Curves of low genus on surfaces and applications to Diophantine problems

by

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Abstract

We describe in detail a technique due to Vojta for finding the explicit set of curves of low genus on certain algebraic surfaces of general type, and refine some of its aspects. We then provide applications of this method to three Diophantine problems.

We prove under the Bombieri-Lang Conjecture that there are finitely many non-trivial sequences of integers of length 11 whose squares have constant second differences, and we prove unconditionally the analogous result for function fields of characteristic zero.

We prove under the Bombieri-Lang Conjecture that there are finitely many integer sequences of length 8 whose k-th powers have second differences equal to 2, we give an unconditional result for function fields of characteristic zero. Moreover, this gives new examples of surfaces having no curves of genus 0 or 1.

The third application is related to the surface parametrizing perfect cuboids. We give some new properties about their curves of genus 0 or 1 and we give new bounds for the degree of curves in this surface, in terms of their genus.

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Statement of Originality

I hereby certify that all of the work described within this thesis is the original work of the author. Any published (or unpublished) ideas and/or techniques from the work of others are fully acknowledged in accordance with the standard referencing practices.

Contents

Chapter 1

Introduction

In this thesis, we consider the problem of finding the complete list of curves of low genus on certain algebraic surfaces. More specifically, we focus on three problems of arithmetic relevance that naturally lead to the geometric task of finding curves on surfaces, and we make significant progress on these matters. This progress is achieved by building on previous work by Vojta, of which we give a detailed exposition and we refine on some aspects.

1.1 Motivation

A *Diophantine equation* is a polynomial equation (in several variables) with integer coefficients. For instance, $2x^2 + yz = 1$ is a Diophantine equation in three variables. Given a system of Diophantine equations, one is interested in its integer (or rational) solutions. Most of the time it is too difficult to explicitly describe all solutions, so one tries to study qualitative aspects of the integer or rational solutions of the system. For instance, one can ask: Is there some solution? Are there finitely many solutions? If there are infinitely many solutions, do they come in parametric families?

Closely related to the previous questions is the following problem posed by Hilbert:

Problem 1.1 (Hilbert's tenth problem)**.** *Does there exist an algorithm which decides, in a finite number of steps, whether a Diophantine equation does or does not have integer solutions?*

This is the tenth of the list of 23 problems proposed by Hilbert in the year 1900 at the International Congress of Mathematicians. In 1970, Matiyasevich [Mat70] (after the work of Davis, Putnam and Robinson) proved that such an algorithm does not exist. In particular, we cannot hope to understand *all* Diophantine equations at the same time, and it is necessary to focus on particular families in order to make some progress.

Although some Diophantine equations can be addressed by elementary methods, most Diophantine equations need deeper theory to obtain information about its solutions. For example, to show that the Fermat equation $x^n + y^n = z^n$ does not have non-trivial solutions for exponents greater than two, took more than 350 years and its first proof is based on a deep understanding of the relation between elliptic curves, Galois representations, and modular forms.

A system of diophantine equations $D_1 = 0, \ldots, D_n = 0$ is equivalent (over the integers) to the single diophantine equation $D_1^2 + \ldots + D_n^2 = 0$. Thus, a system of diophantine equations is equivalent to a single diophantine equation of a very particular type.

A system of Diophantine equations gives rise to an algebraic variety, which classically is defined to be the set of solutions of this system over the complex numbers. In this thesis, we will only treat the case where this system defines a *surface* (that is, when the variety has dimension 2 over the complex numbers). Rational points on a surface naturally give a rational solution of the system of equations. Hence, we are

interested in finding rational points on surfaces.

The general philosophy is that the geometry of the algebraic variety attached to a system of Diophantine equations should tell us valuable information about the rational solutions of the original system of Diophantine equations. For the case of curves, Faltings [Fal83] proved that a curve of genus greater than or equal to 2 has finitely many rational points. In the case of surfaces, a conjecture due to Bombieri and Lang (cf. Subsection 2.1.1) implies that on a surface of general type, all but finitely many rational points lie on the curves of genus 0 or 1 of the surface. This conjecture is a generalization of Faltings' theorem.

Moreover, it is conjectured that surfaces of general type can only have finitely many curves of genus 0 or 1. Hence the Bombieri-Lang conjecture shows that if we find all the curves of genus 0 or 1 of this surface (conjecturally, they are finitely many), then we will know that all but finitely many rational points are in those *finitely many* curves.

Regarding the geometric conjecture of the finiteness of curves of genus 0 or 1 on general type surfaces, there is important progress due to Bogomolov. He proved that if a general type surface satisfies a certain inequality regarding two numerical invariants of the surface (cf. Theorem 2.6), then there are only finitely many curves of genus 0 or 1 on that surface.

In this thesis, we are concerned about the problem of explicitly finding curves on surfaces which goes beyond the finiteness statement proved by Bogomolov. We study a method given by Vojta [Voj00] (inspired by work of Bogomolov), and we will provide concrete applications of this method for some relevant Diophantine problems.

The problem studied by Vojta is the following one, first raised by Büchi in 1970.

Büchi proved that a positive answer to it implies a strong improvement of Matiyasevich's negative solution to Hilbert's tenth problem.

Problem 1.2 (n-squares problem)**.** *Does there exist a positive integer M such that every sequence of M or more integer squares, whose second differences are constant and equal to 2, is necessarily a sequence of squares of the form $(x+i)^2$, with $i = 1, 2, \ldots$ for some integer x?*

By an integer sequence whose squares have *second differences equal to* 2, we mean a sequence $a_1, \ldots, a_n$ of integers such that for $3 \leq i \leq n$ we have

$$(a_i^2 - a_{i-1}^2) - (a_{i-1}^2 - a_{i-2}^2) = a_{i-2}^2 - 2a_{i-1}^2 + a_i^2 = 2.$$

If a sequence satisfies $|a_i| = b + i$ for b an integer, then it trivially satisfies that its squares have second differences equal to 2. These sequences are called *trivial*.

Although Matiyasevich's solution of Hilbert's Tenth Problem shows that there is no algorithm for the class of all diophantine equations, it is not clear if an algorithm exists for the smaller class corresponding to diagonal quadratic systems, and Büchi conjectured that there is no such algorithm; that is, that diagonal quadratic systems are undecidable. Moreover, Büchi showed that a positive solution to the n-Squares Problems implies this undecidability (cf. [Lip90], [Maz94]).

Remark 1.3. In general, it might be useful to make the following informal observation: if a class of problems C_1 contains a class of problems C_2, an algorithm to solve all problems in C_1 also solves all problems in C_2. Hence, if C_2 is undecidable, so is C_1. In this sense, undecidability for a smaller class of problems is stronger than undecidability for a larger class.

The n-squares problem is open, but Vojta [Voj00] solved it under the Bombieri-Lang conjecture. He did this by finding the complete set of curves of genus 0 or 1 on the projective surfaces $B_n \subseteq \mathbb{P}^n$ (for $n \geq 8$) defined by the following system of Diophantine equations

$$B_n : \begin{cases} 2x_0^2 = x_1^2 - 2x_2^2 + x_3^2 \\ \quad \vdots \\ 2x_0^2 = x_{n-2}^2 - 2x_{n-1}^2 + x_n^2 \end{cases}$$

1.2 Contribution of the thesis

The main theme in this thesis consists of concrete applications of Vojta's method. After presenting Vojta's method discussed above (filling in details and clarifying some arguments), we give some technical improvements, which allow us to achieve better results for surfaces defined by equations of higher degree. We apply Vojta's technique to three Diophantine problems, which are explained below and for which we give a more detailed historical background in Section 2.1.

The first Diophantine problem (cf. Chapter 4) is a generalization of the n-squares problem which has received significant attention in the literature (see [BB10], [GX11]). Here we deal with sequences of integers whose squares have constant second differences, that is, sequences $a_1, \ldots, a_n$ such that for each $i \geq 3$ one has

$$a_{i-2}^2 - 2a_{i-1}^2 + a_i^2 = D,$$

for D an integer independent of i. Sequences satisfying $(a_i)^2 = (b + ci)^2$ for some integers b, c are called trivial sequences, because their squares always have constant

second differences. The question is

Problem 1.4. *Does there exist a positive integer M such that every sequence of M or more coprime integers, whose squares have constant second differences, is necessarily a trivial sequence?*

This problem has been completely understood in the symmetric case (when the sequence satisfies $a_i = a_{n-i+1}$ for $i = 1, \ldots, n$), but it remains open in the general case. There is extensive numerical evidence supporting the existence of the bound M (see for example [Bre03]). Browkin and Brzezinski [BB10], and independently Gonzalez-Jimenez and Xarles [GX11], conjectured around 2010 that the ideas from Vojta's work should give information for long enough sequences under Bombieri-Lang.

Consider the following surfaces $X_n \subseteq \mathbb{P}^n$ constructed by Browkin and Brzezinski [BB10]:

$$X_n : \begin{cases} x_1^2 - 3x_2^2 + 3x_3^2 = x_4^2 \\ \vdots \\ x_{n-3}^2 - 3x_{n-2}^2 + 3x_{n-1}^2 = x_n^2 \end{cases} \tag{1.1}$$

In Chapter 4, we use Vojta's method to find all curves of genus 0 or 1 on these surfaces for $n \geq 11$. We get the following

Theorem 1.5. *For $n \geq 11$, the only curves of genus 0 or 1 on X_n are the trivial lines $[s + t : \pm(2s + t) : \cdots : \pm(ns + t)]$ with $[s : t] \in \mathbb{P}^1$.*

By $[x_0 : \ldots : x_n]$ we mean the homogeneous coordinates of a point in $\mathbb{P}^n$.

This implies the following conditional result.

Corollary 1.6. *If the Bombieri-Lang conjecture for the surfaces X_n with $n \geq 11$ holds, then there are finitely many non-trivial coprime sequences of 11 integers whose squares have constant second differences. Moreover, there exists an $M > 0$ such that if $x_1, \ldots, x_M$ is a coprime sequence of integers whose squares have constant second differences, then the sequence is trivial.*

Thus, we have solved Problem 1.4 under the Bombieri-Lang conjecture for these surfaces. Moreover, we find all curves of genus up to a certain bound (not only those of genus 0 or 1), so we solve unconditionally the analogue of Problem 1.4 for function fields of genus g over fields of characteristic zero (see Theorem 4.1 and Corollary 4.3).

The second application of the method (cf. Chapter 5) generalizes the n-squares problem in a different direction. Here we consider higher powers instead of squares, and we want to study sequences of integers whose k-th powers have second differences equal to 2. The question is the following.

Problem 1.7. *Let $k > 2$. Does there exist a positive integer M such that there are no sequences of k-th powers having second differences constant and equal to 2?*

The following surfaces $X_{n,k} \subseteq \mathbb{P}^n$ are related to this problem

$$X_{n,k} : \begin{cases} 2x_0^k = x_1^k - 2x_2^k + x_3^k \\ 2x_0^k = x_2^k - 2x_3^k + x_4^k \\ \quad \vdots \\ 2x_0^k = x_{n-2}^k - 2x_{n-1}^k + x_n^k. \end{cases} \tag{1.2}$$

In Chapter 5, we use Vojta's method to find all curves of genus 0 or 1 on these surfaces (for n large enough depending on k, but greater than or equal to 4 for $k \geq 6$). In

fact, we prove the following.

Theorem 1.8. *There are no curves of geus* 0 *or* 1 *on the surface* $X_{n,k}$ *when either*

(a) $k > 2$ *and* $n \geq 8$*, or*

(b) $k \geq 6$ *and* $n \geq 4$*.*

Some of these surfaces are new examples of surfaces of general type with finitely many curves of genus 0 or 1 (in fact, with no curves of this type) and not covered by Bogomolov's general theorem.

This implies the following conditional result.

Corollary 1.9. *Assume the Bombieri-Lang conjecture for the surfaces* $X_{n,k}$*.*

- *For* $k > 2$*, there are finitely many sequences of* 8 *integers whose* k*-th powers have second differences equal to* 2*.*
- *For* $k \geq 6$*, there are finitely many sequences of* 4 *integers whose* k*-th powers have second differences equal to* 2*.*

Moreover, for any $k > 2$*, there is a constant* $M_k > 0$ *such that there are no sequences of* M_k *integers whose* k*-th powers have constant second differences equal to* 2*.*

Moreover, we find all curves of low genus (not only for genus 0 or 1) on $X_{n,k}$, so we solve (unconditionally) the analogue of Problem 1.7 for function fields of genus g over fields of characteristic zero (see Theorem 5.1 and Corollary 5.3).

If in addition we assume the n-term ABC conjecture (formulated by Browkin and Brzezinski [BB94]) over the integers, we obtain a result for sequences of powers, possibly of different exponents, having second differences equal to 2.

Corollary 1.10. *Assume the Bombieri-Lang conjecture for surfaces and the* 4*-term ABC conjecture. There exists an* $M > 0$ *such that there are finitely many non-trivial sequences of length* M *consisting of integer powers (of exponents greater or equal to* 2*) which have constant differences equal to* 2*.*

By *non-trivial sequences* we mean sequences which are not of the form $(x + yi)^2$ for $i = 1, \ldots, n$.

This generalization of the n-squares problem has received some attention in the literature (see [Pas13]), and it was only known under the assumption of a strong analogue of the ABC conjecture for algebraic numbers.

Finally, the third application of the method is to investigate the classical perfect cuboid problem (cf. Chapter 6). A *perfect cuboid* is a rectangular box having rational sides, face diagonals and main diagonal. The search for perfect cuboids with sides x_0, x_1, x_2 leads to finding positive rational solutions of the following system of Diophantine equations

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 = x_3^2 \\ \qquad x_1^2 + x_2^2 = x_4^2 \\ \qquad x_0^2 + x_2^2 = x_5^2 \\ \qquad x_0^2 + x_1^2 = x_6^2. \end{cases} \tag{1.3}$$

Several classical questions remain open about perfect cuboids, for instance: are there any perfect cuboids? Is there a parametric family of perfect cuboids?

The existence of cuboids whose sides and face diagonals are rational (that is, solutions of the last three equations of (1.3)) has been studied since the 18th century, but no example of a perfect cuboid has been found. Euler (1770) and Sanderson (1740)

independently gave a systematic method for finding cuboids whose sides and face diagonals are rational. Several authors have investigated the problem of finding perfect cuboids, either by trying to find solutions [Kor92] or, more recently, by studying the geometry behind it ([vLu00], [ST10], [FS13]).

The system of equations (1.3) defines a surface of general type with 48 singular points. We denote this surface by Y_4. While we are not able to find all curves of genus 0 or 1 on Y_4 by using the techniques of this thesis, nevertheless Vojta's method gives new information about these curves. We show the following

Proposition 1.11. *Every curve of genus* 0 *or* 1 *on the surface* Y_4 *contains at least two singular points of* Y_4.

Moreover, in the case of genus 0 curves we can give more detailed information. We also give bounds on the degree of curves (in $\mathbb{P}^6$) depending on their genus and how they pass through the singular points of Y_4, namely

Theorem 1.12. *Let* C *be a curve in* Y_4 *which does not contain any singularity of* Y_4. *Then*

$$\deg C \leq 4g(C) - 4.$$

Corollary 1.13. *Let* C *be a curve in* Y_4, *and suppose that* C *is smooth at each of the singularities of this surface that belong to* C, *if any. Then*

$$\deg C \leq 4g(C) + 44.$$

This is closely related to results given by Freitag and Salvati Manni [FS13].

1.3 Organization of the thesis

This thesis is structured as follows.

In the second chapter (Background) we explain the Bombieri-Lang Conjecture and its relation with curves of genus 0 or 1. We discuss the historical background for the number-theoretical problems treated in this thesis, and we include several sections containing preliminary material that will be used repeatedly in the next chapters.

In the third chapter we begin by sketching Vojta's method. Then we discuss in detail functorial properties of ω-integral curves, a concept studied by Vojta, which is key to the theory. This allows us to give a more detailed treatment of Vojta's work. After this, we will discuss in detail some theoretical tools that we introduce in order to make systematic verifications in applications. Finally, we improve a result of Vojta, which will allow us to deal with higher degree equations in a more efficient way, taking advantage of higher order ramification (cf. Theorem 3.87).

The remaining chapters consider the applications discussed above. More precisely: in the fourth chapter we apply the method of Chapter 3 to the problem of sequences of squares with constant second differences. In Chapter 5 we consider the problem of higher powers with constant second difference equal to 2. Finally, in Chapter 6 we focus on the perfect cuboid problem.

Chapter 2

Background

2.1 Diophantine equations and rational points on surfaces

2.1.1 The Bombieri-Lang conjecture

In this section, we formulate a Diophantine conjecture due (in this generality) to Lang. Let us begin by briefly defining some geometric notions. For the basic geometric definitions we refer to [Har77].

Definition 2.1. Let k be an algebraically closed field of characteristic zero. Given a smooth projective variety X over k, the *Kodaira dimension* $\kappa(X)$ is the transcendence degree over k of the quotient field of the ring $\oplus_{n\geq 0} H^0(X, \mathcal{L}(nK_X))$ minus 1, where K_X is a canonical divisor of X and $\mathcal{L}(nK_X)$ is the sheaf associated to the divisor nK_X.

It is known that $\kappa(X)$ can take on every value from -1 to $\dim(X)$. For more information about this definition in the case of surfaces, we refer to [BPV84], p. 22.

Definition 2.2. Let X be a smooth variety. If $\kappa(X) = \dim(X)$, then X is called a *variety of general type.*

An example of a variety of general type is any smooth projective curve of genus greater or equal to 2. Here the genus $g(X)$ of a smooth curve X is defined as $\dim_k \Gamma(X, \omega_X)$.

Let us recall some Diophantine statements, with $X(K)$ being the set of K-rational points of X:

Theorem 2.3 (Faltings)**.** *If X is a curve of genus $g \geq 2$ (i.e. a curve of general type) defined over a number field K_0, then $X(K)$ is finite for every number field K containing K_0.*

The following conjecture is a generalization to higher dimension of the previous theorem.

Conjecture 2.4 (Bombieri-Lang)**.** *If X is a smooth projective algebraic variety of general type defined over a number field K_0, then there exists a proper Zariski-closed subset Z of X such that for all number fields K containing K_0, the set $X(K) \setminus Z(K)$ is finite.*

We will only use this conjecture for surfaces of general type, that is, the particular case when $\dim(X) = 2$. The following remark is the reason why we are interested in finding curves of genus 0 or 1 on surfaces of general type.

Remark 2.5. Let X be a surface of general type defined over a number field K_0. Then by the Bombieri-Lang Conjecture, we expect that all but finitely many K_0-rational points of X lie in a Zariski-closed subset Z of X, that is, in a finite union of curves.

Using Faltings' Theorem, we can assume that Z is a finite union of curves of genus 0 or 1, since the curves of genus greater than or equal to 2 on Z only contain finitely many K_0-rational points.

From this we conclude that (under the Bombieri-Lang Conjecture) there are only finitely many K_0-rational points outside the genus 0 and 1 curves of X.

Let $c_1^2(X)$, $c_2(X)$ denote the Chern numbers of a surface X (where $c_1^2 = K_X^2$ and c_2 is defined as in [Har77], p. 433). Bogomolov [Bog10], proved a theorem which implies (cf. [Des79]) the following finiteness result for curves of genus 0 or 1 on certain surfaces of general type:

Theorem 2.6. *Let X be a surface of general type such that $c_1^2(X) > c_2(X)$. Then there are only finitely many curves of genus 0 or 1 on X.*

2.1.2 The n-Squares Problem

The following is a stronger version of Hilbert's Tenth Problem.

Problem 2.7 (Büchi's Problem)**.** *Does there exist an algorithm which decides, in a finite number of steps, whether a system of diagonal quadratic Diophantine equations does or does not have integer solutions?*

By a *diagonal quadratic Diophantine equation*, we mean an equation of the form $\sum_{i=1}^{n} a_i x_i^2 = a_{n+1}$, with $a_i \in \mathbb{Z}$ and variables $x_1, \ldots, x_n$. Recall from Chapter 1 the n-Squares Problem, which implies Büchi's Problem (cf. [Lip90], [Maz94]):

Problem 2.8 (n-Squares Problem)**.** *Does there exist a positive integer M such that every sequence of M or more integers, whose squares have second differences equal to 2, is necessarily a sequence whose squares are of the form $(x+i)^2$, $i = 1, 2, \ldots$ for some integer x?*

We say that a sequence $a_1, a_2, \ldots, a_n$ has *second differences equal to* 2 if for all $3 \leq i \leq n$ we have $a_{i-2} - 2a_{i-1} + a_i = 2$, and that a sequence having as squares

$(x+i)^2$, $i = 1, 2 \ldots$ is called *trivial*, because

$$(x+i-2)^2 - 2(x+i-1)^2 + (x+i)^2 = 2.$$

For each n, the rational points on the smooth complete intersection surface

$$B_n : \begin{cases} 2x_0^2 = x_1^2 - 2x_2^2 + x_3^2 \\ \quad \vdots \\ 2x_0^2 = x_{n-2}^2 - 2x_{n-1}^2 + x_n^2 \end{cases}$$

(in $\mathbb{P}^n$) defined in [Voj00] give sequences of length n of rational numbers whose squares have second differences equal to 2.

An example of a non-trivial sequence of length 4 is $6, 23, 32, 39$ (their squares have first differences $493, 495, 497$), but there is no known example of length 5. However, for rational numbers Lipman [Maz94] found examples of length 5, using the fact that the surface B_5 has many curves of genus 1. It is believed that for $M \geq 6$ there are few sequences of length M, because the surfaces B_n with $n \geq 6$ are of general type; see [Maz94].

In 2000, Vojta [Voj00], inspired by work of Bogomolov, proved the following result.

Theorem 2.9 (Vojta)**.** *For $n \geq 8$, the only curves of genus 0 or 1 on B_n are the 2^n lines*

$$\pm x_1 = \pm x_2 - x_0 = \ldots = \pm x_n - (n-1)x_0.$$

Under the Bombieri-Lang conjecture for the surface B_n, we thus obtain that all but finitely many rational points on B_n lie on these curves. Moreover, Vojta proved under Bombieri-Lang that for n large enough, all rational points will be in one of

these curves.

If a rational point $[x_0 : \ldots : x_n]$ is on one of these curves, then by dehomogenizing we obtain that $\pm x_i = \pm x_1 + (i-1)$, and taking squares we see that

$$x_i^2 = (\pm x_1 + (i-1))^2,$$

so $x_1, \ldots, x_n$ form a trivial sequence. Hence, the unconditional geometric result obtained by Vojta shows that the Bombieri-Lang Conjecture implies that the assertion of Problem 2.8 is true. This is the first conditional result for the n-Squares Problem. (A second conditional result is in [Pas13].) Vojta's result also gives unconditionally a solution for the analogue of the n-Squares Problem for function fields over a field of characteristic zero.

Shlapentokh and Vidaux [SV11] prove an analogue of the n-Squares Problem for function fields in any characteristic, following the method of Pheidas and Vidaux from [PV06], [PV10], where they treat the case of function fields of genus 0 over fields of any characteristic. (The case of function fields of positive characteristic is interesting as it involves some exceptional solutions that do not occur in characteristic zero. However, we will not consider this case in this thesis.)

2.1.3 Squares with constant second differences

One generalization of the n-Squares Problem is to consider sequences of squares having constant differences not necessarily equal to 2.

Definition 2.10. A sequence $a_1, \ldots, a_n$ of rational numbers has *constant second differences* if for all $3 \leq i \leq n$

$$a_{i-2} - 2a_{i-1} + a_i = D, \tag{2.1}$$

with D a constant.

Taking third differences, the previous condition is equivalent to the following condition which does not depend on D

$$a_{i-3} \;=\; 3a_{i-2} - 3a_{i-1} + a_i,$$

for $4 \leq i \leq n$.

Problem 2.11. *Does there exist a positive integer M such that every sequence of M or more integers, whose squares have constant second differences, is necessarily proportional to a sequence whose squares are of the form $(x + yi)^2$, $i = 1, 2, \ldots, M$ for some integers x, y?*

The sequences $(x+i)^2$ with $i = 1, 2 \ldots, n$ are the *trivial* sequences of the n-Squares Problem.

Proposition 2.12. *A sequence of integers $x_1, \ldots, x_n$ having constant second differences gives rise to a quadratic polynomial $f \in \mathbb{Z}[t]$ which satisfies the condition that $f(0), \ldots, f(n-1)$ are integer squares. Conversely, a quadratic polynomial f, with the property that $f(0), \ldots, f(n)$ are integer squares, gives a sequence $x_1, \ldots, x_n$ having constant second differences. A quadratic polynomial which is a square corresponds to a trivial sequence.*

Proof. If we have a quadratic polynomial $f(t) = at^2 + bt + c \in \mathbb{Z}[t]$ with the property that $f(0), f(1), \ldots, f(n-1)$ are integer squares, then by the appendix in [Brz12], the first differences $f(i) - f(i-1)$ are $2ai + a + b$, hence the sequence $f(0), \ldots, f(n-1)$ has constant second differences equal to $2a$.

Now, suppose that we have a sequence $a_1, \ldots, a_n$ of integers whose squares have constant second differences equal to D, and consider the quadratic polynomial

$$f(t) = 2D(t^2 - t) + 4(a_2^2 - a_1^2)t + 4a_1^2 \in \mathbb{Z}[t].$$

Using Equation (2.1), we have that $f(i) = a_{i+1}^2$, hence this polynomial satisfies that $f(0), \ldots, f(n-1)$ are all squares.

If the quadratic polynomial f is of the form $(a + bt)^2$, then it is clear that it corresponds to a trivial sequence $(a + bi)^2$. □

Hence, the problem of finding sequences of length M of integers whose squares have constant second differences is equivalent to the problem of finding a quadratic polynomial with integer coefficients and such that it is a square when evaluated at $0, \ldots, M-1$. In the n-Squares Problem we have to deal with *monic* polynomials.

The problem of integer squares with constant second differences has been studied by several people. In 1986, Allison [All86] found infinitely many non-trivial examples of sequences of length 8; these are polynomials of the form $a(x^2 + x) + c$ and they are evaluated at $-3, -2, \ldots, 3, 4$. Subsequent work by Bremner [Bre03] and by Gonzalez-Jimenez and Xarles [GX11] gives a complete answer in the symmetric case, and it turns out that the bound M in this case is 9.

In the general case (when symmetry is not assumed), Browkin and Brzezinski

[BB06] proved that there are infinitely many non-trivial sequences of six integer squares with constant second differences. Moreover, in [BB10] they define the following surfaces in $\mathbb{P}^{n-1}$ which are related to Problem 2.11:

$$X_n : \begin{cases} x_1^2 - 3x_2^2 + 3x_3^2 = x_4^2 \\ \vdots \\ x_{n-3}^2 - 3x_{n-2}^2 + 3x_{n-1}^2 = x_n^2, \end{cases}$$

and they conjecture (inspired by Vojta's work [Voj00]) that the only curves of genus 0 or 1 in X_7 are the curves $[s+t : \pm(2s+t) : \ldots : \pm(ns+t)]$ with $[s:t] \in \mathbb{P}^1$ and an elliptic curve giving the examples found by Allison.

The curves $[s+t : \pm(2s+t) : \ldots : \pm(ns+t)]$ are called trivial because taking squares of the coordinates of a point in such a curve one gets a trivial sequence $(s+t)^2, (2s+t)^2, \ldots, (ns+t)^2$.

Brzezinski [Brz12] conjectures that every sequence of integer squares of length 8 with constant second differences is trivial or symmetric.

Gonzalez-Jimenez and Xarles [GX11] speculate that Vojta's argument in [Voj00] for Büchi's Problem could give an answer under the Bombieri-Lang Conjecture to Problem 2.11, which is what we do in Chapter 4 of this thesis.

We remark that for Büchi's Problem (i.e. for *monic* quadratic polynomials) there is a completely different approach by Pasten [Pas13] using a version of the ABC conjecture, which yields unconditional results over function fields. However, despite the strength of the results in [Pas13], Pasten remarks on p. 2967 that it is not clear that his method can be adapted to the general case (non-monic quadratic polynomials) that we consider here.

2.1.4 An extension of the n-Squares Problem for higher powers

In Chapter 5, we study the problem of sequences of k-th powers with second differences equal to 2, namely

Problem 2.13. *Let $k > 2$. Does there exist a positive integer M such that there are no sequences of k-th powers having second differences equal to 2?*

An example of a sequence of this type is $64, -1, -64, -125$, which is a sequence of cubes having second differences equal to 2, hence for $k = 3$ the bound M has to be at least 5. The analogous case $k = 2$ is the n-Squares Problem and there are (trivial) sequences of any length.

This problem gives rise to the smooth complete intersection surfaces $X_{n,k} \subseteq \mathbb{P}^n$, which are given by the equations

$$X_{n,k} : \begin{cases} 2x_0^k = x_1^k - 2x_2^k + x_3^k \\ \quad \vdots \\ 2x_0^k = x_{n-2}^k - 2x_{n-1}^k + x_n^k \end{cases}$$

We study these surfaces using Vojta's method in order to give an answer under the Bombieri-Lang Conjecture of this problem, and an unconditional solution for function fields of characteristic zero. This problem was conditionally solved under Vojta's general ABC conjecture for algebraic numbers by Pasten [Pas13]. He also gives a solution for the analogue of Problem 2.13 for function fields over fields of characteristic zero using completely different methods (Nevanlinna theory), but his bounds are much weaker than ours due to the methods that he uses. In fact, our bounds are sufficiently sharp so as to give new examples of surfaces having no curves

of genus 0 or 1, which do not satisfy Bogomolov's hypothesis on Chern numbers.

There is another generalization for the n-Squares Problem in the literature, proposed by Pheidas and Vidaux [PV05], considering k-th differences of k-th powers. Although we will not consider this generalization, let us briefly comment on the existing results for it. Pasten [Pas13] also solves this under Vojta's general ABC conjecture for algebraic numbers, and unconditionally for function fields over fields of characteristic zero. Moreover, An, Huang, and Wang [AHW13] give an alternative solution for function fields of characteristic zero following the methods of [PV06], [PV10].

2.1.5 Perfect cuboids

Recall from the introduction that the surface parametrizing perfect cuboids is given by the equations:

$$Y_4 : \begin{cases} x_0^2 + x_1^2 + x_2^2 = x_3^2 \\ x_1^2 + x_2^2 = x_4^2 \\ x_0^2 + x_2^2 = x_5^2 \\ x_0^2 + x_1^2 = x_6^2 \end{cases} \tag{2.2}$$

and that perfect cuboids correspond to positive rational solutions of (2.2).

As of today, no perfect cuboid has been found. Using computer search, Korec [Kor92] showed in 1992 that a perfect cuboid with integer sides x_0, x_1, x_2 must satisfy $\max\{x_0, x_1, x_2\} > 4 \times 10^9$. However, there are infinitely many cuboids satisfying any three of the four equations in (2.2) (see [Guy04] for details). Euler and Sanderson (independently) gave a systematic method for finding cuboids satisfying the last three equations of (2.2). Bremner [Bre88] gives a geometric argument (by finding rational

curves on this surface) to explain why there are lots of solutions for the last three equations of (2.2).

The surface Y_4 was studied by van Luijk in his undergraduate thesis [vLu00]. He studies in detail the geometry of this surface. He shows that Y_4 is a surface of general type having 48 isolated singularities. He also found 92 curves of genus 0 or 1 on this surface, but we do not know if these are *all* the curves of genus 0 or 1. The known curves are the following:

(a) The irreducible components of $x_0x_1x_2x_3 = 0$ (32 rational curves).

(b) The irreducible components of $x_4x_5x_6 = 0$ (12 elliptic curves).

(c) The curve defined by the equations

$$\begin{cases} x_0 = x_1 \\ x_4 = x_5 \\ \sqrt{2}x_0 = x_6 \\ x_2^2 + x_6^2 = x_3^2 \\ 2x_5^2 + x_6^2 = 2x_3^2. \end{cases}$$

Applying $\mathrm{Aut}(Y_4)$ one gets 48 elliptic curves on Y_4.

Stoll and Testa in [ST10] conjecture the following

Conjecture 2.14. *All curves of genus at most* 1 *on* Y_4 *are contained in the previous list.*

They prove that all curves of genus 0 or 1 which have degree less than or equal to 4 are in the list.

In 2013, Beauville [Bea13] showed that the surface Y_4 is a *diagonal quotient surface* (see Section 6.2 for details on his proof), hence we can use the theory of diagonal quotient surfaces from [KS97] in order to obtain more information about Y_4.

The same year, Freitag and Salvati-Manni [FS13] parametrized the surface Y_4 using theta functions. They deduced that Y_4 has an explicit modular parametrization, i.e. is a *modular diagonal quotient surface* in the sense of [KS98]. They also give a bound on the degree of the curves having *mild* singularities:

Proposition 2.15. *Let $C \subset Y_4$ be a curve such that the normalization map $\tilde{C} \to C$ is bijective. Let g be the genus of $\tilde{C}$ and d be the degree of C. Then the inequality*

$$d \leq 176 + 16g$$

holds.

They also observe (from [Bea13]) that a curve of genus 0 or 1 must intersect at least one of the 48 singularities of Y_4.

Finally we would like to remark that the surface parametrizing perfect cuboids is defined by diagonal quadratic equations. Therefore if one could find an algorithm as the one in Büchi's Problem, then one would be able to decide if there exists a perfect cuboid.

2.2 Divisors

We will present some well known facts regarding divisors which will be used in the subsequent chapters of this thesis. The basic references are [Har77], §II.6 and [Mum66].

2.2.1 Cartier divisors

Definition 2.16. Let X be a scheme. For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$, which are not zero divisors in each local ring $\mathcal{O}_{X,x}$, for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}_X$ we call the *sheaf of total quotient rings* of $\mathcal{O}_X$. We denote by $\mathcal{K}_X^*$ the sheaf of invertible elements in $\mathcal{K}_X$, i.e. for $U \subset X$ an open set, $\Gamma(U, \mathcal{K}_X^*)$ are the invertible elements of $\Gamma(U, \mathcal{K}_X)$.

Definition 2.17. A *Cartier divisor* D on X is a section over X of $\mathcal{K}_X^*/\mathcal{O}_X^*$.

Concretely, a Cartier divisor is given by a collection of elements $D_x \in (\mathcal{K}_X^*/\mathcal{O}_X^*)_x$ such that, for every $x \in X$, there is an open neighborhood U of X, and an element $f \in \Gamma(U, \mathcal{K}_X^*)$ which induces D_x for all $x \in U$. The element f is called a *local equation* of D on U. The group of Cartier divisors on X forms a group (because these are sections of a sheaf of abelian groups), which will be denoted by $\mathrm{Cdiv}(X)$. If $D \in \mathrm{CDiv}(X)$ then we can write $D = \{(U_i, g_i)\}$, where $\{U_i\}$ is an open cover of X and each g_i is a local equation for D on U_i.

Definition 2.18. A Cartier divisor D is *effective* ($D \geq 0$) if $D = \{(U_i, f_i)\}$ with $f_i \in \mathcal{O}_X(U_i)$.

Definition 2.19. A *sheaf of ideals* on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{J}$ which is a subsheaf of $\mathcal{O}_X$. Thus, $\mathcal{J}(U)$ is an ideal in $\mathcal{O}_X(U)$ for every open set $U \subseteq X$.

Definition 2.20. Let Y be a closed subscheme of a scheme X, and let $i_Y : Y \to X$ be the inclusion morphism. We define the *ideal sheaf* of Y, denoted $\mathcal{J}_Y$, to be the kernel of the morphism $i_Y^{\#} : \mathcal{O}_X \to i_{Y*}\mathcal{O}_Y$.

Proposition 2.21. *For any closed subscheme Y of X, the corresponding ideal sheaf $\mathcal{J}_Y$ is a quasi-coherent sheaf of ideals on X. Conversely, if $\mathcal{J}$ is a quasicoherent sheaf of ideals on X, then $\mathcal{J}$ is the ideal sheaf of a uniquely determined closed subscheme Y of X. The scheme Y is determined by* $\operatorname{supp}(\mathcal{O}_X/\mathcal{J})$ *with structure sheaf $\mathcal{O}_X/\mathcal{J}$.*

Proof. [Har77] Proposition II.5.9. □

Proposition 2.22. *For any Cartier divisor $D = \{(U_i, g_i)\}$ over X, there exists a unique invertible subsheaf $\mathcal{L}(D)$ (also denoted by $\mathcal{O}_X(D)$) of $\mathcal{K}_X$, defined by the rule*

$$\mathcal{L}(D)_{|U_i} = g_i^{-1}\mathcal{O}_{U_i}$$

Proof. [Har77], proof of Proposition II.6.13. □

Remark 2.23. From [Mum66] p. 63, a Cartier divisor $D = \{(U_i, f_i)\}$ is effective if and only if the sheaf $\mathcal{L}(-D)$ is an ideal sheaf, which will be denoted by $\mathcal{J}_D$. If D is effective, then let Y_D be the associated subscheme of $\mathcal{J}_D = \mathcal{L}(-D)$.

Example 2.24. Let $X = \mathbb{P}^n_k = \operatorname{Proj}(k[x_0, \ldots, x_n])$ with open affine cover $\{U_i\}_{0\leq i\leq n}$, where $U_i = D_+(x_i)$ (cf. [Har77], Proposition II.2.5). If $f \in k[x_0, \cdots, x_n]$ is homogeneous of degree d, then f defines an effective Cartier divisor $\left\{\left(U_i, \frac{f}{x_i^d}\right)\right\}_{0\leq i\leq n}$ on $\mathbb{P}^n_k$. Its associated subscheme is $\mathbb{V}(f)$, the subscheme associated to the homogeneous ideal (f).

If $D \geq 0$ is an effective Cartier divisor on X, then we have the following exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to i_{Y_D *}\mathcal{O}_{Y_D} \to 0. \tag{2.3}$$

Thus, if D is represented by $\{(U_i, f_i)\}$, then $(i_{Y_D *}\mathcal{O}_{Y_D})_{|U_i} \cong \mathcal{O}_{U_i}/f_i\mathcal{O}_{U_i}$. Tensoring with a *locally free* sheaf $\mathcal{F}$ on X is exact, and we get the exact sequence

$$0 \to \mathcal{L}(-D) \otimes \mathcal{F} \xrightarrow{\mu} \mathcal{F} \to i_{Y_D *}\mathcal{O}_{Y_D} \otimes \mathcal{F} \to 0, \tag{2.4}$$

where

$$\mu : \mathcal{L}(-D) \otimes \mathcal{F} \to \mathcal{F}$$

is the composition of $i_{\mathcal{L}(-D)} \otimes \mathrm{Id}_{\mathcal{F}}$ (where $i_{\mathcal{L}(-D)} : \mathcal{L}(-D) \to \mathcal{O}_X$ is the inclusion homomorphism) with the canonical isomorphism $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ (that is, multiplication of sections).

Taking global sections we obtain the exact sequence

$$0 \to H^0(X, \mathcal{L}(-D) \otimes \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F} \otimes i_{Y_D *}\mathcal{O}_{Y_D}). \tag{2.5}$$

From the previous exact sequence, we obtain the following.

Proposition 2.25. *Let X be a scheme, let $\mathcal{F}$ be a locally free sheaf, and let D be an effective Cartier divisor with associated subscheme Y_D. If the section $s \in H^0(X, \mathcal{F})$ maps to zero under the homomorphism $H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F} \otimes i_{Y_D *}\mathcal{O}_{Y_D})$, then there exists a unique section $s' \in H^0(X, \mathcal{L}(-D) \otimes \mathcal{F})$ which maps to s.*

2.2.2 Functorial properties of effective divisors

Definition 2.26. If $D = \{(U_i, f_i)\}$ is a Cartier divisor, then its *support* is the closed set $\mathrm{supp}(D) := \bigcup_i \left\{x \in U_i : f_i \notin \mathcal{O}^*_{X,x}\right\}$.

If $D \geq 0$ is effective, then $\mathrm{supp}(D) = \mathrm{supp}(\mathcal{O}_X/\mathcal{J}_D)$, i.e. $\mathrm{supp}(D)$ is just the

support of the subscheme Y_D.

Proposition 2.27. *Let $f : X \to Y$ be a morphism of integral schemes, and let $D = \{U_i, g_i\} \geq 0$ be an effective Cartier divisor on Y such that $f(X) \not\subseteq \mathrm{supp}(D)$. Then the rule $f^*(\{(U_i, g_i)\}) = \{(f^{-1}U_i, f^*g_i)\}$ defines an effective Cartier divisor on X, and we have that*

$$Y_{f^*D} = f^{-1}(Y_D) := Y_D \times_Y X.$$

Proof. The first assertion is a special case of [Mum66], p. 69, and the second is explained in [EGA] (IV,21.4.7). □

Lemma 2.28. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of integral schemes and let $D \geq 0$ be a Cartier divisor on Z such that $g(Y) \not\subseteq \mathrm{supp}(D)$ and such that $f(X) \not\subseteq \mathrm{supp}(g^*D)$. Then $(g \circ f)^*D$ is defined and $(g \circ f)^*D = f^*g^*D$.*

Proof. See [EGA] (IV,21.4.4). □

Proposition 2.29. *If $f : X \to Y$ is as before, then $\mathcal{L}(f^*D) \cong f^*\mathcal{L}(D)$, for all Cartier divisors $D \geq 0$ where f^*D is defined.*

Proof. [EGA] IV,21.4.2. □

The following proposition will be used several times in Chapters 4, 5 and 6.

Proposition 2.30. *Let $\tilde{\pi}_n : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be the projection map from the point $P = [0 : \ldots : 0 : 1]$, and let $X_n \subseteq \mathbb{P}^n$ be a closed subvariety of $\mathbb{P}^n$ with $P \notin X_n$, so $\tilde{\pi}_n$ induces a morphism $\pi_n : X_n \to X_{n-1} := \tilde{\pi}_n(X_n)$.*

Let $f \in k[x_0, \ldots, x_{n-1}] \subset k[x_0, \ldots, x_n]$ be a homogeneous polynomial and suppose that $X_n \not\subseteq \mathbb{V}_{\mathbb{P}^n}(f)$. Then $\pi_n^*\mathrm{div}_{X_{n-1}}(f) = \mathrm{div}_{X_n}(f)$, and hence its associated subscheme is given by the system of equations defining X_n and $f = 0$. In particular, if X_n is a complete intersection, then so is $Y_{\mathrm{div}_{X_n}(f)}$.

Proof. Let $i_n : X_n \to \mathbb{P}^n$ be the inclusion morphism. Let $U = \mathbb{P}^n \setminus \{[0 : \ldots : 0 : 1]\}$. We have $\mathrm{div}_{\mathbb{P}^n}(f)_{|U}$ is represented by

$$\left\{(U \cap U_i, \frac{f}{x_i^d})\right\}$$

(where $U_i = \{[x_0 : \ldots : x_n] : x_i \neq 0\}$). We also have that $(\tilde{\pi}_{n|U})^* \mathrm{div}_{\mathbb{P}^{n-1}}(f)$ is represented by $\left\{(U \cap U_i, \frac{f}{x_i^d})\right\}$, hence in U they are equal (i.e. $\mathrm{div}_U(f) = (\tilde{\pi}_{n|U})^* \mathrm{div}_{\mathbb{P}^{n-1}}(f)$).

We have the commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{i_n} & U \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{n|U}} \\ X_{n-1} & \xrightarrow[i_{n-1}]{} & \mathbb{P}^{n-1}. \end{array}$$

Thus, by Lemma 2.28 we obtain

$$\begin{aligned} \pi_n^* \mathrm{div}_{X_{n-1}}(f) &= \pi_n^* i_{n-1}^* \mathrm{div}_{\mathbb{P}^{n-1}}(f) \\ &= (i_{n-1} \circ \pi_n)^* \mathrm{div}_{\mathbb{P}^{n-1}}(f) \\ &= (\tilde{\pi}_{n|U} \circ i_n)^* \mathrm{div}_{\mathbb{P}^{n-1}}(f) \\ &= i_n^* \tilde{\pi}_{n|U}^* \mathrm{div}_{\mathbb{P}^{n-1}}(f) \\ &= i_n^* \mathrm{div}_U(f) \\ &= \mathrm{div}_{X_n}(f). \end{aligned}$$

From Example 2.31, we know that $\mathrm{div}_{X_n}(f) = i_{X_n}^* \mathrm{div}_{\mathbb{P}^n}(f)$, hence it is the intersection of X_n with $\mathbb{V}(f)$. Therefore, $\mathrm{supp}(\mathrm{div}_{X_n}(f))$ is given by the system of equations defining X_n and $f = 0$. □

Example 2.31. Let $f \in k[x_0, \ldots, x_n]$ be a non-zero homogeneous polynomial of

degree d, and let $X \subseteq \mathbb{P}^n$ be a closed subvariety with $X \not\subseteq \mathbb{V}(f)$. Then by Example 2.24 and Proposition 2.30 we see that $\mathrm{div}_X(f) := i_X^* \mathrm{div}_{\mathbb{P}^n}(f)$ defines an effective Cartier divisor on X such that its associated subscheme is $Y_{\mathrm{div}_X(P)} = X \cap \mathbb{V}(f)$ the (scheme-theoretic) intersection of X with $\mathbb{V}(f)$.

2.2.3 Weil divisors

Let X be a noetherian integral separated scheme of finite type over a field k which is regular in codimension 1. (This means that every local ring $\mathcal{O}_{X,P}$ of dimension one is regular, i.e. a discrete valuation ring.) An example of a scheme of this type is any normal variety over k (cf. [Har77], p. 130).

Definition 2.32. A *prime divisor on* X is a closed integral subscheme Y of codimension one. A *Weil divisor* is an element of the free abelian group $\mathrm{Div}X$ generated by the prime divisors. We write a Weil divisor as $D = \Sigma n_i C_i$, where the C_i are prime divisors.

Definition 2.33. A Weil divisor D is *effective* ($D \geq 0$) if $D = \sum n_i C_i$ with $n_i \geq 0$.

Proposition 2.34. *If X is locally factorial, then the group* $\mathrm{Div}(X)$ *of Weil divisors on X is isomorphic to the group of Cartier divisors* $\mathrm{Cdiv}(X)$.

Proof. See [Har77], Proposition II.6.11. □

Remark 2.35. Note that for X as in Proposition 2.34, effective Cartier divisors correspond to effective Weil divisors.

Remark 2.36. From the proof of [Har77], Proposition II.6.11, we see that if X is a noetherian integral separated normal scheme of finite type over a field k, then the

group of Cartier divisors $\mathrm{Cdiv}(X)$ is embedded in the group of Weil divisors $\mathrm{Div}(X)$. We need X to be locally factorial in order to have equality.

Definition 2.37. If $D = \Sigma n_i C_i$ is a Weil divisor, then its *support* is

$$\mathrm{supp}(D) := \cup_{n_i \neq 0} C_i.$$

From [EGA] IV,21.6.6.2, we have that this notion is equivalent to that of Definition 2.26, when D is an effective Cartier divisor on a scheme X satisfying the conditions of Proposition 2.34.

Definition 2.38. Let Y be a prime divisor on X, and let $\eta \in Y$ be its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field $k(X)$, the function field of X. We call the corresponding (surjective) valuation $v_Y : k(X)^* \to \mathbb{Z}$ the *valuation* of Y.

Let $f \in k(X)^*$ be any nonzero rational function on X. Then $v_Y(f)$ is an integer. If it is positive, we say that f has a *zero* along Y, of that order; if it is negative, we say that f has a *pole* along Y, of order $-v_Y(f)$.

Lemma 2.39. *Let X be as in the beginning of this subsection, and let $f \in k(X)^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y.*

Proof. See [Har77], Lemma II.6.1. □

Definition 2.40. Let X be as in the beginning of this subsection and let $f \in k(X)^*$. We define the *divisor* of f by $(f) := \sum v_Y(f) Y$.

2.2.4 The degree of invertible sheaves on curves

Fix an arbitrary algebraically closed field k. All schemes in this section are projective over k.

Let $\mathcal{F}$ be a coherent sheaf on a scheme X of dimension d. Let

$$\chi(\mathcal{F}) := \sum_{i=1}^{d} (-1)^i \dim_k H^i(X, \mathcal{F})$$

be the Euler characteristic of $\mathcal{F}$.

Definition 2.41. Let C be an integral projective scheme of dimension 1 over an algebraically closed field k. Using [Har77], Ex.IV.1.9, we define the *degree* $\deg_C(\mathcal{F})$ of an invertible sheaf $\mathcal{F}$ on C as follows: Let C_{reg} be the open set of regular points of C (we delete from C the singular points). By [Har77] Ex.IV.1.9(c) there is a Cartier divisor D on C with support contained in C_{reg}, such that $\mathcal{F} \cong \mathcal{L}(D)$. Since C_{reg} is regular, the restriction of D to C_{reg} is given by a Weil divisor $\sum_i n_i P_i$ (finite sum, $n_i \in \mathbb{Z}$ and P_i closed points of C_{reg}). We define $\deg_C(\mathcal{F}) := \deg(D) = \sum n_i$.

This definition is independent of the choice of the divisor D (provided that D is supported on C_{reg} and that $\mathcal{F} \cong \mathcal{L}(D)$). Indeed, from [Har77], Ex.IV.1.9(a), we have that $\deg(D) = \chi(\mathcal{L}(D)) - 1 + p_a(C)$, and the right hand side is invariant under sheaf isomorphisms. Note that $\deg_C(\mathcal{O}_C) = 0$ and that $\deg_C(\mathcal{F} \otimes \mathcal{G}) = \deg_C(\mathcal{F}) + \deg_C(\mathcal{G})$ (as the same property holds for divisors supported on C_{reg}). In particular, we have $\deg_C(\mathcal{F}^{-1}) = -\deg_C(\mathcal{F})$.

Theorem 2.42 (Riemann-Roch for curves)**.** *Let C be an integral projective scheme*

of dimension 1 *over* k*, and let* $\mathcal{F}$ *be an invertible sheaf on* C*. We have*

$$\chi(\mathcal{F}) = \deg_C(\mathcal{F}) + \chi(\mathcal{O}_C)$$

and $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ *is linear.*

Proof. See [Har77], Exercise IV.1.9 and the above discussion. □

Notation 2.43. We write $k(X)$ for the function field of an integral variety X. Note that $k(X)$ is the local ring $\mathcal{O}_{X,\eta_X}$, where η_X is the generic point of X.

Proposition 2.44. *If* $\varphi : C_1 \to C_2$ *is a surjective morphism of projective integral curves, then*

$$\deg_{C_1}(f^*\mathcal{L}) = \deg(\varphi)\deg_{C_2}(\mathcal{L})$$

for all $\mathcal{L} \in \mathrm{Pic}(C_2)$*, where* $\deg(\varphi) = [k(C_1) : \varphi^* k(C_2)]$.

Proof. See [Bad01] Lemma 1.18. □

Remark 2.45. For a closed subvariety $X \subseteq \mathbb{P}^n$, we write $\deg(X)$ for the degree of X in $\mathbb{P}^n$ as defined in [Har77], p. 52. Let $i : C \to \mathbb{P}^n$ be the inclusion morphism. Then by Example IV.3.3.2 in [Har77], we get

$$\deg(C) = \deg_C(i^*\mathcal{O}_{\mathbb{P}^n}(1)) = \deg_C(\mathcal{O}_C(1)).$$

2.2.5 Intersection numbers on surfaces

If X/k is a smooth projective surface, then we have an intersection pairing on the group $\mathrm{Div}(X)$; cf. [Har77], §V.1. More generally, on any projective surface we can define an intersection product as follows.

Notation 2.46. Let X be a projective scheme of dimension 2 over k, and let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(X)$. Write

$$(\mathcal{L}_1.\mathcal{L}_2)_X := \chi(\mathcal{L}_1 \otimes \mathcal{L}_2) - \chi(\mathcal{L}_1) - \chi(\mathcal{L}_2) + \chi(\mathcal{O}_X).$$

Remark 2.47. This defines a bilinear pairing on $\mathrm{Pic}(X)$; cf. Lemma 1.6 in [Bad01].

Proposition 2.48. *If X is a smooth projective surface over k, then $(-.-)_X$ coincides with the usual intersection product on divisors:*

$$(\mathcal{L}(C).\mathcal{L}(D))_X = (C.D).$$

Proof. By [Har77], Ex.V.1.1. □

Proposition 2.49 (Weak projection formula)**.** *If $f : X \to Y$ is a surjective morphism of projective integral surfaces over K, then we have*

$$(f^*\mathcal{L}_1.f^*\mathcal{L}_2)_X = \deg(f)(\mathcal{L}_1.\mathcal{L}_2)_Y$$

for $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(Y)$.

Proof. See [Bad01] Lemma 1.18. □

The following result, which we will need in Section 3.8, relates the intersection pairing with the degree map on curves.

Proposition 2.50. *Let X be a smooth projective surface over k. Let $\mathcal{F}$ be an invertible sheaf on X, and let $C \subseteq X$ be an integral projective curve. Then we have*

$$\deg_C(i_C^*\mathcal{F}) = (\mathcal{L}(C).\mathcal{F})_X.$$

Proof. From (2.3) with $D = C$ we have $\chi(O_X) - \chi(\mathcal{L}(-C)) = \chi(\mathcal{O}_C)$. From [Liu02], Ex. 5.1.1(c) we know $\mathcal{F}^{-1} \otimes i_{C*}\mathcal{O}_C \cong i_{C*}i_C^*\mathcal{F}^{-1}$. From (2.4) replacing $\mathcal{F}$ by $\mathcal{F}^{-1}$ we have $\chi(\mathcal{F}^{-1}) - \chi(\mathcal{F}^{-1} \otimes \mathcal{L}(-C)) = \chi(i_C^*\mathcal{F})$. From Proposition 2.42 we obtain $\chi(i_C^*\mathcal{F}^{-1}) = \deg_C(i_C^*\mathcal{F}^{-1}) + \chi(\mathcal{O}_C)$. We know by Remark 2.47 that

$$\begin{aligned}
(\mathcal{F}.\mathcal{L}(C))_X &= (\mathcal{F}^{-1}.\mathcal{L}(-C))_X \\
&= \chi(\mathcal{F}^{-1} \otimes \mathcal{L}(-C)) - \chi(\mathcal{F}^{-1}) - \chi(\mathcal{L}(-C)) + \chi(\mathcal{O}_X) \\
&= -\chi(i_C^*\mathcal{F}^{-1}) + \chi(\mathcal{O}_C) \\
&= -\deg_C(i_C^*\mathcal{F}^{-1}) \\
&= \deg_C(i_C^*\mathcal{F})
\end{aligned}$$

The last equality holds by Proposition 2.42. □

2.2.6 The desingularization of a normal singular surface

In this section, *surface* means projective variety of dimension 2 over an algebraically closed field k of characteristic 0.

Following [Mum61], we will discuss pullbacks of $\mathbb{Q}$-divisors on a normal surface X to its desingularization Y. Define the group of $\mathbb{Q}$-divisors on a normal surface X as the group $\mathrm{Div}(X) \otimes \mathbb{Q}$. In other words, this is the group of formal $\mathbb{Q}$-linear combinations of curves (prime divisors) on X. Note that $\mathrm{Div}(X)$ is included in the group of $\mathbb{Q}$-divisors.

Normal surfaces can be singular. For example, the quadric surface $\{xy = z^2\} \subseteq \mathbb{P}^3$ is normal (cf. Exercise I.3.17(b) [Har77]) and it is singular at $[1 : 0 : 0 : 0]$, where we use the coordinates $[w : x : y : z]$ in $\mathbb{P}^3$. However, normal surfaces are regular in codimension 1 and therefore the only possible singularities of a normal surface occur

in codimension 2, that is, singular points.

By [Zar39], every normal surface X has a desingularization $\delta : Y \to X$, which is obtained by a sequence of normalizations and quadratic transformations. Thus, Y is smooth and δ is birational and proper. It is known that δ can be chosen in such a way that for each singular point $P \in X$ (if any) the fibre of P is the union of a finite set of smooth curves $E_1, \ldots, E_n$ such that for all $i \neq j$, either $E_i \cap E_j = \emptyset$ or E_i, E_j intersect transversely in exactly one point (so that $(E_i.E_j)_Y = 1$), which does not belong to any other E_ℓ.

Following [Mum61] p. 17 we define the pullback C' (or 'total transform' in the terminology of [Mum61]) of a $\mathbb{Q}$-divisor C in X by δ, which is going to be a $\mathbb{Q}$-divisor of Y. Since δ is an isomorphism away from the singular points of X, it suffices to define the pull-back C' near a singular point P, with fibre $\cup_{i=1}^n E_i$, where the E_i are as before. For a $\mathbb{Q}$-divisor C, write $C = \sum q_i C_i$ where C_i are irreducible curves (prime divisors) and $q_i \in \mathbb{Q}$. The *strict transform* of C is defined as $\sum_i q_i D_i$ where D_i is the strict transform (cf. [Har77] p. 30) of the curve C_i.

Notation 2.51. Let U be a neighborhood of P such that P is the only singular point of X in U. Let D be the strict transform of C, and define

$$C'_{|U} = D_{|U} + \sum_{i=0}^{n} r_i E_i$$

where the $r_i \in \mathbb{Q}$ are subject to the conditions (for $1 \leq j \leq n$)

$$(D.E_j)_Y + \sum_i r_i (E_i.E_j)_Y = 0.$$

These conditions uniquely determine the values of the $r_i \in \mathbb{Q}$ (cf. [Mum61] p.17).

We remark that the collection of data C'_U (as P varies) is compatible because it only modifies D on the fibres of singular points, which are disjoint. Thus, this process uniquely defines a $\mathbb{Q}$-divisor C'.

On a normal variety every Cartier divisor is a Weil divisor (see Remark II.6.11.2 in [Har77]), and therefore every Cartier divisor is also a $\mathbb{Q}$-divisor. If C is a Cartier divisor on X, then we can also consider the pullback δ^*C as Cartier divisor because δ is dominant.

Proposition 2.52. *With the previous notation, we have $\delta^*C = C'$.*

Proof. We follow the idea of property (i), [Mum61] p. 17. Let $C = \{(U_i, f_i)\}$ be a Cartier divisor. The functions $g_i := \delta^* f_i$ are rational functions on Y and therefore $((g_i).E_j)_Y = 0$ (Y is smooth, so the intersection pairing is defined on divisors). The curves E_i live in fibres of δ, therefore we have that each E_j is either completely contained in $\delta^{-1}U_i$ or they don't meet. It follows that

$$(\delta^*C.E_j)_Y = (\{\delta^{-1}U_i, g_i\}_i, E_j)_Y = 0.$$

Now we compare δ^*C and C'. Away from the fibres of singular points, we have that both $\mathbb{Q}$-divisors agree, as they are equal to the strict transform D of C away from fibres of singular points. Therefore $\delta^*C = D+S$, where S is a $\mathbb{Q}$-linear combination of the E_i. The property $(\delta^*C.E_j)_Y = 0$ becomes in this notation $(D.E_j)_Y + (S.E_j)_Y = 0$ which is exactly the same condition defining the coefficients r_i for C'. By uniqueness of the solution of the linear equations defining the r_i, we conclude that $\delta^*C = C'$. □

Therefore, we can unambiguously write $\delta^*C := C'$ for C a Cartier, Weil, or a $\mathbb{Q}$-divisor. Moreover, the definition of C' (using linear equations and intersection

numbers in Y to find r_i) can be used to compute the pull-back by δ of Cartier divisors.

2.3 Branches

Branches help us to study the local structure of a curve. In particular, we can study a curve in neighborhoods of a singularity. The following discussion is based on the work of Seidenberg in [Sei68a], [Sei68b] and the notes of [Kan14a].

2.3.1 Basic definitions

Let k be an algebraically closed field of characteristic zero.

Definition 2.53. A *branch* is a non-zero, non-maximal prime ideal of $k[[\mathbf{X}, \mathbf{Y}]]$.

In Seidenberg's work (see [Sei68b]), a branch is defined as the equivalence class (up to multiplication by units) of an irreducible element F of $k[[\mathbf{X}, \mathbf{Y}]]$. These notions are equivalent because the equivalence classes defined by Seidenberg are in bijection with nonzero and non maximal prime ideals of $k[[\mathbf{X}, \mathbf{Y}]]$ (i.e. branches), since $k[[\mathbf{X}, \mathbf{Y}]]$ is a unique factorization domain of dimension 2.

Definition 2.54. A *branch representation* is a non-zero local k-algebra homomorphism

$$\phi : k[[\mathbf{X}, \mathbf{Y}]] \to k[[t]].$$

Given a branch $\mathfrak{P}$, a *branch representation associated to* $\mathfrak{P}$ is a branch representation $\phi : k[[\mathbf{X}, \mathbf{Y}]] \to k[[t]]$ satisfying $\ker(\phi) = \mathfrak{P}$. A branch representation

$$\phi : k[[\mathbf{X}, \mathbf{Y}]] \to k[[t]]$$

is *primitive* if $\phi(k[[\mathbf{X},\mathbf{Y}]]) \not\subseteq k[[\tau]]$ for any $\tau \in k[[t]]$ with $\mathrm{ord}_{k[[t]]}(\tau) > 1$.

With $m_{k[[t]]}$ the maximal ideal of $k[[t]]$, Seidenberg ([Sei68a], p. 89) defines a branch representation at $(0,0)$ as a pair $(x(t), y(t)) \in (m_{k[[t]]})^2 \setminus (0,0)$.

Each pair $(x(t), y(t)) \in (m_{k[[t]]})^2 \setminus (0,0)$ determines a unique nonzero local ring homomorphism $f : k[[\mathbf{X},\mathbf{Y}]] \to k[[t]]$ such that $f(\mathbf{X}) = x(t)$, $f(\mathbf{Y}) = y(t)$, so Definition 2.54 corresponds to the definition of branch representation from [Sei68a] and conversely.

Definition 2.55. Two primitive branch representations f_1, f_2 are *equivalent* if there is a continuous $\alpha \in \mathrm{Aut}_k(k[[t]])$ such that $\alpha \circ f_1 = f_2$.

Proposition 2.56. *The rule $\phi \to \ker(\phi)$ defines a bijection between the equivalence classes of primitive branch representations $\phi : k[[\boldsymbol{X}, \boldsymbol{Y}]] \to k[[t]]$ and the set of branches of $k[[\boldsymbol{X}, \boldsymbol{Y}]]$.*

Proof. (Sketch, details in [Kan14a], p. 8). A key point here is that if ϕ is a branch representation, then the integral closure $\mathrm{Im}(\phi)^{\sim}$ of $\mathrm{Im}(\phi)$ (in its quotient field) is equal to $k[[\tau]]$, for some $\tau \in k[[t]]$. This follows easily from the Weierstrass preparation and division theorems; cf. Theorem 21.1 and 21.2 in [Sei68a]. Moreover, $k[[\tau]] = k[[t]]$ if and only if ϕ is primitive.

From this we see that the given map is injective because if $\ker(\phi_1) = \ker(\phi_2)$, then $im(\phi_1) \cong im(\phi_2)$, and this isomorphism extends to the integral closures, so ϕ_1 and ϕ_2 are equivalent.

Moreover, the map is surjective because if $\mathfrak{P}$ is a branch, then

$$\phi_{\mathfrak{P}} : A \to A/\mathfrak{P} \hookrightarrow (A/\mathfrak{P})^{\sim} \cong k[[t]]$$

is a primitive branch representation (cf. [Kan14a], p. 8). □

Now we present some useful definitions. Let $F \in k[[\mathbf{X}, \mathbf{Y}]]$ be a power series. It can be written as

$$F(\mathbf{X}, \mathbf{Y}) = F_r(\mathbf{X}, \mathbf{Y}) + F_{r+1}(\mathbf{X}, \mathbf{Y}) + \cdots$$

with F_n homogeneous polynomial of degree n and with $F_r \neq 0$. We call the number r the *order of* F, and we denote it by $\mathrm{ord}(F)$.

Write

$$F_{\mathrm{ord}(F)}(\mathbf{X}, \mathbf{Y}) = c \prod_{i=1}^{t} (m_0^{(i)}\mathbf{X} - m_1^{(i)}\mathbf{Y})^{s_i},$$

with $c \in k^\times$, $m_j^{(i)} \in k$ and $m_0^{(i)} m_1^{(j)} \neq m_0^{(j)} m_1^{(i)}$ for $i \neq j$. If F is irreducible, then by using Hensel's Lemma we obtain that t must be equal to 1 , but even in that case s_1 can be different from 1, by [Sei68b], p. 249.

The linear factors $m_0^{(i)}\mathbf{X} - m_1^{(i)}\mathbf{Y}$ of $F_{\mathrm{ord}(F)}$ are called the *tangents* of F. If F is irreducible we can define the *tangents* of the branch (F) as the tangents of F. A *vertical tangent* is a tangent of the form $m_0^{(i)}\mathbf{X}$, and a *horizontal tangent* one of the form $m_1^{(i)}\mathbf{Y}$.

2.3.2 Branches and curves

Let S be an irreducible algebraic surface defined over $k = \mathbb{C}$, and let P be a smooth point on it. We want to relate branches to the curves on a surface which pass through a point P. For this, it is very important that the completion $\hat{\mathcal{O}}_{S,P}$ of the local ring of S at P is isomorphic to the power series ring $k[[\mathbf{X},\mathbf{Y}]]$ (cf. [Har77], Theorem I.5.5.A).

Since the local ring $\mathcal{O}_{S,P}$ of S at P is a unique factorization domain of dimension

2, the prime ideals of height 1 will be non-zero principal prime ideals and conversely. Hence we have a bijection between irreducible curves through P and non-zero principal prime ideals in $\mathcal{O}_{S,P}$.

Definition 2.57. Let C be an irreducible algebraic curve on S passing through P with principal prime ideal $(f) = \mathfrak{p} \in \mathcal{O}_{S,P}$. We define a *branch of C at P* to be an ideal $\mathfrak{P}$ in the $m_{S,P}$-adic completion $\hat{\mathcal{O}}_{S,P}$ of $\mathcal{O}_{S,P}$ satisfying $\mathfrak{P} \cap \mathcal{O}_{S,P} = \mathfrak{p}$.

A *branch representation associated to* $\mathfrak{P}$ is a morphism $\phi : \hat{\mathcal{O}}_{S,P} \to k[[t]]$ such that $\ker(\phi) = \mathfrak{P}$. It is *primitive* if $\phi(\hat{\mathcal{O}}_{S,P}) \not\subseteq k[[\tau]]$ for any $\tau \in k[[t]]$ with $\mathrm{ord}_{k[[t]]}(\tau) > 1$.

Since S is smooth at P, its local ring at P is regular. By Theorem I.5.5.A in [Har77], its completion satisfies $\hat{\mathcal{O}}_{S,P} \cong k[[\mathbf{X}, \mathbf{Y}]]$. (This isomorphism depends on a choice of local parameters.) Hence the definitions given in Definition 2.57 are compatible to the ones in Definition 2.53 and Definition 2.54.

Remark 2.58. Denote by $\hat{f}$ the image of $f \in \mathcal{O}_{S,P}$ under the completion map $c_{\mathcal{O}_{S,P}} : \mathcal{O}_{S,P} \to \hat{\mathcal{O}}_{S,P}$. Note that since $\hat{\mathcal{O}}_{S,P}$ is a unique factorization domain, we can write $\hat{f} := c_{\mathcal{O}_{S,P}}(f) = \alpha g_1^{r_1} \cdots g_n^{r_n}$ with α invertible, and the g_i non-associated irreducible elements and for all $1 \leq i \leq n$ the number $r_i \geq 1$. Moreover, this factorization is essentially unique. Hence $(\hat{f}) = (g_1^{r_1} \cdots g_n^{r_n}) = (g_1^{r_1}) \cdots (g_n^{r_n}) = (g_1)^{r_1} \cdots (g_n)^{r_n}$.

By Theorem 32 in [ZS60] Ch. VIII, §13, if f is irreducible, then the ring $\mathcal{O}_{S,P}/(f)$ is analytically unramified, so its completion $\hat{\mathcal{O}}_{S,P}/(\hat{f})$ has no nilpotents. Therefore the ideal $(\hat{f})$ factors into distinct prime ideals in $\hat{\mathcal{O}}_{S,P}$, or equivalently, $r_1 = \cdots = r_n = 1$. Thus, if f is a local equation of C at P, then the branches of C at P are $(g_1), \ldots, (g_n)$ and $(\hat{f}) = (g_1) \cdots (g_n) = (g_1) \cap \ldots \cap (g_n)$.

Proposition 2.59. *Given a branch $\mathfrak{P} \in \hat{\mathcal{O}}_{S,P}$, there is at most one curve of S having it as branch at P.*

Proof. Suppose that there are two curves C_1, C_2 with ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ in $\mathcal{O}_{S,P}$, which have $\mathfrak{P}$ as branch. Then $\mathfrak{p}_1 = \mathfrak{P} \cap \mathcal{O}_{S,P} = \mathfrak{p}_2$, and hence $C_1 = C_2$. □

Remark 2.60. There exist branches of $\hat{\mathcal{O}}_{S,P} \cong k[[\mathbf{X},\mathbf{Y}]]$ which do not belong to any curve on S. For example, let $S = \mathbb{A}^2$ and $P = (0,0)$. Let $f(t) = \sum_{n\geq 1} t^{n!} \in k[[t]]$. In [ZS60] p. 220, it is shown that $1 + f(t)$ is transcendental over $k(t)$, hence $f(t)$ is transcendental over $k(t)$. Let $F(\mathbf{X},\mathbf{Y}) = \mathbf{Y} - f(\mathbf{X}) \in k[[\mathbf{X},\mathbf{Y}]]$. Note that F is irreducible in $k[[\mathbf{X},\mathbf{Y}]]$ since $\mathrm{ord}(F) = 1$ and ord is an additive function. Thus (F) is a branch. Let

$$\begin{aligned} \phi : k[[\mathbf{X},\mathbf{Y}]] &\to k[[t]] \\ X &\mapsto t \\ Y &\mapsto f(t). \end{aligned}$$

Since $(F) \subseteq \ker(\phi)$ and (F) is a branch, we obtain that ϕ is a branch representation of (F).

Suppose that (F) is a branch of a curve at $P = (0,0) \in \mathbb{A}^2$. Then there is a non-zero polynomial $h \in k[\mathbf{X},\mathbf{Y}]$ such that $h \in (F)$, hence $\phi(h) = 0$. Note that h is non-constant in Y because otherwise we have

$$0 = \phi(h(\mathbf{X},\mathbf{Y})) = \phi(h(\mathbf{X},0)) = h(t,0) = h(t,\mathbf{Y}) \neq 0$$

(in $k[t,\mathbf{Y}] \cong k[\mathbf{X},\mathbf{Y}]$). Hence $\phi(t, f(t)) = \phi(h) = 0$ is a nontrivial equation of $f(t)$ over $k(t)$. This contradicts the fact that $f(t)$ is transcendental over $k(t)$. Therefore (F) is not a branch of a curve at P.

Definition 2.61 ([Har77], p. 388)**.** Let C be an effective Cartier divisor on a surface S which is smooth at a point P. Then we define the *multiplicity of C at P*, denoted by $\mu_P(C)$, to be the largest integer r such that $f \in m_{S,P}^r$, where f is a local equation for C at the point P.

Proposition 2.62. *Let $\mathfrak{p} = (f)$ be the ideal associated to C in $\mathcal{O}_{S,P}$, and let $\hat{\mathfrak{p}}$ be its completion. Then we have*

$$\mu_P(C) = \sum_{\mathfrak{P}|\hat{\mathfrak{p}}} \mathrm{ord}_P(\mathfrak{P}),$$

where $\mathrm{ord}_P(\mathfrak{P}) = \max\left\{r : \mathfrak{P} \subseteq \hat{m}_{S,P}^r\right\}$*, and the sum runs over the branches $\mathfrak{P}$ in $\hat{O}_{S,P}$ with $\hat{\mathfrak{p}} \subset \mathfrak{P}$.*

Proof. We first prove that $\mu_P(C) = \mathrm{ord}_P(\hat{f})$. From [Bou72], p. 208, we have that $m_{S,P}^r \hat{A} = \hat{m}_{S,P}^r$ and $\hat{m}_{S,P}^r \cap A = m_{S,P}^r$. Thus $f \in m_{S,P}^r$ if and only if $\hat{f} \in \hat{m}_{S,P}^r$. Therefore $\mu_P(C) = \mathrm{ord}_P(\hat{f})$.

Let $\mathfrak{P}$ be a prime ideal in $\hat{\mathcal{O}}_{S,P}$. By Remark 2.58 we have $(\hat{f}) = (g_1) \cdots (g_r)$, so if $\mathfrak{P} \supseteq (\hat{f})$, then $\mathfrak{P} \supseteq (g_i)$ for some i, and since (g_i) is prime we obtain $\mathfrak{P} = (g_i)$. We have

$$\mu_P(C) = \mathrm{ord}_P(\hat{f}) = \sum_{i=1}^{n} \mathrm{ord}_P(g_i) = \sum_{\mathfrak{P}|\hat{\mathfrak{p}}} \mathrm{ord}_P(\mathfrak{P}),$$

with (g_i) the branches of C. The second equality holds by Remark 2.58 and because ord_P is a valuation (since $\hat{\mathcal{O}}_{S,P} \cong k[[\mathbf{X},\mathbf{Y}]]$ is a regular local ring and ord_P in $k[[\mathbf{X},\mathbf{Y}]]$ is a valuation, cf. [Bou90] IV.25). The last equality holds because

$$\mathrm{ord}_P(g_i) = \max\left\{r : g_i \in \hat{m}_{S,P}^r\right\} = \max\left\{r : (g_i) \subseteq \hat{m}_{S,P}^r\right\}.$$

□

Definition 2.63. Let $\mathfrak{P} \subseteq \hat{\mathcal{O}}_{S,P}$ be a branch. Let $\phi : \hat{\mathcal{O}}_{S,P} \to k[[t]]$ be a branch representation of $\mathfrak{P}$. We say that ϕ is a *linear branch representation of* $\mathfrak{P}$ if it is surjective. We say that $\mathfrak{P}$ is a *linear branch* if it has a linear branch representation.

In [Sei68a], a linear branch representation is a branch representation

$$\phi : k[[\mathbf{X}, \mathbf{Y}]] \to k[[t]]$$

satisfying $\phi(\mathbf{X}) = c_1 t + \cdots$, $\phi(\mathbf{Y}) = d_1 t + \cdots$ with $c_1 \neq 0$ or $d_1 \neq 0$.

Proposition 2.64. *If $\mathfrak{P}$ is a linear branch, then all primitive branch representations ϕ of $\mathfrak{P}$ are linear.*

Proof. Recall from Definition 2.55 that two primitive branch representations are associated to the same branch if there is $\alpha \in \mathrm{Aut}_k(k[[t]])$ such that $\alpha \circ f_1 = f_2$. Thus, if f_1 is surjective, then f_2 is surjective. □

Proposition 2.65. *Let $\mathfrak{P}$ be a branch and ϕ a primitive branch representation of $\mathfrak{P}$. Let x, y be a system of local parameters at P. The branch $\mathfrak{P}$ is linear if and only if at least one of $\phi(x)$, $\phi(y)$ has order 1 (as power series in t).*

Proof. If both $\phi(x)$, $\phi(y)$ have orders greater than one, and ord is a valuation, we obtain that for any $g \in k[[x, y]]$ we have either $\mathrm{ord}\phi(g) = 0$ (when g has constant term), or $\mathrm{ord}\phi(g) > 1$ (when g has no constant term). Then ϕ cannot be surjective, since t will never be an image of ϕ. Hence if $\mathfrak{P}$ is linear then at least one of $\phi(x)$ or $\phi(y)$ must have order 1. Now suppose that $\phi(x) = t + \cdots$. (The case for $\phi(y) = t + \cdots$ is similar.) Then there is $f_0 \in k[[t]]$ such that $f_0(\phi(x)) = t$ [Bou90] p. A.IV.30. Hence the element $f_0(x) \in \hat{\mathcal{O}}_{S,P}$ has image $\phi(f_0(x)) = f_0(\phi(x)) = t$ (recall that ϕ is continuous). Therefore ϕ is surjective and $\mathfrak{P}$ is a linear branch. □

Corollary 2.66. *A branch $\mathfrak{P}$ is linear if and only if* $\operatorname{ord}(\mathfrak{P}) = 1$.

Proof. Suppose that $\mathfrak{P}$ is linear and choose a primitive branch representation ϕ of $\mathfrak{P}$. After choosing an isomorphism $\hat{\mathcal{O}}_{S,P} \cong k[[\mathbf{X},\mathbf{Y}]]$, we know that at least one of $\phi(X)$, $\phi(Y)$ has order 1, hence $\mathfrak{P}$ is of the form (F) with $F \in k[[\mathbf{X},\mathbf{Y}]]$ of order 1.

Let (F) be a branch of order 1. From [Sei68a], Theorem 11.3, there is a unique power series $c_1\mathbf{X} + c_2\mathbf{X}^2 + \cdots$ such that $F(\mathbf{X}, c_1\mathbf{X} + c_2\mathbf{X}^2 + \cdots) = 0$. Taking the branch $\phi(\mathbf{X}) = t$, $\phi(\mathbf{Y}) = c_1t + c_2t^2 + \cdots$ we obtain that F has only one branch representation and it is linear. □

Corollary 2.67. *A curve passing through a point P is smooth at P if and only if it has only one branch at P, and it is linear.*

Proof. If C is a curve which has only one linear branch at P, then by Proposition 2.62 we obtain that $\mu_P(C) = 1$, hence it is smooth at P. Let C be a curve which is smooth at P, so $\mu_P(C) = 1$. Then by Proposition 2.62 we get that $\sum_{\mathfrak{P}|\hat{\mathfrak{p}}} \operatorname{ord}_P(\mathfrak{P}) = 1$. Therefore C has only one branch $\mathfrak{P}$ and it has order 1. By Corollary 2.66 it is a linear branch representation. □

Notation 2.68. Let $C \subseteq S$ be a curve and let P be a point on C. Let $\nu_C : \tilde{C} \to C$ be the desingularization of C (which can be constructed as the normalization of the curve) and $i_C : C \to S$ the inclusion morphism. Let $\varphi_C = i_C \circ \nu_C$.

The points Q in the preimage of P under ν_C are given by the maximal ideals m_Q of the ring $\tilde{\mathcal{O}}_{C,P}$, the integral closure of $\mathcal{O}_{C,P}$ in its field of fractions. Given $Q \in \nu_C^{-1}(P)$, we have $\mathcal{O}_{\tilde{C},Q} = (\tilde{\mathcal{O}}_{C,P})_{m_Q}$ and the map $\nu_{C,Q}^{\#} : \mathcal{O}_{C,P} \to \mathcal{O}_{\tilde{C},Q} = (\tilde{\mathcal{O}}_{C,P})_{m_Q}$ (cf. [Har77], p. 72) is the inclusion map.

Moreover, if f is a local equation for C at P, then we have $\mathcal{O}_{C,P} = \mathcal{O}_{S,P}/(f)$, hence $i^{\#}_{C,P} : \mathcal{O}_{S,P} \to \mathcal{O}_{S,P}/(f) = \mathcal{O}_{C,P}$. Therefore we have that $\varphi^{\#}_{C,Q} = \nu^{\#}_{Q,C} \circ i^{\#}_{C,P}$ is the composition of a quotient map and an inclusion map.

Taking completions we have that $\hat{\varphi}^{\#}_{C,Q}$ is given by the following composition

$$\hat{\mathcal{O}}_{S,P} \overset{\hat{i}^{\#}_{C,P}}{\to} \hat{\mathcal{O}}_{S,P}/(\hat{f}) \cong \hat{\mathcal{O}}_{C,P} \overset{\hat{\nu}^{\#}_{C,Q}}{\to} (\tilde{\mathcal{O}}_{C,P})^{\wedge}_{m_Q} = \hat{\mathcal{O}}_{\tilde{C},Q}$$

where the isomorphism $\hat{\mathcal{O}}_{S,P}/(\hat{f}) \cong \hat{\mathcal{O}}_{C,P}$ holds because $\mathcal{O}_{S,P}/(f) \cong \mathcal{O}_{C,P}$ and taking completions is exact (cf. [AM69], Proposition 10.12).

The next result relates branches of a curve at a point P to the preimage of P in the normalization of the curve.

Theorem 2.69. *The rule $Q \mapsto \ker(\hat{\varphi}^{\#}_{C,Q})$ gives a bijection between points Q in the preimage of P by the normalization map $\nu_C : \tilde{C} \to C$, and the branches of C at P.*

Proof. The points Q in the preimage of P by the normalization map are in correspondence with the maximal ideals m_Q of $\tilde{\mathcal{O}}_{C,P}$.

From [CKPU13], Proposition 1.12, the maximal ideals of $\tilde{\mathcal{O}}_{C,P}$ are in correspondence with the minimal prime ideals of $\hat{\mathcal{O}}_{C,P}$, given by

$$m_Q \mapsto \ker(\hat{\mathcal{O}}_{C,P} \overset{\hat{\nu}^{\#}_{C,Q}}{\to} (\tilde{\mathcal{O}}_{C,P})^{\wedge}_{m_Q}).$$

Since the minimal prime ideals of $\hat{\mathcal{O}}_{C,P}$ are in bijection with the branches of C at P (by $\mathfrak{P} \mapsto (\hat{i}^{\#}_{C,P})^{-1}(\mathfrak{P})$ for a minimal prime ideal $\mathfrak{P}$), we obtain that the branches of C at P are in bijection with the maximal ideals m_Q of $\tilde{\mathcal{O}}_{C,P}$, which are in bijection

with the preimages Q of P by φ_C. Finally, observe that

$$(\hat{i}^{\#}_{C,P})^{-1}(\ker(\hat{\nu}^{\#}_{C,Q})) = \ker(\hat{\varphi}^{\#}_{C,Q}).$$

□

2.4 Stalks

In this section, we give some useful remarks and definitions about stalks of sheaves and morphisms induced on stalks, which we will use in other parts of this thesis, especially in Section 3.5.

Let X be a scheme. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and a point $P \in X$, we recall from [Har77], p. 62, that the stalk $\mathcal{F}_P$ is defined as

$$\mathcal{F}_P := \varinjlim_{P \in U} \mathcal{F}(U),$$

where the limit is taken over the open neighborhoods of P, using the restriction maps between the various $\mathcal{F}(U)$ (when we have inclusions of open sets). In particular, for any open neighborhood U of P we have a map

$$\begin{array}{rcl} \mathcal{F}(U) & \to & \mathcal{F}_P \\ s & \mapsto & s_P = \langle s, U \rangle \end{array}$$

(see notation in [Har77], p. 62.) The special case $U = X$ (which is always a neighborhood of P) gives the map $H^0(X, \mathcal{F}) \to \mathcal{F}_P$.

Proposition 2.70. *Let X be an integral scheme, and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. Let $U \subseteq X$ be a non-empty open set. Then for all non-empty open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective.*

Proof. Suppose that there exists $V \subseteq U$ with $\mathcal{F}(U) \to \mathcal{F}(V)$ not injective. Then there is a section $s \in \mathcal{F}(U)$ different from zero such that $s_{|V} = 0$. Let $\cup_i \mathrm{Spec}(A_i)$ be an affine cover of U such that for all i, we have that $\mathcal{F}_{|\mathrm{Spec}(A_i)}$ is free. Since $s \neq 0$, there is j such that $s_{|\mathrm{Spec}(A_j)} \neq 0$ (by the sheaf axioms). Let $\eta \in X$ be the generic point (X integral). Then $\eta \in U \cap \mathrm{Spec}(A_j) \cap V$. Since $\eta \in V$, and $s_{|V} = 0$, we have $s_\eta = (s_{|V})_\eta = 0$.

On the other hand, A_j is an integral domain (since X is integral) and we have $\mathcal{F}_{|\mathrm{Spec}(A_j)} \cong (A_j^r)^\sim$. As $\eta \in \mathrm{Spec}(A_j)$ we have $\mathcal{F}(\mathrm{Spec}(A_j)) \cong A_j^r$ and

$$\mathcal{F}_\eta = (\mathcal{F}_{|\mathrm{Spec}(A_j)})_\eta \cong A_j^r \otimes_{A_j} \mathrm{Quot}(A_j).$$

Both isomorphisms are compatible in the sense that the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathrm{Spec}(A_j)) & \longrightarrow & \mathcal{F}_\eta \\ \cong\downarrow & & \downarrow\cong \\ A_j^r & \longrightarrow & A_j^r \otimes \mathrm{Quot}(A_j) \end{array}$$

commutes, where the top map takes a section t to t_η, and the bottom map takes v to $v \otimes 1$. Since A_j is an integral domain, we have that the bottom map is injective, hence $\mathcal{F}(\mathrm{Spec}(A_j)) \to \mathcal{F}_\eta$ is injective. As $s_{|\mathrm{Spec}(A_j)} \neq 0$ we obtain $s_\eta = (s_{|\mathrm{Spec}(A_j)})_\eta \neq 0$, which contradicts the fact that $s_\eta = 0$. □

Corollary 2.71. *Suppose that X is an integral scheme and that $\mathcal{F}$ is locally free. If U is a neighborhood of P in X, then the map $\mathcal{F}(U) \to \mathcal{F}_P$ is injective.*

Proof. If $\mathcal{F}(U) \to \mathcal{F}_P$ is not injective, then there exists a section $s \neq 0$ in $\mathcal{F}(U)$ with $s_P = 0$. So there is a neighborhood $V \subseteq U$ of P such that $s_{|V} = 0$. This contradicts Proposition 2.70. □

Remark 2.72. Given a morphism of $\mathcal{O}_X$-modules $u : \mathcal{F} \to \mathcal{G}$ and given any inclusion of open sets $U \subseteq V$ of X, we have a commutative diagram using the restriction maps

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U). \end{array}$$

For a point $P \in V$, we vary U over the neighborhoods of P. Taking direct limits we get a commutative diagram

$$\begin{array}{ccc} H^0(V, \mathcal{F}) & \longrightarrow & H^0(V, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{F}_P & \longrightarrow & \mathcal{G}_P \end{array}$$

where the lower morphism is the map $u_P : \mathcal{F}_P \to \mathcal{G}_P$ induced on stalks by u.

Let $f : Y \to X$ be a morphism of schemes. Let $Q \in Y$ be a point of Y and let $P = f(Q) \in X$. Recall that by definition of morphism of schemes (see [Har77], p. 72-73) we have an induced ring homomorphism $f_Q^{\#} : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ (given by taking direct limits on $\mathcal{O}_Y(V) \mapsto \mathcal{O}_X(f^{-1}(V))$), and moreover $f_Q^{\#}$ is *local* in the sense that $(f_Q^{\#})^{-1} m_{X,P} = m_{Y,Q}$. In particular, using the morphism f we have that $\mathcal{O}_{Y,Q}$ is an $\mathcal{O}_{X,P}$-algebra.

We can adapt the construction of $f_Q^{\#}$ given in [Har77] p. 72-73 for the case of sheaves of $\mathcal{O}_X$-modules.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, then $f^*\mathcal{F}$ is an $\mathcal{O}_Y$-module and one has the canonical

morphism

$$\rho^f_{\mathcal{F}} : \mathcal{F} \to f_* f^* \mathcal{F}$$

(See Subsection 3.2.1 for a study of the pair $(f^*, \rho^f_{\mathcal{F}})$.) Consider the $\mathcal{O}_{X,P}$-module map on stalks induced by the canonical morphism $\rho^f_{\mathcal{F}}$

$$\rho^f_{\mathcal{F},P} : \mathcal{F}_P \to (f_* f^* \mathcal{F})_P.$$

As V varies over neighborhoods of $P = f(Q)$, we have that $f^{-1}(V)$ varies over neighborhoods of Q, so for any $\mathcal{O}_Y$-module $\mathcal{G}$ we have a natural map

$$f^\circ_{\mathcal{G},Q} : (f_* \mathcal{G})_P \to \mathcal{G}_Q$$

defined as follows

$$f^\circ_{\mathcal{G},Q} : (f_* \mathcal{G})_P = \varinjlim_{P \in V} f_* \mathcal{G}(V) = \varinjlim_{P \in V} \mathcal{G}(f^{-1}(V)) \to \varinjlim_{Q \in U} \mathcal{G}(U) = \mathcal{G}_Q.$$

Notation 2.73. Denote by

$$f^\#_{\mathcal{F},Q} : \mathcal{F}_P \to (f^* \mathcal{F})_Q$$

the homomorphism of $\mathcal{O}_{X,P}$-modules obtained from the composition $f^\circ_{f^*\mathcal{F},Q} \circ \rho^f_{\mathcal{F},P}$.

Remark 2.74. In the particular case that $\mathcal{F} = \mathcal{O}_X$, we have $f^\#_Q : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ is equal to the map $f^\#_{\mathcal{O}_X,Q} : \mathcal{O}_{X,P} \to (f^* \mathcal{O}_X)_Q = \mathcal{O}_{Y,Q}$ because $f^* \mathcal{O}_X = \mathcal{O}_Y$.

Proposition 2.75. *Let $f : Y \to X$ be a morphism of schemes, and let $\epsilon : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Moreover, let $Q \in Y$ and put $P = f(Q)$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{F}_P & \xrightarrow{f^{\#}_{\mathcal{F},Q}} & (f^*\mathcal{F})_Q \\ {\scriptstyle \epsilon_P}\downarrow & & \downarrow{\scriptstyle (f^*\epsilon)_Q} \\ \mathcal{G}_P & \xrightarrow[f^{\#}_{\mathcal{G},Q}]{} & (f^*\mathcal{G})_Q \end{array}$$

Proof. We have $f^{\#}_{\mathcal{F},Q} = f^{\circ}_{f^*\mathcal{F},Q} \circ \rho^f_{\mathcal{F},P}$ and $f^{\#}_{\mathcal{G},Q} = f^{\circ}_{f^*\mathcal{G},Q} \circ \rho^f_{\mathcal{G},P}$. We have the following commutative diagram (cf. Subsection 3.2.1)

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\rho^f_{\mathcal{F}}} & f_*f^*\mathcal{F} \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle f_*f^*\epsilon} \\ \mathcal{G} & \xrightarrow[\rho^f_{\mathcal{G}}]{} & f_*f^*\mathcal{G}. \end{array}$$

Evaluating at an open set U containing P we have

$$\begin{array}{ccccccc} \mathcal{F}(U) & \xrightarrow{\rho^f_{\mathcal{F}}} & (f_*f^*\mathcal{F})(U) & = & (f^*\mathcal{F})(f^{-1}U) & \longrightarrow & (f^*\mathcal{F})(V) \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle f_*f^*\epsilon} & & \downarrow{\scriptstyle f^*\epsilon} & & \downarrow{\scriptstyle f^*\epsilon} \\ \mathcal{G}(U) & \xrightarrow[\rho^f_{\mathcal{G}}]{} & (f_*f^*\mathcal{G})(U) & = & (f^*\mathcal{G})(f^{-1}U) & \longrightarrow & (f^*\mathcal{G}(V) \end{array}$$

where V is any neighborhood of Q satisfying $V \subseteq f^{-1}(U)$. Taking direct limits on U in the neighborhoods of P and $V \subseteq f^{-1}(U)$ in the neighborhoods of Q we obtain Diagram (2.75). □

2.5 Smooth projective varieties

In this section, we will show that a projective scheme X/k defined over an algebraically closed field k is smooth at a point P if and only if the rank of the Jacobian matrix of the *homogeneous* equations defining X is not zero when "evaluated" at P. This result will be used in several proofs of Chapters 4, 5 and 6.

Given polynomials $F_1, \ldots, F_r \in k[x_0, \ldots, x_n]$ of total degrees $d_1, \ldots, d_r \geq 1$ respectively, we define their *Jacobian matrix* as

$$J_{F_1,\ldots,F_r} = \left[\frac{\partial F_i}{\partial x_j}\right]_{\substack{1\leq i\leq r \\ 0\leq j\leq n}}.$$

In the particular case when the F_i are homogeneous, the non-zero entries of the i-th row of $J_{F_1,\ldots,F_r}$ are homogeneous polynomials of degree $d_i - 1$.

Lemma 2.76. *Let $F_1, \ldots F_r \in k[x_0, \ldots x_n]$ be homogeneous polynomials of degrees $d_1, \ldots, d_r \geq 1$, respectively. Let $P_1 = (a_0, \ldots, a_n), P_2 = (b_0, \ldots, b_n) \in k^{n+1}\backslash\{0\}$ be such that $P_2 = cP_1$ for some $c \in k^\times$. Then the matrices $J_{F_1,\ldots,F_r}(P_1)$ and $J_{F_1,\ldots,F_r}(P_2)$ (obtained by evaluating the entries of $J_{F_1,\ldots,F_r}$ at P_1 and P_2, respectively) have the same rank.*

Proof. Since $P_2 = cP_1$ and since each F_i is homogeneous of degree $d_i \geq 1$, we get that the i-th row of $J_{F_1,\ldots,F_r}(P_2)$ is equal to the i-th row of $J_{F_1,\ldots,F_r}(P_1)$ multiplied by $c^{d_i-1} \neq 0$, and therefore these matrices have the same rank. □

From this lemma we obtain that given homogeneous polynomials $F_1, \ldots F_r \in k[x_0, \ldots x_n]$, the *rank of $J_{F_1,\ldots F_r}$ at a point $P \in \mathbb{P}^n_k(k)$* is a well-defined integer, which can be computed as the rank of $J_{F_1,\ldots F_r}(\tilde{P})$ for any affine representative $\tilde{P} \in k^{n+1}$ of

P. Denote this number by

$$\mathrm{rk} J_{F_1,\dots F_r}(P).$$

Note that the following proposition is given in [Liu02], Ex. 4.2.10. I was unable to find a proof in the literature.

Proposition 2.77. *Let $F_1, \dots, F_r \in k[x_0, \dots, x_n]$ be non-zero homogeneous polynomials. Let $X \subseteq \mathbb{P}^n_k$ be the projective scheme defined by the polynomials F_j. Let $P \in X$ be a closed point and let d be the dimension of X at P (i.e. the Krull dimension of the local ring $\mathcal{O}_{X,P}$). The following are equivalent:*

(i) X is smooth at P (i.e. the ring $\mathcal{O}_{X,P}$ is a regular local ring).

(ii) $\mathrm{rk} J_{F_1,\dots F_r}(P) = n - d$.

Proof. Property (i) is invariant under invertible linear changes of coordinates in $\mathbb{P}^n_k$ because it is preserved under isomorphism. Now, if $A : k^{n+1} \to k^{n+1}$ is an invertible linear map (up to scalar it corresponds to a linear invertible change of variables in $\mathbb{P}^n$) then we have by the chain rule (with $\tilde{P} \in k^{n+1}$ an affine representative of P, and $\tilde{Q} \in k^{n+1}$ such that $\tilde{P} = A(\tilde{Q})$)

$$J_{F_1 \circ A,\dots,F_r \circ A}(\tilde{Q}) = J_{F_1,\dots,F_r}(A(\tilde{Q}))J_{A_0,\dots,A_n}(\tilde{Q}) = J_{F_1,\dots,F_r}(A(\tilde{Q}))[A]$$

where $[A]$ is the matrix of A and $A_0, \dots, A_n$ the composition of A with the coordinate projections $k^{n+1} \to k$. Since A is invertible, we get that invertible linear changes of variables do not affect the rank condition from item (ii). Therefore (ii) is invariant under invertible linear changes of coordinates in k^{n+1}.

Therefore we can assume that $P = [1 : 0 : \ldots : 0]$. We fix the affine representative $\tilde{P} = (1, 0, \ldots, 0) \in k^{n+1}$ of P.

Let $f_i \in k[x_1, \ldots, x_k]$ be the polynomial obtained by setting $x_0 = 1$ in F_i (the dehomogenization of F_i with respect to x_0). Let $U_0 = \{x_0 \neq 0\} \subseteq \mathbb{P}^n_k$ so that $U_0 \cong \mathrm{Spec}(k[x_1, \ldots, x_n])$, and observe that under this last isomorphism the point P corresponds to the maximal ideal $m = (x_1, x_2, \ldots, x_n)$ in $k[x_1, \ldots, x_n]$, or equivalently, to the point $\tilde{P}' = (0, \ldots, 0)$ in $\mathbb{A}^n$. Let $U = X \cap U_0$ which is an open neighborhood of P in X, which satisfies $U \cong \mathrm{Spec}(k[x_1, \ldots, x_n])/(f_1, \ldots, f_r)$.

Note that

$$\mathcal{O}_{X,P} \cong \mathcal{O}_{U,P} \cong (k[x_1, \ldots, x_n]/(f_1, \ldots, f_r))_{\bar{m}}$$

where $\bar{m}$ is the image of the maximal ideal m in $k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. By [Mat80] p. 213-214, we know that the local ring $(k[x_1, \ldots, x_n]/(f_1, \ldots, f_r))_{\bar{m}}$ is regular if and only if $\mathrm{rk} J_{f_1,\ldots,f_r}(\tilde{P}') = n - d$, where this Jacobian only involves partial derivatives with respect to $x_1, \ldots, x_n$.

Therefore we deduce that Property (i) holds if and only if $\mathrm{rk} J_{f_1,\ldots,f_r}(\tilde{P}') = n - d$. So we only need to show that $\mathrm{rk} J_{F_1,\ldots,F_r}(P) = \mathrm{rk} J_{f_1,\ldots,f_r}(\tilde{P}')$. By Lemma 2.76 this is equivalent to $\mathrm{rk} J_{F_1,\ldots,F_r}(\tilde{P}) = \mathrm{rk} J_{f_1,\ldots,f_r}(\tilde{P}')$.

Thus, we can assume that $P = [1 : 0 : \ldots : 0] \in X$ and that the affine chart is $U_0 = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_0 \neq 0\}$. Since $P \in X$ we must have $F_j(1, 0, \ldots, 0) = 0$ for each $1 \leq j \leq r$, thus the polynomials F_j do not have monomials of the form αx_0^m with $\alpha \in k$ (otherwise we would get $\alpha = 0$ by evaluating at $\tilde{P} = (1, 0, \ldots, 0)$). Hence all the partial derivatives $\partial F_j / \partial x_0$ vanish at $\tilde{P} = (1, 0, \ldots, 0)$ since all the monomials of F_j are divisible by some of $x_1, \ldots, x_n$. Thus the first column of the $J_{F_1,\ldots,F_r}(\tilde{P})$ is the zero column. The remaining part of the matrix $J_{F_1,\ldots,F_r}(\tilde{P})$ after

deleting the first column is equal to the matrix $J_{f_1,\ldots,f_r}(\tilde{P}')$ (because $\tilde{P} = (1, 0, \ldots, 0)$ and $\tilde{P}' = (0, \ldots, 0)$). We thus obtain that the ranks are the same. □

2.6 Complete intersection schemes

The purpose of this section is to give a result about complete intersection schemes that will be used several times in Chapters 4, 5 and 6.

Definition 2.78. A closed subscheme Y of $\mathbb{P}^n_k$ is called a *complete intersection* if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n_k)$ elements.

Proposition 2.79. *Let $X \subseteq \mathbb{P}^n$ be a smooth complete intersection scheme of dimension $r \geq 1$ defined by $F_1 \ldots F_{n-r}$ homogeneous equations of degree $d_1, \ldots, d_r$. Then X is irreducible and its canonical sheaf is $\mathcal{O}_X(\sum_{i=1}^{n-r} d_i - n - 1)$.*

Proof. Since X is smooth, we have that it is normal. By [Har77], Ex.II.8.4(c) we obtain that X is connected. Since X is smooth and connected we obtain that it is irreducible. Then by [Har77], Ex.II.8.4(e) we have that the canonical sheaf of X is $\mathcal{O}_X(\sum_{i=1}^{n-r} d_i - n - 1)$. □

Proposition 2.80. *Let $C \subseteq \mathbb{P}^n$ be a smooth complete intersection curve defined by $F_1 \ldots F_{n-1}$ homogeneous equations of degree $d_1, \ldots, d_{n-1}$. Then*

$$\begin{aligned} \deg(C) &= \prod_{i=1}^{n-1} d_i, \\ g(C) &= \frac{1}{2}\deg(C)\left(\sum_{i=1}^{n-1} d_i - n - 1\right) + 1. \end{aligned}$$

Proof. Let $i : C \to \mathbb{P}^n$ be the inclusion morphism. Then we get $\deg_C i^*\mathcal{O}_{\mathbb{P}^n}(1) = \deg(C)$ by Remark 2.45. From Proposition 2.79 we have that the canonical sheaf of C is $i^*\mathcal{O}_{\mathbb{P}^n}(\sum_{i=1}^{n-1} d_i - n - 1)$. By [Har77], Example IV.1.3.3, the degree of the canonical sheaf of C is $2g(C) - 2$, so we obtain

$$2g(C) - 2 = (\sum_{i=1}^{n-1} d_i - n - 1)\deg(C).$$

Finally, by [EH00], Theorem III-71 we have $\deg(C) = \prod_{i=1}^{n-1} d_i$, hence proving the proposition. □

Lemma 2.81. *Let k be a field of characteristic zero. For $n \geq 1$ the polynomials $c_0x^n + c_1y^n + c_2 \in k[x, y]$ with $c_0, c_1, c_2 \in k^\times$ are irreducible.*

Proof. The curve C in $\mathbb{P}^2$ defined by $c_0x_0^n + c_1x_1^n + c_2x_2^n = 0$ is a complete intersection. Moreover its matrix $J_F(P)$ (cf. Section 2.5) at a point $[a_0 : a_1 : a_2] \in \mathbb{P}^2$ is $\begin{pmatrix} c_0na_0^{n-1} & c_1na_1^{n-1} & c_2na_2^{n-1} \end{pmatrix}$, which is not zero. Hence C is irreducible by Proposition 2.79, therefore $c_0x_0^n + c_1x_1^n + c_2x_2^n$ is irreducible. Dehomogenizing we obtain that $c_0x^n \pm c_1y^n + c_2$ is irreducible. □

2.7 Ramification

Let k be an algebraically closed field of characteristic 0. Let $\pi : Y \to X$ be a proper dominant morphism of normal integral varieties over k and assume that $\dim(Y) = \dim(X)$. Identify the function field $k(X)$ with the subfield $\pi^*k(X)$ of $k(Y)$. Since Y and X have the same dimension (which equals the transcendence degree of their function fields over k), we obtain that $[k(Y) : k(X)]$ is a finite field extension. Then $\deg(\pi) := [k(Y) : k(X)]$ is defined (and is finite).

Let C be a prime divisor of X and η_C the generic point of C. Then $\mathcal{O}_{X,C} := \mathcal{O}_{X,\eta_C}$ is a discrete valuation of $k(Y)$. Let $\breve{\mathcal{O}}_{X,C}$ be the integral closure of $\mathcal{O}_{X,C}$ in $k(Y)$. The non-zero prime ideals of $\breve{\mathcal{O}}_{X,C}$ are maximal and correspond to the curves D in Y dominating C. Indeed, if D is a prime divisor of Y with $\pi(D) = C$, then we have $\mathcal{O}_{Y,D} \cap k(X) = \mathcal{O}_{X,C}$, and hence $\mathcal{O}_{Y,D} \supseteq \breve{\mathcal{O}}_{X,C}$. Thus $m_{Y,D} \cap \breve{\mathcal{O}}_{X,C} =: \breve{m}_{Y,D}$ is a maximal ideal of $\breve{\mathcal{O}}_{X,C}$. It follows from the valuative criterion of properness [Har77], Theorem II.4.7 (together with the Approximation Theorem, see [Bou72] IV.7.1 Proposition 1) that the map $D \mapsto \tilde{m}_{Y,D}$ is bijective.

Definition 2.82. Let C be a prime divisor of X, and let D be a prime divisor of Y such that $\pi(D) = C$. The *ramification index* of D over C is the number $e_{D/C}(\pi)$ that satisfies

$$m_{X,C}\mathcal{O}_{Y,\eta_D} = m_{Y,D}^{e_{D/C}(\pi)}$$

with $m_{X,C}$ the maximal ideal of $\mathcal{O}_{X,C}$ and $m_{Y,D}$ the maximal ideal of $\mathcal{O}_{Y,D}$.

The *residue degree* of D in the extension $k(Y)/k(X)$ is

$$f_{D/C}(\pi) = [\mathcal{O}_{Y,D}/m_{D,\eta_D} : \mathcal{O}_{X,C}/m_{C,\eta_C}].$$

Note that $f_{D/C}(\pi) = [k(D) : k(C)] = \deg(\pi_{|D})$ because $k(C) = \mathcal{O}_{X,C}/m_{C,\eta_C}$ and similarly for $k(D)$.

Let $m_{Y,D}$ be generated by an element $t \in \mathcal{O}_{Y,\eta_D}$ and let $m_{X,C}$ be generated by an element $s \in \mathcal{O}_{X,\eta_C}$. By definition of $e_{D/C}(\pi)$, we have that $e_{D/C}(\pi) = v_{D|k(X)}(s)$. Then for an element $r = s^a r' \in \mathcal{O}_{X,C}$, (with $r \notin \mathcal{O}_{X,C}^\times$) we have $v_C(r) = a$ and

$$v_{D|k(X)}(r) = v_{D|k(X)}(s^a r') = a v_{D|k(X)}(s) = v_C(r) e_{D/C}(\pi),$$

hence $v_{D|k(X)^\times} = e_{D/C}(\pi)v_C$ as valuations on $k(X)^\times$.

Definition 2.83. If $\pi(D) = C$ is a curve, then we say that π is *unramified* at D if $e_{D/C}(\pi) = 1$. We say that π is *ramified at* D and D is a *ramification divisor* of π if $e_{D/C}(\pi) > 1$. We say that π is *totally ramified* at D if $f_{D/C}(\pi) = 1$ and there is no other curve D' such that $\pi(D') = C$.

Remark 2.84. Note that it follows from Proposition 2.86 below that π is totally ramified at D if and only if $e_{D/C}(\pi) = [k(Y) : k(X)] = \deg(\pi)$.

Proposition 2.85. *Let $\pi : V \to U$ be a finite surjective morphism of smooth surfaces over k. Let $Q \in V$ and put $P = \pi(Q)$. If C is a curve passing through P, then there exists a curve D passing through Q such that $\pi(D) = C$.*

Proof. By [Har77] Ex.III.9.3(a), we know that π is flat. By definition of a flat morphism, we have that $\pi_Q^\# : \mathcal{O}_{U,P} \to \mathcal{O}_{V,Q}$ is flat. From Exercise 11 in Chapter 5 of [AM69], we get that $\pi_Q^\#$ has the going down property. Consider the ideal $\mathfrak{P}_C$ of C in $\mathcal{O}_{U,P}$. By the going down property, since $\pi^{\#-1}(m_{V,Q}) = m_{U,P}$ and $m_{U,P} \subset \mathfrak{P}_C$, then there is a prime ideal $\mathfrak{Q} \subset m_{V,Q}$ in $\mathcal{O}_{V,Q}$ such that $\pi_Q^{\#-1}(\mathfrak{Q}) = \mathfrak{P}_C$. Since $(0) \subset \mathfrak{P}_C \subset m_{U,P}$, we have $(0) \subset \mathfrak{Q} \subset m_{V,Q}$, hence there is a curve D passing through Q and having ideal $\mathfrak{Q}$. Since $\pi_Q^{\#-1}(\mathfrak{Q}) = \mathfrak{P}_C$ we have that the generic point of D is mapped to the generic point of C, thus $\pi(D) = C$ because π is finite, hence closed, by [Har77], Ex.II.3.5(c). □

Proposition 2.86. *Let $\pi : Y \to X$ be a proper dominant morphism of normal varieties, and let $C \subset X$ be a prime divisor in X. Let $D_1, \ldots, D_n \in Y$ be all the*

prime divisors in Y such that $\pi(D_i) = C$. Then

$$\deg(\pi) = \sum_i e_{D_i/C} f_{D_i/C}.$$

Proof. From the discussion at the beginning of this section, we know that since π is a proper and dominant morphism of normal varieties, the curves $D_1, \ldots, D_n$ are in bijection with the maximal ideals of $\breve{\mathcal{O}}_{X,\eta_C}$. Now the result follows from [Ser79] Chap.I, §4 Proposition 10, applied to the ring extension $\mathcal{O}_{X,C} \subseteq \breve{\mathcal{O}}_{X,\eta_C}$. □

Lemma 2.87. *Let $f : Y \to X$ be a dominant generically finite morphism of varieties over k and suppose that* $\mathrm{char}(k) = 0$. *Let $n = \deg(f)$. Then there is a non-empty open set $U \subseteq X$ such that we have $\#f^{-1}(P) = n$, for every closed point $P \in X$.*

Proof. Let $V \subseteq X$ be an open set such that V is smooth and $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is finite. (This is possible by [Har77], Cor.II.8.16 and [Har77] Ex.II.3.7.) Note that $k(Y)/k(X)$ is a separable field extension because $\mathrm{char}(k) = 0$, and note also that V is normal. So we can use Theorem 2.29 in [Sha13] to get a non-empty $U \subseteq V$ which satisfies the conditions. □

Proposition 2.88. *Let $\pi : Y \to X$ be a proper dominant generically finite morphism of surface over k. Let D be an irreducible curve on Y and let $C = \pi(D)$ be a curve on X. Suppose that on a nonempty open set $U \subset C$ we have that for any closed point in U the preimage $\pi^{-1}(P)$ has only one element. Then π is totally ramified at C.*

Proof. We know that D lies in the preimage of C by π, because $C = \pi(D)$. Note that both D and C are irreducible. Since each closed point of U has only one preimage by π, we obtain that D is the only preimage of C. Indeed, if $D' \neq D$ is another curve

mapping onto C, then $D \cap D'$ is finite, so $U \backslash \pi(D \cap D') \neq \emptyset$ and each $P \in U \backslash \pi(D \cap D')$ would have at least two distinct pre-images.

From Lemma 2.87 we obtain that $\deg(\pi_{|D}) = 1$. Moreover $[k(D) : k(C)] = 1$ because $\deg(\pi_{|D}) = 1$. From this we get $f_{D/C}(\pi) = 1$. From Proposition 2.86, we have $\deg(\pi) = e_{D/C}(\pi) f_{D/C}(\pi) = e_{D/C}(\pi)$, thus π is totally ramified at D. □

Proposition 2.89. *Let $\pi : Y \to X$ be a proper dominant morphism of normal surfaces over k. Let U be an open set of X such that for $V = \pi^{-1}U$ we have that $\pi_{|V}$ is finite and $\#\pi^{-1}(P) = \deg(\pi)$ for each P in U. Then π is unramified at any curve whose image intersects U.*

Proof. Let C be a curve which intersects U and let $P \in U \cap C$ be a point on C. Let $D_1, \ldots, D_n$ be all the curves on Y such that $\pi(D_i) = C$. By definition, we have $f_{D_i/C}(\pi) = [k(D_i) : k(C)]$, which is also the number of preimages of P contained in D_i, for general $P \in U \cap C$ (cf. Lemma 2.87 applied to $\pi_{|D_i}$). Note that each preimage of P is at least in one of these curves, by Proposition 2.85 applied to the open set $U \subset X$, $V = \pi^{-1}(U)$. Hence we obtain $\sum_{D_i} f_{D_i/C}(\pi) \geq \#\pi^{-1}(P) = \deg(\pi)$. From Proposition 2.86 we have $\deg(\pi) = \sum_{D_i} e_{D_i/C}(\pi) f_{D_i/C}(\pi)$ which can only hold if $e_{D_i/C}(\pi) = 1$ for all D_i. Therefore π is unramified at each D_i. □

2.8 Diagonal quotient surfaces

In this section, we give some properties of a special case of the diagonal quotient surfaces defined in [KS97]. The surface parametrizing cuboids (see Equation 2.2) is a concrete example of these surfaces.

Definition 2.90. Let X be a variety and $G \leq \mathrm{Aut}(X)$. A *quotient of X by the*

action of G is a pair (Y, π) with Y a variety and $\pi : X \to Y$ a morphism satisfying the following conditions.

(i) For all $g \in G$, $\pi \circ g = \pi$.

(ii) If $\pi' : X \to Y'$ satisfies (i), then there exists a unique morphism $\phi : Y \to Y'$ such that $\pi' = \phi \circ \pi$.

Notation 2.91. If Y exists, then we denote it by $G\backslash X$.

Remark 2.92. A quotient $G\backslash X$ exists when X is a quasi-projective variety and G is a finite group (see [Ser56], §3 and [Gro71] V.1). It is unique up to isomorphism by conditions (i) and (ii), and the morphism π is finite.

Let C be a amooth projective curve, let $G \leq \mathrm{Aut}(C)$ be a subgroup of order $m = |G|$, and let $\pi : C \to G\backslash C$ be the quotient map.

Let $Y := C \times C$ be the product surface, let $Z_G := \Delta_G\backslash Y$ where

$$\Delta_G = \{(g, g) : g \in G\} \leq G \times G$$

is the diagonal subgroup. We call Z_G a *diagonal quotient surface*. Note that Z_G is normal since $C \times C$ is normal (cf. [MF82], p. 5).

Let $\phi : Y \to Z_G$ be the quotient map. Note that

$$\pi \times \pi : C \times C \to G\backslash C \times G\backslash C =: \bar{Y}$$

is the quotient map associated to $(G \times G)\backslash(C \times C)$. From the universal property of ϕ we can define $\psi : Z_G \to \bar{Y} := G\backslash C \times G\backslash C$ as the unique map such that $\pi \times \pi = \psi \circ \phi$.

Put $\psi_i := \mathrm{pr}_i^{\bar{Y}} \circ \psi$ for $i = 1, 2$, with $\mathrm{pr}_i^{\bar{Y}} : \bar{Y} \to G\backslash C$ the i-th projection map.

For $y = (x_1, x_2) \in Y$, put

$$\begin{aligned} G_{x_i} &= \{g \in G : gx_i = x_i\} \\ G_y &= G_{x_1} \cap G_{x_2} \\ S_Y &= \{y \in Y : |G_y| > 1\} \\ S_{Z_G} &= \phi(S_Y). \end{aligned}$$

Proposition 2.93. *The set of singularities of Z_G is exactly S_{Z_G} and each singularity s is a cyclic quotient singularity of type $A_{(n,q)}$ (as defined in [BPV84], p. 82), with $n = |G_s|$ and $1 \leq q < n$ satisfying* $\gcd(n, q) = 1$.

Proof. See [KS97], Theorem 2.3 (a). Note that q can be explicitly computed, but we will only need the fact that $1 \leq q < n$ ($q \in \mathbb{Z}$). □

Proposition 2.94. *Let $x \in C$ and $\bar{x} = \pi(x) \in G\backslash C$. Then*

$$C_{\bar{x},i} := \psi_i^*(\bar{x})_{red} = \phi((\mathrm{pr}_i^{\bar{Y}})^{-1}(\bar{x}))$$

with $i = 1, 2$ is a smooth irreducible curve on Z_G. Moreover, $C_{\bar{x},i} \cong G_x\backslash C$ for $x \in \pi^{-1}(\bar{x})$ and $i = 1, 2$. In particular, $C_{\bar{x},i} \cong C$ if $|G_x| = 1$.

Proof. See [KS97], Proposition 2.1 (b). □

2.9 Varieties over function fields

In this section we discuss the relation between morphisms from curves to projective varieties, and solutions to homogeneous equations with coordinates in the function

field of a curve.

Let C/k be a smooth projective curve defined over an algebraically closed field k of characteristic 0, and let $K = k(C)$ be its function field.

Let n be a positive integer. Given any non-zero $\mathbf{h} = (h_0, \ldots, h_n) \in K^{n+1}\backslash\{0\}$ we have a rational map

$$\Psi_{n,C,\mathbf{h}} : C \dashrightarrow \mathbb{P}^n_k, \quad P \mapsto [h_0(P) : \ldots : h_n(P)].$$

Lemma 2.95. *For each $\mathbf{h} \in K^{n+1}\backslash\{0\}$, the rational map $\Psi_{n,C,\mathbf{h}} : C \to \mathbb{P}^n$ defines a morphism. Moreover, for any $\mathbf{h}, \mathbf{g} \in K^{n+1}\backslash\{0\}$ we have $\Psi_{n,C,\mathbf{h}} = \Psi_{n,C,\mathbf{g}}$ if and only if there is $u \in K^*$ with $\mathbf{h} = u\mathbf{g}$.*

Proof. The map $\Psi_{n,C,\mathbf{h}}$ is defined (at least) on the non-empty open set $U_{\mathbf{h}} \subseteq C$ where none of the h_i has a pole, and not all the h_i are zero simultaneously. Hence, $\Psi_{n,C,\mathbf{h}}$ is defined at the generic point of C (which is contained in every non-empty open set of C because C is irreducible). Let $P \in C$ be a closed point. Since C is a smooth curve we see that $\mathcal{O}_{C,P}$ is a regular local ring of dimension 1, hence a discrete valuation ring. Note that $\mathbb{P}^n_k$ is proper because it is projective. Therefore, using Theorem II.4.7 in [Har77] as in the proof of Lemma V.5.1 in [Har77], we get that $\Psi_{n,C,\mathbf{h}}$ is defined at P. This proves that $\Psi_{n,C,\mathbf{h}}$ is defined everywhere on C, hence it is a morphism.

For the second part, suppose first that there is $u \in K^*$ with $\mathbf{h} = u\mathbf{g}$. Then $\Psi_{n,C,\mathbf{h}}$ and $\Psi_{n,C,\mathbf{g}}$ agree at least on the non-empty open set $U = U_{\mathbf{h}} \cap U_{\mathbf{g}} \cap U_u$ where U_u is the open set where u does not have poles or zeros, because for $P \in U$ we have that $u(P) \in k^*$, hence

$$[h_0(P) : \ldots : h_n(P)] = [u(P)g_0(P) : \ldots : u(P)g_n(P)] = [g_0(P) : \ldots : g_n(P)].$$

Since C is irreducible and $\Psi_{n,C,\mathbf{h}}$ and $\Psi_{n,C,\mathbf{g}}$ agree on a non-empty open set of C, they are equal by [Har77], Lemma I.4.1.

Conversely, suppose that $\Psi_{n,C,\mathbf{h}}$ and $\Psi_{n,C,\mathbf{g}}$ are equal. Then at every closed point $P \notin U_{\mathbf{h}} \cap U_{\mathbf{g}}$ we have

$$(h_0(P), \ldots, h_n(P)) = (\lambda_P g_0(P), \ldots, \lambda_P g_n(P))$$

(in k^{n+1}) for some $\lambda_P \in k^*$. Let j be an index for which $h_j \neq 0$ (and hence $g_j \neq 0$). Then the rational function $u = g_j/h_j \in K^*$ satisfies what we want, because $u(P) = \lambda_P$ at every closed point $P \in U_{\mathbf{h}} \cap U_{\mathbf{g}}$, thus $h_i(P) = u(P)g_i(P)$ on $U_{\mathbf{h}} \cap U_{\mathbf{g}}$, and since C is integral we get by [Har77], Remark I.3.1.1 that $h_i = ug_i$ on C for all i. □

Lemma 2.96. *If $f : C \to \mathbb{P}^n$ is a morphism, then $f = \Psi_{n,C,\mathbf{h}}$ for some $\mathbf{h} \in K^{n+1} \setminus \{0\}$.*

Proof. Let $0 \leq i \leq n$ be such that $f(C)$ is not contained in $\{x_i = 0\}$. To simplify the notation we can assume $i = 0$. Then f restricts to a rational function $f' : C \dashrightarrow \mathbb{A}^n$. Let $h_1, \ldots, h_n \in k(C)$ be the rational functions obtained by composing f' with the coordinate projections of $\mathbb{A}^n$. Let $h_0 = 1$, then $\Psi_{n,C,\mathbf{h}}$ agrees with f on the domain of f', which is a non-empty open subset of C. Since $\Psi_{n,C\mathbf{h}}$ is a morphism, we have that $f = \Psi_{n,C,\mathbf{h}}$ by [Har77], Remark I.3.1.1. □

Proposition 2.97. *Let C/k be a smooth projective curve defined over k, and let $K = k(C)$ be its function field. Let $X \subseteq \mathbb{P}^n_k$ be a projective variety defined over k by homogeneous polynomials $F_1, \ldots, F_r \in k[x_0, \ldots, x_n]$. The rule $\mathbf{h} \mapsto \Psi_{n,C,\mathbf{h}}$ induces a bijection $\tilde{\Psi}_{n,C}$ between the following sets:*

- $(n+1)$-*tuples* $\mathbf{h} = (h_0, \ldots, h_n) \in K^{n+1} \setminus \{(0, \ldots, 0)\}$ *(up to simultaneous multiplication by an element of* K^**), such that*

$$\begin{aligned} F_1(h_0, \ldots, h_n) &= 0 \\ &\vdots \\ F_r(h_0, \ldots, h_n) &= 0. \end{aligned}$$

- *Morphisms* $f : C \to X$*.*

Moreover, under this bijection, $(n+1)$*-tuples that are of the form* $(\lambda_0 u, ..., \lambda_n u)$ *with* $u \in K^*$ *and* $\lambda_i \in k$ *(constants) correspond to constant morphisms* $f : C \to X$*.*

Proof. If $r = 0$, then $X = \mathbb{P}^n$, thus Lemma 2.95 shows that this rule is injective, and by Lemma 2.96 it is surjective.

If $\mathbf{h} = (\mathbf{h_0}, \ldots, \mathbf{h_n}) \in \mathbf{K^n}$ is a solution of the system of equations defining X, then for every P in an open set of C we have that the point $[h_1(P) : \ldots : h_n(P)]$ is in X, therefore $\Psi_{n,C,\mathbf{h}}$ has image in X, and since C and X are reduced, it induces a unique morphism from C to X by Exercise II.3.11(d) in [Har77].

Conversely, if $f = \Psi_{n,C,\mathbf{X}} : C \to X$, then for every P in an open set (away from the poles of the h_i and the common zeroes of the h_i) of C we have that $h_0(P), \ldots, h_n(P)$ is a solution of the system of equations defining X. Therefore $h_0, \ldots, h_n$ is a solution of $F_1 = 0, \ldots, F_n = 0$.

The final part about constant morphisms is clear. □

2.10 Tangent spaces

Let X be a variety over an algebraically closed field k and let $P \in X$ be a closed point. The *tangent space* of X at P is defined as (cf. [Har77] Ex.II.2.8)

$$T_P X = \mathrm{Hom}_k(m_{X,P}/m_{X,P}^2, k).$$

Recall that $m_{X,P}/m_{X,P}^2$ is a k-vector space and that moreover $T_P X \cong (\Omega^1_{X/k,P} \otimes k)^\vee$ (take duals in [Har77] Proposition II.8.7). Therefore Theorem II.8.8 in [Har77] gives:

Proposition 2.98. *The k-vector space $T_P X$ has dimension $\dim X$ if and only if P is a smooth point of X.*

Morphisms of varieties induce maps in tangent spaces in a covariant way, as follows: Let $f : X \to Y$ be a morphism of varieties and let $Q = f(P) \in Y$. Since f is a morphism, we get a map $f_Q^\# : m_{Y,Q} \to m_{X,P}$ on the maximal ideals by restricting to the maximal ideals the induced map $f_Q^\# : \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$. Concretely, if $\phi \in m_{Y,Q} \subseteq \mathcal{O}_{Y,Q}$ is a regular function near Q which vanishes at Q, then $f_Q^\#(\phi) = \phi \circ f$ is a regular function near P which vanishes at P. Thus $f_Q^\#(\phi)$ is an element of $m_{X,P}$. Taking the quotient by the squares of these ideals we get an induced map of k-vector spaces

$$\bar{f}_Q^\# : m_{Y,Q}/m_{Y,Q}^2 \to m_{X,P}/m_{X,P}^2$$

and taking duals as k-vector spaces, we obtain the *differential map of f* or the *induced map in tangent spaces*

$$T_P[f] : T_P X \to T_Q Y.$$

Note that $T_P[f]$ is usually denoted by df, but in this context this can lead to confusions

so we denote it differently. Given $t \in T_P X$, the element $T_P[f](t) \in T_Q Y$ is $t \circ \bar{f}_Q^{\#}$. In other words, $T_P[f](t)$ is the linear functional that takes $\phi \mod m_{Y,Q}^2$ to $t(\phi \circ f \mod m_{X,P}^2)$. The following property follows from the definition and from the properties of $f_Q^{\#}$.

Proposition 2.99. *Given $f : X \to Y$ and $g : Y \to Z$, let $P \in X$, $Q = f(P) \in Y$ and $R = g(Q) \in Z$. Then $T_P[g \circ f] = T_Q[g] \circ T_P[f]$ as k-linear maps $T_P X \to T_R Z$.*

It is trivial to see that $T_P[\mathrm{Id}_X] = \mathrm{Id}_{T_P X}$. From this we have the following

Proposition 2.100. *If $f : X \to Y$ is an isomorphism of varieties, then we have that $T_P[f] : T_P X \to T_Q Y$ is an isomorphism of k-vector spaces.*

Proposition 2.101. *Suppose that $f : X \to Y$ is the inclusion of X in Y as a subvariety, so that $Q = P \in X \subseteq Y$ in this case. Then $T_P[f] : T_P X \to T_P Y$ is injective.*

Proof. The inclusion of the closed subvariety $i : Y \to X$ corresponds to a closed immersion with the structure of reduced scheme (cf. [Har77], Examples II.3.2.5 and II.3.2.6) hence it induces a surjective sheaf morphism $f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$. Given $P \in Y \subseteq X$ we then obtain a surjective induced map of local rings $f_P^{\#} : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,P}$ hence it induces a surjection of the corresponding maximal ideals. Thus the induced map $m_{X,P}/m_{X,P}^2 \to m_{Y,P}/m_{Y,P}^2$ is surjective, and hence its dual is injective. □

Proposition 2.102. *Suppose that $f : X \to Y$ is a constant map, that is, $f(X) = \{Q\}$ is a point. Then the map $T_P[f] : T_P X \to T_Q Y$ is the zero map of vector spaces.*

Proof. Note that f factors over $T_Q f(X) = \{0\}$. □

Recall from [Har77] p. 357, that if X is a smooth surface and $P \in X$, we say that two curves C, D in X *meet transversally at* P if the local equation $f, g \in \mathcal{O}_{X,P}$ of C, D generate the ideal $m_{X,P}$. In this case $\mathcal{O}_{C,P} = \mathcal{O}_{X,P}(f)$ and $\mathcal{O}_{D,P} = \mathcal{O}_{X,P}/(g)$, and the inclusions i, j of C, D in X induce the corresponding quotient maps on local rings. We say that C, D *meet transversally* if they meet transversally at every point $P \in C \cap D$.

By [Har77], if C, D meet transversally then $(C.D) = \#C \cap D$.

Lemma 2.103. *We have that C, D meet transversally at a point $P \in C \cap D$ if and only if $T_P X = T_P[i](T_P C) \oplus T_p[j](T_P D)$.*

Proof. The direct implication is proved in [Sha13] Theorem 2.4, and the converse follows from Exercise 2 in [Sha13] p. 111. We remark that in [Sha13], the condition on tangent spaces is taken as definition of transverse intersection and they prove the equivalence with the condition on local equations (which is what [Har77] takes as definition). □

2.11 Intersection with a contracted curve

When working with desingularizations, we will need to compute the intersection numbers of a strict transform and exceptional divisors. This will be useful in Chapter 6. For this purpose, the next result will be helpful. I thank Cesar Lozano Huerta for giving me the idea of the proof.

Proposition 2.104. *Let $f : S \to S'$ be a birational morphism of projective surfaces, and suppose that S is smooth and irreducible, and S' is normal. Let $D \subseteq S$ be an irreducible curve on S such that $C = f(D)$ is a curve on S'. Let $P \in C$ be a point such that*

(i) $E_P = f^{-1}(P)$ *is a smooth irreducible curve, and*

(ii) P *is smooth on* C*.*

Then E_P *and* D *meet transversely at a unique single point, and hence* $(D.E_P) = 1$*.*

Proof. Since S' and S are projective, we get that the morphism f is proper. We have $1 = \deg(f) = e_{D/C}(f)f_{D/C}(f)$, by Proposition 2.86. From this we get $f_{D/C}(f) = 1$, thus $f_D := f_{|D} : D \to C$ is birational. Since C is smooth at P, there exists an open neighbourhood $U_P \subset C$ of P such that U_P is normal, and then $f_{|f_D^{-1}(U_P)} : f_D^{-1}(U_P) \to U_P$ is an isomorphism (cf. the proof of [Har77], Proposition I.6.7). Thus, since

$$E_P \cap D = f^{-1}(P) \cap D = f_D^{-1}(P),$$

we see that $E_P \cap D = \{Q\}$ is a single point.

Write $E := E_P$. From the previous paragraph it follows that E and D meet only at one point.

We only need to show that the intersection of E and D at Q is transverse.

Note that D, E, S are all smooth at Q so the tangent spaces at Q have dimensions $1, 1, 2$ respectively. By Lemma 2.103, it is enough to show that

$$T_Q S = T_Q[i](T_Q D) \oplus T_Q[j](T_Q E)$$

where $i : D \to S$ and $j : E_P \to S$ are the inclusions. (Note that by Proposition 2.101 each summand is isomorphic to $T_Q D, T_Q E$ respectively.) Since we know the dimensions (namely, 2, 1, 1 respectively) of these spaces, if the intersection is *not*

transverse, then the previous direct sum fails and the only thing that can happen is

$$T_Q[i](T_QD) = T_Q[j](T_QE)$$

inside T_QS. We will show that actually these two subspaces *are different*, which will conclude the proof.

Since E is contracted to a point P, the map $fj : E \to S'$ is constant and we get that $T_Q[fj]$ is the zero map. Hence

$$T_Q[f](T_Q[j](T_QE)) = T_Q[fj](T_QE) = \{0\}.$$

On the other hand, $f \circ i : D \to C$ is an isomorphism near Q and we get that $T_Q[fi] : T_QD \to T_QC$ is an isomorphism, in particular surjective. Thus

$$T_Q[f](T_Q[i](T_QD)) = T_Q[fi](T_QD) = T_QC \neq \{0\},$$

where the inequality holds because C is smooth at P, hence the space T_QC has dimension $\dim C = 1$. Therefore the vector spaces $T_Q[i](T_QD)$ and $T_Q[j](T_QE)$ are different 1-dimensional subspaces of T_QS because they have different images under $T_Q[f]$. □

Chapter 3

Explanation of the technique

The purpose of this chapter is to present a general method from Vojta's work [Voj00] and to provide complete proofs for some of his claims. We do this in an algebraic setting, in a level of generality that will be convenient for improvements and concrete new applications in later chapters of this thesis.

3.1 Sketch of Vojta's method

In 1977, F. Bogomolov [Bog10] (see [Des79]) proved that on a surface of general type, which satisfies the inequality $c_1^2 > c_2$ of Chern numbers, there are only finitely many curves of genus 0 or 1.

In 2000, P. Vojta [Voj00] was able to determine the complete list of curves of genus zero and one for Büchi's surfaces $B_n \subseteq \mathbb{P}^n$ for $n \geq 8$. We recall from Subsection 2.1.2 that the surface B_n is defined by the following equations

$$B_n : \begin{cases} 2x_0^2 = x_1^2 - 2x_2^2 + x_3^2 \\ \quad \vdots \\ 2x_0^2 = x_{n-2}^2 - 2x_{n-1}^2 + x_n^2 \end{cases}$$

These surfaces are smooth, and are of general type for $n \geq 6$. Moreover, for $n \geq 10$ they satisfy $c_1^2 > c_2$ (cf. [Voj00], p. 264). What he proves is the following:

Theorem 3.1. *Let $n \geq 8$. The only curves of genus 0 or 1 on B_n are the 2^n curves given by the equations*

$$\pm x_1 = \pm x_2 - x_0 = \cdots = \pm x_n - (n-1)x_0.$$

The strategy used by Vojta is related to that of Bogomolov in that both study symmetric powers $S^r\Omega^1_{X/\mathbb{C}}$ of the canonical sheaf of a surface X. But whereas Bogomolov studies curves on the 3-fold $\mathbb{P}(\Omega^1_{X/\mathbb{C}}) = \mathrm{Proj}(\oplus_{r\geq 0}\Omega^r_{X/\mathbb{C}})$, Vojta studies curves directly on X. This has the advantage of leading to more effective results. We give a brief sketch of his strategy.

He works with the notion of an ω-integral curve which is defined as follows.

Definition 3.2. Let X be a smooth variety over a field of characteristic zero, let $\mathcal{L}$ be an invertible sheaf on X and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_X)$, where r is an integer. An irreducible curve C on X is said to be *ω-integral* if the image of the section $\varphi_C^*\omega$ in $H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}})$ is zero, where $\varphi_C : \tilde{C} \to X$ is the normalization of $C \subset X$.

By choosing a particularly convenient $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_X)$, Vojta is able to compute the *complete* list of ω-integral curves in X by using local information (cf. Section 3.7). Roughly speaking, he translates the condition of ω-integrality into solutions of differential equations. Then he finds some solutions for the equations and uses a local analysis to show that there are no other solutions. Then he shows, using global cohomological arguments, that every curve of genus 0 or 1 is ω-integral, when ω has been chosen suitably (cf. Section 3.3). Since the complete list of ω-integral

curves on X is known, one knows the complete list of curves of genus 0 or 1 on X.

In the particular case of the surfaces B_n, Vojta has an infinite tower of morphisms

$$\mathbb{P}^2 = B_2 \overset{\pi_2}{\leftarrow} B_3 \overset{\pi_3}{\leftarrow} B_4 \overset{\pi_4}{\leftarrow} \cdots$$

He finds the complete list of ω-integral curves on $B_2 \cong \mathbb{P}^2$ for a suitable section ω. By using pullbacks of ω, he is able to find a suitable section ω_n in B_n (for n large enough) such that every curve of genus 0 or 1 is ω_n-integral. By calculating the components of the pullbacks of the ω-integral curves in B_2, he finds the complete list of ω_n-integral curves on B_n without using local arguments on B_n.

Remark 3.3. The image of $\varphi_C^*\omega$ in $H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}})$ will be denoted by $\varphi^{\bullet}_{C,r,\mathcal{L}}\omega$ (cf. Section 3.2).

3.2 Functoriality and ω-integral curves

Let $\pi : X' \to X$ be a dominant morphism of surfaces. In this section, we will study the relation between ω-integral curves on X and "$\pi^*\omega$-integral curves" on X'. This explains the first statement in the proof of Lemma 2.9 in [Voj00].

I thank E. Kani for suggesting to me the current presentation based on functorial properties, and for explaining to me many valuable ideas. Several intermediate results in Subsections 3.2.1, 3.2.2 and 3.2.4 are due to Kani [Kan14c], [Kan14d] and [Kan15].

3.2.1 The functors f_* and f^*

Proposition 3.4. *Let X, Y, Z be schemes, and let $f : X \to Y$ and $g : Y \to Z$ be morphisms. The functors $g_* \circ f_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Z}$ and $(g \circ f)_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Z}$*

are equal.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By definition,

$$(g \circ f)_* \mathcal{F}(U) = \mathcal{F}((g \circ f)^{-1}(U)) = \mathcal{F}(f^{-1}g^{-1}(U)) = f_* \mathcal{F}(g^{-1}(U)) = g_* f_* \mathcal{F}.$$

□

Remark 3.5. Let $f : X \to Y$ be a morphism of schemes. From [EGA] (0_{I},4.3) and (0_{I},4.4) we obtain that if $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then the covariant functor

$$H^f_{\mathcal{G}} : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Ab}$$

defined by $H^f_{\mathcal{G}}(\mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_* \mathcal{F})$ is representable, i.e. there exists a pair $(f^* \mathcal{G}, \rho^f_{\mathcal{G}})$, where $f^* \mathcal{G}$ is an $\mathcal{O}_X$-module and $\rho^f_{\mathcal{G}} : \mathcal{G} \to f_* f^* \mathcal{G}$ is an $\mathcal{O}_Y$-homomorphism such that for any $\mathcal{O}_X$-module $\mathcal{F}$, the rule $v \mapsto H^f_{\mathcal{G}}(v)(\rho^f_{\mathcal{G}}) = f_*(v) \circ \rho^f_{\mathcal{G}}$ defines an isomorphism

$$\begin{array}{rccl} h^*_{f,\mathcal{G}} : \mathrm{Hom}_X(f^* \mathcal{G}, \mathcal{F}) & \to & \mathrm{Hom}_Y(\mathcal{G}, f_* \mathcal{F}) \\ u & \mapsto & f_*(u) \circ \rho^f_{\mathcal{G}}. \end{array}$$

We obtain a covariant functor

$$f^* : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_X}$$

by defining for $v \in \mathrm{Hom}_Y(\mathcal{G}_1, \mathcal{G}_2)$ the homomorphism $f^* v$ being the unique one making

the following diagram commutative:

$$\begin{array}{ccc} f_*f^*\mathcal{G}_1 & \xrightarrow{f_*(f^*v)} & f_*f^*\mathcal{G}_2 \\ {\scriptstyle \rho^f_{\mathcal{G}_1}}\uparrow & & \uparrow{\scriptstyle \rho^f_{\mathcal{G}_2}} \\ \mathcal{G}_1 & \xrightarrow[v]{} & \mathcal{G}_2. \end{array} \tag{3.1}$$

For $\mathcal{F}$ a $\mathcal{O}_X$-module, we call $\rho^f_{\mathcal{F}} : \mathcal{F} \to f_*f^*\mathcal{F}$ the *canonical map*.

Remark 3.6. From [EGA] (0_{I},4.3.3), one can construct $(f^*\mathcal{G}, \rho^f_{\mathcal{G}})$ in such a way that: if s is a section of $\mathcal{G}$ on an open set V of Y, then $\rho_{\mathcal{G}}(s)$ is the section $s' \otimes 1$ of $f^*\mathcal{G}$ on $f^{-1}(V)$, that is in $f_*f^*\mathcal{G}(V)$, with s' such that $s'_x = s_{f(x)}$ for all $x \in f^{-1}(V)$.

Lemma 3.7. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{H}_i$ be sheaves of abelian groups on X. Then $f^*(\oplus_i \mathcal{H}_i) = \oplus_i f^*\mathcal{H}_i$.*

Proof. See [EGA] (0_{I},4.3.2.1). □

Lemma 3.8. *Let $f : X \to Y$ be a morphism of schemes and suppose that X and Y have noetherian topological spaces. Let $\mathcal{H}_i$ be sheaves of abelian groups on X. Then $f_*(\oplus_i \mathcal{H}_i) = \oplus_i f_*\mathcal{H}_i$.*

Proof. By [Har77], Ex.II.1.11, the sheaf direct sum is equal to the presheaf direct sum for noetherian topological spaces. Thus for any $U \subset Y$ we have

$$(f_* \oplus_i \mathcal{H}_i)(U) = (\oplus_i \mathcal{H}_i)(f^{-1}(U)) = \oplus_i(\mathcal{H}_i)(f^{-1}(U)) = (\oplus f_*\mathcal{H}_i)(U) = \oplus_i(f_*\mathcal{H}_i)(U).$$

□

From this point, we assume in this Thesis that all schemes are noetherian.

Proposition 3.9. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $\mathcal{H}$ is an $\mathcal{O}_Z$-module, then we have that $H^{g\circ f}_{\mathcal{H}} = H^g_{\mathcal{H}} \circ f_*$. There is a unique isomorphism $\varphi^{f,g}_{\mathcal{H}} : (g \circ f)^*\mathcal{H} \to f^*g^*\mathcal{H}$ of $\mathcal{O}_X$-modules such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\rho^{g\circ f}_{\mathcal{H}}} & (g\circ f)_*(g\circ f)^*\mathcal{H} \\ {\scriptstyle \rho^g_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle (g\circ f)_*\varphi^{f,g}_{\mathcal{H}}} \\ g_*g^*\mathcal{H} & \xrightarrow[g_*\rho^f_{g^*\mathcal{H}}]{} & (g\circ f)_*f^*(g^*\mathcal{H}). \end{array}$$

Moreover, $\left\{\varphi^{f,g}_{\mathcal{H}}\right\}_{\mathcal{H}} : (g \circ f)^* \to f^*g^*$ defines an isomorphism of functors.

Proof. From Proposition 3.4, for a $\mathcal{O}_X$-module $\mathcal{F}$ we have that

$$H^{g\circ f}_{\mathcal{H}}(\mathcal{F}) = \mathrm{Hom}_Z(\mathcal{H}, (g\circ f)_*\mathcal{F}) = \mathrm{Hom}_Z(\mathcal{H}, g_*f_*\mathcal{F}) = H^g_{\mathcal{H}}(f_*\mathcal{F}) = (H^g_{\mathcal{H}} \circ f_*)(\mathcal{F}).$$

This proves the first assertion.

We know that $H^{g\circ f}_{\mathcal{H}}$ is represented by $((g \circ f)^*\mathcal{H}, \rho^{g\circ f}_{\mathcal{H}})$. We want to prove that $H^{g\circ f}_{\mathcal{H}}$ is also represented by $(f^*g^*\mathcal{H}, g_*\rho^f_{g^*\mathcal{H}} \circ \rho^g_{\mathcal{H}})$. Since for $v \in \mathrm{Hom}_X(f^*g^*\mathcal{H}, \mathcal{F})$ we have

$$\begin{aligned} H^{g\circ f}_{\mathcal{H}}(v)(g_*\rho^f_{g^*\mathcal{H}} \circ \rho^g_{\mathcal{H}}) &= (g\circ f)_*(v)(g_*\rho^f_{g^*\mathcal{H}} \circ \rho^g_{\mathcal{H}}) \\ &= g_*(f_*v) \circ (g_*\rho^f_{g^*\mathcal{H}}) \circ \rho^g_{\mathcal{H}} \\ &= h^*_{g,\mathcal{H}}(h^*_{f,g^*\mathcal{H}}(v)), \end{aligned}$$

we obtain that $H^{g\circ f}_{\mathcal{H}}(v)(g_*\rho^f_{g^*\mathcal{H}} \circ \rho^g_{\mathcal{H}})$ is an isomorphism because it is a composition of the isomorphisms $h^*_{g,\mathcal{H}}$ and $h^*_{f,g^*\mathcal{H}}$. Therefore the pair $(f^*g^*\mathcal{H}, g_*\rho^f_{g^*\mathcal{H}} \circ \rho^g_{\mathcal{H}})$ also represents $H^{g\circ f}_{\mathcal{H}}$. By properties of representable functors (cf. [EGA] (0_I,1.1.8) we get

that there is a unique isomorphism

$$\varphi_{\mathcal{H}}^{f,g} : (g \circ f)^* \mathcal{H} \to f^* g^* \mathcal{H}$$

which satisfies the commutativity condition.

Since $(g \circ f)^*$ and $f^* g^*$ are both left adjoints of $(g \circ f)_*$, we get that

$$\varphi^{f,g} : (g \circ f)^* \xrightarrow{\sim} f^* g^*$$

defines an isomorphism of functors. Thus for $u \in \mathrm{Hom}_Z(\mathcal{H}_1, \mathcal{H}_2)$ we have the following commutative diagram

$$\begin{array}{ccc} (g \circ f)^* \mathcal{H}_1 & \xrightarrow{\varphi_{\mathcal{H}_1}^{f,g}} & f^* g^* \mathcal{H}_1 \\ {\scriptstyle (g\circ f)^*(u)}\downarrow & & \downarrow{\scriptstyle g^*(f^*(u))} \\ (g \circ f)^* \mathcal{H}_2 & \xrightarrow[\varphi_{\mathcal{H}_2}^{f,g}]{} & f^* g^* \mathcal{H}_2. \end{array}$$

□

3.2.2 The symmetric algebra

Definition 3.10. Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. The *symmetric algebra of* $\mathcal{E}$ is a pair $(S_X(\mathcal{E}), \phi_{\mathcal{E}})$, where $S_X(\mathcal{E})$ is a sheaf of commutative $\mathcal{O}_X$-algebras on X and $\phi_{\mathcal{E}} : \mathcal{E} \to S_X(\mathcal{E})$ is a sheaf homomorphism of $\mathcal{O}_X$-modules, such that it satisfies the following universal property: For each commutative $\mathcal{O}_X$-algebra $\mathcal{B}$ and $\mathcal{O}_X$-module homomorphisms $f : \mathcal{E} \to \mathcal{B}$, there exists a unique $\mathcal{O}_X$-ring homomorphism $\alpha_f : S_X(\mathcal{E}) \to \mathcal{B}$ such that $f = \alpha_f \circ \phi_{\mathcal{E}}$.

The existence and uniqueness (up to isomorphism) of the symmetric algebra come from [EGA] (II,1.7.4) or (I,9.4.3). Moreover, from [EGA] (II,1.7.4) we have a direct

sum decomposition

$$S(\mathcal{E}) = \oplus_{r=0}^{\infty} S^r\mathcal{E}.$$

Remark 3.11. Let $u : \mathcal{E} \to \mathcal{F}$ be an $\mathcal{O}_X$-module homomorphism. By applying the universal property to the homomorphism $\phi_{\mathcal{F}} \circ u : \mathcal{E} \to S_X(\mathcal{F})$, we obtain that there is a unique $\mathcal{O}_X$-ring homomorphism

$$S_X(u) : S_X(\mathcal{E}) \to S_X(\mathcal{F}).$$

Moreover, $S_X(u)$ is a graded homomorphism of $\mathcal{O}_X$-modules.

Proposition 3.12. *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then there exists a unique $\mathcal{O}_X$-ring isomorphism*

$$\alpha^f_{\mathcal{G}} : S_X(f^*\mathcal{G}) \xrightarrow{\sim} f^*S_Y(\mathcal{G})$$

such that

$$\alpha^f_{\mathcal{G}} \circ \phi_{f^*\mathcal{G}} = f^*\phi_{\mathcal{G}}.$$

Proof. Since $S_Y(\mathcal{G})$ is a $\mathcal{O}_Y$-algebra, $f^*S_Y(\mathcal{G})$ is an $\mathcal{O}_X$-algebra by [EGA] (0_{I},4.3.4). Moreover, since $\phi_{\mathcal{G}} : \mathcal{G} \to S_Y(\mathcal{G})$ is an $\mathcal{O}_Y$-module homomorphism, we obtain that $f^*\phi_{\mathcal{G}} : f^*\mathcal{G} \to f^*S_Y(\mathcal{G})$ is an $\mathcal{O}_X$-module homomorphism, and so the existence of $\alpha^f_{\mathcal{G}}$ follows from the universal property of the symmetric algebra applied to $f^*\phi_{\mathcal{G}}$. The homomorphism $\alpha^f_{\mathcal{G}}$ is an isomorphism by [EGA] (I,9.4.5). □

Proposition 3.13. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of schemes. If $\mathcal{H}$ is an $\mathcal{O}_Z$-module, then the following diagram commutes*

$$\begin{array}{ccccc} S_X((g\circ f)^*\mathcal{H}) & & \xrightarrow{\quad\alpha^{g\circ f}_{\mathcal{H}}\quad} & & (g\circ f)^*S_Z(\mathcal{H}) \\ \Big\downarrow{\scriptstyle S_X(\varphi^{f,g}_{\mathcal{H}})} & & & & \Big\downarrow{\scriptstyle \varphi^{f,g}_{S_Z(\mathcal{H})}} \\ S_X(f^*g^*\mathcal{H}) & \xrightarrow[\alpha^f_{g^*\mathcal{H}}]{} & f^*S_Y(g^*\mathcal{H}) & \xrightarrow[f^*\alpha^g_{\mathcal{H}}]{} & f^*g^*S_Z(\mathcal{H}) \end{array} \tag{3.2}$$

where $\varphi^{f,g}_{\mathcal{H}}$ is the isomorphism of Proposition 3.9.

Proof. By the universal property of the symmetric algebra, it suffices to show that

$$f^*\alpha^g_{\mathcal{H}} \circ \alpha^f_{g^*\mathcal{H}} \circ S_X(\varphi^{f,g}_{\mathcal{H}}) \circ \phi_{(g\circ f)^*\mathcal{H}} = \varphi^{f,g}_{S_Z(\mathcal{H})} \circ \alpha^{g\circ f}_{\mathcal{H}} \circ \phi_{(g\circ f)^*\mathcal{H}}.$$

This is verified as follows. We have:

$$\begin{aligned} f^*\alpha^g_{\mathcal{H}} \circ \alpha^f_{g^*\mathcal{H}} \circ S_X(\varphi^{f,g}_{\mathcal{H}}) \circ \phi_{(g\circ f)^*\mathcal{H}} &= f^*\alpha^g_{\mathcal{H}} \circ \alpha^f_{g^*\mathcal{H}} \circ \phi_{f^*g^*\mathcal{H}} \circ \varphi^{f,g}_{\mathcal{H}} \\ &= f^*\alpha^g_{\mathcal{H}} \circ f^*\phi_{g^*\mathcal{H}} \circ \varphi^{f,g}_{\mathcal{H}} \\ &= f^*(\alpha^g_{\mathcal{H}} \circ \phi_{g^*\mathcal{H}}) \circ \varphi^{f,g}_{\mathcal{H}} \\ &= f^*(g^*\phi_{\mathcal{H}}) \circ \varphi^{f,g}_{\mathcal{H}} \\ &= \varphi^{f,g}_{S_Z(\mathcal{H})} \circ (g\circ f)^*(\phi_{\mathcal{H}}) \\ &= \varphi^{f,g}_{S_Z(\mathcal{H})} \circ \alpha^{g\circ f}_{\mathcal{H}} \circ \phi_{(g\circ f)^*\mathcal{H}}, \end{aligned}$$

where the second, fourth and sixth equalities come from Proposition 3.12. The first equality comes from Remark 3.11, and the fifth by Proposition 3.9. □

Proposition 3.14. *Let $u : \mathcal{G}_1 \to \mathcal{G}_2$ be an $\mathcal{O}_Y$-module homomorphism, then the*

following diagram commutes

$$\begin{array}{ccc} S_X(f^*\mathcal{G}_1) & \xrightarrow{\alpha^f_{\mathcal{G}_1}} & f^*S_Y(\mathcal{G}_1) \\ {\scriptstyle S_X(f^*u)}\downarrow & & \downarrow{\scriptstyle f^*S_Y(u)} \\ S_X(f^*\mathcal{G}_2) & \xrightarrow[\alpha^f_{\mathcal{G}_2}]{} & f^*S_Y(\mathcal{G}_2). \end{array}$$

Proof. Again, by the functorial property of the symmetric algebra, to verify the commutativity we only need to check

$$f^*S_Y(u) \circ \alpha^f_{\mathcal{G}_1} \circ \phi_{f^*\mathcal{G}_1} = \alpha^f_{\mathcal{G}_2} \circ S_X(f^*u) \circ \phi_{f^*\mathcal{G}_1}.$$

We have

$$\begin{aligned} f^*S_Y(u) \circ \alpha^f_{\mathcal{G}_1} \circ \phi_{f^*\mathcal{G}_1} &= f^*S_Y(u) \circ f^*\phi_{\mathcal{G}_1} \\ &= f^*(S_Y(u) \circ \phi_{\mathcal{G}_1}) \\ &= f^*(\phi_{\mathcal{G}_2} \circ u) \\ &= f^*f^*(\phi_{\mathcal{G}_2}) \circ f^*(u) \\ &= \alpha^f_{\mathcal{G}_2} \circ \phi_{f^*\mathcal{G}_2} \circ f^*(u) \\ &= \alpha^f_{\mathcal{G}_2} \circ S_X(f^*(u)) \circ \phi_{f^*\mathcal{G}_1} \end{aligned}$$

where the second and fifth equalities hold by Proposition 3.12, the first and last equalities hold by Remark 3.11. □

Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Since $f_*S_X(\mathcal{G}) = \oplus_{r=0}^{\infty} f_*S^r_X(\mathcal{G})$, we have a canonical morphism $f_*\mathcal{G} \to f_*S_X(\mathcal{G})$ (inclusion in degree one). Therefore we obtain a canonical $\mathcal{O}_Y$-homomorphism $S_Y(f_*\mathcal{G}) \to f_*S_X(\mathcal{G})$

(by the universal property of the symmetric algebra).

Proposition 3.15. *Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. The following diagram is commutative:*

$$\begin{array}{ccc} S_X\mathcal{F} & \xrightarrow{\rho^f_{S_X\mathcal{F}}} & f_*f^*S_X\mathcal{F} \\ {\scriptstyle S_X\rho^f_{\mathcal{F}}}\downarrow & & \uparrow{\scriptstyle f_*\alpha^f} \\ S_X(f_*f^*\mathcal{F}) & \longrightarrow & f_*S_Y(f^*\mathcal{F}). \end{array}$$

Proof. By the universal property of the symmetric algebra, it suffices to prove

$$\rho^f_{S\mathcal{F}} \circ \phi_{\mathcal{F}} = f_*\alpha^f \circ c^f_{f^*\mathcal{F}} \circ S\rho^f_{\mathcal{F}} \circ \phi_{\mathcal{F}}.$$

We have that

$$\begin{aligned} f_*\alpha^f \circ c^f_{f^*\mathcal{F}} \circ S\rho^f_{\mathcal{F}} \circ \phi_{\mathcal{F}} &= f_*\alpha^f \circ c^f_{f^*\mathcal{F}} \circ (\oplus_r S^r\rho^f_{\mathcal{F}}) \circ \phi_{\mathcal{F}} \\ &= f_*\alpha^f \circ c^f_{f^*\mathcal{F}} \circ \phi_{f^*\mathcal{F}} \circ \rho^f_{\mathcal{F}} \\ &= f_*\alpha^f \circ f_*\phi_{f^*\mathcal{F}} \circ \rho^f_{\mathcal{F}} \\ &= f_*(\alpha^f \circ \phi_{f^*\mathcal{F}}) \circ \rho^f_{\mathcal{F}} \\ &= f_*f^*\phi_{\mathcal{F}} \circ \rho^f_{\mathcal{F}} \\ &= \rho_{S\mathcal{F}} \circ \rho^f_{\mathcal{F}}. \end{aligned}$$

□

We will later work with stalks and we will need to understand symmetric algebras of modules over rings, so let us briefly discuss some useful facts.

Let B be an A-algebra and let N be a B-module, then N is also an A-module and we have an A-module map $\phi_{B,N} : N \to S_BN$ (inclusion into S^1N, which is also a B-module map). The universal property of symmetric algebras then gives a canonical

morphism of graded A-algebras $\theta_N := \theta_{A,B,N} : S_A N \to S_B N$ which is functorial in the sense that if $f : N \to N'$ is a B-module map then the following diagram commutes

$$\begin{array}{ccc} S_A N & \xrightarrow{\theta_N} & S_B N \\ {\scriptstyle S_A f}\downarrow & & {\scriptstyle S_B f}\downarrow \\ S_A N' & \xrightarrow{\theta_{N'}} & S_B N' \end{array}$$

(which is verified in degree 1 using ϕ, similar to all previous verifications in this subsection). In particular, given a morphism $h : M \to M'$ of A-modules and a morphism $f : N \to N'$ of B-modules, and given an A-module map $u : M \to N$ and an A-module map $u' : M' \to N'$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow \\ M' & \xrightarrow{u'} & N' \end{array}$$

commutes, then we get a commutative diagram

$$\begin{array}{ccc} S_A M & \xrightarrow{Su} & S_B N \\ {\scriptstyle S_A h}\downarrow & & {\scriptstyle S_B f}\downarrow \\ S_A M' & \xrightarrow{Su'} & S_B N' \end{array}$$

where $Su := \theta_N \circ S_A u$ and $Su := \theta_{N'} \circ S_A u'$.

Let us consider a slightly more general construction. Let B be an A-algebra, let M be an A-module and let N be a B module, so that it is also an A-module. Let $f : M \to N$ be an A-module morphism. Then we have an A-module map $u_f : M \to S_B N$ which is given by f composed with the canonical inclusion $N \to S_B N$. In particular u_f maps M to the part of degree 1 in $S_B N$. The A-module map u_f

induces a unique map of A-algebras $S_AM \to S_BN$ which we denote by $\theta_{A,B,f}$, and this map respects the grading because it maps $S^1_AM = M$ to S^1_BN (and S_AM is generated in degree 1). By construction, we have that the map induced by $\theta_{A,B,f}$ in degree 1 gives again the map $f : M = S^1_AM \to S^1_BN = N$. It is also clear that $\Theta_{A,B,f} = \Theta_{A,B,N} \circ S_Af$ which was denoted before as Sf. Moreover, we have:

Lemma 3.16. *Let A be a ring, B an A-algebra and C a B-algebra (so that it is also an A-algebra). Let M, N, P be modules over A, B, C respectively. Suppose that we have A-module homomorphisms $f : M \to N$, $h : M \to P$ and a B-module homomorphism $g : N \to P$ such that $h = gf$. Then the following diagram commutes*

$$\begin{array}{ccc} S_AM & \xrightarrow{\Theta_{A,B,f}} & S_BN \\ & {\scriptstyle \Theta_{A,C,h}} \searrow \quad \swarrow {\scriptstyle \Theta_{B,C,g}} & \\ & S_CP & \end{array}$$

Proof. Since S_AM is generated in degree 1 as A-algebra and the morphisms θ are morphisms of A-algebras, it suffices to check commutativity on elements of $M = S^1_AM$. Now the result follows from the hypothesis $h = gf$ because by the discussion before the lemma, we know that the morphisms θ induce the original module maps on degree 1. □

Given a scheme X, a point $P \in X$ and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a canonical map induced on stalks, constructed as follows. If $A = \mathcal{O}_{X,P}$, then we have the A-module morphism $\phi_{\mathcal{F},P} : \mathcal{F}_P \to (S_X\mathcal{F})_P$ that maps the A-module $\mathcal{F}_P$ to the part of degree 1 of the graded commutative A-algebra $(S_X\mathcal{F})_P$ (that is, to $(S^1_X\mathcal{F})_P$). By the universal property of the symmetric A-algebra $S_A\mathcal{F}_P$, we then get a morphism of A-algebras

$S_A\mathcal{F}_P \to (S_X\mathcal{F})_P$ induced by $\phi_{\mathcal{F},P}$, which we denote by $s_{\mathcal{F},P}$.

Lemma 3.17. *With the previous notation, the map $s_{\mathcal{F},P}$ respects the grading, is an isomorphism, and it maps the homogeneous element $t_{1,P}\cdots t_{r,P}$ to $(t_1\cdots t_r)_P$.*

Proof. First we observe that the construction of $s_{\mathcal{F},P}$ is the same as the construction of the canonical isomorphism $S_A\mathcal{F}_P \cong (S_X\mathcal{F})_P$ in [EGA] (II,1.7.4), which refers to [EGA] (II,1.7.2). Therefore, $s_{\mathcal{F},P}$ is an isomorphism of graded A-algebras.

From the construction we have that $s_{\mathcal{F},P}$ maps an element $t_P \in \mathcal{F}_P = S^1_A\mathcal{F}_P$ to $t_P \in (S^1_X\mathcal{F})_P$. As $s_{\mathcal{F},P}$ respects multiplication, we obtain that it maps elements as claimed. □

We will also need the following:

Proposition 3.18. *Let $f : X \to Y$ be a morphism of schemes. Let $Q \in X$ and $P = f(Q) \in Y$. Let $\mathcal{F}$ be a locally free sheaf on Y. Write $B = \mathcal{O}_{X,Q}$ and $A = \mathcal{O}_{Y,P}$. Then the following diagram commutes:*

$$\begin{array}{ccccc} S_A\mathcal{F}_P & & \xrightarrow{\quad s_{\mathcal{F},P} \quad} & & (S_Y\mathcal{F})_P \\ {\scriptstyle Sf^{\#}_{\mathcal{F},Q}}\downarrow & & & & \downarrow{\scriptstyle f^{\#}_{S_Y\mathcal{F},Q}} \\ S_B(f^*\mathcal{F})_Q & \xrightarrow[s_{f^*\mathcal{F},Q}]{} & (S_Xf^*\mathcal{F})_Q & \xrightarrow[\alpha^f_{\mathcal{F},Q}]{} & (f^*S_Y\mathcal{F})_Q \end{array}$$

where $f^{\#}_{\mathcal{F},Q} : \mathcal{F}_P \to (f^*\mathcal{F})_Q$ is as in Notation 2.73.

Proof. The morphisms in the diagram are morphisms of graded A-algebras, and $S_A\mathcal{F}_P$ is generated in degree 1 as an A-algebra (see [EGA] (II,1.7.1)), so we have to prove that the diagram commutes in degree 1.

First we note that given $\mathcal{O}_X$-modules $\mathcal{H}_i$ the following diagram commutes:

$$\begin{array}{ccccccc}
(f_* \oplus_i \mathcal{H}_i)_P & = & \lim_{P\in U}(\oplus_i \mathcal{H}_i)(f^{-1}U) & \xrightarrow{f^\circ_{\oplus_i \mathcal{H}_i,Q}} & \lim_{Q\in V}(\oplus_i \mathcal{H}_i)(V) & = & (\oplus_i \mathcal{H}_i)_Q \\
\| & & & & & & \| \\
\oplus(f_*\mathcal{H}_i)_P & = & \oplus_i \lim_{P\in U} \mathcal{H}_i(f^{-1}U) & \xrightarrow[\oplus_i f^\circ_{\mathcal{H}_i,Q}]{} & \oplus_i \lim_{Q\in V} \mathcal{H}_i(V) & = & \oplus_i \mathcal{H}_{i,Q}
\end{array}$$

because taking stalks and f_* commute with direct sums (cf. Lemma 3.8). Hence $f^\circ_{\oplus_i \mathcal{H}_i,Q} = \oplus_i f^\circ_{\mathcal{H}_i,Q}$, where $f^\circ_{\mathcal{H},Q} : (f_*\mathcal{H})_P \to \mathcal{H}_Q$ is as before Notation 2.73.

From Remark 3.6 we have that $\oplus_i \rho^f_{\mathcal{F}_i} = \rho^f_{\oplus_i \mathcal{F}_i}$, hence $f^\#_{\oplus_i \mathcal{G}_i,Q} = \oplus_i f^\#_{\mathcal{G}_i,Q}$ for $\mathcal{O}_Y$-modules $\mathcal{G}_i$. In particular, we have

$$f^\#_{S_Y\mathcal{F},Q} = f^\#_{\oplus_r S^r_Y\mathcal{F},Q} = \oplus_r f^\#_{S^r_Y\mathcal{F},Q}.$$

Then the following diagram commutes

$$\begin{array}{ccccc}
S^1_A\mathcal{F}_P & & \longrightarrow & & (S^1_Y\mathcal{F})_P \\
{\scriptstyle \theta \circ (S^1_A f^\#_{\mathcal{F},Q})}\downarrow & & & & \downarrow{\scriptstyle f^\#_{S^1_Y\mathcal{F},Q}} \\
S^1_B(f^*\mathcal{F})_Q & \longrightarrow & (S^1_X f^*\mathcal{F})_Q & \xrightarrow[\alpha^f_{1,\mathcal{F},Q}]{} & (f^*S^1_Y\mathcal{F})_Q
\end{array}$$

because it is exactly the same as the diagram

$$\begin{array}{ccccc}
\mathcal{F}_P & & \longrightarrow & & \mathcal{F}_P \\
{\scriptstyle f^\#_{\mathcal{F},Q}}\downarrow & & & & \downarrow{\scriptstyle f^\#_{\mathcal{F},Q}} \\
(f^*\mathcal{F})_Q & \longrightarrow & (f^*\mathcal{F})_Q & \longrightarrow & (f^*\mathcal{F})_Q
\end{array}$$

where the unlabeled arrows are the respective identity maps (and this diagram commutes). As $f^{\#}_{S^1_Y\mathcal{F},Q}$ is the component of $f^{\#}_{S_Y\mathcal{F},Q}$ in degree 1, and $f^{\#}_{\mathcal{F},Q} = \theta \circ (S^1_A f^{\#}_{\mathcal{F},Q})$ is the component of degree 1 of $Sf^{\#}_{\mathcal{F},Q}$ (as $\theta = \theta_{A,\mathcal{F}_P}$ is a graded morphism), this proves what we want. □

3.2.3 Functoriality for tensor products

Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Recall from [EGA] (0_{I},4.3.3) that we have an isomorphism

$$f^*\mathcal{F} \otimes f^*\mathcal{G} \to f^*(\mathcal{F} \otimes \mathcal{G}),$$

which is explicitly constructed. In this section, we give an alternative construction for such an isomorphism, whose functorial properties are easier to understand. The approach below using bigraded symmetric algebras was suggested to me by Hector Pasten, and I thank him for this suggestion which simplified the exposition.

Let us consider the following variation of the notion of graded $\mathcal{O}_X$-algebra.

A bigraded $\mathcal{O}_X$-algebra is a commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ with a direct sum decomposition $\mathcal{A} = \oplus_{i,j\geq 0}\mathcal{A}_{ij}$ where each $\mathcal{A}_{ij}$ is an $\mathcal{O}_X$-module, such that for all open sets $U \subseteq X$ this decomposition gives to $\mathcal{A}(U)$ the structure of a bigraded $\mathcal{O}_X(U)$-algebra. For example, if $\mathcal{B} = \oplus_i \mathcal{B}_i$ and $\mathcal{C} = \oplus_j \mathcal{C}_j$ are graded $\mathcal{O}_X$-algebras, then the algebra $\mathcal{B} \otimes \mathcal{C} = \oplus_{ij} \mathcal{B}_i \otimes \mathcal{C}_j$ is bigraded.

In particular, given $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, we have that $S(\mathcal{F} \oplus \mathcal{G}) = S(\mathcal{F}) \otimes S(\mathcal{G})$ (cf. [EGA] (II,1.7.4)) hence $S(\mathcal{F} \oplus \mathcal{G}) = \oplus_{ij} S^i\mathcal{F} \otimes S^j\mathcal{G}$ is bigraded. Moreover, this bigraded $\mathcal{O}_X$-algebra comes with two canonical maps $b_{\mathcal{F}} := \phi_{\mathcal{F}\oplus\mathcal{G}} \circ i_{\mathcal{F}} : \mathcal{F} \to S(\mathcal{F}\oplus\mathcal{G})$ and $b_{\mathcal{G}} := \phi_{\mathcal{F}\oplus\mathcal{G}} \circ i_{\mathcal{G}}$ where $i_{\mathcal{F}}, i_{\mathcal{G}}$ are the canonical inclusions into $\mathcal{F} \oplus \mathcal{G}$. Note that

$b_{\mathcal{F}}$ is an isomorphism of $\mathcal{F}$ with the part of bidegree $(1,0)$ and $b_{\mathcal{G}}$ is an isomorphism of $\mathcal{G}$ with the part of bidegree $(0,1)$, because $\phi_{\mathcal{F}\oplus\mathcal{G}}$ identifies $\mathcal{F}\oplus\mathcal{G}$ with $S^1(\mathcal{F}\oplus\mathcal{G})$ in $S(\mathcal{F}\otimes\mathcal{G})$.

The universal property of the symmetric algebra (cf. Definition 3.10) gives:

Lemma 3.19. *The triple $(S(\mathcal{F}\oplus\mathcal{G}), b_{\mathcal{F}}, b_{\mathcal{G}})$ has the following universal property:*

Given a bigraded $\mathcal{O}_X$-algebra $\mathcal{A}$ and $\mathcal{O}_X$-module morphisms $\ell_{\mathcal{F}} : \mathcal{F} \to \mathcal{A}$ and $\ell_{\mathcal{G}} : \mathcal{G} \to \mathcal{A}$, there is a unique morphism of $\mathcal{O}_X$-algebras $\xi : S(\mathcal{F}\oplus\mathcal{G}) \to \mathcal{A}$ such that $\ell_{\mathcal{F}} = \xi \circ b_{\mathcal{F}}$ and $\ell_{\mathcal{G}} = \xi \circ b_{\mathcal{G}}$. Moreover, if $\ell_{\mathcal{F}}$ maps $\mathcal{F}$ to $\mathcal{A}_{10}$ and $\ell_{\mathcal{G}}$ maps $\mathcal{G}$ to $\mathcal{A}_{01}$ then ξ respects the bigrading.

Proof. From the universal property of the symmetric algebra we can take $\xi = \alpha_{\ell_{\mathcal{F}}\oplus\ell_{\mathcal{G}}}$, then $\xi \circ b_{\mathcal{F}} = \alpha_{\ell_{\mathcal{F}}\oplus\ell_{\mathcal{G}}} \circ \phi_{\mathcal{F}\oplus\mathcal{G}} \circ i_{\mathcal{F}} = (\ell_{\mathcal{F}} \oplus \ell_{\mathcal{G}}) \circ i_{\mathcal{F}} = \ell_{\mathcal{F}}$ and similarly for $\mathcal{G}$. Uniqueness follows because given ξ' with the same property, we have $\xi' \circ \phi_{\mathcal{F}\oplus\mathcal{G}} = (\ell_{\mathcal{F}} \oplus \ell_{\mathcal{G}})$ which implies $\xi' = \alpha_{\ell_{\mathcal{F}}\oplus\ell_{\mathcal{G}}}$ by the universal property of the symmetric algebra.

Suppose that $\ell_{\mathcal{F}}$ maps $\mathcal{F}$ to $\mathcal{A}_{10}$ and $\ell_{\mathcal{G}}$ maps $\mathcal{G}$ to $\mathcal{A}_{01}$. Then we need to show that $\xi = \alpha_{\ell_{\mathcal{F}}\oplus\ell_{\mathcal{G}}}$ respects the bigrading, that is, the direct sum decomposition

$$S(\mathcal{F}\oplus\mathcal{G}) = \oplus_{ij} S^i\mathcal{F} \otimes S^j\mathcal{G},$$

and $\mathcal{A} = \oplus_{ij}\mathcal{A}_{ij}$, so we have to show that the image of $S^i\mathcal{F} \otimes S^j\mathcal{G}$ is in the subsheaf $\mathcal{A}_{ij}$ of $\mathcal{A}$ for all i, j.

We claim that this can be checked on stalks. This is because if the direct sum decompositions are respected on stalks, then let $U \subseteq X$ be an open set and $s \in H^0(U, S^i\mathcal{F} \otimes S^j\mathcal{G})$, then the image t of s is in $H^0(U, \mathcal{A})$ and we have to show that it is in $H^0(U, \mathcal{A}_{ij})$. Note that $t_P \in (\mathcal{A}_{ij})_P$ for all $P \in U$ by hypothesis, so there is

$\langle t', V\rangle \in (\mathcal{A}_{ij})_P$ with $P \in V \subseteq U$ such that $t'_P = t_P$ so there is $P \in V' \subseteq V$ with $t_{|V'} = t'_{|V'} \in H^0(V', \mathcal{A}_{ij})$. This holds at every $P \in U$ so by sheaf axioms this means that $t \in H^0(U, \mathcal{A}_{ij})$ as we claimed.

Finally we have to prove that given any $P \in X$, ξ_P respects the bigrading of the stalks. Note that $\mathcal{A}_P = \oplus_{ij}(\mathcal{A}_{ij})_P$ and $S(\mathcal{F} \oplus \mathcal{G})_P = S(\mathcal{F}_P \oplus \mathcal{G}_P) = S(\mathcal{F}_P) \otimes S(\mathcal{G}_P)$ (by [EGA] (II,1.7.4)). We have that for a ring A, and an A-module M, the graded algebra $S(M)$ is generated as an A-algebra in degree 1 because $S^i(M)$ consists of A-linear combinations of products of i elements of M (see [EGA] (II,1.7.1)). Hence the bigraded $\mathcal{O}_{X,P}$-algebra $S(\mathcal{F}_P) \otimes S(\mathcal{G}_P)$ is generated by the elements of bidegree $(1,0)$ and $(0,1)$ (because $S(\mathcal{F}_P)$ and $S(\mathcal{G}_P)$ are generated in degree 1).

By hypothesis, ξ respects the bigrading in bidegree $(1,0)$ and $(0,1)$ because $\ell_{\mathcal{F}}$ maps $\mathcal{F}$ to $\mathcal{A}_{10}$ and $\ell_{\mathcal{F}} = \xi \circ b_{\mathcal{F}}$, but the part of bidegree $(1,0)$ is the image of $b_{\mathcal{F}}$ (see the discussion before this lemma) and similarly for $\mathcal{G}$. Therefore the homomorphism $\xi_P : S(\mathcal{F}_P) \otimes S(\mathcal{G}_P) \to \mathcal{A}_P$ respects the bigrading in bidegree $(1,0)$ and $(0,1)$. These elements generate $S(\mathcal{F}_P) \otimes S(\mathcal{G}_P)$ as $\mathcal{O}_{X,P}$-algebra, and ξ_p respects the $\mathcal{O}_{X,P}$-algebra structure, therefore elements of bidegree (i,j) (which can be written in terms of elements of bidegree $(1,0),(0,1)$) are mapped to elements of bidegree (i,j). □

Recall from Proposition 3.12 that we have the isomorphism

$$\alpha^f_{\mathcal{F}\oplus\mathcal{G}} : S_X f^*(\mathcal{F} \oplus \mathcal{G}) \xrightarrow{\sim} f^* S_Y(\mathcal{F} \oplus \mathcal{G}).$$

We have

$$S_X f^*(\mathcal{F} \oplus \mathcal{G}) = S_X(f^*\mathcal{F} \oplus f^*\mathcal{G}) = \oplus_n \oplus_{i+j=n} S^i_X f^*\mathcal{F} \otimes S^j_X f^*\mathcal{G}, \tag{3.3}$$

where the first equality follows from Lemma 3.7. The second equality holds by the fact that $S_X(\mathcal{F}_1) \otimes S_X(\mathcal{F}_2) = (\oplus_i S^i_X \mathcal{F}_1) \otimes (\oplus_j S^j_X \mathcal{F}_2) = \oplus_n \oplus_{i+j=n} S^i_X \mathcal{F}_1 \otimes S^j_X \mathcal{F}_2$ (cf. [EGA] (II,1.7.4)).

We also have

$$f^* S_X(\mathcal{F} \oplus \mathcal{G}) = f^* \oplus_n \oplus_{i+j=n} S^i_X \mathcal{F} \otimes S^j_X \mathcal{G} = \oplus_n \oplus_{i+j=n} f^*(S^i_X \mathcal{F} \otimes S^j_X \mathcal{G})$$

where the second equality holds by Lemma 3.7.

The isomorphism $\alpha^f_{\mathcal{F}\oplus\mathcal{G}}$ respects the previous direct sum decomposition by Lemma 3.19.

Notation 3.20. If we restrict to $i = 1, j = 1$ we obtain a functorial isomorphism

$$T^f_{\mathcal{F},\mathcal{G}} : f^*\mathcal{F} \otimes f^*\mathcal{G} \xrightarrow{\sim} f^*(\mathcal{F} \otimes \mathcal{G}).$$

This isomorphism is functorial on $\mathcal{F}$ and $\mathcal{G}$ in the sense that if we have $\mathcal{O}_X$-module homomorphisms $u : \mathcal{F}_1 \to \mathcal{F}_2$ and $v : \mathcal{G}_1 \to \mathcal{G}_2$, then we get a commutative diagram

$$\begin{array}{ccc} f^*\mathcal{F}_1 \otimes f^*\mathcal{G}_1 & \xrightarrow{T^f_{\mathcal{F}_1,\mathcal{G}_1}} & f^*(\mathcal{F}_1 \otimes \mathcal{G}_1) \\ \downarrow{\scriptstyle f^*u \otimes f^*v} & & \downarrow{\scriptstyle f^*(u\otimes v)} \\ f^*\mathcal{F}_2 \otimes f^*\mathcal{G}_2 & \xrightarrow{T^f_{\mathcal{F}_2,\mathcal{G}_2}} & f^*(\mathcal{F}_2 \otimes \mathcal{G}_2) \end{array}$$

This is deduced from the analogous property for the symmetric algebra $S(\mathcal{F}_i \oplus \mathcal{G}_i)$.

Now we will study the functorial properties of this isomorphism.

Proposition 3.21. *Let X, Y, Z be schemes and let $f : X \to Y$ and $g : Y \to Z$. Let $\mathcal{H}_i$ be $\mathcal{O}_Z$-modules. The following diagram commutes:*

$$\begin{array}{ccc} f^*g^* \oplus_i \mathcal{H}_i & = & \oplus_i f^*g^*\mathcal{H}_i \\ {\scriptstyle \varphi^{f,g}_{\oplus\mathcal{H}_i}}\uparrow & & \uparrow{\scriptstyle \oplus\varphi^{f,g}_{\mathcal{H}_i}} \\ (g\circ f)^* \oplus_i \mathcal{H}_i & = & \oplus (g\circ f)^*\mathcal{H}_i. \end{array}$$

Proof. It suffices to show that the following diagram commutes (cf. Proposition 3.9)

$$\begin{array}{ccc} \oplus_i \mathcal{H}_i & \xrightarrow{\rho^{g\circ f}_{\oplus_i\mathcal{H}_i}} & (g\circ f)_*(g\circ f)^* \oplus_i \mathcal{H}_i \\ {\scriptstyle \rho^{g}_{\oplus_i\mathcal{H}_i}}\downarrow & & \downarrow{\scriptstyle (g\circ f)_*\oplus_i\varphi^{f,g}_{\mathcal{H}_i}} \\ g_*g^* \oplus_i \mathcal{H}_i & \xrightarrow[g_*\rho^{f}_{\oplus_i\mathcal{H}_i}]{} & (g\circ f)_*f^*(g^* \oplus_i \mathcal{H}_i). \end{array}$$

From Lemma 3.7, pullbacks commute with direct sums (since $\mathcal{O}_Z$-modules are sheaves of abelian groups). From Remark 3.6, we have that $\oplus_i \rho^f_{\mathcal{F}_i} = \rho^f_{\oplus_i\mathcal{F}_i}$. Since pushforwards also commute with direct sums in the noetherian case (see Lemma 3.8), the previous diagram becomes

$$\begin{array}{ccc} \oplus_i \mathcal{H}_i & \xrightarrow{\oplus_i\rho^{g\circ f}_{\mathcal{H}_i}} & \oplus_i(g\circ f)_*(g\circ f)^*\mathcal{H}_i \\ {\scriptstyle \oplus_i\rho^{g}_{\mathcal{H}_i}}\downarrow & & \downarrow{\scriptstyle \oplus_i(g\circ f)_*\varphi^{f,g}_{\mathcal{H}_i}} \\ \oplus_i g_*g^*\mathcal{H}_i & \xrightarrow[\oplus_i g_*\rho^{f}_{\mathcal{H}_i}]{} & \oplus_i(g\circ f)_*f^*(g^*\mathcal{H}_i), \end{array}$$

which commutes by the universal property of $\varphi^{f,g}_{\mathcal{H}_i}$ (cf. Proposition 3.9). Therefore $\oplus_i\varphi^{f,g}_{\mathcal{H}_i} \cong \varphi^{f,g}_{\oplus\mathcal{H}_i}$. □

Proposition 3.22. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_Z$-modules. Then the following diagram commutes:*

$$\begin{array}{ccc}
(g\circ f)^*\mathcal{F}\otimes(g\circ f)^*\mathcal{G} & \xrightarrow{T^{g\circ f}_{\mathcal{F},\mathcal{G}}} & (g\circ f)^*(\mathcal{F}\otimes\mathcal{G}) \\
\downarrow{\scriptstyle \varphi^{f,g}_{\mathcal{F}}\otimes\varphi^{f,g}_{\mathcal{G}}} & & \\
f^*g^*\mathcal{F}\otimes f^*g^*\mathcal{G} & & \downarrow{\scriptstyle \varphi^{f,g}_{\mathcal{F}\otimes\mathcal{G}}} \\
\downarrow{\scriptstyle T^f_{g^*\mathcal{F},g^*\mathcal{G}}} & & \\
f^*(g^*\mathcal{F}\otimes g^*\mathcal{G}) & \xrightarrow[f^*T^g_{\mathcal{F},\mathcal{G}}]{} & f^*g^*(\mathcal{F}\otimes\mathcal{G})
\end{array}$$

Proof. Diagram (3.2) applied to $\mathcal{F}\oplus\mathcal{G}$ is

$$\begin{array}{ccccc}
S_X((g\circ f)^*(\mathcal{F}\oplus\mathcal{G})) & & \xrightarrow{\alpha^{g\circ f}_{\mathcal{F}\oplus\mathcal{G}}} & & (g\circ f)^*S_Z(\mathcal{F}\oplus\mathcal{G}) \\
\downarrow{\scriptstyle S_X(\varphi^{f,g}_{\mathcal{F}\oplus\mathcal{G}})} & & & & \downarrow{\scriptstyle \varphi^{f,g}_{S_Z(\mathcal{F}\oplus\mathcal{G})}} \\
S_X(f^*g^*(\mathcal{F}\oplus\mathcal{G})) & \xrightarrow[\alpha^f_{g^*(\mathcal{F}\oplus\mathcal{G})}]{} & f^*S_Y(g^*(\mathcal{F}\oplus\mathcal{G})) & \xrightarrow[f^*\alpha^g_{\mathcal{F}\oplus\mathcal{G}}]{} & f^*g^*S_Z(\mathcal{F}\oplus\mathcal{G}),
\end{array}$$

which by using Equation (3.3) and Proposition 3.21 is equal to

$$\begin{array}{ccccc}
S_X((g\circ f)^*\mathcal{F}\oplus(g\circ f)^*\mathcal{G}) & & \xrightarrow{\alpha^{g\circ f}_{\mathcal{F}\oplus\mathcal{G}}} & & (g\circ f)^*S_Z(\mathcal{F}\oplus\mathcal{G}) \\
\downarrow{\scriptstyle S_X(\varphi^{f,g}_{\mathcal{F}}\oplus\varphi^{f,g}_{\mathcal{G}})} & & & & \downarrow{\scriptstyle \varphi^{f,g}_{S_Z(\mathcal{F}\oplus\mathcal{G})}} \\
S_X(f^*g^*\mathcal{F}\oplus f^*g^*\mathcal{G})) & \xrightarrow[\alpha^f_{g^*(\mathcal{F}\oplus\mathcal{G})}]{} & f^*S_Y(g^*\mathcal{F}\oplus g^*\mathcal{G}) & \xrightarrow[f^*\alpha^g_{\mathcal{F}\oplus\mathcal{G}}]{} & f^*g^*S_Z(\mathcal{F}\oplus\mathcal{G}).
\end{array}$$

The latter is also equal to

$$\begin{array}{ccc}
(g\circ f)^*\bigoplus_n\bigoplus_{i+j=n} S^i_Z\mathcal{F}\otimes S^j_Z\mathcal{G} & \xrightarrow{\varphi^{f,g}_{\bigoplus_n\bigoplus_{i+j=n} S^i_Z\mathcal{F}\otimes S^j_Z\mathcal{G}}} & f^*g^*\bigoplus_n\bigoplus_{i+j=n} S^i_Z\mathcal{F}\otimes S^j_Z\mathcal{G} \\
\uparrow{\scriptstyle \alpha^{g\circ f}_{\mathcal{F}\oplus\mathcal{G}}} & & \uparrow{\scriptstyle f^*\alpha^g_{\mathcal{F}\oplus\mathcal{G}}} \\
 & & f^*\bigoplus_n\bigoplus_{i+j=n} S^i_Yg^*\mathcal{F}\otimes S^j_Yg^*\mathcal{G} \\
 & & \uparrow{\scriptstyle \alpha^f_{g^*(\mathcal{F}\oplus\mathcal{G})}} \\
\bigoplus_n\bigoplus_{i+j=n} S^i_X(g\circ f)^*\mathcal{F}\otimes S^j_X(g\circ f)^*\mathcal{G} & \xrightarrow[\bigoplus_n\bigoplus_{i+j=n} S^i_X\varphi^{f,g}_{\mathcal{F}}\otimes S^j_X\varphi^{f,g}_{\mathcal{G}}]{} & \bigoplus_n\bigoplus_{i+j=n} S^i_Xf^*g^*\mathcal{F}\otimes S^j_Xf^*g^*\mathcal{G}
\end{array}$$

Restricting the component for $i=1$, $j=1$ we obtain Diagram (3.22). □

The canonical homomorphism $Sf_*(\mathcal{G}_1\oplus\mathcal{G}_2)\to f_*S(\mathcal{G}_1\oplus\mathcal{G}_2)$ induces a homomorphism

$$f_*\mathcal{G}_1\otimes f_*\mathcal{G}_2\to f_*(\mathcal{G}_1\otimes\mathcal{G}_2)$$

which is not in general injective or surjective (cf. [EGA] (0_{I},4.2.2.1)).

Proposition 3.23. *Let $f: X\to Y$ and let $\mathcal{F},\mathcal{G}$ be two $\mathcal{O}_Y$-modules. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{F}\otimes\mathcal{G} & \xrightarrow{\rho^f_{\mathcal{F}\otimes\mathcal{G}}} & f_*f^*(\mathcal{F}\otimes\mathcal{G}) \\
{\scriptstyle \rho^f_{\mathcal{F}}\otimes\rho^f_{\mathcal{G}}}\downarrow & & \uparrow{\scriptstyle f_*T^f} \\
f_*f^*\mathcal{F}\otimes f_*f^*\mathcal{G} & \longrightarrow & f_*(f^*\mathcal{F}\otimes f^*\mathcal{G}).
\end{array}$$

Proof. As in the proof of Proposition 3.22, the diagram of Proposition 3.15 applied

to $\mathcal{F} \oplus \mathcal{G}$ is equal to:

$$\begin{array}{ccc} \oplus_n \oplus_{i+j=n} S^i_X\mathcal{F} \otimes S^j_X\mathcal{G} & \xrightarrow{\rho^f_{\oplus_n \oplus_{i+j=n} S^i_X\mathcal{F}\otimes S^j_X\mathcal{G}}} & \oplus_n \oplus_{i+j=n} f_*f^*S^i_X\mathcal{F} \otimes f_*f^*S^j_X\mathcal{G} \\ \downarrow{\scriptstyle \oplus_n\oplus_{i+j=n} S^i_X\rho^f_{\mathcal{F}}\otimes S^j_X\rho^f_{\mathcal{G}}} & & \downarrow{\scriptstyle f_*\alpha^f} \\ \oplus_n \oplus_{i+j=n} S^i_X f_*F^*\mathcal{F} \otimes S^j_X f_*f^*\mathcal{G} & \longrightarrow & \oplus_n \oplus_{i+j=n} S^i_Y f_*f^*\mathcal{F} \otimes S^j_Y f_*f^*\mathcal{G} \end{array}$$

The proposition follows by restricting to the component $i = 1$, $j = 1$. □

Proposition 3.24. *Let $f : X \to Y$ be a morphism of schemes. Let $Q \in X$ and $P = f(Q) \in Y$, and let $A = \mathcal{O}_{Y,P}$, $B = \mathcal{O}_{X,Q}$. Let $\mathcal{G}, \mathcal{H}$ be locally free sheaves on Y. Then the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{G}_P \otimes_A \mathcal{H}_P & & \xrightarrow{\cong} & & (\mathcal{G}\otimes\mathcal{H})_P \\ \downarrow{\scriptstyle f^\#_{\mathcal{G},Q}\otimes f^\#_{\mathcal{H},Q}} & & & & \downarrow{\scriptstyle f^\#_{\mathcal{G}\otimes\mathcal{H},Q}} \\ (f^*\mathcal{G})_Q \otimes_B (f^*\mathcal{H})_Q & \xrightarrow[\cong]{} & (f^*\mathcal{G}\otimes f^*\mathcal{H})_Q & \xrightarrow[T^f_{\mathcal{G},\mathcal{H},Q}]{} & (f^*(\mathcal{G}\otimes\mathcal{H}))_Q \end{array}$$

where the isomorphisms are the canonical isomorphisms. Here we write $f^\#_{\mathcal{G},Q} \otimes f^\#_{\mathcal{H},Q}$ for the composition

$$\mathcal{G}_P \otimes_A \mathcal{H}_P \to (f^*\mathcal{G})_Q \otimes_A (f^*\mathcal{H})_Q \to (f^*\mathcal{G})_Q \otimes_B (f^*\mathcal{H})_Q$$

where the first arrow is $f^\#_{\mathcal{G},Q} \otimes_A f^\#_{\mathcal{H},Q}$ and the second is induced by the universal property of the tensor product (or equivalently, induced by $\theta_{\mathcal{G}_P\oplus\mathcal{H}_P}$ cf. Section 3.2.2).

Proof. As before, we deduce this by taking the parts of bidegree $i = 1, j = 1$ in Proposition 3.18 applied to $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$. The bidegrees are respected by Lemma 3.19. □

3.2.4 Functoriality for symmetric powers of differentials

Definition 3.25. Let $p : X \to S$ be an S-scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *S-derivation of $\mathcal{O}_X$ into $\mathcal{F}$* is a homomorphism $D : \mathcal{O}_X \to \mathcal{F}$ of sheaves of additive groups such that

- For all open $U \subseteq X$ and $t_1, t_2 \in \Gamma(U, \mathcal{O}_X)$ we have

$$D(t_1 t_2) = t_1 D(t_1) + D(t_1) t_2;$$

- For every open $U \subseteq X$, $t \in \Gamma(U, \mathcal{O}_X)$ and every $s \in \Gamma(V, \mathcal{O}_S)$, where $V \subseteq S$ is open with $p^{-1}(V) \supseteq U$, we have

$$D(p^{\#} s_{|U} t) = p^{\#} s_{|U} D(t).$$

  with $p^{\#} : \mathcal{O}_S \to p_* \mathcal{O}_X$ defined as in [Har77], p. 72.

Remark 3.26. By [EGA] (IV,16.5.1) we have that the set of S-derivations of $\mathcal{O}_X$ into $\mathcal{F}$ forms a $\Gamma(X, \mathcal{O}_X)$-module $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$.

Let $f : X \to Y$ be a morphism of S-schemes, and let $\Omega^1_{X/S}$ be the sheaf of differentials on X/S, with S-derivation $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$, as defined in [EGA] (IV,16.3.6).

Proposition 3.27. *The functor* $\mathrm{Der}_S(\mathcal{O}_X, -) : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\Gamma(X, \mathcal{O}_X)}$ *is represented by the pair* $(\Omega^1_{X/S}, d_{X,S})$.

Proof. [EGA] (IV,16.5.3). □

Proposition 3.28. *If $f : X \to Y$ is a morphism of S-schemes, then there is a unique homomorphism*

$$f_{X/Y/S} : f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

which makes the following diagram commutative:

$$\begin{array}{ccccc} \mathcal{O}_Y & & \xrightarrow{\quad f^\# \quad} & & f_*\mathcal{O}_X \\ {\scriptstyle d_{Y/S}}\downarrow & & & & \downarrow{\scriptstyle f_*(d_{X/S})} \\ \Omega^1_{Y/S} & \xrightarrow[\rho^f_{\Omega^1_{Y/S}}]{} & f_*f^*\Omega^1_{Y/S} & \xrightarrow[f_*(f_{X/Y/S})]{} & f_*\Omega^1_{X/S} \end{array} \tag{3.4}$$

Proof. The homomorphism $f_*(d_{X/S}) \circ f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*\Omega^1_{X/S}$ is an S-derivation of $\mathcal{O}_Y$ into $f_*\Omega^1_{X/S}$. By Proposition 3.27, we get that there is a unique morphism $\bar{f}_{X/Y/S} : \Omega^1_{Y/S} \to f_*\Omega^1_{X/S}$ such that $\bar{f}_{X/Y/S} \circ d_{Y/S} = f_*(d_{X/S}) \circ f^\#$. By the adjoint property of f^*, we get that there is a unique homomorphism $f_{X/Y/S} : f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ such that $f_*(f_{X/Y/S}) \circ \rho^f_{\Omega^1_{X/S}} = \bar{f}_{X/Y/S}$. Thus applying $d_{Y/S}$ to both sides we get

$$f_*(f_{X/Y/S}) \circ \rho^f_{\Omega^1_{X/S}} \circ d_{Y/S} = \bar{f}_{X/Y/S} \circ d_{Y/S} = f_*(d_{X/S}) \circ f^\#.$$

If another homomorphism $h : f^*\Omega^1_{X/S} \to \Omega^1_{Y/S}$ also makes diagram (3.4) commutative, then we get $f_*(h) \circ \rho^f_{\Omega^1_{Y/S}} \circ d_{Y/S} = f_*(f_{X/Y/S}) \circ \rho^f_{\Omega^1_{Y/S}} \circ d_{Y/S}$, hence by [EGA] (IV,16.5.3) we have $f_*(h) \circ \rho^f_{\Omega^1_{Y/S}} = f_*(f_{X/Y/S}) \circ \rho^f_{\Omega^1_{Y/S}}$. By the adjoint property of f^*, and because $\rho^f_{\Omega^1_{X/S}}$ is an isomorphism, we obtain that $h = f_{X/Y/S}$. □

Proposition 3.29. *Let S be a scheme and let $f : X \to Y$, $g : Y \to Z$ be morphisms of S-schemes. Then the following diagram commutes:*

$$\begin{array}{ccc} (g\circ f)^*\Omega^1_{Z/S} & \xrightarrow{(g\circ f)_{X/Z/S}} & \Omega^1_{X/S} \\ {\scriptstyle \varphi^{f,g}_{\Omega^1_{Z/S}}}\big\downarrow & & \big\uparrow{\scriptstyle f_{X/Z/S}} \\ f^*g^*\Omega^1_{Z/S} & \xrightarrow[f^*(g_{Y/Z/S})]{} & f^*\Omega^1_{Y/S}. \end{array} \tag{3.5}$$

Proof. Recall from Proposition 3.28 that $(g\circ f)_{X/Z/S}$ is the unique $\mathcal{O}_X$-homomorphism such that

$$(g\circ f)_*((g\circ f)_{X/Z/S})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\circ d_{Z/S} = (g\circ f)_*(d_{X/S})\circ(g\circ f)^{\#}.$$

If we prove that $f_{X/Y/S}\circ f^*(g_{Y/Z/S})\circ\varphi^{f,g}_{\Omega^1_{Z/S}}$ satisfies the same condition, then we obtain the conclusion. Recall that from Proposition 3.28 we have

$$g_*(g_{Y/Z/S})\circ\rho^{g}_{\Omega^1_{Z/S}}\circ d_{Z/S} = g_*(d_{Y/S})\circ g^{\#}.$$

We also recall that if $u : \mathcal{G}_1 \to \mathcal{G}_2$ is an $\mathcal{O}_Y$-module homomorphism, then we have that $f^*(u) : f^*\mathcal{G}_1 \to f^*\mathcal{G}_2$ is the unique homomorphism of $\mathcal{O}_X$-modules such that we have $f_*f^*(u)\circ\rho^f_{\mathcal{G}_1} = \rho^f_{\mathcal{G}_2}\circ u$. Thus, applying this to $u = g_{Y/Z/S} : g^*\Omega^1_{Z/S} \to \Omega^1_{Y/S}$, we obtain $f_*f^*(g_{Y/Z/S})\circ\rho^f_{g^*\Omega^1_{Z/S}} = \rho^f_{\Omega^1_{Y/S}}\circ g_{Y/Z/S}$. Moreover we have

$$(g\circ f)_*(\varphi_{\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}} = g_*\rho^f_{g^*\Omega^1_{Z/S}}\circ\rho^g_{\Omega^1_{Z/S}}.$$

Thus,

$$\begin{aligned}(g\circ f)_*(f^*(g_{Y/Z/S})\circ\varphi_{\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}} &= (g\circ f)_*(f^*(g_{Y/Z/S}))\circ(g\circ f)_*(\varphi_{\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\\ &= g_*(f_*f^*(g_{Y/Z/S}))\circ g_*(\rho^f_{g^*\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\\ &= g_*(f_*f^*(g_{Y/Z/S})\circ\rho^f_{g^*\Omega^1_{Z/S}})\circ\rho^g_{\Omega^1_{Z/S}}\\ &= g_*(\rho^f_{\Omega^1_{Y/S}}\circ g_{Y/Z/S})\circ\rho^g_{\Omega^1_{Z/S}}.\end{aligned}$$

and so

$$\begin{aligned}(g\circ f)_*(f^*(g_{Y/Z/S})\circ\varphi_{\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\circ d_{Z/S} &= g_*(\rho^f_{\Omega^1_{Y/S}})\circ g_*(g_{Y/Z/S})\circ\rho^g_{\Omega^1_{Z/S}}\circ d_{Z/S}\\ &= g_*(\rho^f_{\Omega^1_{Y/S}})\circ g_*(d_{Y/S})\circ g^{\#}.\end{aligned}$$

It thus follows that

$$\begin{aligned}&(g\circ f)_*(h)\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\circ d_{Z/S}\\ &\qquad= (g\circ f)_*(f_{X/Y/S})\circ(g\circ f)_*(f^*(g_{Y/Z/S})\circ\varphi_{\Omega^1_{Z/S}})\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\circ d_{Z/S}\\ &\qquad= g_*(f_*(f_{X/Y/S}))\circ g_*(\rho^f_{\Omega^1_{Y/S}})\circ g_*(d_{Y/S})\circ g^{\#}\\ &\qquad= g_*(f_*(f_{X/Y/S})\circ\rho^f_{\Omega^1_{Y/S}}\circ d_{Y/S})\circ g^{\#}\\ &\qquad= g_*(f_*(d_{X/S})\circ f^{\#})\circ g^{\#}\\ &\qquad= (g\circ f)_*(d_{X/S})\circ(g\circ f)^{\#}.\end{aligned}$$

This proves that

$$(g\circ f)_*(h)\circ\rho^{g\circ f}_{\Omega^1_{Z/S}}\circ d_{Z/S} = (g\circ f)_*(d_{X/S})\circ(g\circ f)^{\#},$$

and so by the uniqueness of $(g\circ f)_{X/Z/S}$ it follows that the diagram is commutative. $\square$

Notation 3.30. For a morphism $f : X \to Y$ of S-schemes, let

$$\phi^f = (S_X f_{X/Y/S}) \circ (\alpha^f_{\Omega^1_{Y/S}})^{-1} : f^* S_Y \Omega^1_{Y/S} \to S_X \Omega^1_{X/S}.$$

Recall from Proposition 3.12 that $\alpha^f_{\Omega^1_{Y/S}}$ is an isomorphism.

Proposition 3.31. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of S-schemes, then the following diagram commutes:*

$$\begin{array}{ccc} (g\circ f)^* S_Z \Omega^1_{Z/S} & \xrightarrow{\phi^{g\circ f}} & S_X \Omega^1_{X/S} \\ {\scriptstyle \varphi_{S_Z\Omega^1_{Z/S}}}\downarrow & & \uparrow{\scriptstyle \phi^f} \\ f^* g^* S_Z \Omega^1_{Z/S} & \xrightarrow[f^*\phi^g]{} & f^* S_Y \Omega^1_{Y/S}. \end{array} \tag{3.6}$$

Proof. We want to prove that the following diagram commutes, since its outer diagram is exactly Diagram 3.6.

$$\begin{array}{ccccc} (g\circ f)^* S_Z \Omega^1_{Z/S} & \xrightarrow{(\alpha^{g\circ f}_{\Omega^1_{Z/S}})^{-1}} & S_X (g\circ f)^* \Omega^1_{Z/S} & \xrightarrow{S_X (g\circ f)_{X/Z/S}} & S_X \Omega^1_{X/S} \\ & & {\scriptstyle S_X \varphi^{f,g}_{\Omega^1_{Z/S}}}\downarrow & & \uparrow{\scriptstyle Sf_{X/Y/S}} \\ {\scriptstyle \varphi^{f,g}_{S_Z\Omega^1_{Z/S}}}\downarrow & & S_X (f^* g^* \Omega^1_{Z/S}) & \xrightarrow{S_X (f^* g_{Y/Z/S})} & S_X f^* \Omega^1_{Y/S} \\ & & {\scriptstyle (\alpha^f_{g^*\Omega^1_{Z/S}})^{-1}}\uparrow & & \uparrow{\scriptstyle (\alpha^f_{\Omega^1_{Y/S}})^{-1}} \\ f^* g^* S_Z \Omega^1_{Z/S} & \xrightarrow[f^*(\alpha^g_{\Omega^1_{Z/S}})^{-1}]{} & f^* S_Y g^* \Omega^1_{Z/S} & \xrightarrow{f^* S_Y (g_{Y/Z/S})} & f^* S_Y \Omega^1_{Y/S} \end{array}$$

Applying the functor S_X to diagram (3.5) gives us commutativity of the top right hand square. The bottom right hand square commutes by Proposition 3.14 applied

to $\mathcal{G}_1 = g^*\Omega^1_{Z/S}$, $\mathcal{G}_2 = \Omega^1_{Y/S}$ and $u = g_{Y/X/S}$, and because $\alpha^f_{\Omega^1_{Z/S}}$, $\alpha^f_{\Omega^1_{Y/S}}$ are isomorphisms (cf. Proposition 3.12). The left hand square commutes by Proposition 3.13 and because $\alpha^{g\circ f}_{\Omega^1_{Z/S}}$, $\alpha^f_{g^*\Omega^1_{Z/S}}$ and $\alpha^g_{\Omega^1_{Z/S}}$ are isomorphisms. Therefore Diagram (3.6) commutes. □

Notation 3.32. We denote by

$$\phi^f_r : f^*S^r\Omega^1_{Y/S} \to S^r\Omega^1_{X/S}$$

the induced map on S^r by ϕ^f (cf. Notation 3.30 and Lemma 3.7).

Moreover, for $f : X \to Y$, and $\mathcal{L}$ an invertible sheaf on Y, let

$$\kappa_{f,\mathcal{L},r} = (\mathrm{Id} \otimes \phi^f_r) \circ (T^f_{\mathcal{L},S^r\Omega^1_{Y/S}})^{-1} : f^*(\mathcal{L} \otimes S^r\Omega^1_{Y/S}) \to f^*\mathcal{L} \otimes S^r\Omega^1_{X/S}$$

be the composition of $(T^f_{\mathcal{L},S^r\Omega^1_{Y/S}})^{-1} : f^*(\mathcal{L} \otimes S^r\Omega^1_{Y/S}) \xrightarrow{\sim} f^*\mathcal{L} \otimes f^*S^r\Omega^1_{Y/S}$ (defined in Notation 3.20) with $\mathrm{Id} \otimes \phi^f_r : f^*\mathcal{L} \otimes f^*S^r\Omega^1_{Y/S} \to f^*\mathcal{L} \otimes S^r\Omega^1_{X/S}$.

Proposition 3.33. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of S-schemes and let $\mathcal{L}$ be an invertible sheaf on Z. The following diagram commutes.*

$$\begin{array}{ccccc} (g\circ f)^*(\mathcal{L} \otimes S^r\Omega^1_{Z/S}) & \xrightarrow{\kappa_{g\circ f,\mathcal{L},r}} & (g\circ f)^*\mathcal{L} \otimes S^r\Omega^1_{X/S} & \xrightarrow{\varphi^{f,g}_{\mathcal{L}} \otimes \mathrm{Id}} & f^*g^*\mathcal{L} \otimes S^r\Omega^1_{X/S} \\ {\scriptstyle \varphi^{f,g}_{\mathcal{L}\otimes S^r\Omega^1_{Z/S}}}\downarrow & & & & \uparrow{\scriptstyle \kappa_{f,g^*\mathcal{L},r}} \\ f^*g^*(\mathcal{L} \otimes S^r\Omega^1_{Z/S}) & & \xrightarrow[f^*\kappa_{g,\mathcal{L},r}]{} & & f^*(g^*\mathcal{L} \otimes S^r\Omega^1_{Y/S}) \end{array} \tag{3.7}$$

Proof. We want to prove that the following diagram is commutative, since its outer

diagram is exactly Diagram 3.7.

$$\begin{array}{ccccc}
(g\circ f)^*(\mathcal{L}\otimes S^r\Omega^1_{Z/S}) & \xrightarrow{(T^{g\circ f}_{\mathcal{L})^{-1},S^r\Omega^1_{Z/S}}} & (g\circ f)^*\mathcal{L}\otimes (g\circ f)^*S^r\Omega^1_{Z/S} & \xrightarrow{\mathrm{Id}\otimes\phi_r^{g\circ f}} & (g\circ f)^*\mathcal{L}\otimes S^r\Omega^1_{X/S} \\
\downarrow{\scriptstyle \varphi^{f,g}_{\mathcal{L}\otimes S^r\Omega^1_{Z/S}}} & & \downarrow{\scriptstyle \varphi^{f,g}_{\mathcal{L}}\otimes\varphi^{f,g}_{S^r\Omega^1_{Z/S}}} & & \downarrow{\scriptstyle \varphi^{f,g}_{\mathcal{L}}\otimes\mathrm{Id}} \\
 & & f^*g^*\mathcal{L}\otimes f^*g^*S^r\Omega^1_{Z/S} & \xrightarrow{\mathrm{Id}\otimes f^*\phi_r^g} & f^*g^*\mathcal{L}\otimes f^*S^r\Omega^1_{Y/S} \\
 & & \uparrow{\scriptstyle (T^f_{g^*\mathcal{L},g^*S^r\Omega^1_{Z/S}})^{-1}} & & \uparrow{\scriptstyle \mathrm{Id}\otimes\phi_r^f} \\
f^*g^*(\mathcal{L}\otimes S^r\Omega^1_{Z/S}) & \xrightarrow[f^*(T^g_{\mathcal{L},S^r\Omega^1_{Z/S}})^{-1}]{} & f^*(g^*\mathcal{L}\otimes g^*S^r\Omega^1_{Z/S}) & \xrightarrow{f^*(\mathrm{Id}\otimes\phi_r^g)} & f^*(g^*\mathcal{L}\otimes S^r\Omega^1_{Y/S}) \\
 & & & & \uparrow{\scriptstyle (T^f_{g^*\mathcal{L},S^r\Omega^1_{Y/S}})^{-1}}
\end{array}$$

We have that the top right hand square commutes, because

$$\begin{aligned}
(\mathrm{Id}\otimes\phi_r^f)\circ(\mathrm{Id}\otimes f^*\phi_r^g)\circ(\varphi^{f,g}_{\mathcal{L}}\otimes\varphi^{f,g}_{S^r\Omega^1_{Z/S}}) &= \varphi^{f,g}_{\mathcal{L}}\otimes(\phi_r^f\circ f^*\phi_r^g\circ\varphi^{f,g}_{S^r\Omega^1_{Z/S}}) \\
&= \varphi^{f,g}_{\mathcal{L}}\otimes\phi_r^{g\circ f} \\
&= (\varphi^{f,g}_{\mathcal{L}}\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\phi_r^{g\circ f}),
\end{aligned}$$

where the second equality holds by Proposition 3.31.

The bottom right hand square commutes because T^f is a functorial isomorphism. The left hand square commutes by Proposition 3.22. Therefore Diagram (3.7) commutes. □

3.2.5 ω-integrality of pullbacks

We now come to the main aim of this section, which is to relate the ω-integral curves of a smooth surface X to the "$\pi^*\omega$-integral curves" of a smooth surface X'. Here $\pi : X' \to X$ is a dominant morphism. For this, however, we first have to clarify

what we mean by $\pi^*\omega$-integral curves. Let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. By the canonical morphism

$$\rho^{\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} : \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}} \to \pi_*\pi^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$$

we obtain an induced global section $\pi^*\omega \in H^0(X', \pi^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}))$. But since $\pi^*\omega$ is a section of the sheaf $\pi^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$ (which is not of the form $\mathcal{L}' \otimes S^r\Omega^1_{X'/\mathbb{C}}$), we cannot talk about $\pi^*\omega$-integral curves as in Definition 3.2.

We work with the image $\pi^{\bullet}_{r,\mathcal{L}}\omega$ of ω in $H^0(X', \pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}})$ instead, where the homomorphism

$$\pi^{\bullet}_{r,\mathcal{L}} : H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \to H^0(X', \pi^*\mathcal{L} \otimes S^r\Omega^1_{X'\mathbb{C}})$$

is the induced map in global sections coming from the sheaf homomorphism

$$\eta_{\pi,\mathcal{L},r} := \pi_*\kappa_{\pi,\mathcal{L},r} \circ \rho^{\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} : \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}} \to \pi_*(\pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}}).$$

Definition 3.34. An irreducible curve $C' \subseteq X'$ is said to be $\pi^*\omega$-*integral* if we have $i'^{\bullet}_{r,\pi^*\mathcal{L}}\pi^{\bullet}_{r,\mathcal{L}}\omega = 0$ in $H^0(X', i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{X'})$, where $i' : \tilde{C}' \to X'$ is the composition of the normalization of C' with the inclusion to X'.

We want to prove the following:

Theorem 3.35. *Let $\pi : X' \to X$ be a morphism of smooth surfaces. Let $C' \subseteq X'$ be an irreducible curve and $C = \pi(C')$ be an irreducible curve on X. Let $\mathcal{L}$ be an invertible sheaf on X and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. The following are equivalent*

(i) The curve C is ω-integral;

(ii) The curve C' is $\pi^*\omega$-integral.

In order to prove this, we first prove some technical lemmas. For these, note that the situation of Theorem 3.35 is the following. We have the commutative diagram

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow[i']{} & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ \tilde{C} & \xrightarrow[i]{} & X \end{array}$$

where $i : \tilde{C} \to X$ is the normalization of C, the map $i' : \tilde{C}' \to X'$ is the desingularization of C' and p is the morphism from the universal property of the normalization of $\tilde{C}$ applied to $\pi \circ i'$.

The following two lemmas will help us to prove the equivalence of (i) and (ii) in Theorem 3.35.

Lemma 3.36. *There is an isomorphism* $i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}} \cong p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}$ *such that the diagram*

$$\begin{array}{ccc} H^0(X', \pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}}) & \xrightarrow{i'^{\bullet}_{r,\pi^*\mathcal{L}}} & H^0(\tilde{C}', i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\ & & \downarrow\cong \\ \uparrow{\scriptstyle \pi^{\bullet}_{r,\mathcal{L}}} & & H^0(\tilde{C}', p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\ & & \uparrow{\scriptstyle p^{\bullet}_{r,i^*\mathcal{L}}} \\ H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow[i^{\bullet}_{r,\mathcal{L}}]{} & H^0(\tilde{C}, i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \end{array} \tag{3.8}$$

commutes.

Lemma 3.37. *The map* $p^{\bullet}_{r,i^*\mathcal{L}} : H^0(\tilde{C}, i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \to H^0(\tilde{C}', p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}})$ *is injective.*

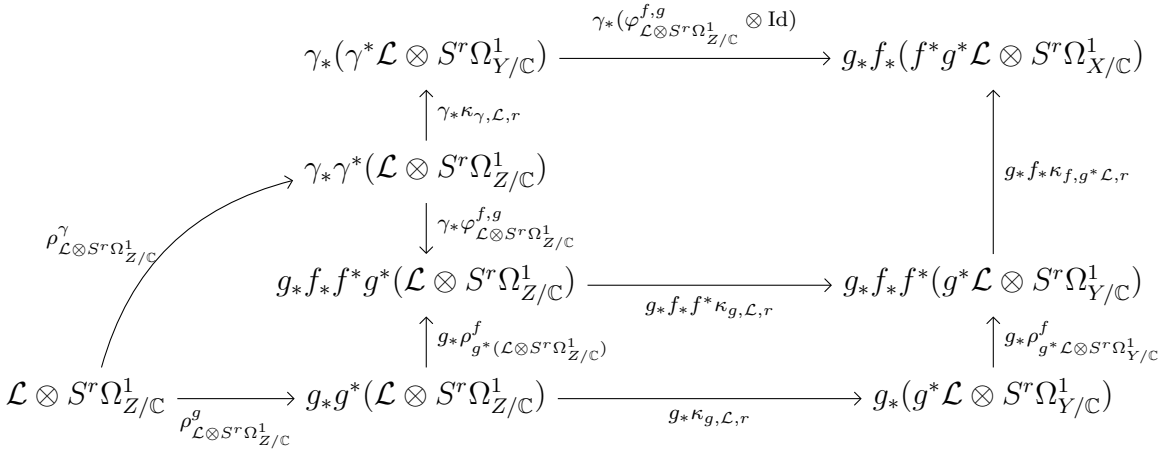
Before proving these two lemmas, we will show how they prove Theorem 3.35.

Proof of Theorem 3.35. Suppose first that C is ω-integral. Then $i^{\bullet}_{r,\mathcal{L}}\omega = 0$, hence we have $p^{\bullet}_{r,i^*\mathcal{L}} i^{\bullet}_{r,\mathcal{L}}\omega = 0$. From the commutativity of Diagram (3.8) we deduce that $i'^{\bullet}_{r,\pi^*\mathcal{L}}\pi^{\bullet}_{r,\mathcal{L}}\omega = 0$, and hence $\tilde{C}$ is $\pi^*\omega$-integral.

Now suppose that $\tilde{C}$ is $\pi^*\omega$-integral. Then $i'^{\bullet}_{r,\pi^*\mathcal{L}}\pi^{\bullet}_{r,\mathcal{L}}\omega = 0$. By Diagram (3.8) we obtain that $p^{\bullet}_{r,i^*\mathcal{L}} i^{\bullet}_{r,\mathcal{L}}\omega = 0$. By Lemma 3.37 we have that $p^{\bullet}_{r,i^*\mathcal{L}}$ is injective, and hence $i^{\bullet}_{r,\mathcal{L}}\omega = 0$. Therefore C is ω-integral. □

In the proof of Lemma 3.36 we will use the following result:

Lemma 3.38. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms and let $\gamma = g \circ f$. Let $\mathcal{L}$ be an $\mathcal{O}_Z$-module. The following diagram is commutative.*



(3.9)

Proof. The left hand side square commutes by Proposition 3.9 applied to $\mathcal{L}\otimes S^r\Omega^1_{Z/\mathbb{C}}$. The bottom left hand side square commutes by Remark 3.5. The top right hand side square commutes by Proposition 3.33. Therefore Diagram (3.9) is commutative. □

Proof of Lemma 3.36. Let $\gamma = i \circ p = \pi \circ i'$. Since $\eta_{\pi,\mathcal{L},r} = \pi_*\kappa_{\pi,\mathcal{L},r} \circ \rho^{\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}}$, we

have the following from the outer diagram of (3.9).

$$\eta_{\gamma,\mathcal{L},r} = (\gamma_*(\varphi^{p,i}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id}))^{-1} \circ i_*\eta_{p,i^*\mathcal{L},r} \circ \eta_{i,\mathcal{L},r}.$$

Similarly, since $\pi \circ i' = i \circ p$ we obtain from Diagram (3.9):

$$\eta_{\gamma,\mathcal{L},r} = (\gamma_*(\varphi^{i',\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id}))^{-1} \circ \pi_*\eta_{i',\pi^*\mathcal{L},r} \circ \eta_{\pi,\mathcal{L},r}$$

From these equalities and noting that $\gamma_*(\varphi^{p,i}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id})$ and $\gamma_*(\varphi^{i',\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id})$ are isomorphisms, we obtain the commutative diagram

$$\begin{array}{ccc}
\pi_*(\pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}}) & \xrightarrow{\pi_*\eta_{i',\pi^*\mathcal{L},r}} & \pi_*i'_*(i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\
 & & \downarrow \cong \\
\uparrow \eta_{\pi,\mathcal{L},r} & & i_*p_*(p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\
 & & \uparrow i_*\eta_{p,i^*\mathcal{L},r} \\
\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}} & \xrightarrow[\eta_{i,\mathcal{L},r}]{} & i_*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}})
\end{array}$$

where the isomorphism is

$$(\gamma_*(\varphi^{p,i}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id})) \circ (\gamma_*(\varphi^{i',\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \otimes \mathrm{Id}))^{-1} = \gamma_*(\varphi^{p,i}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}} \circ (\varphi^{i',\pi}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}})^{-1} \otimes \mathrm{Id}).$$

Taking global sections we obtain the commutative diagram

$$\begin{array}{ccc} H^0(X, \pi_*(\pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}})) & \longrightarrow & H^0(X, \pi_* i'_*(i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}})) \\ \uparrow & & \downarrow \cong \\ & & H^0(X, i_* p_*(p^* i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}})) \\ & & \uparrow \\ H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \longrightarrow & H^0(X, i_*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}})) \end{array}$$

Hence the diagram

$$\begin{array}{ccc} H^0(X', \pi^*\mathcal{L} \otimes S^r\Omega^1_{X'/\mathbb{C}}) & \xrightarrow{i'^\bullet_{r,\pi^*\mathcal{L}}} & H^0(\tilde{C}', i'^*\pi^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\ \uparrow{\scriptstyle \pi^\bullet_{r,\mathcal{L}}} & & \downarrow \cong \\ & & H^0(\tilde{C}', p^* i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}) \\ & & \uparrow{\scriptstyle p^\bullet_{r,i^*\mathcal{L}}} \\ H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow[i^\bullet_{r,\mathcal{L}}]{} & H^0(\tilde{C}, i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \end{array} \tag{3.10}$$

commutes. □

Now we want to prove Lemma 3.37. For this, we prove the following three auxiliary results.

Lemma 3.39. *Let $f : X \to Y$ be a dominant morphism of integral schemes, and let $\mathcal{L}$ be an invertible sheaf on Y. The sheaf homomorphism $\rho_{\mathcal{L}} : \mathcal{L} \to f_* f^*\mathcal{L}$ is injective.*

Proof. The homomorphism $\rho^f_{\mathcal{K}_Y} : \mathcal{K}_Y \to f_* f^*\mathcal{K}_Y = \mathcal{K}_X$ is induced from the inclusion $f^* : k(Y) \hookrightarrow k(X)$ of function fields, hence it is injective. Since $\mathcal{L}$ is invertible, it is a

subsheaf of $\mathcal{K}_Y$, thus we have an injection $i : \mathcal{L} \to \mathcal{K}_Y$. From Diagram 3.1 we obtain

$$\begin{array}{ccc} f_*f^*\mathcal{L} & \xrightarrow{f_*f^*i} & f_*f^*\mathcal{K}_Y \\ {\scriptstyle \rho^f_{\mathcal{L}}}\uparrow & & \uparrow{\scriptstyle \rho^f_{\mathcal{K}_Y}} \\ \mathcal{L} & \xhookrightarrow[i]{} & \mathcal{K}_Y \end{array}$$

Thus $\rho^f_{\mathcal{L}}$ is injective. □

Lemma 3.40. *Let $f : X \to Y$ be a non-constant morphism of smooth curves. Then*

$$0 \to f^*\Omega^1_{Y/\mathbb{C}} \xrightarrow{f_{X/Y/\mathbb{C}}} \Omega^1_{X/\mathbb{C}} \to \Omega^1_{X/Y} \to 0$$

is exact.

Proof. See [Har77], Proposition IV.2.1. Note that the proof of this result does not require X and Y to be complete. □

Lemma 3.41. *Let X be a scheme, let $\mathcal{F}, \mathcal{G}$ be invertible sheaves on X, and let $u : \mathcal{F} \to \mathcal{G}$ be a morphism. Suppose that u is injective.*

(a) We have a functorial isomorphism $S^r\mathcal{F} \cong \mathcal{F}^{\otimes r}$, for all $r \geq 1$.

(b) $\forall \mathcal{L}$ invertible sheaf, the map $\mathrm{Id} \otimes u : \mathcal{L} \otimes \mathcal{F} \to \mathcal{L} \otimes \mathcal{G}$ is injective.

(c) $\forall r \geq 1$, $S^r u : S^r\mathcal{F} \to S^r\mathcal{G}$ is injective.

Proof. Since $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of rank 1, we obtain $S^r\mathcal{F} \cong \mathcal{F}^{\otimes r}$. This proves (a).

Since tensoring with an invertible sheaf is exact, we obtain that $\mathrm{Id} \otimes u$ is injective, hence proving (b).

By part (a), $S^r(u) = u^{\otimes r} : \mathcal{F}^{\otimes r} \to \mathcal{G}^{\otimes r}$. By (b) and induction, we have that $u^{\otimes r}$ is injective. $\square$

Proof of Lemma 3.37. The morphism $p : \tilde{C}' \to \tilde{C}$ is a dominant morphism of smooth curves. Thus, by Lemma 3.39 we see that the homomorphism

$$\rho^p_{i^*\mathcal{L}\otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}} : i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}} \to p_*p^*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}})$$

is injective.

Now we prove that $\kappa_{p,i^*\mathcal{L},r} : p^*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \to p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}$ is injective. We have that $(T^p_{i^*\mathcal{L},S^r\Omega^1_{\tilde{C}/\mathbb{C}}})^{-1} : p^*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \to p^*i^*\mathcal{L} \otimes p^*S^r\Omega^1_{\tilde{C}/\mathbb{C}}$ is an isomorphism, hence is injective. By Proposition 3.12, the homomorphism

$$(\alpha^p_{S^r\Omega^1_{\tilde{C}/\mathbb{C}}})^{-1} : p^*S^r\Omega^1_{\tilde{C}/\mathbb{C}} \to S^rp^*\Omega^1_{\tilde{C}/\mathbb{C}}$$

is injective. The homomorphism $p_{\tilde{C}'/\tilde{C}/\mathbb{C}} : p^*\Omega^1_{\tilde{C}/\mathbb{C}} \to \Omega^1_{\tilde{C}'/\mathbb{C}}$ is injective by Lemma 3.40, since $p : \tilde{C}' \to \tilde{C}$ is a non-constant morphism of smooth curves. By Lemma 3.41(c), with $\mathcal{F} = p^*\Omega^1_{\tilde{C}/\mathbb{C}}$, $\mathcal{G} = \Omega^1_{\tilde{C}'/\mathbb{C}}$ we get that $S^rp_{\tilde{C}'/\tilde{C}/\mathbb{C}}$ is injective. By Lemma 3.41(b), with sheaves $\mathcal{F} = S^rp^*\Omega^1_{\tilde{C}/\mathbb{C}}$, $\mathcal{G} = S^r\Omega^1_{\tilde{C}'/\mathbb{C}}$ we obtain that the morphism $p^*i^*\mathcal{L} \otimes S^rp^*\Omega^1_{\tilde{C}/\mathbb{C}} \to p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}$ is injective, and hence the morphism

$$\kappa_{p,i^*\mathcal{L},r} : p^*(i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) \to p^*i^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}'/\mathbb{C}}$$

is injective.

Therefore the composition $\kappa_{p,i^*\mathcal{L},r} \circ \rho^p_{i^*\mathcal{L}\otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}}$ is injective. Since taking global sections is a left exact functor, we obtain that $p^\bullet_{r,i^*\mathcal{L}}$ is injective. $\square$

3.3 Curves of low genus and ω-integrality

The following result allows us to find conditions on an invertible sheaf on a surface to say that all curves of genus less than or equal to g on the surface are ω-integral for any $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. This generalizes a result in [Voj00].

Proposition 3.42. *Let $N \geq 0$, let X be a smooth projective surface and let $C \subseteq X$ be an irreducible curve of genus $g \leq N$ with $\varphi_C : \tilde{C} \to X$ the normalization of C. Let $r \in \mathbb{N}$ and $\mathcal{L}$ be an invertible sheaf on X such that*

$$\deg_{\tilde{C}}(\varphi_C^*\mathcal{L}) < 2r - 2rN.$$

Then C is ω-integral for any $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$.

Proof. Note that $\deg_{\tilde{C}}(\varphi_C^*\mathcal{L}) = \deg_C(\mathcal{L}_{|C}) = (C.\mathcal{L})$ by Proposition 2.44 and Proposition 2.50. Let $\omega \in H^0(X, \mathcal{L} \otimes \Omega^1_{X/\mathbb{C}})$. We have with the notation as in Subsection 3.2.5, that

$$\varphi^{\bullet}_{C,r,\mathcal{L}}\omega \in H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}).$$

We have $S^r\Omega^1_{\tilde{C}/\mathbb{C}} \cong K_{\tilde{C}}^{\otimes r}$ by the definition of a canonical sheaf (cf. [Har77], p. 180), and by Lemma 3.41(a) because $\Omega^1_{\tilde{C}/\mathbb{C}}$ is invertible. We obtain

$$\begin{aligned}
\deg_{\tilde{C}}(\varphi_C^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) &= \deg_{\tilde{C}}(\varphi_C^*\mathcal{L}) + \deg(K_{\tilde{C}}^{\otimes r}) \\
&= \deg_{\tilde{C}}(\varphi_C^*\mathcal{L}) + r(2g-2) \\
&\leq \deg_{\tilde{C}}(\varphi_C^*\mathcal{L}) + r(2N-2) \\
&< 0.
\end{aligned}$$

Then $H^0(\tilde{C}, \varphi_C^*\mathcal{L} \otimes S^r\Omega^1_{\tilde{C}/\mathbb{C}}) = 0$, so $\varphi^{\bullet}_{C,r,\mathcal{L}}\omega$ is forced to be zero. Thus C is ω-integral. □

3.4 Differential equations and branches

The connection between branches and ω-integral curves will be made explicit via their relations with solutions of differential equations. In this section we show under what conditions one is able to count branch solutions of differential equations.

In this section, let k be an algebraically closed field of characteristic zero.

Definition 3.43 ([Sei68b], p. 251)**.** Let $A(\mathbf{X}, \mathbf{Y}), B(\mathbf{X}, \mathbf{Y}) \in k[[\mathbf{X}, \mathbf{Y}]]$. A *solution of the differential equation*

$$A(\mathbf{X}, \mathbf{Y})d\mathbf{Y} = B(\mathbf{X}, \mathbf{Y})d\mathbf{X}$$

is a branch representation $\phi : k[[\mathbf{X}, \mathbf{Y}]] \to k[[t]]$ such that

$$A(x(t), y(t))y'(t) = B(x(t), y(t))x'(t),$$

where $x(t) = \phi(\mathbf{X})$, $y(t) = \phi(\mathbf{Y})$. Two solutions are *equivalent* if they are equivalent as branch representations (see Definition 2.55).

Remark 3.44. If ϕ_1, ϕ_2 are equivalent branch representations (i.e. if there is a continuous automorphism $\alpha \in \mathrm{Aut}_k k[[t]]$ such that $\alpha \circ \phi_1 = \phi_2$), then ϕ_1 is a solution if and only if ϕ_2 is a solution. Indeed, if $\alpha(t) = a$, with $\phi_i(X) = x_i(t)$, and $\phi_i(Y) = y_i(t)$

and if $0 = A(x_1, y_1)dy_1 - B(x_1, x_1)dx_1$ then we have

$$\begin{aligned} 0 &= A(x_1(a), y_1(a))dy_1(a) - B(x_1(a), y_1(a))dx_1(a) \\ &= A(\alpha(x_1(t)), \alpha(y_1(t)))d\alpha(y_1(t)) - B(\alpha(x_1(t)), \alpha(y_1(t)))d\alpha(x_1(t)) \\ &= A(x_2(t), y_2(t))dy_2(t) - B(x_2(t), y_2(t))dx_2(t). \end{aligned}$$

The following theorem is Theorem 2 of [Sei68b].

Theorem 3.45. *Suppose that $A(0,0) \neq 0$. Then the differential equation*

$$A(\boldsymbol{X}, \boldsymbol{Y})d\boldsymbol{Y} = B(\boldsymbol{X}, \boldsymbol{Y})d\boldsymbol{X}$$

has only one solution up to equivalence. It is a linear branch representation associated to a branch $\mathfrak{P}$ whose tangent is non-vertical (cf. Section 2.3.1).

Now we want to relate solutions of differential equations in the sense of Definition 3.43 with solutions of elements in certain modules of differentials, in a sense that we will explain below. More specifically, branch representations $\phi : k[[\mathbf{X},\mathbf{Y}]] \to k[[t]]$ induce maps on differential k-algebras.

Definition 3.46. A *differential algebra of R/R_0* is an associative (not necessarily commutative) graded R-algebra $\Omega = \oplus_{n\in\mathbb{N}}\Omega^n$, on which an R_0-linear map $d : \Omega \to \Omega$ of degree 1 is given (i.e. $d\Omega^n \subseteq \Omega^{n+1}$) such that the following axioms are satisfied

- $\Omega^0 = R$ and R is contained in the center of Ω.
- $\Omega = R(dR)$ (i.e. as an R-algebra Ω is generated by the elements dr $(r \in R)$).
- For all $r, r' \in R$ we have $d(rr') = rdr' + r'dr$.

- For all $r, r_1, \ldots, r_m \in R$ we have $d(rdr_1 \cdots dr_m) = drdr_1 \cdots dr_m$.
- $drdr = 0$ for each $r \in R$.

The mapping d is called the *differentiation* of Ω and the elements of Ω^n are called n-forms.

Definition 3.47 ([Kun86], p. 171)**.** Let R_0 be a ring and R be an R_0-algebra. A *universally finite differential algebra of* R/R_0 is a finite differential R-algebra Ω of R/R_0 such that each finite differential R-algebra of R/R_0 is a homomorphic image of Ω (with respect to an R-homomorphism). If it exists, it is unique up to a canonical R-isomorphism. It will be denoted by $\tilde{\Omega}_{R/R_0}$.

Remark 3.48. By [Kun86], Example 12.7, we have that the universally finite differential algebra $\tilde{\Omega}_{k[[\mathbf{X},\mathbf{Y}]]/k}$ exists, and that its degree one part satisfies

$$\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} = k[[\mathbf{X},\mathbf{Y}]]d\mathbf{X} \oplus k[[\mathbf{X},\mathbf{Y}]]d\mathbf{Y}.$$

Similarly, $\tilde{\Omega}_{k[[t]]/k}$ exists and its degree one part satisfies

$$\tilde{\Omega}^1_{k[[t]]/k} = k[[t]]dt.$$

Taking symmetric powers we obtain

$$S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} = S^r(k[[\mathbf{X},\mathbf{Y}]]d\mathbf{X} \oplus k[[\mathbf{X},\mathbf{Y}]]d\mathbf{Y}) = \oplus_{i=0}^r k[[\mathbf{X},\mathbf{Y}]](d\mathbf{X})^{r-i}(d\mathbf{Y})^i,$$

$$S^r\tilde{\Omega}^1_{k[[t]]/k} = k[[t]](dt)^r.$$

Therefore, for $\tilde{\omega} \in S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k}$ we have

$$\tilde{\omega} = \sum_{i=0}^{r} A_i (d\mathbf{X})^{r-i} (d\mathbf{Y})^i,$$

with $A_i \in k[[\mathbf{X},\mathbf{Y}]]$. We call this expression a *representation of* $\tilde{\omega}$.

Notation 3.49. Given a local map $\sigma : k[[\mathbf{X},\mathbf{Y}]] \to k[[t]]$, we have by [Kun86], Def. 11.3, an induced homomorphism of differential algebras $\sigma : \tilde{\Omega}_{k[[\mathbf{X},\mathbf{Y}]]/k} \to \tilde{\Omega}_{k[[t]]/k}$, and this map induces a map on the degree one component $\sigma_1 : \tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \to \tilde{\Omega}^1_{k[[t]]/k}$. Taking symmetric powers we obtain a continuous homomorphism of algebras

$$\tilde{\sigma} : S\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \to S\tilde{\Omega}^1_{k[[t]]/k},$$

which restricts to a continuous map on r-th symmetric powers

$$\sigma_r : S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \to S^r\tilde{\Omega}^1_{k[[t]]/k}.$$

Definition 3.50. Given $\tilde{\omega} \in S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k}$, a *solution of* $\tilde{\omega}$ is a branch $\mathfrak{P}$ such that for any primitive branch representation $\phi_{\mathfrak{P}} : k[[\mathbf{X},\mathbf{Y}]] \to k[[t]]$ of $\mathfrak{P}$, the induced map $\phi_{\mathfrak{P},r} : S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \to S^r\tilde{\Omega}^1_{k[[t]]/k}$ satisfies $\phi_{\mathfrak{P},r}(\tilde{\omega}) = 0$.

Remark 3.51. Note that the property that $\sigma_{\mathfrak{P},r}(\tilde{\omega}) = 0$ does not depend on the branch representation we use. Indeed, let $\sigma : k[[\mathbf{X},\mathbf{Y}]] \to k[[t]]$ be a branch representation of a solution $\mathfrak{P}$ of $\tilde{\omega}$. Then $\sigma_{\mathfrak{P},r}(\tilde{\omega}) = 0$. Let σ' be another branch representation of $\mathfrak{P}$, so there is an $\alpha \in \mathrm{Aut}(k[[t]])$ such that $\alpha \circ \sigma = \sigma'$. Then α induces an isomorphism $\bar{\alpha} : S^r\tilde{\Omega}^1_{k[[t]]/k} \xrightarrow{\sim} S^r\tilde{\Omega}^1_{k[[t]]/k}$, and we have

$$\begin{aligned}
\sigma'_{\mathfrak{P},r}(\tilde{\omega}) &= \sigma'_{\mathfrak{P},r}\left(\sum_{i=0}^{r} A_i(d\mathbf{X})^{r-i}(d\mathbf{Y})^i\right) \\
&= \sum_{i=0}^{r} \sigma'_{\mathfrak{P}}(A_i)(d\sigma'_{\mathfrak{P}}(\mathbf{X}))^{r-i}(d\sigma'_{\mathfrak{P}}(\mathbf{Y}))^i \\
&= \sum_{i=0}^{r} \alpha\circ\sigma_{\mathfrak{P}}(A_i)(d(\alpha\circ\sigma_{\mathfrak{P}}(\mathbf{X})))^{r-i}(d(\alpha\circ\sigma_{\mathfrak{P}}(\mathbf{Y})))^i \\
&= \bar{\alpha}(\sum_{i=0}^{r} \sigma_{\mathfrak{P}}(A_i)(d\sigma_{\mathfrak{P}}(\mathbf{X}))^{r-i}(d\sigma_{\mathfrak{P}}(\mathbf{Y}))^i) \\
&= \bar{\alpha}(\sigma_{\mathfrak{P},r}(\tilde{\omega})) = 0
\end{aligned}$$

where the fourth equality holds by the definition of $\bar{\alpha}$.

Theorem 3.52. *Let $\tilde{\omega} \in S^r\tilde{\Omega}^1_{k[[\boldsymbol{X},\boldsymbol{Y}]]/k}$, and let*

$$\sum_{i=0}^{r} A_i(d\boldsymbol{X})^{r-i}(d\,\boldsymbol{Y})^i$$

be a representation of $\tilde{\omega}$. Let $\mathfrak{P}$ be a branch and let $\sigma_{\mathfrak{P}}$ be a branch representation of $\mathfrak{P}$. Then $\mathfrak{P}$ is a solution of $\tilde{\omega}$ if and only if

$$\sum_{i=0}^{r} A_i(x,y)(x')^{r-i}(y')^i = 0$$

for $x(t) = \sigma_{\mathfrak{P}}(\boldsymbol{X})$, $y(t) = \sigma_{\mathfrak{P}}(\boldsymbol{Y})$.

Proof. We have

$$\begin{aligned}
\sigma_{\mathfrak{P},r}(\tilde{\omega}) &= \sigma_{\mathfrak{P},r}\left(\sum_{i=0}^{r} A_i(d\mathbf{X})^{r-i}(d\mathbf{Y})^i\right) \\
&= \sum_{i=0}^{r} \sigma_{\mathfrak{P}}(A_i)(d\sigma_{\mathfrak{P}}(\mathbf{X}))^{r-i}(d\sigma_{\mathfrak{P}}(\mathbf{Y}))^i
\end{aligned}$$

$$\begin{aligned} &= \sum_{i=0}^{r} A_i(x,y)(dx)^{r-i}(dy)^i \\ &= \sum_{i=0}^{r} A_i(x,y)(x'dt)^{r-i}(y'dt)^i \\ &= \sum_{i=0}^{r} A_i(x,y)(x')^{r-i}(y')^i(dt)^r, \end{aligned}$$

where the second equality comes from the definition of $\sigma_{\mathfrak{P},r}$, the third by the continuity of $\sigma_{\mathfrak{P}}$, and the fourth by the continuity of d (see [Kun86], p. 182). Since $(dt)^r$ is a basis of $S^r\Omega^1_{k[[t]]/k}$, we get that $\mathfrak{P}$ is a solution of $\tilde{\omega}$ (i. e. $\sigma_{\mathfrak{P},r}(\tilde{\omega}) = 0$) if and only if

$$\sum_{i=0}^{r} A_i(x,y)(x')^{r-i}(y')^i = 0.$$

□

Note that when $r = 1$, Theorem 3.52 shows that Definition 3.43 and Definition 3.50 are equivalent.

3.4.1 Discriminants and counting solutions

Recall the definition of the discriminant of a polynomial from [Bou90], p. A.IV.81 and [Bou89] §III.9.5.

Definition 3.53. Let R be a commutative ring, let f be a monic polynomial of $R[T]$ of degree m, and denote by E the R-algebra $R[T]/(f)$ and by x the canonical image of T in E. We define the *discriminant* of f, written $\mathrm{disc}(f)$, as the discriminant $D_{E/R}(1, x, \ldots, x^{m-1})$ of the basis $(1, x, \ldots, x^{m-1})$ of the R-algebra E. Thus

$$\mathrm{disc}(f) = \det(\mathrm{Tr}_{E/R}(x^{i+j-2})_{1\leq i,j\leq m}),$$

where $\mathrm{Tr}_{E/R} : E \to R$ denotes the trace.

Proposition 3.54. *Let $f \in R[T]$ be of degree m. Let $\alpha_1, \ldots, \alpha_m$ be elements of R. If $f = (T - \alpha_1) \cdots (T - \alpha_m)$, then we have*

$$\mathrm{disc}(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

Proof. This is Proposition 11 in [Bou90], p. A.IV.83. □

We now show a way to count the number of solutions of a differential equation coming from a representation of $\tilde{\omega} \in S^r \tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k}$ under certain conditions.

Theorem 3.55. *Let*

$$\sum_{i=0}^{r} A_i(X, Y)(d\boldsymbol{X})^{r-i}(d\,\boldsymbol{Y})^i$$

be a representation of $\tilde{\omega}$ in $S^r \tilde{\Omega}^1_{k[[\boldsymbol{X},\boldsymbol{Y}]]/k}$. Suppose that $A_0(0,0) \neq 0$ and $\delta(0,0) \neq 0$, with $\delta = \mathrm{disc}(\sum_{i=0}^{r} \frac{A_i}{A_0} T^{r-i})$. Then $\tilde{\omega}$ has at most r distinct solutions and they are linear branch representations that have non-vertical tangents.

Proof. The module $\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k}$ is a free module on $k[[\mathbf{X},\mathbf{Y}]]$ with basis $\{d\mathbf{X}, d\mathbf{Y}\}$, so there is an isomorphism of graded rings $S\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \cong k[[\mathbf{X},\mathbf{Y}]][Z_1, Z_2]$ by [Kun86] p. 171. We can associate to $\tilde{\omega}$ the polynomial $\sum_{i=0}^{r} A_i Z_1^{r-i} Z_2^i$. Dividing by $A_0 Z_2^r$ (and writing $T = \frac{Z_2}{Z_1}$) we obtain the monic polynomial $\sum_{i=0}^{r} \frac{A_i}{A_0} T^{r-i}$. Since A_0 is invertible the elements $\frac{A_i}{A_0}$ are power series. Since $\delta(0,0) \neq 0$ we get that the solutions of this polynomial modulo $(\mathbf{X},\mathbf{Y})$ are distinct in $k \cong k[[\mathbf{X},\mathbf{Y}]]/(\mathbf{X},\mathbf{Y})$, thus by Hensel's Lemma (Theorem 17 in [AM69] Ex.9, p. 115 repeatedly) this polynomial can be factored into $\sum_{i=0}^{r} \frac{A_i}{A_0} T^{r-i} = (T - \alpha_1) \cdots (T - \alpha_r)$ with α_i distinct elements in $k[[\mathbf{X},\mathbf{Y}]]$. Using again the isomorphism $S^r \tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \cong k[[\mathbf{X},\mathbf{Y}]][Z_1, Z_2]_r$ we obtain

that $(Z_2 - Z_1\alpha_1)\cdots(Z_2 - Z_1\alpha_r)$ is associated to $(d\mathbf{Y} - \alpha_1 d\mathbf{X})\cdots(d\mathbf{Y} - \alpha_r d\mathbf{X})$. Hence $\sum_{i=0}^{r} A_i(d\mathbf{X})^{r-i}(d\mathbf{Y})^i = A_0(d\mathbf{Y} - \alpha_1 d\mathbf{X})\cdots(d\mathbf{Y} - \alpha_r d\mathbf{X})$.

Let $\mathfrak{P}$ be a solution of $\tilde{\omega}$ with branch representation $\sigma_{\mathfrak{P}}$. Then we have the induced maps $\sigma_{\mathfrak{P},r} : S^r\tilde{\Omega}^1_{k[[\mathbf{X},\mathbf{Y}]]/k} \to S^r\tilde{\Omega}^1_{k[[t]]/k}$. Moreover, $\mathfrak{P}$ is a solution of $\tilde{\omega}$ if and only if $\sigma_{\mathfrak{P},r}(\tilde{\omega}) = 0$. Then

$$\begin{aligned} 0 &= \sigma_{\mathfrak{P},r}(\tilde{\omega}) = \tilde{\sigma}_{\mathfrak{P}}(\tilde{\omega}) = \tilde{\sigma}_{\mathfrak{P}}(\sum_{i=0}^{r} A_i(\mathbf{X},\mathbf{Y})(d\mathbf{X})^{r-i}(d\mathbf{Y})^i) \\ &= \tilde{\sigma}_{\mathfrak{P}}(d\mathbf{Y} - \alpha_1 d\mathbf{X})\cdots(d\mathbf{Y} - \alpha_r d\mathbf{X}) \\ &= \tilde{\sigma}_{\mathfrak{P}}(d\mathbf{Y} - \alpha_1 d\mathbf{X})\cdots\tilde{\sigma}_{\mathfrak{P}}(d\mathbf{Y} - \alpha_r d\mathbf{X}) \in S\tilde{\Omega}^1_{k[[t]]/k}. \end{aligned}$$

Since $S\tilde{\Omega}^1_{k[[t]]/k} \cong k[[X,Y]][Z_1,Z_2]$ is an integral domain, we obtain that

$$\sigma_{\mathfrak{P},r}(d\mathbf{Y} - \alpha_i d\mathbf{X}) = \tilde{\sigma}_{\mathfrak{P}}(d\mathbf{Y} - \alpha_i d\mathbf{X}) = 0$$

for some i. Hence $\mathfrak{P}$ is a solution of some $d\mathbf{X} - \alpha_i d\mathbf{Y} = 0$ (i.e $x' - \alpha_i y' = 0$). Since $d\mathbf{X} - \alpha_i d\mathbf{Y} = 0$ has exactly one solution (by Theorem 3.45), we get that there are at most r solutions of the equation, and that they are linear branch representations, each having non-vertical tangent by Theorem 3.45. □

3.5 Connection between ω-integral curves and differential equations

In this section we explore the relation between the notion of ω-integral curves and (local) solutions of formal differential equations in the sense of branches. The main results of this section are Theorems 3.66 and 3.67.

Let X and Y be smooth varieties (not necessarily projective) defined over $\mathbb{C}$, and let $f : Y \to X$ be a morphism. Let $\mathcal{L}$ be an invertible sheaf on X, let $r \geq 1$ and let

$\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$.

As in Subsection 3.2.5, from the map $\eta_{f,\mathcal{L},r} : \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}} \to f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}})$ we obtain by taking global sections the homomorphism

$$f^\bullet_{r,\mathcal{L}} : H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \;\; \to \;\; H^0(X, f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}}) = H^0(Y, f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}}).$$

Let $Q \in Y$ be a point on Y, and let $P := f(Q)$. Taking stalks at P we obtain the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow{f^\bullet_{r,\mathcal{L}}} & H^0(X, f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}})) \\ \downarrow & & \downarrow \\ (\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[(\eta_{f,\mathcal{L},r})_P]{} & (f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}}))_P. \end{array}$$

Recall from Subsection 3.2.5 that the morphism $\eta_{f,\mathcal{L},r}$ decomposes as follows

$$\begin{array}{lll} \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}} & \overset{\rho^f_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}}}{\to} & f_*f^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \\ & \overset{f_*(T^f_{\mathcal{L},S^r\Omega^1_{X/\mathbb{C}}})^{-1}}{\to} & f_*(f^*\mathcal{L} \otimes f^*S^r\Omega^1_{X/\mathbb{C}}) \\ & \overset{f_*(\mathrm{Id}\otimes\phi^f_r)}{\to} & f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}}). \end{array}$$

Notation 3.56. Let

$$\bar{\beta}_f := \phi^f_{r,Q} \circ f^{\#}_{S^r\Omega^1_{X/\mathbb{C}},P} : (S^r\Omega^1_{X/\mathbb{C}})_P \to (S^r\Omega^1_{Y/\mathbb{C}})_Q,$$

where $f^{\#}_{S^r\Omega^1_{X/\mathbb{C}},P} : (S^r\Omega^1_{X/\mathbb{C}})_P \to (f^*(S^r\Omega^1_{X/\mathbb{C}}))_Q$ is defined in Notation 2.73.

Proposition 3.57. *There exist injective vertical maps which make the following diagram commutative:*

$$\begin{array}{ccc} H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow{f^\bullet_{r,\mathcal{L}}} & H^0(X, f_*(f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}})) \\ \downarrow & & \downarrow \\ (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[\bar{\beta}_f]{} & (S^r\Omega^1_{Y/\mathbb{C}})_Q \end{array}$$

Therefore, given $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$, one has that $f^\bullet_{r,\mathcal{L}}(\omega) = 0$ if and only if $\bar{\beta}_f(\omega_P) = 0$, where ω_P is the image of ω in $(S^r\Omega^1_{X/\mathbb{C}})_P$.

For this we will need the following lemmas.

Lemma 3.58. *We have the commutative diagram*

$$\begin{array}{ccc} H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow{f^\bullet_{r,\mathcal{L}}} & H^0(Y, f^*\mathcal{L} \otimes S^r\Omega^1_{Y/\mathbb{C}}) \\ \downarrow & & \downarrow \\ \mathcal{L}_P \otimes (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[\beta_f]{} & (f^*\mathcal{L})_Q \otimes (S^r\Omega^1_{Y/\mathbb{C}})_Q, \end{array}$$

where $\beta_f = f^\#_{\mathcal{L},P} \otimes \bar{\beta}_f$, and the vertical arrows map sections to the respective stalks. Moreover, both vertical arrows are injective.

Proof. Recall Notation 2.73. We have the commutative diagram

$$\begin{array}{ccc} (\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow{(\rho^f_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}})_P} & (f_*f^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}))_P \\ & \searrow^{f^\#_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}},Q}} & \downarrow f^\circ_{f^*(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}),Q} \\ & & (f^*(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}))_Q \end{array}$$

Let $\gamma_1 = f^\circ_{f^*(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}),Q}$, $\gamma_2 = f^\circ_{f^*\mathcal{L}\otimes\varphi_C^* S^r\Omega^1_{X/\mathbb{C}},Q}$ and $\gamma_3 = f^\circ_{f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}},Q}$.

From the definition of γ_1, γ_2 and γ_3 the following diagram commutes:

$$\begin{array}{ccccc}
(f_*f^*(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}))_P & \xrightarrow{(f_*(T^f_{\mathcal{L},S^r\Omega^1_{X/\mathbb{C}}})^{-1})_P} & (f_*(f^*\mathcal{L}\otimes f^*S^r\Omega^1_{X/\mathbb{C}}))_P & \xrightarrow{(f_*(\mathrm{Id}\otimes\phi^f_{r,Q}))_P} & (f_*(f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}))_P \\
\downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} \\
(f^*(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}))_Q & \xrightarrow[((T^f_{\mathcal{L},S^r\Omega^1_{X/\mathbb{C}}})^{-1})_Q]{} & (f^*\mathcal{L}\otimes f^*S^r\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow[(\mathrm{Id}\otimes\phi^f_{r,Q})_Q]{} & (f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})_Q
\end{array}$$

From Proposition 3.24 (with $\mathcal{G}=\mathcal{L}$, $\mathcal{H}=S^r\Omega^1_{X/\mathbb{C}}$), we obtain the following commutative diagram:

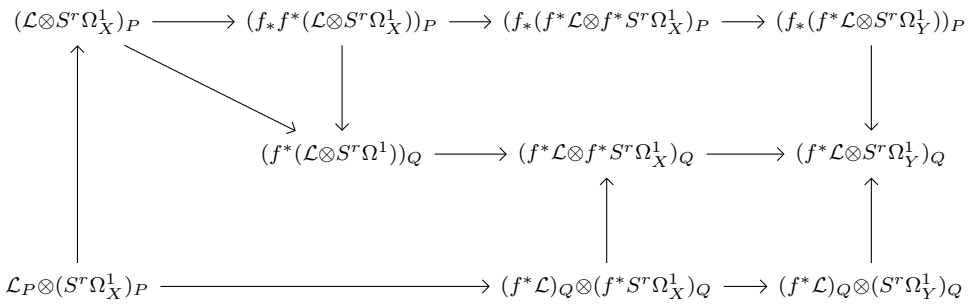
$$\begin{array}{ccccc}
(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow{f^{\#}_{\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}},Q}} & (f^*(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}))_Q & \xrightarrow{(T^f_{\mathcal{L},S^r\Omega^1_{X/\mathbb{C}}})^{-1}_Q} & (f^*\mathcal{L}\otimes f^*S^r\Omega^1_{X/\mathbb{C}})_Q \\
\uparrow{\scriptstyle\cong} & & & & {\scriptstyle\cong}\uparrow \\
\mathcal{L}_P\otimes(S^r\Omega^1_{X/\mathbb{C}})_P & & \xrightarrow[f^{\#}_{\mathcal{L},P}\otimes f^{\#}_{S^r\Omega^1_{X/\mathbb{C}},P}]{} & & (f^*\mathcal{L})_Q\otimes(f^*S^r\Omega^1_{X/\mathbb{C}})_Q
\end{array}$$

Distributing stalks over the tensor product we obtain the commutative diagram

$$\begin{array}{ccc}
(f^*\mathcal{L}\otimes f^*S^r\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow{(\mathrm{Id}\otimes\phi^f_{r,Q})_Q} & (f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})_Q \\
{\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\cong} \\
(f^*\mathcal{L})_Q\otimes(f^*S^r\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow[\mathrm{Id}\otimes(\phi^f_{r,Q})_Q]{} & (f^*\mathcal{L})_Q\otimes(S^r\Omega^1_{Y/\mathbb{C}})_Q.
\end{array}$$

Putting the previous diagrams together we deduce (with simplified notation, where

the arrows are the morphisms indicated in the previous four diagrams):



Hence, by definition of $\eta_{f,\mathcal{L},r}$ and by definition of $f^{\bullet}_{r,\mathcal{L}}$, the previous commutative diagram together with Remark 2.72 gives the commutative diagram

$$\begin{array}{ccc}
H^0(X, \mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow[f^{\bullet}_{r,\mathcal{L}}]{} & H^0(X, f_*(f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})) \\
\downarrow & & \downarrow{\scriptstyle \lambda} \\
(\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[(\eta_{f,\mathcal{L},r})_P]{} & (f_*(f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}))_P \\
 & & \downarrow{\scriptstyle \gamma_3} \\
\cong\uparrow & & (f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})_Q \\
 & & \uparrow{\scriptstyle \cong} \\
\mathcal{L}_P\otimes (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[f^{\#}_{\mathcal{L},P}\otimes(\bar{\beta}_f)]{} & (f^*\mathcal{L})_Q\otimes (S^r\Omega^1_{Y/\mathbb{C}})_Q
\end{array}$$

We claim that $\gamma_3 \circ \lambda$ is injective. Indeed, from the definition of γ_3 we have the following commutative diagram

$$\begin{array}{ccc}
H^0(X, f_*(f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})) & = & H^0(Y, f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}) \\
\lambda\downarrow & & \downarrow \\
(f_*(f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}))_P & \xrightarrow[\gamma_3]{} & (f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}})_Q
\end{array}$$

where both vertical arrows are injective by Corollary 2.71 (because $S^r\Omega^1_{Y/\mathbb{C}}$ is locally free).

Therefore we get the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow{f^\bullet_{r,\mathcal{L}}} & H^0(Y, f^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}) \\ \downarrow & & \downarrow \\ \mathcal{L}_P\otimes (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[\beta_f]{} & (f^*\mathcal{L})_Q\otimes (S^r\Omega^1_{Y/\mathbb{C}})_Q. \end{array}$$

where $\beta_f = f^\#_{\mathcal{L},P}\otimes(\bar\beta_f)$, the vertical arrows map sections to the respective stalks. Moreover the vertical arrows are injective. □

Lemma 3.59. *There is a neighborhood U of P and an isomorphism $\epsilon : \mathcal{L}_{|U}\to\mathcal{O}_{X|U}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{L}_P\otimes (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow{\beta_f} & (f^*\mathcal{L})_Q\otimes (S^r\Omega^1_{Y/\mathbb{C}})_Q \\ {\scriptstyle \epsilon_P\otimes \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle (f^*_{|U}\epsilon)_Q\otimes\mathrm{Id}} \\ \mathcal{O}_{X,P}\otimes_{\mathcal{O}_{X,P}}(S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[f^\#_Q\otimes\bar\beta_f]{} & \mathcal{O}_{Y,Q}\otimes_{\mathcal{O}_{Y,Q}}(S^r\Omega^1_{Y/\mathbb{C}})_Q \\ \downarrow & & \downarrow \\ (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[\bar\beta_f]{} & (S^r\Omega^1_{Y,\mathbb{C}})_Q, \end{array} \tag{3.11}$$

where the lower vertical arrows are the canonical isomorphisms and the upper vertical arrows are also injective, and β_f is as defined in Lemma 3.58.

Proof. Since $\mathcal{L}$ is invertible, there is an open neighborhood $U\subset X$ of P such that $\mathcal{L}_{|U}\cong\mathcal{O}_{X|U}$. Denote by ϵ this isomorphism. We have an induced isomorphism $\epsilon_P : \mathcal{L}_P\to\mathcal{O}_{X,P}$. Since $f^*_{|U}$ is a functor we have $(f^*_{|U}\epsilon)_Q : (f^*_{|U}\mathcal{L}_{|U})_Q\to(f^*_{|U}\mathcal{O}_{X|U})_Q$ is

also an isomorphism. From Proposition 2.75 we have the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_P & \xrightarrow{f^{\#}_{\mathcal{L},Q}} & (f^*\mathcal{L})_Q \\ {\scriptstyle \epsilon_P}\downarrow & & \downarrow{\scriptstyle (f^*_{|U}\epsilon)_Q} \\ \mathcal{O}_{X,P} & \xrightarrow[f^{\#}_{\mathcal{O}_X,Q}]{} & (f^*\mathcal{O}_X)_Q. \end{array}$$

We have that the homomorphism $f^{\#}_{\mathcal{O}_X,Q} : \mathcal{O}_{X,P} \to (f^*\mathcal{O}_X)_Q$ and the homomorphism $f^{\#}_Q : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ are equal by Remark 2.74. From this we obtain that the following diagram commutes (by checking on elements)

$$\begin{array}{ccc} (\mathcal{L})_P \otimes_{\mathcal{O}_{X,P}} (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow{\beta_f} & (f^*\mathcal{L})_Q \otimes_{\mathcal{O}_{Y,Q}} (S^r\Omega^1_{Y/\mathbb{C}})_Q \\ {\scriptstyle \epsilon_P\otimes \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle (f^*_{|U}\epsilon)_Q\otimes \mathrm{Id}} \\ \mathcal{O}_{X,P} \otimes_{\mathcal{O}_{X,P}} (S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[f^{\#}_Q\otimes\bar{\beta}_f]{} & \mathcal{O}_{Y,Q} \otimes_{\mathcal{O}_{Y,Q}} (S^r\Omega^1_{Y,\mathbb{C}})_Q. \end{array}$$

Using the canonical isomorphisms

$$\mathcal{O}_{X,P} \otimes_{\mathcal{O}_{X,P}} (S^r\Omega^1_{X/\mathbb{C}})_P \to (S^r\Omega^1_{X/\mathbb{C}})_P$$

and

$$\mathcal{O}_{Y,Q} \otimes_{\mathcal{O}_{Y,Q}} (S^r\Omega^1_{Y,\mathbb{C}})_Q \to (S^r\Omega^1_{Y,\mathbb{C}})_Q$$

we obtain that Diagram (3.11) commutes.

Since ϵ_P and $(f^*_{|U}\epsilon)_Q$ are isomorphisms we obtain that the vertical maps are injective. □

Proof of Proposition 3.57. From Lemma 3.58 and Lemma 3.59 we obtain the result.

□

Write $A := \mathcal{O}_{X,P}$ and $B := \mathcal{O}_{Y,Q}$. The homomorphism $f_Q^{\#} : A \to B$ induces a unique homomorphism of A-modules

$$v_{1,f,Q} : \Omega^1_{A/\mathbb{C}} \to \Omega^1_{B/\mathbb{C}}$$

such that $v_{1,f,Q} \circ d_A = d_B \circ f_Q^{\#}$ with d_A and d_B the respective associated derivations.

From the universal property of $S_A\Omega^1_{B/\mathbb{C}}$ applied to the inclusion homomorphism $\Omega^1_{B/\mathbb{C}} \to S_B\Omega^1_{B/\mathbb{C}}$, we obtain the A-algebra map $S_A\Omega^1_{B/\mathbb{C}} \to S_B\Omega^1_{B/\mathbb{C}}$, which induces the map

$$v_{r,f,Q} : S^r_A\Omega^1_{A/\mathbb{C}} \to S^r_B\Omega^1_{B/\mathbb{C}}.$$

Consider the isomorphism

$$\begin{aligned} w_{X,P} : \Omega^1_{A/\mathbb{C}} &\xrightarrow{\sim} (\Omega^1_{X/\mathbb{C}})_P \\ f_P dg_P &\mapsto (fdg)_P \end{aligned}$$

where $f, g \in \mathcal{O}_X(U)$ for some neighborhood U of P.

Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. The isomorphism $s_{\mathcal{F},P}$ (cf. Lemma 3.17) induces the following isomorphism in degree r:

$$\begin{aligned} s_{r,\mathcal{F},P} : S^r_A\mathcal{F}_P &\xrightarrow{\sim} (S^r_X\mathcal{F})_P \\ t_{1,P} \cdots t_{r,P} &\mapsto (t_1 \cdots t_r)_P \end{aligned}$$

where $t_1, \ldots, t_r$ are in $\mathcal{F}(U)$.

By [EGA] (II,1.7.4), we have the isomorphism

$$\begin{aligned} \mu_{r,X,P} := s_{r,\Omega^1_{X/\mathbb{C}},P} \circ S^r_A w_{X,P} : S^r_A \Omega^1_{A/\mathbb{C}} &\xrightarrow{\sim} (S^r_X \Omega^1_{X/\mathbb{C}})_P \\ f_P dg_{1,P} \cdots dg_{r,P} &\mapsto (f dg_1 \cdots dg_r)_P. \end{aligned}$$

The following two Lemmas are easily verified by applying the maps to elements of the stalks.

Lemma 3.60. *The following diagram is commutative:*

$$\begin{array}{ccccc} \Omega^1_{\mathcal{O}_{X,P}/\mathbb{C}} & & \xrightarrow{\quad v_{1,r,Q} \quad} & & \Omega^1_{\mathcal{O}_{Y,Q}/\mathbb{C}} \\ \downarrow{\scriptstyle w_{X,P}} & & & & \downarrow{\scriptstyle w_{Y,Q}} \\ (\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[f^{\#}_{\Omega^1_{X/\mathbb{C}},Q}]{} & (f^*\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow[f_{X/Y/\mathbb{C},Q}]{} & (\Omega^1_{Y/\mathbb{C}})_Q \end{array}$$

Lemma 3.61. *Let Y be a scheme and let $Q \in Y$. Let $u : \mathcal{F} \to \mathcal{G}$ be a morphism of locally free $\mathcal{O}_Y$-modules. Then the following diagram commutes:*

$$\begin{array}{ccc} (S^r_Y \mathcal{F})_Q & \xrightarrow{(S^r_Y u)_Q} & (S^r_Y \mathcal{G})_Q \\ \uparrow{\scriptstyle s_{r,\mathcal{F},Q}} & & \uparrow{\scriptstyle s_{r,\mathcal{G},Q}} \\ S^r_Y(\mathcal{F}_Q) & \xrightarrow[S^r_Y u_Q]{} & S^r_Y(\mathcal{G}_Q). \end{array}$$

We also need the following lemma:

Lemma 3.62. *Let $f : Y \to Z$ be a morphism of schemes. Let Q be a point on Y and*

$P = f(Q) \in Z$. Let $\mathcal{F}$ be a locally free sheaf on Z. The following diagram commutes:

$$\begin{array}{ccccc} S^r_{\mathcal{O}_{Z,P}}\mathcal{F}_P & & \xrightarrow{\quad s_{r,\mathcal{F},P} \quad} & & (S^r_Z\mathcal{F})_P \\ {\scriptstyle S^r f^{\#}_{\mathcal{F},Q}}\downarrow & & & & \downarrow{\scriptstyle f^{\#}_{S^r_Z\mathcal{F},Q}} \\ S^r_{\mathcal{O}_{Y,Q}}(f^*\mathcal{F})_Q & \xrightarrow[s_{r,f^*\mathcal{F},Q}]{} & (S^r_Y f^*\mathcal{F})_Q & \xrightarrow[(\alpha^f)_Q]{} & (f^* S^r_Z\mathcal{F})_Q \end{array}$$

Proof. This follows from Proposition 3.18. □

Proposition 3.63. *The following diagram commutes:*

$$\begin{array}{ccc} S^r_{\mathcal{O}_{X,P}}\Omega^1_{\mathcal{O}_{X,P}/\mathbb{C}} & \xrightarrow{v_{r,f,Q}} & S^r_{\mathcal{O}_{Y,Q}}\Omega^1_{\mathcal{O}_{Y,Q}/\mathbb{C}} \\ {\scriptstyle \mu_{r,X,P}}\downarrow & & \downarrow{\scriptstyle \mu_{r,Y,Q}} \\ (S^r_X\Omega^1_{X/\mathbb{C}})_P & \xrightarrow[\bar{\beta}_f]{} & (S^r_Y\Omega^1_{Y/\mathbb{C}})_Q. \end{array}$$

Moreover, the vertical maps are isomorphisms. Thus $v_{r,f,Q}(\omega_P) = 0$ if and only if $\bar{\beta}_f(\omega_P) = 0$.

Proof. We will prove that the following diagram is commutative

$$\begin{array}{ccccccc} S^r_{\mathcal{O}_{X,P}}\Omega^1_{\mathcal{O}_{X,P}/\mathbb{C}} & \xrightarrow{S^r_{\mathcal{O}_{X,P}}(w_{X,P})} & S^r_{\mathcal{O}_{X,P}}(\Omega^1_{X/\mathbb{C}})_P & & \xrightarrow{\quad s_{r,\Omega^1_{X/\mathbb{C}},P} \quad} & & (S^r_X\Omega^1_{X/\mathbb{C}})_P \\ & & {\scriptstyle S^r(f^{\#}_{\Omega^1_{X/\mathbb{C}},Q})}\downarrow & & & & \downarrow{\scriptstyle f^{\#}_{S^r_X\Omega^1_{X,/\mathbb{C}},Q}} \\ {\scriptstyle v_{r,f,Q}}\downarrow & & S^r_{\mathcal{O}_{Y,Q}}(f^*\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow{s_{r,f^*\Omega^1_{X/\mathbb{C}},Q}} & (S^r_Y f^*\Omega^1_{X/\mathbb{C}})_Q & \xrightarrow{(\alpha^f)_Q} & (f^* S^r_X\Omega^1_{X/\mathbb{C}})_Q \\ & & {\scriptstyle S^r_Y(f_{X/Y/\mathbb{C}},Q)}\downarrow & & \downarrow{\scriptstyle (S^r_{\mathcal{O}_{Y,Q}}f_{X/Y/\mathbb{C}})_Q} & & \\ S^r_{\mathcal{O}_{Y,Q}}\Omega^1_{\mathcal{O}_{Y,Q}/\mathbb{C}} & \xrightarrow[S^r_{\mathcal{O}_{Y,Q}}(w_{Y,Q})]{} & S^r_{\mathcal{O}_{Y,Q}}(\Omega^1_{Y/\mathbb{C}})_Q & \xrightarrow[s_{r,\Omega^1_{Y/\mathbb{C}},Q}]{} & (S^r_Y\Omega^1_{Y/\mathbb{C}})_Q & & \end{array}$$

The left hand side diagram commutes by Lemma 3.60 and Lemma 3.16. The top right hand side diagram commutes by Lemma 3.62. The bottom right hand side

square commutes by Lemma 3.61. Thus, the above diagram commutes. Note that $\bar{\beta}_f = (S^r_Y f_{X/Y/\mathbb{C}})_Q \circ (\alpha^f)^{-1}_Q \circ f^{\#}_{S^r_X \Omega^1_{X/k},Q}$. The vertical maps are $\mu_{r,X,P}$ and $\mu_{r,Y,Q}$, which are isomorphisms. □

Taking completions of $v_{1,f,Q}$ we obtain a homomorphism

$$\hat{v}_{1,f,Q} : \hat{\Omega}^1_{A/\mathbb{C}} \to \hat{\Omega}^1_{B/\mathbb{C}}.$$

Thus, if $c_{\Omega^1_{A/\mathbb{C}}}$ and $c_{\Omega^1_{B/\mathbb{C}}}$ are the respective completion maps, then $\hat{v}_{1,f,Q}$ is the unique map satisfying

$$\hat{v}_{1,f,Q} \circ c_{\Omega^1_{A/\mathbb{C}}} = c_{\Omega^1_{B/\mathbb{C}}} \circ v_{1,f,Q}.$$

We write

$$\hat{v}_{r,f,Q} : S^r \hat{\Omega}^1_{A/\mathbb{C}} \to S^r \hat{\Omega}^1_{B/\mathbb{C}}.$$

Proposition 3.64. *We have a commutative diagram*

$$\begin{array}{ccc} S^r_A \Omega^1_{A/\mathbb{C}} & \xrightarrow{v_{r,f,Q}} & S^r_B \Omega^1_{B/\mathbb{C}} \\ {\scriptstyle S^r c_{\Omega^1_{A/\mathbb{C}}}}\downarrow & & \downarrow{\scriptstyle S^r c_{\Omega^1_{B/\mathbb{C}}}} \\ S^r_{\hat{A}} \hat{\Omega}^1_{A/\mathbb{C}} & \xrightarrow[\hat{v}_{r,f,Q}]{} & S^r_{\hat{B}} \hat{\Omega}^1_{B/\mathbb{C}}. \end{array}$$

Moreover, the vertical maps are injective.

Proof. We have $\hat{v}_{1,f,Q} \circ c_{\Omega^1_{A/\mathbb{C}}} = c_{\Omega^1_{B/\mathbb{C}}} \circ v_{1,f,Q}$. Taking r-th symmetric powers we obtain the commutativity of the diagram.

Since B is Noetherian, we have from [AM69], Corollary 10.20 that $S^r \Omega^1_{B/\mathbb{C}} \to$

$(S^r\Omega^1_{B/\mathbb{C}})^\wedge$ is injective. Given a finitely generated R-module M, we have

$$\widehat{S^r M} \cong \hat{R} \otimes_R S^r M \cong S^r(\hat{R} \otimes_R M) \cong S^r \hat{M}.$$

Indeed, the first isomorphism holds by Proposition 10.13 in [AM69], as the ring R is noetherian, and the second isomorphism holds by [Bou89], Proposition 7 III §6.4. From this we obtain that the map $S^r c_{\Omega^1_{B/\mathbb{C}}}$ is injective. □

The homomorphism $f_Q^\#$ induces a unique homomorphism $\hat{f}_Q^\# : \hat{A} \to \hat{B}$. The homomorphism $\hat{f}_Q^\#$ induces a unique homomorphism on the universally finite differential algebras

$$\sigma_{C,f,1} : \tilde{\Omega}^1_{\hat{A}/\mathbb{C}} \to \tilde{\Omega}^1_{\hat{B}/\mathbb{C}}$$

with associated derivations $\tilde{d}_{\hat{A}}$ and $\tilde{d}_{\hat{B}}$ respectively. Thus $\sigma_{C,f,1}$ is the unique map satisfying $\sigma_{C,f,1} \circ \tilde{d}_{\hat{A}} = \tilde{d}_{\hat{B}} \circ \hat{f}_Q^\#$.

By the universal property of $\Omega^1_{A/\mathbb{C}}$, the completion homomorphism $c_A : A \to \hat{A}$ induces a unique homomorphism

$$\alpha_A : \Omega^1_{A/\mathbb{C}} \xrightarrow{\sim} \tilde{\Omega}^1_{\hat{A}/\mathbb{C}}$$

which satisfies $\tilde{d}_{\hat{A}} \circ c_A = \alpha_A \circ d_A$. Similarly there exists a unique α_B such that $\tilde{d}_{\hat{B}} \circ c_B = \alpha_B \circ d_B$. By [Kun86], Cor.12.5 (b), the homomorphism α_A induces an isomorphism

$$\hat{\alpha}_A : \hat{\Omega}^1_{A/\mathbb{C}} \to \tilde{\Omega}^1_{\hat{A}/\mathbb{C}}$$

such that $\hat{\alpha}_A \circ c_{\Omega^1_{A/\mathbb{C}}} = \alpha_A$. Similarly, we get a map $\hat{\alpha}_B$ such that $\hat{\alpha}_B \circ c_{\Omega^1_{B/\mathbb{C}}} = \alpha_B$.

Proposition 3.65. *The following diagram commutes:*

$$\begin{array}{ccc} S^r_{\hat{A}}\hat{\Omega}^1_{A/\mathbb{C}} & \xrightarrow{\hat{v}_{r,f,Q}} & S^r_{\hat{B}}\hat{\Omega}^1_{B/\mathbb{C}} \\ {\scriptstyle S^r\hat{\alpha}_A}\downarrow & & \downarrow{\scriptstyle S^r\hat{\alpha}_B} \\ S^r_{\hat{A}}\tilde{\Omega}^1_{\hat{A}/\mathbb{C}} & \xrightarrow[\sigma_{C,f,r}]{} & S^r_{\hat{B}}\tilde{\Omega}^1_{\hat{B}/\mathbb{C}}. \end{array} \tag{3.12}$$

Proof. It suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc} \hat{\Omega}^1_{A/\mathbb{C}} & \xrightarrow{\hat{v}_{1,f,Q}} & \hat{\Omega}^1_{B/\mathbb{C}} \\ {\scriptstyle \hat{\alpha}_A}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}_B} \\ \tilde{\Omega}^1_{\hat{A}/\mathbb{C}} & \xrightarrow[\sigma_{C,f,1}]{} & \tilde{\Omega}^1_{\hat{B}/\mathbb{C}}. \end{array}$$

For this it suffices to prove that the following diagram commutes

$$\begin{array}{ccc} \Omega^1_{A/\mathbb{C}} & \xrightarrow{v_{1,f,Q}} & \Omega^1_{B/\mathbb{C}} \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ \tilde{\Omega}^1_{\hat{A}/\mathbb{C}} & \xrightarrow[\sigma_{C,f,1}]{} & \tilde{\Omega}^1_{\hat{B}/\mathbb{C}}. \end{array}$$

because $\Omega^1_{A/\mathbb{C}}$ is dense on $\hat{\Omega}^1_{A/\mathbb{C}}$ and the homomorphisms are continuous.

Given $a \in A$, we have

$$\begin{aligned} \sigma_{C,f,1} \circ \alpha_A(d_A a) &= \sigma_{C,f,1}(\tilde{d}_{\hat{A}}(c_A(a))) \\ &= \tilde{d}_{\hat{B}}(\hat{f}^{\#}_Q(c_A(a))) \\ &= \tilde{d}_{\hat{B}}(c_B(f^{\#}_Q(a))) \\ &= \alpha_B(d_B(f^{\#}_Q(a))) \\ &= \alpha_B(v_{1,f,Q}(d_A(a))), \end{aligned}$$

hence proving the commutativity. □

We denote by $\hat{\omega}_P$ the image of ω_P under the map $S^r c_{\Omega^1_{A/\mathbb{C}}} : S^r\Omega^1_{A/\mathbb{C}} \to S^r\hat{\Omega}^1_{A/\mathbb{C}}$.

By [Kun86], p. 171, since $\Omega^1_{A/\mathbb{C}}$ is a finitely generated A-module, we have that $\Omega_{A/\mathbb{C}} = \oplus\Omega^r_{A/\mathbb{C}}$ is a finite differential A-algebra. Thus by [Kun86], Cor. 12.5 (b), $\tilde{\Omega}_{\hat{A}/\mathbb{C}}$ exists and it satisfies $\tilde{\Omega}_{\hat{A}/\mathbb{C}} \cong \hat{\Omega}_{A/\mathbb{C}}$. We denote by $\tilde{\omega}_P$ the image of $\hat{\omega}_P$ under the isomorphism $S^r\hat{\alpha}_A$, that is, $\tilde{\omega}_P = S^r\hat{\alpha}_A(\hat{\omega}_P) = S^r\alpha_A(\omega_P)$.

Theorem 3.66. *Let X be a smooth surface, let $\mathcal{L}$ be an invertible sheaf and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. Let C be a curve on X, let $P \in C$ and let Q be a fixed preimage of P via the map $\varphi_C : \tilde{C} \to X$. The following are equivalent:*

(i) The curve C is ω-integral (i.e., $\varphi^{\bullet}_{C,r,\mathcal{L}}(\omega) = 0$),

(ii) $\bar{\beta}_{\varphi_C}(\omega_P) = 0$,

(iii) $v_{r,\varphi_C,Q}(\omega_P) = 0$,

(iv) $\hat{v}_{r,\varphi_C,Q}(\hat{\omega}_P) = 0$,

(v) The branch $\ker(\hat{\varphi}^{\#}_{C,Q})$ of C is a solution of $\tilde{\omega}_P$.

Proof. From Proposition 3.57 we get that $\varphi^{\bullet}_{C,r,\mathcal{L}}(\omega) = 0$ if and only if $\bar{\beta}_{\varphi_C}(\omega_P) = 0$, thus proving the equivalence between (i) and (ii).

From Proposition 3.63 we obtain that $\bar{\beta}_{\varphi_C}(\omega_P) = 0$ if and only if $v_{r,\varphi_C,Q}(\omega_P) = 0$. Hence (ii) and (iii) are equivalent.

From Proposition 3.64 we get that $\hat{v}_{r,\varphi_C,Q}(\hat{\omega}_P) = 0$ if and only if $v_{r,\varphi_C,Q}(\omega_P) = 0$. Therefore (iii) and (iv) are equivalent.

Since diagram (3.12) is commutative and the right vertical arrow is injective we obtain the equivalence of item (iv) with $\sigma_{C,\varphi_C,r}(\tilde{\omega}_P) = 0$. The latter is equivalent to

item (v) by definition of $\sigma_{C,\varphi_C,r}$ (recall Definition 3.50 and see the discussion after Proposition 3.64). Hence (iv) and (v) are equivalent. □

Since the point Q in Theorem 3.66 was arbitrarily chosen in $\varphi_C^{-1}(P)$, the theorem works for any $Q \in \varphi_C^{-1}(P)$, so we obtain:

Theorem 3.67. *Let X be a smooth surface, let $\mathcal{L}$ be an invertible sheaf and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. Let C be a curve on X, and $P \in C$. The following are equivalent:*

- *The curve C is ω-integral (i.e., $\varphi^{\bullet}_{C,r,\mathcal{L}}(\omega) = 0$),*
- *For one/any $Q \in \varphi_C^{-1}(P)$, the branch $\ker(\hat{\varphi}^{\#}_{C,Q})$ of C is a solution of $\hat{\omega}_P$.*

3.6 ω-integrality from equations

In this section we will show how to verify that a curve on a surface is ω-integral from the equations that define it.

Let k be a field. From Proposition II.8.4A in [Har77], we have, for a k-algebra A and an ideal $I = (f_1, \ldots, f_r)$ of A, the exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_A A/I \to \Omega^1_{(A/I)/k} \to 0 \tag{3.13}$$

where for $\bar{b} \in I/I^2$ we have $\delta(\bar{b}) = db \otimes 1$, and the homomorphism

$$\Omega^1_{A/k} \otimes_A A/I \to \Omega^1_{(A/I)/k}$$

maps an element in the generating set $adb \otimes [c]$ to $[ac]d[b]$, where $[a]$ is the class of a in R/I.

Lemma 3.68. *If $u : M \to N$ is a surjective A-linear mapping, then the homomorphism $S(u) : S_A(M) \to S_A(N)$ is surjective and its kernel is the ideal of $S(M)$ generated by the kernel $P \subset M \subset S(M)$ of u.*

Proof. See Proposition 4 in [Bou89], III.6.2. □

From this lemma, the surjective homomorphism of A/I-modules

$$\Omega^1_{A/k} \otimes_A A/I \to \Omega^1_{(A/I)/k}$$

induces a surjective A/I-algebra homomorphism

$$\gamma : S_{A/I}(\Omega^1_{A/k} \otimes_A A/I) \to S_{A/I}(\Omega^1_{(A/I)/k}),$$

with kernel generated by $\mathrm{Im}(\delta)$ in $S(\Omega^1_{A/k} \otimes_A A/I)$. It induces a surjective map

$$\gamma_r : S^r(\Omega^1_{A/k} \otimes_A A/I) \to S^r(\Omega^1_{(A/I)/k})$$

with kernel $\ker(\gamma_r) = \mathrm{Im}(\delta) S^{r-1}(\Omega^1_{A/k} \otimes_A A/I)$. Let $S^r_A \Omega^1_{A/\mathbb{C}} \to S^r_{A/I}(\Omega^1_{A/\mathbb{C}} \otimes A/I)$ be induced by $\Omega^1_{A/\mathbb{C}} \to \Omega^1_{A/\mathbb{C}} \otimes A/I$ and let

$$q_r : S^r_A \Omega^1_{A/\mathbb{C}} \to S^r_{A/I}(\Omega^1_{A/\mathbb{C}} \otimes A/I)/\ker(\gamma_r)$$

be the composition with the quotient map.

We will be using repeatedly the fact that if $U = \mathrm{Spec}(A)$ is an affine k-variety, then

$$\Omega^1_{U/k} = \tilde{\Omega}^1_{A/k}$$

$$S_U\Omega^1_{U/k} = (S_A\Omega^1_{A/k})^\sim$$

which follow from [EGA] (IV,16.3.7) and [EGA] (II,1.7.6), respectively.

Proposition 3.69. *Let X be a smooth surface over $\mathbb{C}$, let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$ and let $U = \operatorname{Spec}(A)$ be an affine open set of X such that $\mathcal{L}_{|U} \cong \mathcal{O}_U$. Let C be an irreducible curve in X which meets U. Let $I = (f_1, \ldots, f_r)$ be an ideal in A such that $C \cap U = \mathbb{V}_U(I)$, the subscheme associated to the ideal I. Denote by $\omega_0 \in S^r\Omega^1_{A/\mathbb{C}}$ the image of ω by the maps*

$$H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \to H^0(U, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \xrightarrow{h} H^0(U, S^r\Omega^1_{U/\mathbb{C}}) = S^r\Omega^1_{A/\mathbb{C}}$$

where the first is the restriction to U, and h is the isomorphism induced by $\mathcal{L}_{|U} \cong \mathcal{O}_U$. Suppose that the image of ω_0 by the map

$$q_r : S^r_A\Omega^1_{A/\mathbb{C}} \to S^r_{A/I}(\Omega^1_{A/\mathbb{C}} \otimes A/I)/\ker(\gamma_r)$$

is zero. Then C is ω-integral.

Proof. Let P be *any* smooth point of $C \cap U$. Then locally near P the normalization of C is an isomorphism, because we have that $\mathcal{O}_{C,P}$ is a regular local ring, and hence $\tilde{\mathcal{O}}_{C,P} = \mathcal{O}_{C,P}$. So $\mathcal{O}_{\tilde{C},Q} = \mathcal{O}_{C,P}$ where Q is the preimage of P in the normalization of C. Note that $C \cap U = \operatorname{Spec}(A/I)$. If m is the maximal ideal of A/I corresponding to the point $P \in C$, then $\mathcal{O}_{\tilde{C},Q} = \mathcal{O}_{C,P} = (A/I)_m$. By Theorem 3.66, C is ω-integral if and only if $\bar{\beta}_{\varphi_C,Q}(\omega_P) = 0$.

We need the following two propositions

Proposition 3.70. *Let $\lambda : S^r_A\Omega^1_{A/\mathbb{C}} \to (S^r\Omega^1_{A/\mathbb{C}})_P \cong S^r_{A_P}\Omega^1_{A_P/\mathbb{C}}$ be the localization*

map. The following diagram commutes:

$$\begin{array}{ccccccc}
 & & H^0(X, \mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}) & & & & \\
 & & \downarrow & & & & \\
a & & H^0(U, \mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}}) & \xrightarrow{h} & H^0(U, S^r\Omega^1_{X/\mathbb{C}}) & \xleftarrow{=} & S^r\Omega^1_A \\
 & & \downarrow & & \downarrow & & \downarrow\lambda \\
 & \searrow & (\mathcal{L}\otimes S^r\Omega^1_{X/\mathbb{C}})_P & \xrightarrow{h_P} & (S^r\Omega^1_{X/\mathbb{C}})_P & \xleftarrow{\mu_{r,X,P}} & S^r\Omega^1_{\mathcal{O}_{X,P}} = S^r\Omega^1_{A_p} \\
 & & & & \bar\beta_{\varphi_C}\downarrow & & \downarrow v_{r,\varphi_C,Q} \\
 & & & & (S^r\Omega^1_{\tilde C/\mathbb{C}})_Q & \xleftarrow{\mu_{r,\tilde C,Q}} & S^r\Omega^1_{\mathcal{O}_{\tilde C/\mathbb{C}}}
\end{array}$$

Therefore for C to be ω-integral it suffices to prove that $v_{r,\varphi_C,Q}\circ\lambda(\omega_0)=0$.

Proof. The left hand side square commutes by Remark 2.72, the bottom right hand side square commutes by Proposition 3.63. The top right hand side diagram commutes by using the formula of $\mu_{r,X,P}$ (from the statement before Lemma 3.60) and checking on elements $f dg_1\cdots dg_r \in S^r\Omega^1_{A/k} = H^0(U, S^r\Omega^1_{X/\mathbb{C}})$, which are generators. Therefore the diagram commutes. Note that the image of ω by $a\circ h_P$ is ω_P, and C is ω-integral if and only if $\bar\beta_{\varphi_C}(\omega_P)=0$. From the commutativity of the diagram and the fact that $\mu_{r,\tilde C,Q}$ is an isomorphism (cf. Proposition 3.63) it suffices to prove that $v_{r,\varphi_C,Q}\circ\lambda(\omega_0)=0$. □

Proposition 3.71. *If ω_0 maps to zero under the map $S^r_A\Omega^1_{A/k}\to S^r_{A/I}\Omega^1_{(A/I)/k}$, then C is ω-integral.*

Proof. The diagram

$$\begin{array}{ccc}
S^r_A\Omega^1_{A/k} & \longrightarrow & S^r_{A/I}\Omega^1_{(A/I)/k} \\
\lambda\downarrow & & \downarrow \\
S^r_{\mathcal{O}_{X,P}}\Omega^1_{\mathcal{O}_{X,P}} & \xrightarrow[v_{r,\varphi_C,Q}]{} & S^r_{\mathcal{O}_{\tilde C,Q}}\Omega^1_{\mathcal{O}_{\tilde C,Q}}
\end{array}$$

commutes because the diagram

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A_P & \longrightarrow & (A/I)_m = A_P/I_P \end{array}$$

commutes. Thus, if the image of ω_0 is zero under the morphism $S^r_A\Omega^1_{A/k} \to S^r_{A/I}\Omega^1_{(A/I)/k}$, then $\lambda \circ v_{r,\varphi_C,Q}(\omega_P) = 0$ and thus by Proposition 3.70 we have that C is ω-integral. $\square$

We continue with the proof of Proposition 3.69. We have the commutative diagram

$$\begin{array}{ccc} \Omega^1_{A/k} & \longrightarrow & \Omega^1_{A/k} \otimes A/I \\ & \searrow \quad \swarrow & \\ & \Omega^2_{(A/I)/k} & \end{array}$$

where the right arrow comes from [Mat80], p. 186, the left arrow is induced by the ring homomorphism $A \to A/I$, and the horizontal arrow is extension of scalars (commutativity is verified on elements). Hence the following diagram commutes

$$\begin{array}{ccc} S^r_A\Omega^1_A & \longrightarrow & S^r_{A/I}(\Omega^1_A \otimes A/I) \\ \downarrow & & \downarrow \\ S^r_{A/I}\Omega^1_{A/I} & \xleftarrow[\bar{\gamma}_r]{\sim} & S^r_{A/I}(\Omega^1_A \otimes A/I)/\ker(\gamma_r), \end{array}$$

by Lemma 3.16 and because $S^r_{A/I}(\Omega^1_A \otimes A/I) \to S^r_{A/I}\Omega^1_{A/I}$ factors through the isomorphism $\bar{\gamma}_r$. We deduce that the map of Proposition 3.71 is exactly $\bar{\gamma}_r \circ q_r$. This proves the result. $\square$

Corollary 3.72. *Let X be a smooth surface, let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. Let C be an irreducible curve in X, let $U = \mathrm{Spec}(A)$ be an open set in X such that $C \cap V \neq \emptyset$,*

and $\mathcal{L}_{|U} \cong \mathcal{O}_U$. Let $I = (g)$ be an ideal in A such that $C \cap U = \mathbb{V}_U(I)$. If $\omega_0 \in S^r\Omega^1_{A/\mathbb{C}}$ (as defined in Proposition 3.69) lies in $gS^r(\Omega^1_{A/\mathbb{C}}) + dgS^{r-1}\Omega^1_{A/\mathbb{C}}$, then C is ω-integral.

Proof. Note that $gS^r(\Omega^1_{A/\mathbb{C}}) + dgS^{r-1}\Omega^1_{A/\mathbb{C}} \subseteq \ker(q_r)$ because $dg \in \mathrm{Im}(\delta)$ and we are tensoring by $A/(g)$. If $\omega_0 \in gS^r(\Omega^1_{A/\mathbb{C}}) + dgS^{r-1}\Omega^1_{A/\mathbb{C}}$, we get that $\omega_0 \in \ker(q_r)$. By Proposition 3.69 we get that C is ω-integral. □

3.7 Showing that there are no more ω-integral curves on a surface

Given a set of ω-integral curves on a smooth surface X, we want to know when that set consists of *all* ω-integral curves of X.

We will use the following proposition for obtaining functions on an open set from local information.

Proposition 3.73. *Let Y be a smooth variety over K. If $u_1, \ldots, u_n$ is any system of local parameters at a point $P \in Y$, then there exists an open affine neighborhood V of P such that the u_i extend to regular functions on V and $du_1, \ldots, du_n$ generate $\Omega^1_{Y/K}(V)$ as an $\mathcal{O}_Y(V)$-module. Moreover, $du_1, \ldots, du_n$ are a basis of the $\mathcal{O}_Y(V)$-module $\Omega^1_{Y/k}(V)$.*

Proof. See [Sha13], p. 193. □

Let X be a smooth surface over $\mathbb{C}$, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})$. The purpose of this subsection is to give a criterion (cf. Theorem 3.76) for testing if a list of ω-integral curves consists of *all* ω-integral curves.

Let $V \subseteq X$ be a non-empty affine open set such that there are regular functions $u, v \in \mathcal{O}_X(V)$ with the property that du, dv are a basis of $\Omega^1_{X/\mathbb{C}}(V)$ as $\mathcal{O}_X(V)$-module

(they exist by Proposition 3.73). Since V is affine we have by [EGA], (IV.16.3.7), that

$$\Omega^1_{V/\mathbb{C}} = (\Omega^1_{V/\mathbb{C}}(V))^{\sim} = (\Omega^1_{X/\mathbb{C}}(V))^{\sim},$$

Writing $V = \mathrm{Spec}A$, for each non-empty basic affine open subset $V' = D(f) \subseteq V$ (for $f \in A$) we have that the restrictions of du, dv to V' form a basis for the A_f-module

$$\Omega^1_{V/\mathbb{C}}(V') = (\Omega^1_{X/\mathbb{C}}(V))^{\sim}(D(f)) = (\Omega^1_{X/\mathbb{C}}(V))_f \cong \Omega^1_{X/\mathbb{C}}(V) \otimes A_f$$

because extension of scalars (to a non-zero ring) takes a basis to a basis (note that A_f is not the zero ring V' was taken non-empty).

Hence for all non-empty basic affine open sets $V' = D(f) \subseteq V$, the $\mathcal{O}_X(V')$-module $S^r\Omega^1_{X/\mathbb{C}}(V')$ has $\{(du)^i(dv)^{r-i}\}_{0\leq i\leq r}$ as a basis, because on affine open sets we have

$$(S^r\Omega^1_{X/\mathbb{C}})(V') = S^r(\Omega^1_{X/\mathbb{C}}(V')).$$

Lemma 3.74. *Using the previous notation, for any point $P \in V$ write $a_P = u(P)$ and $b_P = v(P)$. Then $\bar{u}_P := u - a_P$ and $\bar{v}_P := v - b_P$ are local parameters at P and they satisfy $d(\bar{u}_P) = du$ and $d(\bar{v}_P) = dv$.*

Proof. We have that $\bar{u}_P, \bar{v}_P \in m_P$ because they vanish at the point P, and we also have $d(\bar{u}_P) = d(u - a_P) = du$ and $d(\bar{v}_P) = d(v - b_P) = dv$ so $d(\bar{u}_P)$ and $d(\bar{v}_P)$ generate $\Omega^1_{\mathcal{O}_{X,P}}$ (because du, dv generate). From Proposition II.8.7 in [Har77], we have $\Omega^1_{\mathcal{O}_{X,P}} \otimes \mathcal{O}_{X,P}/m_P \cong m_P/m_P^2$. Thus, the images of $\bar{u}_P$, $\bar{v}_P$ generate m_P/m_P^2, hence $\bar{u}_P$, $\bar{v}_P$ generate m_P (by [AM69], Prop. 2.8). Therefore $\bar{u}_P, \bar{v}_P$ are local parameters and satisfy the required equation. □

Let $U \subseteq V$ be a non-empty basic affine open set such that $\mathcal{L}_{|U} \cong \mathcal{O}_U$ (it exists because such open sets form a basis for the topology of V); we fix this choice for the rest of this subsection. Then under the isomorphism

$$H^0(U, \mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}}) \cong H^0(U, S^r\Omega^1_{X/\mathbb{C}})$$

induced by $\mathcal{L}_{|U} \cong \mathcal{O}_U$, we have that the image of $\omega_{|U}$ in $H^0(U, S^r\Omega^1_{X/\mathbb{C}})$ can be written as

$$\sum_{i=0}^{r} A_i (du)^{r-i} (dv)^i \tag{3.14}$$

with $A_i \in \mathcal{O}_U(U)$ and u, v the regular functions on V obtained by Proposition 3.73. Since $\{(du)^i (dv)^{r-i}\}_{0 \leq i \leq r}$ is a basis for the free $\mathcal{O}_U(U)$-module $S^r\Omega^1_{X/\mathbb{C}}(U)$, the coefficients A_i are uniquely determined by the fixed choice of u, v and the isomorphism $\mathcal{L} \cong \mathcal{O}_U$.

Let $\delta \in K = k(X)$ be the discriminant of the monic polynomial

$$\sum_{i=0}^{r} \frac{A_i}{A_0} T^{r-i} \in K[T],$$

where $K = k(X)$ is the function field of X and T is a transcendental variable, and note that $\delta \in \mathcal{O}_X(U \backslash \mathbb{V}_U(A_0))$ (that is, δ is a regular function on U away from the zero set of A_0).

Notation 3.75. Let

$$\Delta_U := (X \backslash U) \cup \mathbb{V}_U(A_0) \cup \mathbb{V}_{U \backslash \mathbb{V}_U(A_0)}(\delta) \subseteq X,$$

where, for a scheme W and a regular function $h \in \mathcal{O}_W(W)$, we write $\mathbb{V}_W(h)$ for the

vanishing set of h (which is closed in W).

Write $D_U(A_0) := U\backslash\mathbb{V}_U(A_0)$ and $D_{U\backslash\mathbb{V}_U(A_0)}(\delta) := (U\backslash\mathbb{V}_{A_0})\backslash\mathbb{V}_{U\backslash\mathbb{V}_U(A_0)}(\delta)$, and note that they are open in X. We have

$$\begin{aligned}
\Delta_U^c &= U \cap ((X\backslash U) \cup D_U(A_0)) \cap ((X\backslash(U\backslash\mathbb{V}_U(A_0))) \cup D_{U\backslash\mathbb{V}_U(A_0)}(\delta) \\
&= (U \cap D_U(A_0)) \cap ((X\backslash U) \cup \mathbb{V}_U(A_0) \cup D_{U\backslash\mathbb{V}_U(A_0)}(\delta) \\
&= D_U(A_0) \cap ((X\backslash U) \cup \mathbb{V}_U(A_0) \cup D_{U\backslash\mathbb{V}_U(A_0)}(\delta) \\
&= \emptyset \cup \emptyset \cup (D_U(A_0) \cap D_{U\backslash\mathbb{V}_U(A_0)}(\delta)).
\end{aligned}$$

which is a union of open sets of X. Thus, the set Δ_U is Zariski closed in X.

In practice (in the applications in the next chapters), we will be able to compute the closed set Δ_U explicitly for suitable choice of u, v and isomorphism $\mathcal{L}_{|U} \cong \mathcal{O}_U$.

Now we will prove Theorem 3.76, which states that for a point P outside Δ_U, then P belongs to at most r ω-integral curves of X.

Let $P \in X\backslash\Delta_U$ and use the local parameters $\bar{u} = \bar{u}_P$ and $\bar{v} = \bar{v}_P$ from Lemma 3.74. Recall that the image of $\omega_{|U}$ in $H^0(U, S^r\Omega^1_{X/\mathbb{C}})$ can be written as $\sum_{i=0}^r A_i(du)^{r-i}(dv)^i$. Following notation of Section 3.5, the image of ω in $(S^r\Omega^1_{X/\mathbb{C}})_P$ is ω_P, and the image of ω_P under the morphism $S^r c_{\Omega^1_{A/\mathbb{C}}} : S^r\Omega^1_{A/\mathbb{C}} \to S^r\hat{\Omega}^1_{A/\mathbb{C}}$ is $\hat{\omega}_P$.

From Lemma 3.74 we can replace du and dv by $d\bar{u}_P$ and $d\bar{v}_P$ in (3.14). Since ω is mapped to $\hat{\omega}_P$ via localization and completion, we obtain the expression

$$\hat{\omega}_P = \sum_{i=0}^{r} \hat{A}_i(\widehat{d\bar{u}_P})^{r-i}(\widehat{d\bar{v}_P})^i$$

where $\hat{A}_i$ is the image of $A_i \in \mathcal{O}_U(U)$ in $\hat{\mathcal{O}}_{U,P} = \hat{\mathcal{O}}_{X,P}$.

Consider the isomorphism $c : \hat{\mathcal{O}}_{X,P} \to k[[\mathbf{X}, \mathbf{Y}]]$ given by this choice of local

parameters, with $c(\hat{\bar{u}}_P) = \mathbf{X}$ and $c(\hat{\bar{v}}_P) = \mathbf{Y}$. The isomorphism c induces the isomorphism

$$c_r : S^r\tilde{\Omega}^1_{\hat{\mathcal{O}}_{X,P}/\mathbb{C}} \to S^r\tilde{\Omega}^1_{\mathbb{C}[[\mathbf{X},\mathbf{Y}]]/\mathbb{C}} = S^r(\mathbb{C}[[\mathbf{X},\mathbf{Y}]]d\mathbf{X} \oplus \mathbb{C}[[\mathbf{X},\mathbf{Y}]]d\mathbf{Y}),$$

which is given by the rule $c_r(F(d\hat{\bar{u}}_P)^{r-i}(d\hat{\bar{v}}_P)^i) = c(F)(d\mathbf{X})^{r-i}(d\mathbf{Y})^i$, for $F \in \hat{\mathcal{O}}_{X,P}$.

We have the canonical map $S^r\Omega^1_{\mathcal{O}_{X,P}/\mathbb{C}} \to S^r\tilde{\Omega}^1_{\hat{\mathcal{O}}_{X,P}/\mathbb{C}}$ given by

$$A(d\bar{u}_P)^{r-i}(d\bar{v}_P)^i \mapsto \hat{A}(d\hat{\bar{u}}_P)^{r-i}(d\hat{\bar{v}}_P)^i$$

which induces an isomorphism $f : S^r\hat{\Omega}^1_{\mathcal{O}_{X,P}/\mathbb{C}} \xrightarrow{\sim} S^r\tilde{\Omega}^1_{\hat{\mathcal{O}}_{X,P}/\mathbb{C}}$ (from Cor.12.5(b) in [Kun86]) with the property that

$$\hat{A}(\widehat{d\bar{u}_P})^{r-i}(\widehat{d\bar{v}_P})^i \mapsto \hat{A}(d\hat{\bar{u}}_P)^{r-i}(d\hat{\bar{v}}_P)^i.$$

We have that the image of $\hat{\omega}_P \in S^r\hat{\Omega}^1_{\mathcal{O}_{X,P}/\mathbb{C}}$ in $S^r\tilde{\Omega}^1_{\mathbb{C}[[\mathbf{X},\mathbf{Y}]]/\mathbb{C}}$ under these maps is

$$c_r f(\hat{\omega}_P) = c_r\left(\sum_{i=0}^{r} \hat{A}_i(d\hat{\bar{u}}_P)^{r-i}(d\hat{\bar{v}}_P)^i\right) = \sum_{i=0}^{r} c(\hat{A}_i)(d\mathbf{X})^{r-i}(d\mathbf{Y})^i.$$

Theorem 3.76. *Using the previous notation, for any given point $P \in X\backslash\Delta_U$ there are at most r ω-integral curves passing through P. More precisely, the sum of the multiplicities $\mu_P(C)$ for all ω-integral curves C passing through P is at most r.*

Proof. We apply Theorem 3.55 to $\tilde{\omega} := c_r f(\hat{\omega}_P)$. Then the power series A_0 from the statement of Theorem 3.55 is $c(\hat{A}_0)$, and since $F \mapsto c(\hat{F})$ is a ring homomorphism, the power series δ in the statement of Theorem 3.55 is $c(\hat{\delta})$. Hence, the condition that

$A_0(0,0) \neq 0$ and $\delta(0,0) \neq 0$ in Theorem 3.55 (i.e. that these power series are not in the maximal ideal $(\mathbf{X},\mathbf{Y})$) is satisfied, because in our context $A_0(P) \neq 0$ and $\delta(P) \neq 0$ (which holds since $P \notin \Delta_U$) which means that $A_0, \delta \notin m_{X,P}$. We conclude from Theorem 3.55 that $c_r f(\hat{\omega}_P)$ has at most r distinct solutions in the sense of Definition 3.50 and that they are linear branches (in $\mathbb{C}[[\mathbf{X},\mathbf{Y}]]$) with non-vertical tangents.

Finally, let C be an ω-integral curve passing through P. By Theorem 3.67, all branches of C are solutions of $fc(\hat{\omega}_P)$ in the sense of Definition 3.50. From Proposition 2.62 the number $\mu_P(C)$ is the sum of the orders of the branches of C at P. Since the branch solutions of $fc(\hat{\omega}_P)$ are all linear (of order 1 by the proof of Corollary 2.65), we obtain that $\mu_P(C)$ is the number of branches of C at P.

Since different curves C cannot have a common branch by Proposition 2.59, we get that the sum of $\mu_P(C)$ for all ω-integral curves C passing through P is at most the number of solution branches of $\hat{\omega}_P$, which is r. □

3.8 Improving bounds via ramification

The main goal of this section is to improve Lemma 2.10 in [Voj00]. This leads to better bounds when we deal with Diophantine equations of larger degrees. Moreover, we extend it to morphism of surfaces that are not necessarily finite. Such morphisms will appear when we resolve singularities in certain applications, as in chapter 6.

3.8.1 Sections vanishing identically

Definition 3.77. Given a smooth irreducible surface X, an effective Cartier divisor D of X with associated subscheme $Y = Y_D$, a locally free sheaf $\mathcal{F}$ and a section $s \in H^0(X,\mathcal{F})$, we say that s *vanishes identically along* D if the image of s under the

map

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)$$

(defined in Subsection 2.2.1) is zero.

Lemma 3.78. *Let $\mathcal{F}$ be a locally free sheaf on X, let D be an effective Cartier divisor on X. For each $P \in X$, let $f_P \in \mathcal{O}_{X,P}$ be a local equation for D at P. If $s \in H^0(X, \mathcal{F})$, then s vanishes identically along D if and only if $s_P \in f_P\mathcal{F}_P$ for all $P \in X$.*

Proof. Let Y be the closed subscheme of X associated to D. Let $\mathcal{I}$ be the ideal sheaf of Y. Then for all $P \in X$ we have $\mathcal{I}_P = f_P\mathcal{O}_{X,P}$. Consider the homomorphism $\mu : \mathcal{I} \otimes \mathcal{F} \to \mathcal{F}$ from (2.4), which is defined on open sets to be the multiplication of sections. From the exact sequence (2.4) we have that the section s vanishes identically along D if and only if $s \in H^0(X, \mathrm{im}(\mu))$.

If $s \in H^0(X, \mathrm{im}(\mu))$, then for all P in X we have $s_P \in \mathrm{im}(\mu)_P$. Conversely, suppose that for all $P \in X$ we have $s_P \in \mathrm{im}(\mu)_P$, then there is a neighborhood U_P of P and $t \in \mathrm{im}(\mu)(U_P)$ such that $s_P = t_P$ in $\mathrm{im}(\mu)_P$, so in $\mathcal{F}$. Thus there exists $V_P \subseteq U_P$ such that $s_{|V_P} = t_{|V_P}$ in $\mathcal{F}(V_P)$, and since t is a section of $\mathrm{im}(\mu)$ we get $s_{|V_P} \in \mathrm{im}(\mu)(V_P)$. Since this holds for every P in X we obtain that $s \in H^0(X, \mathrm{im}(\mu))$.

By [Har77], Ex.II.1.2(a) we have that $s_P \in \mathrm{im}(\mu)_P$ if and only if $s_P \in \mathrm{im}(\mu_P)$. Since $\mathrm{im}(\mu_P) = \mathcal{I}_P\mathcal{F}_P = f_P\mathcal{O}_{X,P}\mathcal{F}_P = f_P\mathcal{F}_P$, we obtain that $s_P \in \mathrm{im}(\mu_P)$ if and only if $s_P \in f_P\mathcal{F}_P$. □

Proposition 3.79. *Let X be a smooth variety and let D, D' be effective Cartier divisors without common components. Moreover, let $\mathcal{F}$ be a locally free sheaf and let $s \in H^0(X, \mathcal{F})$. Then s vanishes identically along D and along D' if and only if s*

vanishes identically along $D + D'$.

Proof. For every $P \in X$, we let f_P and g_P in $\mathcal{O}_{X,P}$ be local equations for D and D', respectively. Note that $f_P g_P$ is a local equation for $D + D'$ at P.

Since X is smooth, $\mathcal{O}_{X,P}$ is a unique factorization domain and since D and D' do not share components, we see that f_P and g_P do not have a common irreducible factor. Thus for all $h \in \mathcal{O}_{X,P}$ we have that $f_P g_P$ divides h if and only if f_P divides h and g_P divides h.

Now choose an $\mathcal{O}_{X,P}$-module isomorphism $\mathcal{F}_P \cong \mathcal{O}_{X,P}^r$ (recall that $\mathcal{F}$ is locally free). Write $t_P = (h_{P,1}, \ldots, h_{P,r}) \in \mathcal{O}_{X,P}^r$ for the image of s_P under this isomorphism. Then by Lemma 3.78, s vanishes identically along $D + D'$ if and only if for all P, the germ $s_P \in f_P g_P \mathcal{F}_P$. This holds if and only if for all P we have $t_P \in f_P g_P \mathcal{O}_{X,P}^r$. By definition this holds if and only if $f_P g_P$ divides $h_{p,j}$ for each $j = 1, \ldots, r$ and for all P. Since f_P and g_P do not have a common irreducible factor this holds if and only if f_P divides $h_{P,j}$ and g_P divides $h_{P,j}$ for each $j = 1, \ldots, r$. Moreover, this holds if and only if s vanishes identically along D and along D' by Lemma 3.78. □

Now we will give a local criterion for proving that a section vanishes identically along an effective Cartier divisor.

Proposition 3.80. *Let D be a Cartier divisor on a smooth variety X which is of the form nD', with D' prime and $n > 0$. Let Y be the associated subscheme of D, let U be an affine open subset of X intersecting Y, let $\mathcal{F}$ be a locally free sheaf of X and let $s \in H^0(X, \mathcal{F})$. If the image of $s_{|U}$ in $H^0(U, (\mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)_{|U})$ is zero, then s vanishes identically along D.*

For this, we will use the following three lemmas.

Lemma 3.81. *Let A be a ring such that $A \cong B/f^n$, where B is a unique factorization domain, $f \in B$ is an irreducible element and n is a positive integer. Then every zero divisor of A is nilpotent.*

Proof. Suppose that $xy = 0$ with $x, y \in A$ not zero. Let u, v in B be such that $x = \bar{u}$ and $y = \bar{v}$ in the quotient $A = B/(f^n)$. Then f^n divides uv in B.

Since B is unique factorization domain, we get that f divides u or v. If $n = 1$ we get a contradiction because x, y are non-zero. If $n > 1$ then (because x, y are non-zero) we have that f^n cannot only divide u and cannot only divide v. Thus f divides u and f divides v. Thus f^n divides u^n and divides v^n, so both x, y are nilpotent. □

Lemma 3.82. *Let $V = \mathrm{Spec}(A)$ with A satisfying the conditions of Lemma 3.81. Let $\mathcal{G}$ be a free $\mathcal{O}_V$-module, let $P \in V$ and let $V' \subset V$ be a neighborhood of P such that $V' = D(g)$ with $g \in A$. Let $s \in H^0(V, \mathcal{G})$. If $s_{|V'} = 0$, then $s = 0$.*

Proof. Since $\mathcal{G}$ is free, then it is quasi-coherent. By Lemma II.5.3 in [Har77], we obtain that there exists $n > 0$ such that $g^n \cdot s = 0$.

Since $V' = D(g) \cong \mathrm{Spec}(A_g)$ is not empty, the ring A_g is not zero, hence by [Har77] Ex.II.2.18 we have g is not nilpotent, and so $g^n \neq 0$.

We have that $\mathcal{G}(V)$ is free, and isomorphic to A^r for some $r \geq 1$. If $s \neq 0$, then for some non-zero coordinate t of $s \in A^r$ we have $g^n t = 0$ with $g^n \neq 0$. From Lemma 3.81 we obtain that g^n is nilpotent, but we know that g is not nilpotent. Thus $t = 0$, which contradicts the fact that t was a non-zero coordinate. Therefore $s = 0$. □

Lemma 3.83. *Let D be an effective Cartier divisor on a smooth variety X such that D is of the form nD' with $n \geq 1$ and D' a prime divisor. Let $i_Y : Y \to X$ be the associated closed subscheme of D and let $\mathcal{G}$ be an $\mathcal{O}_X$-module such that ${i_Y}^*\mathcal{G}$ is*

a locally free $\mathcal{O}_Y$-module. Let U be a non-empty open set of X which intersects Y. Then the restriction map $H^0(Y, i_Y^*\mathcal{G}) \to H^0(U \cap Y, (i_Y^*\mathcal{G})_{|U\cap Y})$ is injective.

Proof. Let $s \in H^0(Y, i_Y^*\mathcal{G})$ and suppose that $s_{|U\cap Y} = 0$ in $H^0(U \cap Y, (i_Y^*\mathcal{G})_{|U\cap Y})$. We want to show that $s = 0$.

Let $\{U_i, q_i\}_i$ represent the divisor D', with $\{U_i\}_i$ a covering of X by affine open sets $U_i = \mathrm{Spec}B_i$ such that $i_Y^*\mathcal{G}$ is free on each $U_i \cap Y$ (this is possible because $i_Y^*\mathcal{G}$ is locally free on Y, and Y has the subspace topology induced from X) and such that each $B_i = \mathcal{O}_X(U_i)$ is a unique factorization domain (this is possible, possibly after refining the covering $\{U_i\}_i$, because X is smooth). Note that each $q_i \in B_i$ is irreducible when U_i intersects Y, and is invertible when U_i does not intersect Y, because D' is a prime divisor. Write $f_i = q_i^n$; then D is represented by $\{(U_i, f_i)\}_i$ because $D = nD'$.

The sheaf of ideals $\mathcal{I}_Y$ is locally generated by $f_i = q_i^n$, and so the subscheme Y is given in U_i by $Y \cap U_i = \mathrm{Spec}B_i/(q_i^n)$ for all i with U_i intersecting Y.

Let i be such that U_i has non-empty intersection with Y. Since Y is irreducible (although it can be non-reduced), the non-empty open sets $U_i \cap Y$ and $U \cap Y$ meet, and so there exists $P \in U_i \cap U \cap Y$. Let $W \subseteq U_i \cap U \cap Y$ be an open neighborhood of P in Y such that $W = D_{U_i\cap Y}(g)$ for some $g \in \mathcal{O}_X(U_i \cap Y) = B_i/(f_i)$. Since $i_Y^*\mathcal{G}$ is free on $Y \cap U_i$, we see that all the conditions of Lemma 3.82 are satisfied. Note that $s_{|W} = 0$ because $s_{|Y\cap U} = 0$ and $W \subseteq Y \cap U$. Therefore Lemma 3.82 gives that $s_{|U_i\cap Y} = 0$.

We conclude that for every i with U_i intersecting Y, we have that $s_{|U_i\cap Y} = 0$. Since these $U_i \cap Y$ form an open cover of Y, we conclude that $s = 0$ in $H^0(Y, i_Y^*\mathcal{G})$. □

Proof of Proposition 3.80. From Exercise II.5.1(d) in [Har77], there exists a canonical

isomorphism $\iota : \mathcal{F} \otimes i_{Y*}\mathcal{O}_Y \xrightarrow{\sim} i_{Y*}i_Y^*\mathcal{F}$.

Since ι is an isomorphism of sheaves, we get the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{F} \otimes i_*\mathcal{O}_Y) & \longrightarrow & H^0(U, \mathcal{F} \otimes i_*\mathcal{O}_Y) \\ {\scriptstyle \iota(X)}\downarrow & & {\scriptstyle \iota(U)}\downarrow \\ H^0(X, i_*i^*\mathcal{F}) & \longrightarrow & H^0(U, i_*i^*\mathcal{F}) \\ \| & & \| \\ H^0(Y, i^*\mathcal{F}) & \longrightarrow & H^0(U \cap Y, i^*\mathcal{F}) \end{array}$$

where the horizontal arrows are restriction maps and the top vertical arrows are isomorphisms. The bottom map is injective by Lemma 3.83, and so the top map is injective.

By definition of a morphism of sheaves, we have the following commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F} \otimes i_*\mathcal{O}_Y) \\ \downarrow & & \downarrow \\ H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{F} \otimes i_*\mathcal{O}_Y) \end{array}$$

where the vertical maps are the restriction maps, and the right vertical map is injective by the previous discussion. By commutativity of this last diagram, if $s_{|U}$ maps to zero on $H^0(U, \mathcal{F} \otimes i_*\mathcal{O}_Y)$, then s maps to zero on $H^0(X, \mathcal{F} \otimes i_*\mathcal{O}_Y)$, so s vanishes identically along D. □

Corollary 3.84. *Let D be a Cartier divisor on a smooth variety X which is of the form nD', with D' prime and $n > 0$. Let Y be the associated subscheme of D, let $P \in Y$, let $\mathcal{F}$ be a locally free sheaf of X and let $s \in H^0(X, \mathcal{F})$. If the image of $s_P \in \mathcal{F}_P$ of the map $\mathcal{F}_P \to (\mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)_P$ is zero, then s vanishes identically along D.*

Proof. If s_P is mapped to 0 under the map $\mathcal{F}_P \to (\mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)_P$, then there is a neighborhood U of P such that $s_{|U} \in H^0(U, \mathcal{F})$ has image 0 under the map

$$\mathcal{F}(U) = H^0(U, \mathcal{F}) \to (\mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)(U) = H^0(U, (\mathcal{F} \otimes i_{Y*}\mathcal{O}_Y)_{|U})$$

(by definition of the image of a section in a stalk). The open set U contains P, hence it is non-empty and it intersects Y. Therefore we can apply Proposition 3.80. □

3.8.2 Higher order ramification

Let X be an integral smooth surface defined over $\mathbb{C}$, let C be an irreducible curve on X, let P be a point in C such that C is smooth at P. Let $\nu_C : \tilde{C} \to C$ be the normalization map, $i : C \to X$ the inclusion and $\varphi_C = i_C \circ \nu_C : \tilde{C} \to X$. Let Q be the unique preimage of P under φ_C. (Since C is smooth at P, there is only one preimage of P.)

Lemma 3.85. *There are local parameters $x, y \in m_{X,P} \subseteq \mathcal{O}_{X,P}$ and $t \in m_{\tilde{C},Q} \subseteq \mathcal{O}_{\tilde{C},Q}$ such that y is a local equation for C at P, and such that $\varphi^{\#}_{C,Q}(x) = t$ and $\ker(\varphi^{\#}_{C,Q}) = (y)$.*

Proof. Since P is a smooth point of X, we can take a local equation $y \in m_{X,P}$ for C at P. The morphism $i : C \to X$ induces the isomorphism

$$\overline{i^{\#}_P} : \mathcal{O}_{X,P}/(y) \to \mathcal{O}_{C,P} = \mathcal{O}_{\tilde{C},Q}.$$

Let $t \in m_{\tilde{C},Q}$ local parameter and let $x \in m_{X,P}$ such that $\nu^{\#}_{C,Q}\overline{i^{\#}_P}(\bar{x}) = t$, then $m_{X,P}/(y) = (\bar{x})$ in $\mathcal{O}_{X,P}/(y)$ (because $m_{\tilde{C},Q} = (t)$ and $\nu^{\#}_{C,Q}$ is isomorphism since C

is smooth at P). Therefore we get $m_{X,P} = (x, y)$ and since $\mathcal{O}_{X,P}$ is a local ring of dimension 2 we get that x, y are local parameters at P.

Finally

$$\begin{aligned} \varphi^{\#}_{C,Q}(x) &= \nu^{\#}_{C,Q} i^{\#}_P(x) = \nu^{\#}_{C,Q} \overline{i^{\#}_P}(\bar{x}) = t \\ \varphi^{\#}_{C,Q}(y) &= \nu^{\#}_{C,Q} i^{\#}_P(y) = \nu^{\#}_{C,Q} \overline{i^{\#}_P}(\bar{y}) = 0. \end{aligned}$$

Since $\mathcal{O}_{X,P}$ has dimension two, and $(y) \subseteq \ker(\varphi^{\#}_{C,Q}) \neq (x, y)$, we obtain that $(y) = \ker(\varphi^{\#}_{C,Q})$. □

Let $\pi' : Y \to X$ be a morphism of surfaces. Recall the notation $\pi^{\bullet}_{r,\mathcal{L}} : H^0(X, \mathcal{L} \otimes S^r\Omega^1_X) \to H^0(Y, \pi^*\mathcal{L} \otimes S^r\Omega^1_Y)$ from Subsection 3.2.5.

Lemma 3.86. *Let X, Y be smooth integral surfaces defined over $\mathbb{C}$. Let $\pi : Y \to X$ be a dominant morphism and let $D \subseteq Y$ be a prime divisor such that $C = \pi(D)$ is a curve (i.e. $\pi(D)$ is not a point). Let $P' \in D$ be such that D is the only component of $(\pi^*C)_{\mathrm{red}}$ passing through P', and put $P = \pi(P')$. Let $y \in \mathcal{O}_{X,P}$ be a local equation for C and let $z \in \mathcal{O}_{Y,P'}$ be a local equation for D. (They exist because X, Y are smooth.) Then there is an $\alpha \in \mathcal{O}^{\times}_{Y,P'}$ such that $\pi^{\#}_{P'}(y) = \alpha z^e$ where $e = e_{D/C}(\pi)$.*

Proof. We have that C is a Cartier divisor (X is a smooth variety). By the definition of pullback of Cartier divisors (cf. Subsection 2.2.2) we have that $\pi^{\#}_{P'}(y)$ is a local equation for π^*C at P'. The only component of π^*C at P' is supported on D and z is local equation for D. Since $\mathcal{O}_{Y,P'}$ is a unique factorization domain and z is irreducible, there exists $\alpha \in \mathcal{O}^{\times}_{Y,P'}$ such that $\pi^{\#}_{P'}(y) = \alpha z^r$ for some $r \geq 1$ (r is the multiplicity of D in π^*C).

We have $\mathcal{O}_{Y,P'} \subseteq \mathcal{O}_{Y,\eta_D}$ (with η_D the generic point of D), so $\alpha \in \mathcal{O}^{\times}_{Y,P'} \subseteq \mathcal{O}^{\times}_{Y,\eta_D}$.

($s \in \mathcal{O}_{Y,P'}$ if and only if s is regular in U which is a neighborhood of η_D). Since $\alpha \in \mathcal{O}_{Y,\eta_D}^{\times}$, we have $v_D(\alpha) = 0$. Therefore

$$\begin{aligned} e_{D/C}(\pi) &= v_{D|k(X)}(y) \\ &= v_D(\pi_{P'}^{\#}(y)) \\ &= v_D(\alpha z^r) \\ &= v_D(\alpha) + r v_D(z) = 0 + r \cdot 1 = r \end{aligned}$$

where the first equality holds because y generates m_{X,η_C}, the second because $\pi_P^{\#}(y)$ is the image of y under $k(X) \hookrightarrow (Y)$, and the fifth because z generates m_{Y,η_D}. □

Theorem 3.87. *Let X and Y be smooth integral surfaces defined over $\mathbb{C}$. Let $\pi : Y \to X$ be a dominant morphism and let $D \subseteq Y$ be a prime divisor such that $C = \pi(D)$ is a curve (i.e. $\pi(D)$ is not a point). Suppose that π has ramification index $e = e_{D/C}(\pi) > 1$ at D. Let $\mathcal{L}$ be an invertible sheaf on X, let r be a positive integer, and let $\omega \in H^0(X, \mathcal{L} \otimes S^r\Omega_X^1)$. If C is ω-integral, then $\pi_{r,\mathcal{L}}^{\bullet}\omega \in H^0(Y, \pi^*\mathcal{L} \otimes S^r\Omega_Y^1)$ vanishes identically along $(e-1)D$.*

Proof. Let Z be the subscheme associated to $(e-1)D$ (see Definition 2.21). By Corollary 3.84, it is enough to prove that the image of $(\pi_{r,\mathcal{L}}^{\bullet}\omega)_{P'}$ under the map $(\pi^*\mathcal{L} \otimes S^r\Omega_Y^1)_{P'} \to (\pi^*\mathcal{L} \otimes S^r\Omega_Y^1 \otimes i_{Z*}\mathcal{O}_Z)_{P'}$ is zero for any $P' \in Z$.

Choose $P' \in D$ such that $(\pi^*C)_{\text{red}}$ is smooth at P'. This ensures that D is the only component of $(\pi^*C)_{\text{red}}$ containing P' and that D is smooth at P'. There is an open non-empty set of D satisfying this requirement, so we can further require that $P = \pi(P')$ is a smooth point of C.

Let $Q \in \tilde{C}$ be the unique preimage of P by the normalization $\varphi_C : \tilde{C} \to C$. Let

$z \in m_{Y,P'}$ be a local equation of D at P'. By Lemma 3.85 we have that there are local parameters x_P, y_P coming from rational functions x, y such that $m_{X,P} = (x_P, y_P)$ where $\ker(\varphi^{\#}_{C,Q}) = (y_P)$ and $\varphi^{\#}_{C,Q}(x_P) = m_{\tilde{C},Q}$.

From Lemma 3.86 we obtain that $\pi^{\#}_{P'}(y_P) = \alpha z^e$ with $e = e_{D/C}(\pi)$.

Since P is a smooth point of X and x_P, y_P are local parameters at P, we have that $(\Omega^1_{X/\mathbb{C}})_P$ has basis dx_P, dy_P as $\mathcal{O}_{X,P}$-module, that is $(\Omega^1_{X/\mathbb{C}})_P = \mathcal{O}_{X,P}dx_P \oplus \mathcal{O}_{X,P}dy_P$. Note that $\mathcal{L}_P \cong \mathcal{O}_{X,P}$, that is, it is a free $\mathcal{O}_{X,P}$-module of rank 1. It follows that under the identification $(\mathcal{L} \otimes S^r\Omega^1_{X/\mathbb{C}})_P \cong \mathcal{L}_P \otimes S^r(\Omega^1_{X/\mathbb{C}})_P$, there are unique $a_i \in \mathcal{L}_P$ such that the image of ω in this stalk can be written as

$$\omega_P = \sum_{i=0}^{r} a_i \otimes dx_P^i dy_P^{r-i}$$

because the elements $dx_P^i dy_P^{r-i}$ form a basis of $S^r(\Omega^1_{X/\mathbb{C}})_P$. It follows that

$$\begin{aligned} v_{r,\varphi_C,Q}(\omega_P) &= \sum_{i=0}^{r} \varphi^{\#}_{C,\mathcal{L},Q}(a_i) \otimes d\varphi^{\#}_{C,\mathcal{L},Q}(x_P)^i d\varphi^{\#}_{C,\mathcal{L},Q}(y_P)^{r-i} \\ &= \sum_{i=0}^{r} \varphi^{\#}_{C,\mathcal{L},Q}(a_i) \otimes dt^i d0^{r-i} \\ &= \varphi^{\#}_{C,\mathcal{L},Q}(a_r) \otimes dt^r. \end{aligned}$$

Since C is ω-integral, we obtain by Theorem 3.52 that $v_{r,\varphi_C,Q}(\omega_P) = 0$, hence $\varphi^{\#}_{C,\mathcal{L},Q}(a_r) = 0$. By our choice of x_P, y_P, t, the kernel of $\varphi^{\#}_{C,\mathcal{L},Q}$ is (y_P), hence $a_r = y_P h$ for some $h \in \mathcal{L}_P$.

Thus we have $\pi^{\#}_{\mathcal{L},P'}(a_r) = \alpha z^e \pi^{\#}_{\mathcal{L},P'}(h)$. Moreover,

$$d\pi^{\#}_{P'}(y_P) = d(z^e\alpha) = z^{e-1}(e\alpha dz + z d\alpha) \in (z)^{e-1} \subseteq \mathcal{O}_{Y,P'}.$$

From Lemma 3.58 we have

$$\begin{aligned}(\pi^{\bullet}_{r,\mathcal{L}}\omega)'_P &= \beta_\pi(\omega_P)\\ &= \beta_\pi(\sum_{i=0}^{r} a_i \otimes (dx)^i(dy)^{r-i})\\ &= \sum_{i=0}^{r} \pi^{\#}_{\mathcal{L},P'}(a_i)\otimes \bar{\beta}_\pi((dx)^i(dy)^{r-i})\\ &= \sum_{i=0}^{r-1} \pi^{\#}_{\mathcal{L},P'}(a_i)\otimes \bar{\beta}_\pi((dx)^i(dy)^{r-i}),\end{aligned}$$

where the last equality holds because $\pi^{\#}_{\mathcal{L},P'}(a_r)=0$.

From Lemma 3.63 we obtain that

$$\begin{aligned}\bar{\beta}_\pi((dx)^i(dy)^{r-i})_P &= \mu_{r,C,Q}\circ v_{r,f,Q}\circ \mu^{-1}_{r,X,P}((dx)^i(dy)^{r-i})_P\\ &= \mu_{r,C,Q}\circ v_{r,f,Q}((dx_P)^i(dy_P)^{r-i})\\ &= \mu_{r,C,Q} d(\pi^{\#}_Q x_P)^i d(\pi^{\#}_Q y_P)^{r-i}\\ &= \mu_{r,C,Q} d(\pi^{\#}_Q x_P)^i d(z^e\alpha)^{r-i}\\ &= \mu_{r,C,Q} d(\pi^{\#}_Q x_P)^i (z^{e-1}(e\alpha dz + z d\alpha))^{r-i}.\end{aligned}$$

Hence $(\pi^{\bullet}_{r,\mathcal{L}}\omega)_{P'} = \sum_{i=0}^{r-1}\pi^{\#}_{\mathcal{L},P'}(a_i)\otimes d(\pi^{\#}_Q x_P)^i(z^{e-1}(e\alpha dz + z d\alpha))^{r-i}$ which is zero modulo (z^{e-1}), thus the image of $(\pi^{\bullet}_{r,\mathcal{L}}\omega)_P$ in $(\pi^*\mathcal{L}\otimes S^r\Omega^1_{Y/\mathbb{C}}\otimes i_{Z*}\mathcal{O}_Z)_{P'}$ is zero, and thus $\pi^{\bullet}_{r,\mathcal{L}}\omega$ vanishes identically along $(e-1)D$. □

In later sections, the previous theorem will be used combined with Proposition 3.79 and with the following:

Proposition 3.88. *Let Y be a smooth integral surface and let $\omega_0 \in H^0(Y, \mathcal{L}\otimes S^r\Omega^1_Y)$. Let D be an effective divisor on Y. Suppose that ω_0 vanishes identically along D.*

Then there is a symmetric differential $\omega_0' \in H^0(Y, \mathcal{L}(-D) \otimes \mathcal{L} \otimes S^r\Omega^1_Y)$ *such that all* ω_0'*-integral curves are among the* ω_0*-integral curves.*

Proof. The exact sequence (2.5) (cf. Subsection 2.2.1) gives the exact sequence

$$0 \to H^0(Y, \mathcal{L}(-D) \otimes \mathcal{L} \otimes S^r\Omega^1_Y) \xrightarrow{\mu_0} H^0(Y, \mathcal{L} \otimes S^r\Omega^1_Y) \to H^0(Y, \mathcal{L} \otimes S^r\Omega^1_Y \otimes i_*\mathcal{O}_D)$$

where μ_0 is induced by $\mu : \mathcal{L}(-D) \otimes \mathcal{L} \otimes S^r\Omega^1_Y \to \mathcal{L} \otimes S^r\Omega^1_Y$ on global sections (cf. the discussion after (2.4) in Subsection 2.2.1).

As ω_0 vanishes along D, it is in the image of the injective map μ_0, and we let ω_0' be the unique pre-image of ω_0 under μ_0.

Let $\mu' : \mathcal{L}(-D) \otimes \mathcal{L} \to \mathcal{L}$ be the map from Subsection 2.2.1. Then $\mu' \otimes Id_{S^r\Omega^1_Y} = \mu$ because the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{L}(-D) \otimes \mathcal{L}) \otimes S^r\Omega^1_{Y/k} & \longrightarrow & (\mathcal{O}_Y \otimes \mathcal{L}) \otimes S^r\Omega^1_{Y/k} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{L}(-D) \otimes (\mathcal{L} \otimes S^r\Omega^1_{Y/k}) & \longrightarrow & \mathcal{O}_Y \otimes (\mathcal{L} \otimes S^r\Omega^1_{Y/k}) \end{array}$$

as the tensor product is associative (see the definition of the morphisms μ in Subsection 2.2.1).

Let $C \in Y$ be a curve with normalization $i : \tilde{C} \to Y$. The following diagram commutes because $T^i_{\mathcal{L}, S^r\Omega^1_Y}$ is functorial on the sheaves (see Subsection 3.2.3)

$$\begin{array}{ccc} i^*(\mathcal{L}(-D) \otimes \mathcal{L}) \otimes i^*S^r\Omega^1_{Y/k} & \xrightarrow{T^i_{\mathcal{L}(-D)\otimes\mathcal{L}, S^r\Omega^1_{Y/k}}} & i^*(\mathcal{L}(-D) \otimes \mathcal{L} \otimes S^r\Omega^1_{Y/k}) \\ {\scriptstyle i^*\mu' \otimes i^*Id}\downarrow & & \downarrow{\scriptstyle i^*(\mu' \otimes Id) = i^*\mu} \\ i^*\mathcal{L} \otimes i^*S^r\Omega^1_{Y/k} & \xrightarrow{T^i_{\mathcal{L}, S^r\Omega^1_{Y/k}}} & i^*(\mathcal{L} \otimes S^r\Omega^1_{Y/k}) \end{array}$$

and the following diagram clearly commutes

$$\begin{array}{ccc} i^*(\mathcal{L}(-D)\otimes\mathcal{L})\otimes i^*S^r\Omega^1_{Y/k} & \xrightarrow{Id\otimes\phi^i_r} & i^*(\mathcal{L}(-D)\otimes\mathcal{L})\otimes S^r\Omega^1_{C/k} \\ \downarrow{\scriptstyle i^*\mu'\otimes i^*Id} & & \downarrow{\scriptstyle (i^*\mu')\otimes Id} \\ i^*\mathcal{L}\otimes i^*S^r\Omega^1_{Y/k} & \xrightarrow{Id\otimes\phi^i_r} & i^*\mathcal{L}\otimes S^r\Omega^1_{C/k} \end{array}$$

so the following diagram commutes

$$\begin{array}{ccc} i^*(\mathcal{L}(-D)\otimes\mathcal{L}\otimes S^r\Omega^1_{Y/k}) & \xrightarrow{\kappa_{i,\mathcal{L}(-D)\otimes\mathcal{L},r}} & i^*(\mathcal{L}(-D)\otimes\mathcal{L})\otimes S^r\Omega^1_{C/k} \\ \downarrow{\scriptstyle i^*\mu} & & \downarrow{\scriptstyle (f^*\mu')\otimes Id} \\ i^*(\mathcal{L}\otimes S^r\Omega^1_{Y/k}) & \xrightarrow{\kappa_{i,\mathcal{L},r}} & i^*\mathcal{L}\otimes S^r\Omega^1_{C/k} \end{array}$$

Also, we know that the following commutes by properties of the canonical map

$$\begin{array}{ccc} \mathcal{L}(-D)\otimes\mathcal{L}\otimes S^r\Omega^1_{Y/k} & \xrightarrow{\rho^i_{\mathcal{L}(-D)\otimes\mathcal{L}\otimes S^r\Omega^1_{Y/k}}} & i_*i^*(\mathcal{L}(-D)\otimes\mathcal{L}\otimes S^r\Omega^1_{Y/k}) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle i_*i^*\mu} \\ \mathcal{L}\otimes S^r\Omega^1_{Y/k} & \xrightarrow{\rho^i_{\mathcal{L}\otimes S^r\Omega^1_{Y/k}}} & i_*i^*(\mathcal{L}\otimes S^r\Omega^1_{Y/k}) \end{array}$$

Therefore the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}(-D)\otimes\mathcal{L}\otimes S^r\Omega^1_{Y/k} & \xrightarrow{\eta_{i,\mathcal{L}(-D)\otimes\mathcal{L},r}} & i_*(i^*(\mathcal{L}(-D)\otimes\mathcal{L})\otimes S^r\Omega^1_{C/k}) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle (f^*\mu')\otimes Id} \\ \mathcal{L}\otimes S^r\Omega^1_{Y/k} & \xrightarrow{\eta_{i,\mathcal{L},r}} & i_*(i^*\mathcal{L}\otimes S^r\Omega^1_{C/k}) \end{array}$$

and taking global sections we get that the following commutative diagram

$$\begin{array}{ccc} H^0(Y, \mathcal{L}(-D) \otimes \mathcal{L} \otimes S^r\Omega^1_{Y/k}) & \xrightarrow{i^\bullet_{\mathcal{L}(-D)\otimes\mathcal{L},r}} & H^0(C, i^*(\mathcal{L}(-D) \otimes \mathcal{L}) \otimes S^r\Omega^1_{C/k}) \\ \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle (f^*\mu')\otimes Id} \\ H^0(Y, \mathcal{L} \otimes S^r\Omega^1_{Y/k}) & \xrightarrow{i^\bullet_{\mathcal{L},r}} & H^0(C, i^*\mathcal{L} \otimes S^r\Omega^1_{C/k}) \end{array}$$

As μ_0 is injective and $\mu_0(\omega_0') = \omega_0$, we conclude that if C is ω_0'-integral (that is $i^\bullet_{\mathcal{L}(-D)\otimes\mathcal{L},r}(\omega_0') = 0$), then it is ω_0-integral (that is $i^\bullet_{\mathcal{L},r}(\omega_0) = 0$). □

Chapter 4

Squares with constant second differences

4.1 Introduction and main results

Recall the surfaces

$$X_n : \begin{cases} x_1^2 - 3x_2^2 + 3x_3^2 = x_4^2 \\ \vdots \\ x_{i-3}^2 - 3x_{i-2}^2 + 3x_{i-1}^2 = x_i^2 \\ \vdots \\ x_{n-3}^2 - 3x_{n-2}^2 + 3x_{n-1}^2 = x_n^2 \end{cases} \tag{4.1}$$

which were defined in Subsection 2.1.2. For each n, the surface X_n is in $\mathbb{P}^{n-1}$, and it is defined by $n-3$ equations. By convention, $X_3 = \mathbb{P}^2$. It is well known that these are surfaces, see [BB10]. A point $[x_1 : \cdots : x_n] \in X_n(\mathbb{Q})$ corresponds to a sequence $x_n, \ldots, x_1$ of rational numbers (up to scaling) satisfying $x_{i-3}^2 - 3x_{i-2}^2 + 3x_{i-1}^2 = x_i^2$ (i.e. whose squares have constant second differences, cf. Subsection 2.1.3). Equivalently, it corresponds to a polynomial $P \in \mathbb{Q}[t]$ of degree less than or equal to 2 (up to scaling) with all $P(0), \ldots, P(n-1)$ square rational numbers.

In this chapter, we use Vojta's technique to prove the following:

Theorem 4.1. *Let $g \geq 0$. If $n > \max\{9, 4g+6\}$, then the only curves of genus at most g on X_n are the trivial lines $[s+t : \pm(2s+t) : \cdots : \pm(ns+t)]$ with $[s:t] \in \mathbb{P}^1$.*

This theorem specializes in the case $g \leq 1$ as follows:

Theorem 4.2. *For $n \geq 11$, the only curves of genus 0 or 1 on X_n are the trivial lines $[s+t : \pm(2s+t) : \cdots : \pm(ns+t)]$ with $[s:t] \in \mathbb{P}^1$.*

Hence we are able to find *all* rational or elliptic curves on X_n for $n \geq 11$.

Note that a point $[x_1 : \cdots : x_n]$ on a curve $[s+t : \pm(2s+t) : \cdots : \pm(ns+t)]$ satisfies that $x_i^2 = (is+t)^2$, that is, it gives rise to a sequence of squares of elements in arithmetic progression.

Theorem 4.1 gives us a result in the arithmetic of function fields.

Theorem 4.3. *Let K be a function field of genus g with constant field $\mathbb{C}$, and let $n > \max\{9, 4g+6\}$. Let $f_1, \ldots, f_n \in K$ be such that the squares of this sequence have second differences equal to a fixed $f \in K$. Then the sequence $(f_1, \ldots, f_n)$ is proportional to a sequence of complex numbers, or it is of the form $f_j = \epsilon_j(aj+b)$ for some $a, b \in K$ and $\epsilon_j \in \{-1, 1\}$.*

Browkin and Brzezinski [BB10] observed that if one is able to prove a theorem like Theorem 4.2, we get the following arithmetic consequence (which we prove in Subsection 4.8) regarding sequences of integers having constant second differences.

Theorem 4.4. *Assume the Bombieri-Lang Conjecture (Conjecture 2.4) for the surfaces X_n with $n \geq 11$. Then there are (up to scaling) finitely many sequences of 11 integers whose squares have constant second differences, but which are not of the form $(ns+t)^2$, with $n = 1, 2, \ldots$ and $s, t \in \mathbb{Z}$. Moreover, there exists an $M > 0$ such that*

if $x_1, \ldots, x_M$ is a coprime sequence of integers whose squares have constant second differences, then the sequence is trivial.

Theorem 4.4 formulated in terms of square values of polynomials is as follows:

Theorem 4.5. *Assume the Bombieri-Lang Conjecture for the surfaces X_n, with $n \geq 11$. Then, up to scaling, there are only finitely many polynomials $P(t) \in \mathbb{Q}[t]$ which are not the square of a polynomial, and the values $P(1), P(2), \ldots, P(11)$ are all squares. Moreover, there exists an integer $M > 0$ such that if $P(t) \in \mathbb{Q}[t]$ is a quadratic polynomial for which the values $P(1), P(2), \ldots, P(M)$ are squares, then $P(t)$ is the square of a polynomial.*

4.2 The geometry of the surfaces X_n

Recall that the surface X_n is in $\mathbb{P}^{n-1}$.

Lemma 4.6. *If $[x_1 : \ldots : x_n] \in X_n$, then we have*

$$\frac{(m-3)(m-2)}{2}x_1^2 - ((m-2)^2 - 1)x_2^2 + \frac{(m-2)(m-1)}{2}x_3^2 = x_m^2,$$

for any $1 \leq m \leq n$.

Proof. Fix $n \geq 4$. We will prove this lemma by induction on m. It can be seen that the lemma is true for $m = 1, 2, 3$. We know that in X_n we have $x_1^2 - 3x_2^2 + 3x_3^2 = x_4^2$, so the lemma is also true for $m = 4$. Now let $4 < m \leq n$ and suppose that the statement is true for all $i < m$. From the formula $x_{m-3}^2 - 3x_{m-2}^2 + 3x_{m-1}^2 = x_m^2$ of (4.1) we obtain by the induction hypothesis that

$$\begin{aligned} x_m^2 &= x_{m-3}^2 - 3x_{m-2}^2 + 3x_{m-1}^2 \\ &= \left(\frac{(m-6)(m-5)}{2}x_1^2 - ((m-5)^2-1)x_2^2 + \frac{(m-5)(m-4)}{2}x_3^2\right) \\ &\quad -3\left(\frac{(m-5)(m-4)}{2}x_1^2 - ((m-4)^2-1)x_2^2 + \frac{(m-4)(m-3)}{2}x_3^2\right) \\ &\quad +3\left(\frac{(m-4)(m-3)}{2}x_1^2 - ((m-3)^2-1)x_2^2 + \frac{(m-3)(m-2)}{2}x_3^2\right) \\ &= \frac{(m-3)(m-2)}{2}x_1^2 - ((m-2)^2-1)x_2^2 + \frac{(m-2)(m-1)}{2}x_3^2. \end{aligned}$$

□

Proposition 4.7. *Let $f_i = x_i^2 - 3x_{i+1}^2 + 3x_{i+2}^2 - x_{i+3}^2$ be the generators of the ideal defining X_n,and let*

$$g_k = \frac{k(k+1)}{2}x_1^2 - ((k+1)^2-1)x_2^2 + \frac{(k+1)(k+2)}{2}x_3^2 - x_{k+3}^2.$$

Then we have the equality of ideals $(f_1, \ldots, f_{n-3}) = (g_1, \ldots, g_{n-3})$ in $k[x_1, \ldots, x_n]$. In particular, the g_i for $1 \leq i \leq n-3$ are also defining equations for X_n.

Proof. First we show that for $n \geq 4$ we have $I_n := (g_1, \ldots, g_{n-3}) \subseteq J_n := (f_1, \ldots, f_{n-3})$. It suffices to show that $g_j \in J_n$ for each $1 \leq k \leq n-3$, and we will do this by induction on k. Note that we have $g_k = 0$ for $k = -2, -1, 0$, so for these values of k we have $g_k \in J_n$. Let $1 \leq k \leq n-3$ and suppose (as induction hypothesis) that $g_j \in J_n$ for all $-2 \leq j < k$. Then, working modulo J_n and using that $g_j \in J_n$ for $-2 \leq j < k$ we obtain

$$\begin{aligned} x_k^2 &\equiv x_{k-3}^2 - 3x_{k-2}^2 + 3x_{k-1}^2 \mod J_n \\ &\equiv \left(\frac{(k-6)(k-5)}{2}x_1^2 - ((k-5)^2-1)x_2^2 + \frac{(k-5)(k-4)}{2}x_3^2\right) \\ &\quad -3\left(\frac{(k-5)(k-4)}{2}x_1^2 - ((k-4)^2-1)x_2^2 + \frac{(k-4)(k-3)}{2}x_3^2\right) \\ &\quad +3\left(\frac{(k-4)(k-3)}{2}x_1^2 - ((k-3)^2-1)x_2^2 + \frac{(k-3)(k-2)}{2}x_3^2\right) \mod J_n \\ &\equiv \frac{k-3}{2}x_1^2 - ((k-2)^2-1)x_2^2 + \frac{(k-2)(k-1)}{2}x_3^2 \mod J_n. \end{aligned}$$

This proves that $g_k \equiv 0 \mod J_n$, that is, $g_k \in J_n$. Therefore $I_n \subseteq J_n$.

Now we prove the inclusion $J_n \subseteq I_n$. Working modulo I_n we get the relations

$$x_{k+3}^2 \equiv \frac{k(k+1)}{2}x_1^2 - ((k+1)^2-1)x_2^2 + \frac{(k+1)(k-2)}{2}x_3^2 \mod I_n$$

for $-2 \leq k \leq n-3$, so for $1 \leq k \leq n-3$ we get

$$\begin{aligned} f_k &= x_k^2 - 3x_{k+1}^2 + 3x_{k+2}^2 - x_{k+3}^2 \\ &\equiv \left(\frac{(k-3)(k-2)}{2}x_1^2 - ((k-2)^2-1)x_2^2 + \frac{(k-2)(k-1)}{2}x_3^2\right) \\ &\quad -3\left(\frac{(k-2)(k-1)}{2}x_1^2 - ((k-1)^2-1)x_2^2 + \frac{(k-1)k}{2}x_3^2\right) \\ &\quad +3\left(\frac{(k-1)k}{2}x_1^2 - (k^2-1)x_2^2 + \frac{k(k+1)}{2}x_3^2\right) \\ &\quad -\left(\frac{k(k+1)}{2}x_1^2 - ((k+1)^2-1)x_2^2 + \frac{(k+1)(k+2)}{2}x_3^2\right) \mod I_n \\ &\equiv 0 \cdot x_1^2 + 0 \cdot x_2^2 + 0 \cdot x_3^2 \equiv 0 \mod I_n. \end{aligned}$$

which proves that $f_k \in I_n$ for $1 \leq k \leq n-3$. Therefore $I_n = J_n$. □

Lemma 4.8. *If $[x_1 : \cdots : x_n]$ is a point on X_n, then no three of $x_1, \ldots, x_n$ can be simultaneously zero.*

Proof. If x_1, x_2, x_3 are all zero, then by Lemma 4.6 every $x_i = 0$, which contradicts the fact that $[x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}$. Now view

$$\frac{(j-3)(j-2)}{2}x_1^2 - ((j-2)^2 - 1)x_2^2 + \frac{(j-1)(j-2)}{2}x_3^2 = 0$$

as an equation in j. Note that for $j = 1$ this gives $x_1^2 = 0$, for $j = 2$ it gives $x_2^2 = 0$ and for $j = 3$ this gives $x_3^2 = 0$. This equation can be written in the form

$$(x_1^2 - 2x_2^2 + x_3^2)j^2 - (-5x_1^2 + 8x_2^2 - 3x_3^2)j + (6x_1^2 - 6x_2^2 + 2x_3^2) \;=\; 0. \qquad (4.2)$$

If all $x_1^2 - 2x_2^2 + x_3^2$, $-5x_1^2 + 8x_2^2 - 3x_3^2$ and $6x_1^2 - 6x_2^2 + 2x_3^2$ are zero, then it can be computed that $x_1 = x_2 = x_3 = 0$, which is not possible. Hence Equation (4.2) has at most two solutions in j. Therefore there are at most two values of j such that $x_j = 0$. □

The following observation will allow us to prove that the surfaces X_n are smooth.

Observation 4.9. Let $\alpha \neq \beta$ be different from $1, 2, 3$. Then the matrix

$$\begin{pmatrix} (\alpha-3)(\alpha-2)x_1 & 2((\alpha-2)^2-1)x_2 \\ (\beta-3)(\beta-2)x_1 & 2((\beta-2)^2-1)x_2 \end{pmatrix}$$

has determinant $2x_1x_2(\alpha-3)(\beta-3)(\alpha-\beta)\neq 0$, if $x_1x_2\neq 0$. The matrix

$$\begin{pmatrix} (\alpha-3)(\alpha-2)x_1 & (\alpha-2)(\alpha-1)x_3 \\ (\beta-3)(\beta-2)x_1 & (\beta-2)(\beta-1)x_3 \end{pmatrix}$$

has determinant $2x_1x_3(\alpha-2)(\beta-2)(\alpha-\beta)\neq 0$, if $x_1x_3\neq 0$, and the matrix

$$\begin{pmatrix} 2((\alpha-2)^2-1)x_2 & (\alpha-2)(\alpha-1)x_3 \\ 2((\beta-2)^2-1)x_2 & (\beta-2)(\beta-1)x_3 \end{pmatrix}$$

has determinant $2x_2x_3(\alpha-1)(\beta-2)(\alpha-\beta)\neq 0$, if $x_2x_3\neq 0$.

Lemma 4.10. *For each $n\geq 3$, the surface X_n is smooth.*

Proof. Since $X_3\cong\mathbb{P}^2$ we know that X_3 is smooth. Now let $n\geq 4$. By Proposition 4.7, we have that the surface X_n is defined by the equations $g_1,\ldots,g_n$. From Proposition 2.77, we only need to show that the Jacobian matrix of the homogeneous equations $g_1,\ldots,g_n$ of X_n has rank $n-3$ when evaluated at $[x_1:\ldots:x_n]\in X_n$. This is the following $(n-3)\times n$ matrix

$$\begin{pmatrix} 2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (n-3)(n-2)x_1 & 2((n-2)^2-1)x_2 & (n-2)(n-1)x_3 & 0 & \cdots & 0 & -2x_n \end{pmatrix}$$

We will prove by induction on i that this matrix has maximal rank equal to $n-3$. We know from Lemma 4.8 that for $i=4$ this matrix has maximal rank (equal to 1). Suppose by the induction hypothesis that the following $(i-3)\times i$ matrix with

$4 \leq i \leq n-1$ has maximal rank.

$$M_i = \begin{pmatrix} 2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (i-3)(i-2)x_1 & 2((i-2)^2-1)x_2 & (i-2)(i-1)x_3 & 0 & \cdots & 0 & -2x_i \end{pmatrix}$$

and consider the $(i-2) \times (i+1)$ matrix

$$M_{i+1} = \begin{pmatrix} 2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (i-2)(i-1)x_1 & 2((i-1)^2-1)x_2 & (i-1)(i)x_3 & 0 & \cdots & 0 & -2x_{i+1} \end{pmatrix}$$

If $x_{i+1} \neq 0$ then the matrix M_{i+1} has rank $M_i + 1 = i - 2$. Now suppose that $x_{i+1} = 0$. By Lemma 4.8, at most one among $x_1, \ldots, x_i$ can be zero. If all x_j with $4 \leq j \leq i$ are non-zero, then we are done. If $x_j = 0$ for some $4 \leq j \leq i$, then we only have to prove that the $(j-3)$-th row is not a multiple of the $(i-2)$-nd row. By Lemma 4.8 we have that at least two of x_1, x_2, x_3 are different from zero. Then the j-th row is not a multiple of the first row by Observation 4.9, so the matrix M_{i+1} has maximal rank $i-2$.

Therefore the Jacobian matrix of X_n has rank $n-3$ and thus by Proposition 2.77 the surface X_n is smooth. □

Proposition 4.11. *For $n \geq 3$, the surface X_n is smooth and irreducible, and it is of general type for $n \geq 7$.*

Proof. Since X_n is a smooth complete intersection by Lemma 4.10, we get from

Proposition 2.79, that X_n is irreducible and its canonical sheaf is $\mathcal{O}(n-6)$. By Example II.7.6.1 in [Har77], the sheaf $\mathcal{O}(n-6)$ is ample for $n \geq 7$, thus a multiple of $\mathcal{O}(n-6)$ determines an embedding $X_n \to \mathbb{P}^N$, and hence X_n is a surface of general type, by Theorem V.6.5 in [Har77]. □

For each $n \geq 4$, define the map $\pi_n : X_n \to X_{n-1}$ as the restriction to X_n of the morphism

$$\begin{aligned} \tilde{\pi}_n : \mathbb{P}^{n-1} \setminus \{[0:\cdots:0:1]\} &\to \mathbb{P}^{n-2} \\ [x_1:\cdots:x_n] &\mapsto [x_1:\cdots:x_{n-1}]. \end{aligned}$$

The rational map $\tilde{\pi}_n$ corresponds to the inclusion map $k[x_1,\ldots,x_{n-1}] \to k[x_1,\ldots,x_n]$ (which respects the grading) in the sense of [Har77] II, Exercise 2.14(b), and the morphism π_n corresponds to the induced map

$$k[x_1,\ldots,x_{n-1}]/(f_1,\ldots,f_{n-4}) \to k[x_1,\ldots,x_n]/(f_1,\ldots,f_{n-3}),$$

which exists because $(f_1,\ldots,f_{n-4}) \subseteq k[x_1,\ldots,x_{n-1}] \cap (f_1,\ldots,f_{n-3})$. Therefore $\pi_n(X_n) \subseteq X_{n-1}$.

Proposition 4.12. *For each n, the map π_n is a finite surjective morphism of degree 2, with ramification curve $R_n := \{x_n = 0\} \cap X_n$.*

Proof. If $\pi_n([x_1:\cdots:x_n])$ is undefined, then we get $x_0 = \cdots = x_{n-1} = 0$. By Lemma 4.6 we obtain that $x_n = 0$, contradicting the fact that $[x_1:\cdots:x_n] \in \mathbb{P}^{n-1}$. Therefore π_n is a morphism.

Now let $P = [x_1:\cdots:x_{n-1}] \in X_{n-1}$, and let $\tilde{P} := [x_1:\ldots:x_n] \in \mathbb{P}^{n-1}$ be a

preimage of P with respect to $\tilde{\pi}_n$. Then $\tilde{P}$ lies on X_n if and only if

$$x_n^2 = \frac{(n-2)(n-1)}{2}x_3^2 - ((n-2)^2-1)x_2^2 + \frac{(n-3)(n-2)}{2}x_1^2,$$

by Lemma 4.6. Since we can always solve this, we have that π_n is a surjective quasi-finite morphism. Moreover, it is finite because it is projective. It is of degree 2 by Lemma 2.87.

The curve

$$\frac{(n-2)(n-1)}{2}x_3^2 - ((n-2)^2-1)x_2^2 + \frac{(n-3)(n-2)}{2}x_1^2 = 0$$

in $\mathbb{P}^2$ is irreducible by Lemma 2.81. Since each point on this curve has only one preimage under $\tilde{\pi}_n$ in X_n, it follows that π_n is totally ramified at each component of R_n by Proposition 2.88. From Lemma 4.6 we have that the pullback of this curve on X_n has equation $x_n^2 = 0$. □

We have a tower of finite morphisms between the surfaces X_n:

$$\mathbb{P}^2 = X_3 \overset{\pi_4}{\leftarrow} X_4 \overset{\pi_5}{\leftarrow} X_5 \overset{\pi_6}{\leftarrow} \cdots$$

Define $\rho_n = \pi_4 \circ \cdots \circ \pi_n$. The morphism ρ_n has degree 2^{n-3}. Note that by Lemma 4.6, the image under ρ_n of the ramification curve $R_n : x_n = 0$ in $X_3 = \mathbb{P}^2$ is

$$C_n : \frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 = 0,$$

so $\rho_n(R_n) = C_n$.

Lemma 4.13. *For $n \geq 4$ we have $\rho_n^*(C_n) = 2R_n$.*

Proof. Recall that

$$C_n = \mathrm{div}_{X_3}\left(\frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2\right).$$

Then by Proposition 2.30 and Lemma 4.6 we have

$$\begin{aligned}\rho_n^*(C_n) &= \mathrm{div}_{X_n}\left(\frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2\right)\\ &= \mathrm{div}_{X_n}(x_n^2) = 2R_n.\end{aligned}$$

□

4.3 A convenient differential form

Let $\{U_i\} \subseteq \mathbb{P}^2$ be the usual affine open cover of $\mathbb{P}^2$ with $U_i = D_+(X_i)$. Working on the open set U_3 with affine coordinates $x_1 = \frac{X_1}{X_3}$, $x_2 = \frac{X_2}{X_3}$, we have the following symmetric differential

$$x_1^2x_2^2dx_1dx_1 + (x_1x_2 - 4x_1x_2^3 - x_1^3x_2)dx_1dx_2 + 4x_1^2x_2^2dx_2dx_2 \in S^2\Omega^1_{U_3/\mathbb{C}}.$$

Proposition 4.14. *This differential form in U_3 can be extended to a form*

$$\omega \in H^0(\mathbb{P}^2, \mathcal{O}(7) \otimes S^2\Omega^1_{\mathbb{P}^2}).$$

Proof. Write

$$\begin{aligned} A &= x_1^2 x_2^2 dx_1 dx_1, \\ B &= (x_1 x_2 - 4x_1 x_2^3 - x_1^3 x_2) dx_1 dx_2, \\ C &= 4x_1^2 x_2^2 dx_2 dx_2. \end{aligned}$$

In the open set U_1 with affine coordinates $x_2 := \frac{X_2}{X_1}$, $x_3 := \frac{X_3}{X_1}$, these forms become

$$\begin{aligned} A &= x_2^2 x_3^{-4} d(x_3^{-1}) d(x_3^{-1}), \\ B &= (x_2 x_3^{-2} - 4x_2^3 x_3^{-4} - x_2 x_3^{-4}) d(x_3^{-1}) d(x_2 x_3^{-1}), \\ C &= 4x_2^2 x_3^{-4} d(x_2 x_3^{-1}) d(x_2 x_3^{-1}), \end{aligned}$$

hence in U_1

$$A + B + C = \frac{1}{x_3^7}(4x_2^2 x_3 dx_2 dx_2 + (x_2 - 4x_2^3 - x_2 x_3^2) dx_2 dx_3 + x_2^2 x_3 dx_3 dx_3).$$

Similarly, in the open set U_2 with affine coordinates $x_1 := \frac{X_1}{X_2}$, $x_3 := \frac{X_3}{X_2}$

$$\begin{aligned} A &= x_1^2 x_3^{-4} d(x_1 x_3^{-1}) d(x_1 x_3^{-1}), \\ B &= (x_1 x_3^{-2} - 4x_1 x_3^{-4} - x_1^3 x_3^{-4}) d(x_1 x_3^{-1}) d(x_3^{-1}), \\ C &= 4x_1^2 x_3^{-4} d(x_3^{-1}) d(x_3^{-1}), \end{aligned}$$

hence in U_2,

$$A + B + C = \frac{1}{x_3^7}(x_1^2 x_3 dx_1 dx_1 + (-x_1 x_3^2 - x_1^3 + 4x_1) dx_1 dx_3 + x_1^2 x_3 dx_3 dx_3).$$

Therefore our symmetric differential extends to a section $\omega \in H^0(\mathbb{P}^2, \mathcal{O}(7) \otimes S^2\Omega^1_{\mathbb{P}^2})$.

□

Remark 4.15. We have carefully chosen this differential in such a way that the curves C_i are ω-integral. This is a *critical* part of the argument, see for instance Appendix A in [Voj00], where Vojta explains how he found his differential for the application considered in [Voj00], using computer search in positive characteristic.

4.4 Finding all ω-integral curves in X_3

In this section we will find the complete set of ω-integral curves in $X_3 = \mathbb{P}^2$.

Lemma 4.16. *The following curves in X_3 are ω-integral:*

(i) $C_\alpha : \frac{(\alpha-3)(\alpha-2)}{2}x_1^2 - ((\alpha-2)^2-1)x_2^2 + \frac{(\alpha-2)(\alpha-1)}{2}x_3^2 = 0$, $\alpha \in \mathbb{C}\setminus\{1,2,3\}$;

(ii) $C_\infty : x_1^2 - 2x_2^2 + x_3^2 = 0$;

(iii) *The coordinate axes* $x_1 = 0$, $x_2 = 0$, $x_3 = 0$;

(iv) *The four curves* $x_1 \pm 2x_2 \pm x_3 = 0$.

Proof. The curves of type (i) and type (ii) are irreducible by Lemma 2.81. Let C_α be a curve of type (i). The curve C_α restricted to U_3 has equation $x_1^2 = 2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}$. Taking differentials we obtain $dx_1 = 2\frac{\alpha-1}{\alpha-2}\frac{x_1}{x_2}dx_2$.

$$\begin{aligned} A_{|C_\alpha} &= x_1^2x_2^2dx_1dx_1 \\ &= x_1^2x_2^2\left(2\frac{\alpha-1}{\alpha-2}\frac{x_1}{x_2}\right)^2 dx_2dx_2 \\ &= \left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right)^2\left(2\frac{\alpha-1}{\alpha-2}\right)^2 dx_2dx_2, \end{aligned}$$

$$\begin{aligned}
B_{|C_\alpha} &= \left(x_1x_2 - 4x_1x_2^3 - x_1^3x_2\right) dx_1dx_2 \\
&= \left(x_1x_2 - 4x_1x_2^3 - x_1^3x_2\right) \left(2\frac{\alpha-1}{\alpha-2}\frac{x_1}{x_2}\right) dx_2dx_2 \\
&= x_1^2\left(1 - 4x_2^2 - x_1^2\right) \left(2\frac{\alpha-1}{\alpha-2}\right) dx_2dx_2 \\
&= \left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) \left(1 - 4x_2^2 - 2\frac{\alpha-1}{\alpha-2}x_2^2 + \frac{\alpha-1}{\alpha-3}\right) 2\frac{\alpha-1}{\alpha-2}dx_2dx_2,
\end{aligned}$$

$$\begin{aligned}
C_{|C_\alpha} &= 4x_1^2x_2^2dx_2dx_2 \\
&= 4\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) x_2^2dx_2dx_2.
\end{aligned}$$

Thus the restriction of ω to C_α has equation

$$\begin{aligned}
&\left(\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right)^2 \left(2\frac{\alpha-1}{\alpha-2}\right)^2 + 4\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) x_2^2 + \right. \\
&\left.\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) \left(1 - 4x_2^2 - \left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right)\right) \left(2\frac{\alpha-1}{\alpha-2}\right)\right) dx_2dx_2 \\
= &\left(4\left(\frac{\alpha-1}{\alpha-2}\right)^2 x_2^4 + 2\frac{\alpha-1}{\alpha-2}x_2^2 - 8\frac{\alpha-1}{\alpha-2}x_2^4 - 2\frac{\alpha-1}{\alpha-2}\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) x_2^2\right. \\
&\left.+4\left(2\frac{\alpha-1}{\alpha-2}x_2^2 - \frac{\alpha-1}{\alpha-3}\right) x_2^2\right) dx_2dx_2 \\
= &\ 0.
\end{aligned}$$

Hence by Corollary 3.72, the curves of type (i) are ω-integral.

The curve of type (ii) restricted to U_3 has equation $x_1^2 - 2x_2^2 + 1 = 0$. Taking differentials we get $dx_1 = 2\frac{x_2}{x_1}dx_2$. Therefore

$$\begin{aligned} A_{|C_\infty} &= 4x_2^4 dx_2 dx_2, \\ B_{|C_\infty} &= 2(2x_2^2 - 6x_2^4) dx_2 dx_2, \\ C_{|C_\infty} &= 4(2x_2^4 - x_2^2) dx_2 dx_2, \end{aligned}$$

hence the restriction of ω to the curve C_∞ has equation

$$\begin{aligned} & (4x_2^4 + 2(2x_2^2 - 6x_2^4) + 4(2x_2^4 - x_2^2)) dx_2 dx_2 \\ = \; & (2x_2^2 - 4x_2^4 + 2x_1^2 x_2^2) dx_2 dx_2 \\ = \; & (x_1^2 + 1 - (x_1^2 + 1)^2 + x_1^2(x_1^2 + 1)) dx_2 dx_2 \\ = \; & 0. \end{aligned}$$

Thus by Corollary 3.72, the curve of type (ii) is ω-integral.

For the coordinate axes (curves of type (iii)), it is easy to prove that ω vanishes along them. (Write ω in appropriate coordinates for each case.) For example, on U_3 we have

$$\omega_{|x_1=0} = x_1^2 x_2^2 dx_1 dx_1 + (x_1 x_2 - 4x_1 x_2^2 - x_1^3 x_2) dx_1 dx_2 + 4x_1^2 x_2^2 dx_2 dx_2 = 0$$

because the curve $x_1 = 0$ satisfies $dx_1 = 0$. Hence by Corollary 3.72 they are ω-integral.

Let $C_{\epsilon_2,\epsilon_2} := x_1 + 2\epsilon_2 x_2 + \epsilon_3 x_3 = 0$, with $\epsilon_2, \epsilon_3 \in \{\pm 1\}$. In U_3 it has equation $x_1 = -2\epsilon_2 x_2 - \epsilon_3$. Taking differentials we get $dx_1 = -2\epsilon_2 dx_2$. Hence

$$\begin{aligned}
A_{|C_{\epsilon_2,\epsilon_3}} &= x_1^2x_2^2(-2\epsilon_2)^2dx_2dx_2 = (16x_2^4 + 4x_2^2 + 16\epsilon_2\epsilon_3x_2^3)dx_2dx_2,\\
B_{|C_{\epsilon_2,\epsilon_3}} &= (x_1x_2 - 4x_1x_2^3 - x_1^3x_2)(-2\epsilon_2)dx_2dx_2 = (-32x_2^4 - 32\epsilon_2\epsilon_3x_2^3 - 8x_2^2)dx_2dx_2,\\
C_{|C_{\epsilon_2,\epsilon_3}} &= 4x_1^2x_2^2dx_2dx_2 = (16x_2^4 + 4x_2^2 + 16\epsilon_2\epsilon_3x_2^3)dx_2dx_2.
\end{aligned}$$

Thus $\omega_{|C_{\epsilon_2,\epsilon_3}}$ has equation

$$A_{|C_{\epsilon_2,\epsilon_3}} + B_{|C_{\epsilon_2,\epsilon_3}} + C_{|C_{\epsilon_2,\epsilon_3}} = 0.$$

Therefore from Corollary 3.72, the curves of type (iv) are ω-integral. □

Lemma 4.17. *The curves from Lemma 4.16 are the only ω-integral curves on the surface $X_3 = \mathbb{P}^2$.*

Proof. The restriction of ω to U_3 is

$$x_1^2x_2^2dx_1dx_1 + (x_1x_2 - 4x_1x_2^3 - x_1^3x_2)dx_1dx_2 + 4x_1^2x_2^2dx_2dx_2.$$

We have from the Notation 3.75

$$\Delta = (\mathbb{P}^2\backslash U_3) \cup \left\{P \in U_3 : A_0(P) = 0 \text{ or } A_1^2(P) - 4A_0(P)A_2(P) = 0\right\} \subseteq \mathbb{P}^2.$$

Restricting to our case the last condition becomes (for $P = (x_1, x_2)$):

$$\begin{aligned}
A_2^2(P) - 4A_1(P)A_3(P) &= (x_1x_2 - 4x_1x_2^3 - x_1^3x_2)^2 - 16x_1^4x_2^4\\
&= x_1^2x_2^2 \prod_{\epsilon_2,\epsilon_3\in\{-1,1\}} (x_1 + \epsilon_2 2x_2 + \epsilon_3). \qquad (4.3)
\end{aligned}$$

We also have $A_0 = x_1^2 x_2^2$. Therefore Δ is the union of the three coordinate axes and the curves $x_1 \pm 2x_2 \pm 1$. All of them are ω-integral curves.

From Theorem 3.76 we only need to prove that for any $P \notin \Delta$ there are at least two ω-integral curves passing through P. Let $P = [x_1 : x_2 : x_3]$ be a point outside Δ. The point P lies on a curve C_α if and only if $\frac{(\alpha-3)(\alpha-2)}{2}x_1^2 - ((\alpha-2)^2-1)x_2^2 + \frac{(\alpha-2)(\alpha-1)}{2}x_3^2 = 0$, and it is in C_∞ if and only if $x_1^2 - 2x_2^2 + x_3^2 = 0$. Since $P \notin \Delta$, we have that $x_i \neq 0$ for $i = 1, 2, 3$ (because Δ contains the coordinate axes). The discriminant of the quadratic equation (with α the unknown)

$$\frac{(\alpha-3)(\alpha-2)}{2}x_1^2 - ((\alpha-2)^2-1)x_2^2 + \frac{(\alpha-2)(\alpha-1)}{2}x_3^2 = 0 \tag{4.4}$$

is

$$\begin{aligned} & \left(-\frac{3}{2}x_3^2 + 4x_2^2 - \frac{5}{2}x_1^2\right)^2 - 4\left(x_3^2 - 3x_2^2 + 3x_1^2\right)\left(\frac{x_3^2}{2} - x_2^2 + \frac{x_1^2}{2}\right) \\ = \; & \frac{1}{4}(x_1 + 2x_2 + x_3)(x_1 + 2x_2 - x_3)(x_1 - 2x_2 + x_3)(x_1 - 2x_2 - x_3), \end{aligned}$$

which is different from zero at P because it is outside Δ (by Equation (4.3)). The leading coefficient from the quadratic equation (4.4) is $\frac{1}{2}(x_1^2 - 2x_2^2 + x_3^2)$. Suppose that P is not in C_∞, then equation 4.4 has degree 2 in α. Noting that Equation (4.4) is the Equation of C_α (see Lemma 4.16), we get that there are precisely two values of α such that the curve C_α passes through P. On the other hand, if $P \in C_\infty$, then Equation 4.4 is linear in α and it has exactly one solution α_0 and we get that C_∞ and C_{α_0} pass through P. Since this holds for every P outside Δ we know that there are no more ω-integral curves in X_3. $\square$

4.5 Pullbacks of ω-integral curves

Now that we have the complete list of ω-integral curves of $X_3 \cong \mathbb{P}^2$, we will use them to find $\rho_n^*\omega$-integral curves on the other surfaces X_n.

Lemma 4.18. *The pull-back under ρ_n of a curve of type (iii) of Lemma 4.16 is a smooth complete intersection, and it is given by the equations*

$$\begin{aligned} x_i &= 0 \\ x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\ &\vdots \\ \frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2, \end{aligned}$$

with $i \in \{1,2,3\}$.

Proof. Let C be the pullback of a curve of type (iii). By Proposition 2.30 the equations of C are

$$\begin{aligned} x_i &= 0 \\ x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\ &\vdots \\ \frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2, \end{aligned}$$

with i one of $0,1,2$, hence it is a complete intersection on $\mathbb{P}^{n-1}$. Its Jacobian matrix

evaluated at the point $[x_1 : \ldots : x_n] \in X_n$ is a $(n-2) \times n$ matrix of the form

$$\begin{pmatrix} a & b & c & 0 & 0 & 0 & 0 \\ 2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (n-3)(n-2)x_1 & 2((n-2)^2-1)x_2 & (n-2)(n-1)x_3 & 0 & \cdots & 0 & -2x_n \end{pmatrix}$$

with exactly one of a, b, c different from zero (and equal to 1).

If none of $x_4, \ldots, x_n$ is zero, then the $(n-2) \times n$ matrix has maximal rank (because one of a, b, c is equal to 1). If exactly two of $x_4, \ldots, x_n$ are zero, then by Lemma 4.8 we obtain that none of x_1, x_2, x_3 is zero. The determinant of the submatrix

$$\begin{pmatrix} a & b \\ (i-3)(i-2)x_1 & 2((i-2)^2-1)x_2 \end{pmatrix} \tag{4.5}$$

with $i \geq 4$ is $b(i-3)(i-2)x_1 - 2a((i-2)^2-1)x_2$, which is different from zero when $x_1, x_2 \neq 0$ and either $(a, b) = (0, 1)$ or $(a, b) = (1, 0)$. The determinant of the submatrix

$$\begin{pmatrix} a & c \\ (i-3)(i-2)x_1 & (i-2)(i-1)x_3 \end{pmatrix} \tag{4.6}$$

with $i \geq 4$ is $c(i-3)(i-2)x_1 - a(i-2)(i-1)x_3$, which is different from zero when $x_1, x_2 \neq 0$ and either $(a, c) = (0, 1)$ or $(a, c) = (1, 0)$. The determinant of the submatrix

$$\begin{pmatrix} b & c \\ 2((i-2)^2-1)x_2 & (i-2)(i-1)x_3 \end{pmatrix} \tag{4.7}$$

with $i \geq 4$ is $2c((i-2)^2-1)x_2 - b(i-2)(i-1)x_3$, which is different from zero when $x_1, x_2 \neq 0$ and either $(b,c) = (0,1)$ or $(b,c) = (1,0)$. hence if $x_0 x_1 \neq 0$ we get that the $(n-2) \times n$ matrix has maximal rank. If one of $x_4, \ldots, x_n$ is zero, then at least two of x_1, x_2, x_3 must be different from zero. Suppose that $x_1 = 0$. If either b or c is different from zero, then Matrix (4.7) has non-zero determinant. If $a \neq 0$, then Matrix (4.5) and Matrix (4.6) will have non-zero determinant. Therefore, by a reasoning similar to the proof of Lemma 4.10, the $(n-2) \times n$ matrix will have maximal rank. The cases $x_2 = 0$ and $x_3 = 0$ are proved similarly. □

Lemma 4.19. *The pull-back under ρ_n of the curve C_α of type (i) of Lemma 4.16 with $\alpha \neq 4, \ldots, n$ is a smooth complete intersection curve given by the system of equations*

$$\begin{aligned}
\frac{(\alpha-3)(\alpha-2)}{2}x_1^2 - ((\alpha-2)^2-1)x_2^2 + \frac{(\alpha-2)(\alpha-1)}{2}x_3^2 &= 0 \\
x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\
&\vdots \\
\frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2.
\end{aligned}$$

The pull-back under ρ_n of the curve C_∞ of type (ii) is a smooth complete intersection given by the equations

$$\begin{aligned}
x_1^2 - 2x_2^2 + x_3^2 &= 0 \\
x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\
&\vdots \\
\frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2.
\end{aligned}$$

Proof. Let $C = C_\alpha$ be the pullback of a curve of type (i). By Proposition 2.30 the

equations of C are

$$\begin{aligned}
\frac{(\alpha-3)(\alpha-2)}{2}x_1^2-((\alpha-2)^2-1)x_2^2+\frac{(\alpha-2)(\alpha-1)}{2}x_3^2 &= 0\\
x_1^2-3x_2^2+3x_3^2 &= x_4^2\\
&\vdots\\
\frac{(n-3)(n-2)}{2}x_1^2-((n-2)^2-1)x_2^2+\frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2
\end{aligned}$$

Therefore, this curve is a complete intersection in $\mathbb{P}^{n-1}$. The Jacobian matrix of C evaluated at $[x_1:\ldots:x_n]\in X_n$ is the $(n-2)\times n$ matrix

$$\begin{pmatrix}
(\alpha-3)(\alpha-2)x_1 & 2((\alpha-2)^2-1)x_2 & (\alpha-2)(\alpha-1)x_3 & 0 & 0 & 0 & 0\\
2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0\\
6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0\\
(n-3)(n-2)x_1 & 2((n-2)^2-1)x_2 & (n-2)(n-1)x_3 & 0 & \cdots & 0 & -2x_n
\end{pmatrix}.$$

A computation similar to the proof of Lemma 4.10 (using Observation 4.9) gives us that the curve C_α is smooth.

The equations defining the curve of type (ii) are

$$\begin{aligned}
x_1^2-2x_2^2+x_3^2 &= 0\\
x_1^2-3x_2^2+3x_3^2 &= x_4^2\\
&\vdots\\
\frac{(n-3)(n-2)}{2}x_1^2-((n-2)^2-1)x_2^2+\frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2.
\end{aligned}$$

Hence this curve is a complete intersection in $\mathbb{P}^{n-1}$. The Jacobian matrix of C evaluated at $[x_1 : \ldots : x_n] \in X_n$ is the $(n-2) \times n$ matrix

$$\begin{pmatrix} 2x_1 & -4x_2 & 2x_3 & 0 & 0 & 0 & 0 \\ 2x_1 & -6x_2 & 6x_3 & -2x_4 & 0 & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & -2x_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (n-3)(n-2)x_1 & 2((n-2)^2-1)x_2 & (n-2)(n-1)x_3 & 0 & \cdots & 0 & -2x_n \end{pmatrix}.$$

The submatrix

$$\begin{pmatrix} 2x_1 & -4x_2 \\ (i-3)(i-2)x_1 & 2((i-2)^2-1)x_2 \end{pmatrix}$$

has determinant $4(i-3)(2i-3)x_1x_2 \neq 0$ when $i \neq 3$ and $x_1x_2 \neq 0$. The submatrix

$$\begin{pmatrix} 2x_1 & 2x_3 \\ (i-3)(i-2)x_1 & (i-2)(i-1)x_3 \end{pmatrix}$$

has determinant $4(i-2)x_1x_3 \neq 0$ when $i \neq 2$ and $x_1x_3 \neq 0$. The submatrix

$$\begin{pmatrix} -4x_2 & 2x_3 \\ 2((i-2)^2-1)x_2 & (i-2)(i-1)x_3 \end{pmatrix}$$

has determinant $-4(2i-5)(i-1)x_2x_3 \neq 0$ when $i \neq 1$ and $x_2x_3 \neq 0$.

Hence a similar computation proves that this matrix has maximal rank. Therefore the curve of type (ii) is smooth. □

Lemma 4.20. *For $3 \leq i \leq n$, the curves $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i$ are smooth complete*

intersection curves given by the system of equations

$$\begin{aligned} x_i &= 0 \\ x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\ &\vdots \\ \frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2. \end{aligned}$$

Proof. By Proposition 2.30 the equations of $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i$ are

$$\begin{aligned} x_i &= 0 \\ x_1^2 - 3x_2^2 + 3x_3^2 &= x_4^2 \\ &\vdots \\ \frac{(n-3)(n-2)}{2}x_1^2 - ((n-2)^2-1)x_2^2 + \frac{(n-2)(n-1)}{2}x_3^2 &= x_n^2. \end{aligned}$$

Therefore R_i is a complete intersection in $\mathbb{P}^{n-1}$. The Jacobian matrix of the curve R_i evaluated at the point $[x_0 : \ldots : x_n] \in X_n$ is the $(n-2) \times n$ matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 2x_1 & -6x_2 & 6x_3 & 2x_4 & 0 & \cdots & \cdots & 0 \\ 6x_1 & -16x_2 & 12x_3 & 0 & 2x_5 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ (n-3)(n-2)x_1 & 2((n-2)^2-1)x_2 & (n-2)(n-1)x_3 & 0 & \cdots & \cdots & 0 & 2x_n \end{pmatrix}$$

where the non-zero component on the first row is on the i-th column, and $2x_i = 0$. Interchange the $(i-2)$-nd row and the first row. Then a computation similar to the proof of Lemma 4.10 gives that this matrix has maximal rank, hence the curve R_i is

smooth. □

Lemma 4.21. *The pullbacks of the curves* $x_1 \pm 2x_2 \pm x_3 = 0$ *are the following* 2^{n-1} *curves:*

$$[s+t : \pm(2s+t) : \cdots : \pm(ns+t)] \text{ with } [s:t] \in \mathbb{P}^1.$$

These are smooth irreducible curves of genus 0.

Proof. First note that $[s+t : \pm(2s+t) : \ldots : \pm(ns+t)]$ is in X_n because taking squares we obtain a sequence consisting of squares of elements in arithmetic progression (which have constant second differences).

The image under π_n of the curve $[s+t : \epsilon_2(2s+t) : \ldots : \epsilon_n(ns+t)]$ (with $\epsilon_i \in \{\pm 1\}$) is the curve $[s+t : \epsilon_2(2s+t) : \ldots : \epsilon_{n-1}((n-1)s+t)]$. Since π_n is of degree 2 and there are two different curves mapping onto $[s+t : \epsilon_2(2s+t) : \ldots : \epsilon_{n-1}((n-1)s+t)]$, namely $[s+t : \epsilon_2(2s+t) : \ldots : (ns+t)]$ and $[s+t : \epsilon_2(2s+t) : \ldots : -(ns+t)]$, we obtain from Proposition 2.87 that these two curves are all the preimage curves of $[s+t : \epsilon_2(2s+t) : \ldots : \epsilon_{n-1}((n-1)s+t)]$.

Write $C_{\epsilon_2,\ldots,\epsilon_n}$ for the curve in X_n given by $[s+t : \epsilon_2(2s+t) : \ldots : \epsilon_n(ns+t)]$ and $\epsilon_i \in \{\pm 1\}$. The image under ρ_n of $C_{\epsilon_2,\ldots,\epsilon_n}$ is $[s+t : \epsilon_2(2s+t) : \epsilon_3(3s+t)]$. The points on this curve satisfy the equation $x_1 - \epsilon_2 2x_2 + \epsilon_3 x_3 = 0$, which is the equation of one of the curves of type (iv). The curves $C_{\epsilon_2,\ldots,\epsilon_n}$ are irreducible smooth curves of genus zero because they are isomorphic images of $\mathbb{P}^1$ (the inverse of this map is the morphism $\eta : C_{\epsilon_2,\ldots,\epsilon_n} \to \mathbb{P}^1$, mapping $[x_1 : x_2 : \cdots : x_n]$ to $[\epsilon_2 x_2 - x_1 : 2x_1 + \epsilon_2 x_2]$). □

Lemma 4.22. *Let* C *be a curve of type (i), (ii) or (iii) in Lemma 4.16. Then* $(\rho_n^*(C))_{\text{red}}$ *is smooth and irreducible for every* $n \geq 4$.

Proof. The pullbacks of the curves of type (i) with $\alpha \neq 1, \ldots n$, of type (ii) and of

type (iii) are smooth and complete intersections by Lemma 4.18 and Lemma 4.19. From the proofs of these Lemmas, we know that the pullbacks $\rho_n^*(C_\alpha)$ of curves of type (i) (with $\alpha \neq 4, \ldots, n$), (ii) and (iii) are smooth complete intersections. From Proposition 2.79 we have that these curves are all irreducible. By Lemma 4.13 we have that

$$\rho_n^* C_i = (\pi_{i+1} \circ \cdots \circ \pi_n)^* \rho_i^*(C_i) = 2(\pi_{i+1}^* \circ \cdots \circ \pi_n)^*(R_i)$$

with $4 \leq i \leq n$. Since the pullbacks $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i$ are smooth and complete intersections by Lemma 4.20, they are irreducible by Proposition 2.79. Thus $(\rho_n^*(C_i))_{\text{red}} = (\pi_{i+1} \circ \cdots \circ \pi_n)^*(R_i)$ is smooth and irreducible. □

4.6 Integral curves on X_n

Recall that $\rho_n = \pi_4 \circ \cdots \circ \pi_n$. Using the notation of Subsection 3.2.5, let

$$\omega_n = (\rho_n)^{\bullet}_{2,\mathcal{O}(7)} \omega \in H^0(X_n, \mathcal{O}_{X_n}(7) \otimes S^2\Omega^1_{X_n/\mathbb{C}}).$$

Lemma 4.23. *The following curves on X_n are smooth, irreducible and ω_n-integral. Moreover, every ω_n-integral curve is one of these curves:*

(a) $\rho_n^* C_\alpha$, *with* $\alpha \in \mathbb{C} \setminus \{1, \cdots, n\}$. *They have genus* $2^{n-3}(n-4)+1$;

(a') $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i = (\rho_n^*(C_i))_{\text{red}}$, *with* $4 \leq i \leq n$. *They have genus* $2^{n-4}(n-5)+1$.

(b) $\rho_n^* C_\infty$. *It has genus* $2^{n-3}(n-4)+1$.

(c) The pullbacks under ρ_n *of the coordinate axes* $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ *of* $X_3 = \mathbb{P}^2$. *They have genus* $2^{n-4}(n-5)+1$;

(d) The irreducible components of the pullbacks of the curves $x_1 \pm 2x_2 \pm x_3 = 0$. *These are the* 2^{n-1} *curves* $[s+t : \pm(2s+t) : \cdots : \pm(ns+t)]$ *with* $[s:t] \in \mathbb{P}^1$ *and have genus* 0.

Proof. Let $C \subset X_n$ be a curve. By Theorem 3.35 the curve C is ω_4-integral if and only if its image $D = \rho_n(C)$ is ω-integral. By Lemma 4.16, this means that D is one of the curves of type (i)-(iv) of Lemma 4.16, and hence C is a component of $\rho_n^* D$.

If D has type (iv), then by Lemma 4.21, the components of D are the curves of type (d), which are smooth and irreducible of genus 0 by Lemma 4.21.

Now suppose that D is of type (i), (ii) or (iii). Then $(\rho_n^*(D))_{\text{red}}$ is irreducible by Lemma 4.22.

Let D be a curve of type (i). Then $C = (\rho_n^* D)_{\text{red}} = \rho_n^* D$ is of type (a) and from Proposition 4.19 we know the equations defining C, thus by Proposition 2.79 we have $K_C = \mathcal{O}(2(n-2)-n) = \mathcal{O}(n-4)$. From Proposition 2.80 we have that the genus of C is $2^{n-3}(n-4)+1$. Similarly, if C is a curve of type (a'), we have $K_C = \mathcal{O}(n-5)$ by Proposition 4.20, hence the genus of C is $2^{n-4}(n-5)+1$. If C is the curve of type (b), then by Proposition 4.19 we have $K_C = \mathcal{O}(n-4)$ and its genus is $2^{n-3}(n-4)+1$. If C is a curve of type (c), then by Proposition 4.18 $K_C = \mathcal{O}(n-5)$ and $g(C) = 2^{n-4}(n-5)+1$. □

4.7 Curves of low genus on X_n

Now we will prove that all the curves in X_n, whose genus is bounded by a certain explicit constant (depending on n) must be ω_n-integral, and hence they are of type (a), (a'), (b), (c) or (d).

Lemma 4.24. *For each $n \geq 4$, the section $\omega_n \in H^0(X_n, \mathcal{O}_{X_n}(7) \otimes S^2\Omega^1_{X_n})$ determines a unique section $\omega'_n \in H^0(X_n, \mathcal{O}_{X_n}(10-n) \otimes S^2\Omega^1_{X_n})$. Moreover, the ω'_n-integral curves are ω_n-integral curves.*

Proof. For $4 \leq i \leq n$, the curves C_i are ω-integral and ramified with respect to ρ_n by Lemma 4.13. The section ω_n vanishes along $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i = (\rho_n^*(C_i))_{\mathrm{red}}$ by Theorem 3.87 (which is irreducible by Lemma 4.22).

Since $(\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i$ is the intersection of X_n and $\{x_n = 0\}$, it is a hyperplane section in $X_n \subseteq \mathbb{P}^{n-1}$, and hence its ideal sheaf is $\mathcal{O}(-1)$.

By Proposition 3.79, we get that ω_n vanishes along $\sum_{i=3}^{n} (\pi_{i+1} \circ \cdots \circ \pi_n)^* R_i$. Thus by Proposition 2.25, we get that for each n, the section $\omega_n \in H^0(X_n, \mathcal{O}(7) \otimes S^2\Omega^1_{X_n})$ determines a unique section $\omega'_n \in H^0(X_n, \mathcal{O}(10-n) \otimes S^2\Omega^1_{X_n})$. By Theorem 3.88 we obtain that the ω'_n-integral curves are ω_n-integral curves. □

Lemma 4.25. *Let $n > \max\{9, 4g+6\}$, and let $C \subseteq X_n$ be an irreducible curve of genus g. Then C is ω_n-integral, and hence is one of the curves of type (a), (a'), (b), (c) or (d).*

Proof. For $n > \max\{9, 4g+6\}$, and an irreducible curve $C \subseteq X_n$ of genus g, let $\varphi_C : \tilde{C} \to X$ be the normalization of C.

We know from Remark 2.45 that $\deg_{\tilde{C}} \varphi_C^* \mathcal{O}(1) = \deg(C) > 0$. Thus, since $n > 9$, we have that

$$\deg_{\tilde{C}} \varphi_C^* \mathcal{O}(10-n) = (10-n) \deg_{\tilde{C}} \varphi_C^* \mathcal{O}(1) \leq 10 - n < 4 - 4g.$$

Therefore by Proposition 3.42 we obtain that C is ω'_n-integral. By Lemma 4.24, we get that C is ω_n-integral. The last statement holds by Lemma 4.23. □

4.8 Proof of the main results

Proof of Theorem 4.1. From Lemma 4.25 we have that all the curves of genus g in X_n are ω_n'-integral. Hence by Lemma 4.24, these curves are ω_n-integral. Since $g < \frac{n-6}{4}$ and $n > 9$, we know by Lemma 4.23 that the only ω_n-integral curves of genus less than or equal to g are the curves of type (d). Therefore the only curves of genus less than or equal to g in X_n are the curves of type (d), hence proving Theorem 4.1. □

Proof of Theorem 4.2. Let C be a curve of genus 0 or 1. By Theorem 4.1 we obtain that for $n \geq 11$ all the curves of genus 0 or 1 are the curves of type (d). □

Proof of Theorem 4.3. Let K be a function field of genus g, and let C_K is the unique curve (up to isomorphism) with function field K. Let $n > 4g + 6$. By Proposition 2.97, the solutions over K (up to scaling) of the system of equations (4.1) are in bijection with the morphisms $\{f : C_K \to X/\mathbb{C}\}$.

By Riemann-Hurwitz, these morphisms are either constant, or must map the curve C_K to curves in X with genus less than or equal to g. By Theorem 4.1, the only curves with genus less than or equal to g are the curves of type (d).

Hence each non-constant map $f : C_K \to X$ must have image contained in one of the curves of type (d), which means that the corresponding solution $[a_1 : \ldots : a_n]$ to the previous system is not proportional to a constant solution, and it corresponds to a K-rational point in a curve of type (d). Hence it is of the form

$$[a_1 : \ldots : a_n] = [\pm(s + t) : \ldots : \pm(ns + t)].$$

Solving for s and t we see that they are in K, so we get $a, b \in K$ satisfying

$$[a_1 : \ldots : a_n] = [\pm(a+b) : \ldots : \pm(na+b)].$$

□

Proof of Theorem 4.4. Let $n \geq 11$ and let $a_n, \ldots, a_1$ be a sequence of n integers whose squares have constant second differences, but which is not an arithmetic progression (up to signs), hence for all $3 \leq i \leq n$ it satisfies $a_{i-2}^2 - 2a_{i-1}^2 + a_i^2 = D$, for some D not depending on i. By transitivity of the equality, the following holds for $4 \leq i \leq n$: $a_{i-3}^2 - 3a_{i-2}^2 + 3a_{i-1}^2 = a_i^2$. Since it is not an arithmetic progression (up to signs), there are no $s, t \in \mathbb{Z}$ such that $a_i^2 = (si+t)^2$. Thus, the point $[a_1 : \ldots : a_n]$ is in X_n, and it is not in one of the curves of type (d). If there are infinitely many non-proportional sequences satisfying this, we would obtain infinitely many rational points on X_n which are not on the curves of genus 0 or 1 of X_n. This contradicts the Bombieri-Lang conjecture on X_n.

To deduce an absolute bound (possibly larger than 11) from the finiteness, we follow an elementary combinatorial idea of Vojta [Voj00] adapted to our case.

Suppose that there are, up to scaling, exactly N sequences of squares $x_1^2, \ldots, x_{11}^2$ with $x_i \in \mathbb{Q}$ having constant second differences and such that the sequence is non-trivial (i.e. the x_i are not an arithmetic progression up to sign). We claim that there is no nontrivial sequence of rational $M = N + 11$ squares with constant second differences.

Indeed, suppose that $a_1^2, \ldots, a_M^2$ is such a non-trivial sequence. Then for all $1 \leq i \leq N+1$ we have that $a_i^2, \ldots, a_{i+10}^2$ also has constant second differences and we claim

that it is non-trivial. Suppose that it is trivial, then in particular there are signs $\epsilon_1, \epsilon_2 \in \{1, -1\}$ with $a_i - 2\epsilon_2 a_{i+1} + \epsilon_3 a_{i+2} = 0$. Note that $[a_i : a_{i+1} : a_{i+2}] \in X_3 = \mathbb{P}^2$ lies on a curve of type (iv) and $[a_i : \ldots : a_M] \in X_{M-i+1}$, so it is in a curve of type (d), so $a_i, a_{i+2}, a_{i+3}, \ldots, a_M$ is an arithmetic progression up to sign. Similarly $a_1, \ldots, a_i, a_{i+2}, a_{i+3}$ is an arithmetic progression up to sign (using the same signs for a_1, a_2, a_3), and we get that the sequence $a_1, \ldots, a_M$ is trivial, a contradiction with the fact that it is non-trivial.

Thus, we obtain non-trivial sequences $a_i^2, \ldots, a_{i+10}^2$ for all $1 \leq i \leq N+1$. We claim that they are non-proportional. Indeed, there is a polynomial $f(t) = ut^2 + vt + w$ such that $f(n) = a_n^2$ for all $1 \leq n \leq M$ because the a_n have constant second differences, and our sequence is non-trivial so f is non-constant. It is easy to check that the function $F : \mathbb{A}^1 \to \mathbb{P}^2$ defined by $t \mapsto [f(t) : f(t+1) : f(t+3)]$ is injective, proving our claim.

Finally, this is a contradiction because there are at most N non-proportional non-trivial sequences of length 11 and we have produced $N+1$ of them. This proves that there is no non-trivial sequence of length $M = N + 11$.

□

Chapter 5

An extension of Büchi's problem

5.1 Introduction and main results

For $k \geq 2$ and $n \geq 2$ define $X_{n,k}$ by the equations

$$\begin{aligned} 2x_0^k &= x_1^k - 2x_2^k + x_3^k \\ 2x_0^k &= x_2^k - 2x_3^k + x_4^k \\ &\vdots \\ 2x_0^k &= x_{n-2}^k - 2x_{n-1}^k + x_n^k. \end{aligned} \tag{5.1}$$

These are smooth surfaces in $\mathbb{P}^n$ (cf. Proposition 5.10). By convention $X_{2,k} = \mathbb{P}^2$. For $n \geq 2 + \frac{4}{k-1}$, the surface $X_{n,k}$ is of general type (cf. Proposition 5.13). We will find all the genus 0 and 1 curves in $X_{n,k}$ (for n large enough). In particular, we prove:

Theorem 5.1. *Let $k \geq 3$, let $g \geq 0$ and let $n > \frac{4g}{k-1} + 3$. If C is a curve in $X_{n,k}$, then $g(C) > g$.*

This theorem specializes as follows for $g = 0, 1$:

Theorem 5.2. *(a) For $k > 2$ and $n \geq 8$, there are no curves of genus 0 or 1 on $X_{n,k}$.*

(b) For $k \geq 6$ and $n \geq 4$, there are no curves of genus 0 or 1 on $X_{n,k}$.

Note that the case $k = 2$ is considered by Vojta in [Voj00], and for every $n \geq 2$ the surface $X_{n,2}$ has rational curves (see Theorem 2.9 in Section 2.1.2).

Theorem 5.2 gives us examples of regular surfaces (i.e. $q := \dim H^0(X, \Omega^1_{X/k}) = 0$) without rational or elliptic curves. Moreover, we obtain examples of surfaces which do not satisfy Bogomolov's condition $c_1^2 > c_2$ (see Theorem 2.6), but which have *no* curves of genus 0 or 1. The following examples were computed using *Magma* (recall that $c_1^2 = K^2_{X_{n,k}}$):

n	k	c_1^2	c_2	$c_1^2 - c_2$
4	8	7744	7808	-64
6	3	2025	2187	-162
4	6	1764	2088	-324
4	7	3969	4263	-294

In the case that $k = 2$, Vojta notes that Bogomolov's inequality holds for $n \geq 10$.

From Theorem 5.1 we obtain the following result on the arithmetic of function fields.

Theorem 5.3. *Let K be a function field of genus g with constant field $\mathbb{C}$, let $k \geq 3$, and let $n > \frac{4g}{k-1} + 3$. Let $f_1, \ldots, f_n \in K$ be such that the k-th powers of this sequence have second differences equal to 2. Then the sequence $(f_1, \ldots, f_n)$ is a sequence of complex numbers.*

From Theorem 5.2 the following result on Diophantine equations follows:

Theorem 5.4. *Assume the Bombieri-Lang Conjecture (Conjecture 2.4) for the surface $X_{n,k}$. Let L be a number field.*

- *If $k > 2$, there are only finitely many sequences of 8 elements of L whose k-th powers have second differences equal to 2.*
- *If $k \geq 6$, there are only finitely many sequences of 4 elements of L whose k-th powers have second differences equal to 2.*

Moreover for any $k \geq 3$, there exists $M_{k,L} > 0$ such that there are no sequences of $M_{k,L}$ elements of L whose k-th powers have constant second differences equal to 2.

Concrete examples of sequences as considered in this theorem do in fact exist. For instance

$$64, -1, -64, -125$$

is an example of 4 cubes with second differences equal to 2.

More generally, we can ask about sequences of powers of *possibly different* exponents. We also obtain a result in this direction, see Theorem 5.34 below. This was motivated by a question of R. Murty about the n-term ABC Conjecture in our context. I thank him for asking that question.

5.2 The geometry of the surfaces $X_{n,k}$

Recall that for each n, the scheme $X_{n,k}$ is in $\mathbb{P}^n_{\mathbb{C}}$.

Lemma 5.5. *Let $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n$ with $n \geq 3$, then $P \in X_{n,k}$ if and only if*

$$(i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k = x_i^k, \text{ for } 1 \leq i \leq n.$$

This clearly follows from the following much stronger proposition.

Proposition 5.6. *Let* $f_i = 2x_0^k - x_{i-2}^k + 2x_{i-1}^k - x_i^k$ *be the generators of the ideal defining* $X_{n,k}$*, and let*

$$g_i = (i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k - x_i^k.$$

Then we have the equality of ideals $(f_1, \ldots, f_n) = (g_1, \ldots, g_n)$ *in* $k[x_1, \ldots, x_n]$*. In particular, the* g_i *for* $1 \leq i \leq n$ *are also defining equations for* $X_{n,k}$*.*

Proof. First we show that for $n \geq 1$ we have $I_n := (g_1, \ldots, g_n) \subseteq J_n := (f_1, \ldots, f_n)$. It suffices to show that $g_i \in J_n$ for each $1 \leq i \leq n$, and we will do this by induction on i. Note that we have $g_i = 0$ for $i = 1, 2$, so for these values of i we have $g_i \in J_n$. Let $1 \leq i \leq n$ and suppose (as induction hypothesis) that $g_j \in J_n$ for all $1 \leq j < i$. Then, working modulo J_n and using that $g_j \in J_n$ for $1 \leq j < i$ we obtain

$$\begin{aligned} x_i^2 &\equiv 2x_0^k - x_{i-2}^k + 2x_{i-1}^k - x_i^k \mod J_n \\ &\equiv 2x_0^k - ((i-3)(i-4)x_0^k - (i-4)x_1^k + (i-3)x_2^k) \\ &\quad +2((i-2)(i-3)x_0^k - (i-3)x_1^k + (i-2)x_2^k) \\ &\quad -((i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k) \mod J_n \\ &\equiv (i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k \mod J_n. \end{aligned}$$

This proves that $g_i \equiv 0 \mod J_n$, that is, that $g_i \in J_n$. Therefore $I_n \subseteq J_n$.

Now we prove the inclusion $J_n \subseteq I_n$. Working modulo I_n we get the relations

$$x_i^k \equiv (i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k \mod I_n$$

for $1 \leq i \leq n$, so for $1 \leq i \leq n$ we get

$$\begin{aligned} f_k &= 2x_0^k - x_{i-2}^k + 2x_{i-1}^k - x_i^k \\ &\equiv 2x_0^k - ((i-3)(i-4)x_0^k - (i-4)x_1^k + (i-3)x_2^k) \\ &\quad +2((i-2)(i-3)x_0^k - (i-3)x_1^k + (i-2)x_2^k) \\ &\quad -((i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k) \mod I_n \\ &\equiv 0 \cdot x_0^k + 0 \cdot x_1^k + 0 \cdot x_2^k \equiv 0 \mod I_n, \end{aligned}$$

which proves that $f_i \in I_n$ for $1 \leq i \leq n-3$. Therefore $I_n = J_n$. □

Lemma 5.7. *If $[x_0 : \cdots : x_n]$ is a point on $X_{n,k}$, then no three of $x_0, \ldots, x_n$ are zero.*

Proof. We cannot have $x_0 = x_1 = x_2 = 0$ because of Lemma 5.5. Now view $(j-1)(j-2)x_0^k - (j-2)x_1^k + (j-1)x_2^k = 0$ as an equation in j. It can be written in the form

$$x_0^k j^2 + (-3x_0^k - x_1^k + x_2^k)j + 2x_0^k + 2x_1^k - x_2^k = 0 \tag{5.2}$$

If $x_0 \neq 0$, then Equation (5.2) has at most two solutions, hence there are at most two values of j for which $x_j = 0$.

If $x_0 = 0$, then Equation (5.2) becomes a linear equation in j. If also both $-3x_0^k - x_1^k + x_2^k = 0$ and $2x_1^k - x_2^k = 0$, we get $x_0 = x_1 = x_2 = 0$, which is not possible. Hence Equation (5.2) has at most one solution when $x_0 = 0$. □

For each $n \geq 3$, let $\pi_n : X_{n,k} \to X_{n-1,k}$ be the restriction to $X_{n,k}$ of the morphism

$$\tilde{\pi}_n : \mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\} \to \mathbb{P}^{n-1}$$

$$[x_0 : \cdots : x_n] \quad \mapsto \quad [x_0 : \cdots : x_{n-1}].$$

The rational map $\tilde{\pi}_n$ corresponds to the inclusion map $k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_{n-1}]$ (which respects the grading) in the sense of [Har77] Exercise II.2.14(b), and the morphism π_n corresponds to the induced map

$$k[x_0, \ldots, x_n]/(f_1, \ldots, f_n) \to k[x_0, \ldots, x_{n-1}]/(f_1, \ldots, f_{n-1}),$$

which exists because

$$(f_1, \ldots, f_n) \subseteq k[x_0, \ldots, x_{n-1}] \cap (f_1, \ldots, f_{n-1}).$$

Therefore $\pi_n(X_{n,k}) \subseteq X_{n-1,k}$.

Lemma 5.8. *For each $n \leq 3$, the map $\pi_n : X_{n,k} \to X_{n-1,k}$ is finite and surjective.*

Proof. Let $P = [x_0 : \ldots : x_{n-1}]$ be in $X_{n-1,k}$. Then for any $x_n \in k$, the point $\tilde{P} = [x_0 : \ldots : x_n]$ is a preimage of P under $\tilde{\pi}_n$. By Lemma 5.5, $\tilde{P}$ lies on $X_{n,k}$ if and only if

$$x_n^k = (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k.$$

Since this equation always has a solution $x_n \in k$, we see that π_n is surjective. Moreover, we see that $|\pi_n^{-1}(P)| \leq k$, so the map is quasi-finite, hence finite, because π_n is projective. □

Lemma 5.9. *Each irreducible component of $X_{n,k}$ has dimension greater than or equal to 2.*

Proof. First we will prove by induction that each irreducible component in the intersection of m hypersurfaces in $\mathbb{P}^n$ has dimension greater than or equal to $n-m$. An hypersurface in $\mathbb{P}^n$ has dimension $n-1$. Suppose that the intersection (denoted by Y) of m hypersurfaces has all irreducible components having dimension $d \geq n-m$. If we intersect Y with another hypersurface, we get by Theorem I.7.2 in [Har77] that the dimension of all irreducible components of this intersection is greater than or equal to $d+n-1-n = d-1 = n-m-1$. Therefore the intersection of $m+1$ hypersurfaces has all its irreducible components with dimension greater than or equal to $n-(m+1)$.

Since the variety $X_{n,k}$ is the intersection of $n-2$ hypersurfaces in $\mathbb{P}^n$, we obtain that each irreducible component of $X_{n,k}$ has dimension greater than or equal to 2. □

Proposition 5.10. *The dimension of each irreducible component of $X_{n,k}$ is exactly 2.*

Proof. Let Y be an irreducible component of $X_{n,k}$. By Lemma 5.8, the morphism $\rho_{n,k} = \pi_3 \circ \cdots \circ \pi_n : X_{n,k} \to \mathbb{P}^2$ is finite and surjective, hence $\dim Y \leq \dim \mathbb{P}^2 = 2$. By Proposition 5.9 we obtain that $\dim Y = 2$. □

The following observation will be useful for several subsequent lemmas.

Observation 5.11. Let $\alpha, \beta \neq 1, 2$, with $\alpha \neq \beta$. The matrix

$$\begin{pmatrix} -(\alpha-2)(\alpha-1)x_0^{k-1} & (\alpha-2)x_1^{k-1} \\ -(\beta-2)(\beta-1)x_0^{k-1} & (\beta-2)x_1^{k-1} \end{pmatrix}$$

has determinant $x_0^{k-1}x_1^{k-1}(\alpha-2)(\beta-2)(\beta-\alpha) \neq 0$ when $x_0x_1 \neq 0$. The matrix

$$\begin{pmatrix} -(\alpha-2)(\alpha-1)x_0^{k-1} & -(\alpha-1)x_2^{k-1} \\ -(\beta-2)(\beta-1)x_0^{k-1} & -(\beta-1)x_2^{k-1} \end{pmatrix}$$

has determinant $x_0^{k-1}x_2^{k-1}(\alpha-1)(\beta-1)(\alpha-\beta) \neq 0$ when $x_0x_2 \neq 0$. The matrix

$$\begin{pmatrix} (\alpha-2)x_1^{k-1} & -(\alpha-1)x_2^{k-1} \\ (\beta-2)x_1^{k-1} & -(\beta-1)x_2^{k-1} \end{pmatrix}$$

has determinant $x_1^{k-1}x_2^{k-1}(\beta-\alpha) \neq 0$ when $x_1x_2 \neq 0$.

Proposition 5.12. *For each $n \geq 2$, the scheme $X_{n,k}$ is smooth.*

Proof. Note that $X_{2,k} \cong \mathbb{P}^2$, thus it is smooth and irreducible. Let $[x_0 : \cdots : x_n] \in X_{n,k}$ with $n \geq 3$. By Proposition 2.77 we only need to check that the Jacobian matrix of the homogeneous equations defining $X_{n,k}$ evaluated at $[x_0 : \cdots : x_n] \in X_{n,k}$ has rank $n-2$, because $X_{n,k}$ is equidimensional of dimension 2 by Proposition 5.10. By Proposition 5.6 we know that the ideal $(g_1, \ldots, g_n)$ defines $X_{n,k}$. We thus get the $(n-2) \times (n+1)$ matrix

$$k \begin{pmatrix} -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -2x_0^{k-1} & 0 & x_2^{k-1} & -2x_3^{k-1} & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ -2x_0^{k-1} & 0 & \cdots & 0 & x_{n-2}^{k-1} & -2x_{n-1}^{k-1} & x_n^{k-1} \end{pmatrix}.$$

After applying suitable row operations we obtain

$$\begin{pmatrix} -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -(n-2)(n-1)x_0^{k-1} & (n-2)x_1^{k-1} & -(n-1)x_2^{k-1} & 0 & \cdots & 0 & x_n^{k-1} \end{pmatrix}.$$

We will prove by induction on i that this matrix has rank $n-2$. We know from Lemma 5.7 that no three of $x_0, \ldots, x_n$ are zero, hence

$$\begin{pmatrix} -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} \end{pmatrix}$$

is not the zero vector. Let $3 \leq i \leq n-1$ and suppose by the induction hypothesis that the following $(i-2) \times (i+1)$ matrix has rank $i-2$.

$$M_i = \begin{pmatrix} -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -(i-2)(i-1)x_0^{k-1} & (i-2)x_1^{k-1} & -(i-1)x_2^{k-1} & 0 & \cdots & 0 & x_i^{k-1} \end{pmatrix}.$$

and consider the $(i-1) \times (i+2)$ matrix

$$M_{i+1} = \begin{pmatrix} -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -(i-1)(i)x_0^{k-1} & (i-1)x_1^{k-1} & -ix_2^{k-1} & 0 & \cdots & 0 & x_{i+1}^{k-1} \end{pmatrix}.$$

If $x_{i+1} \neq 0$ then the matrix M_{i+1} has maximal rank $\mathrm{rk}M_{i+1} = \mathrm{rk}M_i + 1 = i - 1$.

Now suppose that $x_{i+1} = 0$. If none of $x_3, \ldots, x_i$ is zero, then M_{i+1} has maximal rank. By Lemma 5.7, at most one among $x_0, \ldots, x_i$ can be zero. If $x_j = 0$, then we only have to prove that the j-th row is not a multiple of the $(i+1)$-st row. By Lemma 5.7 we have that at least two of x_0, x_1, x_2 are different from zero. Then by Observation 5.11, we obtain that the j-th row is not a multiple of the $(i+1)$-st row, so the matrix M_{i+1} has maximal rank. Therefore the Jacobian matrix has rank $n-2$ and thus by Proposition 2.77 the surface $X_{n,k}$ is smooth. □

Proposition 5.13. *The surface $X_{n,k}$ is smooth and irreducible and its canonical sheaf is*

$$\mathcal{O}(k(n-2) - n - 1).$$

The surface $X_{n,k}$ is of general type for $n \geq \frac{2k+2}{k-1} = 2 + \frac{4}{k-1}$. Moreover, when $k \geq 5$, the surface $X_{n,k}$ is of general type for $n \geq 3$.

Proof. Since $X_{n,k}$ is defined by $n-2$ equations, which is equal to its codimension in $\mathbb{P}^n$ (by Proposition 5.10), we obtain that $X_{n,k}$ is a complete intersection. From Proposition 5.12 and Proposition 2.79 we have that $X_{n,k}$ is irreducible and its canonical sheaf is $\mathcal{O}(k(n-2) - n - 1)$. We have that $k(n-2) - n - 1 \geq 1$ when $n \geq \frac{2k+2}{k-1}$. From Example II.7.6.1 in [Har77] we have that for $n \geq \frac{2k+2}{k-1}$ the sheaf $\mathcal{O}(k(n-2) - n - 1)$ is ample, thus a multiple of it determines an embedding $X_{n,k} \to \mathbb{P}^N$ and by Theorem V.6.5 in [Har77] we have that $X_{n,k}$ is a surface of general type. □

Corollary 5.14. *We have*

$$c_1^2 = (k(n-2) - (n+1))^2 k^{n-2}.$$

Proof. From Proposition 5.13, a canonical divisor of $X_{n,k}$ is $(k(n-2)-n-1)H$ with H a hyperplane section, hence $c_1^2 = (k(n-2)-n-1)^2(H.H)$.

Let L be an hyperplane in $\mathbb{P}^n$ such that its intersection with X is a smooth curve C (this exists by [Har77], Theorem II.8.18). Then $(H.H) = (H.C)$ because C is an hyperplane section. Since H is also an hyperplane section we get $(H.C) = \deg(C)$ by [Har77] Theorem I.7.7. Since C is a complete intersection, we have by Proposition 2.80 that $\deg(C) = \prod_{i=1}^{n-1} d_i$ where d_i are the degrees of the equations F_i defining C in $\mathbb{P}^n$. By Proposition 2.30, we have $\deg(C) = k^{n-2}$, thus $c_1^2 = (k(n-2)-(n+1))k^{n-2}$. □

Lemma 5.15. *The morphism $\pi_n : X_{n,k} \to X_{n-1,k}$ is ramified only at the components of the divisor*

$$C_n = \mathrm{div}_{X_{n,k}}(x_n).$$

Moreover, we have

$$\rho_{n,k}(C_n) = D_{n-2},$$

where $\rho_{n,k} = \pi_3 \circ \cdots \circ \pi_n : X_{n,k} \to \mathbb{P}^2$ and

$$D_{n-2} : (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k = 0.$$

In addition,

$$\rho_{n,k}^* D_{n-2} = kC_n.$$

Proof. From the proof of Lemma 5.8, we see that $|\pi_n^{-1}(P)| = k$ for all $P \in X_{n-1,k}$ except when $P = [x_0 : \ldots : x_{n-1}]$ lies on the curve

$$\tilde{D}_{n-2} : (n-1)x_2^k - (n-2)x_1^k - (n-1)(n-2)x_0^k = 0.$$

Thus $\deg(\pi_n) = k$ by Lemma 2.87, and π_n is unramified at any curve $C \not\subset \tilde{D}_{n-2}$. Moreover, since $|\pi_n^{-1}(P)| = 1$ for each $P \in \mathrm{supp}(\tilde{D}_{n-2})$, we see by Lemma 2.88 that π_n is totally ramified of degree k at each component of $\tilde{D}_{n-2}$. Now

$$\begin{aligned}\pi_n^*\tilde{D}_{n-2} &= \mathrm{div}_{X_{n,k}}((n-1)x_2^k - (n-2)x_1^k - (n-1)(n-2)x_0^k)\\ &= \mathrm{div}_{X_{n,k}}(x_n^k) = kC_n,\end{aligned}$$

Thus, π_n is ramified precisely the components of C_n. Finally

$$\rho_{n,k}^* D_{n-2} = \pi_n^*\rho_{n-1,k}^* D_{n-2} = \pi_n^*\tilde{D}_{n-2} = kC_n.$$

□

We have by Lemma 5.5 that the image under $\rho_{i,k}$ of the curve $C_i : x_i = 0$ in $X_{2,k} = \mathbb{P}^2$ is

$$D_{i-2} : (i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k = 0.$$

5.3 Finding all ω_2-integral curves in $X_{2,k}$

Let us prove the following lemma, which will be useful later.

Lemma 5.16. *Let*

$$P(x_1, x_2) := 1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k = ((x_1^k - x_2^k) - 1)^2 - 4x_2^k.$$

- *If k is an even integer, then P factors in irreducible factors as follows:*

$$P = (-x_1^{\frac{k}{2}} - x_2^{\frac{k}{2}} - 1)(x_1^{\frac{k}{2}} + x_2^{\frac{k}{2}} - 1)(-x_1^{\frac{k}{2}} + x_2^{\frac{k}{2}} - 1)(x_1^{\frac{k}{2}} - x_2^{\frac{k}{2}} - 1);$$

- *If k is an odd integer, then $P(x_1, x_2)$ is irreducible.*

Proof. That $P(x_1, x_2)$ factors in that form for k even is easily checked. Note that each $(x^{\frac{k}{2}} \pm y^{\frac{k}{2}} \pm 1)$ is irreducible by Lemma 2.81.

When k is an odd integer, we want to prove that $P(x_1, x_2)$ is irreducible. Let $F : k[x_1, x_2] \to k[u, v]$ be the homomorphism of k-algebras defined by $F(x_1) = u^2$, $F(x_2) = v^2$. If $P(x_1, x_2) = Q(x_1, x_2)R(x_1, x_2)$, then $P(u^2, v^2) = Q(u^2, v^2)R(u^2, v^2)$. Note that

$$P(u^2, v^2) = (-u^k - v^k - 1)(u^k + v^k - 1)(-u^k + v^k - 1)(u^k - v^k - 1)$$

by the previous assertion, and this factorization into irreducible factors is unique (up to constants) because $k[u, v]$ is a unique factorization domain.

Therefore (without loss of generality, because of the symmetry of Q and R) we have, from the above factorization of $P(u^2, v^2)$, that either $Q(u^2, v^2)$ is irreducible or a product of two irreducible factors.

In the first case $Q(u^2, v^2) = \epsilon_1 u^k - \epsilon_2^k - 1$, with $\epsilon_i \in \{\pm 1\}$, and in the second case we have that

$$\begin{aligned} Q(u^2, v^2) &= (\epsilon_1 u^k + \epsilon_2 v^k - 1)(\epsilon_3 u^k + \epsilon_4 v^k - 1) \\ &= \epsilon_1\epsilon_3 u^{2k} + \epsilon_1\epsilon_4 u^k v^k - \epsilon_1 u^k + \epsilon_2\epsilon_3 u^k v^k + \epsilon_2\epsilon_4 v^{2k} \\ &\quad -\epsilon_2 v^k - \epsilon_3 u^k - \epsilon_4 v^k + 1. \end{aligned}$$

but neither of these two polynomials is in the image of F, because k is odd and in both cases we obtain an exponent equal to k. Therefore $P(x_1, x_2)$ is irreducible. □

Let $\{U_i\} \subseteq \mathbb{P}^2$ be the usual affine open cover of $\mathbb{P}^2$. In U_0 with affine coordinates

$x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$, consider the following symmetric differential form

$$x_1^{k-1}x_2dx_1dx_1 + (1 - x_1^k - x_2^k)dx_1dx_2 + x_1x_2^{k-1}dx_2dx_2.$$

Remark 5.17. Choosing this particularly convenient symmetric differential is a non-trivial key step in the argument. See Remark 4.15.

Proposition 5.18. *This differential form in U_i can be extended to a form*

$$\omega_{2,k} \in H^0(\mathbb{P}^2, \mathcal{O}(k+3) \otimes S^2\Omega^1_{\mathbb{P}^2}).$$

Proof. Write

$$\begin{aligned} A &= x_1^{k-1}x_2dx_1dx_1 \\ B &= (1 - x_1^k - x_2^k)dx_1dx_2 \\ C &= x_1x_2^{k-1}dx_2dx_2. \end{aligned}$$

In the open set U_1 with affine coordinates $x_0 = \frac{X_0}{X_1}$, $x_2 = \frac{X_2}{X_1}$, these forms become

$$\begin{aligned} A &= x_0^{-k}x_2d(x_0^{-1})d(x_0^{-1}) = x_0^{-k-4}x_2dx_0dx_0 \\ B &= (1 - x_0^{-k} - x_0^{-k}x_2^k)d(x_0^{-1})d(x_0^{-1}x_2) = (1 - x_0^{-k} - x_0^{-1}x_2^k)(-x_0^{-3}dx_0dx_2 \\ &\quad +x_0^{-4}x_2dx_0dx_0) \end{aligned}$$

$$\begin{aligned} C &= x_0^{-k}x_2^{k-1}d(x_0^{-1}x_2)d(x_0^{-1}x_2) = x_0^{-k-4}x_2^{k-1}(x_0^2dx_2dx_2 + x_2^2dx_0dx_0 \\ &\quad -2x_0x_2dx_0dx_2). \end{aligned}$$

Hence

$$A+B+C = \frac{1}{x_0^{k+3}}(x_0^{k-1}x_2dx_0dx_0+(1-x_0^{k+1}-x_2^k)dx_0dx_2+x_0x_2^{k-1}dx_2dx_2).$$

Similarly on U_2 we have (with affine coordinates $x_0=\frac{X_0}{X_2}$, $x_1=\frac{X_1}{X_2}$)

$$\begin{aligned} A &= x_0^{-k}x_1^{k-1}d(x_0^{-1}x_1)d(x_0^{-1}x_1) \\ B &= (1-x_0^{-k}x_1^k-x_0^{-k})d(x_0^{-1}x_1)d(x_0^{-1}) \\ C &= x_0^{-k}x_1d(x_0^{-1})d(x_0^{-1}). \end{aligned}$$

thus

$$A+B+C=\frac{1}{x_0^{k+3}}(x_0^{k-1}x_1dx_0dx_0+(-x_1^k-x_0^k+1)dx_0dx_1+x_0x_1^{k-1}dx_1dx_1).$$

□

Lemma 5.19. *For a natural number $k\geq 1$, the following irreducible curves are $\omega_{2,k}$-integral curves on $X_{2,k}=\mathbb{P}^2$:*

(i) $x_0=0$, $x_1=0$, $x_2=0$;

(ii) $D_c: c(c+1)x_0^k=cx_1^k-(c+1)x_2^k$, *with* $c\in\mathbb{C}\setminus\{-1,0\}$.

If k is an even natural number, the following are also $\omega_{2,k}$-integral curves on $X_{2,k}$:

(iii) $x_0^{\frac{k}{2}}\pm x_1^{\frac{k}{2}}=\pm x_2^{\frac{k}{2}}$.

If k is an odd natural number, the following (irreducible) curve is also $\omega_{2,k}$-integral:

(iv) $1+x_1^{2k}+x_2^{2k}-2x_1^k-2x_2^k-2x_1^kx_2^k=0$.

Proof. For curves of type (i), the curve $x_i = 0$ satisfies $dx_i = 0$ (writing $\omega_{2,k}$ in appropriate coordinates for each case, note that we are working in $\mathcal{O}(k+3)\otimes S^r\Omega^1_{X_{2,k}}$, where we tensor by x_0^{k+3}). Hence by Corollary 3.72 one gets that curves of type (i) are $\omega_{2,k}$-integral.

Curves of type (ii) are irreducible by Lemma 2.81. For a curve D_c of type (ii), we have $c(c+1) = cx_1^k - (c+1)x_2^k$ in U_0. Taking differentials we obtain $dx_1 = \left(\frac{c+1}{c}\right)\left(\frac{x_2}{x_1}\right)^{k-1} dx_2$, hence on that curve

$$\begin{aligned}
A_{|D_c} &= x_1^{k-1}x_2\left(\frac{c+1}{c}\right)^2\left(\frac{x_2}{x_1}\right)^{2k-2} dx_2dx_2 = x_2^k\left(\frac{c+1}{c}\right)^2\frac{x_2^{k-1}}{x_1^{k-1}}dx_2dx_2,\\
B_{|D_c} &= (1-x_1^k-x_2^k)\frac{c+1}{c}\left(\frac{x_2}{x_1}\right)^{k-1} dx_1dx_2 = \frac{c+1}{c}(-c-\frac{2c+1}{c}x_2^k)\frac{x_2^{k-1}}{x_1^{k-1}}dx_2dx_2,\\
C_{|D_c} &= x_1x_2^{k-1}dx_2dx_2 = c+1+\frac{c+1}{c}x_2^k\frac{x_2^{k-1}}{x_1^{k-1}}dx_2dx_2.
\end{aligned}$$

Hence

$$\begin{aligned}
A_{|D_c}+B_{|D_c}+C_{|D_c} &= \left(x_2^k\left(\frac{c+1}{c}\right)^2+\frac{c+1}{c}(-c-\frac{2c+1}{c}x_2^k)\right.\\
&\quad \left.+c+1+\frac{c+1}{c}x_2^k\right)\frac{x_2^{k-1}}{x_1^{k-1}}dx_2dx_2\\
&= 0\frac{x_2^{k-1}}{x_1^{k-1}}dx_2dx_2.
\end{aligned}$$

By Corollary 3.72 we get that D_c is $\omega_{2,k}$-integral.

For an even integer k, the curves of type (iii) are irreducible by Lemma 2.81. Let $C^k_{\epsilon_1,\epsilon_2} := x_0^{\frac{k}{2}}\epsilon_1 x_1^{\frac{k}{2}} = \epsilon_2 x_2^{\frac{k}{2}}$ with $\epsilon_1,\epsilon_2 \in \{\pm 1\}$, then we have $1+\epsilon_1 x_1^{\frac{k}{2}} = \epsilon_2 x_2^{\frac{k}{2}}$ in U_0.

Taking differentials we obtain $\epsilon_1 x_1^{\frac{k}{2}-1} dx_1 = \epsilon_2 x_2^{\frac{k}{2}-1} dx_2$, hence on that curve

$$\begin{aligned}
A_{|C^k_{\epsilon_1,\epsilon_2}} &= x_1^{k-1} x_2 \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \left(\frac{x_2}{x_1}\right)^{k-2} = x_1 x_2^{k-1} dx_2 dx_2, \\
B_{|C^k_{\epsilon_1,\epsilon_2}} &= (1 - x_1^k - x_2^k)\frac{\epsilon_2}{\epsilon_1}\left(\frac{x_2}{x_1}\right)^{\frac{k}{2}-1} dx_2 dx_2 \\
&= \left(1 - \left(\frac{\epsilon_2}{\epsilon_1} x_2^{k/2} - \frac{1}{\epsilon_1}\right)^2 - x_2^k\right)\left(\frac{\epsilon_2}{\epsilon_1}\right) x_2^{k/2} \left(\frac{\epsilon_2}{\epsilon_1} x_2^{k/2} - \frac{1}{\epsilon_1}\right)^{-1} \frac{x_2}{x_1} \\
&= -2\epsilon_2 x_2^{k/2-1}(\epsilon_2 x_2^{k/2} - 1)\epsilon_2 x_2^{k/2}(\epsilon_2 x_2^{k/2} - 1)^{-1}\frac{x_2}{x_1} \\
&= -2x_2^{k-1} x_1 dx_2 dx_2, \\
C_{|C^k_{\epsilon_1,\epsilon_2}} &= x_1 x_2^{k-1} dx_2 dx_2.
\end{aligned}$$

Therefore $A_{|C^k_{\epsilon_1,\epsilon_2}} + B_{|C^k_{\epsilon_1,\epsilon_2}} + C_{|C^k_{\epsilon_1,\epsilon_2}} = 0$ and from Corollary 3.72 we have that $C^k_{\epsilon_1,\epsilon_2}$ is $\omega_{2,k}$-integral.

Now we consider the case k odd and the curve of type (iv): Note that this curve is irreducible by Lemma 5.16. Taking differentials of $1+x_1^{2k}+x_2^{2k}-2x_1^k-2x_2^k-2x_1^k x_2^k = 0$ we obtain

$$(x_1^{2k-1} - x_1^{k-1} - x_1^{k-1}x_2^k)dx_1 - (x_1^k x_2^{k-1} + x_2^{k-1} - x_2^{2k-1})dx_2 = 0$$

hence (by a similar computation) $\omega_{2,k}$ restricted to the curve of type (iv) on the open set $U_3 \cap D_+(x_1^k - x_2^k - 1)$ has equation

$$-\frac{x_1^{1-k}x_2^{k-1}(1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k)}{(x_1^k - x_2^k - 1)^2} dx_2 dx_2 = 0.$$

We have that $U_3 \cap D_+(x_1^k - x_2^k - 1)$ intersects the irreducible curve

$$1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k = 0$$

because $x_1^k - x_2^k - 1$ is not a multiple of this curve. Therefore applying Corollary 3.72 to this open set, we have that the curve

$$1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k = 0$$

is $\omega_{2,k}$-integral when k is odd.

So the curves of type (i), (ii) are $\omega_{2,k}$-integral for any k, curves of type (iii) are $\omega_{2,k}$-integral for k even, and the curve of type (iv) is $\omega_{2,k}$-integral for k odd. □

Lemma 5.20. *If k is an odd natural number, the curves of type (i), (ii) and (iv) are the only $\omega_{2,k}$-integral curves on $X_{2,k} = \mathbb{P}^2$. If k is even, the curves of type (i), (ii) and (iii) are the only $\omega_{2,k}$-integral curves on $X_{2,k}$.*

Proof. The restriction of $\omega_{2,k}$ to U_0 has equation

$$\omega_{2,k} = x_1^{k-1} x_2 dx_1 dx_1 + (1 - x_1^k - x_2^k) dx_1 dx_2 + x_1 x_2^{k-1} dx_2 dx_2.$$

We have from the definition of Δ in Subsection 3.7

$$\Delta = (\mathbb{P}^2 \backslash U_0) \cup \left\{ P \in U_0 : A_0(P) = 0 \text{ or } A_1^2(P) - 4A_0(P)A_2(P) = 0 \right\} \subseteq \mathbb{P}^2.$$

In our case the last condition becomes (for $P = (x_1, x_2)$)

$$A_2^2(P) - 4A_1(P)A_3(P) \;=\; (1 - x_1^k - x_2^k)^2 - 4(x_1^{k-1} x_2)(x_1 x_2^{k-1})$$

$$= \quad 1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k.$$

Therefore

$$\Delta = \left\{ [x_0 : x_1 : x_2] : x_0 x_1 x_2 (1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k) = 0 \right\}.$$

Note that by Lemma 5.19, when k is odd, Δ is the union of the curves of type (i) and (iv), and when k is even, Δ is the union of the curves of type (i) and (iii).

Now we want to prove that the $\omega_{2,k}$-integral curves not contained in Δ are exactly the curves of type (ii). Let $P = [x_1 : x_2 : x_3]$ be a point ouside Δ. From Theorem 3.76 we only need to prove that there are at least two $\omega_{2,k}$-integral curves of type (ii) passing through P. The point P lies on D_c if and only if $c(c+1)x_0^k = cx_1^k - (c+1)x_2^k$. The discriminant of the equation $cx_1^k - (c+1)x_2^k = c(c+1)$ (with c the variable) is

$$1 + x_1^{2k} + x_2^{2k} - 2x_1^k - 2x_2^k - 2x_1^k x_2^k,$$

which is different from zero because P is outside Δ. Therefore there are two values of c for which D_c pass through P. □

5.4 Pullbacks of $\omega_{2,k}$-integral curves

Now that we have the complete list of $\omega_{2,k}$-integral curves of $X_{2,k} \cong \mathbb{P}^2$ for any $k \geq 1$, we will use them to find integral curves on the other surfaces $X_{n,k}$.

We have a chain of finite surjective morphisms:

$$X_{2,k} \xleftarrow{\pi_3} X_{3,k} \xleftarrow{\pi_4} X_{4,k} \xleftarrow{\pi_5} \cdots .$$

and recall that for each $n \geq 3$ we denote by $\rho_{n,k}$ the composition $\pi_3 \circ \pi_4 \circ \cdots \pi_n$. We have $\deg(\rho_{n,k}) = k^{n-2}$.

Lemma 5.21. *The pull-backs under $\rho_{n,k}$ of the curves of type (i) and (ii) with $c \neq -1, \ldots, n-2$ of Lemma 5.19 are smooth complete intersection curves. The pull-back of a curve of type (i) is given by the equations*

$$\begin{aligned} x_j &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k. \end{aligned}$$

The pullback of a curve of type (ii) with $c \neq -1, \ldots, n-2$ is given by the equations

$$\begin{aligned} c(c+1)x_0^k - cx_1^k + (c+1)x_2^k &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k. \end{aligned}$$

Proof. Let C be the pullback of a curve of type (ii) with $c \neq -1, \ldots, n-2$. By Proposition 2.30 and Proposition 5.6, the equations of C are

$$\begin{aligned} c(c+1)x_0^k - cx_1^k + (c+1)x_2^k &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k \end{aligned}$$

with $c \neq -1, \ldots, n-2$ and it is a complete intersection in $\mathbb{P}^n$. Writing $d = c+2$ we have $d \neq 1, \ldots, n$, and the Jacobian matrix of C evaluated at the point $[x_0 : \ldots : x_n] \in X_{n,k}$ is the $(n-1) \times (n+1)$ matrix

$$k \begin{pmatrix} -(d-2)(d-1)x_0^{k-1} & (d-2)x_1^{k-1} & -(d-1)x_2^{k-1} & 0 & 0 & \cdots & 0 \\ -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -(n-2)(n-1)x_0^{k-1} & (n-2)x_1^{k-1} & -(n-1)x_2^{k-1} & 0 & \cdots & 0 & x_n^{k-1} \end{pmatrix}.$$

The last n rows are linearly independent by Lemma 5.12. If at least two of x_0, x_1, x_2 are zero, then by Observation 5.11 we obtain that this matrix has maximal rank (since $d \neq 1, \ldots, n$). If only one of x_0, x_1, x_2 is different from zero, then $x_3 \cdots x_n \neq 0$ by Lemma 5.7. Therefore the matrix has maximal rank.

Now let C be the pullback of a curve of type (i). By Proposition 2.30 and Proposition 5.6, it has equations

$$\begin{aligned} x_j &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k \end{aligned}$$

with $j = 0, 1, 2$ and it is a complete intersection. The Jacobian matrix of C evaluated

at the point $[x_0 : \ldots : x_n] \in X_{n,k}$ is the $n-1 \times n+1$ matrix

$$k\begin{pmatrix} a & b & c & 0 & 0 & \cdots & 0 \\ -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -(n-2)(n-1)x_0^{k-1} & (n-2)x_1^{k-1} & -(n-1)x_2^{k-1} & 0 & \cdots & 0 & x_n^{k-1} \end{pmatrix}.$$

with exactly one of a, b, c different from 0 (and equal to $\frac{1}{k}$). Note that the submatrix

$$\begin{pmatrix} a & b \\ -(i-2)(i-1)x_0^{k-1} & (i-2)x_1^{k-1} \end{pmatrix}$$

has determinant $(i-2)(x_1^{k-1}a + (i-1)x_0^{k-1}b) \neq 0$ when $i \neq 1, 2$, $x_1x_2 \neq 0$ and either $a \neq 0$ or $b \neq 0$. The submatrix

$$\begin{pmatrix} a & c \\ -(i-2)(i-1)x_0^{k-1} & -(i-1)x_2^{k-1} \end{pmatrix}$$

has determinant $(i-1)(-x_2^{k-1}a + (i-2)x_0^{k-1}c) \neq 0$ when $i \neq 1, 2$, $x_1x_2 \neq 0$ and either $a \neq 0$ or $c \neq 0$. The submatrix

$$\begin{pmatrix} b & c \\ (i-2)x_1^{k-1} & -(i-1)x_2^{k-1} \end{pmatrix}$$

has determinant $-(i-1)x_2^{k-1}b - (i-2)x_1^{k-1}c \neq 0$ when $i \neq 1, 2$, $x_1x_2 \neq 0$ and either $b \neq 0$ or $c \neq 0$. From this observation and a similar computation to the one in Lemma

5.12 we obtain that this matrix has maximal rank, hence C is smooth. Therefore the pullbacks of curves of type (i) are smooth. □

Lemma 5.22. *Let $3 \leq j \leq n$, and let $C_{n,j} = (\pi_{j+1} \circ \cdots \circ \pi_n)^* C_j$ in $X_{n,k}$. The curves $C_{n,j}$ are smooth complete intersection curves given by the following equations:*

$$\begin{aligned} x_j &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k \end{aligned}$$

Proof. By Proposition 2.30 and Proposition 5.6, the equations of $C_{n,j}$ are

$$\begin{aligned} x_j &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k \end{aligned}$$

with $j = 3, \ldots, n$, hence it is complete intersection. The Jacobian matrix of $C_{n,j}$ evaluated at the point $[x_0 : \ldots : x_n] \in X_{n,k}$ is the $(n-1) \times (n+1)$ matrix

$$k \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ -2x_0^{k-1} & x_1^{k-1} & -2x_2^{k-1} & x_3^{k-1} & 0 & \cdots & \cdots & 0 \\ -6x_0^{k-1} & 2x_1^{k-1} & -3x_2^{k-1} & 0 & x_4^{k-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -(n-2)(n-1)x_0^{k-1} & (n-2)x_1^{k-1} & -(n-1)x_2^{k-1} & 0 & \cdots & \cdots & 0 & x_n^{k-1} \end{pmatrix}.$$

where the non-zero component of the first row is on the $j+1$-st column. The last n rows are linearly independent by Lemma 5.12. If $x_j = 0$, then the matrix has maximal rank. If $x_j \neq 0$, then since no three of $x_1, \ldots, x_n$ are zero, we obtain that the $(j+1)$-st row and the first row are linearly independent. Therefore the matrix has maximal rank and hence the curve $C_{n,j}$ is smooth. □

Lemma 5.23. *Let k be an even integer. The irreducible components of the pullbacks of the curves of type (iii) are the smooth complete intersection curves $C^k_{\epsilon_1,\ldots,\epsilon_n} \subseteq X_{n,k}$ given by the equations (with $\epsilon_i \in \{\pm 1\}$):*

$$\begin{aligned} x_0^{\frac{k}{2}} + \epsilon_1 x_1^{\frac{k}{2}} &= \epsilon_2 x_2^{\frac{k}{2}} \\ &\vdots \\ (n-1)x_0^{\frac{k}{2}} + \epsilon_1 x_1^{\frac{k}{2}} &= \epsilon_n x_n^{\frac{k}{2}} \end{aligned} \tag{5.3}$$

Proof. Let $C^k_{\epsilon_1,\epsilon_2}$ be of type (iii).

From the first equation of $C^k_{\epsilon_1,\ldots,\epsilon_n}$ we have $2\epsilon_1 x_0^{\frac{k}{2}} x_1^{\frac{k}{2}} = -x_0^k - x_1^k + x_2^k$. Squaring $(i-1)x_0^{\frac{k}{2}} + \epsilon_1 x_1^{\frac{k}{2}} = \epsilon_i x_i^{\frac{k}{2}}$ and replacing $2\epsilon_1 x_0^{\frac{k}{2}} x_1^{\frac{k}{2}}$ by $-x_0^k - x_1^k + x_2^k$ one gets

$$(i-1)(i-2)x_0^k - (i-2)x_1^k + (i-1)x_2^k = x_i^k.$$

Since this holds for every $i \geq 3$ we obtain that $C^k_{\epsilon_1,\ldots,\epsilon_n} \subseteq X_{n,k}$.

We have $\pi_n(C^k_{\epsilon_1,\ldots,\epsilon_n}) \subseteq C^k_{\epsilon_1,\ldots,\epsilon_{n-1}} \subseteq X_{n,k}$. Since for any

$$P = [x_0 : \ldots : x_{n-1}] \in C^k_{\epsilon_1,\ldots,\epsilon_{n-1}} \subseteq X_{n,k}$$

we have that $Q = [x_0 : \ldots : x_n] \in C^k_{\epsilon_1,\ldots,\epsilon_n}$ with $\epsilon_n x_n^{k/2} = \epsilon_1 x_1^{k/2} + (n-1)x_0^{k/2}$ is a

preimage of P, we obtain that $\pi_n(C^k_{\epsilon_1,\ldots,\epsilon_n}) \supseteq C^k_{\epsilon_1,\ldots,\epsilon_{n-1}}$, thus $\pi_n(C^k_{\epsilon_1,\ldots,\epsilon_n}) = C^k_{\epsilon_1,\ldots,\epsilon_{n-1}}$. From this we also get that every component of $C^k_{\epsilon_1,\ldots,\epsilon_n}$ has dimension less than or equal to 1 since $\pi_n(C^k_{\epsilon_1,\ldots,\epsilon_n}) \neq \pi_n(X_{n,k}) = X_{n-1,k}$ and π_n is finite. On the other hand, every component of $C^k_{\epsilon_1,\ldots,\epsilon_n}$ has dimension at least 1 because it is defined by $n-1$ equations in $\mathbb{P}^n$ and we conclude by Theorem I.7.2 in [Har77]. Therefore, $C^k_{\epsilon_1,\ldots,\epsilon_n}$ has all its irreducible components of dimension exactly 1 (it is equidimensional).

Since $C^k_{\epsilon_1,\ldots,\epsilon_n}$ is equidimensional of dimension 1 in $\mathbb{P}^n$ and is defined by $n-1$ equations, it is a complete intersection. The Jacobian matrix of $C^k_{\epsilon_1,\ldots,\epsilon_n}$ evaluated at $[x_0 : \ldots : x_n]$ is the following $(n-1)\times(n+1)$ matrix:

$$\frac{k}{2}\begin{pmatrix} x_0^{\frac{k}{2}-1} & \epsilon_1 x_1^{\frac{k}{2}-1} & \epsilon_2 x_2^{\frac{k}{2}-1} & 0 & \ldots & 0 \\ 2x_0^{\frac{k}{2}-1} & \epsilon_1 x_1^{\frac{k}{2}-1} & 0 & \epsilon_3 x_3^{\frac{k}{2}} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ldots \\ (n-1)x_0^{\frac{k}{2}-1} & \epsilon_1 x_1^{\frac{k}{2}-1} & 0 & \ldots & 0 & \epsilon_n x_n^{\frac{k}{2}-1} \end{pmatrix}$$

If none of $x_2,\ldots,x_n$ is zero, then this matrix has maximal rank. If one of $x_2,\ldots,x_n$ is zero, then at least one of x_0, x_1 is not zero, hence the matrix has maximal rank. If two of $x_2,\ldots,x_n$ are zero, then we have $x_0x_1 \neq 0$. Noting that for any $2 \leq i \neq j \leq n$ we have that

$$\begin{pmatrix} (i-1)x_0^{\frac{k}{2}-1} & \epsilon_1 x_1^{\frac{k}{2}-1} \\ (j-1)x_0^{\frac{k}{2}-1} & \epsilon_1 x_1^{\frac{k}{2}-1} \end{pmatrix}$$

has non-zero determinant, we obtain that the $(n-1)\times(n+1)$ matrix has maximal rank. Since $C^k_{\epsilon_1,\ldots,\epsilon_n}$ is a smooth complete intersection, we obtain that it is irreducible.

For fixed ϵ_1, ϵ_2 the image of any $C^k_{\epsilon_1,\ldots,\epsilon_n}$ under $\rho_{n,k}$ is $C^k_{\epsilon_1,\epsilon_2}$. Now we want to prove that these irreducible curves are all the preimages of $C^k_{\epsilon_1,\epsilon_2}$. The restriction of the

morphism $\mathbb{P}^n \setminus [0 : \ldots : 0 : 1] \to \mathbb{P}^{n-1}$ to $C^k_{\epsilon_1,\ldots,\epsilon_n}$ has degree $\frac{k}{2}$ (by Lemma 2.87). The composition of these restrictions has degree $\left(\frac{k}{2}\right)^{n-2}$, hence $\deg(\rho_{n,k|C^k_{\epsilon_1,\ldots,\epsilon_n}}) = \left(\frac{k}{2}\right)^{n-2}$. Let ζ be a primitive $\frac{k}{2}$-th root of -1. For $2 \leq j \leq n$ let

$$\alpha_j = \begin{cases} 0 & \text{if } \epsilon_j\epsilon_1 = 1 \\ 1 & \text{if } \epsilon_j\epsilon_1 = -1 \end{cases}$$

Then the point $P_{\epsilon_3,\ldots,\epsilon_n} := [0 : 1 : \zeta^{\alpha_2} : \zeta^{\alpha_3} : \ldots : \zeta^{\alpha_n}]$ will satisfy the equations $(j-1)x_0^{k/2} + \epsilon_1 x_1^{k/2} = \epsilon_j x_j^{k/2}$ for $2 \leq j \leq n$ because $\epsilon_1\epsilon_j = (\zeta^{\alpha_j})^{k/2}$ holds. Note that if we choose a point $P_{\epsilon'_3,\ldots,\epsilon'_n}$, then not all the equations $(j-1)x_0^{k/2} + \epsilon_1 x_1^{k/2} = \epsilon_j x_j^{k/2}$ will be satisfied. Hence the point $P_{\epsilon'_3,\ldots,\epsilon'_n}$ only belongs to $C_{\epsilon'_1,\ldots,\epsilon'_n}$. Therefore all the components of the preimage of $C^k_{\epsilon_1,\epsilon_2}$ under $\rho_{n,k}$ are distinct.

Since there are 2^{n-2} distinct curves $C^k_{\epsilon_1,\ldots,\epsilon_n}$ in the preimage of $C^k_{\epsilon_1,\epsilon_2}$, we get $\left(\frac{k}{2}\right)^{n-2} 2^{n-2} = k^{n-2}$, hence the curves $C^k_{\epsilon_1,\ldots,\epsilon_n}$ are all the preimages of $C^k_{\epsilon_1,\epsilon_2}$. □

Lemma 5.24. *Let k be odd. If $C^{(iv)}_{n,k}$ is defined as the pull-back to $X_{n,k}$ of the curve of type (iv) of $X_{2,k}$, then $C^{(iv)}_{n,k}$ is irreducible and is given by the equations*

$$\begin{aligned} x_0^{2k} + x_1^{2k} + x_2^{2k} - 2x_0^k x_1^k - 2x_0^k x_2^k - 2x_1^k x_2^k &= 0 \\ 2x_0^k - x_1^k + 2x_2^k &= x_3^k \\ &\vdots \\ (n-1)(n-2)x_0^k - (n-2)x_1^k + (n-1)x_2^k &= x_n^k. \end{aligned}$$

Moreover, the 2^n curves $C^{2k}_{\epsilon_1,\ldots,\epsilon_n} \subseteq X_{n,2k}$ are isomorphic to each other, and are birational to $C^{(iv)}_{n,k}$.

Proof. The rule $[x_0 : \ldots : x_n] \mapsto [x_0^2 : \ldots : x_n^2]$ defines a surjective morphism

$F_n : X_{n,2k} \to X_{n,k}$. We have the commutative diagram (as can be seen from the equations)

$$\begin{array}{ccc} X_{n,2k} & \xrightarrow{F_n} & X_{n,k} \\ {\scriptstyle \rho_{n,2k}}\downarrow & & \downarrow{\scriptstyle \rho_{n,k}} \\ X_{2,2k} & \xrightarrow[F_2]{} & X_{2,k} \end{array}$$

Note that $C_{2,k}^{(iv)} = \mathrm{div}_{\mathbb{P}^2}(\bar{P}_k)$ with $\bar{P}_k = x_0^{2k} + x_1^{2k} + x_2^{2k} - 2x_0^k x_1^k - 2x_0^k x_2^k - 2x_1^k x_2^k$ which is the homogenization of the polynomial P from Lemma 5.16. By that lemma, we have $F_2^* C_{2,k}^{(iv)} = \sum_{\epsilon_1,\epsilon_2} C_{\epsilon_1,\epsilon_2}^{2k}$.

By Lemma 5.23 we obtain $\rho_{n,2k}^* \sum_{\epsilon_1,\epsilon_2} C_{\epsilon_1,\epsilon_2}^{2k} = \sum_{\bar{\epsilon}\in G} C_{\bar{\epsilon}}^{2k}$, where $G = \{\pm 1\}^n$. By definition, we have $C_{n,k}^{(iv)} := \rho_{n,k}^* C_{2,k}^{(iv)}$ so we get

$$F_n^* C_{n,k}^{(iv)} = F_n^* \rho_{n,k}^* C_{2,k}^{(iv)} = \rho_{n,2k}^* F_2^* C_{2,k}^{(iv)} = \sum_{\bar{\epsilon}\in G} C_{\bar{\epsilon}}^{2k}.$$

In particular $C_{n,k}^{(iv)}$ is reduced because the curves $C_{\bar{\epsilon}}^{2k}$ are reduced by Lemma 5.23.

Let $G = \{\pm 1\}^n \cong (\mathbb{Z}/2\mathbb{Z})^n$ act on $\mathbb{P}^n$ via

$$\tau([x_0 : \ldots : x_n]) = [x_0 : \tau_1 x_1 : \ldots : \tau_n x_n].$$

Then $\tau(X_{n,2k}) = X_{n,2k}$, for all $\tau \in G$ and $F_n \circ \tau = F_n$. Moreover $\tau C_{\bar{\epsilon}}^{2k} = C_{\tau\bar{\epsilon}}^{2k}$, where $\bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in G$, and so $C_{\bar{\epsilon}}^{2k} = \bar{\epsilon} C_{(1,\ldots,1)}^{2k}$. Thus $F_n(C_{\bar{\epsilon}}^{2k}) = F_n(C_{(1,\ldots,1)}^{2k})$, for all $\bar{\epsilon} \in G$.

Thus, since F_n is surjective, we have

$$C_{n,k}^{(iv)} = F_n F_n^{-1}(C_{n,k}^{(iv)}) = F_n(\cup C_{\bar{\epsilon}}^{2k}) = F_n(\tau C_{(1,\ldots,1)}^{2k}) = F_n(C_{(1,\ldots,1)}^{2k}).$$

Since $C^{2k}_{(1,\dots,1)}$ is irreducible, so is $C^{(iv)}_{n,k} = F_n(C^{2k}_{\bar{\epsilon}})$. Thus $C^{(iv)}_{n,k}$ is an integral curve, and $F_n(C^{2k}_{\bar{\epsilon}}) = C^{(iv)}_{n,k}$, for all $\bar{\epsilon}$. Since $\deg(F_n) = 2^n$, we have by Proposition 2.86 that $f_{C^{2k}_{\bar{\epsilon}}/C^{(iv)}_{n,k}} = 1$, for all $\bar{\epsilon}$, and hence $F_{n|C^{2k}_{\bar{\epsilon}}} : C^{2k}_{\bar{\epsilon}} \to C^{(iv)}_{n,k}$ is birational □

Lemma 5.25. *The pullbacks under $\rho_{n,k}$ of curves of type (i) and the curves of type (ii) with $c \neq -1, \dots, n-2$ are irreducible and reduced. If k is even, then the pullbacks of the curves of type (iii) is the union of the curves $C^k_{\bar{\epsilon}}$ with $\bar{\epsilon} \in G$ which are irreducible and reduced. If k is odd the pullback of the curve of type (iv) is irreducible and reduced. The pullback of curves of type (ii) with $c = 1, \dots, n-2$ is $\rho^*_{n,k}(D_c) = kC_{n,c+2}$ where $C_{n,j}$ are irreducible and reduced.*

Proof. From Lemma 5.24 we know that the curves of type (iv) are irreducible. The pullbacks of the curves of type (i), (iii) and (ii) with $c \neq -1, \dots, n-2$ are smooth and complete intersection curves by Lemmas 5.21 and 5.23. From Lemma 5.22 we know that the curves C_i are smooth complete intersections. From Proposition 2.79 we get that all these curves are irreducible. The curves of type (ii) with $c = 1, \dots, n-2$ are irreducible (but not reduced) because they are k times a curve $C_{n,c+2}$ (see Lemma 5.22 and Lemma 5.5). □

5.5 Integral curves on $X_{n,k}$

Let

$$\omega_{n,k} = (\rho_{n,k})^{\bullet}_{r,\mathcal{O}(k+3)}\omega_{2,k} \in H^0(X_{n,k}, \mathcal{O}(k+3) \otimes S^2\Omega^1_{X_{n,k}}).$$

Proposition 5.26. *For $k \geq 2$, the $\omega_{n,k}$-integral curves in $X_{n,k}$ are the following:*

(a) The pullbacks under $\rho_{n,k}$ of the coordinate axes of $X_{2,k}$. These curves are smooth and irreducible with genus $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

(b) $C_{n,c+2} = (\rho_{n,k}^*(D_c))_{\text{red}}$ *with* $c \neq \{-1, 0\}$. *These curves are smooth and irreducible. When* $c \notin \{1, \ldots, n-2\}$, *they have genus* $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$, *and when* $c \in \{1, \ldots, n-2\}$, *they have genus* $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

Moreover, the following curves are also $\omega_{n,k}$*-integral:*

(c) *If* k *is odd, the pullback of the curve of type (iii). It is irreducible and it has genus* $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$.

(d) *If* k *is even, the* 2^n *curves* $C_{\epsilon_1,\ldots,\epsilon_n}$*:*

$$\begin{aligned} \epsilon_1 x_1^{\frac{k}{2}} &= \epsilon_2 x_2^{\frac{k}{2}} - x_0^{\frac{k}{2}} \\ &\vdots \\ \epsilon_1 x_1^{\frac{k}{2}} &= \epsilon_n x_n^{\frac{k}{2}} - (n-1)x_0^{\frac{k}{2}}. \end{aligned}$$

They are smooth and irreducible and have genus $\frac{1}{2}(\frac{k}{2})^{n-1}(\frac{k}{2}(n-1)-n-1)+1$.

Proof. Let $C \subseteq X_{n,k}$ be an $\omega_{n,k}$-integral curve. By Theorem 3.35 its image $D = \rho_{n,k}(C)$ must be $\omega_{2,k}$-integral. Therefore C is a component of $\rho_{n,k}^*(D)$. Hence by Lemma 5.19 and Lemma 5.25, C is a curve described in this proposition.

Now we compute the genus of these curves. Let C be a curve of type (a). From Proposition 5.21 we have $K_C = \mathcal{O}(k(n-2)-n)$, hence from Proposition 2.80 its genus is $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

Let C be a curve of type (b) with $c \notin \{1, \ldots, n-2\}$. Then from Proposition 5.21 we have $K_C = \mathcal{O}(k(n-1)-n-1)$. From Proposition 2.80 we have that the genus of C is $\frac{k^{n-1}}{2}(k(n-1)-n-1)+1$. If C is a curve of type (b) with $c \in \{1, \ldots, n-2\}$, we have $K_C = \mathcal{O}(k(n-2)-n)$ by Proposition 5.22 and Lemma 5.25, hence the genus of C is $\frac{k^{n-2}}{2}(k(n-2)-n)+1$.

Let k be even and let C be a curve of type (d). By Proposition 5.23 we have $K_C = \mathcal{O}\left((n-1)\frac{k}{2} - n - 1\right)$, and by Proposition 2.80 we obtain that C has genus $\frac{1}{2}(\frac{k}{2})^{n-1}(\frac{k}{2}(n-1) - n - 1) + 1$.

Since the genus of a curve of type (d) in $X_{n,2k}$ is $\frac{1}{2}(\frac{2k}{2})^{n-1}(\frac{2k}{2}(n-1) - n - 1) + 1$, we obtain by Lemma 5.24 that the genus of the curve of type (c) in $X_{n,k}$ is $\frac{k^{n-1}}{2}(k(n-1) - n - 1) + 1$. □

5.6 Curves of low genus on $X_{n,k}$

Now we will show that the curves of bounded genus (with bound depending on n and k) on $X_{n,k}$ are $\omega_{n,k}$-integral.

Lemma 5.27. *The section $\omega_{n,k}$ defines a unique section*

$$\omega'_{n,k} \in H^0(X_{n,k}, \mathcal{O}(k+3-(k-1)(n-2)) \otimes S^2\Omega^1_{X_{n,k}}).$$

Moreover, every $\omega'_{n,k}$-integral curve is $\omega_{n,k}$-integral.

Proof. By Lemma 5.15, we have that $\rho_i^*(D_{i-2}) = kC_i$. Since the curves D_c with $c = 1, \ldots, n-2$ are $\omega_{2,k}$-integral, Theorem 3.87 shows that the section $\omega_{n,k}$ vanishes along $C_{n,c+2} = (\rho_{n,k}^*(D_c))_{\text{red}}$ with multiplicity $k-1$, for each $c = 1, \ldots, n-2$. Since $C_{n,c+2}$ is an hyperplane section in $X_{n,k}$, its ideal sheaf is $\mathcal{O}(-1)$. By Proposition 3.79, we get that the section $\omega_{n,k}$ vanishes along $(k-1)\sum_{i=3}^{n} C_{n,c+2}$. Thus by Proposition 2.25, we get that for each n, the section $\omega_{n,k} \in H^0(X_{n,k}, \mathcal{O}(k+3) \otimes S^2\Omega^1_{X_{n,k}})$ determines a unique section $\omega'_{n,k} \in H^0(X_{n,k}, \mathcal{O}(k+3-(k-1)(n-2)) \otimes S^2\Omega^1_{X_{n,k}})$. From Theorem 3.88 we obtain that the $\omega'_{n,k}$-integral curves are $\omega_{n,k}$-integral. □

Proposition 5.28. *Let $k \geq 3$, let $g \geq 0$ and let $n > \max\left\{\frac{4}{k-1}+3, \frac{4g}{k-1}+3\right\}$. If C is an irreducible curve of genus $g(C) < g$ in $X_{n,k}$, then C is $\omega'_{n,k}$-integral.*

Proof. Recall that $\varphi_C : \tilde{C} \to X_{n,k}$ is the normalization of C. We know that

$$\deg_{\tilde{C}} \varphi_C^* \mathcal{O}(1) = (C.H) > 0,$$

with H an hyperplane section on $X_{n,k}$. Hence since $n > \frac{4}{k-1}+3$ we have

$$\begin{aligned} \deg_{\tilde{C}} \varphi_C^* \mathcal{O}(k+3-(k-1)(n-2)) &= (k+3-(k-1)(n-2)) \deg_{\tilde{C}} \varphi_C^* \mathcal{O}(1) \\ &< (k+3-(k-1)(n-2)) \\ &< 4-4g. \end{aligned}$$

Therefore by Proposition 3.42 we obtain that C is $\omega'_{n,k}$-integral. □

5.7 Proof of the main results

Proof of Theorem 5.1. Let g be fixed and let $n > \frac{4g}{k-1}+3$. Recall that $k \geq 3$.

The curves of type (a) and the curves of type (b) with $c \in \{1, \ldots, n-2\}$ have genus

$$\begin{aligned} \frac{k^{n-2}}{2}(k(n-2)-n)+1 &= \frac{k^{n-2}}{2}(n(k-1)-2k)+1 \\ &> \frac{k^{n-2}}{2}(4g+3(k-1)-2k)+1 \\ &= \frac{k^{n-2}}{2}(4g+k-3)+1 > 4g+2-3+1 \geq g, \end{aligned}$$

hence curves of type (a) and curves of type (b) with $c \in \{1, \ldots, n-2\}$ have genus strictly greater than g.

Curves of type (b) with $c \notin \{1, \ldots, n-2\}$ and curves of type (c) have genus

$$\begin{aligned}\frac{k^{n-1}}{2}(k(n-1)-n-1)+1 &= \frac{k^{n-1}}{2}(n(k-1)-k-1)+1\\ &> \frac{k^{n-1}}{2}(4g+3(k-1)-k-1)+1\\ &= \frac{k^{n-1}}{2}(4g+2k-4)+1 > 4g+4-4 \geq g.\end{aligned}$$

Thus, curves of type (b) with $c \notin \{1, \ldots, n-2\}$ and curves of type (c) have genus strictly greater than g.

Now suppose that $k \geq 4$ is even. Curves of type (d) have genus

$$\begin{aligned}\frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\frac{k}{2}(n-1)\ -n-1\right)+1 &= \frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(n\left(\frac{k}{2}-1\right)-\frac{k}{2}-1\right)+1\\ &> \frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(\left(\frac{4g}{k-1}+3\right)\left(\frac{k}{2}-1\right)-\frac{k}{2}-1\right)+1\\ &= \frac{1}{2}\left(\frac{k}{2}\right)^{n-1}\left(2g\frac{k-2}{k-1}+k-4\right)+1\\ &\geq \frac{27}{16}\left(2g\frac{2}{3}+4-4\right)+1 \geq g.\end{aligned}$$

Hence for $k \geq 4$, curves of type (d) have genus greater than g.

From Proposition 5.28 we get that all curves with genus $g(C) < g$ are $\omega'_{n,k}$-integral. Since for $k \geq 3$ and $n > \frac{4g}{k-1} + 3$ the $\omega'_{n,k}$-integral curves have genus strictly greater than g, we get that there are no curves of genus $g(C) < g$ in $X_{n,k}$. □

Proof of Theorem 5.2. If $k > 2$ and $n \geq 8$, then we have $n \geq \frac{4\cdot 1}{k-1} + 3$. If $k \geq 6$ and $n \geq 4$, then we have $n > \frac{4\cdot 1}{k-1} + 3$. Therefore by Theorem 5.1 we get that there are no curves of genus $g(C) \leq 1$ on $X_{n,k}$ in these cases. □

Proof of Theorem 5.3. Let K be a function field of genus g, let $k \geq 3$ and let

$n > \frac{4g}{k-1} + 3$. By Proposition 2.97, the solutions over K (up to scaling) of the system of equations (5.1) are in bijection with the morphisms $\{f : C_K \to X/\mathbb{C}\}$, with C_K the curve (up to isomorphism) with function field K. By Riemann-Hurwitz, these morphism are either constant, or must map the curve C_K to curves in X with genus less than or equal to g. By Theorem 5.1, there are no curves of genus less than or equal to g. Therefore there are no nonconstant solutions in K of the system of equations (5.1), so there are no sequences of length n of elements in K not all constant whose k-th powers have second differences equal to 2. □

Lemma 5.29. *Let $a_1, \ldots, a_n$ be a sequence in a number field. It has second differences equal to 2 if and only if for all $1 \leq j \leq n$ we have*

$$a_j = -(j-2)a_1 + (j-1)a_2 + (j-1)(j-2).$$

Proof. We first prove by induction that if a sequence has constant second differences, then it satisfies $a_j = -(j-2)a_1 + (j-1)a_2 + (j-1)(j-2)$ for all $1 \leq j \leq n$. The formula is trivially true for $j = 1, 2$. Suppose now that $j \geq 3$ and that the lemma is true for all integers up to $j-1$. Since the sequence has second differences equal to 2, we know that $2 - a_{j-2} + 2a_{j-1} = a_j$. Then we have

$$\begin{aligned} a_j &= 2 - a_{j-2} + 2a_{j-1} \\ &= 2 - (-(j-4)a_1 + (j-3)a_2 + (j-3)(j-4)) \\ &\quad +2(-(j-3)a_1 + (j-2)a_2 + (j-2)(j-3)) \\ &= -(j-2)a_1 + (j-1)a_2 + (j-1)(j-2). \end{aligned}$$

Hence $(j-1)(j-2) - (j-2)a_1 + (j-1)a_2 = a_j$.

To prove the converse, let $1 \leq j \leq n-2$. Then we have

$$\begin{aligned}
-(j-2)a_1 + (j-1)a_2 + (j-1)(j-2) - 2(-(j-1)a_1 + (j)a_2 + (j)(j-1)) \\
&+ (-(j)a_1 + (j+1)a_2 + (j+1)(j)) \\
&= (-(j-2) + 2(j-1) - j)a_1 + ((j-1) - 2j + (j+1))a_2 \\
&+ ((j-1)(j-2) - 2j(j-1) + (j+1)j) \\
&= 0a_1 + 0a_2 + 2.
\end{aligned}$$

Therefore the sequence has constant second differences. □

Proof of Theorem 5.4. Let $a_1, \ldots, a_n$ be a sequence of n elements of L whose k-th powers have second differences equal to 2. Then $[1 : x_1 : \ldots : x_n]$ is an L-rational point on $X_{n,k}$ by Lemma 5.29 and Lemma 5.5. If we have infinitely many sequences of length n satisfying these conditions, then we obtain infinitely many L-rational points on $X_{n,k}$. There are only finitely many L-rational points on $X_{n,k}$ which are not in the curves of genus 0 or 1 of $X_{n,k}$ by the Bombieri-Lang conjecture, since $X_{n,k}$ is of general type for $n \geq 4$ by Proposition 5.13. By Corollary 5.2, we get that there are finitely many sequences of this form for $n \geq 8$ when $k > 2$, and for $n \geq 4$, when $k \geq 6$ (since in these cases the surface $X_{n,k}$ has no curves of genus 0 or 1). This finite number only depends on k and L.

Let $k > 2$, and suppose that there are N sequences of length 8 whose k-th powers have second differences equal to 2 (for $k \geq 6$ we can replace 8 by 4 in this argument). Let $x_1, \ldots, x_{N+8}$ be a sequence of elements of L whose k-th powers have second differences equal to 2. By Lemma 5.5 we have that no term appears three times in the sequence. The $N+1$ sequences $x_i, \ldots, x_{i+7}$ (for $1 \leq i \leq N+1$) are distinct

sequences of length 8 whose k-th powers have second differences equal to 2. This contradicts the fact that there are only N sequences satisfying this condition. □

Lemma 5.30. *If $a_1, \ldots, a_n$ is a sequence in $\mathbb{C}$ having second differences equal to 2, then no three terms are equal.*

Proof. For every $1 \leq i \leq n$ we have $a_i = (i-1)a_2 - (i-2)a_1 + (i-1)(i-2)$, by Lemma 5.29. Suppose that three terms in the sequence are equal to α, which means $(i-1)a_2 - (i-2)a_1 + (i-1)(i-2) = \alpha$ for at least three values of i. Writing this as an equation on i, we obtain that

$$i^2 + (a_2 - a_1 - 3)i + 2a_1 - a_2 + 2 - \alpha = 0,$$

and so we see that there are at most two values of i for which $a_i = \alpha$. □

The following conjecture is due to Browkin and Brzezinski [BB94]:

Conjecture 5.31 (n-term ABC conjecture)**.** *Given any integer $n > 2$ and any $\epsilon > 0$, there exists a constant $C_{n,\epsilon}$ such that for all integers $a_1, \ldots, a_n$ with $a_1 + \cdots + a_n = 0$, $\gcd(a_1, \ldots, a_n) = 1$ and no proper zero subsum, we have*

$$\max(|a_1|, \ldots, |a_n|) \leq C_{n,\epsilon}\mathrm{rad}(a_1 \cdots a_n)^{2n-5+\epsilon}.$$

We will use the following very important theorem:

Theorem 5.32 (Szemerédi's theorem)**.** *Let k be a positive integer and let $0 < \delta < \frac{1}{2}$. There exists a positive integer $N = N(k, \delta)$ such that every subset of $\{1, \ldots, N\}$ of size at least δN contains an arithmetic progression of length k.*

Definition 5.33. Recall (from Chapter 1) that a sequence $a_1, \ldots, a_n$ is *trivial* if there is an arithmetic progression $a + jb$ such that $a_i = (a + ib)^2$. Note that in the case when the second differences are 2, trivial sequences have the form $a_i = (i + a)^2$.

We want to prove the following

Theorem 5.34. *Assume the Bombieri-Lang Conjecture for the surfaces $X_{n,k}$ with $n \geq 2 + \frac{4}{k-1}$ and the 4-term ABC conjecture. There exists an $M > 0$ such that there are no non-trivial sequences of length M consisting of integer powers (of possibly different exponents greater or equal to 2) which have constant differences equal to 2.*

Lemma 5.35. *For $k \geq 2$, define the sets*

$$S_k = \{n \in \mathbb{Z} : n \text{ is a } k\text{-th power}\} = \left\{m^k : m \in \mathbb{Z}\right\},$$

and also define

$$S_\infty = \{n \in \mathbb{Z} : n \text{ is a } k\text{-th power, with } k \geq 13\} = \cup_{k \geq 13} S_k.$$

There exists an N such that for any sequence $a_1, \ldots, a_N$ formed by integer powers with constant second differences equal to 2, there is an arithmetic progression

$$m, m + n, \ldots, m + 20n$$

(of length 21) in $\{1, \ldots, N\}$ such that for all $0 \leq j \leq 20$ we have $a_{m+jn} \in S_k$, for a fixed $k \in \{2, \ldots, 12, \infty\}$. Moreover, $n \leq (N - 1)/20$.

Proof. Let $N = N(21, 1/13)$ be the integer obtained by Szemeréri's theorem. There exists $k \in \{2, \ldots, 12, \infty\}$ such that at least $\frac{1}{13}$ of the elements of $\{a_1, \ldots, a_N\}$ are in

S_k (since the sequence consists of integer powers with exponent at least 2). Hence by Theorem 5.32, there is an arithmetic progression $m, m+n, \ldots, m+20n$ such that $a_m, \ldots, a_{m+20n}$ consists of elements of (the same) S_k. □

Notation 5.36. If α is an algebraic number over $\mathbb{Q}$, then we denote the number field $\mathbb{Q}(\alpha)$ by L_α.

Lemma 5.37. *Fix N as in the previous lemma. There is a* finite *collection $\mathcal{F}$ of integer sequences of length* 21*, depending* only *on the choice of N, with the following property:*

Let $a_1, ..., a_N$ be a sequence of powers with second differences equal to 2*. Suppose that for some $k \in \{2, 3, \ldots, 12\}$ and some $n \in \left\{1, \ldots, \frac{N-1}{20}\right\}$ there is a subsequence $a_m, \ldots, a_{m+20n}$ consisting of k-th powers. If $k > 2$, then this subsequence belongs to $\mathcal{F}$. If $k = 2$, then either the subsequence $a_m, \ldots, a_{m+20n}$ is* trivial *(that is, of the form $(nj+m)^2$) or it belongs to $\mathcal{F}$.*

Proof. Since $a_1, \ldots, a_N$ have second differences equal to 2, there exists a monic polynomial $P(x) = x^2 + bx + c \in \mathbb{Q}[x]$ such that $P(i) = a_i$.

Suppose that $P(m), P(m+n), \ldots, P(m+20n)$ are all k-th powers. Then the monic polynomial

$$\begin{aligned} Q(z) &:= \frac{1}{n^2}((m+zn)^2 + b(m+zn) + c) \\ &= z^2 + \frac{1}{n}(2m+b)z + \frac{1}{n^2}(m^2 + bm + c) \end{aligned}$$

(which is an element in $\mathbb{Q}[z]$) satisfies that $Q(0), \ldots, Q(20)$ are k-th powers in $L_{n^{2/k}}$. (Observe that for $k = 2$ we have $L_{n^{2/k}} = \mathbb{Q}$.)

From Corollary 5.4, and under Bombieri-Lang (for the number field $L_{n^{2/k}}$), we

obtain that there are finitely many sequences of length 21 formed by k-th powers (in $L_{n^{2/k}}$) whose second differences are equal to two.

From Theorem 3.1, and under Bombieri-Lang (for $\mathbb{Q}$) we get that there are finitely many non-trivial sequences of length 21 formed by squares. □

Remark 5.38. Since $k \leq 12$ and $n \leq \frac{N-1}{20}$ can take only finitely many values, we get that there are finitely many subsequences in $a_1, \ldots, a_N$ indexed by 21 elements in arithmetic progression and which consist of elements of the same set S_k, with $2 \leq k \leq 12$.

Lemma 5.39. *Assume the* 4*-term ABC Conjecture. There is a* finite *collection* $\mathcal{F}'$ *of integer sequences of length* 21*, depending* only *on the choice of* N*, with the following property:*

Let $n \in \left\{1, \ldots, \frac{N-1}{20}\right\}$*. If the subsequence* $a_m, \ldots, a_{m+20n}$ *of a sequence of* N *terms with second differences* 2*, consists of elements in* S_∞*, then it belongs to* $\mathcal{F}'$*.*

Proof. From Lemma 5.30 we know that there are at most 2 values of j for which $a_{m+jn} = 0$, that there are at most 2 values of j for which $a_{m+jn} = 2n^2$ and that there are at most 2 values of j for which $a_{m+jn} = -n^2$. Since our subsequence consists of 21 elements, there are three consecutive elements such that they all are different from $0, 2n^2, -n^2$. The elements a_{m+jn} (in our subsequence) satisfy (for $0 \leq j \leq 21$) the relation

$$a_{m+(j+2)n} - 2a_{m+(j+1)n} + a_{m+jn} - 2n^2 = 0, \tag{5.4}$$

because the sequence $a_1, a_2, \ldots, a_n$ has second differences equal to 2. If a subsum of three terms in the relation is equal to zero, then the fourth term has to be equal to

zero, but this cannot hold since all terms are different from zero. We cannot have $a_{m+jn} = 2n^2$, $a_{m+(j+2)n} = 2n^2$ or $a_{m+(j+1)n} = -n^2$, hence no proper subsum consisting of two terms is zero. Therefore no proper subzero sum of

$$a_{m+(j+2)n} - 2a_{m+(j+1)n} + a_{m+jn} - 2n^2$$

is zero, and $a_{m+(j+2)n} - 2a_{m+(j+1)n} + a_{m+jn} - 2n^2 = 0$. If $\gcd(a_{m+(l+2)n}, 2a_{m+(l+1)n}, a_{m+ln}, 2n^2) \neq 1$ we divide by the common term. From Conjecture 5.31 with $\epsilon = \frac{1}{5}$, there exists $C_{4,\epsilon} > 0$ such that

$$\max(|a_{m+(l+2)n}|, |2a_{m+(l+1)n}|, |a_{m+ln}|, |2n^2|) \leq C_{4,\epsilon}\mathrm{rad}(a_{m+(l+2)n}a_{m+(l+1)n}a_{m+ln})^{\frac{14}{5}}.$$

(The gcd condition can be omitted because the relevant gcd is at most $2n^2 < N^2$ which can be absorbed in $C_{4,\epsilon}$.) Hence we have

$$\begin{aligned} |a_{m+(l+2)n}a_{m+(l+1)n}a_{m+ln}|^{1/3} &\leq \max(|a_{m+(l+2)n}|, |2a_{m+(l+1)n}|, |a_{m+ln}|, |2n^2|) \\ &\leq C_{4,\epsilon}\mathrm{rad}(a_{m+(l+2)n}a_{m+(l+1)n}a_{m+ln})^{\frac{14}{5}} \\ &\leq C'(|a_{m+(l+2)n}a_{m+(l+1)n}a_{m+ln}|^{1/13})^{14/5} \end{aligned}$$

for an absolute constant C', because $a_{m+(l+2)n}, 2a_{m+(l+1)n}, a_{m+ln}$ are powers of exponent at least 13 (they are in S_∞). As $1/3 > (1/13)(14/5)$ we conclude that $a_{m+(l+2)n}, 2a_{m+(l+1)n}, a_{m+ln}$ are bounded by an absolute constant, hence there are only finitely many possibilities for these three integers. Thus there are only finitely many $a_{m+ln}, a_{m+(l+1)n}, a_{m+(l+2)n}$ satisfying Equation 5.4. Since the polynomial P has degree 2, it can be uniquely determined by the finitely many choices of a_{m+ln}, $a_{m+(l+1)n}$,

$a_{m+(l+2)n}$, therefore $\mathcal{F}'$ is finite. □

Proof of Theorem 5.34. A monic quadratic polynomial is completely determined by its values at three given points. Thus, the fact that the sets $\mathcal{F}$ and $\mathcal{F}'$ from the previous lemmas are finite gives the result up to finitely many sequences. We conclude by the same combinatorial argument as in the proof of Corollary 5.4. □

Chapter 6

The surface parametrizing cuboids

6.1 Introduction and main results

Recall from Subsection 2.1.5 that the box variety is the surface $Y_4 \subseteq \mathbb{P}^6$ defined by

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 &= x_3^2 \\ x_1^2 + x_2^2 &= x_4^2 \\ x_0^2 + x_2^2 &= x_5^2 \\ x_0^2 + x_1^2 &= x_6^2. \end{aligned} \tag{6.1}$$

We use the theory of diagonal quotient surfaces and a modification of Vojta's technique to classify all the ω-integral curves in the desingularization $\tilde{Y}_4$ of Y_4 for a specific ω (Theorem 6.39). Moreover, we give a numerical criterion for a curve C' on $\tilde{Y}_4$ to be ω_4'-integral (Theorem 6.48). From Theorem 6.48 we deduce the following:

Theorem 6.1. *Let C be a curve in Y_4 which does not contain any singularity of Y_4. Then*

$$\deg C \leq 4g(C) - 4.$$

Here and below, the degree $\deg(C)$ of a curve $C \subseteq Y_4 \subseteq \mathbb{P}^1$ is as defined in Remark 2.45. From Theorem 6.48 we also deduce the following:

Theorem 6.2. *Every curve of genus* 0 *or* 1 *on the surface* Y_4 *contains at least two singular points of* Y_4.

Theorem 6.3. *An irreducible curve of genus* 0 *on* Y_4 *must contain singularities of* Y_4 *of at least two of the following sets:*

$$\begin{aligned} A_1 &= \{[0 : \pm i : 1 : 0 : 0 : \pm 1 : \pm i]\} \cup \{[1 : 0 : 0 : \pm 1 : 0 : \pm 1 : \pm 1]\} \\ A_2 &= \{[0 : 1 : 0 : \pm 1 : \pm 1 : 0 : \pm 1]\} \cup \{[\pm i : 0 : 1 : 0 : \pm 1 : 0 : \pm i]\} \\ A_3 &= \{[0 : 0 : 1 : \pm 1 : \pm 1 : \pm 1 : 0]\} \cup \{[\pm i : 1 : 0 : 0 : \pm 1 : \pm i : 0]\}. \end{aligned}$$

Theorem 6.4. *Let* C *be an irreducible curve in* Y_4*, smooth at the singularities of this surface. (*C *can have singularities outside the* 48 *singular points of* Y_4*.) Then*

$$\deg C \leq 4g(C) + 44.$$

This result can be compared with a result in [FS13] (see Proposition 2.15 in this thesis for their result). By using other methods, Kani also obtains the inequality $\deg C \leq 4g(C) + 44$ for smooth curves.

6.2 An example of a diagonal quotient surface

The box variety Y_4 is an example of a diagonal quotient surface (cf. Subsection 2.8). This is discussed in [Bea13], using canonical embeddings. Here we prove this using a slightly different approach based on the universal property of diagonal quotient surfaces. This approach was suggested by Kani.

Let $\bar{Y}_4 \subseteq \mathbb{P}^6$ (with coordinates X, Y, Z, T, U, V, W) be defined by the equations

$$\begin{aligned} 0 &= XT - YZ \\ U^2 &= XT \\ V^2 &= X^2 - Y^2 - Z^2 + T^2 \\ W^2 &= X^2 + Y^2 + Z^2 + T^2, \end{aligned}$$

and let $\bar{Y}_1 \subseteq \mathbb{P}^3$ (with coordinates z_0, z_1, z_2, z_3) be defined by the equation

$$z_0 z_3 = z_1 z_2.$$

Consider the isomorphism $\eta_1 : \mathbb{P}^3 \to \mathbb{P}^3$ (the first $\mathbb{P}^3$ with coordinates x_0, x_1, x_2, x_3, the second with z_0, z_1, z_2, z_3) defined by

$$\eta_1([x_0 : x_1 : x_2 : x_3]) = [x_0 + ix_1 : x_3 + x_2 : x_3 - x_2 : x_0 - ix_1]$$

This is an isomorphism because it is an invertible linear change of variables, with inverse given by

$$\eta_1^{-1}([z_0 : z_1 : z_2 : z_3]) = \left[\frac{z_0 + z_3}{2} : \frac{z_0 - z_3}{2i} : \frac{z_1 - z_2}{2} : \frac{z_1 + z_2}{2}\right]$$

Similarly we define the isomorphism $\eta_4 : \mathbb{P}^6 \to \mathbb{P}^6$ by

$$\eta_4([x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6]) = [x_0 + ix_1 : x_3 + x_2 : x_3 - x_2 : x_0 - ix_1 : x_4 : x_5 : x_6]$$

whose inverse is given by

$$\eta_4^{-1}([z_0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6]) = \left[\frac{z_0 + z_3}{2} : \frac{z_0 - z_3}{2i} : \frac{z_1 - z_2}{2} : \frac{z_1 + z_2}{2} : z_4 : z_5 : z_6\right]$$

Define $Y_1 \subseteq \mathbb{P}^3$ by

$$x_0^2 + x_1^2 + x_2^2 = x_3^2$$

and recall the definition (6.1) of Y_4 from the previous section. Then directly from the equations we see that η_i (for $i = 1, 4$) induces an isomorphism $\eta_i' : Y_i \to \bar{Y}_i$.

Lemma 6.5. *We have that $Y_4, \bar{Y}_4$ are normal complete intersection projective surfaces, and $Y_1, \bar{Y}_1$ are smooth projective surfaces.*

Proof. In [vLu00], Lemma 3.2.15, it is proved that the box variety Y_4 is a normal complete intersection, and one can directly check that Y_1 is a smooth surface. Since η_i' (for $i = 1, 4$) are isomorphisms we obtain the result. □

Define $p : Y_4 \to Y_1$ by

$$p([x_0 : \ldots : x_6]) = [x_0 : x_1 : x_2]$$

This is a morphism from Y_4 to Y_1 because from the equations of Y_4 we see that the conditions $x_0 = x_2 = x_3 = 0$ imply that $x_i = 0$ for all $0 \leq i \leq 6$ (so p is defined) and the equation defining Y_1 is one of the equations defining Y_4 (thus p maps Y_4 to Y_1).

Similarly we define $\bar{p} : \bar{Y}_4 \to \bar{Y}_1$ as

$$\bar{p}([z_0 : \ldots : z_6]) = [z_0 : z_1 : z_2]$$

and similarly this is a morphism from $\bar{Y}_4$ to $\bar{Y}_1$. Moreover, from the definitions we see that

$$\eta_1' \circ p = \bar{p} \circ \eta_4'. \tag{6.2}$$

Lemma 6.6. *The morphisms $p, \bar{p}$ are finite of degree* 8.

Proof. By Equation (6.2) we only have to prove the result for p. Given a point $P = [a : b : c] \in Y_1$, we see that the preimages of P by p are given by

$$[a : b : c : \pm\sqrt{b^2 + c^2} : \pm\sqrt{a^2 + c^2} : \pm\sqrt{a^2 + b^2}]$$

which are at most 8 points, all of them in Y_4, and in general (more precisely, when the last 3 coordinates are non-zero), they are exactly 8. Therefore p is quasi finite and by Proposition 2.87 of degree 8, and since p is projective we obtain that it is finite. □

In $\mathbb{P}^4$ with homogeneous coordinates x, y, u, v, w we consider the scheme D defined by the equations

$$\begin{aligned} xy &= u^2 \\ x^2 - y^2 &= v^2 \\ x^2 + y^2 &= w^2. \end{aligned}$$

One sees that D has dimension 1 using the same method that we used in Chapter 5

for the surfaces $X_{n,k}$. The Jacobian matrix of D is

$$\begin{pmatrix} y & x & -2u & 0 & 0 \\ 2x & 2y & 0 & -2v & 0 \\ 2x & 2y & 0 & 0 & -2w \end{pmatrix}$$

and so it follows that D is smooth. As it is a smooth complete intersection, we deduce that D is a smooth irreducible curve with canonical sheaf $\mathcal{O}(6-4-1) = \mathcal{O}(1)$ and genus $\frac{1}{2}2^3 \cdot 1 + 1 = 5$ (cf. Section 2.6).

Following [Bea13], we let $G = (\mathbb{Z}/2\mathbb{Z})^3$ act on D by changing the sign of the coordinates u, v, w. This action is faithful, so we have a quotient map $\pi_+ : D \to G\backslash D$ of degree $\#G = 8$. (Note that Beauville writes Γ_+ instead of G.)

Lemma 6.7. *We have $G\backslash D = \mathbb{P}^1$ with quotient map given by the map*

$$\pi_D : [x : y : u : v : w] \mapsto [x : y].$$

Proof. We have that D is a smooth projective curve, thus $G\backslash D$ is again a smooth projective curve (cf. [KS97] Section 1.1).

As in the proof of Lemma 6.6 (computing pre-images) we see that π_D is a finite morphism of degree 8. Moreover, for all $\gamma \in G$ and $Q \in D$ we have $\pi_D(\gamma \cdot Q) = \pi_D(Q)$ because γ only changes the signs of the last three coordinates of Q, so by the universal property of the quotient $(G\backslash D, \pi_+)$ there is a map $t : G\backslash D \to \mathbb{P}^1$ such that $t \circ \pi_+ = \pi_D$.

Since $\deg(\pi_+) = 8 = \deg(\pi_D)$ we get that t is birational. Since $\mathbb{P}^1$ is a smooth projective curve, and $G\backslash D$ is projective, and $t : G\backslash D \to \mathbb{P}^1$ is birational, we get that t is an isomorphism. Since the pair $(G\backslash D, \pi_+)$ is unique up to isomorphism, we can

choose it so that it equals $(\mathbb{P}^1, \pi_D)$. □

Let $Y = D \times D \subseteq \mathbb{P}^4 \times \mathbb{P}^4$. Then Y is a smooth surface. (We denote the homogeneous coordinates in the first copy of $\mathbb{P}^4$ by x, y, u, v, w and those of the second by x', y', u', v', w'.)

Let $\Delta_G \cong G$ be the diagonal subgroup of $H := G \times G$, and let H act on $Y = D \times D$ as $(\gamma, \gamma') \cdot (Q, Q') := (\gamma \cdot Q, \gamma' \cdot Q')$. Then Δ_G has the diagonal action on Y. (These actions are faithful.) Consider the diagonal quotient surface $Z := \Delta_G \backslash Y$ and the surface $H \backslash Y$. Let $q_{\Delta_G} : Y \to Z$ and $q_H : Y \to H \backslash Y$ be the respective quotients. By Lemma 6.7 we have that

$$H \backslash Y = (G \backslash D) \times (G \backslash D) = \mathbb{P}^1 \times \mathbb{P}^1$$

with quotient map explicitly given by

$$q_H = \pi_D \times \pi_D : ([x : y : u : v : w], [x' : y' : u' : v' : w']) \mapsto ([x : y], [x', y']).$$

Note that the universal property of (Z, q_{Δ_G}) gives a unique map $q : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $q_{\Delta_G} \circ q = q_H$. From this and the theory of diagonal quotient surfaces (cf. Section 2.8) we deduce:

Lemma 6.8. *The surface Z is normal and projective. The morphisms*

$$q_{\Delta_G} : Y \to Z$$
$$q_H : Y \to \mathbb{P}^1 \times \mathbb{P}^1$$

are finite of degrees $\#\Delta_G = 8$*, and* $\#H = 64$*, respectively. The morphism*

$$q : Z \to \mathbb{P}^1 \times \mathbb{P}^1$$

has degree $8 = 64/8$*.*

Lemma 6.9. *Let* $\bar{p}_+ : D \times D \to \bar{Y}_4$ *be given by*

$$([x : y : u : v : w], [x' : y' : u' : v' : w']) \mapsto [xx' : xy' : yx' : yy' : uu' : vv' : ww']$$

Then $\bar{p}_+$ *defines a morphism from* $D \times D$ *to* $\bar{Y}_4$*.*

Proof. Given a point $([x : y : u : v : w], [x' : y' : u' : v' : w']) \in D \times D$, suppose that $xx' = 0, xy' = 0, yx' = 0, yy' = 0, uu' = 0, vv' = 0, ww' = 0$. Without loss of generality we suppose that $x = 0$. From the equations of D we get $u = 0$, $y = \pm iv$, $y = \pm w$, so none of y, v, w can be zero. Then $y' = v' = w' = 0$ and from the equations of D we get $x' = 0, u' = 0$, which is not possible. Therefore $\bar{p}_+$ is defined everywhere in $D \times D$, so it is a morphism. Substituting the formula for $\bar{p}_+$ in the equations of $\bar{Y}_+$ and using the equations defining D, we get that it maps $D \times D$ to $\bar{Y}_4$. □

By composition we obtain the morphism $\bar{p} \circ \bar{p}_+ : Y \to \bar{Y}_1$. It is given by

$$([x : y : u : v : w], [x' : y' : u' : v' : w']) \mapsto [xx' : xy' : yx' : yy'].$$

The following is the main result of this section. In particular, it shows that the box variety Y_4 is a diagonal quotient surface.

Proposition 6.10. *There exist unique morphisms $\varphi_1 : H\backslash Y = \mathbb{P}^1 \times \mathbb{P}^1 \to \bar{Y}_1$ and $\varphi_4 : Z \to \bar{Y}_4$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
Y & = & Y & & \\
{\scriptstyle q_{\Delta_G}}\downarrow & & \downarrow{\scriptstyle \bar{p}_+} & & \\
Z & \xrightarrow[\varphi_4]{} & \bar{Y}_4 & \xleftarrow[\eta_4']{} & Y_4 \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\varphi_1]{} & \bar{Y}_1 & \xleftarrow[\eta_1']{} & Y_1
\end{array}$$

Moreover, φ_1 is the Segre map

$$([x, y], [x', y']) \mapsto [xx' : xy' : yx' : yy'],$$

which is an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\bar{Y}_1$. Moreover, φ_4 is also an isomorphism.

Proof. Let $\gamma \in G$ and $(Q, Q') = ([x : y : u : v : w], [x' : y' : u' : v' : w']) \in Y = D \times D$. Recall that γ changes signs at most in u, v, w and u', v', w' (simultaneously, because we use the diagonal action of Δ_G). Then from the explicit formula defining $\bar{p}_+$ we see that

$$\bar{p}_+((\gamma Q, \gamma Q')) = \bar{p}_+((Q, Q')).$$

The existence and uniqueness of φ_4 follows from the universal property of the quotient (Z, q_{Δ_G}).

The existence and uniqueness of φ_1 is proved similarly, using the action of H and the formula for $\bar{p} \circ \bar{p}_+$.

All other parts of the diagram commute by the previous results in this section.

Now recall that the Segre morphism $s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ (cf. [Har77] Ex. I.2.14) is

given by

$$([x,y],[x',y']) \mapsto [xx' : xy' : yx' : yy']$$

and by the formulas of q_H and $\bar{p} \circ \bar{p}_+$ we have that $s \circ q_H = \bar{p} \circ \bar{p}_+$. By the uniqueness of φ_1 with respect to this property, we get $\varphi_1 = s$. In particular φ_1 is an isomorphism because the Segre map is an isomorphism with its image $\bar{Y}_1 = \{z_0 z_3 = z_1 z_2\}$.

Note that $\bar{p} \circ \bar{p}_+ = \varphi_1 \circ q_H$ is finite and surjective because q_H is finite and surjective, and because φ_1 is an isomorphism. We have that $\bar{p}_+$ has finite fibres because q_H is finite, therefore $\bar{p}_+$ is quasifinite. Since Y, $\bar{Y}_4$ and $\bar{Y}_1$ are irreducible of dimension 2, and the morphism $\bar{p}_+$ is quasifinite (in particular it is not constant), it follows that $\bar{p}_+$ is dominant, hence it is surjective because it is projective.

By Lemma 6.6 we have $\deg(\bar{p}) = 8$. Since φ_1 has degree 1 (because it is an isomorphism) and $\deg(\bar{p} \circ \bar{p}_+) = \deg(\varphi_1 q_H) = \deg(q_H) = 64$, we get that $\deg(\bar{p}_+) = 8$. Since $\bar{p}_+ = \varphi_4 q_{\Delta_G}$ and $\deg(q_{\Delta_G}) = 8$ we get that $\deg \varphi_4 = 1$ so φ_4 is a birational morphism.

The morphism φ_4 is surjective and has finite fibres because $\bar{p}_+$ is surjective and has finite fibres, because $\bar{p}_+ = \varphi_4 q_{\Delta_G}$ and because q_{Δ_G} is surjective. Since φ_4 is birational and quasifinite, and $\bar{Y}_4$ is normal, we obtain that φ_4 is an isomorphism by [Liu02] Corollary 4.4.3. □

6.3 Singularities of Y_4

Recall from the previous section that $Y_4 \cong Z = \Delta_G \backslash (D \times D)$ is a diagonal quotient surface. Let $\delta : \tilde{Y}_4 \to Y_4$ (respectively $\delta_Z : \tilde{Z} \to Z$) be the minimal desingularization of Y_4 (respectively, of Z).

See Subsection 2.2.6 for the notation about diagonal quotient surfaces.

Proposition 6.11. *If $y = (x_1, x_2) \in S_Y$, then $|G_y| = 2$.*

Proof. Since $y \in S_Y$ we know $|G_y| > 1$, hence both G_{x_1} and G_{x_2} are nontrivial. Since we are working in an algebraically closed field of characteristic zero, by [Ser79], p. 67, both G_{x_1}, G_{x_2} are cyclic and they are subgroups of $G \cong (\mathbb{Z}/2\mathbb{Z})^3$. Thus $|G_{x_1}| = |G_{x_2}| = |\mathbb{Z}/2\mathbb{Z}| = 2$. Therefore $|G_y| \leq 2$, and hence $|G_y| = 2$. □

Proposition 6.12. *Let $s \in S_Z$. Then $\delta_Z^{-1}(s) = E_s \cong \mathbb{P}^1$ and $E_s^2 = -2$.*

Proof. We have that s is a quotient singularity of type $A_{(n,q)}$, cf. [KS97], p. 16. From Theorem 2.3 (a) in [KS97] we know that $n = |G_y| = 2$ with $y = \phi^{-1}(s)$, hence $q = 1$ (because $1 \leq q < n$). Since s is of type $A_{(2,1)}$, we have by [BPV84] III (2.3)(ii) that the desingularization of s is irreducible and isomorphic to $\mathbb{P}^1$, and we also have $(E_s.E_s) = -2$ for E_s the exceptional divisor of s. □

Note that we also have $(E_s.E_{s'}) = 0$ for $s, s' \in S_Z$, when $s \neq s'$.

Given $\bar{x} \in \mathbb{P}^1 = G\backslash D$, denote by $\tilde{D}_{\bar{x},i}$ the strict transform of $D_{\bar{x},i}$ (cf. Proposition 2.94 for notation).

Proposition 6.13. *For $\bar{x} \in \mathbb{P}^1 = G\backslash D$, the curve $D_{\bar{x},i}$ is irreducible and smooth, and $\tilde{D}_{\bar{x},i} \cong D_{\bar{x},i}$ for $i = 1, 2$.*

Proof. This assertion comes from Proposition 2.94 and Proposition 2.1(b) in [KS97]. □

Proposition 6.14. *Let $s \in S_Z$, $\bar{x} \in \mathbb{P}^1 = G\backslash D$. Then*

$$(\tilde{D}_{\bar{x},1}.E_s) = \begin{cases} 1 & \text{if } s \in S_Z \cap \psi_1^{-1}(\bar{x}) \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{D}_{\bar{x},2}.E_s) \;=\; \begin{cases} 1 & \text{if } s \in S_Z \cap \psi_2^{-1}(\bar{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\psi_i = \mathrm{pr}_i \circ q : Z \to \mathbb{P}^1$, *and* $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ *is the* i*-th projection.*

Proof. We will consider $\tilde{D}_{\bar{x},1}$, the other case is similar. If $s \notin S_Z \cap \psi_1^{-1}(\bar{x})$, then $s \notin D_{\bar{x},1}$. Therefore $\tilde{D}_{\bar{x},1}$ and E_s do not meet. Hence $(\tilde{D}_{\bar{x},1}.E_s) = 0$. Now suppose that $s \in S_Z \cap \psi_1^{-1}(\bar{x}) \subseteq D_{\bar{x},1}$. Then by Proposition 2.104, since $D_{\bar{x},1} = \delta(\tilde{D}_{\bar{x},1})$ is smooth, we obtain that $(\tilde{D}_{\bar{x},1}.E_s) = 1$. □

Proposition 6.15. *The total transform* $\delta_Z^*(D_{\bar{x},i})$ *is equal to* $\tilde{D}_{\bar{x},i} + \sum_{s\in S_Z} m_{\bar{x},i,s}E_s$, *where* $m_{\bar{x},i,s} = \frac{1}{2}$ *when* $s \in D_{\bar{x},i}$, *and* $m_{\bar{x},i,s} = 0$ *otherwise.*

Proof. By Proposition 2.52 and Notation 2.51, we have $\delta_Z^* D_{\bar{x},i} = \tilde{D}_{\bar{x},i} + \sum_{s\in S_Z} m_s E_s$, where the $E_s \cong \mathbb{P}^1$ are the exceptional divisors obtained by desingularization of the 48 singularities of Y_4. The m_s can be computed by the equation

$$(\tilde{D}_{\bar{x},i}.E_s) + \sum_{t\in S_Z} m_s(E_t.E_s) = 0.$$

Since $(E_s.E_s) = -2$ and $(E_t.E_s) = 0$ for $E_t \neq E_s$, we have $(\tilde{D}_{\bar{x},i}.E_s) = 2m_s$.

From Proposition 6.14 we get that $m_s = \frac{1}{2}$ when $s \in D_{\bar{x},i}$, and $m_s = 0$ otherwise.

□

Proposition 6.16. *The genus of* $D_{\bar{x},i}$ *is* 5 *when* $\bar{x} = \pi_D(x)$ *and* $|G_x| = 1$. *The genus of* $D_{\bar{x},i}$ *is* 1 *when* $\bar{x} = \pi_D(x)$ *and* $|G_x| = 2$.

Proof. The case $|G_x| = 1$ comes from Proposition 2.94, because the genus of D is 5, see Section 6.2.

Now suppose that $|G_x| = 2$. By Proposition 2.94 it is sufficient to prove that $g(G_x\backslash D) = 1$. Let $G_x = \{\mathrm{Id}, \gamma\}$, with $\gamma \neq \mathrm{Id}$. Then $\gamma \cdot x = x$. Since $x \in D$ is fixed by a nontrivial element of G, it has to be of the form $x = [a : b : u_0 : v_0 : w_0]$ with at least one of u_0, v_0, w_0 equal to zero. From the equations of D we cannot have more than one of u_0, v_0, w_0 equal to zero, so $\gamma \in \{(1,0,0),(0,1,0),(0,0,1)\} \subset G$ (where $(i,j,k) \in G$ acts by $(i,j,k) \cdot [a : b : u : v : w] = [a : b : (-1)^i u : (-1)^j v : (-1)^k w]$). A verification with the equations of D shows that in these three cases γ has exactly 8 fixed points (for example, if $w = 0$ then either $x + iy = 0$ or $x - iy = 0$, and thus the 8 points $[-iy : y : \pm(-iy^2) : \pm((-iy)^2 - y^2) : 0]$, $[iy : y : \pm iy^2 : \pm((iy)^2 - y^2) : 0]$ are fixed by $(0,0,1)$). Since $D \to G_x\backslash D$ has degree $2 = |G_x|$, and 8 ramification points, the Riemann-Hurwitz formula gives $8 = 2(2g(G_x\backslash D) - 2) + 8$, thus $g(G_x\backslash D) = 1$. □

Proposition 6.17. *The set S_Z is the union of the following sets:*

$$\begin{aligned}
S_1 &= \{[0:0:1:\pm1:\pm1:\pm1:0]\} \\
S_2 &= \{[0:\pm i:1:0:0:\pm1:\pm i]\} \\
S_3 &= \{[0:1:0:\pm1:\pm1:0:\pm1]\} \\
S_4 &= \{[\pm i:0:1:0:\pm1:0:\pm i]\} \\
S_5 &= \{[\pm i:1:0:0:\pm1:\pm i:0]\} \\
S_6 &= \{[1:0:0:\pm1:0:\pm1:\pm1]\},
\end{aligned}$$

giving a total of 48 *points. These singularities are ordinary double points, the exceptional divisor of the desingularization of any of these singularities is isomorphic to* $\mathbb{P}^1$ *and has self-intersection number* -2.

Proof. That these points are the 48 singularities of Y_4 is shown in Corollary 3.2.3 in [vLu00]. The properties of these singularities come from Proposition 6.12. □

6.4 Ramification curves

In addition to Y_4, define Y_3 (in $\mathbb{P}^5$) by

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 &= x_3^2 \\ x_1^2 + x_2^2 &= x_4^2 \\ x_0^2 + x_2^2 &= x_5^2, \end{aligned}$$

define Y_2 (in $\mathbb{P}^4$) by

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 &= x_3^2 \\ x_1^2 + x_2^2 &= x_4^2, \end{aligned}$$

and recall that Y_1 is given by

$$x_0^2 + x_1^2 + x_2^2 = x_3^2.$$

Lemma 6.18. *The surfaces $Y_1, \ldots, Y_4$ are normal.*

Proof. We know that Y_1 and Y_4 are normal by Lemma 6.5. We prove that Y_2, Y_3 are normal by using *Magma*, with the codes:

```
P4<x,y,z,w,t> := ProjectiveSpace(Rationals(),4);
X := Surface(P4,[x^2+y^2+z^2-w^2, y^2+z^2-t^2]);
X;
IsNormal(X);
```

giving as result

```
Surface over Rational Field defined by
x^2 + y^2 + z^2 - w^2,
y^2 + z^2 - t^2
true
```

in less than 3 seconds, and

```
P5<x,y,z,w,s,t> := ProjectiveSpace(Rationals(),5);
X := Surface(P5,[x^2+y^2+z^2-w^2, y^2+z^2-s^2,x^2+z^2-t^2]);
IsNormal(X);
```

giving as result

```
Surface over Rational Field defined by
x^2 + y^2 + z^2 - w^2,
y^2 + z^2 - s^2,
x^2 + z^2 - t^2
true
```

for the surface Y_3. □

Lemma 6.19. *For each $1 \leq i \leq 3$, the map $\pi_i : Y_{i+1} \to Y_i$ defined by the restriction to Y_{i+1} of the morphism*

$$\begin{array}{rcl} \tilde{\pi}_i : \mathbb{P}^{i+3} \setminus \{[0 : \cdots : 0 : 1]\} & \to & \mathbb{P}^{i+2} \\ [x_0 : \cdots : x_{i+3}] & \mapsto & [x_0 : \cdots : x_{i+2}] \end{array}$$

is a finite surjective morphism. Thus, each Y_i is a normal algebraic variety and $\deg(\pi_i) = 2$.

Proof. First note that $[0 : \cdots : 0 : 1] \notin Y_{i+1}$. Indeed, suppose that for a point in $Y_{i+1} \subseteq \mathbb{P}^{i+2}$ we have $x_0 = \cdots = x_{i+2} = 0$. Then from the equations of Y_i we obtain that $x_{i+3} = 0$, which contradicts the fact that $[x_0 : \cdots : x_{i+3}] \in \mathbb{P}^{i+3}$. Therefore $[0 : \cdots : 0 : 1] \notin Y_{i+1}$. Hence $\pi_i := (\tilde{\pi}_i)_{|Y_{i+1}}$ is defined. We have $\pi_i(Y_{i+1}) \subseteq Y_i$ because the equations of Y_i also hold in Y_{i+1}.

Let $P = [x_0 : \cdots : x_{i+2}] \in Y_i$. Then $[x_0 : \ldots : x_{i+3}] \in \pi_i^{-1}(P)$ if and only if $x_{i+3}^2 = x_a^2 + x_b^2$. If $\pm x_{i+3} \neq 0$ (which is the general case), then the points having P as image under π_i are $[x_0 : \cdots : x_{i+2} : \pm x_{i+3}]$ of which there are exactly two. If $x_{i+3} = 0$ then the only point having P as image under π_i is $[x_0 : \cdots : x_{i+2} : 0]$. Thus, π_i is a surjective quasi-finite projective morphism, hence is a finite surjective morphism.

Since Y_4 is irreducible by Lemma 6.5, it follows that $Y_i = \pi_i \circ \cdots \circ \pi_3(Y_4)$ $(1 \leq i \leq 3)$ is also irreducible, and so by Lemma 6.18, Y_i is a normal algebraic variety. The above discussion and Proposition 2.87 shows that $\deg(\pi_i) = 2$. □

Proposition 6.20. *For every $1 \leq i \leq 4$, Y_i is a normal complete intersection surface.*

Proof. First of all, we know that Y_1 is a surface because it is a hypersurface in $\mathbb{P}^3$. Since the morphisms π_i are surjective and finite we obtain by [Bou83], p. AC VIII.17

that Y_2, Y_3 and Y_4 are also surface. Counting the number of equations defining each Y_i we get that Y_i is a complete intersection. These surfaces are normal by Lemma 6.18. □

We have a tower of morphisms of surfaces

$$Y_1 \overset{\pi_1}{\leftarrow} Y_2 \overset{\pi_2}{\leftarrow} Y_3 \overset{\pi_3}{\leftarrow} Y_4.$$

Notation 6.21. Define $\rho_i = \pi_1 \circ \cdots \circ \pi_i$, with $1 \leq i \leq 3$. Observe that $\rho_3 = p$ (cf. Section 6.2).

Proposition 6.22. *The morphism π_i is unramified outside $R_i := \mathrm{div}_{Y_{i+1}}(x_{i+3})$ and it is totally ramified over the components of R_i.*

Proof. From the proof of Lemma 6.19, we know that if $P \in Y_{i+1} \backslash R_i$, then

$$\#\pi_i^{-1}(\pi_i(P)) = 2 = \deg(\pi_i).$$

Thus by Proposition 2.89 π_i is unramified on any curve intersecting $Y_{i+1} \backslash R_i$. Since $\#\pi_i^{-1}(\pi_i(P)) = 1$ for $P \in R_i$, we obtain from Proposition 2.88 that π_i is totally ramified on the components of R_i. □

Proposition 6.23. *We have $\rho_i(R_i) = \mathrm{supp}(C_i)$ for $1 \leq i \leq 3$, where*

$$\begin{aligned} C_1 &= \mathrm{div}_{Y_1}(x_1^2 + x_2^2), \\ C_2 &= \mathrm{div}_{Y_1}(x_0^2 + x_2^2), \\ C_3 &= \mathrm{div}_{Y_1}(x_0^2 + x_1^2). \end{aligned}$$

Proof. This holds because we know that in Y_4 we have $x_6^2 = x_0^2 + x_1^2$, in Y_3 we have $x_5^2 = x_0^2 + x_2^2$, and in Y_2 we have $x_4^2 = x_1^2 + x_2^2$. □

Proposition 6.24. *We have* $\rho_i^*(C_i) = 2R_i$.

Proof. We will prove the proposition for $i = 3$. The proof of the other cases is similar. Recall that $C_3 = \mathrm{div}_{Y_1}(x_0^2 + x_1^2)$. Since $Y_i \not\subseteq \mathbb{V}_{\mathbb{P}^{i+2}}(x_0^2 + x_1^2)$, we have by Proposition 2.30 that

$$\rho_3^*\mathrm{div}_{Y_1}(x_0^2 + x_1^2) = \mathrm{div}_{Y_4}(x_0^2 + x_1^2).$$

Since $x_0^2 + x_1^2 = x_6^2$ on Y_4, we have

$$\mathrm{div}_{Y_4}(x_0^2 + x_1^2) = \mathrm{div}_{Y_4}(x_6^2) = 2\mathrm{div}_{Y_4}(x_6).$$

Recalling that $R_3 = \mathrm{div}_{Y_4}(x_6)$ we obtain that $\rho_3^*(C_3) = 2R_3$. □

Notation 6.25. We denote $B_i = \mathrm{div}_{Y_4}(x_{i+3})$ for $i = 1, 2, 3$. Thus $R_3 = B_3$.

Proposition 6.26. *For* $1 \leq i \leq 3$, *we have* $(\rho_3)^*(C_i) = 2B_i$.

Proof. From Proposition 6.24 we know that $\rho_i^*(C_i) = 2R_i$. For $i = 3$ we are done since $B_3 = R_3$. By Proposition 2.30 and using the definition of R_i we have now that

$$\rho_3^*(C_2) = \pi_3^*\rho_2^*(C_2) = \pi_3^*(2R_2) = 2\mathrm{div}_{Y_4}(x_5) = 2B_2$$

$$\rho_3^*(C_1) = \pi_3^*\pi_2^*(2R_1) = 2\mathrm{div}_{Y_4}(x_4) = 2B_1.$$

□

Recall from Section 6.2 that the change of variables $\eta_1 : \mathbb{P}^3 \to \mathbb{P}^3$ is defined by the

following equations

$$\begin{aligned} z_0 &= x_0 + ix_1, \\ z_1 &= x_3 + x_2, \\ z_2 &= x_3 - x_2, \\ z_3 &= x_0 - ix_1, \end{aligned} \tag{6.3}$$

which induces an isomorphism $\eta_1' : Y_1 \to \bar{Y}_1$. For simplicity we now write $\eta := \eta_1'$.

The inverse of η is given by

$$\begin{aligned} x_0 &= \frac{z_0 + z_3}{2} \\ x_1 &= \frac{z_0 - z_3}{2i} \\ x_2 &= \frac{z_1 - z_2}{2} \\ x_3 &= \frac{z_1 + z_2}{2}, \end{aligned}$$

hence the branch curves C_1, C_2, C_3 in $Y_1 \subseteq \mathbb{P}^3$ are mapped to

$$\begin{aligned} \eta C_1 &: (z_1 - z_2)^2 - (z_0 - z_3)^2 = 0 \\ \eta C_2 &: (z_0 + z_3)^2 + (z_1 - z_2)^2 = 0 \\ \eta C_3 &: z_0 z_3 = 0 \end{aligned}$$

in $\bar{Y}_1$.

We denote by $\sigma : \bar{Y}_1 \to \mathbb{P}^1 \times \mathbb{P}^1$ the inverse of the Segre isomorphism

$$\bar{\phi}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \bar{Y}_1$$

(cf. Proposition 6.10).

Proposition 6.27. *The images of C_1, C_2, C_3 by $\sigma \circ \eta$ are the following divisors in $\mathbb{P}^1 \times \mathbb{P}^1$:*

$$\begin{aligned} V_1 &:= \mathbb{P}^1 \times [1:-1] + \mathbb{P}^1 \times [1:1] + [1:-1] \times \mathbb{P}^1 + [1:1] \times \mathbb{P}^1 \\ V_2 &:= \mathbb{P}^1 \times [1:i] + \mathbb{P}^1 \times [1:-i] + [1:i] \times \mathbb{P}^1 + [1:-i] \times \mathbb{P}^1 \\ V_3 &:= [0:1] \times \mathbb{P}^1 + \mathbb{P}^1 \times [0:1] + [1:0] \times \mathbb{P}^1 + \mathbb{P}^1 \times [1:0]. \end{aligned}$$

Proof. The images of the curves $\eta(C_1)$, $\eta(C_2)$, $\eta(C_3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ under σ are (with U_0, U_1 coordinates on the first copy of $\mathbb{P}^1$, and W_0, W_1 coordinates on the second copy of $\mathbb{P}^1$):

$$\begin{aligned} V_1 &: (U_0W_1)^2 + (U_1W_0)^2 - (U_0W_0)^2 - (U_1W_1)^2 = 0 \\ V_2 &: (U_0W_0)^2 + (U_1W_1)^2 + (U_0W_1)^2 + (U_1W_0)^2 = 0 \\ V_3 &: U_0W_0U_1W_1 = 0. \end{aligned}$$

Factoring these equations we get

$$\begin{aligned} V_1 &: (W_0 + W_1)(W_1 - W_0)(U_0 + U_1)(U_0 - U_1) = 0 \\ V_2 &: (W_0 + iW_1)(W_0 - iW_1)(U_0 + iU_1)(U_0 - iU_1) = 0 \\ V_3 &: U_0W_0U_1W_1 = 0. \end{aligned}$$

Thus the irreducible components of V_1, V_2 and V_3 have the form $\{p\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{p\}$ with $p \in \mathbb{P}^1$, explicitly,

$$\begin{aligned} V_1 &= \mathbb{P}^1 \times [1:-1] + \mathbb{P}^1 \times [1:1] + [1:-1] \times \mathbb{P}^1 + [1:1] \times \mathbb{P}^1 \\ V_2 &= \mathbb{P}^1 \times [1:i] + \mathbb{P}^1 \times [1:-i] + [1:i] \times \mathbb{P}^1 + [1:-i] \times \mathbb{P}^1 \\ V_3 &= [0:1] \times \mathbb{P}^1 + \mathbb{P}^1 \times [0:1] + [1:0] \times \mathbb{P}^1 + \mathbb{P}^1 \times [1:0]. \end{aligned}$$

□

Corollary 6.28. *The curves $C_i \subseteq \mathbb{P}^3$ are reduced. Each C_i is the union of 4 irreducible components. No two C_i's share a common irreducible component.*

Proof. From Proposition 6.27 we know that each V_i is reduced and it has 4 irreducible components. Moreover, V_i and V_j do not share common components when $i \neq j$. Since V_i is the image of C_i under the isomorphism $\sigma \circ \eta$, we obtain the result. □

Proposition 6.29. *For each $1 \leq i \leq 3$ the divisor B_i of Notation 6.25 is reduced.*

Proof. We will prove the case $i = 2$; the other cases are proved similarly. From Proposition 6.22 we have that π_1 is only ramified along R_1, hence it is only branched along C_1 and it is not branched along any irreducible component of C_2. Since C_2 is reduced, we get that $\pi_1^*(C_2)$ is reduced. By Proposition 2.86, we get that the pullback under π_2 of each irreducible component of $\pi_1^*(C_2)$ is either twice an irreducible curve or the sum of two distinct irreducible curves. By Proposition 6.24 we have that $\pi_2^* \circ \pi_1^*(C_2) = 2R_2$, hence the pullback of each irreducible component of C_2 is twice an irreducible component of R_2. Since π_2 is of degree two and ramified along each irreducible component of $\pi_1^*(C_2)$, it follows that R_2 is reduced. We have $2B_2 = \rho_3^*(C_2) = \pi_3^*(2R_2) = 2\pi_3^*(R_2)$. Since π_3 only ramifies along the irreducible

components of R_3, and R_3, $\pi_3^*(R_2)$ do not share components (since $\rho_3(R_3) = C_3$, $\rho_2(R_2) = C_2$ do not have common components), we obtain that B_2 is reduced. □

Including η and σ, we now have the sequence of morphisms

$$\mathbb{P}^1 \times \mathbb{P}^1 \overset{\sigma}{\leftarrow} \bar{Y}_1 \overset{\eta}{\leftarrow} Y_1 \overset{\pi_1}{\leftarrow} Y_2 \overset{\pi_2}{\leftarrow} Y_3 \overset{\pi_3}{\leftarrow} Y_4.$$

We denote the composition of these maps by $\Psi : Y_4 \to \mathbb{P}^1 \times \mathbb{P}^1$. Note that by Proposition 6.26 and Proposition 6.27, we have $\Psi(B_i) = \sigma \circ \eta(C_i) = V_i$.

Proposition 6.30. *Let D' be an irreducible component of V_i. Then $(\Psi^*(D'))_{\mathrm{red}}$ is irreducible and smooth.*

Proof. We obtain this from Proposition 2.94 and Proposition 6.10 by observing that $\rho_3 = p$, because an irreducible component of V_i is the image of C_i under $\sigma \circ \eta$. □

Proposition 6.31. *The divisors B_1, B_2 and B_3 in Y_4 have exactly 4 distinct irreducible smooth components.*

Proof. Recall that $\Psi(B_i) = V_i$ and that each B_i is reduced. The irreducible components of the curves $V_i \in \mathbb{P}^1 \times \mathbb{P}^1$ are of the form $\{p\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{q\}$. From Proposition 6.30 applied to each irreducible component of V_i, we obtain that for $1 \leq i \leq 3$, the divisor B_i contains 4 distinct irreducible components and they are smooth. □

6.5 Image of the singularities under Ψ

Notation 6.32. Let $[x : y] \in \mathbb{P}^1$. If $\mathbb{P}^1 \times [x : y] \in V_i$, then we write $a_{i,[x:y]} = \mathbb{P}^1 \times [x : y]$ and similarly we write $b_{i,[x:y]} = [x : y] \times \mathbb{P}^1$ in the case that $[x : y] \times \mathbb{P}^1 \in V_i$.

Lemma 6.33. *The image of the* 48 *singular points of* Y_4 *under* Ψ *are the* 12 *points* $a_{i,P} \cap b_{i,P}$ *with* $i \in \{1,2,3\}$ *and* $P, Q \in \mathbb{P}^1$ *such that* $a_{i,P} \cup b_{i,Q} \subseteq V_i$.

Proof. The image under ρ_3 of the 48 singular points of Y_4 (see Proposition 6.17) are the 12 points in Y_1 (where $s_i := \rho_3(S_i)$):

$$\begin{aligned} s_1 &= \{[0:0:1:\pm 1]\} \\ s_2 &= \{[0:\pm i:1:0]\} \\ s_3 &= \{[0:1:0:\pm 1]\} \\ s_4 &= \{[\pm i:0:1:0]\} \\ s_5 &= \{[\pm i:1:0:0]\} \\ s_6 &= \{[1:0:0:\pm 1]\}. \end{aligned}$$

where the points in S_i are mapped to points in s_i. Their images under η are

$$\begin{aligned} \eta(s_1) &= \{[0:1:0:0],[0:0:1:0]\} \\ \eta(s_2) &= \{[1:1:-1:-1],[1:-1:1:-1]\} \\ \eta(s_3) &= \{[1:-i:-i:-1],[1:i:i:-1]\} \\ \eta(s_4) &= \{[1:-i:i:1],[1:i:-i:1]\} \\ \eta(s_5) &= \{[1:0:0:0],[0:0:0:1]\} \\ \eta(s_6) &= \{[1:1:1:1],[1:-1:-1:1]\} \end{aligned}$$

Then applying σ we get the 12 points (in $\mathbb{P}^1 \times \mathbb{P}^1$)

$$\begin{aligned} ([1:0],[0:1]) &= a_{3,[1:0]} \cap b_{3,[0:1]} \\ ([0:1],[1:0]) &= a_{3,[0:1]} \cap b_{3,[1:0]} \end{aligned}$$

$$\begin{aligned}
([1:-1],[1:1]) &= a_{1,[1:-1]} \cap b_{1,[1:1]} \\
([1:1],[1:-1]) &= a_{1,[1:1]} \cap b_{1,[1:-1]} \\
([1:-i],[1:-i]) &= a_{2,[1:-i]} \cap b_{2,[1:-i]} \\
([1:i],[1:i]) &= a_{2,[1:i]} \cap b_{2,[1:i]} \\
([1:i],[1:-i]) &= a_{2,[1:i]} \cap b_{2,[1:-i]} \\
([1:-i],[1:i]) &= a_{2,[1:-i]} \cap b_{2,[1:i]} \\
([1:0],[1:0]) &= a_{3,[1:0]} \cap b_{3,[1:0]} \\
([0:1],[0:1]) &= a_{3,[0:1]} \cap b_{3,[0:1]} \\
([1:1],[1:1]) &= a_{1,[1:1]} \cap b_{1,[1:1]} \\
([1:-1],[1:-1]) &= a_{1,[1:-1]} \cap b_{1,[1:-1]}.
\end{aligned}$$

Noting that there are 12 intersections of this type, and they intersect at the 12 images in Y_1 of the singularities, we get that all these intersections are images of singularities of Y_4. □

Remark 6.34. From the proof of Lemma 6.33, we also see that every V_i contains exactly 4 of these 12 points. Moreover, these 4 points which are contained in V_i are contained in $a_{i,P_1} \cup a_{i,P_2}$, and also in $b_{i,P_1} \cup b_{i,P_2}$, when $P_1, P_2 \in \mathbb{P}^1$ are such that

$$a_{i,P_1} \cup a_{i,P_2} \cup b_{i,P_1} \cup b_{i,P_2} = V_i.$$

Proposition 6.35. *We have*

$$\begin{aligned}
S_2 \cup S_6 &\subseteq B_1 \\
S_3 \cup S_4 &\subseteq B_2 \\
S_1 \cup S_6 &\subseteq B_3.
\end{aligned}$$

Moreover, each of these singular points is contained in 2 *irreducible components of the corresponding* B_i.

Proof. It follows from Proposition 6.17 that the singularities of S_2 and S_6 in Y_4 are contained in $B_1 = \mathrm{div}_{Y_4}(x_4)$, the singularities of S_3 and S_4 are contained in $B_2 = \mathrm{div}_{Y_4}(x_5)$ and the singularities of S_1 and S_5 are contained in $B_3 = \mathrm{div}_{Y_4}(x_6)$.

From Proposition 6.30 we obtain that the preimage of an irreducible component of V_i is an irreducible component of $\Psi^*(V_i) = 2B_i$. Take a singular point in Y_4 contained in B_i. From Lemma 6.33, its image is in V_i and it is contained in a component a_{ik} and in one b_{il} in V_i. From Proposition 6.31 we get that the singular point in Y_4 is contained in 2 irreducible components of B_i. □

Note that from the definition of B_i, the irreducible components of $B_1 + B_2 + B_3$ are the elliptic curves of type (b) found by van Luijk (see Section 2.1.5).

6.6 ω-integral curves on $\mathbb{P}^1 \times \mathbb{P}^1$

We are interested in obtaining more information about the curves of genus 0 or 1 in Y_4 by using ω-integral curves. Define the open sets $X_i \times Z_i \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ where

$$X_i = \left\{ [U_0 : U_1] \in \mathbb{P}^1 : U_i \neq 0 \right\},$$

and

$$Z_i = \left\{ [W_0 : W_1] \in \mathbb{P}^1 : W_i \neq 0 \right\}.$$

Let $u_0 := \frac{U_0}{U_1}$ and $u_1 := \frac{U_1}{U_0}$ so $u_0 = u_1^{-1}$, and similarly define w_0, w_1. Consider the following symmetric differential on $X_0 \times Z_0$:

$$du_1 dw_1.$$

Proposition 6.36. *The symmetric differential $du_1 dw_1$ in $X_0 \times Z_0$ extends to a symmetric differential*

$$\omega \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2) \otimes S^2\Omega^1_{\mathbb{P}^1 \times \mathbb{P}^1}),$$

where $\mathcal{O}(a,b) = p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes_{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}} p_2^*\mathcal{O}_{\mathbb{P}^1}(b)$ and $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the i-th component, for $i = 1, 2$.

Proof. In the open set $X_1 \times Z_1$ with affine coordinates u_0, w_0, the symmetric differential form becomes

$$\frac{1}{u_0^2 w_0^2} du_0 dw_0.$$

Similarly, in the open set $X_0 \times Z_1$ with affine coordinates u_1, w_0 (respectively, in $X_1 \times Z_0$ with u_0, w_1) this section becomes $\frac{1}{w_1^2} du_1 dw_0$ (respectively $\frac{1}{u_0^2} du_0 dw_1$). Therefore $du_1 dw_1$ extends to a differential (tensoring by $u_0^2 w_0^2$)

$$\omega \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2) \otimes S^2\Omega^1_{\mathbb{P}^1 \times \mathbb{P}^1}).$$

□

Theorem 6.37. *The ω-integral curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are exactly the ones given by the equations: $aU_0 + bU_1 = 0$, $cW_0 + dW_1 = 0$ (seen as bihomogeneous polynomials in $k[U_0, U_1; W_0, W_1]$ of bidegrees $(1,0)$ and $(0,1)$ respectively). Hence the ω-integral curves are of the form*

(i) $\{P\} \times \mathbb{P}^1$, *with* $P \in \mathbb{P}^1$;

(ii) $\mathbb{P}^1 \times \{P\}$ *with* $P \in \mathbb{P}^1$.

Proof. If $\{P\} \times \mathbb{P}^1$ is a curve of type (i) with $P \neq [0:1]$, then it is given by the formula $aU_0 + bU_1 = 0$ with $b \neq 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$. In $X_0 \times Z_0$ this formula can be written as $a + bu_1 = 0$ with $b \neq 0$. Differentiating we get $bdu_1 = 0$, thus $du_1 dw_1 = 0$ on this curve, hence by Corollary 3.72 we obtain that $\{P\} \times \mathbb{P}^1$ is ω-integral. Similarly, it can be proved that any curve of type (ii) with $P \neq [0:1]$ is ω-integral. We can check that $\{[0:1]\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{[0:1]\}$ are ω-integral by using Corollary 3.72 and the restriction of ω to the open set $X_1 \times Z_1$. Therefore the curves of type (i) and (ii) are ω-integral. Observe that exactly 2 of these ω-integral curves pass through each point on $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we will prove that there are no more ω-integral curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the open affine subset $U := X_0 \times Z_0 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

In our case $A_0 = 0$, so results of Section 3.7 do not apply directly here. Instead, consider the isomorphism

$$\begin{aligned} \theta : \mathbb{A}^2 &\to U \\ (u_0, u_1) &\mapsto (u_0 + u_1, u_0 - u_1). \end{aligned}$$

By Theorem 3.35, a curve $C \subset \mathbb{A}^2$ is $\theta^*\omega$-integral if and only if $\theta(C)$ is ω-integral. We get

$$\theta^*(\omega) = d(u_0 + u_1)d(u_0 - u_1) = 1(du_0)^2 + 0du_0du_1 - 1(du_1)^2$$

giving (in the notation of Section 3.7) $\delta = 4$ and $A_0 = 1$ (they never vanish). Therefore

at most two $\theta^*\omega$-integral curves pass through each point of U.

For $a, b \in \mathbb{C}$, the curve $2a + b(u_0 - u_1) = 0$ in $\mathbb{A}^2$ has image the curve $a + bu_1 = 0$ in U. Similarly, a curve $a(u_0 + u_1) + 2b = 0$ in $\mathbb{A}^2$ has image the curve $au_0 + b = 0$. The curves $2a + b(u_0 - u_1) = 0$ and $a(u_0 + u_1) + 2b = 0$ are $\theta^*\omega$-integral. When we choose a point $P = (u_0, u_1) \in \mathbb{A}^2$, we obtain a pair $(a, b) \in \mathbb{C}$ such that the curves $2a + b(u_0 - u_1) = 0$ and $a(u_0 + u_1) + 2b = 0$ pass through P.

Since at most two $\theta^*\omega$-integral curves pass through each point P, we get that there are no more $\theta^*\omega$-integral curves. Using the bijection between ω-integral curves and $\theta^*\omega$-integral curves, this proves the theorem. □

6.7 ω_4-integral curves in Y_4

Let $\delta : \tilde{Y}_4 \to Y_4$ be the desingularization of Y_4. Let $\tilde{\Psi} = \Psi \circ \delta$. We want to find *all* $\Psi^*\omega$-integral curves in $\tilde{Y}_4$.

Recall from Proposition 6.27 that the images in $\mathbb{P}^1 \times \mathbb{P}^1$ under $\sigma \circ \eta$ of the curves $C_1, C_2, C_3 \subset Y_1$ are

$$\begin{aligned} V_1 &: (v_0 + v_1)(v_1 - v_0)(u_0 + u_1)(u_0 - u_1) = 0 \\ V_2 &: (v_0 + iv_1)(v_0 - iv_1)(u_0 + iu_1)(u_0 - iu_1) = 0 \\ V_3 &: u_0v_0u_1v_1 = 0. \end{aligned}$$

Lemma 6.38. *Let $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of type (i) or (ii) of Theorem 6.37, which is not an irreducible component of $V_1 \cup V_2 \cup V_3$. Then the pull-back of C under Ψ does not contain any singular point of Y_4.*

Proof. Let C be a curve of type (i) (respectively (ii)). Since it is not an irreducible

component of $V_1 \cup V_2 \cup V_3$ we get that it does not intersect any component of type (i) (resp, type (ii)) of $V_1 \cup V_2 \cup V_3$. By Lemma 6.33 we obtain that the pull-back of C under Ψ does not contain any singular point of Y_4. □

Let

$$\omega_4 := \Psi^{\bullet}_{2,\mathcal{O}(2,2)}\omega \in H^0(\tilde{Y}_4, \Psi^*\mathcal{O}(2,2) \otimes S^2\Omega^1_{\tilde{Y}_4}).$$

Since $\sigma^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2) = \mathcal{O}_{\bar{Y}_1}(2)$ by [Har77] Example II.6.6.2, and $\rho_n^*\mathcal{O}_{Y_1}(2) = \mathcal{O}_{Y_4}(2)$ by Proposition 2.30, we see that

$$\Psi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2) = \mathcal{O}_{Y_4}(2).$$

We want to explicitly find all the ω_4-integral curves in $\tilde{Y}_4$. We will not prove that exceptional divisors obtained by the desingularization of Y_4 are ω_4 integral, since the image of these curves in Y_4 are points.

Theorem 6.39. *With ω_4 on $\tilde{Y}_4$ defined as before, the following curves on $\tilde{Y}_4$ are all the ω_4-integral curves on $\tilde{Y}_4$ which are not exceptional divisors.*

(a) The pull-backs (under $\tilde{\Psi}$) of curves of type (i), not contained in $V_1 \cup V_2 \cup V_3$;

(a') The strict transforms of the preimages (under Ψ) of an irreducible component of $V_1 \cup V_2 \cup V_3$ of type (i);

(b) The pull-backs (under $\tilde{\Psi}$) of curves of type (ii), not contained in $V_1 \cup V_2 \cup V_3$;

(b') The strict transforms of preimages (under Ψ) of an irreducible component of $V_1 \cup V_2 \cup V_3$ of type (ii).

The curves of type (a) and (b) have genus 5 *and do not contain singular points of* Y_4*. The curves of type (a') and (b') have genus* 1 *and any of these curves contains* 4 *singular points of* Y_4*.*

Proof. Let $C \in \tilde{Y}_4$ be an ω_4-integral curve which is not an exceptional divisor (in particular C is irreducible). By Theorem 3.35, we have that the curve $C_0 = \tilde{\Psi}(C)$ must be ω-integral, hence C_0 is a curve of type (i) or (ii) of Theorem 6.37.

Let C be of type (a) or (b). By Proposition 6.13, C is a smooth irreducible curve isomorphic to a curve $D_{\bar{x},i}$, with $\bar{x} \in G\backslash D$. Therefore C is the pull-back of C_0. Since C does not contain any singular point of Y_4 we get $|G_x| = 1$ and by Proposition 6.16 that $g_C = 5$. Let C be of type (a') or (b'). By Proposition 6.13, C is a smooth irreducible curve isomorphic to a curve $D_{\bar{x},i}$, with $\bar{x} \in G\backslash D$. Therefore C is the preimage of C_0. By Proposition 6.13 C is isomorphic to a curve $D_{\bar{x},i}$, with $\bar{x} \in \mathbb{P}^1 = G\backslash D$, and since C contains a singular point of Y_4 we get by Proposition 6.16 that $g_C = 1$.

The curves of type (a) and (b) do not contain singular points of Y_4 because the curves of type (i) are not contained in $V_1 \cup V_2 \cup V_3$ (hence they do not contain the image of a singular point of Y_4). A curve of type (a') or (b') contains 8 singular points of Y_4 because it contains the bijective image of 8 fixed points from S_Y (cf. the proof of Proposition 6.16). □

6.8 Curves of low genus on Y_4 and main inequality

Recall that $\delta : \tilde{Y}_4 \to Y_4$ is the minimal desingularization of Y_4 (in this case it is exactly the blow-up of the 48 ordinary double points of Y_4 by [vLu00], Lemma 3.2.8).

Notation 6.40. From now on, we write $B = B_1 + B_2 + B_3$ and denote its strict transform by $\tilde{B}$. Thus $\tilde{B}$ is the sum of the strict transforms of the irreducible components

of $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$.

Remark 6.41. From Proposition 6.29, we know that each B_i is reduced. Since the images of B_1, B_2, B_3 under ρ_3 are C_1, C_2, C_3, and no two C_i share common components, we obtain that B is reduced. Since B is reduced, we have that $\tilde{B}$ is reduced.

Proposition 6.42. *The section* $\omega_4 = \tilde{\Psi}^{\bullet}_{r,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)}\omega \in H^0(\tilde{Y}_4, \delta^*\mathcal{O}(2)\otimes S^2\Omega^1_{\tilde{Y}_4})$ *vanishes identically on* $\tilde{B}$.

Proof. We know from Proposition 6.26 that Ψ ramifies on B_1, B_2 and B_3. Since δ is an isomorphism outside the singularities of Y_4 (which are contained in B) we get that $\tilde{\Psi}$ ramifies on the total transforms of B_1, B_2 and B_3. In particular it ramifies on the irreducible components of $\tilde{B}_1 + \tilde{B}_2 + \tilde{B}_3$. Since the strict transforms of the irreducible components of B_1, B_2, B_3 are ω_4-integral we get from Proposition 3.87 (applied to $\tilde{\Psi}$) that ω_4 vanishes identically along each component of B_i, and from Proposition 3.79 we obtain that ω_4 vanishes identically on B. □

Corollary 6.43. *The section* ω_4 *gives rise to a unique section*

$$\omega_4' \in H^0(\tilde{Y}_4, \mathcal{L}(-\tilde{B})\otimes\delta^*\mathcal{O}(2)\otimes S^2\Omega^1_{\tilde{Y}_4}).$$

Moreover, the ω_4'*-integral curves are also* ω_4*-integral.*

Proof. By Proposition 6.42 we have that ω_4 vanishes identically on $\tilde{B}$. By Proposition 2.25, the section ω_4 defines a unique section ω_4' in $H^0(\tilde{Y}_4, \mathcal{L}(-\tilde{B})\otimes\tilde{\Psi}^*\mathcal{O}(2,2)\otimes S^2\Omega^1_{\tilde{Y}_4})$. From Proposition 3.88 we obtain that the ω_4'-integral curves are also ω_4-integral. □

Let C be a curve in Y_4. To determine whether its strict transform C' in $\tilde{Y}_4$ is

ω_4'-integral, we have to check that $\varphi^{\bullet}_{C',r,\mathcal{L}(-\tilde{B})\otimes\delta^*\mathcal{O}(2)}\omega_4'$ is zero in

$$H^0(\tilde{C}, \varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}}),$$

where $\varphi_{C'} : \tilde{C} \to C'$ is the desingularization of C'. For this, it suffices to show that

$$\deg_{\tilde{C}}(\varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}}) < 0,$$

because then $H^0(\tilde{C}, \varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}}) = 0$. We now give conditions for making this degree negative.

Lemma 6.44. *We have that* $\delta^*B = \tilde{B} + \sum_{s\in S_Z} E_s$.

Proof. From Remark 6.34 we know that the 4 singular points contained in V_i are also contained in $a_{i,P_1} \cup a_{i,P_2}$, hence $\Psi^*(a_{i,P_2}) \cup \Psi^*(a_{i,P_2})$ contains the 16 singular points of Y_4 that are preimages of these 4 points. Similarly, $\Psi^*(b_{i,P_1}) \cup \Psi^*(b_{i,P_2})$ also contains these 16 points.

Consider B_1. Here the irreducible components which are preimages of $[1:1] \times \mathbb{P}^1$ and $[1:-1] \times \mathbb{P}^1$ contain all the singularities of type S_2 and S_6. Similarly the irreducible components $\mathbb{P}^1 \times [1:-1]$ and $\mathbb{P}^1 \times [1:1]$ contain all the singularities of type S_2 and S_6. Therefore from Proposition 6.15 applied to the irreducible components of $\tilde{B}_1$ we have

$$\delta^*B_1 = \tilde{B}_1 + \sum_{s\in S_2\cup S_6} \frac{1}{2}E_s + \sum_{s\in S_2\cup S_6} \frac{1}{2}E_s = \tilde{B}_1 + \sum_{s\in S_2\cup S_6} E_s.$$

Similarly, we also have

$$\delta^*B_2 = \tilde{B}_2 + \sum_{s\in S_3\cup S_4} E_s$$

and

$$\delta^* B_3 = \tilde{B}_3 + \sum_{s \in S_1 \cup S_5} E_s.$$

□

Proposition 6.45. *We have*

$$\mathcal{L}(-\delta^* B) \otimes \mathcal{L}(\sum_{i=1}^{48} E_i) \cong \mathcal{L}(-\tilde{B}). \tag{6.4}$$

Proof. Note that $\delta^* B = \delta^* B_1 + \delta^* B_2 + \delta^* B_3$ is a Cartier divisor because B_1, B_2, B_3 are hyperplane sections of Y_4 (see Subsection 2.2.6). Hence $\mathcal{L}(\delta^* B)$ is defined. From Lemma 6.44, we have

$$-\delta^* B + \sum_{i=1}^{48} E_i = -\tilde{B}.$$

□

Remark 6.46. Since for each $1 \leq i \leq 3$ the divisor B_i is an hyperplane section, we have

$$\mathcal{L}(B_i) \cong \mathcal{O}_{Y_4}(1),$$

thus

$$\mathcal{L}(B) \cong \mathcal{O}_{Y_4}(3).$$

Remark 6.47. Let C be a curve on Y_4 with strict transform $C' \subseteq \tilde{Y}_4$. Let H be a

hyperplane section of Y_4.

$$\begin{aligned}(\delta^*H, C') &= \deg_{C'}(\delta^*\mathcal{O}_{Y_4}(1)_{|C'}) \text{ by Proposition 2.50}\\ &= \deg_C(\mathcal{O}_{Y_4}(1)_{|C}) \quad \text{by Proposition 2.44}\\ &= \deg(C) \geq 1 \quad \text{by Proposition 2.45.}\end{aligned}$$

Theorem 6.48. *Let C be a curve on Y_4 with strict transform $C' \subseteq \tilde{Y}_4$. Then*

$$\deg_{\tilde{C}}(\varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}}) = -\deg(C) + (E.C') + 2(2g-2),$$

where g is the genus of C, H is any hyperplane section in Y_4, $\tilde{C}$ is the desingularization of C' and $E = \sum_{i=1}^{48} E_i$. Therefore, if

$$\deg(C) > (E.C') + 2(2g-2)$$

then C' is ω_4'-integral.

Proof. Since $S^2\Omega^1_{\tilde{C}/\mathbb{C}} \cong K_C^{\otimes 2}$, we obtain from Lemma 6.44 and Remark 6.46 that:

$$\begin{aligned}&\deg_{\tilde{C}}(\varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}})\\ &\qquad = \deg_{\tilde{C}}(\varphi_{C'}^*(\delta^*\mathcal{O}_{Y_4}(-3) \otimes \mathcal{L}(E) \otimes \delta^*\mathcal{O}_{Y_4}(2)) \otimes S^2\Omega^1_{\tilde{C}})\\ &\qquad = \deg_{\tilde{C}}(\varphi_{C'}^*\delta^*\mathcal{O}_{Y_4}(-1)) + \deg_{\tilde{C}}(\varphi_{C'}^*\mathcal{L}(E)) + 2(2g-2)\\ &\qquad = \deg_{C'}((\delta^*\mathcal{O}_{Y_4}(-1))_{|C'}) + \deg_{C'}(\mathcal{L}_{|C'}(E)) + 2(2g-2)\end{aligned}$$

where the last equality comes from Proposition 2.44. We then have

$$\begin{aligned}\deg_{C'}((\delta^*\mathcal{O}_{Y_4}(-1))_{|C'} &+ \deg_{C'}(\mathcal{L}_{|C'}(E)) + 2(2g-2)\\ &= \deg_{C'}(\delta^*\mathcal{L}(-H) + \deg_{C'}(\mathcal{L}(E))) + 2(2g-2)\\ &= -(\delta^*H.C') + (E.C') + 2(2g-2)\\ &= -\deg(C) + (E, C') + 2(2g-2),\end{aligned}$$

where the second equality holds by Proposition 2.50 and the last by Remark 6.47. This proves the asserted degree formula.

If $\deg(C) > (E.C') + 2(2g-2)$, then $\deg_{\tilde{C}}(\varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}/\mathbb{C}}) < 0$, hence $H^0(\tilde{C}, \varphi_{C'}^*(\mathcal{L}(-\tilde{B}) \otimes \delta^*\mathcal{O}(2)) \otimes S^2\Omega^1_{\tilde{C}}) = 0$ and C is therefore ω_4'-integral. □

6.9 Some applications

Now we use Theorem 6.48 to prove the results in Section 6.1.

Lemma 6.49. *Let C be an irreducible curve of genus zero or one on Y_4 that only contains exactly one singular point of Y_4. Then C' (the strict transform of C in $\tilde{Y}_4$) is ω_4'-integral.*

Proof. Suppose without loss of generality that the singularity of Y_4 contained in C is in B_1. Then by Lemma 6.35 we have $(E.C') = (\sum_{s\in S_2\cup S_6} E_s.C')$. Choose $H = B_1$. By the proof of Lemma 6.44 we have $\delta^*B_1 = \tilde{B}_1 + \sum_{s\in S_2\cup S_6} E_s$. Hence

$$\begin{aligned}-(\delta^*H.C') + (E.C') &= -(\tilde{B}_1.C') - \sum_{s\in S_2\cup S_6}(E_s.C') + \sum_{s\in S_2\cup S_6}(E_s.C')\\ &= -(\tilde{B}_1.C').\end{aligned}$$

The curve C' cannot be an irreducible component of $\tilde{B}_1$ because any of these curves contain 4 singular points of Y_4.

We need to prove that $\tilde{B}_1$ intersects C', so that $-(\tilde{B}_1.C') < 0$.

Suppose that C' does not intersect $\tilde{B}_1$. This implies that C does not intersect B_1 at a point which is not a singularity of Y_4.

Let D_1 and D_2 be the two irreducible components of B_1 that do not contain the singularity s (they exist by Lemma 6.35). The images of D_1 and D_2 in $\mathbb{P}^1 \times \mathbb{P}^1$ under Ψ are $\{p_1\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p_2\}$ with $p_1, p_2 \in \mathbb{P}^1$.

We have that $\tilde{\Psi}(C)$ must intersect at least one of $\{p_1\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{p_2\}$. Suppose without loss of generality that $\tilde{\Psi}(C)$ intersects $\{p_1\} \times \mathbb{P}^1$ at a point P. Since P is in $\tilde{\Psi}(C)$, we get that there is a point $Q \in C$ such that $\tilde{\Psi}(Q) = P$. Since D_1 is the pullback of $\{p_1\} \times \mathbb{P}^1$ and this curve contains P, we get that $Q \in D_1$. Thus D_1 and C have nonempty intersection. Similarly, if $\tilde{\Psi}(C)$ intersects $\mathbb{P}^1 \times \{p_2\}$, we obtain that D_2 and C have nonempty intersection.

Therefore C intersects D_1 or D_2, and thus C intersects $\tilde{B}_1$ in another point which is not a singularity (by hypothesis). From this contradiction we get that the intersection between C' and $\tilde{B}_1$ is non-trivial, thus $-(\tilde{B}_1.C') < 0$. By Theorem 6.48 we obtain that C is ω_4'-integral.

One can do the same reasoning for singularities contained only in B_2 or only in B_3 to prove that C' is ω_4'-integral. □

Proof of Theorem 6.2. We know from Theorem 6.48 that if a curve C of genus 0 or 1 on Y_4 does not contain a singular point of Y_4, then it is ω_4'-integral (because such curves give $-\deg(C) < 0$, $(E.C') = 0$ and $2(2g-2) \leq 0$), but by Theorem 6.39, we know that ω_4'-integral curves that do not contain any singular point of Y_4 have genus

5. Suppose that C contains exactly one singular point of Y_4. From Corollary 6.49 we know that C is ω_4'-integral. This contradicts the fact (from Theorem 6.39) that there are no ω_4'-integral curves that contain exactly one singular point on Y_4. Therefore C must contain at least two singular points of Y_4. □

Proposition 6.50. *Let C be an irreducible curve of genus zero in Y_4 satisfying that $C \cap S_Z \subseteq B_i$, for fixed i. Then C' (the strict transform of C in $\tilde{Y}_4$) is ω_4'-integral.*

Proof. Suppose without loss of generality that $C \cap S_Z \subseteq B_1$. Then we have that $(E.C') = (\sum_{s_i \in S_3 \cup S_4} E_i.C')$. Choose $H = B_1$. By the proof of Lemma 6.44 we have $\delta^* B_1 = \tilde{B}_1 + \sum_{s_i \in S_3 \cup S_4} E_i$. Hence

$$\begin{aligned} -(\delta^* H.C') + (E.C') + 4g - 4 &= -(\tilde{B}_1.C') - \sum_{s_i \in S_3 \cup S_4} (E_i.C') + \sum_{s_i \in S_3 \cup S_4} (E_i.C') - 4 \\ &= -(\tilde{B}_1.C') - 4. \end{aligned}$$

If C' and $\tilde{B}_1$ do not have a common component then $(\tilde{B}_1.C') \geq 0$ and thus $-(\tilde{B}_1.C') - 4 < 0$. We get from Theorem 6.48 that C' is ω_4'-integral. If C' is an irreducible common component of $\tilde{B}_1$ then by Theorem 6.39 we have that C' is ω_4'-integral.

One can do the same reasoning for singularities contained only in B_2 or only in B_3 to prove that C' is ω_4'-integral. □

Proof of Theorem 6.3. Let C be an irreducible curve of genus 0 which only contains singularities that belong to B_i, for fixed i. From Corollary 6.50, C' must be ω_4'-integral. This contradicts the fact that there are no ω_4'-integral curves of genus 0. Hence C contains singularities that belong two different B_i, B_j. These points are described in Lemma 6.35. □

From Remark 6.47 and Theorem 6.48 follows the proof of the last two theorems:

Proof of Theorem 6.1. Suppose first that the strict transform C' of C is an ω_4'-integral curve. Since C does not contain a singular point of Y_4, we get that the curve C' has to be of type (a) or (b) (cf. Theorem 6.39). Then C has genus 5, and degree 8 (it is the pull-back of a curve of degree 1 by a morphism of degree 8, with C not in the ramification locus). Therefore $\deg C = 8 < 16 = 4g(C) - 4$.

Now suppose that C' is not ω_4'-integral. Then by Theorem 6.48 it has to satisfy $\deg(C) \leq (E.C') + 2(2g(C) - 2) = 4g(C) - 4$. □

Proof of Theorem 6.4. If C' is an ω_4'-integral curve of type (a) or (b) we know this holds by the proof of Theorem 6.1. By a similar computation, ω_4'-integral curves of type (a') and (b') also satisfy this inequality.

Now suppose that C' is not ω_4-integral. We know that δ is an isomorphism outside the singular points of Y_4. From Proposition 6.12 we have that the preimage of any singular point of Y_4 is isomorphic to $\mathbb{P}^1$, which is a smooth curve. If C is smooth at a singularity s of Y_4, then by Proposition 2.104 we obtain that $(E_s.C') = 1$. Therefore $(E.C') \leq 48$. By Theorem 6.48 the curve C' has to satisfy

$$\deg(C) \leq (E.C') + 2(2g(C) - 2) \leq 48 + 4g(C) - 4.$$

□

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