}} \tilde{V}_{\text{dis}}(\mathbf{q}), \]  

(5.103)

where the Fourier amplitudes \( \tilde{V}_{\text{dis}}(\mathbf{q}) \) are stochastic variables having the average properties given by Eqs. (2.143) and (2.144). Inserting Eq. (5.103) into Eq. (5.102) and performing the disorder averages according to Eqs. (2.143) and (2.144), we find

\[ \frac{d\tilde{E}}{dt} = V_D^2 \int \frac{dq}{(2\pi)^3} i\mathbf{q} \cdot \mathbf{v}(t) \tilde{C}(\mathbf{q}) \int_0^t dt' e^{i(q\mathbf{x}(t)-q\mathbf{x}(t'))} \chi(\mathbf{q}, \mathbf{q}, t - t'), \]  

(5.104)

where \( \tilde{C}(\mathbf{q}) \) is given by Eq. (2.145). Substituting Eq. (5.104) into Eq. (5.99) and using Eq. (3.9) for the response function \( \chi(\mathbf{q}, \mathbf{q}, t - t') \), we obtain

\[ \frac{b_l}{\omega_z} = \frac{V_D^2}{2\pi M v_0} \int_0^{\infty} d\omega \int \frac{dq}{(2\pi)^3} q_z \tilde{C}(\mathbf{q}) S(\mathbf{q}, \omega) \left[ I_l(q_z, \omega) - I_l(q_z, -\omega) \right], \]  

(5.105)

where

\[ I_l(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \int_0^t dt' \cos \omega_z t e^{iq_z z_0 \sin \omega_z t - iq_z z_0 \sin \omega_z t'} e^{-i\omega(t-t')}. \]  

(5.106)

The integral in Eq. (5.106) can be performed analytically (see Appendix C for more details). The result is

\[ I_l(q_z, \omega) = \frac{2\pi i}{q_z v_0 \omega_z} e^{-i(2l-1)\pi \omega} J_{-\omega}(q_z z_0) \left( \frac{\pi \omega}{\sin \pi \omega} J_{-\omega}(q_z z_0) - 1 \right), \]  

(5.107)
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where \( \bar{\omega} = \omega / \omega_z \) and \( J_\nu(x) \) is the so-called Anger function defined as [59]

\[
J_\nu(x) = \frac{1}{\pi} \int_0^\pi d\theta \cos(x \sin \theta - \nu \theta). \tag{5.108}
\]

If \( \nu \) is an integer \( n \), the Anger function \( J_\nu(x) \) becomes the Bessel function of the first kind \( J_n(x) \) of integral order

\[
J_n(x) = \frac{1}{\pi} \int_0^\pi d\theta \cos(x \sin \theta - n\theta). \tag{5.109}
\]

In general \( J_\nu(x) \) is related to the Bessel function \( J_\nu(x) \) of non-integral order through

\[
J_\nu(x) = J_\nu(x) + \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-x \sinh(u) - \nu u} du \tag{5.110}
\]

for positive \( x \) [59].

To simplify Eq. (5.105) we write \( I_l(q_z, \omega) \) in Eq. (5.107) as

\[
I(q_z, \omega) = \frac{2\pi i}{q_z v_0 \omega_z} e^{-i(2l-1)\pi \bar{\omega}} f(q_z, \omega), \tag{5.111}
\]

where

\[
f(q_z, \omega) \equiv J_{-\bar{\omega}}(q_z z_0) \left( \frac{\pi \bar{\omega}}{\sin \pi \bar{\omega}} J_{-\bar{\omega}}(q_z z_0) - 1 \right). \tag{5.112}
\]
Substituting Eq. (5.111) into Eq. (5.105) we get

\[
\frac{b_l}{\omega_z} = \frac{iV_D^2}{M v_0^2} \int_0^\infty d\bar{\omega} \int \frac{dq}{(2\pi)^3} \bar{C}(q) S(q, \omega) \times \left[ e^{-i(2l-1)\pi\bar{\omega}} f(q_z, \omega) - e^{i(2l-1)\pi\bar{\omega}} f(q_z, -\omega) \right]
\]

\[
= \frac{iV_D^2}{M v_0^2} \int_0^\infty d\bar{\omega} \int \frac{dq}{(2\pi)^3} \bar{C}(q) S(q, \omega) \times \left[ e^{-i(2l-1)\pi\bar{\omega}} f(q_z, \omega) - e^{i(2l-1)\pi\bar{\omega}} f(q_z, -\omega) \right],
\]

(5.113)

where we used the fact that \(S(-q, \omega) = S(q, \omega)\) and \(\bar{C}(-q) = \bar{C}(q)\) to change the sign of \(q_z\) in the first term in the square brackets. From its definition in Eq. (5.108), one can see that the Anger function has the property \(J_{-\nu}(x) = J_{\nu}(-x)\). As a result, we have

\[
f(-q_z, \omega) = f(q_z, -\omega).
\]

(5.114)

Using Eq. (5.114) in Eq. (5.113), we finally arrive at

\[
\frac{b_l}{\omega_z} = \frac{2\pi V_D^2}{M v_0^2} \int_0^\infty d\bar{\omega} \int \frac{dq}{(2\pi)^3} \bar{\omega}\bar{C}(q) S(q, \omega) \times \left[ \tilde{\Delta}_l(\bar{\omega}) J_{\nu}^2(2q_z z_0) - \tilde{\delta}_l(\bar{\omega}) J_{\nu}(q_z z_0) \right],
\]

(5.115)

where

\[
\tilde{\Delta}_l(\bar{\omega}) = \frac{\sin(2l-1)\pi\bar{\omega}}{\sin(\pi\bar{\omega})}
\]

(5.116)

and

\[
\tilde{\delta}_l(\bar{\omega}) = \frac{\sin(2l-1)\pi\bar{\omega}}{\pi\bar{\omega}}.
\]

(5.117)

The periodic function defined in Eq. (5.116) is in fact the Dirichlet kernel which arises in Fourier analysis [60]. Equation (5.115) is our main result for the damping
of the centre of mass velocity during the $l$-th period of the oscillation. We emphasize that the only approximation we have made to this point is that of a weak disorder potential.

In the case of a one-dimensional disorder potential of the kind described in [17, 18], the function $\tilde{C}(q)$ is given by Eq. (2.145). Substituting Eq. (2.145) into Eq. (5.115), we find

$$\frac{b_l}{\omega_z} = \frac{\pi^{1/2} V^2_0}{M v_0^2} \int_0^\infty d\omega \int_{-\infty}^{\infty} dq_z \omega e^{-\omega^2 q_z^2/4} S(q_z, \omega)$$

$$\times \left[ \tilde{\Delta}_l(\omega) J_{\tilde{z}}^2(q_z v_0/\omega_z) - \delta_l(\omega) J_{\tilde{z}}(q_z v_0/\omega_z) \right],$$

(5.118)

where we have used $z_0 = v_0/\omega_z$ to display the dependence of the damping rate on the initial velocity $v_0$ of the oscillating disorder potential. As we did earlier, we use the simplified notation $S(q_z, \omega) \equiv S(q = q_z \tilde{z}, \omega)$ for the dynamic structure factor. This function contains all the information regarding the excitations of the system. The way in which these excitations contribute to the damping rate is determined by the frequency- and wavevector-dependent weighting factors appearing in the above integral.

The $l$-dependence of the damping rate is a consequence of suddenly switching on the disorder potential at $t = 0$. As one might expect, there will be transients in the response of the system at early times. However, as we shall see, the $l$-dependence of $b_l$ is in fact very weak and the damping rate quickly reaches a limiting value. This value can be obtained by simply taking the $l \to \infty$ limit. In this situation, we can
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make use of the following properties of the functions $\tilde{\Delta}_l(\bar{\omega})$ and $\tilde{\delta}_l(\bar{\omega})$:

$$
\lim_{l \to \infty} \tilde{\Delta}_l(\bar{\omega}) = \sum_{n=-\infty}^{\infty} \delta(\bar{\omega} - n),
$$

(5.119)

where $n$ is an integer and

$$
\lim_{l \to \infty} \tilde{\delta}_l(\bar{\omega}) = \delta(\bar{\omega}).
$$

(5.120)

Using these limiting forms in Eq. (5.115), we find

$$
b_{\infty}^\omega = \frac{2\pi V^2}{M v_0^2} \sum_{n=1}^{\infty} n \int dq \frac{d}{(2\pi)^3} \tilde{C}(q) J^2_n(qz v_0/\omega_n) S(q, \omega_n),
$$

(5.121)

where $\omega_n = n\omega_z$. We observe that only these discrete frequencies enter the dynamic structure factor. The result for a one-dimensional disorder potential is obtained by substituting Eq. (2.145) into Eq. (5.121). We find

$$
b_{\infty}^\omega = \frac{2\pi^{1/2} V^2}{\sigma M v_0^2} \sum_{n=1}^{\infty} n \int dq z e^{-\sigma^2 q_z^2/4} J^2_n(qz v_0/\omega_n) S(q, \omega_n),
$$

(5.122)

where we used the fact that $J_n(-x) = (-1)^n J_n(x)$ and $S(-q_z, \omega) = S(q_z, \omega)$ to restrict the $q_z$ integration to positive values. Eq. (5.122) is one of the key results of this thesis.

5.4.2 Case (ii)

In this case, as can be seen in Fig. 5.2 (a), the initial position of the centre of mass is displaced by $\hat{Z}(t = 0) = z_0$. With this initial condition and $\hat{Z}(t = 0) = 0$, a damped
oscillator experiences the displacement

\[ \ddot{Z}(t) = z_0 e^{-b t} \left( \cos \Omega t + \frac{b}{\Omega} \sin \Omega t \right), \quad (5.123) \]

where \( \Omega = \sqrt{\omega_0^2 - b^2} \). For \( b \ll \omega_0 \), one finds

\[ \frac{b}{\omega_0} \approx \frac{1}{2\pi} \left| \frac{\Delta \ddot{Z}(t)}{Z(t)} \right|, \quad (5.124) \]

where \( \Delta \ddot{Z}(t) \equiv \ddot{Z}(t + T) - \ddot{Z}(t) \). Defining \( \Delta \ddot{Z}_i = \ddot{Z}(T_i) - \ddot{Z}(T_{i-1}) \), we have

\[ \frac{b}{\omega_0} = \frac{1}{2\pi} \left| \frac{\Delta \ddot{Z}_i}{Z(T_{i-1})} \right| \approx \frac{1}{2\pi} \left| \frac{\Delta \ddot{Z}_i}{z_0} \right|, \quad (5.125) \]

where we used the assumption that \( \ddot{Z}(T_{i-1}) \approx z_0 \) for the first few oscillations. This relationship between the damping rate and change in amplitude over one period can be used to obtain a damping rate from the solution of Eq. (5.54). For the assumed initial conditions, we have

\[ \ddot{Z}(t) = z_0 \cos \omega_0 t + \frac{1}{M \omega_0} \int_0^t dt' \sin \omega_0 (t - t') \ddot{F}(t'), \quad (5.126) \]

which gives rise to

\[ \left| \frac{\Delta \ddot{Z}_i}{z_0} \right| = \frac{1}{M z_0 \omega_0} \int_{T_{i-1}}^{T_i} dt \sin \omega_0 t \ddot{F}(t). \quad (5.127) \]

Substitution of Eq. (5.127) into Eq. (5.125) yields the damping rate

\[ \frac{b_i}{\omega_0} = \frac{1}{2\pi M \omega_0 z_0} \int_{T_{i-1}}^{T_i} dt \sin \omega_0 t \ddot{F}(t). \quad (5.128) \]

The force \( \ddot{F}(t) \) appearing in Eq. (5.128) corresponds to the picture illustrated in
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Fig. 5.2 (a). As explained in Sec. 5.3.2, this force is equal to the force $\tilde{F}(t)$ experienced by the condensate in Fig. 5.2 (b). The latter can again be evaluated by means of the linear response theory. Rather than following the method used in case (i), we evaluate $\tilde{F}(t)$ directly using the equation

$$\tilde{F}(t) = \langle \tilde{\Psi}(t)|\tilde{F}_z|\tilde{\Psi}(t) \rangle = \langle \tilde{\Psi}_1(t)|\tilde{F}_{z,1}|\tilde{\Psi}_1(t) \rangle. \quad (5.129)$$

A perturbative analysis of Eq. (5.83) yields

$$|\tilde{\Psi}_1(t)\rangle \simeq |\Phi_0\rangle - \frac{i}{\hbar} \int_0^t dt' \tilde{V}_{\text{dis},1}(x(t'), t') |\Phi_0\rangle, \quad (5.130)$$

where we recall that $|\Phi_0\rangle$ is the ground state of the system in the presence of the disorder potential, namely the ground state of $\tilde{H}$ in Eq. (5.79). To lowest order in the disorder potential with respect to the ground state of $\tilde{H}$, i.e., $|\Psi_0\rangle$, we have

$$|\tilde{\Phi}_0\rangle \simeq |\Psi_0\rangle - \int_{-\infty}^0 dt' e^{\epsilon t'} \tilde{V}_{\text{dis},1}(x, t') |\Psi_0\rangle, \quad (5.131)$$

where $\epsilon$ is a positive infinitesimal number. Substituting Eq. (5.130) into Eq. (5.129) and using Eq. (5.131), we find to second order in the disorder potential that

$$\tilde{F}(t) = \langle \Psi_0|\tilde{F}_{z,1}(t)|\Psi_0 \rangle - \frac{i}{\hbar} \int_{-\infty}^0 dt' e^{\epsilon t'} \langle \Psi_0|[\tilde{F}_{z,1}(t), \tilde{V}_{\text{dis},1}(x, t')]|\Psi_0 \rangle - \frac{i}{\hbar} \int_0^t dt' \langle \Psi_0|[\tilde{F}_{z,1}(t), \tilde{V}_{\text{dis},1}(x(t'), t')]|\Psi_0 \rangle. \quad (5.132)$$

We now write

$$\tilde{V}_{\text{dis},1}(x(t), t) = \int dr \tilde{V}_{\text{dis}}(r + x(t)) \tilde{n}_1(r, t), \quad (5.133)$$
where \( \hat{n}_1(\mathbf{r}, t) \) is the density operator in the interaction picture. Inserting Eq. (5.133) into Eq. (5.132) and performing the disorder average according to Eq. (2.143) and (2.144), we find that \( \bar{F}(t) = \bar{F}_1(t) + \bar{F}_2(t) \) with

\[
\bar{F}_1(t) = iV^2_D \int_{-\infty}^0 dt' e^{it'} \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})q_z e^{i\mathbf{q}[x(t)-x]} \chi(\mathbf{q}, \mathbf{q}, t-t'),
\]

\[
\bar{F}_2(t) = iV^2_D \int_{0}^t dt' \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})q_z e^{i\mathbf{q}[x(t)-x(t)']} \chi(\mathbf{q}, \mathbf{q}, t-t'),
\]

where the response function \( \chi(\mathbf{q}, \mathbf{q}, t-t') \) is defined according to Eq. (3.9).

Using Eq. (3.9) in Eq. (5.134) we find

\[
\int_{T_{l-1}}^{T_l} dt \sin \omega_z t \bar{F}_1(t) = -V^2_D \int d\omega \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})q_z S(\mathbf{q}, \omega) \left[ I_{l(1)}^{(1)}(q_z, \omega) - I_{l(1)}^{(1)}(q_z, -\omega) \right]
\]

(5.136)

where \( I_{l(1)}^{(1)}(q_z, \omega) \) is given by

\[
I_{l(1)}^{(1)}(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \sin \omega_z t \int_{-\infty}^0 dt' e^{it'} e^{iqz_0 \cos \omega_z (t-1)} e^{-i\omega(t-t')}.
\]

(5.137)

Using the fact that \( \tilde{C}(-\mathbf{q}) = \tilde{C}(\mathbf{q}) \) and \( S(-\mathbf{q}, \omega) = S(\mathbf{q}, \omega) \), we rewrite Eq. (5.136) as

\[
\int_{T_{l-1}}^{T_l} dt \sin \omega_z t \bar{F}_1(t) = -V^2_D \int d\omega \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})q_z S(\mathbf{q}, \omega) \left[ I_{l(1)}^{(1)}(q_z, \omega) + I_{l(1)}^{(1)*}(q_z, \omega) \right].
\]

(5.138)

Using the Jacobi-Anger expansion, we find that \( I_{l(1)}^{(1)}(q_z, \omega) \) can be written as (see
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Appendix C for more details)

\[ I_l^{(1)}(q_z, \omega) = -\frac{1}{q_z v_0 \omega_z} \sum_{n, n' = -\infty}^{\infty} i^{n-n'} n J_n(q_z z_0) J_{n'}(q_z z_0) \frac{e^{-i2l\pi \bar{\omega}} - e^{-i2(l-1)\pi \bar{\omega}}}{(\bar{\omega} - n)(\bar{\omega} - i\epsilon)}. \]  (5.139)

where \( \bar{\omega} \equiv \omega / \omega_z \). We observe that in the above summation the terms with odd values of \( n - n' \) are even in \( q_z \) due to the fact that \( J_n(-x) = (-1)^n J_n(x) \). As a result these terms do not contribute to the integral in Eq. (5.138) since the factor \( \tilde{C}(q)q_z S(q, \omega) \) is an odd function of \( q_z \). Bearing this in mind, we substitute Eq. (5.139) into Eq. (5.138) and find that

\[
\int_{T_{l-1}}^{T_l} dt \sin \omega_z t \bar{F}_1(t) = \frac{2V_D^2}{v_0} \int d\bar{\omega} \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q}) S(\mathbf{q}, \omega) \times \sum_{n, n' = -\infty}^{\infty} i^{n-n'} n J_n(q_z z_0) J_{n'}(q_z z_0) \frac{\cos 2l\pi \bar{\omega} - \cos 2(l-1)\pi \bar{\omega}}{(\bar{\omega} - n)\bar{\omega}}.
\]  (5.140)

Similarly, using Eq. (3.9) in Eq. (5.135) we find

\[
\int_{T_{l-1}}^{T_l} dt \sin \omega_z t \bar{F}_2(t) = -V_D^2 \int d\omega \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q}) q_z S(\mathbf{q}, \omega) \left[ I_l^{(2)}(q_z, \omega) + I_l^{(2)*}(q_z, \omega) \right],
\]  (5.141)

where (see Appendix C for more details)

\[
I_l^{(2)}(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \sin \omega_z t \int_0^t dt' e^{iq_z z_0 (\cos \omega_z t' - \cos \omega_z t')} e^{-i\omega(t-t')} \frac{e^{-i2l\pi \bar{\omega}} - e^{-i2(l-1)\pi \bar{\omega}}}{(\bar{\omega} - n)(\bar{\omega} - n')}.
\]  (5.142)
Substituting Eq. (5.142) into Eq. (5.141) we obtain

\[
\int_{T_{l-1}}^{T_l} dt \sin \omega_z t \tilde{F}_2(t) = \frac{2V_D^2}{v_0} \int d\tilde{\omega} \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})S(\mathbf{q}, \omega) 
\times \sum_{n, n'=-\infty}^{\infty} i^{n-n'} n J_n(q_z z_0) J_{n'}(q_z z_0) \frac{\cos 2(l-1)\pi \tilde{\omega} - \cos 2l\pi \tilde{\omega}}{\tilde{\omega} - n)(\tilde{\omega} - n')}. \tag{5.143}
\]

As in case (i), we will focus on the damping rate \(b_\infty\) in the \(l \to \infty\) limit. In order to take this limit in Eqs. (5.140) and (5.143), we make use of the following identity

\[
\lim_{l \to \infty} \frac{\cos 2(l-1)\pi \tilde{\omega} - \cos 2l\pi \tilde{\omega}}{(\tilde{\omega} - n)(\tilde{\omega} - n')} = \lim_{l \to \infty} \frac{[1 - \cos 2l\pi \tilde{\omega}] - [1 - \cos(2(l-1)\pi \tilde{\omega})]}{(\tilde{\omega} - n')(\tilde{\omega} - n)} 
= \lim_{l \to \infty} (\tilde{\omega} - n') \left\{ \frac{1 - \cos 2l\pi \tilde{\omega}}{(\tilde{\omega} - n')^2} - \frac{1 - \cos 2\pi(l-1)\tilde{\omega}}{(\tilde{\omega} - n)^2} \right\} 
= 2\pi^2 \delta_{nn'} \delta(\tilde{\omega} - n), \tag{5.144}
\]

where we have used

\[
\lim_{\alpha \to \infty} \frac{1 - \cos(\alpha x)}{\pi \alpha x^2} = \lim_{\alpha \to \infty} \frac{\sin^2(\alpha x/2)}{\pi \alpha x^2/2} = \delta(x). \tag{5.145}
\]

Using Eq. (5.144) in Eq. (5.140) and Eq. (5.143), we immediately find

\[
\lim_{l \to \infty} \int_{T_{l-1}}^{T_l} dt \sin \omega_z t \tilde{F}_1(t) = 0 \tag{5.146}
\]

and

\[
\lim_{l \to \infty} \int_{T_{l-1}}^{T_l} dt \sin \omega_z t \tilde{F}_2(t) = \frac{4\pi^2 V_D^2}{v_0} \sum_{n=0}^{\infty} n \int \frac{d\mathbf{q}}{(2\pi)^3} \tilde{C}(\mathbf{q})J_n^2(q_z z_0)S(\mathbf{q}, \omega). \tag{5.147}
\]
Substituting Eq. (5.146) and Eq. (5.147) into Eq. (5.127) we finally find

\[
\frac{b_\infty}{\omega_z} = \frac{2\pi V_{Dz}^2}{M v_0^2} \sum_{n=0}^{\infty} n \int \frac{dq}{(2\pi)^3} \tilde{C}(q) J_n^2(q_z v_0/\omega_z) S(q, \omega), \tag{5.148}
\]

where we have again replaced \( z_0 \) by \( v_0/\omega_z \). In this case, \( v_0 \) is the maximum speed of the oscillating disorder potential. We see that Eq. (5.148) is exactly the same as the relative damping rate given by Eq. (5.121) in case (i). In other words, we find that apart from some initial transients, the damping for the situations in Fig. 5.1 (b) and Fig. 5.2 (a) are the same.

### 5.4.3 Results

First, we investigate the behaviour of the damping as a function of the velocity \( v_0 \). To be specific, we consider a condensate consisting of \( N = 2 \times 10^5 \) \(^7\)Li atoms in an axially symmetrical trap with \( \omega_\perp = 2\pi \times 260\text{Hz} \) and \( \omega_z = 2\pi \times 5.5\text{Hz} \). The condensate is thus highly elongated with an aspect ratio \( \lambda \simeq 0.02 \). The s-wave scattering length of the atoms in this system is \( a_s = 200a_0 \), where \( a_0 \) is the Bohr radius. The system parameters that we have specified here are typical of the experiments carried out in [16, 17, 18].

Using the above parameters, the TF chemical potential given by Eq. (2.23) is found to be \( \mu_{TF} \simeq h \times 1.25\text{kHz} = 4.8h\omega_\perp \) and the axial TF radius of the condensate is \( R_z \simeq 345\mu\text{m} \). This indicates that the density of the condensate can be estimated with sufficient accuracy using the TF approximation. The TF linear density \( \nu(z) \) given in Eq. (3.112) is then used to determine the dynamic structure factor \( S(q_z, \omega) \) in Eq. (5.118) and Eq. (5.122) within cylindrical LDA. Here we have the option of determining the modes using the TF hydrodynamics or the full solution of the BdeG
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To illustrate our numerical calculations in more detail, we focus on the damping rate $b_\infty$ given in Eq. (5.122). We see from this expression that the velocity dependence arises from the $1/v_0^2$ prefactor as well as the $J_n^2(q_z v_0/\omega_z)$ factors. A detailed analysis is required in order to reveal how these dependences manifest themselves in the final velocity dependence of $b_\infty$.

Substituting Eq. (3.106) into Eq. (5.122), we obtain

$$b_\infty = \frac{2\pi^{1/2}V_D^2 \sigma}{M v_0^2} \sum_{n=1}^{\infty} n \int_0^{\infty} dq_z e^{-\sigma^2 q_z^2/4} J_n^2(q_z v_0/\omega_z) S_{\text{cyl}}^{\text{LDA}}(q_z, \omega_n)$$

$$= \frac{2\pi^{1/2}V_D^2 \sigma}{h M v_0^2} \sum_{n=1}^{\infty} n \int_{-R_z}^{R_z} dz \nu(z) \int_0^{\infty} dq_z e^{-\sigma^2 q_z^2/4} J_n^2(q_z v_0/\omega_z)$$

$$\times W_j(q_z; \nu(z)) \delta \left( \omega_j(q_z; \nu(z)) - n\omega_z \right). \quad (5.149)$$

A contribution to $b_\infty$ is thus obtained whenever $\omega_j(q_z; \nu(z)) = n\omega_z$. This condition is illustrated schematically in Fig. 5.4 (left) for $z = 0$ by the intersections of the horizontal lines $n\omega_z$ with the frequencies of the various modes $\omega_j$. Performing the integral with respect to $q_z$ in Eq. (5.149), we find

$$b_\infty = \frac{2\pi^{1/2}V_D^2 \sigma}{h M v_0^2} \int_{-R_z}^{R_z} dz \nu(z) \sum_{j} \sum_{n=1}^{\infty} n e^{-\sigma^2 q_{jn}^2/4} J_n^2(q_{jn} v_0/\omega_z) W_j(q_{jn}; \nu(z))' \left| \omega_j'(q_{jn}; \nu(z)) \right|, \quad (5.150)$$

where $q_{jn}$ are the solutions to the equation $\omega_j(q_z; \nu(z)) - n\omega_z = 0$ and $\omega_j' \equiv \partial \omega_j / \partial q_z$. Although the sums in Eq. (5.150) are infinite, only a limited number of terms contribute significantly. The Gaussian factor $e^{-\sigma^2 q_{jn}^2/4}$ provides a cut-off which is indicated in Fig. 5.4 (left) by the vertical dashed line at $q_c \sim 1/\sigma$. This cut-off restricts the contributions to those intersections occurring to the left of the vertical dashed
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The frequencies \( n\omega_z \) corresponding to these intersections are indicated by the highlighted horizontal lines. With increasing \( \sigma \) it is clear that a given mode \( j \) will contribute a finite number of terms. A similar kind of cut-off with respect to the mode index \( j \) is provided by the factor \( W_j(q_z; \nu(0))/|\bar{\omega}'_j(q_z; \nu(0))| \). This factor is plotted for \( z = 0 \) in Fig. 5.4 (right) as a function of \( q_z \). From this figure it is clear that only a few of the lowest radial modes contribute significantly.

![Figure 5.4](image)

**Figure 5.4:** On the left, the discrete mode frequencies are indicated by the horizontal lines. Due to the cut-off indicated by the vertical line, only the highlighted modes actually contribute to Eq. (5.151). On the right, the factor \( W_j(q_z; \nu(0))/|\bar{\omega}'_j(q_z; \nu(0))| \) plotted as a function of \( \bar{q}_z = q_z a_{\perp} \), where \( \bar{\omega}'_j \equiv \omega'_j/(a_{\perp} \omega_{\perp}) \).

We now define the function

\[
C(z) = \frac{2\pi^{1/2} V_0^2 \sigma}{\hbar M v_0^2} \nu(z) \sum_{n=1}^{\infty} \sum_j e^{-\sigma^2 q_{jn}^2/4} J_n^2(q_{jn} v_0/\omega_z) \frac{W_j(q_{jn}; \nu(z))}{|\bar{\omega}'_j(q_{jn}; \nu(z))|} \\
= \sum_{n=1}^{\infty} C_n(z) \tag{5.151}
\]
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in terms of which

$$\frac{b_\infty}{\omega_z} = \int_{-R_z}^{R_z} dz C(z). \tag{5.152}$$

In Fig. 5.5 (left), we plot the summand $C_n(z = 0)$ in Eq. (5.151) as a function of $n$ for various velocities around $v_0/c_0 = 1$. (Recall that $c_0$ is the sound speed of a uniform cylindrical condensate with linear density $\nu(z = 0)$.) We see that $C_n(0)$ terminates at $n_{\text{max}} \simeq 60$. This behaviour follows from the fact that the $j = 0$ mode makes the dominant contribution to $C_n(0)$ and the cut-off provided by the $e^{-\sigma^2 q_j n^2/4}$ factor. We also observe in Fig. 5.5 (left) a strong dependence on $v_0/c_0$. Details of this dependence are displayed in Fig. 5.5 (right) where $C(z)$ is plotted vs $v_0/c_0$ for various values of $z$. We find that $C(z)$ as a function of $v_0$ exhibits a sharp peak at velocities close to the local speed of sound $c_0(z)$. Since $c_0(z) \propto \nu(z)^{1/4}$, it decreases as one moves towards the ends of the condensate, and the peaks in these curves occur at $v_0/c_0 \simeq c_0(z)/c_0 < 1$. The appearance of a peak is due to the $J_n^2(q_j n v_0/\omega_z)$ factors. For the dominant $j = 0$ mode, the argument of the Bessel function is approximately $nv_0/c_0(z) = n$ for $v_0/c_0(z) = 1$, which occurs near a maximum of the Bessel functions for all orders $n$.

For $l = \infty$, the damping rates are obtained from Eq. (5.152) by integrating $C(z)$ with respect to $z$. For other values of $l$, $b_l/\omega_z$ is obtained from Eq. (5.118). These damping rates are plotted as a function of $v_0/c_0$ in Fig. 5.6 (left). Interestingly, we see that the damping rates $b_l$ depend very weakly on $l$; the results for $l = 1$ are very close to the $l = \infty$ limit except in the vicinity of $v_0/c_0 = 1$, while the $l = 2$ results are virtually indistinguishable from the $l = \infty$ limit for all velocities. This indicates that the transients in the damping of the centre of mass oscillation following its initiation are very weak. In other words, the damping experienced over the first
The period of oscillation is already close to the steady-state value.

The other notable feature of the result in Fig. 5.6 (left) is the sharp peak near \( v_0/c_0 = 1 \). It arises from the average of the curves in Fig. 5.5 (right). As we already pointed out, the peaks of these curves occur roughly at \( v_0/c_0 = c_0(z)/c_0 = \sqrt{1-(z/R_z)^2} \). As a result, they move away from the position of \( v_0/c_0 = 1 \) rather slowly as \( z \) moves away from the centre; for \( z/R_z = 0.8 \), the peak of \( C(z) \) is at \( v_0/c_0 \sim 0.7 \). Thus, aside from the extreme ends of the condensate, the peaks of the curves occur close to \( v_0/c_0 = 1 \), and when averaged over \( z \), \( b_\infty \) exhibits a peak in the neighbourhood of \( v_0/c_0 = 1 \).

The results in Fig. 5.6 (left) have been obtained using the Bogoliubov modes of a uniform condensate. However, the calculations can also be done using the TF hydrodynamic modes and it is of interest to compare the results for the damping rates using these two sets of modes. This is done in Fig. 5.6 (right) for the same set of system parameters. As we can see, the TF calculations, which are numerically
much simpler, are in very good agreement with the Bogoliubov calculations. Thus our previously published results [61] based on the TF modes are in fact quite accurate.

Figure 5.6: On the left, the damping rate $b_l/\omega_z$ plotted as a function of $v_0/c_0$ for $l = 1, 2$ and $l = \infty$. The disorder correlation length in this calculation is $\sigma = 12 \mu m$. On the right, the damping rates calculated using the TF hydrodynamic modes (dashed line) are compared to those using the Bogoliubov modes (solid line).

We now turn to the dependence of the damping rate on other system parameters. In view of the $l$-dependence shown in Fig. 5.6 (left), it is sufficient to consider only $b_\infty$. The proportionality to the square of the strength $V_D$ of the disorder potential is an obvious consequence of being in the linear response regime. It is for this reason that we have plotted $b_\infty/\omega_z$ divided by $V_D^2$ in Fig. 5.6. To reveal more clearly the other dependences it is useful to consider $b_\infty/\omega_z$ as calculated using the TF modes. Using Eq. (3.112) for $\nu(z)$ and Eq. (3.70) for $W_j(q_{jn};\nu(z))$, we find

$$
\frac{b_\infty}{\omega_z} = \frac{15\sqrt{\pi}}{4} \left( \frac{V_D^2}{\mu_{TF}^2} \right) \left( \frac{\sigma^2}{R_z} \right) \int_{-1}^{1} d\bar{z} \sqrt{1 - \bar{z}^2} e^{-\frac{\sigma^2 R_z^2}{4\lambda \sqrt{1 - \bar{z}^2}}} \sum_{n=1}^{\infty} \sum_{j} e^{-\frac{\bar{q}_{j,n}^2}{4\lambda \sqrt{1 - \bar{z}^2}}} \sum_{j} e^{-\frac{\bar{q}_{j,n}^2}{4\lambda \sqrt{1 - \bar{z}^2}}} \\
\times n^2 J_n^2 \left( \frac{\bar{q}_{j,n} v_0}{2\lambda \sqrt{1 - \bar{z}^2} c_0} \right) \frac{\bar{a}_0^2(j, \bar{q}_{j,n})}{|\bar{\omega}_j(q_{jn})|^2}, \tag{5.153}
$$
where \( \bar{z} = z/R_z \), \( \bar{q}_{jn} = q_{jn}R_\perp(z) \) and \( \bar{\omega}_j'(\bar{q}) = \partial \bar{\omega}_j/\partial \bar{q} \). All the \( \bar{z} \)-dependence is now displayed explicitly; the quantities \( \bar{q}_{jn} \) and \( \bar{\omega}_j' \) are \( \bar{z} \)-independent.\(^6\) As discussed earlier, the \( j = 0 \) term makes the dominant contribution. If we assume that the acoustic mode has a strictly linear dispersion, \( \bar{\omega}_0 = \bar{q}/2 \), we get \( \bar{\omega}_0' = 1/2 \) and \( \bar{q}_0n = 2\lambda n \). We then have

\[
\left. \frac{b_\infty}{\omega_z} \right|_{j=0} \approx \frac{15\sqrt{\pi}}{2} \left( \frac{V_0^2}{\mu_\text{TF}^2} \right) \left( \frac{\sigma}{R_z} \right) \left( \frac{c_0^2}{v_0^2} \right) \int_{-1}^{1} d\bar{z} \sqrt{1 - \bar{z}^2} \sum_{n=1}^{\infty} e^{-\frac{x^2}{2(1-\bar{z}^2)}} \\
\times n^2 J_n^2 \left( \frac{n}{\sqrt{1 - \bar{z}^2}} \frac{v_0}{c_0} \right) \bar{a}_0^2(0, \bar{q}_0n). \quad (5.154)
\]

We see from this that the expected dependence on \( \sigma \) appears through the ratio \( \sigma/R_z \).

Figure 5.7: The damping rates of the system with \( N = 10^6 \) and \( \sigma = 16.6 \mu m \) (dashed line) compared to those of the system in Fig. 5.6 (solid line).

Thus, if one increases the disorder correlation length \( \sigma \) and at the same time adjusts the number of the atoms in the system so that the ratio \( \sigma/R_z \) remains fixed, the behaviour of \( (b_\infty/\omega_z)/(V_0^2/\mu_\text{TF}^2) \) as a function of \( v_0/c_0 \) will be largely unchanged. In

\(^6\)We remind the reader that the dimensionless wavevectors defined in TF hydrodynamics calculations are \( \bar{q} = qR_\perp \), as opposed to \( \bar{q} = qa_\perp \) defined in BdeG solutions (see Sec. 3.3.2).
view of the agreement shown in Fig. 5.6 (right) between the damping rates determined by the TF modes and those by the Bogoliubov modes, we expect that this dependence also remains true for the Bogoliubov calculations. To demonstrate this, we compare the damping rate of the system in Fig. 5.6 with \( N = 2 \times 10^5 \) to that of a system with \( N = 10^6 \) and a value of \( \sigma \) chosen to make \( \sigma/R_z \) the same. Since \( R_z \propto N^{1/5} \), the disorder correlation length must be increased to \( \sigma = 5^{1/5} \times 12 \mu m = 16.6 \mu m \). The agreement between these two calculations shown in Fig. 5.7 confirms that the damping rate, when scaled by \( (V_D/\mu_{TF})^2 \), depends on \( \sigma/R_z \) in the way revealed by the TF expression in Eq. (5.154). The small deviations from a strict \( \sigma/R_z \) dependence are mainly due to two factors: the difference between the TF and Bogoliubov modes and the fact that all modes are included in the full Bogoliubov calculation of \( b_\infty \). However, their effects are clearly very weak.

![Figure 5.8](image-url)

Figure 5.8: On the left, the damping rate \( b_\infty/\omega_z \) plotted as a function of \( v_0/c_0 \) for three different correlation lengths. On the right, the damping rate \( b_\infty/\omega_z \) plotted as a function of \( \sigma/R_z \) for four different velocities.

From Eq. (5.154) we also see that, as \( \sigma \to 0 \), the damping rate \( b_\infty \) is proportional to \( \sigma \) and hence vanishes. Likewise, as \( \sigma \to \infty \), the Gaussian factor in Eq. (5.154) becomes
small and \( b_\infty \rightarrow 0 \). In this limit, the vanishing damping rate can be understood from the fact that the speckle grain size eventually becomes larger than the size of the condensate and the condensate experiences an essentially uniform potential. These two limiting cases imply the existence of a maximal damping rate at a finite correlation length \( \sigma_{\text{max}} \). However, as shown in Fig. 5.8, the value of \( \sigma_{\text{max}} \) depends on the velocity \( v_0 \); it becomes smaller as \( v_0 \) increases.

Finally, we comment on the dependence of the damping rate on the scattering length \( a_s \). This dependence can only arise in the quantities \( \mu_{\text{TF}} \), \( R_z \) and \( c_0 \) in Eq. (5.153). However, the latter two are in fact both functions of the chemical potential \( \mu_{\text{TF}} \) which we recall is given by \( \mu_{\text{TF}} = \frac{15^{2/5}}{2} \left( \frac{Na_s}{a} \right)^{2/5} \hbar \bar{\omega} \). This indicates that the damping rate depends on \( a_s \) only in the combination \( Na_s \). In other words, the variation of \( a_s \) in experiments does not provide any information that cannot be gained through variations of \( N \), and vice versa.

### 5.4.4 An alternative approach

In this section, we consider an alternative, heuristic method of calculating the energy dissipation rate of a condensate moving through a disorder potential with a non-uniform velocity \( v(t) \). The basic idea of the method is to approximate the dissipation rate at the instant of time \( t \) by that of a condensate moving at a constant velocity equal to the instantaneous velocity \( v(t) \). Although this method is much less sophisticated than our previously developed theory, it turns out to be a very good approximation to the latter.

The general formula for the energy dissipation rate of a uniformly moving condensate is given by Eq. (4.14) of the previous chapter. Converting the summations
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in Eq. (4.14) into integrals and taking the disorder average according to Eq. (2.144), we find

\[
\frac{dE}{dt} = -V_D^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \mathbf{v} \tilde{C}(\mathbf{q}) \chi(\mathbf{q}, \mathbf{q}, \omega = \mathbf{q} \cdot \mathbf{v})
\]
\[
= V_D^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \mathbf{v} \tilde{C}(\mathbf{q}) \text{Im} \chi(\mathbf{q}, \mathbf{q}, \omega = \mathbf{q} \cdot \mathbf{v}),
\]

(5.155)

where, to obtain the second line, we have used the fact that \(\text{Re} \chi(\mathbf{q}, \mathbf{q}, \omega = \mathbf{q} \cdot \mathbf{v})\) is an even function of \(\mathbf{q}\). Using Eq. (3.15) and the fact that \(S(-\mathbf{q}, \omega) = S(\mathbf{q}, \omega)\), Eq. (5.155) can be written as

\[
\frac{dE}{dt} = 2\pi V_D^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \mathbf{v} \tilde{C}(\mathbf{q}) S(\mathbf{q}, \omega = \mathbf{q} \cdot \mathbf{v})
\]
\[
= \sqrt{\pi} V_D^2 \sigma v \int_0^\infty dq_z q_z e^{-\sigma^2 q_z^2 / 4} S(q_z, \omega = q_z v),
\]

(5.156)

where we have used the expression for \(\tilde{C}(\mathbf{q})\) in Eq. (2.145).

We now suppose that the condensate is undergoing undamped harmonic motion, that is \(\mathbf{v}(t) = \hat{z} v(t) = -\hat{z} v_0 \cos \omega_z t\). The total energy dissipated over one period of oscillation is then given by

\[
\Delta E = \int_0^T \frac{dE}{dt} dt
\]
\[
= 4 \int_0^{T/4} \frac{dE}{dt} dt.
\]

(5.157)

Since \(v(t) = -v_0 \cos \omega_z t\), we have \(dv = v_0 \omega_z \sin \omega_z t dt = \omega_z \sqrt{v_0^2 - v^2} dt\). Changing the
integration variable from \( t \) to \( v \), we find

\[
\Delta E = 4 \int_0^{v_0} dv \frac{1}{\sqrt{v_0^2 - v^2}} \frac{dE}{dt}.
\]  

(5.158)

The \((v_0^2 - v^2)^{-1/2}\) weighting of the instantaneous energy dissipation rate reflects the fact that the condensate spends relatively longer times at the velocity \( v_0 \).

Assuming that the centre of mass energy behaves as \( E_{cm}(t) = E_0 e^{-2bt} \) where \( E_0 = Mv_0^2/2 \), Eq. (5.158) gives the damping rate

\[
\frac{b}{\omega_z} \equiv \frac{\Delta E}{2\pi Mv_0^2}.
\]  

(5.159)

Using Eq. (5.156), Eq. (5.158) and Eq. (5.159), we find

\[
\frac{b}{\omega_z} = \frac{2V_D^2\sigma}{\sqrt{\pi Mv_0^2\omega_z}} \int_0^{v_0} dv \frac{v}{\sqrt{v_0^2 - v^2}} \int_0^{\infty} dq z q z e^{-\sigma^2 q_z^2/4} S(q_z, \omega = q_z v). \]  

(5.160)

Interchanging the order of the integrations and making the variable change \( \bar{\omega} = q_z v/\omega_z \), we have

\[
\frac{b}{\omega_z} = \frac{2\pi^{1/2}V_D^2\sigma}{Mv_0^2} \int_0^{\infty} dq z e^{-\sigma^2 q_z^2/4} \int_0^{\bar{\omega}_0} d\bar{\omega} \frac{\bar{\omega}}{\pi \sqrt{\bar{\omega}_0^2 - \bar{\omega}^2}} S(q_z, \bar{\omega} \omega_z)
\]

\[
= \frac{2\pi^{1/2}V_D^2\sigma}{Mv_0^2} \int_0^{\infty} dq z e^{-\sigma^2 q_z^2/4} \int_0^{\infty} d\bar{\omega} g_1(\bar{\omega}) S(q_z, \bar{\omega} \omega_z),
\]  

(5.161)

where \( \bar{\omega}_0 \equiv q_z v_0/\omega_z \) and

\[
g_1(\bar{\omega}) = \theta(\bar{\omega}_0 - \bar{\omega}) \frac{\bar{\omega}}{\pi \sqrt{\bar{\omega}_0^2 - \bar{\omega}^2}}.
\]  

(5.162)

To compare this result to that obtained by the more rigorous approach, namely
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Figure 5.9: On the left, the functions $g_1(\bar{\omega})$ (dashed line) and $g_2(\bar{\omega})$ (solid line) plotted as a function of $\bar{\omega}$ for $\bar{\omega}_0 = 100$. On the right, the damping rates calculated from Eq. (5.161) (dashed line) and Eq. (5.163) (solid line). In these particular calculations, the density response function is evaluated using the TF modes.

Eq. (5.122), we will approximate the discrete sum in Eq. (5.122) by a continuous integral. This gives the expression

\[
\frac{b_\infty}{\omega_z} \simeq \frac{2\pi^{1/2}V_0^2\sigma}{Mv_0^2} \int_0^\infty dq_z e^{-\sigma^2 q_z^2/4} \int_0^\infty d\bar{\omega} g_2(\bar{\omega}) S(q_z, \bar{\omega} \omega_z),
\]

(5.163)

where

\[
g_2(\bar{\omega}) = \bar{\omega} J_2^2(\bar{\omega}_0).
\]

(5.164)

Eq. (5.161) and Eq. (5.163) have the same form but differ in the way that the dynamic structure factor $S(q_z, \omega)$ is weighted. In Fig. 5.9 (left) we show a comparison of the two weighting function $g_1(\bar{\omega})$ and $g_2(\bar{\omega})$ for a particular value of $\bar{\omega}_0$. We observe that $g_1(\bar{\omega})$ is a smooth average of the oscillating weight function $g_2(\bar{\omega})$. We would therefore expect that the damping rates calculated on the basis of Eq. (5.161) and Eq. (5.122) to be very similar. This in fact is confirmed by the results in Fig. 5.9 (right). The
differences between the two curves near $v_0/c_0$ arise from the square-root singularity in $g_1(\bar{\omega})$ which is smoothed out in $g_2(\bar{\omega})$. We conclude that overall, our heuristic method can be used to obtain reasonably accurate values of the damping rate as a function of velocity.

5.4.5 Comparisons to experiments

Finally we compare our theoretical results to the experimental observations in [17, 18]. Before we do so it is important to point out some noteworthy differences between the experiments in [17] and those in [18]. First, as we have already explained, the experimental protocol used to initiate the centre of mass motion in [17] is somewhat different from that used in [18]. However, our theoretical analysis indicates that this does not lead to a difference in the damping rates within the linear response regime. A more important difference is that the damping rates observed in [18] are considerably smaller than those observed in [17] for condensates with comparable system parameters. This can be seen in Fig. 5.10. In the left figure from [18], the damping is significantly weaker than that observed in the right figure from [17], even though the disorder strengths are comparable. More quantitatively, the initial damping rate for the condensate in the left figure is $(b/\omega_z)_{\text{exp}} \sim 0.004$ for a disorder strength $V_D/\mu_{\text{TF}} = 0.25$, whereas the initial damping rate in the right figure is an order of magnitude larger at $(b/\omega_z)_{\text{exp}} \sim 0.04$ for a disorder potential with a smaller strength $V_D/\mu_{\text{TF}} = 0.08$. The condensate in the latter experiment is almost overdamped at a disorder strength $V_D/\mu_{\text{TF}} = 0.24$.

We find that our theoretical calculations of the damping rates are in reasonably good agreement with the experimental results in [17] but overestimate the damping.

\footnote{These experiments are in fact all conducted by the same experimental group.}
Figure 5.10: Centre of mass damping due to disorder potential. For the figure on the left (taken from [18]), \( \omega_z = 2\pi \times 5.5\text{Hz} \), \( \omega_\perp = 2\pi \times 260\text{Hz} \) and \( \mu_{\text{TF}} = h \times 1.1\text{kHz} \). The strength of the disorder potential is \( V_D/\mu_{\text{TF}} = 0.25 \) and the correlation length is \( \sigma = 3.89\mu\text{m} \). For the figure on the right (taken from [17]), \( \omega_z = 2\pi \times 3.6\text{Hz} \), \( \omega_\perp = 2\pi \times 180\text{Hz} \) and \( \mu_{\text{TF}} = h \times 1\text{kHz} \). The correlation length of the disorder potential is \( \sigma = 10.61\mu\text{m} \).

rates observed in [18]. For the condensate in Fig. 5.10 (right), the initial displacement of the harmonic potential of \( \sim 700\mu\text{m} \) corresponds to \( v_0 \sim 2.9c_0 \). For the weakest disorder given of \( V_D/\mu_{\text{TF}} = 0.08 \), we find \( (b/\omega_z)_{\text{th}} \approx 0.03 \), which agrees well with the experimental result \( (b/\omega_z)_{\text{exp}} \sim 0.04 \). We emphasize that there are no adjustable parameters in our theoretical calculations. If the disorder strength is increased to \( V_D/\mu_{\text{TF}} = 0.16 \), the damping rate is four times larger, namely \( (b/\omega_z)_{\text{th}} \approx 0.12 \), which is also in reasonably good agreement with the experimental observations in Fig. 5.10 (right). However, for such large damping rates, the dynamics may no longer be in the linear response regime.

The situation in regard to the experiments in [18] is completely different. For the parameters corresponding to Fig. 5.10 (right) with \( v_0/c_0 \approx 4 \), we obtain \( (b/\omega_z)_{\text{th}} \approx 0.2 \) as opposed to \( (b/\omega_z)_{\text{exp}} \approx 0.004 \). Even for a somewhat weaker disorder strength of \( V_D/\mu_{\text{TF}} = 0.08 \) (with \( \mu_{\text{TF}} \approx h \times 1.5\text{kHz} \)) and \( v_0/c_0 \approx 2.9 \), our theoretical damping
rate is \((b/\omega_z)_{th} \simeq 0.06\), whereas \((b/\omega_z)_{exp} \simeq 0.002\). In fact, for all the data presented in \([18]\), our theoretical damping rates are significantly larger than the experimental values. We have no explanation for this discrepancy. Of course, one can always question the validity of linear response theory, which must break down when the disorder potential is too strong. However, we find it strange that one and the same theory agrees with one set of experiments and not the other. In our view, the difference between the data of the two experiments as displayed in Fig. 5.10 is very puzzling. We believe that it would be very useful to have additional experiments which explore the dependence of the damping rate on various system parameters such as \(N\), \(\sigma\) and \(a_s\). This should be done in a systematic way whereby only one system parameter is varied while all the others are held fixed. This would facilitate a comparison with the dependences we find in our linear response calculations.

5.5 Another application

Although we have so far focused on disorder potentials, the theory we have developed is applicable to any form of external perturbation. As an illustration of the theory in a different context, we consider the dipole oscillation of a condensate in the presence of a one-dimensional Gaussian impurity potential \(V_{imp}(r) = V_0 e^{-z^2/w_z^2}\) \([18]\). The centre of mass oscillation is once again initiated by a sudden displacement of the trapping potential through a distance \(z_0\). At the instant the condensate reaches its equilibrium position, the impurity potential is suddenly switched on. This method of preparation is thus analogous to case (i) for the disorder potential.

As for the case of the disorder potential, the drag force and the damping rate can be determined from the perspective of an oscillating Gaussian impurity. The energy
dissipation rate can then be calculated starting from Eq. (5.102) with the disorder potential replaced by the impurity potential $V_{\text{imp}}(r)$. To eliminate the transients, we set the lower limit of the time integration in Eq. (5.102) to $-\infty$. The expression for the energy dissipation rate is then given by

$$
\frac{d\tilde{E}}{dt} = \int \frac{d\mathbf{q}}{(2\pi)^3} \int \frac{d\mathbf{q}'}{(2\pi)^3} i\mathbf{q} \cdot \mathbf{v}(t) \tilde{V}_{\text{imp}}^*(\mathbf{q}) \tilde{V}_{\text{imp}}(\mathbf{q}') \int_{-\infty}^{t} dt' e^{i(q \cdot \mathbf{x}(t) - q' \cdot \mathbf{x}(t'))} \chi(q, q', t - t'),
$$

(5.165)

where the term containing the equilibrium density is dropped as it does not contribute to the damping rate. Substituting Eq. (5.165) into Eq. (5.99) and after a considerable amount of algebra, we find

$$
\frac{b_{\infty}}{\omega_z} = \frac{2\pi}{M v_0^2} \sum_{n=1}^{\infty} n \int \frac{d\mathbf{q}}{(2\pi)^3} \int \frac{d\mathbf{q}'}{(2\pi)^3} \tilde{V}_{\text{imp}}^*(\mathbf{q}) \tilde{V}_{\text{imp}}(\mathbf{q}') J_n \left( \frac{q_z v_0}{\omega_z} \right) J_n \left( \frac{q'_z v_0}{\omega_z} \right) S(q, q', \omega_n).
$$

(5.166)

This result is analogous to Eq. (5.121) for the disorder potential but is actually more complicated due to the additional wavevector integral. In the case of the disorder potential, the disorder average yields a $\delta(q - q')$ factor (see Eq. (2.144)) which reduces the two wavevector integrals to one.

The Fourier transform of the one-dimensional Gaussian impurity potential is given by

$$
\tilde{V}_{\text{imp}}(\mathbf{q}) = V_0(2\pi)^2 \delta(q_x) \delta(q_y) \sqrt{\pi w_z} e^{-w_z^2 q_z^2/4}.
$$

(5.167)
Inserting this result into Eq. (5.166), we find

\[
\frac{b_\infty}{\omega_z} = \frac{V_0^2 w_z^2}{2M v_0^2} \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} dq_z \int_{-\infty}^{\infty} dq'_z e^{-w_z^2(q_z^2+q'_z^2)/4} J_n \left( \frac{q_z v_0}{\omega_z} \right) J_n \left( \frac{q'_z v_0}{\omega_z} \right) S(q_z, q'_z, \omega_n). \tag{5.168}
\]

To proceed, we invoke the cylindrical LDA for \( S(q_z, q'_z, \omega) \) as given in Eq. (3.103). Substituting this expression into Eq. (5.168), we get

\[
\frac{b_\infty}{\omega_z} = \frac{V_0^2 w_z^2}{2M v_0^2} \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} dq_z \int_{-\infty}^{\infty} dq'_z e^{-w_z^2(q_z^2+q'_z^2)/4} J_n \left( \frac{q_z v_0}{\omega_z} \right) J_n \left( \frac{q'_z v_0}{\omega_z} \right) \times \int_{-R_z}^{R_z} dz dze^{-i(q_z-q'_z)z} S_{cyl} \left( \frac{q_z + q'_z}{2}, \omega_n; \nu(z) \right). \tag{5.169}
\]

It is now useful to make the variable changes \( q = (q_z + q'_z)/2 \) and \( k = (q_z - q'_z)/2 \) in the above integrals. This yields

\[
\frac{b_\infty}{\omega_z} = \frac{V_0^2 w_z^2}{M v_0^2} \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dk e^{-\frac{1}{2}w_z^2(q^2+k^2)} J_n \left( \frac{q + k}{\omega_z} \right) J_n \left( \frac{q - k}{\omega_z} \right) \times \int_{-R_z}^{R_z} dz e^{-i2kz} S_{cyl} (q, \omega_n; \nu(z)) \nonumber
\]

\[
= \frac{4V_0^2 w_z^2}{M v_0^2} \sum_{n=1}^{\infty} n \int_{0}^{\infty} dq \int_{0}^{\infty} dk e^{-\frac{1}{2}w_z^2(q^2+k^2)} J_n \left( \frac{q + k}{\omega_z} \right) J_n \left( \frac{q - k}{\omega_z} \right) \times \int_{-R_z}^{R_z} dz \cos 2kz S_{cyl} (q, \omega_n; \nu(z)). \tag{5.170}
\]
To simplify the calculations, we next determine $\bar{S}_{\text{cyl}}(q, \omega; \nu(z))$ using the TF hydrodynamic modes. This finally gives the damping rate in the form

$$
\frac{b_{\infty}}{\omega_{\bar{z}}} = \frac{15}{2\lambda} \left( \frac{V_0^2}{\mu_{TF}^2} \right) \left( \frac{w_z^2}{v_0^2} \right) \left( \frac{c_0^2}{v_0^2} \right) \int_{0}^{\infty} d\bar{k} \sum_{n=1}^{\infty} \sum_{j} \int_{-1}^{1} d\bar{z} e^{-\frac{\bar{a}^2_{jn}(j, \bar{q}_j) + \bar{k}_j}{2\lambda \sqrt{1 - \bar{z}^2} c_0}} \cos \left( \frac{2\bar{k}_j \bar{z}}{\sqrt{1 - \bar{z}^2}} \right) \times n^2 J_n \left( \frac{\bar{q}_jn + \bar{k}}{2\lambda \sqrt{1 - \bar{z}^2} c_0} \right) J_n \left( \frac{\bar{q}_jn - \bar{k}}{2\lambda \sqrt{1 - \bar{z}^2} c_0} \right) a_0^2(j, \bar{q}_jn) |\bar{\omega}'(\bar{q}_jn)|,
$$

(5.171)

where $\bar{q}_jn$ is defined in Sec. 5.4.3, $\bar{k} = kR_\perp(z)$ and $\bar{z} = z/R_z$.

The evaluation of Eq. (5.171) is numerically involved. However, the salient features of the calculation can be understood by writing the damping rate as

$$
\frac{b_{\infty}}{\omega_{\bar{z}}} = \int_{0}^{\infty} d\bar{k} f(\bar{k}, v_0/c_0).
$$

(5.172)

The function $f(\bar{k}, v_0/c_0)$ is plotted in Fig. 5.11 as a function both $v_0/c_0$ and $\bar{k}$. In these
calculations we have considered a condensate [18] consisting of \( N = 10^6 \) \(^7\)Li atoms in a trap with \( \omega_\perp = 2\pi \times 360 \text{Hz} \) and \( \omega_z = 2\pi \times 5 \text{Hz} \). The s-wave scattering length of the atoms in this system is taken to be \( a_s = 200a_0 \) and the width of the Gaussian impurity potential is \( w_z = 8.5 \mu\text{m} \). The rapid decay of \( f(\vec{k}, v_0/c_0) \) as a function of \( \vec{k} \) is due mainly to the Gaussian function of \( \vec{k} \) in Eq. (5.171). For \( v_0/c_0 < 1 \), the decay is oscillatory whereas for \( v_0/c_0 > 1 \), it is exponential-like. This same behaviour is also seen in the plots of \( f(\vec{k}, v_0/c_0) \) as a function of \( v_0/c_0 \) for different values of \( \vec{k} \). We note that the behaviour of \( f(\vec{k} = 0, v_0/c_0) \) is very similar to the \( v_0/c_0 \) dependence of \( b_\infty/\omega_z \) found for the disorder potential.

\[ \begin{align*}
\frac{(b_\infty/\omega_z)}{(V^2_0/\mu^2)} & \quad v_0/c_0
\end{align*} \]

Figure 5.12: The damping rate of the centre of mass oscillation induced by a Gaussian impurity plotted as a function of \( v_0/c_0 \).

The final result for the damping rate is obtained by performing the integral in Eq. (5.172) and is shown in Fig. 5.12. Although the damping rate is again peaked at \( v_0 \sim c_0 \), there is a strong asymmetry from small to large velocities. This can be understood in terms of the behaviour of \( f(\vec{k}, v_0/c_0) \) as a function of \( v_0/c_0 \) seen in Fig. 5.11 (left). The oscillating behaviour for \( v_0/c_0 < 1 \) leads to cancellations which
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suppress the value of $b_\infty/\omega_z$ in this range of velocities. As a result, there is a relatively sharp onset of the damping at $v_0 \sim 0.8c_0$.

This behaviour agrees qualitatively with what was observed in the experiments [18]. However, for the weakest Gaussian impurity potential strength of $V_0 = 0.2\mu_{TF}$ used in [18], the observed damping rate at $v_0/c_0 = 1$ is approximately an order of magnitude smaller than our theoretical value. This discrepancy between theory and experiment is in the same direction as was found for the disorder potential, but is smaller in magnitude. This may again be an indication that the perturbation is outside of the linear response regime. This possibility is supported by the fact that the experimental damping rates are not symmetrical with respect to the sign of the impurity potential. Since the linear response damping rates are proportional to $V_0^2$, repulsive and attractive impurity potentials must give the same damping rates for a given value of $|V_0|$. This symmetry can be used as the criterion for establishing whether or not the experiments are in the linear response regime.
Chapter 6

Amplitude and phase modulation spectroscopy

As explained in Chapter 2, strong periodic potentials for cold atoms in the form of optical lattices can be generated by means of standing light waves. This development has significantly expanded the range of physics that is accessible with cold atom systems [62]. In particular, it has made possible the experimental realization of important and widely used theoretical models in condensed matter physics, such as the Bose and Fermi Hubbard models [63, 64]. In this chapter, we discuss two important methods that can be used to probe ultracold gases in optical lattices, namely amplitude [65] and phase modulation spectroscopies [66].

These modulation spectroscopies can be analyzed theoretically using methods we have developed in earlier chapters. In particular, we derive general linear response formulae for the energy absorption due to both kinds of modulations. We also critique an approximation that has often been used to investigate amplitude modulation spectroscopy theoretically and demonstrate how it can be improved. Finally, we discuss a recent proposal [66] to use phase modulation as a probe of the optical conductivity of lattice gas systems. We find that our linear response approach reproduces the main result of [66] but, more importantly, has further implications for this proposal. We
also point out that these results are applicable to both bosonic and fermionic gases.

Finally, we discuss some possible future directions for the work begun in this chapter.

6.1 Energy absorption in amplitude modulation

We begin by writing down the Hamiltonian for a cold atomic gas in a three-dimensional optical lattice

$$\hat{H}_{\text{op}} = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + V_{\text{op}}(\hat{r}_i) \right) + \sum_{i<j}^{N} g \delta(\hat{r}_i - \hat{r}_j), \quad (6.1)$$

where $g = \frac{4\pi\hbar^2 a_s}{m}$. The situation we have in mind corresponds to the superposition of standing waves in three orthogonal directions. The optical lattice potential in this case takes the form

$$V_{\text{op}}(\mathbf{r}) = \sum_{\mu=x,y,z} V_{\mu 0} \cos^2 k_{\mu} r_{\mu}, \quad (6.2)$$

where $k_{\mu}$ are the wavenumbers of the different laser beams. In general, the amplitudes of the beams can also be different which leads to directionally-dependent depth parameters $V_{\mu 0}$.

One of the important ways to probe such a system is the so-called amplitude modulation spectroscopy [65] which corresponds to a periodic modulation of one or more of the depth parameters $V_{\mu 0}$. Such a modulation excites the system and leads to the absorption of energy. If the amplitude of the modulation is sufficiently small, the energy absorption can be determined by linear response theory. Following the experimental protocol in [65], we assume that the potential depth in the $z$ direction is modulated according to $V_{\mu 0} \rightarrow V_{\mu 0} + \delta V_0 \sin \omega t$. The effect of this modulation is
then represented by the perturbation

\[ \hat{H}_{AM}'(t) = \sin \omega t \delta V_0 \sum_{i=1}^{N} \cos^2 k_z \hat{z}_i = \int dr \delta V_{AM}(z, t) \hat{n}(r), \]  

(6.3)

where

\[ \delta V_{AM}(z, t) = \frac{\delta V_0}{2} \sin \omega t \cos 2k_z z. \]  

(6.4)

Since the \( \delta V_0 \sin \omega t / 2 \) term in Eq. (6.4) gives a contribution to \( \hat{H}_{AM}'(t) \) which is proportional to the total number of particles, it does not cause excitations and will be dropped in the following discussion of the energy absorption. Thus \( \delta V_{AM}(z, t) \) is taken to be

\[ \delta V_{AM}(z, t) = \frac{\delta V_0}{2} \sin \omega t \cos 2k_z z. \]  

(6.5)

The rate of energy absorption for the perturbation in Eq. (6.5) is given by

\[ \frac{dE}{dt} = \left\langle \Psi(t) \left| \frac{\partial \hat{H}_{AM}'(t)}{\partial t} \right| \Psi(t) \right\rangle = \int dr \frac{\partial \delta V_{AM}(z, t)}{\partial t} n(r, t), \]  

(6.6)

where \( n(r, t) = \langle \Psi(t)|\hat{n}(r)|\Psi(t) \rangle \) and \( |\Psi(t) \rangle \) is the state of the system at time \( t \). In the linear response regime we have

\[ \frac{dE}{dt} \simeq \int dr \frac{\partial \delta V_{AM}(z, t)}{\partial t} n_{eq}(r) \]

\[ - \int dr \frac{\partial \delta V_{AM}(z, t)}{\partial t} \int dr' \int_{-\infty}^{\infty} dt' \delta V_{AM}(z', t') \chi(r, r', t - t'), \]  

(6.7)

where \( n_{eq}(r) = \langle \Psi_0|\hat{n}(r)|\Psi_0 \rangle \) is the average density in the ground state \( |\Psi_0 \rangle \) of \( \hat{H}_{op} \).
6.1. ENERGY ABSORPTION IN AMPLITUDE MODULATION

The density response function is given by

\[ \chi(r, r', t - t') = \frac{i}{\hbar} \theta(t - t') \langle \Psi_0 | [\hat{n}_1(r, t), \hat{n}_1(r', t')] | \Psi_0 \rangle, \tag{6.8} \]

where \( \hat{n}_1(r, t) = e^{i\hat{H}_{opt} t/\hbar} \hat{n}(r) e^{-i\hat{H}_{opt} t/\hbar} \) is the density operator in the interaction picture.

Substituting Eq. (6.5) into Eq. (6.7), we find

\[ \frac{dE}{dt} = \frac{\delta V_0}{2} \omega \cos \omega t \int dr \cos 2k_z z n_{eq}(r) \]

\[ - \left( \frac{\delta V_0}{2} \right)^2 \omega \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t' \]

\[ \times \int dr \cos 2k_z z \int dr' \cos 2k_z z' \chi(r, r', t - t'). \tag{6.9} \]

We now consider the energy absorption rate averaged over one period of modulation \( T = 2\pi/\omega \), which is the quantity measured in experiments. In this case, the contribution from the first term of Eq. (6.9) vanishes and we find

\[ \overline{\frac{dE}{dt}} = \frac{1}{T} \int_0^T dt \frac{dE}{dt} \]

\[ = - \left( \frac{\delta V_0}{2} \right)^2 \omega \int_0^T dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin(\omega t') \]

\[ \times \int dr \int dr' \cos 2k_z z \cos 2k_z z' \chi(r, r', t - t'). \tag{6.10} \]

The time integral in Eq. (6.10) yields

\[ \frac{1}{T} \int_0^T dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t' \chi(r, r', t - t') = \frac{1}{4i} [\chi(r, r', -\omega) - \chi(r, r', -\omega)] \tag{6.11} \]
Inserting this result into Eq. (6.10), we have
\[
\frac{dE}{dt} = \left( \frac{\delta V_0}{2} \right)^2 \frac{\omega}{4i} \int dr \int dr' \cos 2k_z z \cos 2k_z' z' \left[ \chi(r, r', \omega) - \chi(r, r', -\omega) \right].
\] (6.12)

The spatial integrals give Fourier transforms of the response functions. Using the fact that (see Eq. (3.13))
\[
\chi(q, q', \omega) - \chi(q, q', -\omega) = 2i \chi''(q, q', \omega),
\] (6.13)
we get
\[
\frac{dE}{dt} = \frac{(\delta V_0)^2}{32} \omega \left[ \chi''(2k_z, 2k_z, \omega) + \chi''(-2k_z, -2k_z, \omega) \right. \\
+ \chi''(2k_z, -2k_z, \omega) + \chi''(-2k_z, 2k_z, \omega) \left. \right] \\
= \frac{(\delta V_0)^2}{16} \omega \left[ \chi''(2k_z, 2k_z, \omega) + \chi''(2k_z, -2k_z, \omega) \right],
\] (6.14)
where we denote \(\chi''(2k_z \hat{z}, 2k_z \hat{z}, \omega)\) by \(\chi''(2k_z, 2k_z, \omega)\) for simplicity. In arriving at the last line of Eq. (6.14), we have used the property \(\chi''(-q, -q', \omega) = \chi''(q, q', \omega)\) of the density response function (see Eq. (3.7)).

### 6.1.1 Modulations in the deep lattice systems

In this section, we examine an approximation that has been used widely in studies of amplitude modulation for deep lattice systems. We shall point out a logical deficiency of this approximation and provide a correction. Before we do so, we first review the derivation of the one-band Hubbard model Hamiltonian, which is used to describe the low energy physics of atoms in a deep optical lattice. For simplicity, we shall consider
6.1. ENERGY ABSORPTION IN AMPLITUDE MODULATION

a cubic lattice for which the optical lattice potential is 
\[ V_{\text{op}}(r) = V_0 \sum_{\mu=x,y,z} \cos^2 kr_{\mu}. \]
The lattice spacing is given by \( a = \pi/k. \)

To obtain a second-quantized version of the Hamiltonian in Eq. (6.1), we require an appropriate single-particle basis. The most natural basis functions are the Bloch states \( \psi_{nq}(r) \) which are the eigenstates of the single-particle Hamiltonian

\[ \hat{h}_0 = \frac{\hat{P}^2}{2m} + V_0 \sum_{\mu} \cos^2 k \hat{r}_{\mu}. \] (6.15)

The Bloch state labels are the band index \( n \) and the Bloch quasi-momentum \( q \). We next introduce the localized Wannier states according to the definition

\[ w_{n,i}(r) = \frac{1}{\sqrt{N_L}} \sum_{\mathbf{q} \in \text{B.Z.}} e^{i\mathbf{q} \mathbf{R}_i} \psi_{nq}(r), \] (6.16)

where \( \mathbf{R}_i \) are Bravais lattice vectors and \( N_L \) is the total number of lattice sites. The Wannier functions satisfy the orthonormality condition

\[ \int dr w_{n,i}^*(r) w_{n',j}(r) = \delta_{nn'} \delta_{ij}. \] (6.17)

Using the Wannier state basis, the second-quantized Hamiltonian for the unperturbed system can be expressed as

\[ \hat{H}_{\text{op}} = - \sum_{n} \sum_{ij} J_n(\mathbf{R}_i - \mathbf{R}_j) \hat{a}_{n,i}^\dagger \hat{a}_{n,j} + \frac{1}{2} \sum_{n_1n_2n_3n_4i_1i_2i_3i_4} U_{n_1n_2n_3n_4i_1i_2i_3i_4} J_n(\mathbf{R}_i - \mathbf{R}_j) \hat{a}_{n_1,i_1}^\dagger \hat{a}_{n_2,i_2} \hat{a}_{n_3,i_3} \hat{a}_{n_4,i_4}. \] (6.18)
where

\[ J_n(R_i - R_j) = -\int d\mathbf{r} w^*_{n,i}(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{op}}(\mathbf{r}) \right) w_{n,j}(\mathbf{r}) \] (6.19)

are referred to as the hopping matrix elements and

\[ U_{n_1n_2n_3n_4}^{i_1i_2i_3i_4} = g \int d\mathbf{r} w^*_{n_1,i_1}(\mathbf{r}) w^*_{n_2,i_2}(\mathbf{r}) w_{n_3,i_3}(\mathbf{r}) w_{n_4,i_4}(\mathbf{r}). \] (6.20)

are the interaction constants. For deep lattices the lowest energy s-band is rather flat and the band gap \( \Delta \) between the s-band and the p-band is much larger than the band width \( W_s \) of the the s-band. If one assumes that all the important energy scales are small compared to the band gap \( \Delta \), a lowest s-band model is an appropriate approximation. Furthermore, due to the localization of the s-band Wannier functions on each lattice site, the non-interacting part of the Hamiltonian can be treated in the tight-binding approximation, namely one neglects all but the nearest neighbour hopping matrix elements in Eq. (6.19).\(^1\) Similarly, one neglects all but the onsite interaction constants, which are the dominant terms in Eq. (6.20). By virtue of lattice translational symmetry, these constants are all equal. These approximations lead to the one-band Hubbard model Hamiltonian

\[ \hat{H}_{\text{HB}} = -J \sum_{<ij>} \hat{a}^\dagger_i \hat{a}_j + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1), \] (6.21)

where the s-band index is suppressed for convenience and \(<ij>\) denotes nearest neighbour sites. The nearest neighbour hopping matrix element \( J \) and the on-site

\(^1\)The on-site \((i = j)\) matrix elements in Eq. (6.18) introduce a constant energy shift which is not dynamically significant.
interaction constant $U$ are given by

$$J = - \int d\mathbf{r} w^*_x(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{op}(r) \right) w_{s,i+1}(r), \quad (6.22)$$

$$U = g \int d\mathbf{r} |w_{s,i}(r)|^4, \quad (6.23)$$

where $w_{s,i}(r)$ are the s-band Wannier functions. The energy bands and the corresponding Wannier functions of the Hamiltonian $\hat{h}_0$ are relatively straightforward to obtain. By solving the relevant Schrödinger equation for the band structure (see Appendix D for more details), we have determined the Hubbard model parameters $J$ and $U$ as well as the next dominant hopping ($J'$) and interaction constant ($U'$). These parameters for various lattice depths are shown in Table 6.1. Note that all the energies are given in terms of the so-called recoil energy $E_r = \frac{\hbar^2 k^2}{2m}$. From Table 6.1, we see that the approximations involved in the derivation of the Hubbard Hamiltonian are well justified for lattice depths $V_0/E_r \geq 10$.

<table>
<thead>
<tr>
<th>$V_0/E_r$</th>
<th>$W_s/4E_r$</th>
<th>$J/E_r$</th>
<th>$\Delta/E_r$</th>
<th>$J'/J$</th>
<th>$(U/E_r) \times (a/a_s)$</th>
<th>$U'/U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.112 974</td>
<td>0.111 027</td>
<td>1.489</td>
<td>0.111 027</td>
<td>6.258</td>
<td>$-1.78 \times 10^{-2}$</td>
</tr>
<tr>
<td>7</td>
<td>0.039 465</td>
<td>0.039 417</td>
<td>3.343</td>
<td>0.039 417</td>
<td>15.188</td>
<td>$-7.32 \times 10^{-3}$</td>
</tr>
<tr>
<td>10</td>
<td>0.019 187</td>
<td>0.019 182</td>
<td>4.573</td>
<td>0.019 182</td>
<td>21.472</td>
<td>$-3.41 \times 10^{-3}$</td>
</tr>
<tr>
<td>15</td>
<td>0.006 519</td>
<td>0.006 519</td>
<td>6.281</td>
<td>0.006 519</td>
<td>31.045</td>
<td>$-1.06 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 6.1: Hubbard Model Parameters

The widely used approximation that we have alluded to at the beginning of this section concerns the derivation of an appropriate perturbation term due to the amplitude modulation that is consistent with the use of the single-band Hubbard model
6.1. ENERGY ABSORPTION IN AMPLITUDE MODULATION

Hamiltonian in Eq. (6.21). In the literature, the following argument [66, 67, 68, 69, 70, 71, 72, 73] is used to obtain this perturbation term. Since \( J \) and \( U \) in Eq. (6.21) are determined by the Wannier functions according to Eqs. (6.22) and (6.23), which are functions of the lattice depth \( V_0 \), it is surmised that a small modulation of this depth, \( V_0 \rightarrow V_0(t) = V_0 + \delta V_0 \sin \omega t \), would lead to a modulation of \( J \) and \( U \) in the Hamiltonian in Eq. (6.21), namely,

\[
J [V_0(t)] \simeq J + \delta J \sin \omega t
\]

\[
U [V_0(t)] \simeq U + \delta U \sin \omega t,
\]

where the amplitudes of the modulations are

\[
\delta J = \frac{dJ}{dV_0} \delta V_0, \quad \delta U = \frac{dU}{dV_0} \delta V_0.
\]

The time-dependent Hamiltonian used in [67, 68, 69] is obtained through the prescription of simply replacing the \( J \) and \( U \) parameters in Eq. (6.21) by the time-dependent parameters in Eq. (6.24). This gives

\[
\hat{H}_{HB}(t) = -J [V_0(t)] \sum_{<ij>} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} U [V_0(t)] \sum_i \hat{n}_i (\hat{n}_i - 1).
\]

This Hamiltonian can in fact be simplified by noting that the modulation of the interaction can be rewritten as

\[
\frac{1}{2} \sin \omega t \delta U \sum_i \hat{n}_i (\hat{n}_i - 1) = \frac{\delta U}{U} \sin \omega t \hat{H}_{HB} + J \frac{\delta U}{U} \sin \omega t \sum_{<ij>} \hat{a}_i^\dagger \hat{a}_j.
\]
Using Eq. (6.27), Eq. (6.26) becomes

\[ \hat{H}_{\text{HB}}(t) = \hat{H}_{\text{HB}} - \delta J_{\text{eff}} \sin \omega t \sum_{<ij>} \hat{a}_i^{\dagger} \hat{a}_j + \frac{\delta U}{U} \sin \omega t \hat{H}_{\text{HB}}, \]  

(6.28)

where \( \delta J_{\text{eff}} \) is the effective modulation of the hopping matrix element

\[ \delta J_{\text{eff}} / J = \left( \frac{d \ln J}{dV_0} - \frac{d \ln U}{dV_0} \right) \delta V_0. \]  

(6.29)

For deep lattices, \( d \ln J / dV_0 \) is the dominant contribution to Eq. (6.29). Since the last term in Eq. (6.28) is proportional to the unperturbed Hamiltonian \( \hat{H}_{\text{HB}} \), it does not cause excitations and can therefore be ignored in calculations of the energy absorption induced by the modulation. Thus the time-dependent Hamiltonian used in [67, 68, 69] is given by

\[ \hat{H}_{\text{HB}}(t) = \hat{H}_{\text{HB}} - \delta J_{\text{eff}} \sin \omega t \sum_{<ij>} \hat{a}_i^{\dagger} \hat{a}_j. \]  

(6.30)

We now argue that the foregoing development is not rigorous. First, we recall that to second quantize the static Hamiltonian in Eq. (6.1), we chose our basis to be a complete set of Wannier states \( \{ w_{n,i}(r) \} \) of the \textit{time-independent} single-particle Hamiltonian \( \hat{h}_0 \) in Eq. (6.15). In the deep lattice limit, we then obtained a simplified Hubbard Hamiltonian by retaining only the dominant terms in the full Hamiltonian. When the lattice depth is modulated in the way described previously, the full time-dependent Hamiltonian is given by

\[ \hat{H}_{\text{AM}}(t) = \sum_{i=1}^{N} \left\{ \frac{\hat{p}_i^2}{2m} + V_0(t) \sum_{\mu=x,y,z} \cos^2 k\hat{r}_{i,\mu} \right\} + g \sum_{i<j} \delta(\hat{r}_i - \hat{r}_j), \]  

(6.31)

In the deep lattice limit, there are two ways to obtain an approximate Hamiltonian
from $\hat{H}_{\text{AM}}(t)$. One can use the instantaneous Wannier states $\{w_{n,i}(r;V_0(t))\}$ of $\hat{h}_0(t)$ as the basis to second quantize $\hat{H}_{\text{AM}}(t)$ at each instant of time, where

$$\hat{h}_0(t) = \frac{\hat{p}^2}{2m} + V_0(t) \sum_{\mu=x,y,z} \cos^2 k \hat{r}_\mu.$$  \hspace{1cm} (6.32)

This leads to

$$\hat{H}_{\text{AM}}(t) \simeq -J [V_0(t)] \sum_{<ij>} \hat{a}_i^\dagger(t)\hat{a}_j(t) + \frac{1}{2} U [V_0(t)] \sum_i \hat{n}_i(t)(\hat{n}_i(t) - 1),$$ \hspace{1cm} (6.33)

where $J [V_0(t)]$ and $U [V_0(t)]$ are determined by $\{w_{s,i}(r;V_0(t))\}$ according to Eq. (6.22) and (6.23). The operators $\hat{a}_i(t)$ and $\hat{a}_i^\dagger(t)$, respectively, annihilate and create a single particle in the state $w_{s,i}(r;V_0(t))$. This, in fact, is the procedure implicitly used to obtain the Hamiltonian in Eq. (6.26). However, the basic flaw of this approach is that it completely disregards the time-dependence of the operators $\hat{a}_i(t)$ and $\hat{a}_i^\dagger(t)$. In particular, $[\hat{a}_i(t), \hat{a}_j^\dagger(t')] \neq \delta_{ij}$ if $t \neq t'$. By treating these operators as the original annihilation and creation operators corresponding to the stationary Wannier states $w_{s,i}(r)$, one obtains the Hamiltonian in Eq. (6.26). However, it is no longer clear what dynamical effects are being neglected.

A more rigorous and consistent approach is to work with a fixed basis in the second-quantized formalism, namely the Wannier states $w_{n,i}(r)$. To obtain the second-quantized Hamiltonian in this case, we rewrite $\hat{H}_{\text{AM}}(t)$ as

$$\hat{H}_{\text{AM}}(t) = \hat{H}_{\text{op}} + \hat{H}_{\text{AM}}'(t),$$ \hspace{1cm} (6.34)
where $\hat{H}_{\text{op}}$ is given in Eq. (6.1) and

$$\hat{H}'_{\text{AM}}(t) = \delta V_0 \sin \omega t \sum_{i=1}^{N} \sum_{\mu=x,y,z} \cos^2 k \hat{r}_{i,\mu}$$

$$= \delta V_0 \sin \omega t \int d\mathbf{r} \left( \cos^2 kx + \cos^2 ky + \cos^2 k_z \right) \hat{n}(\mathbf{r}). \quad (6.35)$$

We showed earlier that the static part of the Hamiltonian $\hat{H}_{\text{op}}$ reduces to the Hubbard Hamiltonian Eq. (6.21) in the deep lattice limit. With the same approximations used in this derivation, the perturbation term $\hat{H}'_{\text{AM}}$ becomes

$$\hat{H}'_{\text{AM}} = -\sin \omega t \delta J_{\text{AM}} \sum_{\langle ij \rangle} \hat{a}_i^\dagger \hat{a}_j, \quad (6.36)$$

where $\delta J_{\text{AM}}$ is given by

$$\delta J_{\text{AM}} = -\delta V_0 \int d\mathbf{r} w_{s,i}^*(\mathbf{r}) \left( \cos^2 kx + \cos^2 ky + \cos^2 k_z \right) w_{s,i+1}^1(\mathbf{r}). \quad (6.37)$$

We see that the amplitude modulation does lead to a modulation of the nearest neighbour hopping element in the Hubbard Hamiltonian, but there is no corresponding modulation of the interaction parameter. The value of $\delta J_{\text{AM}}$, however, differs from that of $\delta J_{\text{eff}}$, which is determined by Eq. (6.29). In Table. 6.2, we compare the values of these two parameters calculated for various lattice depths. We see that although the values of these two parameters have the same order of magnitude, the differences are substantial. Since many theories aim at quantitative comparisons to the experimental observations, the differences between the hopping parameter modulations in Table. 6.2 are significant.
6.2. ENERGY ABSORPTION IN PHASE MODULATION

<table>
<thead>
<tr>
<th>$V_0/E_r$</th>
<th>$(\delta J_{\text{eff}}/J)/(\delta V_0/\delta V)$</th>
<th>$(\delta J_{\text{AM}}/J)/(\delta V_0/\delta V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-3.24</td>
<td>-2.30</td>
</tr>
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<td>15</td>
<td>-3.93</td>
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<td>-3.66</td>
</tr>
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<td>25</td>
<td>-5.03</td>
<td>-4.19</td>
</tr>
</tbody>
</table>

Table 6.2: Modulations of Hopping Parameters

6.2 Energy absorption in phase modulation

Another method of probing a cold atomic gas in an optical lattice is to shake the lattice potential periodically along a certain direction [66]. Since this results in a modulation of the phase of the optical lattice, it is referred to as phase modulation spectroscopy. In the following, we derive a linear response formula for the energy absorption when the phase modulation is weak.

Let us consider the system described by the Hamiltonian in Eq. (6.1) and assume that the lattice potential is shaken along the $z$ direction according to $V_{z,0} \cos^2 k_z z \rightarrow V_{z,0} \cos^2 k_z (z - z_0(t))$ with $z_0(t) = z_0 \sin \omega t$. The perturbation of the original Hamiltonian is then

$$\hat{H}'_{\text{PM}}(t) = V_{z,0} \sum_{i=1}^{N} \cos^2 k_z (\hat{z}_i - z_0(t)) - \cos^2 k_z \hat{z}_i = \int d\mathbf{r} \delta V_{\text{PM}}(z,t) \hat{n}(\mathbf{r}), \quad (6.38)$$

where

$$\delta V_{\text{PM}}(z,t) = V_{z,0} [\cos^2 k_z (z - z_0(t)) - \cos^2 k_z z]. \quad (6.39)$$

For the perturbation to be weak, the amplitude of the oscillation $z_0$ has to be small.
compared to the lattice spacing $\pi/k_z$.\footnote{The physics in this situation is different from that in which phase modulation \cite{74, 75} leads to a dynamically induced phase transition. In this case, the amplitude of the oscillation is comparable to the lattice spacing and the modulation frequencies are much higher.} In this case, Eq. (6.39) can be expanded in powers of $z_0$. Retaining the lowest order term, we have

$$\delta V_{PM}(z, t) \simeq -\frac{\partial V_{op}(r)}{\partial z} z_0(t) = V_{z0} z_0 \sin \omega t \sin 2k_z z. \quad (6.40)$$

As for the case of amplitude modulation, linear response theory gives the energy absorption rate

$$\frac{dE}{dt} = \int dr \frac{\partial \delta V_{PM}(z, t)}{\partial t} n_{eq}(r)$$

$$- \int dr \frac{\partial \delta V_{PM}(z, t)}{\partial t} \int dr' \int_{-\infty}^{\infty} dt' \chi(r, r', t - t') \delta V_{PM}(z', t'). \quad (6.41)$$

Averaging the energy absorption rate over one period, we obtain

$$\overline{\frac{dE}{dt}} = \frac{1}{T} \int_{0}^{T} dt \frac{dE}{dt}$$

$$= -\frac{(V_{z0} z_0 k_z)^2 \omega}{16} \int_{0}^{T} dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t'$$

$$\times \int dr \int dr' \sin(2k_z z) \sin(2k_z z') \chi(r, r', t - t'). \quad (6.42)$$

Again, using Eq. (6.11) and Eq. (6.13), we find

$$\overline{\frac{dE}{dt}} = \frac{(V_{z0} z_0 k_z)^2 \omega}{8} \left[ \chi''(2k_z, 2k_z, \omega) + \chi''(-2k_z, -2k_z, \omega) \right.$$

$$\left. - \chi''(2k_z, -2k_z, \omega) - \chi''(-2k_z, 2k_z, \omega) \right]$$

$$= \frac{(V_{z0} z_0 k_z)^2 \omega}{8} \left[ \chi''(2k_z, 2k_z, \omega) - \chi''(2k_z, -2k_z, \omega) \right]. \quad (6.43)$$
Interestingly, this expression is very similar to Eq. (6.14) obtained for the case of amplitude modulation.

### 6.2.1 Phase modulation as a probe for optical conductivity

In [66] the authors propose to probe the optical conductivity of a cold gas in an optical lattice by shaking the lattice periodically along a certain direction. They show that the energy absorption rate averaged over one period of modulation can be expressed in terms of the current-current correlation function (or equivalently, the momentum-momentum correlation function). This is a very interesting proposal since it is not obvious how one might access experimentally the conductivity of a system consisting of neutral atoms. In this section, we rederive the main result in [66] using our linear response theory and discuss one implication of our approach.

The real part of the frequency-dependent optical conductivity for a lattice system at zero temperature is given by the Kubo formula [76]

\[
\text{Re} \sigma_{\mu\nu} = \frac{1}{\omega} \text{Im} \Pi_{\mu\nu}(\omega),
\]

where

\[
\Pi_{\mu\nu}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega \tau} \Pi_{\mu\nu}(\tau).
\]

Here \(\Pi_{\mu\nu}(t-t')\) is the Fourier transform of the ground state current-current correlation function

\[
\Pi_{\mu\nu}(t-t') = \frac{i}{\hbar} \theta(t-t') \langle \Psi_0 | [\hat{J}_{\mu,1}(t), \hat{J}_{\nu,1}(t')] | \Psi_0 \rangle,
\]

where \(\hat{J}_{\mu,1}(t) = e^{i\hat{H}_{\text{op}}t/\hbar} \hat{J}_{\mu} e^{-i\hat{H}_{\text{op}}t/\hbar}\) with \(\hat{J}_{\mu}\) being the \(\mu\)-component of the total current.
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operator. The latter is defined as

$$\hat{\mathbf{J}} = \int d\mathbf{r} \hat{\mathbf{j}}(\mathbf{r}),$$  \hspace{1cm} (6.47)

where

$$\hat{\mathbf{j}}(\mathbf{r}) \equiv \frac{1}{2} \sum_i \left[ \hat{\mathbf{p}}_i \delta(\mathbf{r} - \hat{\mathbf{r}}_i) + \delta(\mathbf{r} - \hat{\mathbf{r}}_i) \hat{\mathbf{p}}_i \right]$$  \hspace{1cm} (6.48)

is the current density operator. We thus see that \(\hat{\mathbf{J}} = \mathbf{P}/m\) where \(\hat{\mathbf{P}}\) is the total momentum operator. The current-current correlation function in Eq. (6.46) is therefore essentially a total momentum-total momentum correlation function. We note that the total momentum is not a conserved quantity because of the optical lattice.

For the cubic lattice we are considering, different Cartesian components of the total current are not correlated, that is, \(\Pi_{\mu\nu}(t - t') = \delta_{\mu\nu}\Pi_{\mu\mu}(t - t')\). The response function \(\Pi_{\mu\mu}(t - t')\) has the spectral representation

$$\Pi_{\mu\mu}(t - t') = \frac{i}{\hbar} \theta(t - t') \sum_{\alpha} |\langle \Psi_0 | J_\mu | \Psi_\alpha \rangle|^2 \left( e^{-iE_{\alpha_0}(t-t')/\hbar} - e^{iE_{\alpha_0}(t-t')/\hbar} \right),$$  \hspace{1cm} (6.49)

where \(|\Psi_\alpha\rangle\) is an eigenstate of \(\hat{H}_{\text{op}}\) with energy \(E_\alpha\), and \(E_{\alpha_0} \equiv E_\alpha - E_0\). Substituting Eq. (6.49) into Eq. (6.45) and using Eq. (3.10), we obtain

$$\Pi_{\mu\mu}(\omega) = -\frac{1}{\hbar} \sum_{\alpha} |\langle \Psi_0 | J_\mu | \Psi_\alpha \rangle|^2 \left( \frac{1}{\omega - E_{\alpha_0}/\hbar + i\epsilon} - \frac{1}{\omega + E_{\alpha_0}/\hbar + i\epsilon} \right),$$  \hspace{1cm} (6.50)

where \(\epsilon\) is a positive infinitesimal number. Using the identity in Eq. (3.12), we obtain

$$\text{Im}\Pi_{\mu\mu}(\omega) = \pi \sum_{\alpha} |\langle \Psi_0 | \hat{J}_\mu | \Psi_\alpha \rangle|^2 [\delta(\hbar \omega - E_{\alpha_0}) - \delta(\hbar \omega + E_{\alpha_0})],$$  \hspace{1cm} (6.51)
We now demonstrate that the energy absorption rate in Eq. (6.42) can be expressed in terms of the current-current correlation function. Substituting Eq. (6.8) into Eq. (6.42), we have

\[
\frac{dE}{dt} = -iz_0^2 \omega \bar{\hbar} T \int_0^T dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t' \theta(t-t') \\
\times \int dr \int dr' \frac{\partial V_{op}(r)}{\partial z} \frac{\partial V_{op}(r')}{\partial z'} \langle \Psi_0 | [\hat{n}_1(r, t), \hat{n}_1(r', t')] | \Psi_0 \rangle
\]

\[
= -iz_0^2 \omega \bar{\hbar} T \int_0^T dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t' \theta(t-t') \langle \Psi_0 | [\hat{V}_1(t), \hat{V}_1(t')] | \Psi_0 \rangle,
\]

where \( \hat{V}_1(t) \equiv e^{i \hat{H}_{op} t / \bar{\hbar}} \hat{V} e^{-i \hat{H}_{op} t / \bar{\hbar}} \) with

\[
\hat{V} = \int dr \frac{\partial V_{op}(r)}{\partial z} \hat{n}(r) = \sum_{i=1}^N \frac{\partial V_{op}(\hat{r}_i)}{\partial \hat{z}_i}.
\]

Defining the response function

\[
\chi_{V}(t - t') = i \frac{\bar{\hbar}}{\hbar} \theta(t-t') \langle \Psi_0 | [\hat{V}_1(t), \hat{V}_1(t')] | \Psi_0 \rangle,
\]

Eq. (6.52) can be written as

\[
\frac{dE}{dt} = -z_0^2 \omega \bar{\hbar} T \int_0^T dt \cos \omega t \int_{-\infty}^{\infty} dt' \sin \omega t' \chi_V(t-t') \\
= \frac{z_0^2 \omega}{4i} \left[ \chi_V(\omega) - \chi_V(-\omega) \right],
\]

where

\[
\chi_V(\omega) = \int d\tau e^{i\omega \tau} \chi_V(\tau).
\]
Using a spectral analysis, it is easy to show that

\[ \chi_V(\omega) - \chi_V(-\omega) = 2i \text{Im} \chi_V(\omega), \]  

(6.57)

where

\[ \text{Im} \chi_V(\omega) = \pi \sum_\alpha |\langle \Psi_0 | \hat{V} | \Psi_\alpha \rangle|^2 [\delta(h\omega - E_{0\alpha}) - \delta(h\omega + E_{0\alpha})]. \]  

(6.58)

Substituting Eq. (6.57) into Eq. (6.55), we find

\[ \frac{dE}{dt} = z_0^2 \omega \text{Im} \chi_V(\omega). \]  

(6.59)

To relate this result to the current-current correlation function, we make use of the identity

\[ [\hat{H}_{\text{op}}, \hat{J}_z] = \frac{1}{m} [\hat{H}_{\text{op}}, \hat{P}_z] \]

\[ = \frac{i\hbar}{m} \sum_{i=1}^N \frac{\partial V_{\text{op}}(\hat{r}_i)}{\partial \hat{z}_i} \]

\[ = \frac{i\hbar}{m} \hat{V}. \]  

(6.60)

The matrix element in Eq. (6.58) can thus be expressed as

\[ \langle \Psi_0 | \hat{V} | \Psi_\alpha \rangle = \frac{m}{i\hbar} \langle \Psi_0 | [\hat{H}_{\text{op}}, \hat{J}_z] | \Psi_\alpha \rangle \]

\[ = -\frac{m}{i\hbar} E_{0\alpha} \langle \Psi_0 | \hat{J}_z | \Psi_\alpha \rangle. \]  

(6.61)
6.2. ENERGY ABSORPTION IN PHASE MODULATION

Inserting this result into Eq. (6.58), we obtain

\[ \text{Im} \chi_V(\omega) = m^2 \omega^2 \text{Im} \Pi_{zz}(\omega). \] (6.62)

Thus, Eq. (6.59) is equivalent to

\[ \frac{dE}{dt} = \frac{m^2 z_0^2 \omega^3}{2} \text{Im} \Pi_{zz}(\omega), \] (6.63)

which is the formula obtained previously in [66] using a complete different method (see Appendix E for details).

As we have seen previously, the lowest-band approximation can be used to describe the dynamics of the system if the lattice potential is sufficiently deep. In this situation, our forgoing analysis on the energy absorption due to phase modulation has an interesting implication. Namely, the energy absorption due to a perturbation of the form given in Eq. (6.40) vanishes completely! To demonstrate this, we second quantize the operator \( \hat{V} \) using the Bloch states \( \{ \psi_{nq}(r) \} \) of the single-particle Hamiltonian \( \hat{h}_0 \). One finds

\[ \hat{V} = \sum_{n'q'nq} V_{n'n}(q',q) \hat{a}_{n'q'}^\dagger \hat{a}_{nq}, \] (6.64)

where

\[ V_{n'n}(q',q) = \int d \mathbf{r} \psi_{n'q'}^*(\mathbf{r}) V_{\text{op}}(\mathbf{r}) \frac{\partial}{\partial z} \psi_{nq}(\mathbf{r}). \] (6.65)

To evaluate this matrix element, it is useful to represent the Bloch state \( \psi_{nq}(\mathbf{r}) \)
by the following Fourier expansion

\[ \psi_{nq}(r) = \sum_G c_n(q + G)e^{i(q+G) \cdot r}, \quad (6.66) \]

where \( G \) is a reciprocal lattice vector. The expansion coefficients \( c_n(q) \) and the corresponding band energy \( \epsilon_{nq} \) are determined by the Schrödinger equation

\[ \left( \frac{\hbar^2}{2m} (q + G)^2 - \epsilon_{nq} \right) c_n(q + G) + \sum_{G'} U(G - G')c_n(q + G') = 0, \quad (6.67) \]

where \( U(G) \) are the Fourier components of the periodic potential defined by

\[ V_{op}(r) = \sum_G U(G)e^{iG \cdot r}. \quad (6.68) \]

Substituting Eq. (6.66) and Eq. (6.68) into Eq. (6.65), we find that

\[ V_{n'n}(q'q) = i\delta_{q'q} \sum_{G,G'} (G_z' - G_z)c^*_n(q + G')c_n(q + G)U(G' - G') \]

\[ = \delta_{q'q} V_{n'n}(q, q). \quad (6.69) \]

We next consider the matrix elements \( V_{nn}(q) \) which are diagonal in the band index. Using the fact that \( U^*(G) = U(-G) \), we have

\[ V_{nn}(q) = i \sum_{G,G'} G_z \{ c^*_n(q + G)c_n(q + G')U(G - G') - \text{c.c.} \}. \quad (6.70) \]
From Eq. (6.67) we find

\[
\sum_{G,G'} G_z c_n^*(q+G)c_n(q+G')U(G-G') = -\sum_G G_z \left( \frac{\hbar^2}{2m} (q+G)^2 - \epsilon_{n,k} \right) |c_n(q+G)|^2.
\]

(6.71)

This shows that the sum on the left hand side of this equation is real and as a result \( V_{nn}(q) \) in Eq. (6.70) is zero. Together with Eq. (6.69), we see that \( V_{n'n}(q',q) \) has the structure

\[
V_{n'n}(q',q) = \delta_{qq'} (1 - \delta_{nn'}) V_{nn}(q).
\]

(6.72)

Substituting Eq. (6.72) into Eq. (6.64), we thus have

\[
\hat{\mathcal{V}} = \sum_{n' \neq n,q} V_{n'n}(q) \hat{a}^\dagger_{n'q} \hat{a}_{nq}.
\]

(6.73)

This implies that a perturbation of the type given in Eq. (6.40) only gives rise to inter-band transitions. This has interesting consequences in the context of the single-band Hubbard model. Within this model, an arbitrary many-body state is constructed from Bloch states of the lowest s-band and can be written as

\[
|\Psi_\alpha\rangle_{LB} = \sum_{\{n_k\}} C_\alpha(\{n_k\}) \prod_k \left( \hat{a}^\dagger_{sk} \right)^{n_k} |0\rangle.
\]

(6.74)

Here \( \hat{a}^\dagger_{sk} \) is the creation operator for the Bloch state \( \psi_{sk}(r) \) and \( \sum_k n_k = N \). For states of this kind, we clearly have

\[
\langle \Psi_\alpha | \hat{\mathcal{V}} | \Psi_0 \rangle_{LB} = 0.
\]

(6.75)

Thus, if these states are used to evaluate Eq. (6.58), we would conclude from Eq. (6.59)
that there is no energy absorption within this approximation. However, the true eigenstates will have admixtures of states in which higher bands are occupied as a result of the particle interactions. A perturbative analysis would suggest that the states take the form

\[ |\Psi_\alpha\rangle = |\Psi_\alpha\rangle_{LB} + |\text{higher band admixtures}\rangle. \] (6.76)

Once these corrections are included, the matrix elements of \( \hat{V} \) will no longer be zero. From this point of view, one must go beyond the lowest band model in order to obtain a finite energy absorption due to phase modulation. The argument given here would suggest that the energy absorption would be proportional to some power of \( U_{\text{rel}}/\Delta \), where \( U_{\text{rel}} \) is the relevant interaction energy parameter and \( \Delta \) is the band gap.

### 6.3 Future directions

Much of the discussion presented in this chapter is preliminary. In closing this chapter, we will briefly discuss some of the possible research directions that can be pursued in the future.

First, it will be of interest to determine the energy absorption for amplitude and phase modulation when the Bose condensate in the optical lattice is treated as a non-homogeneous superfluid. Eqs. (6.14) and (6.43) give the energy absorption in terms of the density response function which can again be evaluated in terms of the Bogoliubov excitations. This calculation would be complementary to that based on the Hubbard model.

Another subject that merits further investigation is the energy absorption due to phase modulation in the deep lattice limit. As we have argued, there is no energy
absorption within the single-band Hubbard model since the matrix elements of the operator $\hat{V}$ in Eq. (6.57) vanish in this approximation. On the other hand, matrix elements of the current operator with respect to the lowest band many-body states do not vanish, namely [66]
\[
\langle \Psi_\alpha | \hat{J}_z | \Psi_0 \rangle_{LB} \neq 0.
\] (6.77)

On the basis of Eq. (6.51) and Eq. (6.63), this implies that the energy absorption is not zero. This contradiction arises since Eq. (6.61) is only valid for the exact eigenstates. Eq. (6.59) and Eq. (6.63) are no longer equivalent when the approximate lowest band many-body states are used to evaluate the spectral densities. This raises the obvious question as to the reliability of Eq. (6.63) within the lowest band approximation. We suspect that a multi-band Hubbard model will be needed to answer this question. This is a problem we plan to investigate in the future.
Chapter 7

Conclusions

In this thesis, linear response theory is applied to understand various dissipative processes that are associated with the flow of an atomic Bose-Einstein condensate at zero temperature. The key physical quantity needed in our linear response calculations is the density response function of the Bose condensate. In Chapter 3, we have evaluated density response functions for both uniform and uniform cylindrical condensates within the Bogoliubov approximation. For the latter condensate, we provide a detailed comparison of the results obtained within the Thomas-Fermi hydrodynamic theory and those obtained by means of the Bogoliubov theory. In addition, we have developed a new local density approximation to determine the density response function for an elongated condensate which makes use of the response properties of a uniform cylindrical condensate. This approximation, which we refer to as the cylindrical LDA, is able to better capture the spectral properties of an elongated condensate than the more commonly used bulk LDA. Importantly, we show that all the response functions obtained in this chapter satisfy the f-sum rule.

In Chapter 4, we have derived in detail general linear response formulae for the
energy dissipation rate of a condensate moving at a constant velocity past a stationary potential. This energy dissipation is directly related to the drag force experienced by the moving condensate. With this approach, we have determined the drag force exerted on both a uniform and a uniform cylindrical condensate by an impurity potential. We also find that knowledge of the drag force allows one to determine the momentum imparted to a stationary condensate by a moving potential. This connection provides us a way to analyze Bragg spectroscopy experiments. Our theoretical results for the momentum imparted to an elongated condensate by a Bragg pulse are in reasonably good agreement with the experimental data. This calculation demonstrated convincingly that our cylindrical LDA is superior to the bulk LDA for the determination of the density response function of a highly elongated condensate.

In Chapter 5, we have studied the centre of mass dynamics of a trapped condensate in the presence of a correlated disorder potential. We have shown that this dynamics can be formulated in terms of a conventional linear response function approach even though the initial state of the system is far out of equilibrium. This is made possible by exploiting an equivalence between the motion of the condensate through a disorder external potential and the harmonic motion of the disorder potential itself relative to the harmonically-confined condensate. Via this reformulation, linear response theory could be used to calculate the damping rate of the centre of mass motion of an elongated condensate. Our theoretical results are found to be in good agreement with one set of experiments but not with another. Although we have no explanation for the discrepancies, we did point out some possible inconsistencies between these two sets of experiments. Our theory further predicts interesting parameter dependences of the damping rate which experiments can verify. With the appropriate response functions,
our theory can be also used to study the dissipation at finite temperatures and in fermionic systems. The former problem is particularly relevant in understanding the centre of mass dynamics at long times for a condensate with a large initial centre of mass energy. In such a situation, the temperature of the condensate increases significantly as the centre of mass energy is dissipated. As a result, the later stages of the dynamics can only be analyzed using finite-temperature response functions. Obtaining these response functions is an important problem for the future.

Finally, in Chapter 6 we have studied some aspects of the amplitude and phase modulation spectroscopy of a cold gas in an optical lattice using linear response theory. In the process, we critique an approximation that has been used extensively in theoretical studies of amplitude modulation spectroscopy. Furthermore, we have shown that linear response theory can shed light on a proposal to measure the optical conductivity of a system using phase modulation. Our analysis suggests a possible inconsistency in the use of a single-band Hubbard model in the calculation of the energy absorption rate.

In conclusion, we hope that we have demonstrated to the reader that linear response theory is a rather versatile tool in dealing with various dissipative dynamical problems encountered in cold atom physics.
Bibliography


Appendix A

Bogoliubov Hamiltonian

In this appendix we outline the procedure used to obtain the Bogoliubov Hamiltonian in Eq. (2.92). Substituting Eq. (2.25) into Eq. (2.24), we find that to second order in field fluctuation operators

\[
\hat{H} = \hat{a}_0^\dagger \hat{a}_0 \int d\mathbf{r} \chi_0^* \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) \right) \chi_0 + \frac{1}{2} g \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \hat{a}_0 \int d\mathbf{r} |\chi_0|^4 \\
+ \hat{a}_0 \int d\mathbf{r} \delta \hat{\psi}^\dagger \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) \right) \chi_0 + g \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \int d\mathbf{r} |\chi_0|^2 \chi_0 \delta \hat{\psi}^\dagger \\
+ \hat{a}_0^\dagger \int d\mathbf{r} \chi_0^* \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) \right) \delta \hat{\psi} + g \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \int d\mathbf{r} |\chi_0|^2 \chi_0^* \delta \hat{\psi}^\dagger \\
+ \int d\mathbf{r} \delta \hat{\psi}^\dagger \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) \right) \delta \hat{\psi} + 2 g \hat{a}_0^\dagger \hat{a}_0 \int d\mathbf{r} |\chi_0|^2 \delta \hat{\psi}^\dagger \delta \hat{\psi} \\
+ \frac{1}{2} g \hat{a}_0^\dagger \hat{a}_0 \int d\mathbf{r} \chi_0^2 \delta \hat{\psi}^\dagger \delta \hat{\psi}^\dagger + \frac{1}{2} g \hat{a}_0^\dagger \hat{a}_0 ^\dagger \int d\mathbf{r} \chi_0^2 \delta \hat{\psi} \delta \hat{\psi}. \tag{A.1}
\]
Using Eq. (2.26) in Eq. (A.1) to eliminate $\hat{a}_0^\dagger \hat{a}_0$, we find that the first line on the right hand side of Eq. (A.1) is given by

$$N \int dr \chi_0^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + \frac{1}{2} g N^2 \int dr |\chi_0(r)|^4$$

$$- \left[ \int dr \chi_0^*(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + g N \int dr |\chi_0(r)|^4 \right] \int dr \delta \hat{\psi}^\dagger(r) \delta \hat{\psi}(r)$$

to second order in field fluctuation operators. Using Eq. (2.14) and Eq. (2.19), the above expression is found to be equal to

$$E_0^{\text{GP}} - \mu \int dr \delta \hat{\psi}^\dagger(r) \delta \hat{\psi}(r). \quad (A.2)$$

Similarly, using Eq. (2.26) in the second line in Eq. (A.1) we find

$$\int dr \delta \hat{\psi}^\dagger(r) \left[ \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(r) \right) \chi_0(r) + g N \int dr |\chi_0(r)|^2 \chi_0(r) \right] \hat{a}_0$$

$$+ \text{third order terms in field fluctuation operators}$$

$$= \mu \int dr \delta \hat{\psi}^\dagger(r) \chi_0(r) \hat{a}_0 + \text{third order terms in field fluctuation operators} \quad (A.3)$$

where Eq. (2.14) is again used. From the orthogonality relation $\int dr \delta \hat{\psi}^\dagger(r) \chi_0(r) = 0$, we see that Eq. (A.3) vanishes to second order in the fluctuation operators. Thus to this order, the second line in Eq. (A.1) can be neglected. The same conclusion is true for the third line in Eq. (A.1) as it is the Hermitian conjugate of the expression in the second line. Thus we have shown that in Eq. (A.1) the terms involving field fluctuation operators to first order vanish. Assuming that effect of the third and fourth order terms is small compared to that of the second order terms, we find that
Hamiltonian is given by

\[
\hat{H}_B = E_0^{GP} + \int d\mathbf{r}\delta\hat{\psi}^\dagger(\mathbf{r})\left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + 2g|\Phi_0(\mathbf{r})|^2 - \mu\right)\delta\hat{\psi}(\mathbf{r})
\]
\[
+ \frac{1}{2}g\hat{b}_0\hat{b}_0\int d\mathbf{r}\Phi_0(\mathbf{r})^2\delta\hat{\psi}^\dagger(\mathbf{r})\delta\hat{\psi}^\dagger(\mathbf{r}) + \frac{1}{2}g\hat{b}_0\hat{b}_0^\dagger\int d\mathbf{r}\Phi_0^*(\mathbf{r})^2\delta\hat{\psi}(\mathbf{r})\delta\hat{\psi}(\mathbf{r}),
\]

(A.4)

where \(\hat{b}_0(\hat{b}_0^\dagger) = \hat{a}_0(\hat{a}_0^\dagger)/\sqrt{N}\).
Appendix B

Dobson’s Harmonic Potential Theorem

In this appendix, we present an alternative derivation of Dobson’s Harmonic Potential Theorem. This theorem concerns a harmonically trapped system in the presence of a time-dependent force $F(t)$ that couples to the centre of mass degree of freedom. The Hamiltonian for this system is

$$\hat{H}(t) = \hat{H} - F(t) \cdot \sum_{i=1}^{N} \hat{r}_i = \hat{H} - N F(t) \cdot \hat{R},$$

(B.1)

where $\hat{H}$ is given in Eq. (5.3) and $\hat{R} = \sum_{i=1}^{N} \hat{r}_i/N$ is the centre of mass coordinate.

In [57], the author constructs the following dynamic state

$$\Psi_{\text{HPT}}(r_1, \ldots, r_N, t) = e^{-iE_{\alpha}t/\hbar - iNS(t) + iNm\dot{x}(t)} R/h \Psi_{\alpha}(r_1 - x(t), \ldots, r_N - x(t)).$$

(B.2)

Here $\Psi_{\alpha}(r_1, \ldots, r_N)$ is an eigenstate of $\hat{H}$ with energy $E_{\alpha}$, $x(t)$ is some time-dependent displacement and $\dot{x}(t) \equiv dx(t)/dt$. The phase angle in Eq. (B.2) is given by

$$S(t) = \frac{1}{\hbar} \int_0^t dt' \sum_{\mu} \left[ \frac{1}{2} m \dot{x}_\mu(t')^2 - \frac{1}{2} m\omega_\mu^2 x_\mu(t)^2 \right].$$

(B.3)
The author then shows that $\Psi_{\text{HPT}}(\{r_j\}, t)$ satisfies the following equation

$$
\left(\frac{i\hbar}{\partial t} - \hat{\mathcal{H}}(t)\right) \Psi_{\text{HPT}}(\{r_j\}, t) = -N \sum_\mu \left[ m\ddot{x}_\mu(t) + m\omega^2_\mu x_\mu(t) - F_\mu(t) \right] R_\mu \Psi_{\text{HPT}}(\{r_j\}, t).$

(B.4)

This implies that $\Psi_{\text{HPT}}$ is a solution to the time-dependent many-body Schrödinger equation

$$
i\hbar \frac{\partial}{\partial t} \Psi_{\text{HPT}}(\{r_j\}, t) = \hat{\mathcal{H}}(t) \Psi_{\text{HPT}}(\{r_j\}, t),$$

(B.5)

if the displacement $x(t)$ satisfies the classical driven harmonic oscillator equation

$$m\ddot{x}_\mu(t) + m\omega^2_\mu x_\mu(t) = F_\mu(t).$$

(B.6)

The above summarizes Dobson’s derivation of the Harmonic Potential Theorem.

We now prove this theorem in a more systematic way. We first observe that Dobson’s wavefunction $\Psi_{\text{HPT}}(\{r_j\}, 0)$ corresponds to the displaced state vector $|\Psi(0)\rangle = e^{i\theta/\hbar} T(x, p)|\Psi_\alpha\rangle$, where $\theta = x \cdot p / 2$, $x = x(0)$ and $p = Nm\dot{x}(0)$. The time evolution of this state is formally given by

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle,$$

(B.7)

where the unitary evolution operator $\hat{U}(t)$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{\mathcal{H}}(t) \hat{U}(t)$$

(B.8)

with the initial condition $\hat{U}(0) = \hat{I}$. To determine $\hat{U}(t)$, we go to the interaction
picture and define

$$\hat{U}_t(t) = e^{i\hat{H}t/\hbar}\hat{U}(t), \quad (B.9)$$

which also has the initial condition $\hat{U}_t(0) = \hat{I}$. This evolution operator satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}_t(t) = -N \sum_{\mu} F_{\mu}(t) \hat{R}_{\mu}(t) \hat{U}_t(t), \quad (B.10)$$

where $\hat{R}_{\mu}(t) = e^{i\hat{H}t/\hbar}\hat{R}_{\mu}e^{-i\hat{H}t/\hbar}$. This operator is given explicitly in Eq. (5.11). The formal solution of Eq. (B.10) can be written as

$$\hat{U}_t(t) = \lim_{N_d \to \infty} e^{i\chi N \sum_{\mu} F_{\mu}(j\Delta t) \hat{R}_{\mu}(j\Delta t) \Delta t}$$

$$= \lim_{N_d \to \infty} \prod_{\mu} e^{i\chi N F_{\mu}(N_d \Delta t) \hat{R}_{\mu}(N_d \Delta t) \Delta t} \cdots e^{i\chi N F_{\mu}(2\Delta t) \hat{R}_{\mu}(2\Delta t) \Delta t} e^{i\chi N F_{\mu}(\Delta t) \hat{R}_{\mu}(\Delta t) \Delta t}, \quad (B.11)$$

where $\Delta t = t/N_d$. This result follows from the fact that $\hat{R}_{\mu}(t)$ and $\hat{R}_{\nu}(t')$ commutes when $\mu \neq \nu$.

The product of operators can be evaluated iteratively by making use of the Baker-Hausdorff formula. From Eq. (5.11), we have

$$[\hat{R}_{\mu}(t), \hat{R}_{\nu}(t')] = \frac{i\hbar}{Nm\omega_{\mu}} \sin \omega_{\mu}(t' - t), \quad (B.12)$$
and as a result,

\[
\exp \left\{ \frac{i}{\hbar} NF_\mu (2\Delta t) \hat{R}_\mu (2\Delta t) \Delta t \right\} \exp \left\{ \frac{i}{\hbar} NF_\mu (\Delta t) \hat{R}_\mu (\Delta t) \Delta t \right\} = \exp \left\{ \frac{i}{\hbar} NF_\mu (2\Delta t) \hat{R}_\mu (\Delta t) \Delta t + \frac{i}{\hbar} NF_\mu (\Delta t) \hat{R}_\mu (\Delta t) \Delta t \right\} \times \exp \left\{ \frac{iN}{2m \hbar \omega_\mu} F_\mu (2\Delta t) F_\mu (\Delta t) \sin \omega_\mu (2\Delta t - \Delta t) (\Delta t)^2 \right\}. 
\]

(B.13)

After repeating these steps \( j - 1 \) times, we must consider in the next step

\[
\exp \left\{ \frac{i}{\hbar} NF_\mu (j\Delta t) \hat{R}_\mu (j\Delta t) \Delta t \right\} \exp \left\{ \frac{i}{\hbar} N \sum_{k=1}^{j-1} F_\mu (k\Delta t) \hat{R}_\mu (k\Delta t) \Delta t \right\} = \exp \left\{ \frac{i}{\hbar} N \sum_{k=1}^{j} F_\mu (k\Delta t) \hat{R}_\mu (k\Delta t) \Delta t \right\} \times \exp \left\{ \frac{iN}{2m \hbar \omega_\mu} F_\mu (j\Delta t) \sum_{k=1}^{j-1} F_\mu (k\Delta t) \sin \omega_\mu (j\Delta t - k\Delta t) (\Delta t)^2 \right\}. 
\]

(B.14)

A phase factor of this kind appears at each step of the process. Accumulating these phase factors, we find that Eq. (B.11) becomes

\[
\mathcal{U}_t(t) = \lim_{N_d \to \infty} \prod_\mu \exp \left\{ \frac{i}{\hbar} N \sum_{j=1}^{N_d} F_\mu (j\Delta t) \cdot \hat{R}_\mu (j\Delta t) \Delta t \right\} \times \exp \left\{ \frac{iN}{2m \hbar \omega_\mu} \sum_{j_1=2}^{N_d} \sum_{j_2=1}^{j_1-1} F_\mu (j_1\Delta t) F_\mu (j_2\Delta t) \sin \omega_\mu (j_1\Delta t - j_2\Delta t) (\Delta t)^2 \right\} = \exp \left\{ \frac{i}{\hbar} \int_0^t dt' NF(t') \cdot \hat{R}(t') \right\} \times \prod_\mu \exp \left\{ \frac{iN}{2m \hbar \omega_\mu} \int_0^t dt' \int_0^{t'} dt'' F_\mu (t') F_\mu (t'') \sin \omega_\mu (t' - t'') \right\}. 
\]

(B.15)

This result for \( \mathcal{U}_t(t) \) can be used to obtain the dynamic state in Eq. (B.7). Using
Eq. (B.9), we have

\[
|\Psi(t)\rangle = e^{i\theta/\hbar}e^{-i\hat{H}t/\hbar}U_1(t)|\Psi(0)\rangle, \\
= e^{i\theta/\hbar}e^{-i\hat{H}t/\hbar}U_\alpha(t)\hat{T}(x, p)|\Psi_\alpha\rangle \\
= e^{i\theta/\hbar}e^{-iE_\alpha t/\hbar}U_1(t)\hat{T}(x, p)e^{i\hat{H}t/\hbar}|\Psi_\alpha\rangle.
\] (B.16)

To evaluate \(U_1(t)\hat{T}(x, p)\), we must consider products of the form

\[
\exp \left\{ \frac{i}{\hbar} \int_0^t dt' NF_\mu(t')\hat{R}_\mu(t') \right\} \exp \left\{ \frac{i}{\hbar}(p_\mu\hat{R}_\mu - x_\mu\hat{P}_\mu) \right\}.
\] (B.17)

Using the Baker-Hausdorff formula, we obtain

\[
\exp \left\{ \frac{i}{\hbar} \int_0^t dt' NF_\mu(t')\hat{R}_\mu(t') \right\} \exp \left\{ \frac{i}{\hbar}(p_\mu\hat{R}_\mu - x_\mu\hat{P}_\mu) \right\} = \exp \left( \frac{c}{2} \right),
\] (B.18)

where

\[
c = \left[ \frac{i}{\hbar} \int_0^t dt' NF_\mu(t')\hat{R}_\mu(t'), \frac{i}{\hbar}(p_\mu\hat{R}_\mu - x_\mu\hat{P}_\mu) \right].
\] (B.19)

Using Eq. (5.11) for \(\hat{R}_\mu(t')\), we find that

\[
c = \frac{i}{\hbar} \int_0^t dt' F_\mu(t')x_\mu(t'),
\] (B.20)

where \(x_\mu(t)\) is given by Eq. (5.30). We thus have

\[
U_1(t)\hat{T}(x, p) = e^{\frac{\hbar x(t)}{i}}\exp \left\{ \frac{i}{\hbar} \left[ p \cdot \hat{R} - x \cdot \hat{P} + \int_0^t dt' NF(t') \cdot \hat{R}(t') \right] \right\},
\] (B.21)
where

\[ \chi(t) = \frac{1}{2} \int_0^t dt' N \mathbf{F}(t') \cdot \mathbf{x}(t') + \sum_\mu \frac{N}{2m\omega_\mu} \int_0^t dt' \int_0^{t'} dt'' F_\mu(t') F_\mu(t'') \sin \omega_\mu(t' - t''). \]

We thus find

\[ |\Psi(t)\rangle = e^{i\hat{\pi}t} e^{-\frac{i}{\hbar} E_\alpha t} e^{i\hat{\pi} \chi(t)} \exp \left\{ \frac{i}{\hbar} \left[ \mathbf{p}(t) \cdot \hat{\mathbf{R}} - \mathbf{x}(t) \cdot \hat{\mathbf{P}} + \int_0^t dt' N \mathbf{F}(t') \cdot \hat{\mathbf{R}}(t') \right] \right\} |\Psi_\alpha\rangle \]

\[ = e^{i\hat{\pi}t} e^{-\frac{i}{\hbar} E_\alpha t} e^{i\hat{\pi} \chi(t)} \hat{T}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{p}}(t)) |\Psi_\alpha\rangle, \]

(B.23)

where

\[ \tilde{x}_\mu(t) = x_\mu(t) + \frac{1}{m\omega_\mu} \int_0^t dt' \sin \omega_\mu(t - t') F_\mu(t'), \]

\[ \tilde{p}_\mu(t) = p_\mu(t) + N \int_0^t dt' \cos \omega_\mu(t - t') F_\mu(t'). \]

(B.24)

Here \( \tilde{x}_\mu(t) \) is the solution of the forced harmonic oscillator equation (B.6) with the initial conditions \( \tilde{x}_\mu(0) = x_\mu(0) \) and \( \dot{\tilde{x}}_\mu(0) = p_\mu(0)/Nm \). In terms of this solution we have

\[ \chi(t) = \frac{N}{2} \int_0^t dt' \mathbf{F}(t') \cdot \tilde{\mathbf{x}}(t'). \]

(B.25)

To show that Eq. (B.23) is indeed the \( \Psi_{\text{HPT}} \) state we rewrite this state as

\[ |\Psi(t)\rangle = e^{i\hat{\pi}t} e^{-\frac{i}{\hbar} E_\alpha t} e^{i\hat{\pi} \chi(t)} e^{-\frac{i}{\hbar} \tilde{\mathbf{x}}(t) \cdot \hat{\mathbf{p}}(t) - \frac{i}{\hbar} \tilde{\mathbf{p}}(t) \cdot \hat{\mathbf{R}} e^{-\frac{i}{\hbar} \tilde{\mathbf{x}}(t) \cdot \hat{\mathbf{p}}} |\Psi_\alpha\rangle. \]

(B.26)
The many-body wavefunction corresponding to this state is simply

$$
\Psi(r_1, \ldots, r_N, t) = e^{-iE_\alpha t/\hbar-iNS'(t)+iNm\tilde{x}(t)} R/\hbar \Psi_\alpha(r_1 - \tilde{x}(t), \ldots, r_N - \tilde{x}(t)), \quad (B.27)
$$

where

$$
S'(t) = \frac{1}{2N\hbar} [\tilde{x}(t) \cdot \tilde{p}(t) - \tilde{x}(0) \cdot \tilde{p}(0)] - \frac{1}{2\hbar} \int_0^t dt' F(t') \cdot \tilde{x}(t'). \quad (B.28)
$$

Using Eq. (B.6) we find

$$
\int_0^t dt' F_\mu(t')\tilde{x}_\mu(t') = m \int_0^t dt'(\tilde{x}_\mu + \omega^2 \tilde{x}_\mu)\tilde{x}_\mu
$$

$$
= m\tilde{x}_\mu(t)\tilde{x}_\mu(t) - m\tilde{x}_\mu(0)\tilde{x}_\mu(0) + \int_0^t dt' \left(m\omega^2 \tilde{x}^2_\mu - m\tilde{x}^2_\mu\right). \quad (B.29)
$$

Substituting this result in Eq. (B.28), we find that $S'$ is the same as the phase angle $S$ given in Eq. (B.3). This thus completes our derivation of Dobson’s HPT.
Appendix C

The $I_l$, $I_l^{(1)}$ and $I_l^{(2)}$ integrals

In this appendix, we provide more details of the evaluations of the integrals $I_l(q_z, \omega)$, $I_l^{(1)}(q_z, \omega)$ and $I_l^{(2)}(q_z, \omega)$. From Eq. (5.106), the integral $I_l(q_z, \omega)$ is given by

$$I_l(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \int_0^t dt' \cos \omega_z t \cos q_z z_0 (\sin \omega_z t - \sin \omega_z t') e^{-i\omega (t-t')} . \tag{C.1}$$

To evaluate this integral we make use of the Jacobi-Anger expansion

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta} , \tag{C.2}$$

where $n$ is an integer and $J_n(x)$ is the integer Bessel function of the first kind

$$J_n(x) = \frac{1}{2\pi} \int_0^\pi d\theta e^{i(x \sin \theta - n\theta)} . \tag{C.3}$$
We find
\[
I_1(q_z, \omega) = \sum_{n,n'=-\infty}^{\infty} J_n(q_z z_0) J_{n'}(q_z z_0) \int_{T_{l-1}}^{T_l} dt \cos \omega_z t e^{-i(\omega - n\omega_z)t} \int_0^t dt' e^{i(\omega - n'\omega_z)t'}.
\] (C.4)

The time integrals give
\[
\int_{T_{l-1}}^{T_l} dt \cos \omega_z t e^{-i(\omega - n\omega_z)t} \int_0^t dt' e^{i(\omega - n'\omega_z)t'}
= \frac{ie^{-i(2l-1)\pi\bar{\omega}} \sin(\pi\bar{\omega})}{\omega_z^2} \left[ \frac{1}{\bar{\omega} - (n + 1)} + \frac{1}{\bar{\omega} - (n - 1)} \right] \frac{1}{\bar{\omega} - n'},
\] (C.5)

where \( \bar{\omega} = \omega/\omega_z \). Inserting this result into Eq. (C.4), we have
\[
I_1(q_z, \omega) = \frac{ie^{-i(2l-1)\pi\bar{\omega}} \sin(\pi\bar{\omega})}{\omega_z^2} \sum_{n,n'=-\infty}^{\infty} \left[ \frac{J_n(q_z z_0)}{\bar{\omega} - (n + 1)} + \frac{J_n(q_z z_0)}{\bar{\omega} - (n - 1)} \right] \frac{J_{n'}(q_z z_0)}{\bar{\omega} - n'}
\]
\[
= \frac{ie^{-i(2l-1)\pi\bar{\omega}} \sin(\pi\bar{\omega})}{\omega_z^2} \sum_{n=-\infty}^{\infty} \frac{J_{n-1}(q_z z_0) + J_{n+1}(q_z z_0)}{\bar{\omega} - n} \sum_{n'=-\infty}^{\infty} \frac{J_{n'}(q_z z_0)}{\bar{\omega} - n'}.
\] (C.6)

Making use of the identity
\[
J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x),
\] (C.7)

Eq. (C.6) becomes
\[
I_1(q_z, \omega) = \frac{2ie^{-i(2l-1)\pi\bar{\omega}} \sin(\pi\bar{\omega})}{q_z v_0 \omega_z} \sum_{n=-\infty}^{\infty} \frac{nJ_n(q_z z_0)}{\bar{\omega} - n} \sum_{n'=-\infty}^{\infty} \frac{J_{n'}(q_z z_0)}{\bar{\omega} - n'}.
\] (C.8)
To simplify Eq. (C.8), we first observe that
\[
\sum_{n=-\infty}^{\infty} \frac{nJ_n(q_0z_0)}{\bar{\omega} - n} = -\sum_{n=-\infty}^{\infty} J_n(q_0z_0) + \bar{\omega} \sum_{n=-\infty}^{\infty} \frac{J_n(q_0z_0)}{\bar{\omega} - n}
\]
\[
= \bar{\omega} \sum_{n=-\infty}^{\infty} \frac{J_n(q_0z_0)}{\bar{\omega} - n} - 1,
\]
(C.9)
where we used that fact that \( \sum_{n=-\infty}^{\infty} J_n(q_0z_0) = 1 \) (see Eq. (C.2)). Thus we finally need to evaluate the expression \( \sum_{n=-\infty}^{\infty} \frac{J_n(q_0z_0)}{\bar{\omega} - n} \). To do so, we note that
\[
\frac{1}{\bar{\omega} - n} \text{ can be written as }
\]
\[
\frac{1}{\bar{\omega} - n} = \frac{e^{i\pi \bar{\omega}}}{2 \sin \pi \bar{\omega}} \int_0^{2\pi} d\theta e^{i\theta(n-\bar{\omega})}.
\]
(C.10)
Using this expression, we have
\[
\sum_{n=-\infty}^{\infty} \frac{J_n(q_0z_0)}{\bar{\omega} - n} = \frac{e^{i\pi \bar{\omega}}}{2 \sin \pi \bar{\omega}} \int_0^{2\pi} d\theta e^{-i\bar{\omega} \theta} \sum_{n=-\infty}^{\infty} J_n(q_0z_0)e^{in\theta}
\]
\[
= \frac{e^{i\pi \bar{\omega}}}{2 \sin \pi \bar{\omega}} \int_0^{2\pi} d\theta e^{-i\bar{\omega} \theta + iq_0z_0 \sin \theta}
\]
\[
= \frac{\pi}{\sin \pi \bar{\omega}} J_{\bar{\omega}}(q_0z_0),
\]
(C.11)
where \( J_{\nu}(x) \) is defined as
\[
J_{\nu}(x) \equiv \frac{e^{-i\pi \nu}}{2\pi} \int_0^{2\pi} d\theta e^{i(x \sin \theta + \nu \theta)}.
\]
(C.12)
This function is real. To show this, we first observe that the variable change \( \theta' = \theta - \pi \) gives

\[
J_\nu(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-i(x \sin \theta - \nu \theta)}. \tag{C.13}
\]

Thus,

\[
J_\nu(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( \cos(x \sin \theta - \nu \theta) - i \sin(x \sin \theta - \nu \theta) \right)
= \frac{1}{\pi} \int_{0}^{\pi} d\theta \cos(x \sin \theta - \nu \theta). \tag{C.14}
\]

This result is identified as the so-called Anger function \([59]\). Combining Eq. (C.6), Eq. (C.9) and Eq. (C.11), we arrive at the final result

\[
I_l(q_z, \omega) = \frac{2\pi i}{q_z \nu \omega z} e^{-i(2l-1)\pi \omega} J_{-\omega}(q_z z_0) \left( \frac{\pi \omega}{\sin \pi \omega} J_{-\omega}(q_z z_0) - 1 \right). \tag{C.15}
\]

We next turn to the integral \( I_{l}^{(1)}(q_z, \omega) \) defined in Eq. (5.137)

\[
I_{l}^{(1)}(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \sin \omega z t \int_{-\infty}^{0} dt' e^{i t t'} e^{i q_z z_0 (\cos \omega z t - 1)} e^{-i \omega (t-t')} \tag{C.16}
\]

Using the Jacobi-Anger expansion

\[
e^{ix \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\theta} \tag{C.17}
\]
we find

\[ I^{(1)}_t(q_z, \omega) = \sum_{n,n'=-\infty}^{\infty} i^{n-n'} J_n(q_z z_0) J_{n'}(q_z z_0) \int_{T_{l-1}}^{T_l} dt \sin \omega z t e^{-i(\omega - n \omega_z) t} \int_0^t dt' e^{i\epsilon t'} e^{i\omega t}. \]

\[ = \frac{e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega}}{2\omega_z^2} \sum_{n,n'=-\infty}^{\infty} i^{n-n'-1} \left[ \frac{J_n(q_z z_0)}{\omega - (n+1)} - \frac{J_n(q_z z_0)}{\omega - (n-1)} \right] \frac{J_{n'}(q_z z_0)}{\omega - i\epsilon} \]

\[ = - \frac{1}{q_z v_0 \omega_z} \sum_{n,n'=-\infty}^{\infty} i^{n-n'} n J_n(q_z z_0) J_{n'}(q_z z_0) e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega} \frac{e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega}}{(\omega - n)(\omega - i\epsilon)}, \]  

(C.18)

where Eq. (C.7) is used to get the last line. This is the expression for \( I^{(1)}_t(q_z, \omega) \) used in Eq. (5.140).

Finally, the integral \( I^{(2)}_t(q_z, \omega) \) defined in Eq. (5.142) is

\[ I^{(2)}_t(q_z, \omega) = \int_{T_{l-1}}^{T_l} dt \sin \omega z t \int_0^t dt' e^{i q_z z_0 (\cos \omega_z t - \cos \omega_z t')} e^{-i\omega (t-t')}. \]  

(C.19)

Using the Jacobi-Anger expansion in Eq. (C.17), we find

\[ I^{(2)}_t(q_z, \omega) = \sum_{n,n'=-\infty}^{\infty} i^{n-n'} J_n(q_z z_0) J_{n'}(q_z z_0) \int_{T_{l-1}}^{T_l} dt \sin \omega z t e^{-i(\omega - n \omega_z) t} \int_0^t dt' e^{i(\omega - n' \omega_z) t'}. \]

\[ = \frac{e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega}}{2\omega_z^2} \sum_{n,n'=-\infty}^{\infty} i^{n-n'-1} \left[ \frac{J_n(q_z z_0)}{\omega - (n+1)} - \frac{J_n(q_z z_0)}{\omega - (n-1)} \right] \frac{J_{n'}(q_z z_0)}{\omega - n'} \]

\[ = \frac{e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega}}{2\omega_z^2} \sum_{n=-\infty}^{\infty} i^n J_{n-1}(q_z z_0) J_{n+1}(q_z z_0) \sum_{n'=-\infty}^{\infty} i^{-n'} J_{n'}(q_z z_0) \frac{e^{-i2l\pi \omega} - e^{-i2(l-1)\pi \omega}}{(\omega - n)(\omega - n')}. \]  

(C.20)

This final result is used in Eq. (5.143). We point out that \( I^{(2)}_t(q_z, \omega) \) can also be
brought into a form that is analogous to that of $I_l(q_z, \omega)$ in Eq. (C.15). To do so, we write

$$I_l^{(2)}(q_z, \omega) = \frac{2ie^{-(2l-1)\pi\bar{\omega}}}{q_z v_0 \omega_z} \sum_{n=-\infty}^{\infty} \frac{i^n J_n(q_z z_0)}{\omega - n} \sum_{n'=-\infty}^{\infty} \frac{i^{-n'} J_{n'}(q_z z_0)}{\bar{\omega} - n'}. \quad (C.21)$$

The first sum can be expressed as

$$\sum_{n=-\infty}^{\infty} \frac{i^n J_n(q_z z_0)}{\omega - n} = -\sum_{n=-\infty}^{\infty} i^n J_n(q_z z_0) + \bar{\omega} \sum_{n=-\infty}^{\infty} \frac{i^n J_n(q_z z_0)}{\omega - n} = \bar{\omega} \sum_{n=-\infty}^{\infty} \frac{i^n J_n(q_z z_0)}{\omega - n} - e^{iq_z z_0}, \quad (C.22)$$

where we used the fact that $\sum_{n=-\infty}^{\infty} i^n J_n(x) = e^{ix}$ (see Eq. (C.17)). Using Eq. (C.10) and Eq. (C.17), we have

$$\sum_{n=-\infty}^{\infty} \frac{i^n J_n(q_z z_0)}{\omega - n} = \frac{e^{i\pi \bar{\omega}}}{2\sin \pi \bar{\omega}} \int_0^{2\pi} d\theta e^{-i\bar{\omega} \theta} \sum_{n=-\infty}^{\infty} i^n J_n(q_z z_0)e^{in\theta}$$

$$= \frac{e^{i\pi \bar{\omega}}}{2\sin \pi \bar{\omega}} \int_0^{2\pi} d\theta e^{-i\bar{\omega} \theta + iq_z z_0 \cos \theta}$$

$$= \frac{\pi}{\sin \pi \bar{\omega}} G_{-\bar{\omega}}(q_z z_0), \quad (C.23)$$

where

$$G_{\nu}(x) \equiv \frac{e^{-i\pi \nu}}{2\pi} \int_0^{2\pi} d\theta e^{i(\nu \theta + x \cos \theta)}. \quad (C.24)$$

From Eq. (C.21-C.23) we find that $I_l^{(2)}(q_z, \omega)$ can be written as

$$I_l^{(2)}(q_z, \omega) = \frac{2\pi i}{q_z v_0 \omega_z} e^{-(2l-1)\pi\bar{\omega}} G^*_{-\bar{\omega}}(q_z z_0) \left( \frac{\pi \bar{\omega}}{\sin \pi \bar{\omega}} G_{-\bar{\omega}}(q_z z_0) - e^{iq_z z_0} \right). \quad (C.25)$$

Unlike $J_{\nu}(x)$, however, $G_{\nu}(x)$ is not to our knowledge a tabulated special function.
Appendix D

Wannier functions

In this appendix, we present some details regarding the calculations of the Wannier functions used in Chapter 6. Since the Hamiltonian of a particle in a 3-dimensional optical lattice of the form given in Eq. (6.15) is separable in Cartesian coordinates, it is sufficient to consider only a one-dimensional optical lattice for which the Hamiltonian is given by

\[ \hat{h}_{0x} = \frac{\hat{p}_x^2}{2m} + V_0 \cos^2 k_x \hat{x}, \]

where \( k_x = \frac{\pi}{a} \) and \( a \) is the lattice spacing.

The Bloch states \( \psi_{nq}(x) \) are obtained by solving the following Schrödinger equation

\[ \hat{h}_{0x} \psi_{nq}(x) = \epsilon_{nq} \psi_{nq}(x). \]

Here \( n \) is the band index and \( q = k \frac{2\pi}{N_La} \), \( k = -N_L/2, -N_L/2 + 1, \ldots, N_L/2 - 1 \), where \( N_L \) is the total number of lattice sites (chosen to be even for convenience). To solve
Eq. (D.2), we expand \( \psi_{nq}(x) \) as a Fourier series

\[
\psi_{nq}(x) = \frac{1}{\sqrt{NLa}} \sum_G c_n(q + G)e^{(q+G)x}, \tag{D.3}
\]

where \( G = lG_0, l = 0, \pm 1, \ldots \). Defining \( G_0 = 2\pi/a, \ G = lG_0 \). The normalization condition of the Bloch wave functions \( \int dx |\psi_{nq}(x)|^2 = 1 \) gives rise to

\[
\sum_G |c_n(q + G)|^2 = 1. \tag{D.4}
\]

Substituting Eq. (D.3) into Eq. (D.2), we find

\[
\left( \frac{\hbar^2}{2m} (q + G)^2 - \epsilon_{nq} \right) c_n(q + G) + \sum_{G'} U(G - G')c_n(q + G') = 0, \tag{D.5}
\]

where \( U(G) \) is the Fourier component of the optical lattice potential

\[
U(G) = \int_{-a/2}^{a/2} dx V(x)e^{-iGx} \\
= V_0 \int_{-a/2}^{a/2} dx \cos^2(\pi x/a)e^{-iGx} \\
= \frac{V_0}{2} \delta_{G,0} + \frac{V_0}{4} (\delta_{G,G_0} + \delta_{G,-G_0}). \tag{D.6}
\]

Since the first term of the last line affects only the zero of energy, it will be dropped hereafter. Writing \( c_n(q + G) = c_n(l, k) \) and \( \epsilon_{nq} = \epsilon_n(k) \), Eq. (D.5) becomes

\[
\left[ \frac{\hbar^2}{2m} \left( \frac{2\pi}{N_La} k + \frac{2\pi}{a} l \right)^2 - \epsilon_n(k) \right] c_n(l, k) + \frac{V_0}{4} \sum_{l'} (\delta_{l-l',0} + \delta_{l-l',-1})c_n(l', k) = 0. \tag{D.7}
\]
Using the recoil energy \( E_r = \frac{k^2}{2m} \left( \frac{\pi}{a} \right)^2 \) as the unit of energy, we have

\[
\left( \frac{2k}{N_L} + 2l \right)^2 - \tilde{e}_n(k) \right) c_n(l, k) + \frac{\tilde{V}_0}{4} \sum_{l'} (\delta_{l-l',1} + \delta_{l-l',-1}) c_n(l', k) = 0, \tag{D.8}
\]

where \( \tilde{e}_n(k) = e_n(k)/E_r \) and \( \tilde{V}_0 = V_0/E_r \). This equation can be written in the form of a matrix equation

\[
\sum_{l'} \mathcal{H}_{ll'} c_n(l', k) = \tilde{\epsilon}_n(k) c_n(l, k), \tag{D.9}
\]

with

\[
\mathcal{H}_{ll'} = \delta_{l,l'} \left( \frac{2k}{N_L} + 2l \right)^2 + \frac{\tilde{V}_0}{4} (\delta_{l-l',1} + \delta_{l-l',-1}). \tag{D.10}
\]

For the lowest few bands the coefficients \( c_n(l, k) \) generally decreases rapidly in \( |l| \) and the matrix equation can be truncated at \( |l| = |l|_{\text{max}} \). In practice we have found that Bloch states of the lowest few bands are given with sufficient accuracy with \( |l| \leq |l|_{\text{max}} = 10 \). Once we have solved Eq. (D.9) for the expansion coefficients, the Bloch wave function is given as

\[
\psi_{nq}(x) \simeq \frac{1}{\sqrt{N_L a}} \sum_{l=|l|_{\text{max}}}^{N_L/2 - 1} c_n(l, k) e^{i \left[ \frac{4\pi}{N_L} l + l \right] \frac{2\pi}{a} x}. \tag{D.11}
\]

The Wannier functions are defined as

\[
w_{n,j}(x) = \frac{1}{\sqrt{N_L}} \sum_{k=-N_L/2}^{N_L/2-1} \psi_{nq}(x) e^{-i \frac{2\pi}{N_L} k j a} = \frac{1}{\sqrt{N_L}} \sum_{k=-N_L/2}^{N_L/2-1} \psi_{nk}(x) e^{-i \frac{2\pi}{N_L} k j}, \tag{D.12}
\]

where \( j \) is the site index. Wannier functions have the property that

\[
w_{n,j}(x) = w_n(x - x_j), \tag{D.13}
\]
where $x_j$ is the coordinate of the $j$-th site. These functions are not unique due to the fact that each term in the sum in Eq. (D.12) can be multiplied by an arbitrary phase factor. Different choices of the phase factors can give rise to distinctively different sets of Wannier functions. Here we use the so-called maximally localized Wannier functions [77] which are generated by choosing $\psi_{nq}(x)$ to be an analytic function of $q$. 

Figure D.1: s-band (left) and p-band (right) Wannier functions (solid lines) for $V_0/E_r = 3$. The dashed lines represent the ground state (left) and the first excited state (right) of the local harmonic potential.

Figure D.2: The same as Fig. D.1 but for $V_0/E_r = 15$. 
with $\psi_{nq}(0)$ real. In practice, these conditions are met if the coefficients $c_n(l,k)$ are real and vary smoothly with $k$ for a fixed $l$.

In Fig. D.1 and D.2 we plot some examples of the maximally localized Wannier functions we have obtained numerically. For comparison we also plot the harmonic oscillator states corresponding to the local curvature of the lattice potential at a lattice site. As the lattice depth increases the harmonic oscillator states become better approximations to the Wannier functions. The oscillatory behaviour of the latter ensures that Wannier functions on different sites are orthogonal.
Appendix E

Alternative derivation of Eq. (6.63)

In this appendix we give a different derivation of the result in Eq. (6.63). The Hamiltonian of the phase-modulated system is

$$\hat{H}_{PM}(t) = \sum_{i=1}^{N} \left[ \frac{\hat{p}_{i}^2}{2m} + V_{op}(\hat{x}_{i}, \hat{y}_{i}, \hat{z}_{i} - z_{0}(t)) \right], \quad (E.1)$$

where we assume that the lattice is displaced in the z-direction. We observe that the Hamiltonian can be expressed as

$$\hat{H}_{PM}(t) = \hat{U}^\dagger(t) \hat{H}_{op} \hat{U}(t), \quad (E.2)$$

where $\hat{H}_{op}$ is given in Eq. (6.1) and $\hat{U}(t)$ is the translational operator

$$\hat{U}(t) = \exp \left\{ -iz_{0}(t) \hat{P}_{z}/\hbar \right\} = \exp \left\{ -imz_{0}(t) \hat{J}_{z}/\hbar \right\}. \quad (E.3)$$
The dynamic state of the system evolves according to the Schrödinger equation

\[ i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} = \hat{H}_{PM}(t)|\Psi(t)\rangle \]
\[ = \hat{U}^\dagger(t)\hat{H}_{op}\hat{U}(t)|\Psi(t)\rangle. \quad (E.4) \]

Defining the state

\[ |\tilde{\Psi}(t)\rangle = \hat{U}(t)|\Psi(t)\rangle, \quad (E.5) \]

we find that \( |\tilde{\Psi}(t)\rangle \) satisfies the equation

\[ i\hbar \frac{\partial |\tilde{\Psi}(t)\rangle}{\partial t} = \tilde{H}(t)|\tilde{\Psi}(t)\rangle, \quad (E.6) \]

where

\[ \tilde{H}(t) = \hat{H}_{op} - m\dot{z}_0(t)\hat{J}_z. \quad (E.7) \]

We see that the Hamiltonian governing the evolution of the state \( |\tilde{\Psi}(t)\rangle \) contains a perturbation proportional to the total current operator. The state \( |\tilde{\Psi}(t)\rangle \) describes the system as seen in the non-inertial frame of reference in which the moving lattice is stationary.

The energy of the phase modulated system is given by

\[ E(t) = \langle \Psi(t)|\hat{H}_{PM}(t)|\Psi(t)\rangle \]
\[ = \langle \tilde{\Psi}(t)|\hat{H}_{op}|\tilde{\Psi}(t)\rangle. \quad (E.8) \]
Using Eq. (E.6) and Eq. (E.8), we find that the energy absorption rate is given by

\[
\frac{dE}{dt} = \frac{1}{i\hbar} \langle \tilde{\Psi}(t) | [\hat{H}_{\text{op}}, \hat{\tilde{H}}(t)] | \tilde{\Psi}(t) \rangle \\
= - \frac{m \dot{z}_0(t)}{i\hbar} \langle \tilde{\Psi}(t) | [\hat{H}_{\text{op}}, \hat{\tilde{J}}_z] | \tilde{\Psi}(t) \rangle
\]  
(E.9)

In the interaction picture, we have

\[
\frac{dE}{dt} = - \frac{m \dot{z}_0(t)}{i\hbar} \langle \tilde{\Psi}_1(t) | [\hat{H}_{\text{op}}, \hat{\tilde{J}}_{z,1}(t)] | \tilde{\Psi}_1(t) \rangle,
\]  
(E.10)

where \( |\tilde{\Psi}_1(t)\rangle \equiv \exp(i\hat{H}_{\text{op}}t/\hbar)|\tilde{\Psi}(t)\rangle \) and \( \hat{\tilde{J}}_{z,1}(t) \equiv \exp(i\hat{H}_{\text{op}}t/\hbar) \hat{\tilde{J}}_z \exp(-i\hat{H}_{\text{op}}t/\hbar) \).

The state \( |\tilde{\Psi}_1(t)\rangle \) evolves according to

\[
i\hbar \frac{\partial}{\partial t} |\tilde{\Psi}_1(t)\rangle = -m \dot{z}_0(t) \hat{\tilde{J}}_{z,1}(t) |\tilde{\Psi}_1(t)\rangle.
\]  
(E.11)

First order perturbation theory gives

\[
|\tilde{\Psi}_1(t)\rangle \simeq |\Psi_0\rangle - \frac{m}{i\hbar} \int_{-\infty}^{t} dt' \dot{z}_0(t') \hat{\tilde{J}}_{z,1}(t') |\Psi_0\rangle.
\]  
(E.12)

Substituting Eq. (E.12) into Eq. (E.10) we have

\[
\frac{dE}{dt} = - \frac{m}{i\hbar} \dot{z}_0(t) \langle \Psi_0 | [\hat{H}_{\text{op}}, \hat{\tilde{J}}_{z,1}(t)] | \Psi_0 \rangle \\
+ \left( \frac{m}{i\hbar} \right)^2 \dot{z}_0(t) \int_{-\infty}^{t} dt' \dot{z}_0(t') \langle \Psi_0 | [\hat{H}_{\text{op}}, \hat{\tilde{J}}_{z,1}(t'), \hat{\tilde{J}}_{z,1}(t')] | \Psi_0 \rangle.
\]  
(E.13)

The first term on the right hand side of this equation vanishes since \( |\Psi_0\rangle \) is the ground state of \( \hat{H}_{\text{op}} \). Using

\[
i\hbar \frac{\partial}{\partial t} \hat{\tilde{J}}_{z,1}(t) = [\hat{\tilde{J}}_{z,1}(t), \hat{H}_{\text{op}}],
\]  
(E.14)
Eq. (E.13) can be written as

\[
\frac{dE}{dt} = -\frac{m^2}{i\hbar} \dot{z}_0(t) \int_{-\infty}^{t} dt' \dot{z}_0(t') \frac{\partial}{\partial t} \langle \Psi_0 | [\dot{J}_{z,1}(t), \dot{J}_{z,1}(t')] | \Psi_0 \rangle. \tag{E.15}
\]

Since

\[
\int_{-\infty}^{t} dt' \dot{z}_0(t') \frac{\partial}{\partial t} \langle \Psi_0 | [\dot{J}_{z,1}(t), \dot{J}_{z,1}(t')] | \Psi_0 \rangle = \frac{\partial}{\partial t} \int_{-\infty}^{t} dt' \dot{z}_0(t') \langle \Psi_0 | [\dot{J}_{z,1}(t), \dot{J}_{z,1}(t')] | \Psi_0 \rangle - \dot{z}_0(t) \langle \Psi_0 | [\dot{J}_{z,1}(t), \dot{J}_{z,1}(t)] | \Psi_0 \rangle
\]

\[
= \frac{\partial}{\partial t} \int_{-\infty}^{t} dt' \dot{z}_0(t') \langle \Psi_0 | [\dot{J}_{z,1}(t), \dot{J}_{z,1}(t')] | \Psi_0 \rangle,
\tag{E.16}
\]

we have

\[
\frac{dE}{dt} = m^2 \dot{z}_0(t) \frac{\partial}{\partial t} \int_{-\infty}^{t} dt' \dot{z}_0(t') \Pi_{zz}(t-t')
\]

\[
= m^2 z_0^2 \omega^3 \cos \omega t \left[ -\sin \omega t \Re \Pi_{zz}(\omega) + \cos \omega t \Im \Pi_{zz}(\omega) \right]. \tag{E.17}
\]

Averaging the energy absorption rate over one period, we obtain

\[
\overline{\frac{dE}{dt}} = \frac{1}{T} \int_{0}^{T} \frac{dE}{dt} dt
\]

\[
= \frac{m^2 z_0^2 \omega^3}{2} \Im \Pi_{zz}(\omega). \tag{E.18}
\]

This is the result given in Eq. (6.63).

We now point out that the above is not in fact the derivation given in [66]. Instead, the authors of [66] suggest that the dynamics of the phase modulated system is equivalent to that of the system described by the Hamiltonian \( \tilde{H}(t) \). From this point
of view, the energy is \( \dot{E}(t) = \langle \tilde{\Psi}(t)|\tilde{H}(t)|\tilde{\Psi}(t) \rangle \) which is not the same as the correct expression in Eq. (E.8). Following their line of reasoning, the energy absorption rate is given by

\[
\frac{d\dot{E}}{dt} = \langle \tilde{\Psi}(t)|\frac{\partial \tilde{H}(t)}{\partial t}|\tilde{\Psi}(t) \rangle,
\]

which is the starting formula in [66]. With \( \tilde{H}(t) \) given by Eq. (E.7), one has

\[
\frac{d\dot{E}}{dt} = -m \ddot{z}_0(t) \langle \tilde{\Psi}(t)|\tilde{J}_z|\tilde{\Psi}(t) \rangle
= -m \ddot{z}_0(t) \langle \tilde{\Psi}_1(t)|\tilde{J}_{z,1}|\tilde{\Psi}_1(t) \rangle.
\]

Substituting Eq. (E.12) into this result, one finds

\[
\frac{d\dot{E}}{dt} = m^2 \ddot{z}_0(t) \int_{-\infty}^{\infty} dt' \dot{z}_0(t') \Pi_{zz}(t-t')
= -m^2 \ddot{z}_0(t) \omega^3 \sin \omega t \left[ \cos \omega t \text{Re} \Pi_{zz}(\omega) + \sin \omega t \text{Im} \Pi_{zz}(\omega) \right].
\]

It is clear from Eq. (E.17) and Eq. (E.21) that \( \frac{d\dot{E}}{dt} \) is different from the correct result \( \frac{dE}{dt} \). However, when averaged over one period, the energy absorption rate of this system is

\[
\frac{\langle d\dot{E} \rangle}{dt} = \frac{m^2 \ddot{z}_0(t) \omega^3}{2} \text{Im} \Pi_{zz}(\omega),
\]

which is the same as \( \frac{dE}{dt} \). In our opinion, the authors of [66] have obtained the correct formula for the averaged energy absorption rate by a fortunate coincidence! In fact, the difference between Eq. (E.17) and Eq. (E.21) can be made more conspicuous if we assume a different \( z_0(t) \), namely \( z_0(t) = \theta(t) v_0 t \). This corresponds to the lattice
moving with a constant velocity \( v_0 \) for \( t > 0 \). For this time-dependent displacement, Eq. (E.19) gives

\[
\frac{d\tilde{E}}{dt} = \langle \tilde{\Psi}(t)|(-m\ddot{z}_0(t)\hat{J}_z)|\tilde{\Psi}(t)\rangle \\
= -mv_0\delta(t)\langle \tilde{\Psi}(0)|\hat{J}_z|\tilde{\Psi}(0)\rangle \\
= 0
\]

(E.23)

since the initial state is one in which there is no current. This conclusion also follows from the first line of Eq. (E.21). On the other hand, Eq. (E.17) gives the following finite result

\[
\frac{dE}{dt} = m^2v_0^2 \frac{\partial}{\partial t} \int_0^t dt' \Pi_{zz}(t-t').
\]

(E.24)

This shows that Eq. (E.21) cannot be the correct result for the energy absorption rate in general.