Variations of Li's criterion for an extension of the Selberg class

by

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Dedication

To my family and friends.

You're amazing.

Abstract

In 1997, Xian-Jin Li gave an equivalence to the classical Riemann hypothesis, now referred to as *Li's criterion*, in terms of the non-negativity of a particular infinite sequence of real numbers. We formulate the analogue of Li's criterion as an equivalence for the generalized quasi-Riemann hypothesis for functions in an extension of the Selberg class, and give arithmetic formulae for the corresponding Li coefficients in terms of parameters of the function in question. Moreover, we give explicit non-negative bounds for certain sums of special values of polygamma functions, involved in the arithmetic formulae for these Li coefficients, for a wide class of functions. Finally, we discuss an existing result on correspondences between zero-free regions and the non-negativity of the real parts of finitely many Li coefficients. This discussion involves identifying some errors in the original source work which seem to render one of its theorems conjectural. Under an appropriate conjecture, we give a generalization of the result in question to the case of Li coefficients corresponding to the generalized quasi-Riemann hypothesis. We also give a substantial discussion of research on Li's criterion since its inception, and some additional new supplementary results, in the first chapter.

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Contents

Chapter 1 Introduction

1.1 Summary and organization

In this thesis, we discuss some generalizations of Xian-Jin Li's 1997 criterion for the classical Riemann hypothesis, referred to as *Li's criterion*. In the classical case, Li's criterion states that the Riemann hypothesis is equivalent to the non-negativity of a particular infinite sequence of real numbers (known as *Li coefficients*). This idea has been extended significantly in the last several years, and there has also been substantial progress in understanding the behaviour of the Li coefficients themselves in special cases (especially that of the classical Riemann zeta function). Our discussion of Li's criterion comprises several aspects, each of which refer to and expand upon different components of the literature work on Li's criterion that has been completed since its inception. Among these are some remarks on certain errors in [3], which pose problems for the proof of one of its main theorems.

Indeed, the subsequent sections of this chapter will be devoted to a review of the ideas motivating Li's criterion, and of work by a number of researchers studying and expanding upon it. Moreover, we will give some preliminary new results and proofs that will inform our later original work. This exposition will lead into our own work in Chapters 2 and 3.

In Chapter 2, our work focuses on giving a generalization of Li's criterion to an equivalence for the generalized quasi-Riemann hypothesis in the Selberg class. The first section gives the generalization itself, along with results that show that the

Li coefficients in this context are well-defined, and that give several equivalent definitions for them. In the second section, we give arithmetic formulas for the Li coefficients in these cases, along with some exposition on the versatility of these formulae. In the last section, we give explicit non-negative bounds on one of the terms in these arithmetic formulae (corresponding to a sum of special values of polygamma functions) that applies with relatively wide generality.

In Chapter 3, we change our focus to studying connections between the non-negativity of *finitely many* Li coefficients and the existence of zero-free regions for the corresponding function. These correspondences were studied by Brown in [3] in the case of Li coefficients corresponding to the full generalized Riemann hypothesis for a wide class of functions. Unfortunately, [3] contains an error which renders one of the main theorems of this correspondence conjectural, without further work. Under an appropriate conjecture, we generalize the problematic theorem to the case of Li coefficients corresponding to a restricted case of the generalized quasi-Riemann hypothesis.

Chapter 4 focuses on a discussion of the significance of our results, and of prospective future research on Li's criterion. Finally, the Appendix gives the application of some of our discussion to the special case of the Riemann zeta function (reproducing work of several other authors).

The main results of the thesis are contained in Theorems 2.1.3, 2.2.1, and 2.3.2 in Chapter 2, and Theorem 3.3.1 in Chapter 3 (which here is proved only under an appropriate conjecture, given in Section 3.2). There are also numerous other original results of some interest throughout the thesis. Some particular examples of these are Proposition 1.5.1 and Theorem 1.7.3 in Chapter 1, Lemma 2.1.2, Proposition 2.2.3, and Corollary 2.3.3 in Chapter 2, and the contents of Sections 1 and 2 of Chapter 3 (including detailed discussion of some problematic errors in the arguments of [3] in Section 3.2).

Lastly, we wish to outline the contents of the present chapter in more detail. The next section will give a general introduction to the theory of the Riemann zeta function and related ideas. In the third section, we introduce a type of complex multiset that will be of interest to us throughout the thesis, and we also define $\star$-convergence for sums over complex multisets. The fourth section introduces the Selberg class and some of its fundamental properties, along with an extension to the Selberg class that we will frequently use in our later work. The fifth section begins with some fundamental equivalences to the Riemann hypothesis, and proceeds to give Li's criterion itself, along with two distinct ways to interpret it on intuitive terms. The sixth section presents a set of results that allow the formulation of Li's criterion for arbitrary complex multisets under appropriate hypotheses, which provide the mechanisms needed to generalize Li's criterion to new classes of functions. For illustrative purposes, we give proofs of most results in this section. The seventh section is divided into three subsections, which give background on separate ideas that we will expand upon later in our own work. These three subsections concern the ideas of generalizing Li's criterion to apply to quasi-Riemann hypotheses, the generalization of Li's criterion to the Selberg class, and correspondences between zero-free regions and the non-negativity of finitely many Li coefficients (and initial discussion of the problems in [3]), respectively. Finally, the last section of this chapter discusses arithmetic formulae for Li coefficients, and the study of particular terms of arithmetic formulae to better understand the behaviour of Li coefficients.

1.2 The Riemann zeta function and the Riemann hypothesis

Since its introduction in 1859 in Riemann's memoir [22], the question of the truth of the Riemann hypothesis is one that has received more attention from mathematicians than nearly any other. This thesis concerns itself with generalizations of a new approach to this question, first presented in 1997 by Xian-Jin Li [17].

The Riemann zeta function is the fundamental object of study of most of classical analytic number theory.

Riemann zeta function: For a complex variable s, we define the function $\zeta(s)$ for $\Re(s) > 1$ by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where the product is over all prime numbers p. The *Riemann zeta function* is defined to be the meromorphic continuation of $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$. Henceforth, the notation $\zeta(s)$ denotes this meromorphic continuation.

The definition can be shown to imply that $\zeta(s)$ does not vanish for $\Re(s) > 1$, and that $\zeta(s)$ has a simple pole at $s = 1$.

Completed zeta function: The *completed Riemann zeta function*, denoted by $\xi(s)$, is defined for all $s \in \mathbb{C}$ by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where Γ is Euler's gamma function.

For a classical discussion of these functions, one may refer, for example, to Davenport [9] or Edwards [10]. We review a few of their properties here, without proof.

Theorem 1.2.1 (Functional equation) *The completed zeta function is entire, and satisfies the functional equation*

$$\xi(s) = \xi(1-s)$$

for all complex s.

In particular, this theorem implies that $s = -2n$ is a simple zero of $\zeta(s)$ for every positive integer n, as a consequence of the existence of simple poles of $\Gamma\left(\frac{s}{2}\right)$ at those points.

For completeness, we recall the statement of the prime number theorem.

Definition: For any real $x \geq 0$, the *prime counting function*, denoted by $\pi(x)$, is equal to $\#\{p \text{ prime} : p \leq x\}$.

Definition: If $f(x)$ and $g(x)$ are real functions of a real variable x, the notation $f(x) \sim g(x)$ as $x \to \infty$ means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Theorem 1.2.2 (Prime Number Theorem) $\pi(x) \sim \frac{x}{\log x}$ *as $x \to \infty$.*

Stronger statements of this theorem give explicit error bounds. The Riemann hypothesis was originally formulated to provide a best-case error bound for the prime number theorem.

Definition: A *non-trivial zero* of the Riemann zeta function is a zero of $\zeta(s)$ which is not equal to $-2n$ for any positive integer n.

It is then clear that the non-trivial zeros of the Riemann zeta function coincide with the zeros of the completed Riemann zeta function.

Definition: The *critical strip* is the region $0 \leq \Re(s) \leq 1$ in the complex plane. The *critical line* is the line $\Re(s) = \frac{1}{2}$ in the complex plane.

It can be shown that the prime number theorem is equivalent to the following result.

Theorem 1.2.3 *Every non-trivial zero ρ of the Riemann zeta function lies in the interior of the critical strip.*

The functional equation also implies the following.

Theorem 1.2.4 *A complex number ρ is a non-trivial zero of the Riemann zeta function if and only if $\bar{\rho}$ and $1 - \rho$ are also non-trivial zeros with the same multiplicity.*

It will prove convenient to define Euler's constant, γ, at this point.

Euler's constant: We define Euler's constant, denoted γ, by

$$\gamma = \lim_{s \to 1} \left[\zeta(s) - \frac{1}{s-1} \right].$$

Finally, we can give the Riemann hypothesis itself.

The Riemann Hypothesis: Every non-trivial zero of the Riemann zeta function lies on the critical line.

1.3 Complex multisets, and other tools

To discuss Li's criterion, we first need to construct some specialized tools. It will prove convenient to define a particular notion of conditional convergence of sums over complex multisets, called $*$-*convergence*. We must also first specify exactly what type of complex multisets we are concerned with.

Definition: A *vertically simple complex multiset* is a set $R \subset \mathbb{C} \times \mathbb{N}$, consisting of pairs of complex numbers ρ with corresponding positive integer multiplicities n_ρ, such that if $(\rho_1, n_{\rho_1}) \neq (\rho_2, n_{\rho_2})$ are both in R, we have $\rho_1 \neq \rho_2$, and such that only finitely many (ρ, n_ρ) in R satisfy $|\Im(\rho)| \leq T$ for any real $T \geq 0$. If (ρ, n_ρ) is in R, we often simply say that $\rho \in R$ with multiplicity n_ρ. Moreover, whenever we discuss complex multisets in this thesis, it is implied that they are vertically simple, unless otherwise specified.

Definition: Let R be a vertically simple complex multiset. Throughout this thesis, the notation $\sum_R^*$ is meant to denote summation over R in the sense that if $a(\rho)$ is a complex number for every ρ in R, then

$$\sum_R^* a(\rho) = \sum_{\rho \in R}^* a(\rho) = \lim_{T \to \infty} \sum_{\substack{\rho \in R \\ |\mathbf{Im}(\rho)| \leq T}} a(\rho),$$

if the limit on the right exists, where the term corresponding to ρ occurs according to the multiplicity of ρ in R. If the limit exists, following the terminology of Bombieri and Lagarias [2], we refer to the sum $\sum_R^* a(\rho)$ as $*$-*convergent*. We may also apply this notation to infinite products, where

$$\prod_R^* a(p) = \lim_{T \to \infty} \prod_{\substack{\rho \in R \\ |\mathbf{Im}(\rho)| \leq T}} a(\rho).$$

Note that in all of our summations and products whose indices are multisets of complex numbers, throughout this proposal, it is always implied that ρ is the variable which runs over the indicated index multiset, unless otherwise specified.

Definition: We denote the complex multiset of non-trivial zeros of the Riemann zeta function by $Z(\xi) = Z$.

We will also need to use the idea of *big-O notation* in our work. As such, we give its definition here.

Definition: Suppose that f is a complex-valued function of a complex variable, and that g is a positive-valued real function of a complex variable. We write $f(x) = O(g(x))$ on a subset S of $\mathbb{C}$ if and only if there exists some positive real constant C such that $|f(x)| \leq Cg(x)$ for every x in S. In other words, we adopt the standard notion of "big O notation."

We can use this idea to define another idea from complex analysis.

Definition: If f is an entire complex function of a single complex variable, we say that f has *finite order* if and only if there exists a real number $\alpha > 0$ such that $f(s) = O(e^{(|s|^{\alpha})})$ as $|s| \to \infty$. If f is of finite order, and $\alpha^* = \inf\{\alpha > 0 \mid f(s) = O(e^{(|s|^{\alpha})})\}$, then we say that *the order of* f is α^*.

One may refer to [9], [18], or [16] for additional discussion of the theory of functions of finite order.

The following theorem (see [9, Chapter 12], for example) states that the completed zeta function is of finite order, and that it may be represented by a *Hadamard product*, a fact that is critical to classical proofs of the prime number theorem, and to the formulation of Li's criterion.

Theorem 1.3.1 *The completed zeta function $\xi(s)$ is an entire function of order 1. As such, it may be represented as a Hadamard product in the form*

$$\xi(s) = \xi(0)e^{Bs} \prod_{Z} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \tag{1.3.1}$$

where

$$B = -\sum_{Z}{}^{*} \frac{1}{\rho}.$$

Here, the product is absolutely convergent, and the sum for B is $\ast$-convergent. Moreover, with our definition of $\xi(s)$, it can be shown that $\xi(0) = 1$.

1.4 The Selberg class

This thesis concerns itself mainly with generalizations of Li's criterion to wider contexts than that in which it was originally formulated. In particular, we work heavily with a general class of functions containing the Selberg class. In this section, we give discussion and definitions relevant to the Selberg class, and state some of its elementary properties.

The Selberg class was introduced in 1991 by Atle Selberg (see [24]), as the conjectural largest context in which it makes sense to talk about the Riemann hypothesis. The Selberg class $\mathcal{S}$ consists of all complex functions F of a single complex variable s satisfying the five axioms (1)-(5) below.

(1) (Dirichlet series) For $\Re(s) > 1$, $F(s)$ may be expressed as an absolutely convergent Dirichlet series, in the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

where $a_F(n)$ is a complex number for every integer $n \geq 1$.

(2) (Analytic continuation) There exists a non-negative integer m such that $G(s) = (s-1)^m F(s)$ extends to an entire function of finite order. The least such integer m is called the *polar order* of F, and is denoted by m_F.

(3) (Functional equation) There exists a non-negative integer r, a real number $Q_F > 0$, positive real numbers $\omega_1, \ldots, \omega_r$, and complex numbers $\nu_1, \ldots, \nu_r$ with $\Re(\nu_j) \geq 0$ for $1 \leq j \leq r$, such that the *completed function*

$$\xi_F(s) = F(s) s^{m_F} (s-1)^{m_F} Q_F^s \prod_{j=1}^{r} \Gamma(\omega_j s + \nu_j)$$

is entire, and satisfies the functional equation

$$\xi_F(s) = w \bar{\xi}_F(1-s),$$

for some complex number w with $|w| = 1$ (where $\bar{\xi}_F(s) = \overline{\xi_F(\bar{s})}$, and Γ is Euler's gamma function).

(4) (Euler product) When $\Re(s) > 1$, we may express $\log F(s)$ as an absolutely convergent Dirichlet series, in the form

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s},$$

where for each n, $b_F(n)$ is a complex number such that $b_F(n) = 0$ whenever $n \neq p^m$ for some prime p and positive integer m, and $b_F(n) = O(n^\theta)$ for some real $\theta < \frac{1}{2}$.

(5) (Ramanujan hypothesis) For every positive real number ϵ, we have

$$a_F(n) = O(n^\epsilon), \tag{1.4.1}$$

where the constant implied in the relation (1.4.1) may depend on the choice of ϵ.

While the values of the numbers ω_j are not unique in axiom **(3)**, it can be shown that the quantity $\deg(F) = d_F = 2\sum_{j=1}^{r}\omega_j$ is a well-defined constant, referred to as the *degree* of the function F, and denoted by $\deg(F)$ (see, for example, [18, p.118]).

There are some related classes of functions whose definitions we will also need. The *extended Selberg class*, denoted by $\mathcal{S}^\sharp$, consists of all functions which satisfy properties **(1)**, **(2)**, and **(3)** above. Another class of functions, denoted by $\mathcal{S}^{\sharp\flat}$ and satisfying $\mathcal{S}^\sharp \supseteq \mathcal{S}^{\sharp\flat} \supseteq \mathcal{S}$, is studied in Lejla Smajlović's paper [25] in terms of the construction of Li's criterion in this context. The class $\mathcal{S}^{\sharp\flat}$ is defined to be the set of all functions F in $\mathcal{S}^\sharp$ such that F also satisfies the following additional property:

(4') (Modified Euler product axiom) For $\Re(s) > 1$, the quantity $\frac{d}{ds}\log F(s) = \frac{F'(s)}{F(s)}$ may be expressed as an absolutely convergent Dirichlet series

$$\frac{d}{ds}\log F(s) = \frac{F'(s)}{F(s)} = -\sum_{n=2}^{\infty}\frac{c_F(n)}{n^s},$$

where the coefficient $c_F(n)$ is a complex number for every integer $n \geq 2$.

The following theorem was proved by Smajlović in [25].

Theorem 1.4.1 *Every function contained in the Selberg class $\mathcal{S}$ is also contained in $\mathcal{S}^{\sharp\flat}$. More precisely, we have $\mathcal{S}^\sharp \supseteq \mathcal{S}^{\sharp\flat} \supseteq \mathcal{S}$.*

We also make some remarks on the zeros of the completed function ξ_F for any function F in $\mathcal{S}^{\sharp\flat}$. Property **(4')** implies that if F is in $\mathcal{S}^{\sharp\flat}$, then $F(s)$ does not vanish for $\Re(s) > 1$, and thus the form of the functional equation shows that ξ_F does not either. Additionally, property **(3)** implies that $1-\bar{\rho}$ is a zero of ξ_F whenever ρ is. Combining these two observations gives us the following theorem.

Theorem 1.4.2 *If F is any function in $\mathcal{S}^{\sharp\flat}$, then every zero of ξ_F lies in the critical strip.*

We remark the contrast with our corresponding theorem for the Riemann zeta function, which states that every non-trivial zero of the Riemann zeta function lies on the *interior* of the critical strip. In the case of the Selberg class and its extension $\mathcal{S}^{\sharp\flat}$, our current knowledge does not allow us to exclude the boundary lines in general (though this exclusion has been proved in the special case of automorphic L-functions [12]). As a result, in most statements concerning the formulation of Li's criterion in this context, we must assume that 0 is not a zero of the function in question.

For convenience, we state our previous observation about the symmetry on the zeros of ξ_F formally in the next theorem.

Theorem 1.4.3 *If F is a function in $\mathcal{S}^{\sharp\flat}$, then ρ is a zero of $\xi_F(s)$ if and only if $1-\bar{\rho}$ is a zero of $\xi_F(s)$ with the same multiplicity.*

We also make the following definition, analogous to that in the previous section for the Riemann zeta function.

Definition: If F is a function in $\mathcal{S}^{\sharp\flat}$, we refer to the zeros of ξ_F as *non-trivial* zeros of F. Moreover, we denote the complex multiset of non-trivial zeros of F by $Z(F)$.

We have the following analogue to the Riemann hypothesis in the context of the Selberg class.

Generalized Riemann hypothesis (GRH): We say that the *generalized Riemann hypothesis* holds for a function F in $\mathcal{S}^{\sharp\flat}$ if and only if all of the non-trivial zeros of F lie on the critical line.

For additional discussion of some properties of the Selberg class, one may refer, for example, to [18, Chapter 8].

1.5 Equivalences to the Riemann hypothesis, and Li's criterion

The Riemann hypothesis is a critical question in analytic number theory. As such, it is interesting to examine different ways to frame it, which may shed more light on its resolution.

The following proposition is a simple result giving an elementary equivalence to the Riemann hypothesis, which we prove here for the sake of interest.

Proposition 1.5.1 *Let S be a vertically simple complex multiset with $0 \notin S$, and such that $(\rho, n_\rho) \in S$ if and only if $(1-\bar{\rho}, n_\rho) \in S$. Suppose that $\sum_S \frac{1}{|\rho|^2}$ converges absolutely and that $\sum_S^* \frac{1}{\rho}$ is $*$-convergent. Then the following two statements are equivalent:*

(1)

$$\sum_S \frac{1}{|\rho|^2} = 2\Re\left(\sum_S^* \frac{1}{\rho}\right).$$

(2)

$$\Re(\rho) = \frac{1}{2} \ \forall \rho \in S.$$

Remark: The proof shows that in fact, condition (1) may be replaced by the weaker assertion that $\sum_S \frac{1}{|\rho|^2} \le 2\Re\left(\sum_S^* \frac{1}{\rho}\right)$.

Proof: First, assume that $\Re(\rho) = \frac{1}{2}$ for all $\rho \in S$. Then

$$\sum_S \frac{1}{|\rho|^2} = \sum_S \frac{1}{\rho\bar{\rho}}$$

$$
\begin{aligned}
&= \sum_S \frac{2\Re(\rho)}{\rho\bar{\rho}} \\
&= \sum_S \frac{\bar{\rho}+\rho}{\rho\bar{\rho}} \\
&= \sum_S \left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right) \\
&= \left(\sum_S{}^* \frac{1}{\rho}\right) + \left(\sum_S{}^* \frac{1}{\bar{\rho}}\right).
\end{aligned}
$$

Since $\sum_S \frac{1}{|\rho|^2}$ is real, and $\Re\left(\frac{1}{\rho}\right) = \Re\left(\frac{1}{\bar{\rho}}\right)$ for every $\rho \in \mathbb{C}^*$, it follows that

$$
\sum_S \frac{1}{|\rho|^2} = 2\Re\left(\sum_S{}^* \frac{1}{\rho}\right),
$$

as we wanted.

Conversely, suppose that $\sum_S \frac{1}{|\rho|^2} = 2\Re\left(\sum^*{}_S \frac{1}{\rho}\right)$. As before, we have

$$
2\Re\left(\sum_S{}^* \frac{1}{\rho}\right) = \sum_S \left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right).
$$

Moreover,

$$
\sum_S \frac{1}{|\rho|^2} = \sum_S \left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right)\frac{1}{2\Re(\rho)}.
$$

It follows that

$$
\begin{aligned}
0 &= \sum_S \frac{1}{|\rho|^2} - 2\Re\left(\sum_S{}^* \frac{1}{\rho}\right) \\
&= \sum_S \left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right).
\end{aligned}
$$

If $\Re(\rho) = \frac{1}{2}$, then $\frac{1}{2\Re(\rho)} - 1 = 0$. It follows that we may re-write our last equation as

$$
0 = \sum_{\rho\in S,\ \Re(\rho)\neq\frac{1}{2}} \left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right)
$$

$$= \left[\sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right)\right] + \left[\sum_{\Re(\rho)>\frac{1}{2}} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right)\right].$$

Now, by assumption, $(\rho, n_\rho) \in S$ if and only if $(1-\bar{\rho}, n_\rho) \in S$. It follows that

$$\sum_{\Re(\rho)>\frac{1}{2}} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right) = \sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{1}{1-\bar{\rho}} + \frac{1}{1-\rho}\right)\left(\frac{1}{2\Re(1-\bar{\rho})} - 1\right).$$

Using our equality above, this shows that

$$0 = \sum_{\Re(\rho)<\frac{1}{2}} \left[\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\left(\frac{1}{2\Re(\rho)} - 1\right) + \left(\frac{1}{1-\bar{\rho}} + \frac{1}{1-\rho}\right)\left(\frac{1}{2\Re(1-\bar{\rho})} - 1\right)\right].$$

Remarking that for any $z \in \mathbb{C}^*$, we have

$$\left(\frac{1}{z} + \frac{1}{\bar{z}}\right)\left(\frac{1}{2\Re(z)} - 1\right) = \frac{1-2\Re(z)}{|z|^2},$$

and writing $\rho \in S$ as $\beta_\rho + i\gamma_\rho$ with $\beta_\rho, \gamma_\rho \in \mathbb{R}$, we obtain

$$\begin{aligned}
0 &= \sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{1-2\Re(\rho)}{|\rho|^2} + \frac{1-2\Re(1-\rho)}{|1-\rho|^2}\right) \\
&= \sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{1-2\beta_\rho}{\beta_\rho^2+\gamma_\rho^2} + \frac{1-2(1-\beta_\rho)}{(1-\beta_\rho)^2+\gamma_\rho^2}\right) \\
&= \sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{(1-2\beta_\rho)((1-\beta_\rho)^2+\gamma_\rho^2) + (2\beta_\rho-1)(\beta_\rho^2+\gamma_\rho^2)}{(\beta_\rho^2+\gamma_\rho^2)((1-\beta_\rho)^2+\gamma_\rho^2)}\right) \\
&= \sum_{\Re(\rho)<\frac{1}{2}} \left(\frac{4\left(\beta_\rho - \frac{1}{2}\right)^2}{(\beta_\rho^2+\gamma_\rho^2)((1-\beta_\rho)^2+\gamma_\rho^2)}\right).
\end{aligned}$$

The summands are all well-defined since $0 \notin S$ by our hypotheses. Moreover, the summand is clearly positive for any real $\beta_\rho \neq \frac{1}{2}$. Thus the equality above shows that in fact, there are no $\rho \in S$ with $\beta_\rho = \Re(\rho) < \frac{1}{2}$ (since otherwise the sum on the right would be strictly positive). Using our hypothesis that $\rho \in S \iff 1-\bar{\rho} \in S$ then establishes that $\Re(\rho) = \frac{1}{2}$ for every $\rho \in S$, as we wanted. ∎

Corollary 1.5.2 *The Riemann hypothesis for the Riemann zeta function is equivalent to the statement that* $\sum_Z \frac{1}{|\rho|^2} = 2 \sum^*{}_Z \frac{1}{\rho} = 2 + \gamma - \log(4\pi)$.

Proof: As we have mentioned previously, the completed zeta function $\xi(s)$ is an entire function of order 1, whose zeros all lie in the critical strip. We also know by earlier discussion that $\xi(0) \neq 0$, and by the functional equation that ρ is a zero of $\xi(s)$ if and only if $1 - \rho$ and $\bar{\rho}$ are a zeros of $\xi(s)$ with the same multiplicity. Well-known results of complex analysis (see [16]) state that the set of zeros of an entire function that is not identically zero does not have any accumulation points on any bounded subset of $\mathbb{C}$, and moreover that for an entire function f of finite order α, the sum

$$\sum_{\rho \neq 0, f(\rho)=0} \frac{1}{|\rho|^\sigma}$$

converges absolutely whenever $\sigma > \alpha$ (see, for example, [9, Ch.12]). Combining these observations shows that the multiset $Z(\xi)$ of zeros of $\xi(s)$ is a vertically simple complex multiset, and that the sum $\sum_{\rho \in Z(\xi)} \frac{1}{|\rho|^2}$ converges absolutely. Finally, it is a consequence of classical analytic number theory (see [9, Chapter 12], for example) that the sum

$$\sum_{\rho \in Z(\xi)}^* \frac{1}{\rho}$$

converges to the real value $1 + \gamma - \frac{1}{4}\log(4\pi)$. All of these facts together allow us to apply Proposition 1.5.1 to the multiset $Z(\xi)$ to obtain the corollary.

Remark: One may also derive an explicit expression for $\sum^*_{\rho \in Z(F)} \frac{1}{\rho}$ for functions F in the Selberg class. This is easily accomplished from the definition of the completed function corresponding to F from Section 1.4, by using the fact (see [25], or Section 1.7.7) that

$$\sum_{\rho \in Z(F)}^* \frac{1}{\rho} = -\frac{\xi_F'(0)}{\xi_F(0)}.$$

Indeed, we find that

$$\sum_{\rho \in Z(F)}^{*} \frac{1}{\rho} = m_F + \log Q_F + \Re\left[\eta_0(F) + \sum_{j=1}^{r} \omega_j \Psi(\omega_j + \nu_j)\right],$$

where $\eta_0(F)$ is the constant term of the Laurent expansion of $\frac{F'}{F}$ about $s = 1$.

Another example of a simple equivalence to the Riemann hypothesis is given in the next theorem.

Theorem 1.5.3 *The Riemann hypothesis for the Riemann zeta function is equivalent to the statement that every non-trivial zero* ρ *of* $\zeta(s)$ *satisfies* $\left|\frac{\rho}{\rho-1}\right| \le 1$.

Proof: Let ρ be any non-trivial zero of $\zeta(s)$. If the Riemann hypothesis holds, then by definition we have $\Re(\rho) = \frac{1}{2}$, which makes it clear that $\left|\frac{\rho}{\rho-1}\right| = 1$. Conversely, suppose that $\left|\frac{\rho}{\rho-1}\right| \le 1$. Writing $\rho = \beta_\rho + i\gamma_\rho$ with β_ρ and γ_ρ real, we see that this inequality states that

$$\frac{\beta_\rho^2 + \gamma_\rho^2}{(\beta_\rho - 1)^2 + \gamma_\rho^2} \le 1,$$

or

$$2\beta_\rho \le 1,$$

which is equivalent to $\Re(\rho) = \beta_\rho \le \frac{1}{2}$. By the functional equation, $1 - \rho$ is also a non-trivial zero of the Riemann zeta function, and by hypothesis it must also satisfy the analogous inequality $1 - \beta_\rho = \Re(1 - \rho) \le \frac{1}{2}$. The only solution to this system of inequalities is $\Re(\rho) = \beta_\rho = \frac{1}{2}$, and so the Riemann hypothesis holds, as we wanted. ∎

From one perspective, Li's criterion, to be described presently, reframes this theorem by combining it with information about the distribution of zeros of $\zeta(s)$

vertically within the critical strip.

The following trivial lemma, which may be interpreted as the essential ingredient in the proof of the previous theorem, will be useful to us in the future.

Lemma 1.5.4 *Let ρ be any complex number not equal to 1. Then* $\left|\frac{\rho}{\rho-1}\right| \leq 1$ *if and only if* $\Re(\rho) \leq \frac{1}{2}$.

To describe Li's criterion itself, we begin by considering a convenient conformal mapping of the complex plane. Recall that the Riemann hypothesis states that all of the zeros of $\xi(s)$ should lie on the line $\Re(s) = \frac{1}{2}$. If we make the replacement $s = \frac{1}{1-w}$, then it is easy to see that $\Re(s) = \frac{1}{2}$ is equivalent to w lying on the unit circle, or $|w| = 1$. This suggests that we should examine the function

$$\varphi(s) = \xi\left(\frac{1}{1-s}\right),$$

since then questions about the behaviour of ξ on the critical line may be transformed into questions about the behaviour of φ on the unit circle. The Riemann hypothesis is equivalent to the statement that all of the zeros of $\varphi(s)$ lie on the unit circle. The equivalent statement that $\xi(s)$ has no zeros in the half-plane $\Re(s) > \frac{1}{2}$ corresponds to $\varphi(s)$ having no zeros in the open unit disc, specified by $|s| < 1$ (as we may readily observe by investigating the mapping $s \mapsto \frac{1}{1-s}$). Since ξ is an entire function, it follows that the Riemann hypothesis is equivalent to the statement that there exists a branch of the complex logarithm such that $\log(\varphi(s))$ is holomorphic on the interior of the unit disc (see [16, p.122]). Moreover, the logarithmic derivative $\frac{\varphi'(s)}{\varphi(s)} = \frac{d}{ds}\log(\varphi(s))$ is analytic at any point $s = s_0 \neq 1$ unless $\varphi(s_0) = 0$. Thus $\frac{\varphi'(s)}{\varphi(s)}$ is holomorphic on the interior of the unit disc if and only if $\varphi(s)$ has no zeros inside the unit disc, where the second statement is equivalent to the truth of the Riemann hypothesis. Defining

$$\psi(s) = \frac{d}{ds}\log(\varphi(s)) = \frac{\varphi'(s)}{\varphi(s)}, \tag{1.5.1}$$

and following our reasoning above, Li's criterion relates the truth of the Riemann hypothesis to the non-negativity of the power series coefficients of $\psi(s)$ about $s_0 = 0$.

More specifically, following [17], we define a sequence of numbers λ_n for non-negative integers n by

$$\lambda_{n+1} = \frac{1}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log(\xi(s))\right]_{s=1}. \tag{1.5.2}$$

We refer to λ_n as a *Li coefficient*. It can be shown that λ_n may also be expressed in the form

$$\lambda_n = \sum_{Z}^{*}\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right). \tag{1.5.3}$$

This form shows that λ_n is real for every n, by the symmetries on the elements of Z imposed by the functional equation for the completed zeta function. Finally, it turns out that the power series expansion of $\psi(s)$ about $s_0 = 0$ is exactly

$$\psi(s) = \frac{\varphi'(s)}{\varphi(s)} = \sum_{n=0}^{\infty}\lambda_{n+1}s^n. \tag{1.5.4}$$

The equivalent interpretations (1.5.2), (1.5.3), and (1.5.4) for the Li coefficients follow from the Hadamard product for the completed zeta function, as we will see when we prove more general versions of these forms in Chapter 3. We also remark that Biane, Pitman, and Yor have interpreted the classical Li coefficients as combinations of cumulants of a certain probability distribution in [1].

Li's criterion for the classical Riemann hypothesis [17, Theorem 1] is the following statement.

Theorem 1.5.5 (Li's criterion for the Riemann zeta function) *The Riemann hypothesis is true if and only if* λ_n *is non-negative for every positive integer* n.

Remark: Our expression in (1.5.3) for the numbers λ_n is not identical to that given by Li in [17]. In fact, Li's definition corresponds to λ_{-n} in ours. However, the symmetry conditions imposed by the functional equation on the zeros of ξ actually show that according to expression (1.5.3), $\lambda_n = \lambda_{-n}$ for all integers n. This follows because $\left(\frac{\rho}{\rho-1}\right) = \left(\frac{\rho-1}{\rho}\right)^{-1} = \left(\frac{(1-\rho)}{(1-\rho)-1}\right)^{-1}$, and $1-\rho$ is always a zero of $\xi(s)$ with the same multiplicity as ρ.

We will prove Li's criterion for the Riemann hypothesis in the next section, using arguments of Bombieri and Lagarias [2]. In Li's original paper, he proved his criterion using special properties of the Riemann zeta function (along with an analogous theorem for Dedekind zeta functions of number fields). Bombieri and Lagarias' work later showed how Li-type criteria could be formulated in much more general contexts. We remark that Keiper also studied some ideas related to Li's criterion in [13].

1.6 The Li criterion for complex multisets

The general framework under which generalizations to Li's criterion for the Riemann hypothesis may be most easily formulated in other contexts is due to work of Bombieri and Lagarias in 2000 [2]. Throughout this section, we let R be a vertically simple complex multiset. We begin with a lemma first stated by Bombieri and Lagarias without proof.

Lemma 1.6.1 *Assume that* $0 \notin R$ *if* $n < 0$, *and that* $1 \notin R$ *if* $n \geq 0$. *Furthermore, assume that*

$$\sum_R \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} < \infty. \tag{1.6.1}$$

Then for every integer n, *the sum*

$$\sum_R \Re\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right) \tag{1.6.2}$$

converges absolutely, where we take $0^n = 0$ *for all* $n > 0$ *and* $0^0 = 1$. *Moreover, if we also have that* $\sum_R^* \frac{1}{\rho}$ *is* $*$*-convergent, then so is* $\sum_R^* \left(1 - \left(\frac{\rho}{\rho-1}\right)^n\right)$ *for every integer* n.

Proof: If $n = 0$, the absolute convergence of both series examined in the lemma is clear since $1 \notin R$. In order to establish the absolute convergence of (1.6.2) for other integers n, we wish to bound the series of absolute values of the summands. From the assumption (1.6.1), by remarking that necessarily $|\rho| \to \infty$, we see that

$$\sum_R \frac{1}{|\rho|^2} < \infty$$

and

$$\sum_R \frac{|\Re(\rho)|}{|\rho|^2} < \infty. \tag{1.6.3}$$

It clearly follows from the first of these facts that

$$\sum_R \frac{1}{|\rho|^k} < \infty \tag{1.6.4}$$

for every integer $k \geq 2$. By identical reasoning we may also conclude that as long as $1 \notin R$, then we have that for integers $k \geq 2$

$$\sum_R \frac{1}{|\rho - 1|^k} < \infty, \tag{1.6.5}$$

and also that

$$\sum_{R} \frac{|\Re(\rho) - 1|}{|\rho - 1|^2} < \infty. \tag{1.6.6}$$

These last several observations follow formally from applications of the limit comparison test for mutual convergence of series [27, §19.2, Theorem III].

Since $\frac{\rho}{\rho-1} = \left(1 - \frac{1}{\rho}\right)^{-1}$, proving that (1.6.2) is absolutely convergent for every nonzero integer n is equivalent to proving that the two series

$$A_n = \sum_{R} \left| \Re\left(1 - \left(1 - \frac{1}{\rho}\right)^n\right)\right|$$

and

$$B_n = \sum_{R} \left| \Re\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right)\right|$$

are both convergent for all *positive* integers n, where we assume that $0 \notin R$ when computing A_n and that $1 \notin R$ when computing B_n.

We first prove the convergence of A_n for positive n under the hypotheses of the theorem. Using a binomial expansion, we obtain

$$\begin{aligned} A_n &= \sum_{R} \left| \Re\left(\sum_{k=1}^{n} \binom{n}{k} \left(-\frac{1}{\rho}\right)^k\right)\right| \\ &\le \sum_{R} \sum_{k=1}^{n} \binom{n}{k} \left| \Re\left(\left(\frac{1}{\rho}\right)^k\right)\right|. \end{aligned}$$

Clearly, for $k \ge 2$, we have $\left|\Re\left(\left(\frac{1}{\rho}\right)^k\right)\right| \le \left|\left(\frac{1}{\rho}\right)^k\right|$ (since for any complex number s, we have $|\Re(s)| \le |s|$, and $0 \notin R$). We may then write

$$\begin{aligned} A_n &\le \sum_{R} \left(n \frac{|\Re(\bar{\rho})|}{|\rho|^2} + \sum_{k=2}^{n} \binom{n}{k} \left| \left(\frac{1}{\rho}\right)^k \right| \right) \\ &= \left(n \sum_{R} \frac{|\Re(\rho)|}{|\rho|^2} \right) + \sum_{k=2}^{n} \binom{n}{k} \sum_{R} \left| \left(\frac{1}{\rho}\right)^k \right|, \end{aligned}$$

where all of the series in the last line converge by (1.6.3) and (1.6.4). Thus the series for A_n converges for every positive integer n.

Now consider B_n, which we may rewrite as

$$B_n = \sum_R \left| \Re\left(1 - \left(1 - \frac{1}{1-\rho}\right)^n\right)\right|.$$

Expanding the binomial as we did for A_n, and then simplifying in an identical manner (and using the fact that now $1 \notin R$), yields

$$B_n \leq n\left(\sum_R \frac{|\Re(\rho) - 1|}{|\rho - 1|^2}\right) + \sum_{k=2}^{n} \binom{n}{k} \sum_R \left|\left(\frac{1}{\rho - 1}\right)^k\right|,$$

where all of the series in this expression are convergent by (1.6.5) and (1.6.6). This shows that the series for B_n is convergent for every positive integer n. Together with our previous results, this also completes the proof that the sum in (1.6.2) converges absolutely for every integer n.

Finally, suppose that

$$\sum_R{}^* \frac{1}{\rho} \tag{1.6.7}$$

is $\star$-convergent. It follows that $0 \notin R$, and that

$$\sum_R{}^* \frac{1}{1-\rho} \tag{1.6.8}$$

is also $\star$-convergent as long as $1 \notin R$, since $\sum_R \left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = \sum_R \frac{1}{\rho(\rho-1)}$ is absolutely convergent (apply the previously mentioned limit comparison test for mutual convergence to this sum and the sum in (1.6.4) or (1.6.5), with $k = 2$, for example). Using the same binomial expansions as in our previous arguments for A_n and B_n, we obtain that for $n > 0$ and $1 \notin R$,

$$\sum_R{}^* \left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right) = \sum_R{}^* \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{1-\rho}\right)^k$$

$$= n\left(-\sum_R{}^{*}\frac{1}{1-\rho}\right) + \sum_{k=2}^{n}\binom{n}{k}(-1)^k\left(\sum_R{}^{*}\left(\frac{1}{1-\rho}\right)^k\right), \qquad (1.6.9)$$

and for $n > 0$ and $0 \notin R$,

$$\sum_R{}^{*}\left(1-\left(1-\frac{1}{\rho}\right)^n\right) = \sum_R{}^{*}\sum_{k=1}^{n}\binom{n}{k}(-1)^k\left(\frac{1}{\rho}\right)^k$$
$$= n\left(-\sum_R{}^{*}\frac{1}{\rho}\right) + \sum_{k=2}^{n}\binom{n}{k}(-1)^k\left(\sum_R{}^{*}\left(\frac{1}{\rho}\right)^k\right), \qquad (1.6.10)$$

where all of the sums in (1.6.9) and (1.6.10) are convergent by our observations in (1.6.7), (1.6.8), (1.6.4), and (1.6.5), and the interchanges in order of summation are justified because the sum over R is the only infinite summation. Thus we have established the last assertion of the lemma, and the proof is complete. ∎

The main theorem of Bombieri and Lagarias stated in [2, Theorem 1] is a version of Li's criterion valid for arbitrary complex multisets. For a complex multiset R, the proof uses the lemma above to ensure convergence of the series $\sum_R \Re\left(1-\left(\frac{\rho}{\rho-1}\right)^n\right)$, in order to establish equivalent conditions for the elements of R all to lie in the half-plane $\Re(\rho) \le \frac{1}{2}$. Moreover, the requirement (1.6.1) also imposes a necessary restriction on the distribution and density of the imaginary parts of the elements of R.

The theorem proved by Bombieri and Lagarias is given below, with proof, since the ideas contained in its proof are essential to understanding Li's crierion itself.

Theorem 1.6.2 (Bombieri and Lagarias) *Suppose that $1 \notin R$, and that*

$$\sum_R \frac{1+|\Re(\rho)|}{(1+|\rho|)^2} < \infty.$$

Then the following statements are equivalent:

(a) $\Re(\rho) \le \frac{1}{2}$ for every element ρ of R.

(b) $\sum_R \Re\left(1 - \left(\frac{\rho}{\rho-1}\right)^n\right) \geq 0$ *for every positive integer* n.

(c) For every $\epsilon > 0$*, there exists a constant* $c(\epsilon)$*, dependent on* ϵ*, such that*

$$\sum_{\rho \in R} \Re\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right) \geq -c(\epsilon)e^{\epsilon n}$$

for every $n \geq 1$.

Proof: It is obvious that (b) implies (c). Moreover, the fact that (a) implies (b) is clear from Lemma 1.5.4. This leaves us only to consider whether or not (c) implies (a). Indeed, let us suppose that (a) does not hold. Then there exists some $\rho \in R$ such that $\Re(\rho) > \frac{1}{2}$. Lemma 1.5.4 shows us that this is equivalent to the existence of some $\rho \in R$ such that $\left|\frac{\rho}{\rho-1}\right| > 1$. For any given $\rho \in R$, write $\rho = \beta_\rho + i\gamma_\rho$ with β_ρ and γ_ρ real. The hypothesis that

$$\sum_R \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} = \sum_{\rho \in R} \frac{1 + |\beta_\rho|}{(1 + \sqrt{\beta_\rho^2 + \gamma_\rho^2})^2}$$

converges implies that by taking sufficiently large $|\gamma_\rho|$, $\frac{|\beta_\rho|}{|\gamma_\rho|}$ may be forced to be arbitrarily small. It follows that

$$\left|\frac{\rho}{\rho - 1}\right|^2 = \frac{\beta_\rho^2 + \gamma_\rho^2}{(\beta_\rho - 1)^2 + \gamma_\rho^2}$$

tends to 1 as $|\gamma_\rho|$ tends to infinity. Thus, the existence of some $\rho \in R$ such that $\left|\frac{\rho}{\rho-1}\right| > 1$ implies further that the quantity $\left|\frac{\rho}{\rho-1}\right|$, for $\rho \in R$, attains its maximum for finitely many elements of R. Suppose now that

$$\max_{\rho \in R} \left|\frac{\rho}{\rho - 1}\right| = r_M$$

with $r_M > 1$, and that this maximum is attained by K elements $\rho_1, \ldots, \rho_K$ of R (repeated according to multiplicity). Then there exists some $\delta > 0$ such that for any $\rho \in R$ with $\rho \neq \rho_1, \ldots, \rho_K$,

$$\left|\frac{\rho}{\rho-1}\right| \leq r_M - \delta.$$

Let us denote the multiset of elements of R not equal to any of $\rho_1, \ldots, \rho_K$ by R_M. Moreover, for $j = 1, \ldots, K$, define ϕ_j to be the phase of $\frac{\rho_j}{\rho_j - 1}$, so that for each such value of j we have

$$\frac{\rho_j}{\rho_j - 1} = r_M e^{i\phi_j}.$$

Then for any $k \geq 1$ we have

$$\begin{aligned}\sum_{R} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] &= \sum_{j=1}^{K} (1 - r_M^n \cos(n\phi_j)) + \sum_{R_M} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] \\ &= \sum_{j=1}^{K} (1 - r_M^n \cos(n\phi_j)) + \sum_{\substack{R_M \\ |\rho|>n}} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] \\ &\quad + \sum_{\substack{R_M \\ 0\leq|\rho|\leq n}} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] \qquad (1.6.11)\end{aligned}$$

We wish to estimate the latter two sums in the last equality above. Indeed, using the same binomial expansion as in the proof of our last lemma, we can see that as n tends to infinity,

$$\begin{aligned}\left|\sum_{\substack{R_M \\ |\rho|>n}} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right]\right| &\leq \sum_{\substack{R_M \\ |\rho|>n}} \left|\Re\left[\sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{\rho-1}\right)^k\right]\right| \\ &\leq \sum_{\substack{R_M \\ |\rho|>n}} n \left|\Re\left(\frac{1}{\rho-1}\right)\right| + \sum_{\substack{R_M \\ |\rho|>n}} \sum_{k=2}^{n} \binom{n}{k} \left|\frac{1}{\rho-1}\right|^k \\ &= \sum_{\substack{R_M \\ |\rho|>n}} n \frac{|\Re(\rho)|}{|\rho-1|^2} + \sum_{\substack{R_M \\ |\rho|>n}} \sum_{k=2}^{n} \binom{n}{k} \left|\frac{1}{\rho-1}\right|^k \\ &\leq \sum_{\substack{R_M \\ |\rho|>n}} n \frac{|\Re(\rho)|}{|\rho-1|^2} + \sum_{\substack{R_M \\ |\rho|>n}} \sum_{k=2}^{n} n^k \left|\frac{1}{\rho-1}\right|^k. \qquad (1.6.12)\end{aligned}$$

Here, since n is tending to infinity and $|\rho| > n$, we have in particular that for $2 \le k \le n$,

$$n^k \left|\frac{1}{\rho - 1}\right|^k = O(1) = O\left(\left|\frac{n}{\rho - 1}\right|^2\right),$$

and so

$$\sum_{k=2}^{n} n^k \left|\frac{1}{\rho - 1}\right|^k = O\left(n\left|\frac{n}{\rho - 1}\right|^2\right).$$

Here, the implied constant may be taken to be independent of ρ. Thus we find that

$$\begin{aligned}\left|\sum_{\substack{R_M\\|\rho|>n}} \Re\left[1 - \left(\frac{\rho}{\rho - 1}\right)^n\right]\right| &\le \sum_{\substack{R_M\\|\rho|>n}} n\frac{|\Re(\rho)|}{|\rho - 1|^2} + \sum_{\substack{R_M\\|\rho|>n}} O\left(n\left|\frac{n}{\rho - 1}\right|^2\right)\\ &= \sum_{\substack{R_M\\|\rho|>n}} O\left(\frac{n|\Re(\rho)| + n^3}{|\rho|^2}\right)\\ &= O(n^3), \end{aligned} \tag{1.6.13}$$

where the last line follows from the hypothesis that

$$\sum_{R} \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} < \infty.$$

To estimate the last term in (1.6.11), notice that based on our previous definitions, as n tends to infinity, we have

$$\left|\sum_{\substack{R_M\\|\rho|\le n}} \Re\left[1 - \left(\frac{\rho}{\rho - 1}\right)^n\right]\right| = C(n)O((r_M - \delta)^n),$$

where for any $n > 0$, we define $C(n)$ to be the number of elements ρ of R_M satisfying $|\rho| < n$, counted with multiplicity. We again invoke the hypothesis

$$\sum_{R} \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} < \infty,$$

which also clearly implies that

$$\sum_{R} \frac{1}{(1+|\rho|)^2} < \infty,$$

from which it follows that $C(n) = O(n^2)$ as n tends to infinity. Thus, in fact, we have that

$$\left| \sum_{\substack{R_M \\ |\rho|\le n}} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] \right| = O(n^2(r_M - \delta)^n) \tag{1.6.14}$$

as n tends to infinity. Now, combining (1.6.13) and (1.6.14) with (1.6.11) shows us that as n tends to infinity,

$$\sum_{R} \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right] = \sum_{j=1}^{K} (1 - r_M^n \cos(n\phi_j)) + O(n^2(r_M - \delta)^n + n^3). \tag{1.6.15}$$

Dirichlet's theorem on simultaneous diophantine approximation (see, for example, [23, Chapter 2]) implies that, for any $\delta' > 0$, there exists some subsequence $\{n_l\}_{l=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that for every l, there exist integers $k_1(l), \ldots k_j(l)$ with the property that

$$|n_l\phi_j - 2\pi k_j(l)| < \delta'$$

for $1 \le j \le K$. In view of (1.6.15), this last statement may be re-interpreted to say precisely that there exists a subsequence $\{n_l\}_{l=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that

$$\lim_{l\to\infty} \frac{\sum_R \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^{n_l}\right]}{-K r_M^{n_l}} = 1,$$

which contradicts statement (c) of the theorem (take any ϵ satisfying $0 < \epsilon < \log r_M$). Thus we have shown that a violation of (a) implies a violation of (c), and so (c) implies (a). This completes the proof of the theorem. ∎

The third equivalent condition (c) in the theorem is striking. It initially appears to be strictly stronger than (b). However, as the proof shows, (c) is actually equivalent to (b). This implies the statements that, in fact, *infinitely many* Li coefficients must be negative for the Riemann hypothesis to be false, and if at least one Li coefficient is negative, then there must exist a subsequence of Li coefficients $\{\lambda_{\alpha_n}\}$, all of whose elements are negative, and where the magnitude of λ_{α_n} grows exponentially with α_n as n tends to infinity.

For completeness, we give the corresponding direct analogue to the version of Li's criterion for the Riemann zeta function, stated in the previous section, below (where taking R to be $Z = Z(\xi)$ yields a stronger version of (1.5.5) itself).

Corollary 1.6.3 (Generalized Li criterion) *We take the same hypotheses as in Theorem 1.6.2. Furthermore, assume that whenever ρ is an element of R, then $1-\rho$ and $\bar{\rho}$ are elements of R with the same multiplicity as ρ. Then the following statements are equivalent:*

(a) $\Re(\rho) = \frac{1}{2}$ *for every element ρ of R.*

(b) $\lambda_n(R) = \sum_R^* \left(1 - \left(\frac{\rho}{\rho-1}\right)^n\right) \geq 0$ *for every positive integer n, where the sum is $\star$-convergent to a real value.*

(c) For every $\epsilon > 0$, there exists a constant $c(\epsilon)$, dependent on ϵ, such that

$$\sum_{\rho\in R} \left(1 - \left(\frac{\rho}{\rho-1}\right)^n\right) \geq -c(\epsilon)e^{\epsilon n}$$

for every $n \geq 1$.

The proof of this corollary is straightforward upon the imposition of its symmetry requirements on R to the statement of Theorem 1.6.2. Indeed, taking R to be Z, and noting that the fact that ξ is an entire function of order 1 implies that

$$\sum_{Z} \frac{1}{|\rho|^2} < \infty,$$

and the fact that $\Re(\rho)$ is uniformly bounded for $\rho \in Z$, we see that our earlier statement of Li's criterion for the Riemann hypothesis is an easy consequence.

As we have mentioned previously, Theorem 1.6.2, and the ideas involved in its proof, are the basic tools permitting the straightforward formulation of generalizations of Li's criterion in wider contexts. We present some of these in the next section.

1.7 Further generalizations of Li's criterion

In this section, we discuss several generalizations of Li's criterion which have been formulated since its introduction. The generalizations of particular interest to us in this section are due to Pedro Freitas [11], Lejla Smajlović [25], and Francis Brown [3], which we present separately in the following three subsections. In the last case, we also mention several errors in the source work, one of which appears to render one of its main theorems conjectural until a corrected argument can be given. Our work in Chapters 2 and 3 of this thesis will relate in large part to combining ideas from the various approaches presented in this section.

1.7.1 The Li criterion for arbitrary half-planes

One natural question to ask, in our present context of discussion, is whether it is possible to formulate Li-type criteria for statements weaker than the Riemann hypothesis itself. We start with the following definition.

δ-Riemann hypothesis (δ-RH) Every non-trivial zero ρ of the Riemann zeta function satisfies $\Re(\rho) \leq \delta$. In particular, when $\delta = \frac{1}{2}$, this is the classical Riemann hypothesis.

Analogously, the δ-generalized Riemann hypothesis (δ-GRH) for a function F in the Selberg class (or the larger class $\mathcal{S}^{\sharp\flat}$) is the same assertion about the non-trivial zeros of F.

Our question now translates to whether we can formulate a Li-type criterion for the δ-Riemann hypothesis with $\delta > \frac{1}{2}$. Freitas considered this question in 2004 [11]. To begin discussing his results, for every $\tau > 0$ and non-negative integer n, we define

$$\alpha_{n+1}(\tau) = \frac{1}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log(\xi(s))\right]_{s=\tau}, \tag{1.7.1}$$

and

$$\lambda_n(\tau) = \lambda_n(\xi, \tau) = \sum_Z \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right). \tag{1.7.2}$$

Freitas then proved the following results, in analogue to Li's discussion of his criterion for the classical Riemann hypothesis.

Lemma 1.7.1 *For every real number $\tau > 0$ such that τ is not a zero of $\xi(s)$, and every positive integer n, we have*

$$\alpha_n(\tau) = \frac{1}{\tau}\sum_Z \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right) = \frac{1}{\tau}\lambda_n(\tau).$$

Furthermore, whenever $\frac{1}{1-z_0}$ is not a zero of $\xi(s)$, we denote the power series coefficients of $\psi(s)$ (defined by (1.5.1)) about $z_0 \neq 1$ by $d_n(z_0)$, so that for s

sufficiently close to z_0 we may write

$$\psi(s) = \frac{\varphi'(s)}{\varphi(s)} = \sum_{n=0}^{\infty} d_n(z_0)(s - z_0)^n.$$

Then we also have that for every positive integer n,

$$\alpha_n(\tau) = \frac{1}{\tau^{n+1}} d_{n-1}\left(1 - \frac{1}{\tau}\right). \tag{1.7.3}$$

Theorem 1.7.2 (Freitas' half-plane criterion for the Riemann ξ function) *Let $\tau \geq 1$. Then $\zeta(s)$ satisfies the $\frac{\tau}{2}$-RH if and only if $\alpha_n(\tau) \geq 0$ for every positive integer n.*

Equivalently, the non-trivial zeros ρ of $\zeta(s)$ all lie in the strip $1 - \frac{\tau}{2} \leq \Re(\rho) \leq \frac{\tau}{2}$ if and only if $\alpha_n(\tau) \geq 0$ for every positive integer n.

Interestingly, Freitas' original work uses arguments analogous to Li's own in [17], rather than the more general framework of Bombieri and Lagarias discussed in the previous section. Since Freitas' results are special cases of theorems we will prove in Chapter 3, we will omit detailed proofs here. It will, however, prove convenient for us to generalize the ideas of Bombieri and Lagarias appropriately to permit discussion of Li's criterion for the δ-GRH. We accomplish this goal with the following two results.

Theorem 1.7.3 (Arbitrary half-plane criterion) *Let $\tau > 0$ be a real number, and suppose that $\tau \notin R$. Assume that*

$$\sum_{R} \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} < \infty. \tag{1.7.4}$$

Then the following statements are equivalent:

(a) $\Re(\rho) \leq \frac{\tau}{2}$ for every element ρ of R.

(b) $\sum_R \Re\left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right) \geq 0$ *for every positive integer* n.

(c) For every $\epsilon > 0$, *there exists a constant* $c(\epsilon)$, *dependent on* ϵ, *such that*

$$\sum_R \Re\left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right) \geq -c(\epsilon)e^{\epsilon n}$$

for every $n \geq 1$.

Proof: Under the hypotheses of the theorem, we define $R_\tau = \{\frac{\rho}{\tau} \mid \rho \in R\}$, where the element $\frac{\rho}{\tau}$ has multiplicity in R_τ equal to that of ρ in R. By assumption, $\tau \notin R$, and so 1 is not an element of R_τ. Moreover, we have that

$$\begin{aligned}\sum_{R_\tau} \frac{1 + |\Re(\rho)|}{(1 + |\rho|)^2} &= \sum_R \frac{1 + \left|\Re\left(\frac{\rho}{\tau}\right)\right|}{(1 + \left|\frac{\rho}{\tau}\right|)^2} \\ &= \tau \sum_R \frac{\tau + |\Re(\rho)|}{(\tau + |\rho|)^2} \\ &< \infty,\end{aligned}$$

and it is clear that $\sum_R \frac{\tau+|\Re(\rho)|}{(\tau+|\rho|)^2}$ converges simultaneously with $\sum_R \frac{1+|\Re(\rho)|}{(1+|\rho|)^2}$. These facts show that the multiset R_τ satisfies the hypotheses of Theorem 1.6.2. The theorem then tells us that the following three conditions are equivalent:

(i) $\Re(\rho) \leq \frac{1}{2}$ for every element ρ of R_τ.

(ii) $\sum_{R_\tau} \Re\left(1 - \left(\frac{\rho}{\rho-1}\right)^n\right) \geq 0$ for every positive integer n.

(iii) For every $\epsilon > 0$, there exists a constant $c(\epsilon)$, dependent on ϵ, such that

$$\sum_{R_\tau} \Re\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right) \geq -c(\epsilon)e^{\epsilon n}$$

for every $n \geq 1$.

Condition (i) is equivalent to saying that $\Re\left(\frac{\rho}{\tau}\right) \leq \frac{1}{2}$ for every element ρ of R, or that $\Re(\rho) \leq \frac{\tau}{2}$ for every element ρ of R (since $\tau > 0$), which is condition (a) in the statement of the theorem. Similarly, condition (iii) is equivalent to saying that for every $\epsilon > 0$, there exists a constant $c(\epsilon)$, dependent on ϵ, such that

$$\begin{aligned}\sum_R \Re\left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right) &= \sum_R \Re\left(1 - \left(\frac{\frac{\rho}{\tau}}{\frac{\rho}{\tau} - 1}\right)^n\right) \\ &= \sum_{R_\tau} \Re\left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right) \\ &\geq -c(\epsilon)e^{\epsilon n},\end{aligned}$$

for every positive integer n, which is condition (c) in the statement of the theorem, and condition (b) may be shown to be equivalent to (ii) by replacing the right hand side of the last inequality by 0. This completes the proof. ∎

Corollary 1.7.4 (Generalized Li criterion for arbitrary strips) *We take the same hypotheses as in Theorem 1.7.3, along with the assumption that $\tau \geq 1$. Additionally, we make the same symmetry assumptions on the elements of R as in Corollary 1.6.3: Assume that whenever ρ is an element of R, then $1-\rho$ and $\bar{\rho}$ are elements of R with the same multiplicity as ρ. Then the following statements are equivalent:*

(a) $1 - \frac{\tau}{2} \leq \Re(\rho) \leq \frac{\tau}{2}$ *for every element ρ of R.*

(b) $\lambda_n(R,\tau) = \sum_R^* \left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right) \geq 0$ *for every positive integer n, where the sum is $\star$-convergent to a real value.*

(c) For every $\epsilon > 0$, there exists a constant $c(\epsilon)$, dependent on ϵ, such that

$$\lambda_n(R,\tau) = \sum_R^* \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right) \geq -c(\epsilon)e^{\epsilon n}$$

for every $n \geq 1$.

Proof: Applying Theorem 1.7.3 to R ensures that the following conditions are equivalent.

(i) $\Re(\rho) \le \frac{\tau}{2}$ for every element ρ of R.

(ii) $\sum_R \Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) \ge 0$ for every positive integer n.

(iii) For every $\epsilon > 0$, there exists a constant $c(\epsilon)$, dependent on ϵ, such that

$$\sum_R \Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) \ge -c(\epsilon)e^{\epsilon n}$$

Since, by assumption, whenever ρ is an element of R, $1-\rho$ is also an element of R with the same multiplicity, condition (i) equivalently says that for every element ρ of R, we have $\Re(1-\rho) \le \frac{\tau}{2}$, or $\Re(\rho) \ge 1-\frac{\tau}{2}$. Putting these two equivalent meanings for (i) together, we see that the statement (i) above may be replaced by condition (a) from the statement of the corollary.

Now consider condition (ii) above. As $\bar{\rho}$ is in R whenever ρ is in R, with the same multiplicity, it follows that for any $T > 0$, the sum

$$\sum_{\substack{\rho\in R \\ |\mathbf{Im}(\rho)|\le T}} \left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right)$$

is real. Since the sum in conditions (ii) and (iii) converges absolutely, we then have

$$\begin{aligned}\sum_R \Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) &= \lim_{T\to\infty} \sum_{\substack{\rho\in R \\ |\mathbf{Im}(\rho)|\le T}} \Re\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) \\ &= \lim_{T\to\infty} \sum_{\substack{\rho\in R \\ |\mathbf{Im}(\rho)|\le T}} \left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right) \\ &= \sum_R{}^{*} \left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right),\end{aligned}$$

and the last sum is $*$-convergent and equal to the sum in condition (ii) for every positive integer n. Thus conditions (ii) and (iii) may equivalently be replaced by conditions (b) and (c) of the corollary, respectively, which completes the proof. ∎

Indeed, applying this corollary to the multiset Z of non-trivial zeros of the Riemann zeta function proves Freitas' generalization of Li's criterion without appealing to special properties of the Riemann zeta function itself.

1.7.2 The Li criterion for the Selberg class

Smajlović has recently generalized Li's criterion to functions in the extension $\mathcal{S}^{\sharp\flat}$ of the Selberg class [25], defined earlier in this chapter. This was accomplished via proofs of some fundamental properties of functions in the class $\mathcal{S}^{\sharp\flat}$, and then applying the framework of Bombieri and Lagarias to this setting. We note that the generalization of Li's criterion to this context has also been studied by Omar and Mazhouda in [20], [21].

Definition: Let F be a function in $\mathcal{S}^{\sharp\flat}$. Then we denote the complex multiset of non-trivial zeros of F by $Z(F)$.

Now, if F is a function in $\mathcal{S}^{\sharp\flat}$ such that $1 \notin Z(F)$, then for every integer n we define

$$\lambda_n(F,1) = \sum_{Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho - 1}\right)^n\right), \tag{1.7.5}$$

as long as the sum on the right is $*$-convergent.

Remark: Our definition of $\lambda_n(F,1)$ is not identical to the definition of the numbers $\lambda_F(n)$ in [25]. In fact, according to our definition, $\lambda_n(F,1) = \lambda_F(-n)$ in the notation of [25]. This deviation is for the purposes of maintaining consistent notation throughout the present thesis. It may easily be observed that in fact,

$\overline{\lambda_n(F,1)} = \lambda_{-n}(F,1)$, since $\overline{\left(\frac{\rho}{\rho-1}\right)^{-1}} = \frac{\bar{\rho}-1}{\bar{\rho}} = \frac{1-\bar{\rho}}{(1-\bar{\rho})-1}$, and $1-\bar{\rho}$ is in $Z(F)$ if and only if ρ is, with the same multiplicity.

The following theorem is proved in [25, Theorem 4.1], and establishes the convergence of the sum in (1.7.5) under appropriate conditions on the function F.

Theorem 1.7.5 (Convergence of $\lambda_n(F,1)$) *Let F be a function in $\mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Then the series on the right of* (1.7.5)*, defining $\lambda_n(F,1)$, is $*$-convergent for every integer n. Moreover, the series*

$$\sum_{Z(F)} \Re\left(1-\left(\frac{\rho}{\rho-1}\right)^n\right) = \Re(\lambda_n(F,1))$$

is absolutely convergent for every integer n.

The proof of this theorem in [25] relies on Lemma 1.6.1 and two other results given below. We state these two results since we will need them for our own proofs in Chapter 3. The first provides a characterization of the behaviour of ξ_F for any function $F \in \mathcal{S}^\sharp$, given in [25, Lemma 3.3].

Lemma 1.7.6 *Let F be a function in $\mathcal{S}^\sharp$. Then ξ_F is an entire function of order 1.*

This lemma allows the construction of Hadamard products for the function ξ_F whenever F is in $\mathcal{S}^\sharp$, according to the Hadamard factorization theorem for entire functions of order 1 (see [18, Theorem 6.2.1] or [16, Theorem 3.5, Ch.X]). If $F \in \mathcal{S}^{\sharp\flat}$ and $0 \notin Z(F)$, then as in [25, (8)], the Hadamard product takes the form

$$\xi_F(s) = \xi_F(0)e^{b_F s}\prod_{Z(F)}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}, \tag{1.7.6}$$

where $b_F = \frac{\xi_F'(0)}{\xi_F(0)}$, and the infinite product is absolutely and uniformly convergent on every disc $|s| \le R$ for $R > 0$ (see [16, Theorem 2.3, Ch.X]). Additionally, under the same conditions on F, we have a simpler $*$-convergent product formula for ξ_F, and a $*$-convergent sum for $\frac{\xi_F'(s)}{\xi_F(s)}$, proved in [25, Theorem 3.4] and stated in the next theorem. This theorem is again crucial to the proof of Theorem 1.7.5 in [25], and will be required in our proofs in Section IV of this proposal, along with the more standard Hadamard product representation (1.7.6).

Theorem 1.7.7 (Simplified Hadamard product for $\mathcal{S}^{\sharp\flat}$) *Let F be a function in $\mathcal{S}^{\sharp\flat}$, and assume that $0 \notin Z(F)$. Then whenever s is a complex number, not in $Z(F)$, we have*

(a) $\frac{\xi_F'(s)}{\xi_F(s)} = \sum_{Z(F)}^{*} \frac{1}{s-\rho}$,

(b) $\xi_F(s) = \xi_F(0) \prod_{Z(F)}^{*} \left(1 - \frac{s}{\rho}\right)$,

where the sum in (a) and the product in (b) are $*$-convergent. Moreover, the constant b_F defined in (1.7.6) *is given by the $*$-convergent sum* $-\sum_{Z(F)}^{*} \frac{1}{\rho}$.

Using these three results, Smajlović was able to prove the following two theorems, which give the direct generalization of Li's criterion to the context of the class of functions $\mathcal{S}^{\sharp\flat}$.

Lemma 1.7.8 *Let F be a function in $\mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Then for every integer $n \ge 0$, the number $\lambda_n(F,1)$ defined by* (1.7.5) *is also given by the expression*

$$\lambda_{n+1}(F,1) = \frac{1}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log \xi_F(s)\right]_{s=1}. \tag{1.7.7}$$

Furthermore, if we define $\varphi_F(s) = \xi_F\left(\frac{1}{1-s}\right)$ for all $s \ne 1$, and also define $\psi_F(s) = \frac{d}{ds}\log\varphi_F(s) = \frac{\varphi_F'(s)}{\varphi_F(s)}$ whenever $\varphi_F(s) \ne 0$, then the numbers $\lambda_{n+1}(F,1)$ are the power series coefficients of $\psi_F(s)$ about $s_0 = 0$. More precisely, for $|s|$ sufficiently small, *we may write*

$$\psi_F(s) = \sum_{n=0}^{\infty} \lambda_{n+1}(F,1)s^n. \tag{1.7.8}$$

Theorem 1.7.9 (Generalized Li criterion for the Selberg class) *Let F be a function in $\mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Then the GRH holds for F if and only if $\Re(\lambda_n(F,1)) \geq 0$ for every positive integer n.*

Once again, these results are special cases of more general theorems which we will prove in Chapter 3, and so we omit proofs here.

1.7.3 Li-type criteria for zero-free regions

Li's criterion states that the generalized Riemann hypothesis for a given function is equivalent to the non-negativity of the real part of each element of the corresponding of sequence of Li coefficients. This invites us to ask whether the non-negativity of *finitely* many of these numbers tells us anything about the function in question.

Francis Brown gave a partial answer to this question in 2005. Indeed, in [3, Theorem 3], Brown proves that the non-negativity of the real parts of the first N Li coefficients implies the existence of zero-free regions for the corresponding function in an effective way. Conversely, [3, Theorem 2] attempts to give a partial converse, showing how the existence of zero-free regions of a particular form implies the non-negativity of the first N Li coefficients, also in an effective way. Unfortunately, as we will discuss in this section (and in more detail in Chapter 3), there are some problems in Brown's proof of a supporting lemma for [3, Theorem 2] which seem to render its statement conjectural without some further work.

Brown's statements are given in sufficient generality to apply to the entire Selberg class, and we outline them (and the motivation for his work) in this section. In Chapter 3, we will consider the problem of generalizing [3, Theorem 2] to the case of Li coefficients for the quasi-Riemann case in the Selberg class, under an appropriate conjecture (which is needed, for the moment, to bypass some problems present in the arguments of [3]).

In [3], Brown studies functions ξ_B satisfying a certain set of assumptions, listed below. The hypotheses required by his arguments, not always simultaneously, are the following.

0. ξ_B is an entire function of order < 2 with $\xi_B(0) \neq 0$ and which satisfies the functional equation

$$\xi_B(s) = w\bar{\xi}_B(1-s)$$

for all $s \in \mathbb{C}$, where w is a fixed complex number of modulus 1.

I. The zeros of ξ_B lie in the critical strip.

II. For $T > 0$, if $2N(T)$ is the number of zeros ρ of $\xi_B(s)$ satisfying $|\Im(\rho)| \leq T$, then we have

$$N(T) = aT\log T + bT + \epsilon(T),$$

where $|\epsilon(T)| \leq c\log^+ T + d$ for some constants $a, b, c, d \in \mathbb{R}$, such that $a, c, d > 0$ and $3a + b \geq 0$, and where $\log^+(T) = \max\{\log T, 0\}$.

Remark: In [3], Brown also includes a third hypothesis (III), which, as stated in the paper, regards the possible presence of a pole of $\xi_B(s)$ at $s = 1$. However, this hypothesis is superfluous. In [3], this extra hypothesis always appears along with (0), which states that ξ_B is an entire function. Indeed, if ξ_B is thought of as a completed function corresponding to some L-function with a potential pole at $s = 1$, Brown's hypothesis (III) becomes relevant in terms of the underlying L-function itself. However, since Brown only considers the function ξ_B (corresponding to a completed function which is entire), the inclusion of this hypothesis in its given form in [3] seems to constitute an oversight.

Let us now consider an arbitrary complex number $s \neq 1$. We remark that the inequality

$$\left|\frac{s}{s-1}\right| \leq 1$$

is equivalent to the statement that $\Re(s) \leq \frac{1}{2}$ (as noted in Lemma 1.5.4).

If we consider a complex multiset S whose elements ρ all satisfy $\left|\frac{z}{z-1}\right| \leq 1$, then it follows that every element of S must lie in the half-plane to the left of the line $\Re(s) = \frac{1}{2}$. Moreover, Bombieri and Lagarias' work on Li's criterion shows that, under an additional assumption on the structure of S (which may, in one sense, be interpreted as a limitation on the number of elements of S with small modulus), both of these facts are equivalent to the non-negativity of the real parts of all of the sums $\sum_S \Re\left[1 - \left(\frac{\rho}{\rho-1}\right)^n\right]$, for $n \geq 1$.

This observation raises the question of what happens if, instead, we require

$$\left|\frac{z}{z-1}\right| \leq r \tag{1.7.9}$$

for some $r \geq 1$, and this question is the starting point for Brown's work [3, p.3]. This condition turns out to be equivalent to statement that the elements of S all lie outside an open disc whose bordering circle is specified by the points $1 \pm i(\sqrt{r^2-1})^{-1}$ and $\frac{r}{r+1}$. Indeed, for any $r > 1$, let us denote this open disc by $D_1(r)$, and moreover let us denote the disc obtained by reflecting $D_1(r)$ about the line $\Re(s) = \frac{1}{2}$ by $D_2(r)$ (i.e. $\rho \in D_2(r)$ if and only if $1 - \rho \in D_1(r)$). We then let $C(r)$ denote the critical strip with its intersection with $D_1(r)$ and with $D_2(r)$ removed.

Interpreting S as the set of non-trivial zeros of a function in the Selberg class, whose elements must all lie in the critical strip and satisfy the symmetry relation that $\rho \in S$ if and only if $1 - \bar{\rho} \in S$, this means that for any given $r > 1$, the truth of the inequality (1.7.9) for every element of S is equivalent to the existence of zero-free regions of the corresponding function inside the critical strip about 0 and 1.

Under hypothesis (II), Brown was able to connect (1.7.9) with the non-negativity of the real parts of finitely many Li coefficients $\lambda_n(\xi_B)$ in an effective sense. Indeed, if $r = 1$, then (1.7.9) is equivalent to every Li coefficient having non-negative real part; analogously, the case that $r > 1$ results in the first N Li coefficients having non-negative real parts for some finite N. This result is one that is naturally expected when performing this analysis, and is suggested by ideas in the proof given by Bombieri and Lagarias for their generalized version of Li's criterion (and reproduced in our proof of Theorem 1.6.2). Here, we state some of Brown's results without proof. Unfortunately, as mentioned previously, the first of these appears to remain conjectural, by virtue of some errors in the proof of a supporting lemma (to be discussed following its statement in Conjecture 3.2.7). In Chapter 3, we give a generalization of this result to the case of Li coefficients corresponding to restricted cases of the generalized quasi-Riemann hypothesis, under an appropriate conjecture.

Definition: For a function ξ_B satisfying (0) above, and for any positive integer k and any real $\tau \geq 1$, define the Li coefficient $\lambda_k(\xi_B, \tau)$ by

$$\lambda_k(\xi_B, \tau) = \sum_{Z(\xi_B)}^{*} \left[1 - \left(\frac{\rho}{\rho - \tau} \right)^k \right].$$

In the following two theorems (given by Brown in [3, Theorem 2] and [3, Theorem 3]), ξ_B is taken to be a function satisfying hypotheses (0), (I), and (II) above.

The first statement we would like to give explains how the existence of zero-free regions of the particular forms discussed above imply the non-negativity of finitely many Li coefficients in an effective sense. It was originally presented as [3, Theorem

2]. Unfortunately, here we must give its statement as a conjecture.

Conjecture 1.7.10 *Let* $r > 1$ *and* $T = \frac{1}{\sqrt{r^2-1}}$ *(so that* $r = \left(1 + \frac{1}{T^2}\right)^{\frac{1}{2}}$*. Then there exists some constant* T_0*, dependent only on* a, b, c, d *(as defined in (II)), such that as long as* $T > T_0$*, we have that if all of the zeros of* ξ_B *lie in the region* $C(r)$*, then* $\Re(\lambda_k(F,1))$ *is non-negative for* $1 \leq k \leq 2T^2 \log T$*.*

Moreover, if $b \geq 0$*, then we can take* $T_0 = 12 \max\left\{1, \frac{2c}{a}, \frac{2d}{3a+b}\right\}$*, and if* $b < 0$*, then we can take* $T_0 = \max\left\{2T_1^2 \log T_1, e^{1-\frac{b}{a}}\right\}$*, where* $T_1 = \max\left\{5, \frac{3c}{a}, \frac{d}{3a+b}\right\}$*.*

The next result is given as [3, Theorem 3]. Its proof in [3] is unaffected by the problem disrupting the validity of [3, Theorem 2], and so it may be stated unconditionally.

Theorem 1.7.11 *There exist real constants* N_0, μ, ν *(dependent only on* a, b, c, d*) such that, if* N *is any integer with* $N > N_0$ *and such that* $\Re(\lambda_k(F,1))$ *is non-negative for* $1 \leq k \leq N$*, then all of the zeros of* ξ_B *are contained in a region of the form* $C(r)$*, where*

$$r = \left(1 + \frac{1}{T^2}\right)^{\frac{1}{2}}$$

with

$$T = \left(\frac{N}{\mu\left(\log(N\nu)\right)^2}\right)^{\frac{1}{3}}$$

Moreover, writing $w = \max\{1, a, b, c, d\}$*, we cn take* $\mu = 27w$*,* $\nu = w^3$*, and* $N_0 = 1200 \log 100w$*.*

Remark: As we have mentioned, Brown's original proof of [3, Theorem 2] (stated above as Conjecture 1.7.10) includes some errors. Notably, the proof of [3, Lemma 5] is erroneous in two ways. One of these is easily resolved, but the other presents more significant difficulties. We elaborate on these problems in Section 3.2.

In Chapter 3, we will show how Conjecture 1.7.10 may be generalized to a conditional theorem in the case of Li coefficients $\lambda_k(F,\tau)$ for $\tau > 1$ (which, in our formulation here, is a natural problem), under an appropriate conjecture.

Note that Brown also applies a similar idea to show that the non-negativity of real part of the second Li coefficient, $\lambda_2(\xi_B, 1)$, is sufficient to show the non-existence for a Siegel zero for ξ_B, under some additional hypotheses (an idea which we also might hope to generalize to the $\tau > 1$ case). It is interesting to note that Biane, Pitman, and Yor's interpretation of Li coefficients for the Riemann zeta function as combinations of cumulants of a probability distribution (mentioned briefly at the end of Section 1.5) allows an indirect argument for the non-negativity of λ_2 in the classical case [1]. It may prove interesting to consider the idea of generalizing this interpretation for Li coefficients corresponding to other functions, in future work.

1.8 Arithmetic formulae for the Li coefficients

In Li's classical case, his original formulation for his criterion gave three equivalent interpretations for the coefficients λ_n. One of these, (1.5.4), was as power series coefficients for the function $\phi_n(s)$ about $s = 0$. The second, (1.5.3), was as a sum over the non-trivial zeros of the zeta function. The last, (1.5.2), was as a derivative of $s^n \log \xi(s)$ evaluated at $s = 1$.

In fact, all of these identities arose as a consequence of the Hadamard product representation (1.3.1) for $\xi(s)$. However, from classical analytic number theory, we also have other information about the structure of $\xi(s)$, and thus about the coefficients λ_n. This raises the question of what other equivalent representations we may give to the Li coefficients which may help us to verify their non-negativity.

This idea was first explored by Bombieri and Lagarias in [2]. Bombieri and Lagarias originally approached the derivation of more useful formulae for Li coefficients from the perspective of Weil's explicit formula for sums over zeros of the zeta function [2]. However, as has been observed by others (see, for example, Coffey [4]), appealing to the explicit formula is not necessary, and in fact one may obtain the arithmetic formula of Bombieri and Lagarias using an elementary argument from the definitions of the Li coefficients and of the completed zeta function. We have

$$\xi(s) = s(s-1)\pi^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

and for any positive integer n and $\tau \geq 1$, by Lemma 1.7.1,

$$\lambda_n(\tau) = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log(\xi(s))\right]_{s=1}.$$

With some manipulation, one can obtain Bombieri and Lagarias' arithmetic formula for λ_n from these identities. To proceed with our discussion of this idea, we first give some necessary definitions and lemmas. We defer proofs of the results stated in this section to later chapters (usually in more general forms), unless otherwise noted.

Lemma 1.8.1 *Let n be a positive integer. Then we have*

$$\lambda_n(\tau) = \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{\tau^k}{(k-1)!}\left[\frac{d^k}{ds^k}\log\xi(s)\right]_{s=1}.$$

Digamma and polygamma functions: We define the digamma function, denoted by Ψ, to be the logarithmic derivative of the gamma function. More precisely, for complex s not equal to a non-positive integer, we have

$$\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

Similarly, for any non-negative integer n, we define the nth polygamma function, denoted by the standard derivative notation $\Psi^{(n)}$, to be the nth derivative of the digamma function.

Hurwitz zeta function: The Hurwitz zeta function of complex variables q and s is defined by

$$\zeta(q,s) = \sum_{m=0}^{\infty} \frac{1}{(q+m)^s}$$

whenever $\Re(q) > 0$ and $\Re(s) > 1$.

The following theorem is a standard result on digamma and polygamma functions (see, for example, [19]), and we omit its proof.

Theorem 1.8.2 (Representations of the digamma and polygamma functions) *Whenever* $|s-1| < 1$, *we have*

$$\Psi(s) = -\gamma - \sum_{m=1}^{\infty} \zeta(m+1)(-1)^m (s-1)^m.$$

Moreover, whenever $s \neq 0, -1, -2, \ldots$, *we have*

$$\Psi(s) = -\gamma + \sum_{m=0}^{\infty} \left(\frac{1}{m+1} - \frac{1}{m+s} \right).$$

It also follows that whenever $|s-1| < 1$ *and* n *is a positive integer, we have that*

$$\Psi^{(n)}(s) = (-1)^{n+1} \sum_{m=0}^{\infty} (-1)^m \frac{(n+m)!}{m!} \zeta(n+m+1)(s-1)^m,$$

and that whenever $s \neq 0, -1, -2, \ldots$ and n is a positive integer,

$$\Psi^{(n)}(s) = (-1)^{n+1} n! \sum_{m=0}^{\infty} \frac{1}{(s+m)^{n+1}} = (-1)^{n+1} n! \zeta(s, n+1).$$

The following definitions will enable us to state the main results of this section.

Laurent expansion of $\frac{\zeta'}{\zeta}$: For $k \geq 0$, we denote the kth Laurent coefficient of $\frac{\zeta'(s)}{\zeta(s)}$ about $s = 1$ by η_k. Formally, this means that we have that for some region around $s = 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{m=0}^{\infty} \eta_m (s-1)^m,$$

and that whenever k is a nonnegative integer, we have

$$\eta_k = \frac{1}{k!} \left[\frac{d^k}{ds^k} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) \right]_{s=1}.$$

Definition: For integers $k \geq 0$, we define $\wp_k$ by

$$\wp_k = \frac{1}{k! 2^{k+1}} \Psi^{(k)} \left(\frac{1}{2} \right).$$

In particular, we have that

$$\wp_0 = \frac{1}{2} \Psi \left(\frac{1}{2} \right) = \frac{-\gamma}{2} - \ln(2),$$

and for positive integers k,

$$\wp_k = \left(\frac{-1}{2} \right)^{k+1} \zeta \left(\frac{1}{2}, k+1 \right).$$

Lemma 1.8.3 *Let s be a complex number with $\Re(s) > 1$. Then we have that*

$$\zeta \left(\frac{1}{2}, s \right) = 2^s (1 - 2^{-s}) \zeta(s).$$

In particular, it follows that for positive integers k,

$$\wp_k = (-1)^{k+1}(1 - \frac{1}{2^{k+1}})\zeta(k+1).$$

Proof: From the definition of the Hurwitz zeta function, we have that

$$\begin{aligned}
\zeta\left(\frac{1}{2}, s\right) &= \sum_{m=0}^{\infty} \frac{1}{\left(\frac{1}{2} + m\right)^s} \\
&= \sum_{m=0}^{\infty} \frac{2^s}{(1+2m)^s} \\
&= 2^s \sum_{m \text{ odd}} \frac{1}{m^s} \\
&= 2^s \left[\zeta(s) - \sum_{m \text{ even}} \frac{1}{m^s}\right] \\
&= 2^s \left[1 - \frac{1}{2^s}\right]\zeta(s),
\end{aligned}$$

as desired. The fact that $\wp_k$ is given by the equality in the lemma then follows immediately from its definition. ∎

We now have the tools to give the Bombieri and Lagarias arithmetic formula, and to properly interpret it.

Theorem 1.8.4 (Arithmetic formula for Li coefficients) *Let n be a non-negative integer. The Li coefficient λ_n is given by the expression*

$$\lambda_n = 1 - \frac{n}{2}(\gamma + \log(4\pi)) + S_1(n) + S_2(n), \tag{1.8.1}$$

where

$$S_1(n) = \sum_{k=2}^{n} \binom{n}{k} \wp_{k-1},$$

and

$$S_2(n) = \sum_{k=1}^{n} \binom{n}{k} \eta_{k-1}.$$

Our notations $S_1(n)$ and $S_2(n)$ are adopted from Coffey [4]. As we can see from Lemma 1.8.3, $S_1(n)$ may be interpreted as a combination of special values of the zeta function at positive integers. On the other hand, $S_2(n)$ is a combination of Laurent coefficients of the logarithmic derivative of the zeta function about $s = 1$.

The theorem is a consequence of combining the expression in Lemma 1.8.1 for λ_n with the definitions of the completed zeta function $\xi(s)$, and of $\wp_k$ and η_k. We will prove a more general version of this theorem in Chapter 2.

The idea of constructing arithmetic formulae for Li coefficients has been generalized significantly. These generalizations have included one for the Selberg class by Smajlović in [25], and one by Freitas to his case of Li coefficients $\lambda_n(\tau)$ with $\tau > 1$ in [11]. Both of these are special cases of a generalization of the idea that we will give in Chapter 2, and so we do not state them explicitly here.

To conclude this section, we will remark on further work on the formula (1.8.1) due to Coffey in [4]. His first result provides a bound on the term $S_1(n)$. We will prove a more general version of this theorem in Chapter 3, which will apply to a wide class of functions within the extension $\mathcal{S}^{\sharp\flat}$ of the Selberg class.

Theorem 1.8.5 (Coffey) *Let $n \geq 2$ be an integer. Then we have that*

$$\frac{1}{2}\left[n(\log(n) + \gamma - 1) + 1\right] \leq S_1(n) \leq \frac{1}{2}\left[n(\log(n) + \gamma + 1) - 1\right].$$

In particular, $S_1(n)$ is nonnegative and real for every positive integer n.

This theorem shows that proving the nonnegativity of λ_n reduces to the investigation of the term $S_2(n)$ in its arithmetic formula. To this end, we have, for example, the following result on the Laurent coefficients η_k involved in the definition of $S_2(n)$, also due to Coffey [5, Proposition 4.2].

Theorem 1.8.6 *The Laurent coefficients η_k have strict sign alternation. That is, we have $\eta_{k+1}\eta_k \leq 0$ for every nonnegative integer k.*

Coffey has continued to study the behaviour of the components $S_1(n)$ and $S_2(n)$ in the arithmetic formulae for $\lambda_n(\tau)$ since the publication of [4], including generalizations to Li coefficients corresponding to quasi-Riemann hypotheses for the Riemann zeta function itself. Moreover, his techniques have been adapted by other authors to give analogous results for more general classes of functions. We refer to [7], [6], [14], [8] for some of Coffey's additional work on this subject. Also, we refer to [15] for Lagarias' application of refined versions of Coffey's arguments to the case of automorphic L-functions, and to [26] for further application of these refined arguments to the case of Li coefficients corresponding to the full GRH in the Selberg class.

Chapter 2 Li's criterion for arbitrary half-planes in the Selberg class

This chapter concerns formulating Li's criterion as an equivalence for the δ-GRH for functions in the class $\mathcal{S}^{\sharp\flat}$. In the first section, we will derive this criterion explicitly and provide several alternate forms for the corresponding Li coefficients. In the second section, we will derive arithmetic formulae for the Li coefficients in this general context, using the definition for completed functions provided by the Selberg class axioms. In the last section, we will give explicit non-negative bounds on polygamma sums which appear in the arithmetic formulae for the Li coefficients in this context, applicable to a wide class of functions within $\mathcal{S}^{\sharp\flat}$. For functions to which these bounds apply, they effectively reduce the question of the δ-GRH (and the full GRH) to the investigation of certain combinations of Laurent coefficients of the logarithmic derivative of the function in question. The results in this chapter arise from combining ideas of Freitas, Smajlović, and Coffey ([11], [25], [4]) discussed in the first chapter.

2.1 Zero-free half-planes for the Selberg Class

In this section, we give a Li-type criterion for the δ-GRH for functions F in the class $\mathcal{S}^{\sharp\flat}$ defined in the first chapter. To this end, we wish to apply Theorem 1.7.3 to the set of non-trivial zeros of $F(s)$, denoted by $Z(F)$. In view of this objective, and the discussion of the preceding chapter, for every real number $\tau \geq 1$

and positive integer n we define $\lambda_n(F,\tau)$ by

$$\lambda_n(F,\tau) = \sum_{Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right), \tag{2.1.1}$$

as long as the sum on the right is $\star$-convergent. This convergence is provided by the following lemma, as long as $0 \notin Z(F)$.

Lemma 2.1.1 (Convergence of $\lambda_n(F,\tau)$) *Let F be a function in $\mathcal{S}^{\sharp\flat}$, and assume that $0 \notin Z(F)$. Let $\tau \geq 1$ be a real number. Then we have the following three results.*

(a) The sum defining $\lambda_n(F,\tau)$ in (2.1.1) is $\star$-convergent for every integer n.

(b) We have

$$\Re(\lambda_n(F,\tau)) = \sum_{Z(F)} \Re\left[1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right],$$

where the sum on the right is absolutely convergent for every integer n.

(c)

$$\sum_{Z(F)} \frac{1+|\Re(\rho)|}{(1+|\rho|)^2} < \infty$$

Proof: Since the zeros of $\xi_F(s)$ lie in the strip $0 \leq \Re(s) \leq 1$, $0 \notin Z(F)$, and $\tau \geq 1$, it follows that $\tau \notin Z(F)$ (if 1 were in $Z(F)$ then $1 - \bar{1} = 0$ would be as well). Define $Z_\tau = \{\frac{\rho}{\tau} \mid \rho \in Z(F)\}$, where the multiplicity of $\frac{\rho}{\tau}$ in Z_τ is the same as that of ρ in $Z(F)$. Then $0, 1 \notin Z_\tau$.

Now, to prove the lemma, note that since 0 is not in $Z(F)$, by part (a) of Theorem 1.7.7, the sum

$$\frac{\xi_F'(0)}{\xi_F(0)} = -\sum_{Z(F)}^{*} \frac{1}{\rho}$$

is $\star$-convergent. It follows that the sum

$$\tau \sum_{Z(F)}^{*} \frac{1}{\rho} = \sum_{Z(F)}^{*} \frac{\tau}{\rho} = \sum_{Z_\tau}^{*} \frac{1}{\rho} \tag{2.1.2}$$

is also $*$-convergent.

Next, we recall a theorem of complex analysis, which says that if f is any entire function of order 1, then

$$\sum_{\substack{Z(f)\\ \rho\neq 0}} \frac{1}{\rho^{\sigma}}$$

is absolutely convergent for any real $\sigma > 1$ (see, for example, [16, Ch.X, Theorem 3.2]). It follows by Lemma 1.7.6 that the sum

$$\sum_{Z(F)} \frac{1}{|\rho|^2}$$

is absolutely convergent. Since every element of $Z(F)$ is in the strip $0 \leq \Re(s) \leq 1$ and $\tau \geq 1$ is real, we see that

$$\begin{aligned} \infty &> (\tau+1) \sum_{Z(F)} \frac{1}{|\rho|^2} \\ &> \sum_{Z(F)} \frac{\tau + |\Re(\rho)|}{(\tau+|\rho|)^2} \qquad (2.1.3) \\ &= \frac{1}{\tau} \sum_{Z(F)} \frac{1+\left|\Re\left(\frac{\rho}{\tau}\right)\right|}{\left(1+\left|\frac{\rho}{\tau}\right|\right)^2} \\ &= \frac{1}{\tau} \sum_{Z_\tau} \frac{1+|\Re(\rho)|}{(1+|\rho|)^2}. \qquad (2.1.4) \end{aligned}$$

In particular, the sums in (2.1.3) and (2.1.4) are both absolutely convergent. The sum in (2.1.3) clearly converges simultaneously with

$$\sum_{Z(F)} \frac{1+|\Re(\rho)|}{(1+|\rho|)^2},$$

and so this completes the proof of part (c) of the lemma.

To prove parts (a) and (b), we remark that the $*$-convergence of the sums in (2.1.2), together with the finiteness of the sum in (2.1.4) and the fact that $0, 1 \notin Z_\tau$, allows us to apply Lemma 1.6.1 to the set Z_τ. The last assertion of Lemma 1.6.1 then shows us that the sum

$$\begin{aligned}\sum_{Z_\tau}^{*}\left(1-\left(\frac{\rho}{\rho-1}\right)^n\right) &= \sum_{Z(F)}^{*}\left(1-\left(\frac{\frac{\rho}{\tau}}{\frac{\rho}{\tau}-1}\right)^n\right)\\ &= \sum_{Z(F)}^{*}\left(1-\left(\frac{\rho}{\rho-\tau}\right)^n\right)\\ &= \lambda_n(F,\tau)\end{aligned}$$

is $*$-convergent for every integer n, which completes the proof of part (a) of the present lemma. Similarly, the first assertion of Lemma 1.6.1 shows us that

$$\sum_{Z_\tau}\Re\left[1-\left(\frac{\rho}{\rho-1}\right)^n\right]$$

is absolutely convergent for every integer n. Re-indexing this sum over the elements of $Z(F)$, exactly as we just did to prove (a), we establish (b). This completes the proof of the lemma. ∎

Next, we prove a result which gives alternate descriptions of the coefficients $\lambda_n(F,\tau)$ for a function F in $\mathcal{S}^{\sharp\flat}$. These reduce to the relations (1.5.2) and (1.5.4) for $\lambda_n(\zeta,1)=\lambda_n$ in the classical case of the Riemann zeta function. The lemma itself is a generalization of Lemma 1.7.1, which is restricted to the case of $\lambda_n(\zeta,\tau)$, and of Lemma 1.7.8, which is restricted to the the case of $\lambda_n(F,1)$. To maintain notational consistency with Lemma 1.7.1, for any real $\tau\ge 1$ and non-negative integer n, we define $\alpha_{n+1}(F,\tau)$ by

$$\alpha_{n+1}(F,\tau)=\frac{1}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n\log\xi_F(s)\right]_{s=\tau}. \tag{2.1.5}$$

Then we have the following lemma.

Lemma 2.1.2 (Alternate forms for $\lambda_n(F,\tau)$) *Let $\tau \geq 1$ be a real number and let F be a function in $\mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$. Define ψ_F and φ_F as in the statement of Lemma 1.7.8. Denote the power series coefficients of ψ_F about any real number $z_0 \neq 1$ which is not a zero of $\varphi_F(s) = \xi_F\left(\frac{1}{1-s}\right)$ by $d_n(z_0, F)$, so that in some neighborhood of z_0 we have*

$$\psi_F(s) = \frac{\varphi'(s)}{\varphi(s)} = \sum_{n=0}^{\infty} d_n(z_0, F)(s - z_0)^n. \tag{2.1.6}$$

Then for every positive integer n,

$$\alpha_n(F,\tau) = \frac{1}{\tau}\lambda_n(F,\tau) = \frac{1}{\tau^{n+1}} d_{n-1}\left(1 - \frac{1}{\tau}, F\right). \tag{2.1.7}$$

Proof: We begin by proving the first equality in (2.1.7). Since $\tau \geq 1$ and $0 \notin Z(F)$, it follows that $\tau \notin Z(F)$. For analytic functions f and g and any positive integer n, we may write $\frac{d^n}{ds^n}(f(s)g(s)) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^k}{ds^k}f(s)\right)\left(\frac{d^{n-k}}{ds^{n-k}}g(s)\right)$. Therefore, for every non-negative integer n, we have

$$\begin{aligned}
\alpha_{n+1}(F,\tau) &= \frac{1}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log \xi_F(s)\right]_{s=\tau} \\
&= \frac{1}{n!}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{d^k}{ds^k}s^n\right)\left(\frac{d^{n+1-k}}{ds^{n+1-k}}\log \xi_F(s)\right)\right]_{s=\tau} \\
&= \frac{1}{n!}\left[\sum_{k=0}^{n}\binom{n+1}{k}\frac{n!}{(n-k)!}s^{n-k}\left(\frac{d^{n+1-k}}{ds^{n+1-k}}\log \xi_F(s)\right)\right]_{s=\tau}.
\end{aligned} \tag{2.1.8}$$

To proceed further, we require an expression for the derivatives of $\log(\xi_F(s))$. We remark that we might naively attempt to apply term-by-term differentiation to part (a) of Theorem 1.7.7, which states that whenever $s \notin Z(F)$,

$$\frac{d}{ds}\log \xi_F(s) = \frac{\xi_F'(s)}{\xi_F(s)} = \sum_{Z(F)}^{*} \frac{1}{s-\rho}, \tag{2.1.9}$$

from which we would obtain that

$$\frac{d^{n+1}}{ds^{n+1}}\log\xi_F(s) = -\sideset{}{^*}\sum_{Z(F)}\frac{n!}{(\rho-s)^{n+1}}.$$

Unfortunately, this term-by-term differentiation is not valid, since we have no information about the uniform convergence of either of these sums (nor is the first sum absolutely convergent). This point is frequently overlooked in related work, as in the proofs in [25], for example. Instead, we begin with the general Hadamard product from (1.7.6), which is absolutely and uniformly convergent on any closed disc of radius $R > 0$. Write

$$\xi_F(s) = \xi_F(0)e^{b_F s}\prod_{Z(F)}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

where $b_F = -\sideset{}{^*}\sum_{Z(F)}\frac{1}{\rho}$ by the last assertion of Theorem 1.7.7. By the absolute and uniform convergence of the infinite product on arbitrary closed discs, it is clear that the factors satisfy the hypotheses of [16, Ch.X, Lemma 1.2] on any open disc. It follows that we may logarithmically differentiate the factors term-by-term to see that

$$\frac{d}{ds}\log\xi_F(s) = \frac{\xi_F'(s)}{\xi_F(s)} = b_F + \sum_{Z(F)}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right), \tag{2.1.10}$$

where the sum over $Z(F)$ is absolutely and uniformly convergent on any closed bounded region not containing an element of $Z(F)$. In some region around any point which is not in $Z(F)$, each of the summands is analytic, and thus we may differentiate term-by-term repeatedly to find all of the derivatives of ξ_F at any such point (see, for example, [16, Ch.V, Theorem 1.6] applied to a sequence of partial sums). Thus we find that for every integer $n \geq 1$,

$$\frac{d^{n+1}}{ds^{n+1}}\log\xi_F(s) = -\sum_{Z(F)}\frac{n!}{(\rho-s)^{n+1}}, \tag{2.1.11}$$

where the sum on the right is also absolutely and uniformly convergent in some region around any point not in $Z(F)$. In any case, whenever $s \notin Z(F)$, we see from (2.1.9) and (2.1.11) that for every positive integer n (using the standard convention $0! = 1$),

$$\frac{d^n}{ds^n} \log \xi_F(s) = -\sum_{Z(F)}^{*} \frac{(n-1)!}{(\rho - s)^n}, \tag{2.1.12}$$

where the indicated $*$-convergence is actually absolute whenever $n \geq 2$.

Since $\tau \geq 1$ is not in $Z(F)$, combining (2.1.8) and (2.1.12) yields

$$\begin{aligned}
\alpha_{n+1}(F,\tau) &= -\frac{1}{n!}\left[\sum_{k=0}^{n}\binom{n+1}{k}\frac{n!}{(n-k)!}s^{n-k}\sum_{Z(F)}^{*}\frac{(n-k)!}{(\rho-s)^{n+1-k}}\right]_{s=\tau} \\
&= -\sum_{k=0}^{n}\binom{n+1}{k}\tau^{n-k}\sum_{Z(F)}^{*}\frac{1}{(\rho-\tau)^{n+1-k}} \\
&= -\frac{1}{\tau}\sum_{Z(F)}^{*}\sum_{k=0}^{n}\binom{n+1}{k}\left(\frac{\tau}{\rho-\tau}\right)^{n+1-k} \\
&= -\frac{1}{\tau}\sum_{Z(F)}^{*}\left[\left(\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{\tau}{\rho-\tau}\right)^{n+1-k}\right)-1\right],
\end{aligned}$$

where the interchange in the order of the summations is justified by the fact that the only infinite summation is over $Z(F)$. The finite summation in the last line above is exactly the binomial theorem expansion of $\left(1+\frac{\tau}{\rho-\tau}\right)^{n+1}$. Thus, for every non-negative integer n, we find that

$$\begin{aligned}
\alpha_{n+1}(F,\tau) &= -\frac{1}{\tau}\sum_{Z(F)}^{*}\left(\left(1+\frac{\tau}{\rho-\tau}\right)^{n+1}-1\right) \\
&= \frac{1}{\tau}\sum_{Z(F)}^{*}\left(1-\left(\frac{\rho}{\rho-\tau}\right)^{n+1}\right) \\
&= \frac{1}{\tau}\lambda_{n+1}(F,\tau),
\end{aligned}$$

which completes the proof of the first equality asserted by the lemma in (2.1.7).

To prove the second equality in (2.1.7), we again start with the expression (2.1.10) for the logarithmic derivative of ξ_F. Whenever $\frac{1}{1-s} \notin Z(F)$ and $s \neq 1$, we may write

$$\begin{aligned}\psi_F(s) &= \frac{d}{ds}\log\left(\xi_F\left(\frac{1}{1-s}\right)\right)\\ &= \frac{1}{(1-s)^2}\frac{\xi_F'\left(\frac{1}{1-s}\right)}{\xi_F\left(\frac{1}{1-s}\right)}\\ &= \frac{1}{(1-s)^2}\left[b_F + \sum_{Z(F)}\left(\frac{1}{\frac{1}{1-s}-\rho}+\frac{1}{\rho}\right)\right],\end{aligned}$$

where the infinite sum is absolutely and uniformly convergent in some region around s. Multiplying the outside factor through, we re-write this as

$$\psi_F(s) = \frac{b_F}{(1-s)^2} + \sum_{Z(F)}\left[\frac{1}{1-s}\cdot\frac{1}{1-\rho+s\rho}+\frac{1}{\rho(1-s)^2}\right]. \tag{2.1.13}$$

Now let $z_0 = 1-\frac{1}{\tau} \neq 1$. Then z_0 is not a zero of $\varphi_F(s)$ because $\frac{1}{1-z_0} = \tau \notin Z(F)$. As in Freitas' proof of [11, Lemma 3.1], we observe that for any $\rho \in Z(F)$

$$\frac{1}{1-\rho+s\rho} = -\left(\frac{1}{\rho(1-z_0)-1}\right)\left(\frac{1}{1-\left[\frac{\rho(s-z_0)}{\rho(1-z_0)-1}\right]}\right) = \left(\frac{\tau}{\rho-\tau}\right)\left(\frac{1}{1-\frac{\rho\tau(s-z_0)}{\rho-\tau}}\right)$$

and

$$\frac{1}{1-s} = \left(\frac{1}{1-z_0}\right)\left(\frac{1}{1-\frac{s-z_0}{1-z_0}}\right) = \tau\left(\frac{1}{1-\tau(s-z_0)}\right).$$

Using these identities in (2.1.13) lets us write

$$\psi_F(s) = \frac{b_F}{(1-s)^2} - \sum_{Z(F)}\left[\frac{\tau^2}{\rho-\tau}\left(\frac{1}{1-\frac{\rho\tau(s-z_0)}{\rho-\tau}}\right)\left(\frac{1}{1-\tau(s-z_0)}\right)-\frac{1}{\rho(1-s)^2}\right].$$

For s sufficiently close to z_0, the two factors involving s in the first part of the infinite summation above may be interpreted as geometric series. Thus for s

sufficiently close to z_0, we obtain

$$\begin{aligned}
\psi_F(s) =& \frac{b_F}{(1-s)^2} - \sum_{Z(F)} \left[\frac{\tau^2}{\rho-\tau} \left(\sum_{n=0}^{\infty} \left(\frac{\rho\tau}{\rho-\tau} \right)^n (s-z_0)^n \right) \left(\sum_{m=0}^{\infty} \tau^m (s-z_0)^m \right) - \frac{1}{\rho(1-s)^2} \right] \\
=& \frac{b_F}{(1-s)^2} - \sum_{Z(F)} \left[\frac{\tau^2}{\rho-\tau} \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{\rho\tau}{\rho-\tau} \right)^m \tau^{n-m} (s-z_0)^n \right) - \frac{1}{\rho(1-s)^2} \right] \\
=& - \sum_{Z(F)} \sum_{n=0}^{\infty} \left[\tau^{n+1} (s-z_0)^n \left(\sum_{m=0}^{n} \frac{\tau}{\rho-\tau} \left(\frac{\rho}{\rho-\tau} \right)^m \right) - \left(\frac{1}{2} \right)^{n+1} \frac{1}{\rho(1-s)^2} \right] \\
& + \frac{b_F}{(1-s)^2},
\end{aligned} \tag{2.1.14}$$

where in the last line we use the fact that $\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} = 1$. The interior finite summation in (2.1.14) is geometric. Furthermore, all of the infinite summations are absolutely convergent for s sufficiently close to z_0, and so we may reverse their order. This shows us that for s sufficiently close to z_0, we may write

$$\begin{aligned}
\psi_F(s) =& - \sum_{n=0}^{\infty} \sum_{Z(F)} \left[\tau^{n+1} \left(\frac{\rho}{\rho-\tau} - 1 \right) \left(\frac{1 - \left(\frac{\rho}{\rho-\tau} \right)^{n+1}}{1 - \frac{\rho}{\rho-\tau}} \right) (s-z_0)^n - \left(\frac{1}{2} \right)^{n+1} \frac{1}{\rho(1-s)^2} \right] \\
& + \frac{b_F}{(1-s)^2} \\
=& \frac{b_F}{(1-s)^2} + \sum_{n=0}^{\infty} \sum_{Z(F)} \left[\tau^{n+1} \left(1 - \left(\frac{\rho}{\rho-\tau} \right)^{n+1} \right) (s-z_0)^n + \left(\frac{1}{2} \right)^{n+1} \frac{1}{\rho(1-s)^2} \right].
\end{aligned}$$

Finally, using the last assertion of Theorem 1.7.7, we see that the sum $\sum_{Z(F)}^{*} \frac{1}{\rho} = -b_F$ is $\star$-convergent. It follows that we may split the absolutely convergent infinite summation over $Z(F)$ above into two $\star$-convergent sums to obtain

$$\begin{aligned}
\psi_F(s) =& \left[\frac{b_F}{(1-s)^2} - \frac{b_F}{2(1-s)^2} \sum_{n=0}^{\infty} \frac{1}{2^n} \right] + \sum_{n=0}^{\infty} \tau^{n+1} \left[\sum_{Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho-\tau} \right)^{n+1} \right) \right] (s-z_0)^n. \\
=& \sum_{n=0}^{\infty} \tau^{n+1} \lambda_{n+1}(F,\tau) (s-z_0)^n \\
=& \sum_{n=0}^{\infty} d_n(z_0,F) (s-z_0)^n,
\end{aligned}$$

where the last two lines use the definitions of $\lambda_n(F,\tau)$ and $d_n(z_0,F)$, and the power series above are absolutely and uniformly convergent for s sufficiently close to z_0. Using the uniqueness of power series expansions and the fact that $z_0 = 1 - \frac{1}{\tau}$, it follows that $\tau^n \lambda_n(F,\tau) = d_{n-1}\left(1-\frac{1}{\tau}, F\right)$ for every integer $n \geq 1$. This establishes the second equality of (2.1.5), and thus completes the proof of the lemma. ∎

With the two previous lemmas in hand, we are now equipped to state and prove the first main theorem of this chapter. This theorem provides the analogue of Freitas' result stated in Theorem 1.7.2, for functions in the class $\mathcal{S}^{\sharp\flat}$. The proof will use Theorem 1.7.3, along with Lemma 2.1.1. This approach lies in contrast to Freitas' proof of [11, Theorem 1], which closely mirrors Li's original work in [17].

Theorem 2.1.3 (Half-plane criterion for the Selberg class) *Let F be a function in $\mathcal{S}^{\sharp\flat}$ such that $0 \notin Z(F)$, and let $\tau \geq 1$ be a real number. Then F satisfies the $\frac{\tau}{2}$-GRH if and only if $\Re(\lambda_n(F,\tau)) \geq 0$ for every positive integer n.*

Equivalently, every zero ρ of $\xi_F(s)$ lies in the strip $1-\frac{\tau}{2} \leq \Re(\rho) \leq \frac{\tau}{2}$ if and only if $\Re(\lambda_n(F,\tau)) \geq 0$ for every positive integer n.

Remark: The proof shows that the theorem is still valid if the statement that $\Re(\lambda_n(F,\tau)) \geq 0$ for every positive integer n is replaced by the weaker assertion that for any $\epsilon > 0$, there exists a real constant $c(\epsilon)$ such that $\Re(\lambda_n(F,\tau)) \geq -c(\epsilon)e^{\epsilon n}$ for every positive integer n.

Proof: By identical arguments to those we have used previously, we see that $\tau \notin Z(F)$. We may apply part (c) of Lemma 2.1.1 to see that

$$\sum_{Z(F)} \frac{1+|\Re(\rho)|}{(1+|\rho|)^2} < \infty.$$

Thus the set $Z(F)$ satisfies the hypotheses of Theorem 1.7.3. Application of the theorem shows that the following conditions are equivalent:

(i) $\mathfrak{R}(\rho) \le \frac{\tau}{2}$ for every element ρ of $Z(F)$.

(ii) $\sum_{Z(F)} \mathfrak{R}\left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right) \ge 0$ for every positive integer n.

(iii) For any $\epsilon > 0$, there exists a real constant $c(\epsilon)$ such that

$$\sum_{Z(F)} \mathfrak{R}\left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right) \ge -c(\epsilon)e^{\epsilon n}$$

for every positive integer n.

Applying part (b) of Lemma 2.1.1 shows that the sum in (ii) and (iii) is absolutely convergent to $\mathfrak{R}(\lambda_n(F,\tau))$, and this proves the first assertion of the theorem, since for $\tau \ge 1$, any zero of F with real part greater than $\frac{\tau}{2}$ must be non-trivial and thus in $Z(F)$.

To establish the confinement of the non-trivial zeros ρ of F to the strip $1 - \frac{\tau}{2} \le \mathfrak{R}(\rho) \le \frac{\tau}{2}$, we use the fact that ρ is in $Z(F)$ if and only if $1 - \bar{\rho}$ is in $Z(F)$. Thus condition (i) equivalently says that $\mathfrak{R}(1 - \bar{\rho}) \le \frac{\tau}{2}$ for every $\rho \in Z(F)$. Rearranging shows that $\mathfrak{R}(\rho) \ge 1 - \frac{\tau}{2}$ for every $\rho \in Z(F)$, and the proof is complete. ∎

With this theorem in hand, we are ready to move on to discussing arithmetic formulae for the Li coefficients in this context.

2.2 Arithmetic formulae for $\lambda_n(F,\tau)$

In this section, we wish to derive some additional representations of the coefficients

$$\lambda_n(F,\tau) = \sum_{Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right)$$

defined in the last section for functions F in $\mathcal{S}^{\sharp\flat}$ and $\tau \geq 1$. As we have proved in Lemma 2.1.2, for $n \geq 0$, we have

$$\lambda_{n+1}(F,\tau) = \tau\alpha_{n+1}(F,\tau) = \frac{\tau}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log \xi_F(s)\right]_\tau$$

whenever $F \in \mathcal{S}^{\sharp\flat}$ and $0 \notin Z(F)$, where ξ_F is the completed function corresponding to F. Henceforth, for positive integers k, functions $F \in \mathcal{S}^{\sharp\flat}$, and $s \notin Z(F)$, we will write

$$\mathcal{Z}_k(F,s) = \sum_{Z(F)}^{*} \frac{1}{(\rho - s)^k},$$

as long as this sum is $*$-convergent. From the definition of $\lambda_n(F,\tau)$, using a binomial expansion, we obtain that

$$\begin{aligned}\lambda_n(F,\tau) &= \sum_{Z(F)}^{*}\left(1 - \left(\frac{\rho}{\rho-\tau}\right)^n\right) \\ &= -\sum_{Z(F)}^{*}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\tau}{\rho-\tau}\right)^k. \\ &= -\sum_{k=1}^{n}\binom{n}{k}\tau^k \mathcal{Z}_k(F,\tau).\end{aligned}$$

It is also convenient at this point to observe that using the symmetry condition that ρ is in $Z(F)$ if and only if $1-\bar{\rho}$ is in $Z(F)$, with the same multiplicities, we may re-write this expression for $\lambda_n(F,\tau)$ as

$$\begin{aligned}\lambda_n(F,\tau) &= -\sum_{k=1}^{n}\binom{n}{k}\tau^k\sum_{Z(F)}^{*}\frac{1}{((1-\bar{\rho})-\tau)^k} \\ &= -\sum_{k=1}^{n}\binom{n}{k}\tau^k\sum_{Z(F)}^{*}\frac{(-1)^k}{(\bar{\rho}-(1-\tau))^k} \\ &= -\sum_{k=1}^{n}\binom{n}{k}(-1)^k\tau^k\overline{\mathcal{Z}_k(F,1-\tau)}.\end{aligned}$$

Another useful identity appears in our proof of Lemma 2.1.2, from which we see that for non-negative integers n,

$$\begin{aligned}\lambda_{n+1}(F,\tau) =& \tau\alpha_{n+1}(F,\tau)\\ =& \frac{\tau}{n!}\frac{d^{n+1}}{ds^{n+1}}\left[s^n \log\xi_F(s)\right]_{s=\tau}\\ =& \frac{\tau}{n!}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{d^{n+1-k}}{ds^{n+1-k}}s^n\right)\left(\frac{d^k}{ds^k}\log\xi_F(s)\right)\right]_\tau\\ =& \frac{\tau}{n!}\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{n!}{(k-1)!}\tau^{k-1}\left[\frac{d^k}{ds^k}\log\xi_F(s)\right]_\tau\\ =& \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{\tau^k}{(k-1)!}\left[\frac{d^k}{ds^k}\log\xi_F(s)\right]_\tau .\end{aligned} \tag{2.2.1}$$

In fact, examining these last two identities, we see that they are simply a consequence of the fact that whenever $s \notin Z(F)$,

$$\frac{d^k}{ds^k}\log\xi_F(s) = -\sum_{Z(F)}^{*}\frac{(k-1)!}{(\rho-s)^k} = -(k-1)!\mathcal{Z}_k(F,s),$$

which we established in our proof of Lemma 2.1.2.

In order to understand the Li coefficients $\lambda_n(F,\tau)$ better, we must examine the function $\xi_F(s)$. What we know about $\xi_F(s)$ is contained in the Selberg class axioms discussed in Section 1.4, and so we must appeal to them. In particular, the functional equation axiom tells us that

$$\xi_F(s) = F(s)s^{m_F}(s-1)^{m_F}Q_F^s\prod_{j=1}^{r}\Gamma(\omega_j s+\nu_j) \tag{2.2.2}$$

for some non-negative integer r, with $\omega_j > 0$ and $v_j \in \mathbb{C}$, $\Re(\nu_j) \geq 0$ for $1 \leq j \leq r$, $Q_F > 0$, and where m_F is the polar order of F (i.e. the order of the pole of F at $s = 1$).

Using this definition for $\xi_F(s)$ gives us another natural way to calculate $\frac{d^k}{ds^k}\log\xi_F(s)$

in terms of Laurent coefficients of $\frac{F'}{F}$ and polygamma functions. Indeed, from (2.2.2), we obtain that for any $k \geq 1$ and $\Re(s) \geq 1$,

$$\frac{d^k}{ds^k}\log \xi_F(s) = \frac{d^{k-1}}{ds^{k-1}}\left(\frac{F'(s)}{F(s)} + \frac{m_F}{s-1}\right) - \frac{(k-1)!m_F}{(-s)^k} + \delta_{k,1}\log Q_F + \sum_{j=1}^{r}\omega_j^k \Psi^{(k-1)}(\omega_j s + \nu_j), \tag{2.2.3}$$

where $\delta_{i,j}$ is the Kronecker delta function (defined by $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise) and we adopt our notation from Chapter 1 for the digamma function Ψ and polygamma functions $\Psi^{(k)}$.

Our next aim is to use the expression (2.2.3) for the logarithmic derivatives of $\xi_F(s)$ to give a new description of the Li coefficients $\lambda_n(F,\tau)$. To this end, for any non-negative integer k, we define

$$\wp_k(F,\tau) = \frac{1}{k!}\sum_{j=1}^{r}\omega_j^{k+1}\Psi^{(k)}(\omega_j\tau + \nu_j), \qquad \tau \geq 1 \tag{2.2.4}$$

$$\ell_k(F,\tau) = \frac{1}{k!}\left[\frac{d^k}{ds^k}\frac{F'(s)}{F(s)}\right]_\tau, \qquad \tau > 1 \tag{2.2.5}$$

$$\eta_k(F,\tau) = \frac{1}{k!}\left[\frac{d^k}{ds^k}\left(\frac{F'(s)}{F(s)} + \frac{m_F}{s-1}\right)\right]_\tau, \qquad \tau \geq 1. \tag{2.2.6}$$

We remark that $\eta_k(F,\tau)$ is well-defined for every $\tau \geq 1$ since the fact that F has a pole of order m_F at $s = 1$ implies that $\frac{F'(s)}{F(s)} + \frac{m_F}{s-1}$ is analytic for $s \geq 1$. Moreover, recall that $F \in \mathcal{S}^{\sharp\flat}$, so $\frac{F'(s)}{F(s)}$ is analytic for $s > 1$ and $\ell_k(F,\tau)$ is well-defined for $\tau > 1$. Finally, since $\omega_j > 0$ and $\Re(\nu_j) \geq 0$ for every j, $\wp_k(F,\tau)$ is well-defined for every $\tau \geq 1$. We also note that we have the Laurent expansion

$$-\frac{F'(s)}{F(s)} = \frac{m_F}{s-1} - \sum_{m=0}^{\infty}\eta_m(F,1)(s-1)^m$$

for s in a neighborhood of 1. Moreover, we have the useful identity

$$\eta_k(F,\tau) = \ell_k(F,\tau) + (-1)^k \frac{m_F}{(\tau-1)^{k+1}},$$

valid for all $k \geq 0$ and $\tau > 1$. We adopt the notational convention that $\eta_k(F) = \eta_k(F,1)$ and $\wp_k(F) = \wp_k(F,1)$, when convenient.

With the definitions above in hand, we state the resulting expansion of $\lambda_n(F,\tau)$ in the following theorem. This theorem gives the most general form for the arithmetic formulae we will discuss for $\lambda_n(F,\tau)$. Indeed, we will see later that by re-expressing the values of η_k, ℓ_k, and $\wp_k$ in convenient ways, we may be able to obtain more transparent interpretations of the coefficients $\lambda_n(F,\tau)$.

Theorem 2.2.1 *Let n be a positive integer and F be a function in $\mathcal{S}^{\sharp\flat}$ with corresponding completed function ξ_F specified by* (2.2.2). *Then whenever $\tau \geq 1$, we have*

$$\lambda_n(F,\tau) = m_F + n\tau\log Q_F + \sum_{k=1}^{n}\binom{n}{k}\tau^k\eta_{k-1}(F,\tau) + \sum_{k=1}^{n}\binom{n}{k}\tau^k\wp_{k-1}(F,\tau), \tag{2.2.7}$$

and moreover, for $\tau > 1$, we have

$$\begin{aligned}\lambda_n(F,\tau) =& m_F\left[2 + (-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right] + n\tau\log Q_F \\ &+ \sum_{k=1}^{n}\binom{n}{k}\tau^k\ell_{k-1}(F,\tau) + \sum_{k=1}^{n}\binom{n}{k}\tau^k\wp_{k-1}(F,\tau).\end{aligned} \tag{2.2.8}$$

Proof: These expansions arise from substituting (2.2.3) into (2.2.1) and using the definitions of $\ell_k(F,\tau)$, $\wp_k(F,\tau)$, and $\eta_k(F,\tau)$ above, as follows. We have

$$\lambda_n(F,\tau) = \sum_{k=1}^{n}\binom{n}{k}\frac{\tau^k}{(k-1)!}\left[\frac{d^k}{ds^k}\log\xi_F(s)\right]_\tau.$$

Substituting (2.2.3), we obtain

$$\lambda_n(F,\tau) = \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \left[\frac{d^{k-1}}{ds^{k-1}} \left(\frac{F'(s)}{F(s)} + \frac{1}{s-1} \right) \right]_\tau$$
$$- m_F \sum_{k=1}^{n} \binom{n}{k} (-1)^k + \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \delta_{k,1} \log Q_F$$
$$+ \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \sum_{j=1}^{r} \omega_j^k \Psi^{(k-1)}(\omega_j \tau + \nu_j).$$

Since $\sum_{k=1}^{n} \binom{n}{k} (-1)^k = 0^n - 1 = -1$, we find

$$\lambda_n(F,\tau) = m_F + n\tau \log Q_F + \sum_{k=1}^{n} \binom{n}{k} \tau^k \eta_{k-1}(F,\tau) + \sum_{k=1}^{n} \binom{n}{k} \tau^k \wp_{k-1}(F,\tau),$$

which is (2.2.7). Now, we remark that for $\tau > 1$,

$$\sum_{k=1}^{n} \binom{n}{k} \tau^k \eta_{k-1}(F,\tau) = \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \left[\frac{d^{k-1}}{ds^{k-1}} \frac{F'(s)}{F(s)} \right]_\tau - m_F \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\tau}{1-\tau} \right)^k$$
$$= -m_F \left(\left(1 + \frac{\tau}{1-\tau} \right)^n - 1 \right) + \sum_{k=1}^{n} \binom{n}{k} \tau^k \ell_{k-1}(F,\tau)$$
$$= m_F \left(1 + (-1)^{n+1} \left(\frac{1}{\tau - 1} \right)^n \right) + \sum_{k=1}^{n} \binom{n}{k} \tau^k \ell_{k-1}(F,\tau).$$

Substituting this into (2.2.7) now yields (2.2.8). ∎

This theorem invites several ways of re-expressing $\lambda_n(F,\tau)$ for a given $\tau \geq 1$ via different interpretations of $\wp_k(F,\tau)$, $\ell_k(F,\tau)$, and $\eta_k(F)$. Indeed, using Theorem 1.8.2, we immediately obtain the following proposition, which gives equivalent interpretations for $\wp_k(F,\tau)$.

Proposition 2.2.2 *We have that for $\tau \geq 1$,*

$$\wp_0(F,\tau) = \sum_{j=1}^{r} \omega_j \Psi(\omega_j \tau + \nu_j),$$

and for $k \geq 1$,

$$\wp_k(F,\tau) = (-1)^{k+1} \sum_{j=1}^{r} \sum_{m=0}^{\infty} \left(\frac{\omega_j}{\omega_j \tau + \nu_j + m} \right)^{k+1}. \tag{2.2.9}$$

Moreover, if we assume that ν_j *is real and choose* τ *such that* $1 \leq \tau < \frac{2-\nu_j}{\omega_j}$ *for every* j, $1 \leq j \leq r$, *then for* $k \geq 1$,

$$\wp_k(F,\tau) = (-1)^{k+1} \sum_{j=1}^{r} \sum_{m=0}^{\infty} (-1)^m \frac{(k+m)!}{m!k!} \zeta(k+m+1)(\omega_j \tau + \nu_j - 1)^m. \tag{2.2.10}$$

Proof: These assertions follow directly from the definition of $\wp_k(F,\tau)$ and Theorem 1.8.2, using the fact that $\omega_j > 0$ and $\Re(\nu_j) \geq 0$ by definition. ∎

Next, let us consider $\ell_k(F,\tau)$ for $\tau > 1$. Indeed, by the modified Euler product axiom of $\mathcal{S}^{\sharp\flat}$, we have that there are complex numbers $c_m(F)$ for $m \geq 2$ such that for $\Re(s) > 1$,

$$\frac{F'(s)}{F(s)} = -\sum_{m=2}^{\infty} \frac{c_m(F)}{m^s}.$$

The following proposition is then clear.

Proposition 2.2.3 *Let* k *be a non-negative integer. Then for any* $\tau > 1$, *we have*

$$\ell_k(F,\tau) = \frac{(-1)^{k+1}}{k!} \sum_{m=2}^{\infty} \frac{c_m(F)(\log m)^k}{m^\tau}. \tag{2.2.11}$$

Recall that our interest is in investigating the non-negativity of the real parts of the coefficients $\lambda_n(F,\tau)$ for a given function F and $\tau \geq 1$ in order to provide insight into whether the GRH, or some weaker version of the GRH, holds for F.

To this end, the next section concentrates on bounding the polygamma sums (i.e. the sum involving $\wp_k(F,\tau)$ terms) involved in the arithmetic formulae for Li coefficients using a generalization of an elementary argument of Coffey in [4].

There also exist ways of investigating these Laurent coefficient sums by methods of contour integration, applied, for example, by Lagarias in [15] for automorphic L-functions, and by Smajlović in [26] for the Selberg class in the case $\tau = 1$. We leave the application of these methods to our present $\tau \geq 1$ context to future work.

2.3 Estimation of the polygamma sums

In this section, we wish to estimate the sums

$$S_1(n, F, \tau) = \sum_{k=2}^{n} \binom{n}{k} \tau^k \wp_{k-1}(F, \tau)$$

involved in the formulae given for $\lambda_n(F, \tau)$ in Theorem 2.2.1, under appropriate hypotheses on the function F. We follow methods of Coffey in [4], which provided effective bounds on $S_1(n, \zeta, 1)$ using the integral test from elementary calculus. Note that the sum defining $S_1(n, F, \tau)$ starts at $k = 2$, since the $k = 1$ term, corresponding to a sum of digamma functions rather than polygamma functions, is not convenient to consider in the same way.

Here, by (2.2.9), we have that for $k \geq 2$,

$$\wp_{k-1}(F, \tau) = (-1)^k \sum_{j=1}^{r} \sum_{m=0}^{\infty} \left(\frac{\omega_j}{\omega_j \tau + \nu_j + m} \right)^k,$$

from which we obtain

$$\begin{aligned} S_1(n, F, \tau) &= \sum_{k=2}^{n} \binom{n}{k} (-\tau)^k \sum_{j=1}^{r} \sum_{m=0}^{\infty} \left(\frac{\omega_j}{\omega_j \tau + \nu_j + m} \right)^k \\ &= \sum_{j=1}^{r} \sum_{m=0}^{\infty} \sum_{k=2}^{n} \binom{n}{k} \left(\frac{-\omega_j \tau}{\omega_j \tau + \nu_j + m} \right)^k \\ &= \sum_{j=1}^{r} \sum_{m=0}^{\infty} \left[\left(1 - \frac{\omega_j \tau}{\omega_j \tau + \nu_j + m} \right)^n - 1 + \frac{n \omega_j \tau}{\omega_j \tau + \nu_j + m} \right] \\ &= \sum_{j=1}^{r} \sum_{m=0}^{\infty} \left[\left(\frac{\nu_j + m}{\omega_j \tau + \nu_j + m} \right)^n - 1 + \frac{n \omega_j \tau}{\omega_j \tau + \nu_j + m} \right]. \end{aligned} \tag{2.3.1}$$

Let us now assume throughout this section that ν_j is real for all j (a hypothesis that holds for most classes of L-functions of interest, for example). Then the summand in the expression above is real. In fact, more is true.

Lemma 2.3.1 *Take any $\tau \geq 1$, and assume that ν_j is real for $1 \leq j \leq r$. Then the summand*

$$s_j(m) = \left[\left(\frac{\nu_j + m}{\omega_j\tau + \nu_j + m}\right)^n - 1 + \frac{n\omega_j\tau}{\omega_j\tau + \nu_j + m}\right]$$

in the expression (2.3.1) *for $S_1(n, F)$ is positive and decreases monotonically with m.*

Proof: Clearly, $s_j(m)$ approaches 0 as m increases to infinity. Thus it is sufficient to prove that $s_j(m)$ is monotonically decreasing as m increases. To this end, we differentiate, to find

$$\begin{aligned}\frac{d}{dm}s_j(m) &= \frac{n(\nu_j + m)^{n-1}}{(\omega_j\tau + \nu_j + m)^n} - \frac{n(\nu_j + m)^n}{(\omega_j\tau + \nu_j + m)^{n+1}} - \frac{n\omega_j\tau}{(\omega_j\tau + \nu_j + m)^2} \\ &= n\left[\frac{\omega_j\tau(\nu_j + m)^{n-1} - \omega_j\tau(\omega_j\tau + \nu_j + m)^{n-1}}{(\omega_j\tau + \nu_j + m)^{n+1}}\right],\end{aligned}$$

which is negative for $m \geq 0$, since by the axioms of the Selberg class, we have $\omega_j > 0$, and $\nu_j \geq 0$ by assumption (and, as usual, $\tau \geq 1$). This establishes the lemma. ∎

We remark that the lemma itself immediately establishes that $S_1(n, F, \tau)$ is non-negative whenever ν_j is real for every j. This conclusion itself is already relevant to resolving when $\Re(\lambda_n(F, \tau))$ is non-negative.

To proceed further, define

$$I_1(n, F, \tau) = \sum_{j=1}^{r} \int_0^\infty \left[\left(\frac{\nu_j + m}{\omega_j\tau + \nu_j + m}\right)^n - 1 + \frac{n\omega_j\tau}{\omega_j\tau + \nu_j + m}\right] dm$$

for $n \in \mathbb{N}$, $F \in \mathcal{S}^{\sharp\flat}$ with $0 \notin Z(F)$, and $\tau \geq 1$. The lemma shows that in the present context, by the integral test, we have

$$\begin{aligned} I_1(n,F,\tau) &\leq S_1(n,F,\tau) \\ &\leq I_1(n,F,\tau) + \sum_{j=1}^{r}\left[\left(\frac{\nu_j}{\omega_j\tau+\nu_j}\right)^n - 1 + n\left(\frac{\omega_j\tau}{\omega_j\tau+\nu_j}\right)\right]. \end{aligned} \tag{2.3.2}$$

In particular, in the case that $\nu_j = 0$ for all j, this reduces to

$$I_1(n,F,\tau) \leq S_1(n,F,\tau) \leq r(n-1) + I_1(n,F,\tau). \tag{2.3.3}$$

In order to establish effective bounds for $S_1(n,F,\tau)$ in the case that ν_j is real, it remains only to evaluate $I_1(n,F,\tau)$. We make the substitution

$$u_j = \frac{\omega_j\tau}{\omega_j\tau+\nu_j+m}$$

for $1 \leq j \leq r$, so that

$$du_j = -\frac{\omega_j\tau}{(\omega_j\tau+\nu_j+m)^2}dm = -\frac{u_j^2}{\omega_j\tau}dm.$$

In this situation, we see that $u_j = \frac{\omega_j\tau}{\omega_j\tau+\nu_j}$ when $m = 0$ and that u_j tends to 0 as m tends to infinity. This shows us that

$$I_1(n,F,\tau) = \sum_{j=1}^{r}\omega_j\tau\int_0^{\frac{\omega_j\tau}{\omega_j\tau+\nu_j}}\frac{(1-u_j)^n - 1 + u_j n}{u_j^2}du_j$$

Noting that

$$\frac{du_j}{u_j^2} = d\left(\frac{-1}{u_j}\right)$$

and

$$d((1-u_j)^n - 1 + u_j n) = n(1-(1-u_j)^{n-1})du_j,$$

we see that by integration by parts, we have

$$I_1(n,F,\tau) = \sum_{j=1}^{r}\omega_j\tau\left(-\left[\frac{(1-u_j)^n-1+u_jn}{u_j}\right]_0^{\frac{\omega_j\tau}{\omega_j\tau+\nu_j}} + n\int_0^{\frac{\omega_j\tau}{\omega_j\tau+\nu_j}}\frac{1-(1-u_j)^{n-1}}{u_j}du_j\right).$$

We can simplify further in the case $\nu_j = 0$ for all j, considered briefly above when we gave the bound (2.3.3) on $S_1(n,F,\tau)$. Indeed, if $\nu_j = 0$ for all j, then our last equality simplifies to

$$\begin{aligned} I_1(n,F,\tau) &= \sum_{j=1}^{r}\omega_j\tau\left(-\left[\frac{(1-u_j)^n-1+u_jn}{u_j}\right]_0^1 + n\int_0^1\frac{1-(1-u_j)^{n-1}}{u_j}du_j\right) \\ &= \left(1+n(\Psi(n)+\gamma-1)\right)\sum_{j=1}^{r}\omega_j\tau, \end{aligned} \tag{2.3.4}$$

where here we have used the identity

$$\Psi(n) = -\gamma + \int_0^1\frac{1-(1-u)^{n-1}}{u}du$$

for $n\in\mathbb{N}$ (see [19], for example), and the fact that

$$\lim_{u\to 0}\frac{(1-u)^n-1}{u} = -n.$$

We summarize the results we have established above in the following theorem.

Theorem 2.3.2 *Assume that ν_j is real for all j. Then $S_1(n,F,\tau)$ is non-negative for every $n\in\mathbb{N}$, $\tau\geq 1$. Moreover, we have*

$$I_1(n,F,\tau) \leq S_1(n,F,\tau) \leq I_1(n,F,\tau) + \sum_{j=1}^{r}\left[\left(\frac{\nu_j}{\omega_j\tau+\nu_j}\right)^n - 1 + n\left(\frac{\omega_j\tau}{\omega_j\tau+\nu_j}\right)\right].$$

In the special case that $\nu_j = 0$ for all j, this result may be stated more simply as

$$[(1-n)+n(\Psi(n)+\gamma)]\,\frac{\tau}{2}\deg(F) \le S_1(n,F,\tau) \le n(\Psi(n)+\gamma)\frac{\tau}{2}\deg(F)+(n-1)\left(r-\frac{\tau}{2}\deg(F)\right).$$

In the statement of the theorem, we have recalled the definition

$$\deg(F) = 2\sum_{j=1}^{r}\omega_j$$

for functions in the Selberg class, given in Chapter 1.

Our last theorem, combined with Theorem 2.2.1, also allows us to give the following bound on the Li coefficients $\lambda_n(F,\tau)$ for functions in $\mathcal{S}^{\sharp\flat}$ such that ν_j is real for every n. In presenting this result, we adopt the notation

$$S_2(n,F,\tau) = \sum_{k=1}^{n}\binom{n}{k}\tau^k\eta_{k-1}(F,\tau)$$

for the Laurent coefficient sums in the arithmetic formulae for the Li coefficients, again following Coffey's example from the classical case.

Corollary 2.3.3 *Suppose that ν_j is real for every j, $1\le j\le r$. Then we have $\lambda_n(F,\tau) \ge m_F + n\tau\log Q_F + I_1(n,F,\tau) - |S_2(n,F,\tau)|$. If $\nu_j = 0$ for every j, this amounts to saying $\lambda_n(F,\tau) \ge m_F + n\tau\log Q_F + \frac{\tau}{2}\left[(1-n)+n(\Psi(n)+\gamma)\right]\deg(F) - |S_2(n,F,\tau)|$.*

Similarly, we have upper bounds on $\lambda_n(F,\tau)$. Indeed, if ν_j is real for every j, we have $\lambda_n(F,\tau) \le m_F + n\tau\log Q_F + I_1(n,F,\tau) + \sum_{j=1}^{r}\left[\left(\frac{\nu_j}{\omega_j\tau+\nu_j}\right)^n - 1 + n\left(\frac{\omega_j\tau}{\omega_j\tau+\nu_j}\right)\right] + |S_2(n,F,\tau)|$ and if $\nu_j = 0$ for every j, $\lambda_n(F,\tau) \le m_F + n\tau\log Q_F + n(\Psi(n)+\gamma)\frac{\tau}{2}\deg(F) + (n-1)\left(r-\frac{\tau}{2}\deg(F)\right) + |S_2(n,F,\tau)|$.

We see that the behaviour of $I_1(n,F,\tau)$ is approximately $n\log n$ in the case that $\nu_j = 0$ for every j, and that its magnitude will be smaller in the case that $\nu_j > 0$ for some j (since then $\frac{\omega_j \tau}{\omega_j \tau + \nu_j} < 1$). In fact, there exist refinements to Coffey's original methods (which we have directly adapted here), applied, for example, by Coffey in [6], Lagarias in [15], and Smajlović in [26] (for the case of the Selberg class, $\tau = 1$ case), which we expect to allow us to extract the same leading behaviour in the $\tau \geq 1$ case without any special hypotheses on the parameters ν_j. However, we leave the careful application of these techniques to the present context to future work.

The strongest form of Li's criterion requires only that there not be any subsequence of $\{\Re(\lambda_n(F,\tau))\}_{n=1}^{\infty}$ that becomes negative and exponentially large in magnitude with increasing n. Our discussion in this section shows that we need only consider estimates of $S_2(n,F,\tau)$ in order to approach questions about the generalized quasi-Riemann hypothesis.

Chapter 3 Generalizing results on zero-free regions

3.1 Generalizing Brown's work to the quasi-Riemann case

In this chapter, we have two goals. The primary goal is to show how it is possible to adapt the arguments of [3], dealing with correspondences between zero-free regions and the non-negativity of finitely many Li coefficients, to apply to Li coefficients corresponding to generalized quasi-Riemann hypotheses. A secondary goal arises in detailing a problem in the proof of [3, Lemma 5], which forces us to state the main theorem of this chapter on a conjectural basis, and assigns the same conjectural status to [3, Theorem 2]. This secondary goal is fulfilled in a remark following the statement of Conjecture 3.2.7 in Section 3.2.

Here, we will consider functions F in the Selberg class $\mathcal{S}$, satisfying an additional hypothesis on the distribution of their zeros, and obtain a generalization of [3, Theorem 2] for such functions under a conjecture to be given in Section 3.2 (here, our results could also be formulated in the setting of Brown's less restrictive hypotheses on the function in question given in Section 1.7.3, but for consistency with our work in Chapter 2, we restrict ourselves to functions F in the Selberg class).

Definition: For any function F in the Selberg class $\mathcal{S}$ and any $T > 0$, we let

$2N_F(T)$ denote the number of non-trivial zeros of F whose imaginary part has absolute value no greater than T. More precisely, we define

$$2N_F(T) = \#\{\rho \in Z(F) \ : \ |\Im(\rho)| \leq T\},$$

where each zero ρ is counted with multiplicity.

Remark: Throughout this section, we let F be a function in $\mathcal{S}$ such that 0 is not in $Z(F)$. We write $N(T) = N_F(T)$ for convenience. Moreover, we assume that $N(T)$ satisfies

$$N_F(T) = aT\log T + bT + \epsilon(T),$$

where $|\epsilon(T)| \leq c\log^+ T + d$ (using the notation $\log^+ T = \sup(\log T, 0)$), with $a, c, d > 0$, and where in addition, we assume that $3a + b > 0$. This hypothesis on $N_F(T)$ is the same as hypothesis (II) taken by Brown in [3]. Note that the fact that $N_F(T)$ takes this form for functions in the Selberg class, less the assumption that $3a + b > 0$, is easily established (see, for example, [18, p.117]). The hypothesis that $3a + b > 0$ is necessary for the proof of the main result of this chapter.

Definition: Under the hypothesis on F given in the remark above, we also define

$$N_+(T) = aT\log T + bT + (c\log^+ T + d)$$

and

$$N_-(T) = aT\log T + bT - (c\log^+ T + d).$$

The following theorem gives a simple criterion for there to exist an explicit zero-free region for a function F in $\mathcal{S}$.

Definition: Let τ and r be real numbers such that $r > 1$ and $1 \le \tau \le \frac{r+1}{r}$. We define $C_r(\tau)$ to be the region in the complex plane consisting of the critical strip with the overlapping portions of the open discs defined by

$$\left(\Re(s) - \frac{\tau r^2}{r^2-1}\right)^2 + \Im(s)^2 < \frac{r^2\tau^2}{(r^2-1)^2}$$

and

$$\left(1 - \Re(s) - \frac{\tau r^2}{r^2-1}\right)^2 + \Im(s)^2 < \frac{r^2\tau^2}{(r^2-1)^2}$$

removed.

Theorem 3.1.1 *Let τ and r be real numbers such that $r > 1$ and $1 \le \tau \le \frac{r+1}{r}$. Then the following statements are equivalent.*

(a) Every non-trivial zero of F lies in the region $C_r(\tau)$.

(b) Every non-trivial zero ρ of F satisfies $\left|\frac{\rho}{\rho-\tau}\right| \le r$.

Proof: Let ρ be any complex number not equal to 1, and write $\rho = \beta_\rho + i\gamma_\rho$ with β_ρ and γ_ρ real. Then the relation

$$\left|\frac{\rho}{\rho-\tau}\right| \le r \tag{3.1.1}$$

is equivalent to

$$\frac{\beta_\rho^2 + \gamma_\rho^2}{(\beta_\rho-\tau)^2 + \gamma_\rho^2} \le r^2.$$

Rearranging, we see that this is equivalent to the inequality

$$\left(\beta_\rho - \frac{\tau r^2}{r^2-1}\right)^2 + \gamma_\rho^2 \ge \frac{r^2\tau^2}{(r^2-1)^2},$$

and so (3.1.1) is equivalent to ρ lying outside the interior of the disc of radius $\frac{r\tau}{r^2-1}$ with center at $\frac{\tau r^2}{r^2-1}$.

Applying the symmetry on the zeros of F enforced by the functional equation axiom, and using the fact that the zeros of F are confined to the critical strip, we see that the argument above shows precisely that if every non-trivial zero ρ of F satisfies (3.1.1), then every non-trivial zero lies in the region $C_r(\tau)$ as defined above, and vice-versa, as we wanted. ∎

This theorem is analogous to the elementary criterion for the Riemann hypothesis given in Theorem 1.5.3, which can be interpreted as a starting point for the formulation of Li's criterion. Mirroring Brown, this equivalence encourages us to ask about finite analogues to Li's criterion.

We begin with the following definitions.

Definition: Given any complex number $\rho \neq 0$ such that $0 \leq \Re(\rho) \leq 1$, any non-negative integer k, and any real $\tau \geq 1$, we define

$$T_\rho(k,\tau) = \Re\left(2 - \left(\frac{\rho}{\rho-\tau}\right)^k - \left(\frac{\rho-1}{\rho+(\tau-1)}\right)^k\right).$$

Definition: Given any complex number $\rho \neq 0$ such that $0 \leq \Re(\rho) \leq 1$ and any $\tau \geq 1$, we define

$$r_\rho(\tau) = \left|\frac{\rho}{\rho-\tau}\right|.$$

Lemma 3.1.2 *Let k be a positive integer and $\tau \geq 1$. Then we have*

$$2\Re(\lambda_k(F,\tau)) = \sum_\rho T_\rho(k,\tau).$$

Proof: From the definition of $\lambda_k(F,\tau)$, we have

$$\begin{aligned}2\Re(\lambda_k(F,\tau)) &= \lambda_k(F,\tau) + \overline{\lambda_k(F,\tau)} \\ &= \sum_\rho \left(1 - \left(\frac{\rho}{\rho-\tau}\right)^k\right) + \sum_\rho \left(1 - \left(\frac{\bar\rho}{\bar\rho-\tau}\right)^k\right) \qquad (3.1.2)\end{aligned}$$

Imposing the functional equation symmetry requirement that ρ is in $Z(F)$ if and only if $1-\bar{\rho}$ is in $Z(F)$ with the same multiplicity means that we may write

$$\sum_{\rho}\left(1-\left(\frac{\bar{\rho}}{\bar{\rho}-\tau}\right)^{k}\right)=\sum_{\rho}\left(1-\left(\frac{1-\rho}{(1-\tau)-\rho}\right)^{k}\right).$$

Combining this last observation with (3.1.2) and using the definition of $T_{\rho}(k,\tau)$ completes the proof of the lemma. ∎

Lemma 3.1.3 *Given any complex number $\rho\neq 0,1$ such that $0\le\Re(\rho)\le 1$, any nonnegative integer k, and any real $\tau\ge 1$, we have the inequality*

$$2+r_{\rho}(\tau)^{k}+r_{\rho}(\tau)^{-k}\ge T_{\rho}(k,\tau)\ge 2-r_{\rho}(\tau)^{k}-r_{\rho}(\tau)^{-k}.$$

Moreover, we also have that

$$T_{\rho}(k,\tau)=T_{1-\bar{\rho}}(k,\tau).$$

Proof: The definition of $T_{\rho}(k,\tau)$ shows us that

$$T_{\rho}(k,\tau)=\Re\left(2-\left(\frac{\rho}{\rho-\tau}\right)^{k}-\left(\frac{\rho-1}{\rho+(\tau-1)}\right)^{k}\right),$$

and so we clearly have

$$T_{\rho}(k,\tau)\ge 2-\left|\frac{\rho}{\rho-\tau}\right|^{k}-\left|\frac{\rho-1}{\rho+(\tau-1)}\right|^{k}.$$

Since $0\le\Re(\rho)\le 1$ and $\tau\ge 1$, we have $|\rho-1|\le|\rho-\tau|$ and $|\rho+(\tau-1)|\ge|\rho|$. It follows that

$$\left|\frac{\rho-1}{\rho+(\tau-1)}\right|^k \leq \left|\frac{\rho-\tau}{\rho}\right|^k,$$

and so

$$T_\rho(k,\tau) \geq 2 - \left|\frac{\rho}{\rho-\tau}\right|^k - \left|\frac{\rho-\tau}{\rho}\right|^k.$$

Recalling the definition of $r_\rho(\tau)$ completes the proof of the lower bound on $T_\rho(k,\tau)$ claimed by the lemma. The upper bound may be established by a completely analogous argument. For the final part of the lemma, we can simply check from the definition of $T_\rho(k,\tau)$ that

$$\begin{aligned}
T_{1-\bar{\rho}}(k,\tau) &= \Re\left[2 - \left(\frac{1-\bar{\rho}}{(1-\bar{\rho})-\tau}\right)^k - \left(\frac{(1-\bar{\rho})-1}{(1-\bar{\rho})+(\tau-1)}\right)^k\right] \\
&= \Re\left[2 - \left(\frac{\bar{\rho}-1}{\bar{\rho}+(\tau-1)}\right)^k - \left(\frac{\bar{\rho}}{\bar{\rho}-\tau}\right)^k\right] \\
&= \Re\overline{\left[2 - \left(\frac{\rho-1}{\rho+(\tau-1)}\right)^k - \left(\frac{\rho}{\rho-\tau}\right)^k\right]} \\
&= \Re\left[2 - \left(\frac{\rho-1}{\rho+(\tau-1)}\right)^k - \left(\frac{\rho}{\rho-\tau}\right)^k\right] \\
&= T_\rho(k,\tau),
\end{aligned}$$

as was claimed. ∎

The next section concerns itself with establishing bounds required for the proof of our main theorem of this chapter. In the course of this discussion we will arrive at an analysis of the problems in Brown's proof of [3, Lemma 5], and state a conjecture that will allow us to proceed with our own generalization.

3.2 Bounds on sums over zeros

To proceed with our generalized (conditional) version of [3, Theorem 2], to be stated in Theorem 3.3.1, we will first need to prove a number of bounds on various components of the sum over the non-trivial zeros of F defining $\lambda_k(F,\tau)$. Some of these results were proved by Brown in [3] and do not need modification for our purposes, while others require special tailoring to the context of Li coefficients with $\tau \geq 1$. Moreover, we will arrive at the necessity of adopting Conjecture 3.2.7, and discuss the problems in Brown's proof of [3, Lemma 5]. In a number of cases, we prove more general versions than are necessary to establish our main theorem of this chapter, since these may prove useful in terms of giving further refinements of its statement in the future, or in terms of giving an unconditional proof of a similar result.

Our first lemma is one that is required in the proofs of most of the others to follow. This result was given by Brown in [3], and for completeness we include its proof here (the proof itself is an application of partial summation, an elementary tool of analytic number theory in estimating sums).

Lemma 3.2.1 *Let $H \geq 1$, and let j be an integer no smaller than 2. Then we have*

$$\left|\frac{1}{2}\left[\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j}\right]-\frac{H^{1-j}}{j-1}\left[a(1+\log H)+b+\frac{a}{j-1}\right]\right|$$
$$\leq H^{-j}\left(\frac{c}{j}+2c\log H+2d\right).$$

Proof: For real numbers x and y, we define the function $\mathbf{1}_x(y)$ by $\mathbf{1}_x(y)=1$ if $y \geq x$ and $\mathbf{1}_x(y)=0$ otherwise.

Suppose that $f(t)$ is a real function of a real variable, differentiable for $t>0$, such that $\lim_{t\to\infty} f(t) = 0$. Let $\{x_i\}_{n\geq 1}$ be a sequence of positive real numbers. We clearly have that for each i,

$$-\int_{x_i}^{\infty} f'(x)dx = f(x_i).$$

It follows that as long as the following sum converges absolutely,

$$\begin{aligned}\sum_{i=1}^{\infty} f(x_i) &= -\sum_{i=1}^{\infty}\int_{x_i}^{\infty} f'(x)dx \\ &= -\sum_{i=1}^{\infty}\int_{0}^{\infty} f'(x)\mathbf{1}_{x_i}(x)dx \\ &= -\int_{0}^{\infty} f'(x)\sum_{i=1}^{\infty}\mathbf{1}_{x_i}(x)dx. \end{aligned} \tag{3.2.1}$$

We remark that for $j \geq 2$, the function $f(t) = \frac{1}{t^j}$ satisfies the hypotheses above. Moreover, by Lemma 1.7.6, the completed function ξ_F is an entire function of order 1, and so $\sum_{|\Im(\rho)|>H} \frac{1}{|\Im(\rho)|^j}$ converges absolutely whenever $j \geq 2$ (see, for example, [16, Ch.X, Theorem 3.1]). Thus if we take our sequence of points $\{x_r\}_{r\geq 1}$ to be the elements of the multiset $\{|\Im(\rho)| : \rho \in Z(F),\ |\Im(\rho)| > H\}$, where the multiplicities of elements are carried over from $Z(F)$, using some arbitrary ordering, then the equality (3.2.1) derived above holds for this choice of f and sequence of points $\{x_r\}_{r\geq 1}$. It follows that we have, for $j \geq 2$,

$$\begin{aligned}\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j} &= -\int_0^{\infty}\left[\frac{d}{dt}\frac{1}{t^j}\right]_{t=x}\sum_{|\Im(\rho)|>H}\mathbf{1}_{|\Im(\rho)|}(x)dx \\ &= -\int_H^{\infty}\left[\frac{d}{dt}\frac{1}{t^j}\right]_{t=x}\#\{\rho\in Z(F) : x \geq |\Im(\rho)| > H\}dx,\end{aligned}$$

where $\#\{\rho \in Z(F) : x \geq |\Im(\rho)| > H\}$ is counted with multiplicity.

Now by definition, we have

$$\#\{\rho \in Z(F) : x \geq |\Im(\rho)| > H\} = 2(N_F(x) - N_F(H)).$$

Thus, again by definition,

$$2(N_-(x) - N_F(H)) \le \#\{\rho \in Z(F) \ : \ x \ge |\Im(\rho)| > H\} \le 2(N_+(x) - N_F(H)).$$

Since $-\left[\frac{d}{dt}\frac{1}{t^j}\right]_{t=x} > 0$ for $x > 0$, our last several observations imply that

$$\frac{1}{2}\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j} \le \int_H^\infty \left[\frac{d}{dt}\frac{1}{t^j}\right]_{t=x}(N_+(x) - N_F(H))dx$$

and

$$\frac{1}{2}\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j} \ge \int_H^\infty \left[\frac{d}{dt}\frac{1}{t^j}\right]_{t=x}(N_-(x) - N_F(H))dx.$$

Now, noting that $N_+(x)x^{-j}, N_-(x)x^{-j} \to 0$ as $x \to \infty$, we may apply integration by parts to the inequalities above to obtain that

$$\frac{1}{2}\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j} \le \frac{1}{H^j}(N_+(H) - N_F(H)) + \int_H^\infty \frac{1}{x^j}N_+'(x)dx \tag{3.2.2}$$

and

$$\frac{1}{2}\sum_{|\Im(\rho)|>H}\frac{1}{|\Im(\rho)|^j} \ge \frac{1}{H^j}(N_-(H) - N_F(H)) + \int_H^\infty \frac{1}{x^j}N_-'(x)dx, \tag{3.2.3}$$

as long as the integrals on the right converge. It remains to evaluate the terms on the right hand sides of these inequalities. Indeed, by definition of $N_+(x)$ and $N_-(x)$ and the fact that $H \ge 1$, we have that for $x \ge H$,

$$N_\pm'(x) = a\log x + a + b \pm \frac{c}{x},$$

and it follows that

$$\int_H^\infty \frac{1}{x^j}N_\pm'(x)dx = \int_H^\infty \left[\frac{a+b}{x^j} + \frac{a\log x}{x^j} \pm \frac{c}{x^{j+1}}\right]dx$$

$$= \left[-\frac{1}{j-1}\frac{(a+b)x}{x^j} - \frac{a}{(j-1)x^j}\left(\frac{x}{j-1} + x\log x\right) \mp \frac{c}{jx^j} \right]_H^\infty$$

$$= \frac{H^{-j}}{j-1}\left[(a+b)H + aH\left(\frac{1}{j-1} + \log H\right)\right] \pm \frac{c}{jH^j}. \qquad (3.2.4)$$

Moreover, we find that

$$\frac{N_+(H) - N_F(H)}{H^j} = \frac{c\log H + d - \epsilon(H)}{H^j} \le 2\left(\frac{c\log H + d}{H^j}\right) \qquad (3.2.5)$$

and

$$\frac{N_-(H) - N_F(H)}{H^j} = \frac{-c\log H - d - \epsilon(H)}{H^j} \ge -2\left(\frac{c\log H + d}{H^j}\right). \qquad (3.2.6)$$

Combining together (3.2.2), (3.2.3), (3.2.4), (3.2.5), and (3.2.6) now establishes the lemma. ∎

The following lemma generalizes a claim proved by Brown in Lemma 3 of [3].

Lemma 3.2.2 *Let ρ be a complex number with $0 \le \Re(\rho) \le 1$ and write $\rho = \beta_\rho + i\gamma_\rho$ with β_ρ and γ_ρ real. Fix a real number τ satisfying $1 \le \tau < 2$. Then we have that*

$$\frac{1}{\tau^2 + \gamma_\rho^2} \le \Re\left(\frac{1}{\tau - \rho} + \frac{1}{\rho + \tau - 1}\right) \le \frac{2\tau - 1}{\gamma_\rho^2}$$

and that

$$\frac{2}{\tau^2 + \gamma_\rho^2} - \frac{2}{\gamma_\rho^4} - \frac{4(\tau^2 - \tau)}{\gamma_\rho^4} \le -\Re\left(\frac{1}{(\rho - \tau)^2} + \frac{1}{(\rho + \tau - 1)^2}\right) \le \frac{2}{\gamma_\rho^2}.$$

Proof: Here, we can easily see that

$$\Re\left(\frac{1}{\tau - \rho} + \frac{1}{\rho + \tau - 1}\right) = \frac{\tau - \beta_\rho}{(\tau - \beta_\rho)^2 + \gamma_\rho^2} + \frac{\beta_\rho + \tau - 1}{(\beta_\rho + \tau - 1)^2 + \gamma_\rho^2}.$$

Since $0 \le \beta_\rho \le 1$ and $1 \le \tau < 2$, we have

$$\begin{aligned}\frac{\tau-\beta_\rho}{(\tau-\beta_\rho)^2+\gamma_\rho^2}+\frac{\beta_\rho+\tau-1}{(\beta_\rho+\tau-1)^2+\gamma_\rho^2} &\le \frac{\tau-\beta_\rho}{\gamma_\rho^2}+\frac{\beta_\rho+\tau-1}{\gamma_\rho^2}\\ &= \frac{2\tau-1}{\gamma_\rho^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\tau-\beta_\rho}{(\tau-\beta_\rho)^2+\gamma_\rho^2}+\frac{\beta_\rho+\tau-1}{(\beta_\rho+\tau-1)^2+\gamma_\rho^2} &\ge \frac{1-\beta_\rho}{\tau^2+\gamma_\rho^2}+\frac{\beta_\rho}{\tau^2+\gamma_\rho^2}\\ &= \frac{1}{\tau^2+\gamma_\rho^2}.\end{aligned}$$

The first part of the lemma is established by these two facts.

For the second part of the lemma, we similarly start with

$$\begin{aligned}-\mathfrak{R}\left(\frac{1}{(\rho-\tau)^2}+\frac{1}{(\rho+\tau-1)^2}\right) &= \frac{\gamma_\rho^2-(\tau-\beta_\rho)^2}{((\tau-\beta_\rho)^2+\gamma_\rho^2)^2}+\frac{\gamma_\rho^2-(\beta_\rho+\tau-1)^2}{((\beta_\rho+\tau-1)^2+\gamma_\rho^2)^2}\\ &= \left[\frac{1}{(\tau-\beta_\rho)^2+\gamma_\rho^2}+\frac{1}{(\beta_\rho+\tau-1)^2+\gamma_\rho^2}\right]\\ &\quad -2\left[\frac{(\tau-\beta_\rho)^2}{((\tau-\beta_\rho)^2+\gamma_\rho^2)^2}+\frac{(\beta_\rho+\tau-1)^2}{((\beta_\rho+\tau-1)^2+\gamma_\rho^2)^2}\right].\end{aligned}$$

Following the reasoning we used in the first case, we observe that since $0 \le \beta_\rho \le 1$ and $1 \le \tau < 2$, we have

$$\begin{aligned}&\left[\frac{1}{(\tau-\beta_\rho)^2+\gamma_\rho^2}+\frac{1}{(\beta_\rho+\tau-1)^2+\gamma_\rho^2}\right]-2\left[\frac{(\tau-\beta_\rho)^2}{((\tau-\beta_\rho)^2+\gamma_\rho^2)^2}+\frac{(\beta_\rho+\tau-1)^2}{((\beta_\rho+\tau-1)^2+\gamma_\rho^2)^2}\right]\\ &\qquad \ge \frac{2}{\tau^2+\gamma_\rho^2}-2\left(\frac{(\tau-\beta_\rho)^2+(\beta_\rho+\tau-1)^2}{\gamma_\rho^4}\right)\\ &\qquad = \frac{2}{\tau^2+\gamma_\rho^2}-2\left(\frac{(2\beta_\rho^2-2\beta_\rho+1)+2(\tau^2-\tau)}{\gamma_\rho^4}\right)\\ &\qquad \ge \frac{2}{\tau^2+\gamma_\rho^2}-\frac{2}{\gamma_\rho^4}-\frac{4(\tau^2-\tau)}{\gamma_\rho^4},\end{aligned}$$

which establishes the lower bound in the second part of the lemma. The upper bound is clear. ∎

Next, we prove the full analogue of Lemma 3 of [3] for our $\tau \geq 1$ context.

Lemma 3.2.3 *Let k be an integer greater than or equal to 2, τ be a real number such that $1 \leq \tau < 2$, and let $\rho = \beta_\rho + i\gamma_\rho$ where β_ρ and γ_ρ are real, $0 \leq \beta_\rho \leq 1$, and $|\gamma_\rho| \geq k\tau$. Then we have that*

$$T_\rho(k,\tau) \leq \left(\frac{3}{2}k^2\tau^2 + k(\tau^2 - \tau)\right)\frac{1}{\gamma_\rho^2},$$

and as long as $k \geq 6$,

$$T_\rho(k,\tau) \geq \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{36}{37} + 5 - 2e - \frac{1}{36\tau^2} - \frac{2(\tau-1)}{9\tau}\right) \geq \frac{2}{5}\frac{\tau^2 k^2}{\gamma_\rho^2}.$$

Moreover, for all $k \geq 2$, we have

$$T_\rho(k,\tau) \geq \frac{3}{10}\frac{\tau^2 k^2}{\gamma_\rho^2}$$

whenever $1 \leq \tau \leq 2$,

$$T_\rho(k,\tau) \geq \frac{2}{5}\frac{\tau^2 k^2}{\gamma_\rho^2}$$

whenever $1 \leq \tau \leq 1.648$, and

$$T_\rho(k,\tau) \geq \frac{1}{2}\frac{\tau^2 k^2}{\gamma_\rho^2}$$

whenever $1 \leq \tau \leq 1.054$.

Proof: Using a binomial expansion of the definition of $T_\rho(k,\tau)$, one may obtain

$$T_\rho(k,\tau) = \Re\left[\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\tau^j\left(\frac{1}{(\tau-\rho)^j} + \frac{1}{(\rho+\tau-1)^j}\right)\right]$$

$$= \Re\Bigg[k\tau\left(\frac{1}{\tau-\rho}+\frac{1}{\rho+\tau-1}\right)+\frac{k(k-1)\tau^2}{2}\left(\frac{1}{(\tau-\rho)^2}+\frac{1}{(\rho+\tau-1)^2}\right)$$
$$-\sum_{j=3}^{k}\binom{k}{j}(-\tau^j)\left(\frac{1}{(\tau-\rho)^j}+\frac{1}{(\rho+\tau-1)^j}\right)\Bigg]. \tag{3.2.7}$$

It follows from the hypotheses $0 \le \beta_\rho \le 1$, $|\gamma_\rho| \ge k\tau$, and $1 \le \tau < 2$, along with the Taylor expansion of e^x about $x = 0$, that

$$\begin{aligned}\left|\sum_{j=3}^{k}\binom{k}{j}(-\tau^j)\left(\frac{1}{(\tau-\rho)^j}+\frac{1}{(\rho+\tau-1)^j}\right)\right| &\le \sum_{j=3}^{k}\binom{k}{j}\tau^j\frac{2}{|\gamma_\rho|^j}\\ &\le \frac{2k^3\tau^3}{|\gamma_\rho|^3}\sum_{j=3}^{k}\frac{1}{j!}\\ &\le (2e-5)\left(\frac{k\tau}{|\gamma_\rho|}\right)^3\\ &\le \frac{1}{2}\left(\frac{k\tau}{|\gamma_\rho|}\right)^3 \end{aligned}\tag{3.2.8}$$

Combining Lemma 3.2.2 with (3.2.7) and (3.2.8), and using the hypothesis that $|\gamma_\rho| \ge k\tau$, gives us the upper bound

$$\begin{aligned}T_\rho(k,\tau) &\le \frac{k\tau(2\tau-1)}{\gamma_\rho^2}+\frac{k(k-1)\tau^2}{\gamma_\rho^2}+\frac{k^3\tau^3}{2|\gamma_\rho|^3}\\ &= \frac{k(\tau^2-\tau)}{\gamma_\rho^2}+\frac{k^2\tau^2}{\gamma_\rho^2}+\frac{k^3\tau^3}{2|\gamma_\rho|^3}\\ &\le \left(\frac{3}{2}k^2\tau^2+k(\tau^2-\tau)\right)\frac{1}{\gamma_\rho^2},\end{aligned}$$

as claimed by the lemma.

For the lower bound in the case that $k \ge 6$, we again combine Lemma 3.2.2 with (3.2.7) and (3.2.8) and use the hypothesis $|\gamma_\rho| \ge k\tau$ to see that

$$\begin{aligned}T_\rho(k,\tau) &\ge \frac{k\tau}{\tau^2+\gamma_\rho^2}+k(k-1)\tau^2\left(\frac{1}{\tau^2+\gamma_\rho^2}-\frac{1}{\gamma_\rho^4}-\frac{2(\tau^2-\tau)}{\gamma_\rho^4}\right)-(2e-5)\frac{k^3\tau^3}{|\gamma_\rho|^3}\\ &\ge \tau^2\left(\frac{k^2}{\tau^2+\gamma_\rho^2}+\frac{k}{\gamma_\rho^4}-\frac{k^2}{\gamma_\rho^4}-(2e-5)\frac{k^2}{\gamma_\rho^2}\right)+(\tau-1)\left(\frac{2k\tau^3}{\gamma_\rho^4}-\frac{k\tau}{\tau^2+\gamma_\rho^2}-\frac{2k^2\tau^3}{\gamma_\rho^4}\right)\end{aligned}$$

$$\geq \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{\gamma_\rho^2}{\tau^2+\gamma_\rho^2} - \frac{1}{\gamma_\rho^2} + 5 - 2e\right) - (\tau-1)\left(\frac{k\tau}{\tau^2+\gamma_\rho^2} + \frac{2k^2\tau^3}{\gamma_\rho^4}\right). \tag{3.2.9}$$

Now, since $\frac{x^2}{a+x^2}$ is an increasing function of x for $x > 0$, the hypotheses that $k \geq 6$ and that $|\gamma_\rho| \geq \tau k$ give us that

$$\frac{-1}{\gamma_\rho^2} \geq -\frac{1}{36\tau^2}$$

and

$$\frac{\gamma_\rho^2}{\tau^2+\gamma_\rho^2} \geq \frac{36}{37}.$$

We also clearly have that

$$-\frac{\gamma_\rho^2}{\tau^2+\gamma_\rho^2} \geq -1.$$

Using these inequalities in 3.2.9 yields

$$\begin{aligned}
T_\rho(k,\tau) &\geq \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{36}{37} - \frac{1}{36\tau^2} + 5 - 2e\right) - (\tau-1)\left(\frac{k\tau}{\tau^2+\gamma_\rho^2} + \frac{2k^2\tau^3}{\gamma_\rho^4}\right) \\
&= \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{36}{37} - \frac{1}{36\tau^2} + 5 - 2e - \frac{\tau-1}{\tau}\left(\frac{\gamma_\rho^2}{k(\tau^2+\gamma_\rho^2)}\right) - \tau(\tau-1)\frac{2}{\gamma_\rho^2}\right) \\
&\geq \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{36}{37} - \frac{1}{36\tau^2} + 5 - 2e - \frac{\tau-1}{6\tau} - \frac{\tau-1}{18\tau}\right) \\
&\geq \frac{\tau^2 k^2}{\gamma_\rho^2}\left(\frac{36}{37} + 5 - 2e - \frac{1}{36\tau^2} - \frac{2(\tau-1)}{9\tau}\right) \\
&\geq \frac{2}{5}\frac{\tau^2 k^2}{\gamma_\rho^2}
\end{aligned}$$

which is the lower bound claimed by the lemma in the $k \geq 6$ case. By minimizing the quantity in brackets as a function of τ for $1 \leq \tau \leq 2$, we see that it is always at least $\frac{2}{5}$, giving the last inequality. Moreover, minimizing the quantity in brackets for $1 \leq \tau \leq 1.054$ shows that it is always at least $\frac{1}{2}$ as long as τ is in this range, and so for $1 \leq \tau \leq 1.054$ and $k \geq 6$, we always have

$$T_\rho(k,\tau) \geq \frac{k^2\tau^2}{2\gamma_\rho^2}.$$

Now, for lower bounds in the cases $k = 2, 3, 4, 5$, we again call on Lemma 3.2.2, along with (3.2.7) and (3.2.8) to write

$$\begin{aligned} T_\rho(k,\tau) &\geq \frac{k\tau}{\tau^2+\gamma_\rho^2} + k(k-1)\tau^2\left(\frac{1}{\tau^2+\gamma_\rho^2} - \frac{1}{\gamma_\rho^4} - \frac{2(\tau^2-\tau)}{\gamma_\rho^4}\right) - 2\sum_{j=3}^{k}\binom{k}{j}\left(\frac{\tau}{|\gamma_\rho|}\right)^j \\ &= \frac{k^2\tau^2}{\gamma_\rho^2}\left[\frac{\gamma_\rho^2}{\tau^2+\gamma_\rho^2}\left(1-\frac{1}{k}\left(1-\frac{1}{\tau}\right)\right) - \frac{k-1}{k\gamma_\rho^2}\left(1+2(\tau^2-\tau)\right) - \frac{2}{k^2}\sum_{j=3}^{k}\binom{k}{j}\left(\frac{\tau}{|\gamma_\rho|}\right)^{j-2}\right]. \end{aligned}$$

Here, set

$$C(k,\gamma_\rho,\tau) = \left[\frac{\gamma_\rho^2}{\tau^2+\gamma_\rho^2}\left(1-\frac{1}{k}\left(1-\frac{1}{\tau}\right)\right) - \frac{1}{\gamma_\rho^2}\left(\frac{k-1}{k}+2(\tau^2-\tau)\right) - \frac{2}{k^2}\sum_{j=3}^{k}\binom{k}{j}\left(\frac{\tau}{|\gamma_\rho|}\right)^{j-2}\right],$$

so that our inequality for $T_\rho(k,\tau)$ above may be written as

$$T_\rho(k,\tau) \geq \frac{k^2\tau^2}{\gamma_\rho^2}C(k,\gamma_\rho,\tau).$$

Examination of $C(k,\gamma_\rho,\tau)$ shows that, for any integer $k \geq 2$, as long as $\gamma_\rho > 0$ and $\tau \geq 1$, it is increasing as a function of $|\gamma_\rho|$. Since $|\gamma_\rho| \geq k\tau$ by assumption, it follows that for any $k \geq 2$ and $\tau \geq 1$, we have

$$T_\rho(k,\tau) \geq \frac{k^2\tau^2}{\gamma_\rho^2}C(k,\tau k,\tau).$$

Calculating, we obtain

$$\begin{aligned} C(2,2\tau,\tau) &= \frac{2}{5} + \frac{2}{5\tau} - \frac{4\tau^2-4\tau+1}{8\tau^2}, \\ C(3,3\tau,\tau) &= \frac{71}{135} + \frac{3}{10\tau} - \frac{6\tau^2-6\tau+2}{27\tau^2}, \\ C(4,4\tau,\tau) &= \frac{1247}{2176} + \frac{4}{17\tau} - \frac{8\tau^2-8\tau+3}{64\tau^2}, \\ C(5,5\tau,\tau) &= \frac{24074}{40625} + \frac{5}{26\tau} - \frac{10\tau^2-10\tau+4}{125\tau^2}. \end{aligned}$$

Each of these is a function decreasing with τ for $1 \le \tau \le 2$. Since we can easily check that $C(k, 2k, 2) \ge \frac{3}{10}$ for $k = 2, 3, 4, 5$ using the expressions above, combining with our earlier result for $k \ge 6$ shows that

$$T_\rho(k, \tau) \ge \frac{3}{10} \frac{k^2 \tau^2}{\gamma_\rho^2}$$

whenever $1 \le \tau \le 2$ and $k \ge 2$, as claimed by the lemma. Moreover, checking that $C(k, 1.648k, 1.648) \ge \frac{2}{5}$ and $C(k, 1.054k, 1.054) \ge \frac{1}{2}$ for $2 \le k \le 5$ yields the remaining results of the lemma, and the proof is complete. ∎

Corollary 3.2.4 *Let k be an integer greater than or equal to 2 and let τ be a real number with $1 \le \tau < 2$. Let*

$$K_- = [2\tau k a \log(\tau k) + 2(2a + b)\tau k - 4c \log(\tau k) - c - 4d]$$

and

$$K_+ = [2\tau k a \log(\tau k) + 2(2a + b)\tau k + 4c \log(\tau k) + c + 4d].$$

Then we have that

$$\sum_{|\Im(\rho)| > k\tau} T_\rho(k, \tau) \le \left(\frac{3}{2} + \frac{\tau - 1}{k\tau}\right) K_+$$

and

$$\sum_{|\Im(\rho)| > k\tau} T_\rho(k, \tau) \ge \frac{3}{10} K_-.$$

Moreover, if $k \ge 6$, we have

$$\sum_{|\Im(\rho)| > k\tau} T_\rho(k, \tau) \ge \left(\frac{36}{37} + 5 - 2e - \frac{1}{36\tau^2} - \frac{2(\tau - 1)}{9\tau}\right) K_-,$$

if $k \geq 6$ *or* $\tau \leq 1.648$*, we have*

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \geq \frac{2}{5}K_-,$$

and if $\tau \leq 1.054$*, we have*

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \geq \frac{1}{2}K_-.$$

Proof: Let C be a positive constant. Then given that for every zero ρ of F, with $|\gamma_\rho| \geq k\tau$ (where we write $\gamma_\rho = \Im(\rho)$), we have

$$T_\rho(k,\tau) \leq C\frac{k^2\tau^2}{\gamma_\rho^2},$$

it is easy to check that Lemma 3.2.1 with $j = 2$ and $H = k\tau$ implies

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \leq CK_+.$$

Similarly, if for every zero ρ of F with $|\gamma_\rho| \geq k\tau$ we have

$$T_\rho(k,\tau) \geq C\frac{k^2\tau^2}{\gamma_\rho^2},$$

it is again easy to check that Lemma 3.2.1 with $j = 2$ and $H = k\tau$ implies

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \geq CK_-.$$

Combining these facts with Lemma 3.2.3 yields the corollary. ∎

Following Brown, we prove a supplementary lemma below, using the same ideas.

Lemma 3.2.5 *Let* k *be an integer greater than or equal to* 2*, let* τ *be a real number* *with* $1 \le \tau \le 2$, *and let* $H \ge \max\left\{k\tau, \frac{5d}{4a+b}, \frac{5c}{a}\right\}$. *Then we have that*

$$\sum_{|\Im(\rho)|>H} T_\rho(k,\tau) \ge \frac{3}{10}\frac{\tau^2 k^2}{H^2} N_+(H).$$

Moreover, if $k \ge 6$ *or* $\tau \le 1.648$, *we have*

$$\sum_{|\Im(\rho)|>H} T_\rho(k,\tau) \ge \frac{2}{5}\frac{\tau^2 k^2}{H^2} N_+(H),$$

and if $\tau \le 1.054$, *we have*

$$\sum_{|\Im(\rho)|>H} T_\rho(k,\tau) \ge \frac{1}{2}\frac{\tau^2 k^2}{H^2} N_+(H).$$

Proof: For every zero ρ of F, let us write $\Im(\rho) = \gamma_\rho$. Let C be a positive constant. Following the reasoning we used in the proof of Corollary 3.2.4, given that whenever ρ is a zero with $|\gamma_\rho| \ge H$ we have

$$T_\rho(k,\tau) \ge C\frac{k^2\tau^2}{\gamma_\rho^2},$$

it follows by Lemma 3.2.1 with $j = 2$ that

$$\sum_{|\Im(\rho)|\ge H} T_\rho(k,\tau) \ge \frac{Ck^2\tau^2}{H^2}\left(2aH\log H + 2(2a+b)H - c - 4c\log H - 4d\right).$$

Let us write

$$K(H) = 2aH\log H + 2(2a+b)H - c - 4c\log H - 4d,$$

so that

$$\sum_{|\Im(\rho)|\ge H} T_\rho(k,\tau) \ge \frac{Ck^2\tau^2}{H^2}K(H).$$

Here, by hypothesis we have $H \geq k \geq 2$, and so by definition,

$$N_+(H) = aH \log H + bH + c \log^+ H + d = aH \log H + bH + c \log H + d.$$

Consider

$$K(H) - N_+(H) = (aH - 5c)\log H + [(4a+b)H - 5d] + [aH - c].$$

Recalling that $3a + b \geq 0$ and $a, c, d \geq 0$ by assumption, and using our hypothesis that

$$H \geq \max\left\{k\tau, \frac{5d}{4a+b}, \frac{5c}{a}\right\},$$

it follows that the difference $K(H) - N_+(H)$ is non-negative. This means that

$$\sum_{|\Im(\rho)| \geq H} T_\rho(k,\tau) \geq \frac{Ck^2\tau^2}{H^2} K(H) \geq \frac{Ck^2\tau^2}{H^2} N_+(H).$$

Now, combining the argument above with the results of Lemma 3.2.3 completes the proof. ∎

We have now arrived at a convenient point to consider the proof of a generalization of [3, Lemma 5]. Unfortunately, in examining Brown's proof of his result in this special case, one arrives at some errors which will require additional future work to resolve. Let us begin by proving the following lemma.

Lemma 3.2.6 *Let k be an integer greater than or equal to 2, and let τ be a real number such that $1 \leq \tau < 2$. Then for every non-trivial zero ρ of F that does not lie on the real line, we have*

$$T_\rho(k,\tau) \geq 2 - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}},$$

where $\gamma_\rho = \Im(\rho) \neq 0$.

Proof: Let ρ be any non-trivial zero of F. Write $\rho = \beta_\rho + i\gamma_\rho$ with β_ρ and γ_ρ real, and recall that necessarily $0 \le \beta_\rho \le 1$ and $|\gamma_\rho| > 0$. We have by definition that

$$r_\rho(\tau) = \left|\frac{\rho}{\rho - \tau}\right|,$$

and we see that

$$\begin{aligned} r_\rho(\tau)^2 &= \frac{\beta_\rho^2 + \gamma_\rho^2}{(\beta_\rho - \tau)^2 + \gamma_\rho^2} \\ &= 1 + \frac{\tau(2\beta_\rho - \tau)}{(\tau - \beta_\rho)^2 + \gamma_\rho^2} \\ &\le 1 + \frac{\tau}{\gamma_\rho^2}. \end{aligned} \tag{3.2.10}$$

Moreover, we have similarly that

$$r_{1-\bar{\rho}}(\tau)^2 \le 1 + \frac{\tau}{\gamma_\rho^2}$$

since $1 - \bar{\rho}$ is also a non-trivial zero of F and $\Im(1 - \bar{\rho}) = \Im(\rho)$.

Next, we remark that if

$$g(x) = x^k + x^{-k},$$

then since $k \ge 2$, for $x \ge 1$ we have that $g'(x) = kx^{k-1} - \frac{k}{x^{k+1}} \ge 0$. Moreover, since $\Re(1 - \bar{\rho}) = 1 - \Re(\rho) = 1 - \beta_\rho$, we necessarily have either $r_\rho(\tau) \ge 1$ or $r_{1-\bar{\rho}}(\tau) \ge 1$. It follows that for any non-trivial zero ρ of F that does not lie on the real line, we have either

$$2 - r_\rho(\tau)^k - r_\rho(\tau)^{-k} \ge 2 - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}}$$

or

$$2 - r_{1-\bar{\rho}}(\tau)^k - r_{1-\bar{\rho}}(\tau)^{-k} \geq 2 - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}}.$$

Now, applying both parts of Lemma 3.1.3, we can see that for any non-trivial zero ρ of F not lying on the real line,

$$T_\rho(k,\tau) \geq 2 - r_\rho(\tau)^k - r_\rho(\tau)^{-k}, \quad 2 - r_{1-\bar{\rho}}(\tau)^k - r_{1-\bar{\rho}}(\tau)^{-k}.$$

Thus our last several observations imply that for any non-trivial zero ρ of T not lying on the real line, we have

$$T_\rho(k,\tau) \geq 2 - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}}, \tag{3.2.11}$$

as claimed. ∎

We will now state a conjecture which will be required for our proof of the main result of this chapter in the next section. This conjecture, combined with the previous lemma, provides a modified, generalized version of [3, Lemma 5].

Conjecture 3.2.7 *Let k be an integer greater than or equal to 2, and let τ and H be real numbers such that $1 \leq \tau < 2$ and $H > e$. Define $r_H(\tau) = \left(1 + \frac{\tau}{H^2}\right)^{\frac{1}{2}}$. Let b be as in our hypothesis on $N(T)$ in the previous section, and define $b_+ = max\{b, 0\}$. Then, if for every zero ρ of F we write $\gamma_\rho = \Im(\rho)$, we have*

$$\begin{aligned}
\sum_{|\Im(\rho)|>H} &\left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} + \left(1 + \frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}} - 2 \\
&\leq 2\left(r_H(\tau)^k + r_H(\tau)^{-k} - 2\right)\left[\frac{1}{3}aH\log H + (4a + 3b_+)\frac{H}{9} + 2c\log H + 2d + \frac{c}{4}\right] \\
&\leq \frac{9}{4}(r_H(\tau)^k + r_H(\tau)^{-k} - 2)\left[aH\log H + b_+H + 2c\log H + 2d\right].
\end{aligned}$$

Remark: The necessity to adopt Conjecture 3.2.7 in the effort to generalize

Brown's statement of [3, Theorem 2] arises from noticing that the proof of [3, Lemma 5] contains two errors, one of which will require additional work to reconcile. These problems also interfere with proving a generalized version suitable to our needs.

The first error in the proof of [3, Lemma 5] is in Brown's statement that the function $f(x) = x^k + x^{-k}$ (where k is a non-negative integer) is increasing for all $x > 0$. In fact, this function is only increasing for $x > 1$. This error is not a significant obstacle, however, and is essentially irrelevant in the special case that Brown is addressing in [3] (though it requires some consideration in our more general context).

The second error is more serious. The proof of [3, Lemma 5] takes the power series expansion of $g(x) = (1+x)^{\frac{k}{2}} + (1+x)^{-\frac{k}{2}} - 2$ about $x = 0$ to be

$$\sum_{j=2}^{\infty} c_j x^j,$$

remarking that $g(0) = g'(0) = 0$, in order to write

$$\left(1 + \frac{1}{\gamma_\rho^2}\right)^{\frac{k}{2}} + \left(1 + \frac{1}{\gamma_\rho^2}\right)^{-\frac{k}{2}} - 2 = \sum_{j=2}^{\infty} \frac{c_j}{\gamma_\rho^{2j}}$$

for a non-trivial zero ρ of F with $|\Im(\rho)| = \gamma_\rho > H$, where $H \geq e$. It follows this by taking the sum over all non-trivial zeros with imaginary part of magnitude at least H on both sides of the equation above, obtaining

$$\sum_{|\gamma_\rho|>H} \left[\left(1 + \frac{1}{\gamma_\rho^2}\right)^{\frac{k}{2}} + \left(1 + \frac{1}{\gamma_\rho^2}\right)^{-\frac{k}{2}} - 2\right] = \sum_{|\gamma_\rho|>H} \sum_{j=2}^{\infty} \frac{c_j}{\gamma_\rho^{2j}} \tag{3.2.12}$$

The problem in the argument arises in the next step. The proof of [3, Lemma 5] interchanges the order of summation on the right hand side, and then applies the upper bound on the sum

$$\sum_{|\gamma_\rho|>H} \frac{1}{\gamma_\rho^{2j}}$$

provided by Lemma 3.2.1 (given originally in [3, Lemma 2]) by direct substitution in order to bound the left hand side of (3.2.12). This step is not valid because not all of the coefficients c_j in the power series expansion of $g(x)$ are non-negative (as one may easily observe). Indeed, Lemma 3.2.1 also gives us a lower bound on

$$\sum_{|\gamma_\rho|>H} \frac{1}{\gamma_\rho^{2j}},$$

and so we expect that with some additional work, a bounding procedure along these lines may yield a useful result. We relegate the proof of variants of Conjecture 3.2.7 to future work.

We now continue with the proof of a final lemma before proceeding to our main result of this chapter in the next section (under the hypothesis that Conjecture 3.2.7 holds). The next lemma was proved by Brown in [3], and does not require modification for our purposes. For completeness, we provide a proof here as well.

Lemma 3.2.8 *If $b \geq 0$, then the quantity $\frac{N_+(T)}{T^2}$ is decreasing as a function of T for $T > e$. If $b < 0$ then $\frac{N_+(T)}{T^2}$ is decreasing as a function of T for $T > e^{1-\frac{b}{a}}$.*

Proof: In either case described by the lemma, we have $T > e$, and so $\log T > 0$. It follows that by definition,

$$\frac{N_+(T)}{T^2} = \frac{a \log T}{T} + \frac{b}{T} + \frac{c \log T}{T^2} + \frac{d}{T^2}.$$

Differentiating, we find that

$$\frac{d}{dT}\frac{N_+(T)}{T^2} = \frac{a}{T^2} - \frac{a \log T}{T^2} - \frac{b}{T^2} + \frac{c}{T^3} - \frac{2c}{T^3} - \frac{2d}{T^3}$$

$$= -\frac{1}{T^3}\left[(\log T - 1)aT + bT + 2d + (2\log T - 1)c\right].$$

Here, recalling that $a, c, d > 0$ by hypothesis, it is clear that if $b \geq 0$, then every term in the square brackets is non-negative as long as $T > e$, and that some of them are strictly positive. This establishes the first part of the lemma. On the other hand, if $b < 0$, then the quantity in square brackets is still positive as long as

$$(\log T - 1)aT + bT \geq 0$$

and

$$\log T \geq \frac{1}{2}.$$

For $T > 0$, the first condition above is equivalent to

$$\log T > 1 - \frac{b}{a}.$$

Moreover, we always have $1 - \frac{b}{a} > \frac{1}{2}$ since $b < 0$ and $a > 0$. This establishes the second part of the lemma. ∎

We are now ready to give the main result of this chapter, in the next section.

3.3 Generalized results on zero-free regions

With the tools of the previous section in hand, we now give a generalization of Conjecture 1.7.10 (stated originally as [3, Theorem 2]) to the case that $\tau \geq 1$, which holds conditionally on the hypothesis of the truth of Conjecture 3.2.7. This theorem gives the first half of an effective correspondence between the existence of effective zero-free regions for the function F and the non-negativity of finitely many Li coefficients, in the $\tau \geq 1$ case, under appropriate restrictions.

As we will discuss at the end of this section and in the next chapter, we expect that it will prove possible to strengthen this result with further refinement of the bounding procedures involved. Moreover, we hope that an unconditional variation on this theorem will be obtainable in the near future.

Theorem 3.3.1 *Assume that Conjecture 3.2.7 holds. Let $r > 1$, and set $T = (r^2-1)^{-\frac{1}{2}}\sqrt{1-r^2(\tau-1)^2}$. Let τ be a real number with $1 \le \tau < \frac{r+1}{r}$. Suppose that every ρ in $Z(F)$ lies in $C_r(\tau)$. Then there exists some absolute constant C_0 with $0 < C_0 < 1$, and some $T_0 > 0$ depending only on a, b, c, d, such that if $\tau < 1 + \frac{C_0}{r}$ and $T > T_0$, then $\Re(\lambda_k(F,\tau)) \ge 0$ for $1 \le k \le \frac{2T^2 \log T}{\tau}$.*

In particular, we may take $C_0 = \frac{1}{20}$. Moreover, when $b \ge 0$, we may take $T_0 = 12\max\left\{1, \frac{2c}{a}, \frac{2d}{3a+b}\right\}$, and when $b < 0$, we may take $T_0 = \max\left\{2T_1^2\log T_1, e^{1-\frac{b}{a}}\right\}$ with $T_1 = \max\left\{5, \frac{3c}{a}, \frac{d}{3a+b}\right\}$.

Proof: If $k = 1$, it is easy to check that for any complex ρ with $0 \le \Re(\rho) \le 1$ and $\rho \ne 1$, and any real $\tau \ge 1$, we have

$$\Re\left(1 - \frac{\rho}{\rho-\tau}\right) = \tau\left(\frac{\tau - \Re(\rho)}{|\rho-\tau|^2}\right) \ge 0,$$

which implies that $\lambda_1(F,\tau) \ge 0$ (independent of any hypotheses about zero-free regions). Thus we look at $\Re(\lambda_k(F,\tau))$ with $k \ge 2$. We will consider the case that $2 \le k\tau \le T$ at the end of the proof. Most of our work will concern the case $T \le k\tau \le 2T^2\log T$.

Lemma 3.1.2 tells us that for any $k \ge 1$ we have

$$2\Re(\lambda_k(F,\tau)) = \sum_\rho T_\rho(k,\tau),$$

where by Lemma 3.1.3 we have

$$T_\rho(k,\tau) \geq 2 - r_\rho(\tau)^k - r_\rho(\tau)^{-k}.$$

Our hypotheses tell us that every ρ in $Z(F)$ is in $C_r(\tau)$, and by Theorem 3.1.1, it follows that $r_\rho(\tau) \leq r$. The last part of Lemma 3.1.3 tells us that we always have $T_\rho(k,\tau) = T_{1-\bar{\rho}}(k,\tau)$ as long as $\rho \neq 0,1$ (and these cases are impossible by hypothesis on F). Since $x^k + x^{-k}$ is an increasing function for $x > 1$, the symmetry of zeros provided by the functional equation satisfied by ξ_F, along with the first part of Lemma 3.1.3, then implies that we always have

$$T_\rho(k,\tau) \geq 2 - r^k - r^{-k}.$$

Suppose henceforth that $H \geq 5$ is a real number. Assume that $k\tau \geq H$. Then our observations so far clearly show that

$$\begin{aligned} 2\Re(\lambda_k(F,\tau)) \geq & \sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \\ & + \sum_{H<|\Im(\rho)|\leq k\tau} T_\rho(k,\tau) + 2N_+(H)(2 - r^k - r^{-k}). \end{aligned} \tag{3.3.1}$$

The results we have proved earlier in this chapter, along with Conjecture 3.2.7, allow us to bound the first two terms in the expression above. Indeed, by Corollary 3.2.4, we have

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \geq \frac{3}{10}\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right].$$

In fact, we will henceforth assume that $\tau = 1 + \frac{C}{r}$ with $0 < C < C_0 = \frac{1}{20}$, which allows us to use the stronger bound

$$\sum_{|\Im(\rho)|>k\tau} T_\rho(k,\tau) \geq \frac{1}{2}\left[2\tau k a\log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right], \qquad (3.3.2)$$

also given in Corollary 3.2.4, valid for $1 \leq \tau < 1.054$. Similarly, since $2 - x^k - x^{-k} \leq 0$ for all $x > 0$ and $H \geq 5 > e$, we have by Conjecture 3.2.7 and Lemma 3.2.6 that

$$\begin{aligned}\sum_{H<|\Im(\rho)|\leq k\tau} T_\rho(k,\tau) &\geq \sum_{H<|\Im(\rho)|\leq k\tau} 2 - \left(1+\frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1+\frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}} \\ &\geq \sum_{|\Im(\rho)|>H} 2 - \left(1+\frac{\tau}{\gamma_\rho^2}\right)^{\frac{k}{2}} - \left(1+\frac{\tau}{\gamma_\rho^2}\right)^{-\frac{k}{2}} \\ &\geq (2 - r_H(\tau)^k - r_H(\tau)^{-k}) \qquad (3.3.3) \\ &\quad\times\left[\frac{2a}{3}H\log H + \frac{8a+6b_+}{9}H + \frac{c}{2} + 4c\log H + 4d\right],\end{aligned}$$

where $r_H(\tau) = \left(1+\frac{\tau}{H^2}\right)^{\frac{1}{2}}$ and $\gamma_\rho = \Im(\rho)$. Suppose now that $H \leq T$. Here, by definition, we have

$$T = (r^2-1)^{-\frac{1}{2}}\sqrt{1-r^2(\tau-1)^2},$$

and recalling that $\tau = 1 + \frac{C}{r}$ shows us that

$$\sqrt{1-r^2(\tau-1)^2} = \sqrt{1-C^2}.$$

It is then clear that for $0 \leq C < 1$ and $\tau \geq 1$,

$$\begin{aligned}r_H(\tau) &= \left(1+\frac{\tau}{H^2}\right)^{\frac{1}{2}} \\ &\geq \left(1+\frac{\tau}{T^2}\right)^{\frac{1}{2}} \\ &= \left(1+\frac{\tau}{1-C^2}(r^2-1)\right)^{\frac{1}{2}} \\ &\geq r > 1, \qquad (3.3.4)\end{aligned}$$

where the last two inequalities follow from the fact that $(1+xA)^{\frac{1}{2}}$ is an increasing function of x for $A, x > 0$, and by hypothesis on r, respectively.

Once more, let us recall the fact that $x^k + x^{-k}$ is increasing for $x > 1$, and combine equations (3.3.1), (3.3.2), (3.3.3), and (3.3.4) to obtain that, as long as $T \geq H \geq 5$, $k\tau \geq H$, and $0 \leq r\tau - 1 = C \leq C_0 = \frac{1}{20}$,

$$\begin{aligned}
2\Re(\lambda_k(F,\tau)) \geq &\ \frac{1}{2}\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right] \\
&+ \left(2 - r_H(\tau)^k - r_H(\tau)^{-k}\right)\left[\frac{2a}{3}H\log H + \frac{8a+6b_+}{9}H + \frac{c}{2} + 4c\log H + 4d\right] \\
&+ 2N_+(H)(2 - r^k - r^{-k}) \\
\geq &\ \frac{1}{2}\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right] \\
&+ \left(2 - r_H(\tau)^k - r_H(\tau)^{-k}\right) \\
&\times \left[\frac{2a}{3}H\log H + \frac{8a+6b_+}{9}H + \frac{c}{2} + 4c\log H + 4d + 2N_+(H)\right].
\end{aligned}$$

Substituting the definition $N_+(H) = aH\log H + bH + c\log H + d$ into the expression in the last square bracket, we obtain that when $b \geq 0$,

$$\begin{aligned}
2\Re(\lambda_k(F,\tau)) \geq &\ \frac{1}{2}\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right] \\
&+ \left(2 - r_H(\tau)^k - r_H(\tau)^{-k}\right)\left[\frac{8}{3}aH\log H + \frac{8}{9}aH + \frac{8}{3}bH + 6c\log H + 6d + \frac{c}{2}\right],
\end{aligned} \tag{3.3.5}$$

and when $b < 0$,

$$\begin{aligned}
2\Re(\lambda_k(F,\tau)) \geq &\ \frac{1}{2}\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right] \\
&+ \left(2 - r_H(\tau)^k - r_H(\tau)^{-k}\right)\left[\frac{8}{3}aH\log H + \frac{8}{9}aH + 2bH + 6c\log H + 6d + \frac{c}{2}\right].
\end{aligned} \tag{3.3.6}$$

At this stage, it is convenient to split our consideration into the cases $b \geq 0$ and $b < 0$. Thus, for now, let us assume $b \geq 0$. The proof in this case follows Brown's quite closely.

In the $b > 0$ case, we will set $H = T$. Assume that $T \le k\tau \le 2T^2 \log T$. Thus we may write

$$k\tau = \lambda T^2 \log T,$$

for some real λ satisfying $\frac{1}{T \log T} \le \lambda \le 2$.

The idea we will use in the $b \ge 0$ case is to find real numbers x_1 and x_2, and an explicit lower bound T_0 for T in terms of a, b, c, d, such that as long as $T \ge T_0$ and $k\tau \ge T$,

$$\left(\frac{8}{3} - x_1\right)(aT \log T + bT) + \frac{8}{9}aT + 6c \log T + 6d + \frac{c}{2} \le 0$$

and

$$(1 - x_2)(\tau k a \log(\tau k) + \tau k b) + 2\tau k a - 2c \log(\tau k) - \frac{c}{2} - 2d \ge 0.$$

Together, these two inequalities would imply that

$$2\tau k a \log(\tau k) + 2(2a + b)\tau k - 4c \log(\tau k) - c - 4d \ge 2x_2 \lambda T^2 (\log T)(a \log T + b)$$

and

$$\frac{8}{3}aT \log T + \frac{8}{9}aT + \frac{8}{3}bT + 6c \log T + 6d + \frac{c}{2} \le x_1(aT \log T + bT),$$

and so (by (3.3.5) with $H = T$),

$$2\Re(\lambda_k(F, \tau)) \ge x_2 \lambda T^2 (\log T)(aT \log T + b) + x_1(aT \log T + bT)\left(2 - r_T(\tau^k - r_T(\tau)^{-k}\right).$$

Given appropriate values for x_1 and x_2, completing the proof in the case $b > 0$, $k\tau \ge T$ (with $C_0 = \frac{1}{20}$) would then reduce to checking the non-negativity of the right hand side as a function of λ.

Indeed, Brown observed that in the $\tau = 1$ case, the choices $T_0 = 12\max\left\{1, \frac{2c}{a}, \frac{2d}{3a+b}\right\}$, $x_1 = \frac{10}{3}$, $x_2 = \frac{11}{12}$ satisfy these properties. It is easy to see that these choices also work in our present context (remark that our choice $C_0 = \frac{1}{20}$ is critical to this point, since otherwise we would not have our present bound on $\Re(\lambda_k(F,\tau))$. With more work, it may be possible to refine our bounding procedures to allow larger values of C_0.) Indeed, using this choice of T_0, we find by simple rearrangement that as long as $T \geq T_0$,

$$\begin{aligned}
&\frac{8}{3}aT\log T + \frac{8}{9}aT + \frac{8}{3}bT + 6c\log T + 6d + \frac{c}{2} \\
&\quad \leq \frac{10}{3}\left(aT\log T + bT\right) + \left[6c - \frac{aT}{2}\right]\log T \\
&\qquad + \left[6d - \frac{3a+b}{3}T\right] + \left[\frac{c}{2} - \left(\frac{\log T}{6} - \frac{1}{9}\right)\right],
\end{aligned}$$

where all of the terms in square brackets are negative by the definition of T_0, and similarly

$$\begin{aligned}
&\tau k a\log(\tau k) + (2a+b)\tau k - 2c\log(\tau k) - \frac{c}{2} - 2d \\
&\quad = \frac{11}{12}(a\tau k\log(\tau k) + b\tau k) + \left[\frac{a\tau k}{12} - 2c\right]\log(\tau k) + \left[\frac{21}{12}a\tau k - \frac{c}{2}\right] + \left[\frac{3a+b}{12}\tau k - 2d\right] \\
&\quad \geq \frac{11}{12}(a\tau k\log(\tau k) + b\tau k) \\
&\quad \geq \frac{11}{12}(a\lambda T^2(\log T)^2 + b\lambda T^2\log T),
\end{aligned}$$

where the last two lines follow from the definition of T_0 and the facts that $k\tau \geq T$, and $k\tau = \lambda T^2\log T$ by our choice of λ (since these imply that the contents of each square bracket is non-negative).

Thus, with the choices $C_0 = \frac{1}{20}$ and $T_0 = \max\{1, \frac{2c}{a}, \frac{2d}{3a+b}\}$, we have that if $b > 0$ and $k\tau = \lambda T^2\log T$, then

$$\begin{aligned}
2\Re(\lambda_k(F,\tau)) \geq\ & \frac{11}{12}\lambda T^2(\log T)(a\log T + b) \\
& + \frac{10}{3}(aT\log T + bT)\left(2 - r_T(\tau)^k - r_T(\tau)^{-k}\right), \qquad (3.3.7)
\end{aligned}$$

and so in this case, it remains only to show that the right side of the inequality above is non-negative for $\frac{1}{\log T} \leq \lambda \leq 2$.

To show this fact, let us first examine the factor $2 - r_T(\tau)^k - r_T(\tau)^{-k}$ in more detail. We have

$$r_T(\tau) = \left(1 + \frac{\tau}{T^2}\right)^{\frac{1}{2}}$$

by definition, and so by using the fact that $\left(1 + \frac{1}{n}\right)^n \leq e$ (where e is Euler's constant) for every $n > 0$, we obtain

$$r_T(\tau)^k \leq e^{\frac{\tau k}{2T^2}} = e^{\frac{\lambda}{2}\log T} = T^{\frac{\lambda}{2}}.$$

Now, we also clearly have $r_T(\tau) \geq 1$, and so $1 - r_T(\tau)^{-k} \geq 0$. It follows that

$$2 - r_T(\tau)^k - r_T(\tau)^{-k} \geq 1 - r_T(\tau)^k \geq 1 - T^{\frac{\lambda}{2}}.$$

Combining this fact with (3.3.7) yields that

$$\begin{aligned} 2\Re(\lambda_k(F,\tau)) &\geq \frac{11}{12}\lambda T^2(\log T)(a\log T + b) - \frac{10}{3}(aT\log T + bT)\left(1 - T^{\frac{\lambda}{2}}\right) \\ &= \frac{11}{12}T(a\log T + b)\left[\lambda T\log T - \frac{40}{11}(T^{\frac{\lambda}{2}} - 1)\right] \end{aligned} \tag{3.3.8}$$

For the case $b \geq 0$ with $k\tau \geq T$ and our present choices of T_0 and C_0, it remains only to show that the factor in square brackets above is non-negative for $\frac{1}{T\log T} \leq \lambda \leq 2$. In fact, this is exactly the term that arises in Brown's paper, and we may use his argument: when $\lambda = \frac{1}{T\log T}$, the quantity in square brackets is

$$\frac{51}{11} - \frac{40}{11}e^{\frac{1}{2T}} \geq 0$$

since $T \geq 12$, and moreover, for $\lambda < 1.5$, we have

$$\frac{\partial}{\partial\lambda}\left[\lambda T\log T - \frac{40}{11}(T^{\frac{\lambda}{2}} - 1)\right] = \left[1 - \frac{40}{22}T^{\frac{\lambda}{2}-1}\right]T\log T \geq 0,$$

which together show that it is non-negative for $\frac{1}{T\log T} \leq \lambda < 1.5$. On the other hand, for $1.5 \leq \lambda \leq 2$, we have

$$\lambda T \log T - \frac{40}{11}(T^{\frac{\lambda}{2}} - 1) \geq T\left(1.5\log 12 - \frac{40}{11}\right) \geq 0.$$

This finishes the case that $b \geq 0$ and $k\tau \geq T$, in the sense that we have shown that those hypotheses imply $\Re(\lambda_k(F,\tau))$ is non-negative as long as $T \geq T_0$ and $\tau \leq 1 + \frac{C_0}{r}$.

Next, let us consider the case that $b < 0$ and $k\tau \geq T$. Again we follow Brown's work quite closely. As in Brown, we now define $T_1 = \max\left\{5, \frac{3c}{a}, \frac{d}{3a+b}\right\}$, choose $T_0 = \max\left\{2T_1^2 \log T_1, e^{1-\frac{b}{a}}\right\}$, and assume that $k\tau \geq T \geq T_0 \geq 2T_1^2 \log T_1$ (remark that this choice of T_0 still keeps our previous hypothesis that $T \geq 12$). Again we only consider $k\tau \leq 2T^2 \log T$. We define H to be the real number such that $k\tau = 2H^2 \log H$.

We certainly have $5 \leq H \leq T$, by the definitions of T_1 and T_0 and the fact that $T \leq k\tau \leq 2T^2 \log T$. Thus equation (3.3.6) holds for this choice of H. Since, by assumption, $b < 0$, we can eliminate the term involving b inside the second square bracket of (3.3.6). Moreover, using the same reasoning as in the $b \geq 0$ case along with the fact that $k\tau = 2H^2 \log H$, we have

$$2 - r_H(\tau)^k - r_H(\tau)^{-k} \geq -r_H(\tau)^k = -\left(1 + \frac{\tau}{H^2}\right)^{\frac{k}{2}} \geq -e^{\frac{\tau k}{2H^2}} = -e^{\log H} = -H.$$

Combining these facts with (3.3.6) we obtain that

$$\begin{aligned} 2\Re(\lambda_k(F,\tau)) \geq \frac{1}{2}&\left[2\tau k a \log(\tau k) + 2(2a+b)\tau k - 4c\log(\tau k) - c - 4d\right] \\ &- H\left[\frac{8}{3}aH\log H + \frac{8}{9}aH + 6c\log H + 6d + \frac{c}{2}\right]. \end{aligned}$$

The proof from this point in the case $b < 0$ is essentially identical to Brown's, once again. Indeed, dividing both sides by $k\tau = 2H^2 \log H$ and using the fact that $\log(k\tau) = \log(2H^2 \log H) = \log\log H + \log 2 + 2\log H \geq 2\log H$ (since $H \geq 5$), we can rearrange the last inequality to show that

$$\begin{aligned}
\frac{2\Re(\lambda_k(F,\tau))}{k\tau} &\geq \left(a - \frac{c}{H^2\log H}\right)\log(\tau k) + 2a + b - \frac{c\log(\tau k)}{H^2\log H} - \frac{c+4d}{4H^2\log H} \\
&\qquad - \frac{4}{3}a - \frac{4a}{9\log H} - \frac{3c}{H} - \frac{3d}{H\log H} + \frac{c}{4H\log H} \\
&\geq a\left[2\log H - \frac{8}{3}H - \frac{4}{9\log H}\right] + \left[(3a+b) - \frac{3d}{H\log H} - \frac{d}{H^2\log H}\right] \\
&\qquad + \left[\frac{a}{3} - \frac{3c}{H} - \frac{c}{4H\log H} - \frac{2c}{H^2} - \frac{c}{4H^2\log H}\right] \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the fact that the definitions of T_1 and T_0, along with the inequality $T_1 \leq H \leq T$, implies that the contents of each square bracket is non-negative. This finishes the case that $b < 0$ and $k\tau \geq T$, in the same sense as the $b \geq 0$ case (with a different choice of T_0).

Finally, we consider $2 \leq k\tau \leq T$ (again, this follows Brown very closely). As previously, let $T_0 = 12\max\left\{1, \frac{2c}{a}, \frac{2d}{3a+b}\right\}$ if $b \geq 0$, and let $T_0 = \max\left\{2T_1^2 \log T_1, e^{1-\frac{b}{a}}\right\}$ with $T_1 = \max\left\{5, \frac{3c}{a}, \frac{d}{3a+b}\right\}$ if $b < 0$. Assume that $T \geq T_0$ in either case. When $2 \leq k\tau \leq T$, Lemma 3.2.5, along with observations from earlier in our proof, allows us to write

$$\begin{aligned}
2\Re(\lambda_k(F,\tau)) &\geq \sum_{|\Im(\rho)|\geq T} T_\rho(k,\tau) + 2N_+(T)(2 - r^k - r^{-k}) \\
&\geq N_+(T)\left[\frac{k^2\tau^2}{2T^2} + 2(2 - r_T(\tau)^k - r_T(\tau)^{-k})\right].
\end{aligned}$$

Since $r^x + r^{-x}$ is an increasing function of x for $x, r \geq 1$, we have

$$\begin{aligned} 2 - r_T(\tau)^k - r_T(\tau)^{-k} &\geq 2 - r_T(\tau)^T - r_T(\tau)^{-T} \\ &\geq 2 - \left(1 + \frac{\tau}{T^2}\right)^{\frac{T}{2}} - \left(1 + \frac{\tau}{T^2}\right)^{\frac{T}{2}} \\ &\geq 2 - e^{\frac{\tau}{2T}} - e^{-\frac{\tau}{2T}} \\ &= -2\sum_{k=1}^{\infty} \frac{1}{(2k)!}\left(\frac{\tau}{2T}\right)^{2k}, \end{aligned}$$

where the last line uses the power series expansion of the exponential function about 0. This last line makes it clear that $2 - r_T(\tau)^k - r_T(\tau)^{-k} \geq -\frac{\tau^2}{T^2}$ for $T \geq 1$, and so we find that in fact,

$$2\Re(\lambda_k(F,\tau)) \geq N_+(T)\left[\frac{k^2\tau^2}{2T^2} + \frac{2\tau^2}{T^2}\right] \geq 0$$

when $k \geq 2$ (since then clearly $\frac{k^2}{2} \geq 2$).

Combining all of our cases, and making note of our choices of T_0 and C_0, now concludes the proof of the theorem. ∎

The fact that Theorem 3.3.1 is conditional on Conjecture 3.2.7 makes the present state of results on zero-free regions in the case that $\tau > 1$ quite unsatisfying. Moreover, even Brown's special case of [3, Theorem 2] seems to be invalidated by problems in the proof of [3, Lemma 5]. We hope to devote additional work in the future to establishing an unconditional result in this respect (by finding a valid and sufficiently powerful bounding procedure for the term involved in Conjecture 3.2.7).

As we have mentioned, we expect it to be possible to give stronger results than Theorem 3.3.1 by refining our bounding procedures (allowing us to take larger values for C_0 and smaller values for T_0, in particular). This possibility is the motivation for many of the the additional, more general, results stated in our lemmas in the previous section, which we have not used directly in the proof of the theorem.

Moreover, we should also naturally hope to obtain an analogue to Theorem 1.7.11 (also given by [3, Theorem 3]) in the $\tau \geq 1$ case, giving the second half of the correspondence between zero-free regions and the non-negativity of the real parts of finitely many Li coefficients in this more general context. Many of the upper bounds proved in Section 3.2 are relevant to this problem. The proof of [3, Theorem 3] also does not appear to be subject to the errors that affect that of [3, Theorem 2], and so we expect such difficulties to be avoidable in attempting a generalization as well. We leave the proof of a generalized version of the second half of the correspondence to future work.

Chapter 4 Summary, Conclusions, and Future Research

This thesis has presented new results on generalizations of Li's criterion, as well as reviewing other research related to its study since its inception. In particular, the focus has been on combining ideas of Freitas [11] relating to extending Li's criterion to an equivalence for the *quasi*-Riemann hypothesis, with work of Smajlović [25], Coffey [4], and Brown [3]. In the last case, our main result (Theorem 3.3.1) remains conditional on Conjecture 3.2.7, a fact that is carried over from some errors in the proof of [3, Lemma 5]. We have elucidated the problems in this proof (and shown that, unfortunately, they seem to invalidate the proof of [3, Theorem 2], as well), and hope to be able to establish an unconditional generalized result in the future.

We have also extended the framework of Bombieri and Lagarias [2] to the context of equivalences for elements of complex multisets to lie in arbitrary half-planes, and given some additional results on equivalences to the Riemann hypothesis and discussion of the general context for the consideration of Li's criterion.

In this chapter, we wish to evaluate the significance of the work in this thesis, to contrast it with other work in the literature, and to discuss promising avenues for future research and for improvements to our results.

4.1 Significance of our results, potential improvements, and comparisons to other research

Here, we wish to summarize our results and their significance in terms of progress towards the study of Li's criterion and its generalizations, along with providing some discussion on the clearest paths to their improvement or further generalization. To this end, we will roughly follow the structure of the thesis itself.

Thus, we begin with the consideration of some of the supporting results that we presented in Chapter 1. In particular, in Section 1.7.1, we provided the direct generalization of the framework of Bombieri and Lagarias from [2] to the context of criteria for elements of a complex multiset to lie in an arbitrary half-plane. While this generalization (Theorem 1.6.2 and Corollary 1.7.4) is quite elementary, it provides a clear bridging step between the work of Freitas in [11] and other research on Li's criterion. Indeed, Freitas' original arguments approached the question of the Li criterion for quasi-Riemann hypotheses from the perspective of Li's methods in [17], which have been generally regarded as subservient to the approach of Bombieri and Lagarias in more recent work. Our discussion in Section 1.7.1 provides the results required to view Freitas' work from the alternate perspective of Bombieri and Lagarias' general framework, in an explicit way.

Chapter 2 contains several of our main results, and these deserve substantial discussion in our present context.

First, in Lemma 2.1.2 and Theorem 2.1.3, we give a general version of Li's criterion for the δ-GRH in an extension of the Selberg class. This generalizes results of Smajlović in [25] and Freitas in [11]. The argument itself is a straightforward application of the framework of Bombieri and Lagarias for the Freitas case (developed in Section 1.7.1), supported by properties of the Selberg class (and the extended class $\mathcal{S}^{\sharp\flat}$) developed by Smajlović in [25]. Its main contribution is simply to show that Freitas' ideas do, indeed, immediately generalize to more general contexts in an extremely straightforward way. It is also worth noting that, in our proof of Lemma 2.1.2, we are careful to establish the absolute and uniform convergence of certain sums with careful rigor, solidifying the work in Smajlović's proof of [25, Theorem A.1] (which does not pay attention to some of these issues).

Next, in Theorem 2.2.1, we gave the general form of arithmetic formulae for the Li coefficients in the Selberg class, quasi-Riemann hypothesis case. Again, this generalizes work of Smajlović [25] and Freitas [11]. Giving such arithmetic formulae for Li coefficients is quite standard, and this theorem gives the general form of such formulae for Li coefficients corresponding to any quasi-Riemann hypothesis for functions in an extension of the Selberg class. Once again, the arguments involved are quite standard, and rely simply on the form of completed functions provided by the Selberg class axioms, and the definition of the Li coefficients themselves. In addition to the arithmetic formula for $\lambda_k(F, \tau)$ itself, Section 2.2 also provides some alternate descriptions for terms involved in the arithmetic formulae. Of particular note is the observation that, in the $\tau > 1$ case, the Laurent coefficients of $\frac{F'}{F}$ involved in the arithmetic formulae for $\lambda_k(F, \tau)$ may be interpreted as Dirichlet series. This observation invites the application of techniques of classical analytic number theory to the study of the Li coefficients via our significant understanding of the theory of Dirichlet series, a strategy that does not appear to have been considered sufficiently in the literature.

It should be noted at this point (as it was briefly in Section 2.1) that there are methods of bounding these Laurent coefficient sums via contour integration that have been applied in the literature already [15], [26]. It is certainly expected that this approach should also yield similar results on these Laurent coefficient sums in the $\tau > 1$ case, but we have left the application of these arguments to this case to future work.

In Theorem 2.3.2, we give an explicit non-negative bound on the polygamma sums involved in arithmetic formulae for $\lambda_k(F, \tau)$ in the case that the parameters ν_j of the completed function in question are real. In this case, we observe the leading behaviour $O(k \log k)$ in these polygamma sums. Our argument was based on methods of Coffey in [4], applying the integral test to estimate the relevant sums. In fact, as also mentioned at the ends of Section 1.8 and of Section 2.3, there also exist refinements of this argument (developed in [15] by Lagarias, and also applied in [26] to the $\tau = 1$ case for the Selberg class, for example) that we expect to allow the extraction of the same leading behaviour without requiring such hypotheses on the parameters ν_j. We have also left the application of these arguments to the $\tau > 1$ case to future work.

Chapter 3 of this thesis concerns attempting to generalize [3, Theorem 2] (stated here as Conjecture 1.7.10) and supporting results to the context of Li coefficients corresponding to quasi-Riemann hypotheses (i.e. the $\tau > 1$) case. This has, unfortunately, involved clarifying some errors which appear in the proof of [3, Lemma 5] and which seem to invalidate the proof of [3, Theorem 2] in its current form. The statement of [3, Theorem 2] represents half of a correspondence given by Brown between zero-free regions and the non-negativity of the real parts of finitely many Li coefficients. The other half of the correspondence, given by [3, Theorem 3], seems unaffected by the problems with the proof of the first half, and appears to remain valid.

In Theorem 3.3.1 we have given a generalization of [3, Theorem 2], conditional on Conjecture 3.2.7, which concerns Li coefficients corresponding to quasi-Riemann hypotheses. There is a great deal of potential for improvement and further progress in this respect. Most obviously, we hope to develop a valid bounding procedure for the terms involved in Conjecture 3.2.7 (and [3, Lemma 5]) which will allow us to prove an unconditional variation of Theorem 3.3.1.

Another obvious natural problem is to give a quasi-Riemann analogue to the second half of the correspondence, proved by Brown as [3, Theorem 3] (and stated here in Theorem 1.7.11). We expect this generalization to be obtainable by relatively straightforward extensions of Brown's arguments (and some results proved in Section 3.2) in an unconditional sense, but we have left its proof to future work. Beyond this, there are several other aspects inviting improvement in our treatment of generalizations of Brown's work. First, by refining bounding arguments, we hope to be able to improve the choices of C_0 and T_0 suggested in the statement of Theorem 3.3.1. Indeed, the bounding arguments used by Brown (and followed quite closely in our work on generalizing his results) are quite crude and certainly invite improvement. We hope to investigate this possibility in the future as well. Second, we also hope, in the future, to prove at least a partial $\tau > 1$ analogue to [3, Theorem 5], which states that the non-negativity of $\lambda_2(F, 1)$ alone is enough to prove the non-existence of a Siegel zero in a very general context.

4.2 Further research

In this final section, we wish to focus specifically on discussion of the most promising avenues of future research on Li's criterion and the Li coefficients. We have mentioned several potential improvements to our work in the thesis itself in the previous section, and we will avoid repeating those ideas again here. Instead, we focus on discussing some ideas that have not been investigated in the literature thus far to our knowledge, but that seem as though they may offer some promise in the further study of Li's criterion and the Li coefficients.

The most important aspect of the study of Li's criterion, in a substantial sense, is that of understanding of the arithmetic formulae for the Li coefficients. Indeed, it is only by gaining insight into the behaviour of the Li coefficients by methods of analysis that we can hope to use Li's criterion to improve our understanding of the Riemann hypothesis and its generalizations.

One natural approach, invited directly by the formulation of Li's criterion for the generalized quasi-Riemann hypothesis in the Selberg class, is to look at the behaviour of Li coefficients $\lambda_k(F,\tau)$ for τ very close to 2. Indeed, as a clear consequence of the axioms of the Selberg class, the zeros of completed functions ξ_F corresponding to functions F in $\mathcal{S}$ must lie in the critical strip. Thus, we know that $\lambda_k(F,2) \geq 0$ for every $k \geq 1$ whenever F is a function in $\mathcal{S}$ with $F(0) \neq 0$. Yet, this fact is not obvious from our arithmetic formulae for the Li coefficients. Indeed, the derivation of arithmetic formulae for Li coefficients in the Selberg class relies essentially solely on the functional equation axiom, while it is clear that both the Euler product axiom and the Ramanujan property are critical to the fact that all of the zeros of ξ_F lie in the critical strip. Thus, we must look at ways to study our arithmetic formulae using the information contained in these axioms.

Thus far, it is not known whether or not functions in the Selberg class may have zeros on the line $\Re(s) = 1$ in general. The Selberg conjectures (see [24] or [18, Chapter 8], for example) may be shown to imply non-vanishing on this line, and in the case of automorphic L-functions, non-vanishing on $\Re(s) = 1$ has been established unconditionally by Jacquet and Shalika [12]. Studying the behaviour of Li coefficients near $\tau = 2$ may shed light on this question more generally without relying explicitly on the Selberg conjectures. An inviting approach in this respect is to look at power series expansions of the arithmetic formulae for Li coefficients about $\tau = 2$. Indeed, investigating the behaviour of the Li coefficients near $\tau = 2$ via expansions of this type may shed some light on quasi-Riemann hypotheses and on the analogue to the Prime Number Theorem in the Selberg class.

Another idea, mentioned briefly in the previous section, involves applying our knowledge of Dirichlet series to the study of the Laurent coefficient sums involved in arithmetic formulae for the Li coefficients. Indeed, for functions F in the Selberg class (or the class $\mathcal{S}^{\sharp\flat}$), we know that the logarithmic derivative $\frac{F'}{F}(s)$ may be expressed as a Dirichlet series for $\Re(s) > 1$. Under appropriate additional hypotheses on the Dirichlet coefficients, applying our knowledge of Dirichlet series from classical analytic number theory along with the axioms of the Selberg class itself may allow us to extract significant information about the behaviour of these Laurent coefficient sums.

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Appendix: Arithmetic formulae for classical Li coefficients

In this section we give the application of our arithmetic formulae of Chapter 2 to the special case of the Riemann zeta function. In large part, this is a reproduction of work of many other authors [2], [4]. We give some additional interpretation of the terms involved, and note an interesting property of the form of these arithmetic formulae. Recall that for $\Re(s) > 1$,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$$

and

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}.$$

Moreover, the completed zeta function $\xi(s)$ is given by

$$\xi(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

In terms of parameters for completed functions in the Selberg class in general (discussed in Section 1.4), we have in this case that $m_\zeta = 1$, $Q_\zeta = \frac{1}{\sqrt{\pi}}$, $r = 1$, $\omega_1 = \frac{1}{2}$, $\nu_1 = 0$.

Let us consider the coefficients $\eta_k(\tau) = \eta_k(\zeta, \tau)$, $\wp_k(\tau) = \wp_k(\zeta, \tau)$, and $\ell_k(\tau) = \ell_k(\zeta, \tau)$ for $k \geq 0$, as defined in Section 2.2. For convenience, we use the notation $\eta_k = \eta_k(1)$ and $\wp_k = \wp_k(1)$. Here, η_k is simply the kth Laurent coefficient of $\frac{\zeta'(s)}{\zeta(s)}$

about $s=1$. We also have

$$\wp_k(\tau) = \frac{1}{k!2^{k+1}}\Psi^{(k)}\left(\frac{\tau}{2}\right) = (-1)^{k+1}\sum_{m=0}^{\infty}\left(\frac{1}{\tau+2m}\right)^{k+1},$$

and so in particular,

$$\begin{aligned}\wp_k &= (-1)^{k+1}\sum_{m=0}^{\infty}\frac{1}{(1+2m)^{k+1}} \\ &= (-1)^{k+1}\zeta(k+1)\left(1-\frac{1}{2^{k+1}}\right).\end{aligned}$$

Finally, to consider $\ell_k(\tau)$, we remark that for $\tau > 1$ using the Dirichlet series for $\frac{F'}{F}$ above, we obtain

$$\ell_k(\tau) = (-1)^{k+1}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{\tau}}(\log m)^k. \tag{.0.1}$$

As a corollary of Theorem 2.2.1, we then have the following results for $\lambda_n(\zeta,\tau) = \lambda_n(\tau)$ and $\lambda_n(\zeta,1) = \lambda_n$.

Corollary .0.1 *For $n \geq 1$ and $\tau > 1$, we have the following identities for $\lambda_n(\tau)$.*

$$\begin{aligned}\lambda_n(\tau) &= 1-\frac{n\tau}{2}\log\pi+\frac{n\tau}{2}\Psi\left(\frac{\tau}{2}\right)+\sum_{k=1}^{n}\binom{n}{k}\tau^k\eta_{k-1}(\tau)+\sum_{k=2}^{n}\binom{n}{k}(-\tau)^k\sum_{m=0}^{\infty}\left(\frac{1}{\tau+2m}\right)^k \\ &= \left(2+(-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right)-\frac{n\tau}{2}\log\pi+\frac{n\tau}{2}\Psi\left(\frac{\tau}{2}\right) \\ &\quad+\sum_{k=1}^{n}\binom{n}{k}\frac{(-\tau)^k}{(k-1)!}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{\tau}}(\log m)^{k-1} \\ &\quad+\sum_{k=2}^{n}\binom{n}{k}(-\tau)^k\sum_{m=0}^{\infty}\left(\frac{1}{\tau+2m}\right)^k.\end{aligned} \tag{.0.2}$$

Moreover, remarking that $\Psi\left(\frac{1}{2}\right) = -\gamma-2\log 2$, we have

$$\lambda_n = 1-\frac{n}{2}\left(\gamma+\log(4\pi)\right)+\sum_{k=1}^{n}\binom{n}{k}\eta_{k-1}+\sum_{k=2}^{n}\binom{n}{k}(-1)^k\left(1-\frac{1}{2^k}\right)\zeta(k). \tag{.0.3}$$

In the case that $\tau = 1$, the arithmetic formula for the Li coefficients of $\zeta(s)$ is quite striking. Indeed, the sum

$$S_1(n) = \sum_{k=2}^{n} \binom{n}{k} (-1)^k \left(1 - \frac{1}{2^k}\right) \zeta(k)$$

is a combination of special values of $\zeta(s)$ at positive integer arguments, while the sum

$$S_2(n) = \sum_{k=1}^{n} \binom{n}{k} \eta_{k-1}$$

is a combination of Laurent coefficients of $\frac{\zeta'}{\zeta}$ about $s = 1$. Thus, the Li coefficients may be given completely in terms of coefficients characteristic of the behaviour of $\zeta(s)$ for $\Re(s) \geq 1$, but by Li's criterion, also contain enough information to give insight into the truth of the Riemann hypothesis.

The main element of these formulas left to study is the sum of coefficients $\eta_{k-1}(\tau)$. For $\tau = 1$, our only option is to examine the behaviour of the Stieltjes constants using analysis. On the other hand, for $\tau > 1$, and in particular for τ very close to 2, we may hope to be able to achieve significant results simply by analysis of the Dirichlet series for $\ell_k(\tau)$, applying tools from classical analytic number theory.