Character Theory and Artin L-functions

by

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Abstract

In the spirit of Artin, Brauer, and Heilbronn, we implement representation theory together with the *Artin formalism* to study L-functions in this thesis.

One of the major themes is motivated by the work of Heilbronn and many others on classical Heilbronn characters. We define the *arithmetic Heilbronn characters* and apply them to study L-functions. In particular, we prove a theorem concerning the analytic ranks of elliptic curves as predicted by the Birch-Swinnerton-Dyer conjecture.

In a different vein, we employ the theory of supercharacters introduced by Diaconis and Isaacs to derive a supercharacter-theoretic analogue of Heilbronn characters. Moreover, we generalise the effective Chebotarev density theorem due to M. R. Murty, V. K. Murty, and Saradha in the context of supercharacter theory.

Lastly, we study the conjectures of Artin and Langlands via group theory and extend the previous work of Arthur and Clozel. For instance, we introduce the notion of *near supersolvability* and *near nilpotency*, and show that Artin's conjecture holds if $\mathrm{Gal}(K/k)$ is nearly supersolvable. As a consequence, the Artin conjecture is true for any solvable Frobenius Galois extension. Also, we derive the automorphy for every nearly nilpotent group. Furthermore, the Langlands reciprocity conjecture has been established for Galois extensions of number fields of either square-free degree or odd cube-free degree as well as all non-A_5 extensions of degree at most 100.

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Statement of Originality

I, Peng-Jie Wong, being a candidate for the degree of Doctor of Philosophy, hereby declare that this dissertation and the work described in it are my own work, unaided except as may be specified below, and that the dissertation does not contain material that has already been used to any substantial extent for a comparable purpose.

Contents

List of Abbreviations and Symbols

M . . . . monomial

SM . . . . subnormally monomial

NM . . . . nearly monomial

NSS . . . . nearly supersolvable

NN . . . . nearly nilpotent

G^i . . . . the i-th derived subgroup of G, inductively defined as the derived subgroup of G^{i-1}, with $G^0 = G$

G' . . . . the derived subgroup of G

G_π . . . . a Hall π-subgroup of G (in particular, if $\pi = \{p\}$, then it is a Sylow p-subgroup of G)

$\mathbf{Z}(G)$ . . . . the centre of G

$\mathbf{F}(G)$ . . . . the Fitting subgroup of G, i.e., the maximal normal nilpotent subgroup of G

$\Phi(G)$ . . . . the Frattini subgroup of G, i.e., the intersection of all maximal subgroups of G

$\langle g \rangle$ . . . . the subgroup generated of g in G

$H \leq G$ . . . . H is a subgroup of G

$N \trianglelefteq G$ . . . . N is a normal subgroup of G

$N \rtimes H$ .... a semi-direct product of N by H

C_m .... the cyclic group of order m

C_m^n .... the direct product of n-copies of C_m

V_4 .... the Klein four-group

Q .... the quaternion group of order 8

S_n .... the symmetric group on n letters

A_n .... the alternating group on n letters

$\mathbb{F}_p$ .... the field of p elements

$GL_n(F)$ .... the group of non-singular $n \times n$ matrices over F

$SL_n(F)$ .... the group of $n \times n$ matrices of determinant 1 over F

$PGL_n(F)$ ...... $GL_n(F)$ modulo the subgroup of scalar matrices

$\mathrm{Irr}(G)$ .... the set of irreducible characters of G

$\mathrm{Irr}(G, \sigma)$ .... the set of irreducible characters of G appearing in σ

$\mathrm{Sup}(G)$ .... the set of supercharacters of G

$\mathbf{C}(G)$ .... the space of class functions of G

$\mathrm{cd}(G)$ ...... $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, the set of character degrees of G

$\mathcal{C}$ .... the class consisting of groups G with $\mathrm{cd}(G) \subseteq \{1, 2\}$

$\mathrm{Ind}_H^G \psi$ .... the induction of ψ from H to G

$\mathrm{SInd}_H^G \tau$ .... the superinduction of τ from H to G

Sym^m .... the m-th symmetric power

$\mathrm{Gal}(K/k)$ ...... the Galois group of a Galois extension K/k

K^H .... the fixed field of $H \leq \mathrm{Gal}(K/k)$

Chapter 1

Introduction

A natural number is said to be *prime* if it has exactly two divisors, namely, 1 and itself. Prime numbers have been studied for a long time as they are building blocks of natural numbers. Indeed, every natural number n admits a *unique factorisation*

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

where p_i's are distinct primes and e_i's are natural numbers. Therefore, in order to understand properties of natural numbers, studying primes is crucial.

It was more than two thousand years ago that Euclid proved the infinitude of primes. His elegant argument is that assuming there were only finitely many primes $p_1, \cdots, p_k$ (say), the number

$$p_1 \cdots p_k + 1$$

is not divisible by any p_i. Thus, there would be a new prime diving $p_1 \cdots p_k + 1$, which leads to a contradiction. In 1737, Euler gave another proof via an *analytic* method.

He considered a special function
$$\sum_{n=1}^{\infty} n^{-x}$$
of a *real* variable $x > 1$. By the unique factorisation, he then wrote

$$\sum_{n=1}^{\infty} n^{-x} = \prod_{p} (1 - p^{-x})^{-1}$$

for $x > 1$, where the product runs over all the primes. Since the series $\sum_{n=1}^{\infty} n^{-1}$ diverges, by taking $x \to 1^+$, Euler proved the existence of infinitely many primes. Motivated by this result, Euler and Legendre asked if there are infinitely many primes in a given *arithmetic progression*. More precisely, for any given two coprime natural numbers a and q, are there infinitely many primes p such that

$$q \mid (p - a)?$$

The affirmative answer was given by Dirichlet in 1837 (for q prime) and in 1840 (for all q). For this purpose, he considered the periodic and completely multiplicative functions defined on integers, what now are called *Dirichlet characters*. Furthermore, he associated the L-function $L(x, \chi)$ to each Dirichlet character χ by defining

$$L(x, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-x} = \prod_{p} (1 - \chi(p) p^{-x})^{-1}$$

of a real variable $x > 1$, where the product is over all primes.

In a different theme, there was a conjecture due to Legendre and Gauss asserting that the prime-counting function

$$\pi(x) = \#\{p \leq x \mid p \text{ is a prime}\}$$

is asymptotic to $x/\log x$ as $x \to \infty$. To answer this, Riemann in 1859 suggested that one should study

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}$$

as a function of a *complex* variable s. Moreover, he showed that $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ which has only a simple pole at $s = 1$ and satisfies a functional equation. Also, Riemann derived an explicit formula for $\pi(x)$ in terms of zeros of his zeta function.

The next step was taken by Hadamard and de la Vallée-Poussin independently in 1896. They proved the conjecture of Legendre and Gauss, which is now the celebrated *prime number theorem*, by using Riemann's idea and showing that $\zeta(s)$ is non-vanishing on $\Re(s) \geq 1$. Shortly after, de la Vallée-Poussin gave a quantitative form of the above-mentioned theorem of Dirichlet. This is now called the *prime number theorem for arithmetic progressions*, which states that if natural numbers a and q have no common prime factors, then the proportion of the primes p congruent to a modulo q is equal to $\frac{1}{\phi(q)}$, where ϕ is the Euler totient function.

Indeed, the analytic properties of L-functions have been utilised to establish results of a purely arithmetic nature. For instance, to study primes, mathematicians were forced to investigate prime ideals in number fields. This led Dedekind to his zeta functions. Furthermore, in light of work of Riemann and many others, Landau derived the *prime ideal theorem*, asserting that in every given number field the number of prime ideals with norm at most x is asymptotic to $x/\log x$. Moreover, in 1922, Chebotarev obtained a vast generalisation of the earlier-mentioned result of Dirichlet. His remarkable result, nowadays called the *Chebotarev density theorem*, describes probabilistically how primes p distribute in a given Galois extension of the field of rational numbers $\mathbb{Q}$ with respect to their *Artin symbols* σ_p (cf. Sections 3.1.1 and 3.1.2). In particular, if the given Galois extension is cyclotomic, the Chebotarev density theorem gives the prime number theorem for arithmetic progressions. More precisely, considering the q-th cyclotomic field $\mathbb{Q}(\zeta_q)$, where ζ_q is a primitive q-th root of unity, one has $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \simeq (\mathbb{Z}/q\mathbb{Z})^\times$. Moreover, for each prime p coprime to q, the Artin symbol σ_p is defined and $\sigma_p(\zeta_q) = \zeta_q^p$. Thus, σ_p only depends on the arithmetic progression to which p belongs modulo q. Hence, by applying the Chebotarev density theorem in this context, for any a coprime to q, the number of primes $p \leq x$ which is congruent to a modulo q is asymptotic to

$$\frac{1}{\phi(q)}\frac{x}{\log x}$$

as $x \to \infty$.

Meanwhile, for every Galois extension K/k of number fields with Galois group G and every representation ρ of G into $GL_n(\mathbb{C})$, Artin defined the L-function attached to ρ and conjectured that his L-function can be extended to an *entire* function whenever ρ is non-trivial and irreducible. Via his celebrated *reciprocity law*, Artin showed that his conjecture is valid when $n = 1$. From this and the *induction-invariance property of Artin L-functions*, Artin established his conjecture when G is an *M-group*, namely, all irreducible characters of G are induced from 1-dimensional characters of subgroups of G. Furthermore, by his *induction theorem*, Brauer proved that all Artin L-functions extend *meromorphically* over $\mathbb{C}$ and indeed satisfy the functional equation predicted by Artin.

In light of Artin reciprocity, Langlands further conjectured that for each representation ρ of G, ρ should be associated to an *automorphic representation* π of $GL_{\dim \rho}(\mathbb{A}_k)$, where $\mathbb{A}_k$ denotes the *adèle ring* of k; and if ρ is irreducible, then π will be *cuspidal*. If such a π exists, then ρ is said to be *automorphic*. This is often called the *Langlands reciprocity conjecture* or the *strong Artin conjecture*. Indeed, Artin's conjecture follows from the Langlands reciprocity conjecture and the theory of automorphic L-functions.

By the works of Iwasawa [29] and Tate [64], one knows that the Langlands conjecture for $GL(1)$ is exactly Artin reciprocity. The next big step was taken by Langlands [37] and Tunnell [66] who proved the Langlands reciprocity conjecture for 2-dimensional ρ with solvable image. In much the same spirit, Ramakrishnan [54] showed that solvable 4-dimensional representations of $GO(4)$-type are all automorphic; and Martin [41, 42] derived the automorphy for ρ with projective image (in $PGL_4(\mathbb{C})$) isomorphic to $E_{2^4} \rtimes C_5$ or an extension of A_4 by V_4. Moreover, the automorphy of odd 2-dimensional icosahedral representations over $\mathbb{Q}$ follows from Khare-Wintenberger's proof of Serre's modularity conjecture (cf. [34]). In a slightly different vein, from their theory of base change and automorphic induction, Arthur and Clozel [2] derived Langlands reciprocity whenever G is *nilpotent*. Moreover, they showed that if G is solvable and ρ is *accessible*, i.e., χ is an *integral* sum of characters induced from linear characters of *subnormal* subgroups of G, then ρ must be automorphic. These results will be discussed and summarised in Section 3.3.

A conjecture of Dedekind asserts that the quotient $\zeta_K(s)/\zeta_k(s)$ is *entire* if K/k is *any* finite extension (not necessarily Galois). By the works of Aramata and Brauer, this conjecture is valid if K/k is a *Galois* extension. Moreover, if K is contained in a *solvable* normal closure of k, Uchida [67] and van der Waall [69] independently proved Dedekind's conjecture in this case. However, this conjecture is still open in general. We remark that Dedekind's conjecture follows from Artin's conjecture and that all these results, in fact, rely on the theories of Artin L-functions and induced representations. On the other hand, to study zeros of Dedekind zeta functions, Heilbronn [23] introduced what are now called the *Heilbronn characters*. His innovation allowed him to give a simple proof of the Aramata-Brauer theorem. This profound idea was also used by Stark [62] to prove that if K/k is Galois and $\zeta_K(s)$ has a simple zero at $s = s_0$, then such a zero must arise from a cyclic extension of k. Moreover, in the spirit of Heilbronn and Stark, Foote and V. K. Murty [19] showed that if K/k is a Galois extension with Galois group G, for fixed $s_0 \in \mathbb{C}$,

$$\sum_{\chi \in \mathrm{Irr}(G)} n(G,\chi)^2 \leq (\mathrm{ord}_{s=s_0}\, \zeta_K(s))^2,$$

where $n(G,\chi)$ denotes the order of the Artin L-function $L(s,\chi,K/k)$ at $s = s_0$. Furthermore, if G is solvable, this result has been improved by M. R. Murty and Raghuram in [49] and later Lansky and Wilson in [39]. In particular, the result of M. R. Murty and Raghuram generalises the works of Uchida and van der Waall.

Following the path illuminated by Heilbronn, Stark, and many others, in Section 5.1, we will introduce the notion of *weak arithmetic Heilbronn characters* that satisfy properties analogous to some properties of the classical Heilbronn characters known by the works of Heilbronn-Stark (Lemma 5.2), Aramata-Brauer (Corollary 5.4), Foote-V. K. Murty (Theorem 5.3), and M. R. Murty-V. K. Murty (Proposition 5.6). Later, in

Section 5.2, more conditions will be imposed on weak arithmetic Heilbronn characters which take them closer to Heilbronn characters. These will be outlined in Lemma 5.7 (Heilbronn-Stark lemma in full strength), Proposition 5.11 (known by the work of M. R. Murty-Raghuram), and Corollary 5.12 (the Uchida-van der Waall Theorem). We will go on to derive several extensions of results of M. R. Murty and Raghuram for arithmetic Heilbronn characters.

In Section 5.3, we will apply the results from Section 5.2 to study Artin-Hecke L-functions and L-functions of CM-elliptic curves. Furthermore, in Section 5.4, by applying the results from Section 5.2, we will study holomorphy of quotients of Rankin-Selberg L-functions arising from certain cuspidal automorphic representations that allows one to investigate holomorphy of quotients of L-functions associated to non-CM elliptic curves. Also, in Section 5.5, we will use properties of weak arithmetic Heilbronn characters along with the celebrated result of Taylor and his school on the *potential automorphy* of symmetric power L-functions of non-CM elliptic curves (cf. Section 3.3.5) to deduce generalisations of the results of Foote, M. R. Murty, and V. K. Murty. In particular, one such consequence, Theorem 5.35, is predicted by the Birch-Swinnerton-Dyer conjecture.

In 1988, under the assumption that the generalised Riemann hypothesis and the Artin conjecture hold, M. R. Murty, V. K. Murty, and Saradha [48] gave an *effective* version of the Chebotarev density theorem, refining the previous work of Lagarias and Odlyzko [40] as well as a result of Serre [58] (cf. Section 3.1.2). More recently, Diaconis and Isaacs in [16] constructed a theory of *supercharacters* and showed that their theory of supercharacters shares similar properties of the classical character theory. For instance, they proved that there is the first orthogonality property in general and derived *super Frobenius reciprocity* for certain matrix groups. We will give a brief review of the supercharacter theory in Section 2.3 and derive super Frobenius reciprocity for all groups. Inspired by the work of Brauer, Heilbronn, and others, we are also interested in studying Artin L-functions via the theory of supercharacters. Indeed, we derive a supercharacter-theoretic analogue of Heilbronn characters in Section 4.1 and give an effective version of the Chebotarev density theorem for Artin L-functions attached to supercharacters in Section 4.3. Furthermore, as will be discussed in Section 4.2, the Artin conjecture is true for Artin L-functions attached to suitable supercharacters. As a consequence, we obtain the same effective estimate given by M. R. Murty-V. K. Murty-Saradha *without* the assumption of Artin's conjecture for these cases.

From the above discussion, it is not hard to see that all the results suggest that the group-theoretic method shall be a key of optimising our understanding of the Artin conjecture. For instance, the Artin conjecture has been derived for certain solvable Frobenius extensions by Zhang [75] by invoking the theory of Frobenius groups. (We recall that a group G is said to be *Frobenius* if it has a non-trivial proper subgroup H such that $g^{-1}Hg \cap H = 1$ for all $g \in G \backslash H$.) Thus, for our purpose of studying Artin's conjecture, the major part of Chapters 2 and 6 are devoted to collecting pure group-theoretic results and developing a method of *low-dimensional groups*, respectively. (We call a group low-dimensional if *all* its irreducible characters are of "small" degree.) Such group-theoretic machinery will allow one to obtain information of representations of groups via their low-dimensional (normal) subgroups.

We will say a finite group G is *nearly supersolvable* if it has an invariant series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G,$$

where each subgroup is normal in G, the quotient N_{i+1}/N_i is cyclic for every $i \geq 1$, and N_1 belongs to the class $\mathcal{C}$ consisting of groups all of whose irreducible representations are of dimension less than or equal to 2. We note that the symmetric group on four letters, S_4, is not supersolvable. However, as it admits a normal subgroup isomorphic to the Klein four-group V_4, S_4 is nearly supersolvable. This, in fact, motivates our notation of nearly supersolvable groups to study Artin's conjecture.

As an application, we derive the following theorem in Sections 6.1 and 6.5 (cf. Theorems 6.5 and 6.29).

Theorem 1.1. *Suppose that K/k is a nearly supersolvable Galois extension or a solvable Frobenius extension. Then the Artin conjecture holds for K/k.*

In light of their work on the automorphy of accessible characters of solvable groups, Arthur and Clozel asked if one can classify the accessible groups (cf. [2, pp. 220-221]). From this question and the accessibility of *subnormally monomial groups*, i.e., the groups all of whose irreducible characters are induced from 1-dimensional characters of subnormal subgroups, one may further want a classification of the subnormally monomial groups. We remark that this desire arises naturally as the symmetric group on three letters, S_3, is not nilpotent, which prevents one from applying Arthur-Clozel's automorphy result on nilpotent extensions. Nevertheless, it can be shown that S_3 is subnormally monomial and hence of automorphic type. As one can see, a general criterion for subnormal monomiality is now crucial for studying the Langlands reciprocity conjecture.

Inspired by the above observation and discussion, we introduce the notation of *nearly nilpotent groups* as follows. A group G is said to be nearly nilpotent if it admits a normal subgroup N in the class $\mathcal{C}$ described above such that G/N is nilpotent.

Furthermore, we will prove the following result that presents an enlargement of Galois extensions of number fields satisfying Langlands reciprocity (cf. Theorem 6.12 and Section 6.6).

Theorem 1.2. *Let K/k be a Galois extension of number fields with Galois group G. If G is a direct product of two nearly nilpotent groups, then Langlands reciprocity is true for K/k. Moreover, Langlands reciprocity holds for all non-A_5 Galois extensions of number fields of degree at most 100.*

Our result covers all *metabelian* Galois extensions as well as Arthur-Clozel's theorem on the automorphy of nilpotent Galois extensions. Also, as all groups of square-free order are *meta-cyclic*, it follows that Langlands reciprocity is valid for all Galois extensions of square-free degree. One may regard this theorem as the "metabelian class field theory" or the "square-free reciprocity" as predicted by the Langlands program.

Chapter 2

Group-Theoretic Preliminaries

2.1 A Little Finite Group Theory

In this section, we will recall some notations and results from the theory of groups. Firstly, G always denotes a finite group, and H and N denote a subgroup and a normal subgroup of G, respectively. A semi-direct product of N by H will be denoted as $N \rtimes H$. Also, G' and $\mathbf{Z}(G)$ will stand for the derived subgroup and the centre of G, respectively. Moreover, we let $\mathbf{F}(G)$ denote the maximal normal nilpotent subgroup of G, i.e., the *Fitting subgroup* of G, and let $\Phi(G)$ stand for the *Frattini subgroup* of G, i.e., the intersection of all maximal subgroups of G. For any finite set π of primes, G_π denotes a *Hall* π*-subgroup* of G. The cyclic group of order m, the Klein four-group, and the quaternion group of order 8 will be denoted as C_m, V_4, and Q, respectively. The direct product of n-copies of C_m will be denoted by C_m^n.

Definition 2.1. *A finite group G is said to be nilpotent if one of the followings holds.*

N1. *G admits a central series, i.e., there is an invariant series of subgroups*

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G,$$

where for each i, H_i is normal in G and $H_{i+1}/H_i \leq \mathbf{Z}(G/H_i)$;

N2. *G is a direct product of its Sylow subgroups; or*

N3. *Every subgroup H of G is subnormal, i.e., there is an invariant series of subgroups*

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G,$$

where each H_i is normal in H_{i+1}.

Definition 2.2. *A finite group G is called supersolvable (resp., solvable) if there exists an invariant series of subgroups*

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G,$$

where each H_i is normal in G (resp., in H_{i+1}) and each H_{i+1}/H_i is cyclic.

Definition 2.3. *A finite group G is said to be meta-cyclic (resp., metabelian), if G has a normal subgroup N such that both N and G/N are cyclic (resp., abelian).*

Let S_n denote the symmetric group on n letters. It is well-known that for $n \leq 4$, S_n is solvable. In particular, S_2 is isomorphic to the cyclic group of order 2. Also, S_3 is not nilpotent but supersolvable. Indeed, S_3 admits a normal cyclic subgroup of order 3 and hence is a meta-cyclic group. However, S_4 is not nilpotent nor supersolvable.

A classical result of Hölder asserts that a (finite) group of square-free order must be meta-cyclic. Moreover, in 2005, Dietrich and Eick [17] studied the class of groups of cube-free order and, in particular, characterised non-solvable groups of cube-free order. Their work has been extended by Qiao and Li [52] who gave a description of the class of solvable groups of cube-free order as follows.

Proposition 2.1. *Let G be a solvable group of cube-free order. Then one of the followings holds.*

1. $G = (C_a \times C_b^2) \rtimes (C_c \times C_d^2)$, *or* $(C_2^2 \times C_a \times C_b^2) \rtimes (C_c \times C_d^2)$*; or*

2. $G = (C_a \times C_b^2) \rtimes (C_c \times C_d^2) \rtimes G_{\{2\}}$,

where a, b, c, and d are suitable odd integers such that $(a, b) = (c, d) = 1$, ac is cube-free, bd is square-free, prime divisors of ab are not less than prime divisors of cd, and C_m denotes a cyclic group of order m.

We remark that Qiao and Li showed that the first case happens if a Hall $\{2, 3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$ of G is non-abelian (cf. [52, Lemma 3.8]).

As mentioned earlier, G is a *Frobenius group* if there is a non-trivial proper subgroup H of G such that $g^{-1}Hg \cap H = 1$ whenever $g \in G \backslash H$. In this case, H is called a *Frobenius complement* of G.

Definition 2.4. *Let G be a finite group. A proper subgroup N is called a CC-subgroup if for every non-trivial element n of N, its centraliser $C_G(n)$ is contained in N, where $C_G(n)$ is defined as*

$$C_G(n) = \{g \in G \mid gn = ng\}.$$

We note that if G admits a non-trivial proper normal CC-subgroup, then G is a Frobenius group. Conversely, a theorem of Frobenius tells us that if G is a Frobenius group with a Frobenius complement H, there is a normal CC-subgroup N of G such that $G = N \rtimes H$, where N is called the *Frobenius kernel* of G. On the other hand, all Sylow subgroups of a Frobenius complement are cyclic or generalised quaternion, and a deep theorem of Thompson asserts that every Frobenius kernel is nilpotent. For more details, we refer the reader to [26, Chapter 7] and [27, Chapter 6].

2.2 Finite-Dimensional Representations and their Characters

Let G be a finite group. A *representation* ρ of G on a finite dimensional vector space V over $\mathbb{C}$ is a group homomorphism from G to $GL(V)$, the general linear group on V. Given a representation ρ of G, the *character* of ρ is a function on G defined by $\chi(g) = \operatorname{tr}\rho(g)$. A linear subspace $W \subset V$ is called G-*invariant* if $\rho(g)w \in W$ for all $g \in G$ and all $w \in W$. In this case, ρ can be seen as a representation of G on W, and we denote such a representation by $\rho|^W$. A representation ρ is said to be *irreducible* if there is no proper and non-trivial G-invariant subspaces W of V. In addition, the character χ of a representation ρ is called irreducible if ρ is irreducible. If $f_1, f_2 \colon G \to \mathbb{C}$ are two functions on G, one can define their inner product by

$$(f_1, f_2)_G = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

If $f \colon G \to \mathbb{C}$ is constant on each conjugacy class in G, then f is called a *class function* on G. We will let $\mathbf{C}(G)$ denote the space of class functions of G. It can be shown that the set $\operatorname{Irr}(G)$ of all irreducible characters of G forms an orthonormal basis for the inner product space of all class functions on G with respect to the inner product defined above.

Let H be a subgroup of G and f a class function on H. The *induction* of f from H to G is defined by

$$\operatorname{Ind}_H^G f(x) = \frac{1}{|H|} \sum_{g \in G} \tilde{f}(g^{-1}xg),$$

where $\tilde{f}$ extends f by setting $\tilde{f}(g) = 0$ for all $g \in G \backslash H$. By using the definition of induction, one can deduce that $\operatorname{Ind}_H^G f$ is a class function on G if f is a class function on H, and one can also show the following reciprocity theorem.

Proposition 2.2 (Frobenius Reciprocity). *For all class functions ϕ on H, a subgroup of G, and all class functions θ of G,*

$$(\operatorname{Ind}_H^G \phi, \theta)_G = (\phi, \theta|_H)_H,$$

where $\theta|_H$ is the restriction of θ from G to H.

We recall that a character is said to be *monomial* if it is induced from a linear character (i.e., a character of degree 1) and that a *monomial group* (or an *M-group* for short) is a group that all of whose irreducible characters are monomial. As will be discussed in the next chapter, monomial characters play a crucial role in studying Artin's conjecture and Galois representations. In light of this, we shall further recall some concepts of *relative M-groups* and *relative SM-groups* (cf. [26, Chapter 6] and [25], respectively).

Definition 2.5. *Let G be a finite group and let N be a normal subgroup of G. A character χ of G is called a relative M-character (resp., a relative SM-character) with respect to N if there exists a subgroup (resp., a subnormal subgroup) H with $N \leq H \leq G$ and an irreducible character $\psi \in \operatorname{Irr}(H)$ such that $\operatorname{Ind}_H^G \psi = \chi$ and $\psi|_N \in \operatorname{Irr}(N)$. If every irreducible character of G is a relative M-character (resp., a relative SM-character) with respect to N, then G is said to be a relative M-group (resp., a relative SM-group) with respect to N.*

We note that if N is normal in G and G/N is nilpotent or supersolvable, then G is a relative M-group with respect to N. In general, one has the following result due to Price (cf. [5, Theorem 7.63] and [26, Theorem 6.22]).

Theorem 2.3. *Let G be a finite group with a normal subgroup N such that G/N is solvable. If every chief factor of every non-trivial subgroup of G/N has order equal to an odd power of some prime, then G is a relative M-group with respect to N.*

From this, one has a result of Huppert (cf. [26, Theorem 6.23]) as stated below.

Proposition 2.4. *Let G be a finite group and let N be a normal subgroup of G such that G/N is supersolvable. If N is solvable and all Sylow subgroups of N are abelian, then G is an M-group.*

Moreover, based on Theorem 2.3, Horváth [25, Proposition 2.7] gave a sufficient condition for groups being relative SM-groups as follows.

Theorem 2.5. *Let G be a finite group and let N be a normal subgroup of G with G/N nilpotent. Then G is a relative SM-group with respect to N.*

We note that Horváth omitted the proof and remarked that it is similar to the proof of Theorem 2.3. However, for the sake of completeness, we give a proof below.

Proof. By Theorem 2.3, we already know that each $\chi \in \operatorname{Irr}(G)$ is a relative M-character with respect to N, i.e., there exists a subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in \operatorname{Irr}(H)$ such that $\operatorname{Ind}_H^G \psi = \chi$ and $\psi|_N \in \operatorname{Irr}(N)$. Now as G/N is nilpotent, all its subgroups are subnormal. In particular, we have

$$H/N = \overline{H_0} \trianglelefteq \overline{H_1} \trianglelefteq \cdots \trianglelefteq \overline{H_{m-1}} \trianglelefteq \overline{H_m} = G/N,$$

where for each i, $\overline{H_i}$ is a normal subgroup of $\overline{H_{i+1}}$. Now by lifting this series (with respect to N), we can see that H is subnormal in G. In other words, each χ is a relative SM-character with respect to N, and hence, G is a relative SM-group with respect to N. □

Recall that a *Dedekind group* is a group G such that every subgroup of G is normal. By an analogous argument as above, we have the following variant that may be of interest.

Proposition 2.6. *Let G be a finite group and let N be a normal subgroup of G such that G/N is a Dedekind group. Then for every $\chi \in \mathrm{Irr}(G)$, there exists a normal subgroup H of G with $N \leq H$ and $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in \mathrm{Irr}(N)$.*

As a consequence, any irreducible character of a metabelian group G is induced from a 1-dimensional character of a normal subgroup of G.

For Frobenius groups, one also has the following theorem that characterises their (induced) irreducible characters (cf. [26, Theorem 6.34]).

Proposition 2.7. *Let G be a Frobenius group with Frobenius kernel N. For any $\chi \in \mathrm{Irr}(G)$ with $N \nsubseteq \mathrm{Ker}\,\chi$, one has $\chi = \mathrm{Ind}_N^G \psi$ for some $\psi \in \mathrm{Irr}(N)$.*

We recall that a group is called *p-elementary* if it is a direct product of a p-group and a cyclic group, and that a group is said to be *elementary* if it is p-elementary for some prime p. Let us state Brauer's theorem on induced characters.

Theorem 2.8 (Brauer Induction Theorem)**.** *Let G be a finite group and χ a character of G. Then there exist integers n_i such that*

$$\chi = \sum_i n_i \,\mathrm{Ind}_{H_i}^G \psi_i,$$

where H_i's are elementary subgroups of G and ψ_i is a linear character of H_i.

To end this section, we collect more results from the representation theory of finite groups.

Lemma 2.9. *Let G be a finite group and $\mathbf{Z}(G)$ its centre. Then for every irreducible character χ of G, one has*

$$\chi(1)^2 \leq [G : \mathbf{Z}(G)].$$

We let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Via the above lemma, Theorem 2.3, Sylow's theory (or the computer algebra package [20]), one has the following.

Lemma 2.10. *If G is of order 1, 2, 4, 3, or 9, then $\mathrm{cd}(G) = \{1\}$. If G is of order 8, 16, 6, or 18, then $\mathrm{cd}(G) \subseteq \{1, 2\}$. If $|G|$ is 12, 24, or 36, then $\mathrm{cd}(G) \subseteq \{1, 2, 3, 4\}$ where $4 \in \mathrm{cd}(G)$ only if $|G'| = 9$.*

We also invoke below a result of Isaacs (cf. [26, Theorems 12.5, 12.6 and 12.15]).

Theorem 2.11. *If G is a finite group with $|\,\mathrm{cd}(G)| \leq 3$, then G must be solvable.*

Let ρ be an irreducible representation of G. As the finite subgroups of $PGL_3(\mathbb{C})$ have been classified by Blichfeldt [6, 44], one has the following.

Lemma 2.12. *If ρ is primitive, 3-dimensional, and with solvable projective image $\overline{G}$ in $PGL_3(\mathbb{C})$, then $\overline{G}$ is of order 36, 72, or 216.*

We let $GO_n(\mathbb{C})$ denote the subgroup of $GL_n(\mathbb{C})$ consisting of orthogonal similitudes, i.e., matrices M such that $M^tM = \lambda_M I$, with $\lambda_M \in \mathbb{C}$. Also, we define the (m-th) symplectic similitude group as

$$GSp_{2m}(\mathbb{C}) = \{M \in GL_{2m}(\mathbb{C}) \mid M^tJM = \lambda_M J, \lambda_M \in \mathbb{C}\},$$

where J is the matrix defined as

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

and I_m is the identity $m \times m$ matrix. We note that irreducible primitive finite subgroups of $SL_4(\mathbb{C})$ were classified by Blichfeldt [6] in 1917. However, as mentioned by Martin [43, Chapter 5], Blichfeldt's list is presented in terms of generating matrices and geometrical invariants, which is not the most convenient form for studying Artin's conjecture. Hence, we will use a classification due to Martin as stated below.

Lemma 2.13. *Suppose that ρ is primitive, 4-dimensional, and with solvable projective image $\overline{G}$ in $PGL_4(\mathbb{C})$, and that the image of ρ is contained in $GSp_4(\mathbb{C})$. Then $\overline{G}$ is isomorphic to $E_{2^4} \rtimes C_5$, $E_{2^4} \rtimes D_{10}$, or $E_{2^4} \rtimes F_{20}$, where E_{2^4} is the elementary abelian group of order 2^4, D_{10} denotes the dihedral group of order 10, and F_{20} is the Frobenius group of order 20.*

2.3 Review of Supercharacter Theory

Recently, Diaconis and Isaacs [16] introduced the theory of *supercharacters* which generalises the classical character theory in a natural way as follows.

Definition 2.6. *Let G be a finite group, let $\mathcal{K}$ be a partition of G, and let $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$. The ordered pair $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory if*

SC1. $\{1\} \in \mathcal{K}$,

SC2. $|\mathcal{X}| = |\mathcal{K}|$, *and*

SC3. *For each $X \in \mathcal{X}$, the character $\sigma_X = \sum_{\sigma \in X} \sigma(1)\sigma$ is constant on each $K \in \mathcal{K}$.*

The characters σ_X *are called supercharacters, and the elements* K *in* $\mathcal{K}$ *are called superclasses. In addition, if* $f\colon G \to \mathbb{C}$ *is constant on each superclass in* G*, then we say* f *is a superclass function on* G*.*

It is clear that the irreducible characters and conjugacy classes of G give a supercharacter theory of G, which will be referred to as the *classical theory* of G. Throughout this section, we will often equip groups with (possible) supercharacter theories without mentioning this.

We remark that Diaconis and Isaacs showed that every superclass is a union of conjugacy classes in G. By the orthogonality property of $\mathrm{Irr}(G)$, the set of all supercharacters, denoted $\mathrm{Sup}(G)$, forms an orthogonal basis for the inner product space of all superclass functions on G with respect to the usual inner product. Moreover, they also defined *superinduction* and then obtained *super Frobenius reciprocity* for certain matrix groups equipped with special supercharacter theories. In general, one can develop such a theory of superinduction as the following.

Definition 2.7 (Compatibility and Superinduction)**.** *Let* G *be a finite group and* H *a subgroup of* G*. If for any* $h \in H$*,* $SCl_H(h)$ *is contained in* $SCl_G(h)$*, where* $SCl_H(h)$ *and* $SCl_G(h)$ *are the superclasses containing* h *in* H *and* G*, respectively, then such supercharacter theories of* H *and* G *are said to be compatible, or* G *and* H *are compatible (with respect to the given supercharacter theories) for short. Moreover, if* G *and* H *are compatible, the superinduction* $\mathrm{SInd}_H^G \phi$ *of* ϕ*, a superclass function of* H*, is defined by*

$$\mathrm{SInd}_H^G \phi(g) = \frac{|G|}{|H|}\frac{1}{|SCl_G(g)|}\sum_{i=1}^{m(g)} |SCl_H(x_{i,g})|\phi(x_{i,g}),$$

where $SCl_G(g)$ is the superclass in G containing g and $\{x_{i,g}\}$ is a set of superclass representatives in H belonging to $SCl_G(g)$.

Since $SCl_G(g') = SCl_G(g)$ for any $g' \in SCl_G(g)$, one can choose $x_{i,g'} = x_{i,g}$ for all i's. Thus, $\mathrm{SInd}_H^G \phi$ is a superclass function of G for any superclass function ϕ of H. On the other hand, Diaconis and Isaacs, in fact, gave an example in which the induction $\mathrm{Ind}_H^G \sigma$ of a supercharacter σ of H is *not* a superclass function of G. Thus, the definition superinduction associated to compatible supercharacter theories of G and H is crucial. We also have below a theorem that generalises Frobenius reciprocity.

Proposition 2.14 (Super Frobenius Reciprocity)**.** *Suppose that G and H are compatible. For all superclass functions ϕ on H and all superclass functions θ of G,*

$$(\mathrm{SInd}_H^G \phi, \theta)_G = (\phi, \theta|_H)_H,$$

where $\theta|_H$ is the restriction of θ from G to H.

Proof. For any $g \in G$,

$$\begin{aligned}\mathrm{SInd}_H^G \phi(g)\overline{\theta(g)} &= \frac{|G|}{|H|}\frac{1}{|SCl_G(g)|}\sum_{i=1}^{m(g)} |SCl_H(x_{i,g})|\phi(x_{i,g})\overline{\theta(g)} \\ &= \frac{|G|}{|H|}\frac{1}{|SCl_G(g)|}\sum_{i=1}^{m(g)} |SCl_H(x_{i,g})|\phi(x_{i,g})\overline{\theta(x_{i,g})},\end{aligned}$$

where $x_{i,g}$'s are superclass representatives in H belonging to $SCl_G(g)$, and the last equality holds provided that $x_{i,g} \in SCl_G(g)$ and θ is a superclass function of G, i.e. θ is constant on each $SCl_G(g)$. Let $g_1, \cdots, g_k$ be distinct superclass representatives of G. Since $\mathrm{SInd}_H^G \phi$ and θ both are superclass functions of G, one has

$$\begin{aligned}(\mathrm{SInd}_H^G \phi, \theta)_G &= \frac{1}{|G|}\sum_{g\in G} \mathrm{SInd}_H^G \phi(g)\overline{\theta(g)} \\ &= \frac{1}{|G|}\sum_{j=1}^{k} |SCl_G(g_j)|\, \mathrm{SInd}_H^G \phi(g_j)\overline{\theta(g_j)} \\ &= \frac{1}{|G|}\sum_{j=1}^{k} |SCl_G(g_j)| \frac{|G|}{|H|}\frac{1}{|SCl_G(g_j)|}\sum_{i=1}^{m(g_j)} |SCl_H(x_{i,g_j})|\phi(x_{i,g_j})\overline{\theta(x_{i,g_j})} \\ &= \frac{1}{|H|}\sum_{j=1}^{k}\sum_{i=1}^{m(g_j)} |SCl_H(x_{i,g_j})|\phi(x_{i,g_j})\overline{\theta(x_{i,g_j})}.\end{aligned}$$

Observe that if $j \neq l$, then for any i and i',

$$x_{i,g_j} \in SCl_H(g_j) \subseteq SCl_G(g_j), \;\; x_{i',g_l} \in SCl_H(g_l) \subseteq SCl_G(g_l),$$

and the intersection of $SCl_G(g_j)$ and $SCl_G(g_l)$ is empty. Thus, $x_{i,g_j} \neq x_{i',g_l}$ if $j \neq l$. From this, one can conclude that x_{i,g_j}'s are all distinct. On the other hand, each superclass K of H is contained in exactly one superclass of G, and so there are i and j such that $K = SCl_H(x_{i,g_j})$. Therefore, $\{x_{i,g_j}\}$ forms a set of (distinct) representatives for all superclasses in H. Therefore,

$$\begin{aligned}(\mathrm{SInd}_H^G \phi, \theta)_G &= \frac{1}{|H|}\sum_{j=1}^{k}\sum_{i=1}^{m(g_j)} |SCl_H(x_{i,g_j})|\phi(x_{i,g_j})\overline{\theta(x_{i,g_j})} \\ &= \frac{1}{|H|}\sum_{h\in H} \phi(h)\overline{\theta|_H(h)} \\ &= (\phi, \theta|_H)_H,\end{aligned}$$

where the second equality holds provided that $\theta(h) = \theta|_H(h)$ for any $h \in H$ and θ is constant on each $SCl_H(x_{i,g_j})$. $\square$

We note that the above results concerning superinduction and super Frobenius reciprocity were also considered by Hendrickson [24], who equips H with the classical theory.

Now, we shall show that superinduction is *unique*. Suppose that there is another arbitrary map $\phi \mapsto \phi^{(G)}$ sending superclass functions of H to superclass functions of G and satisfying super Frobenius reciprocity, i.e., for any superclass function ϕ on H and any superclass function θ of G,

$$(\phi^{(G)}, \theta)_G = (\phi, \theta|_H)_H.$$

Applying the above theorem of super Frobenius reciprocity for SInd_H^G, for any superclass function ϕ on H and any superclass function θ of G, one has

$$(\phi^{(G)}, \theta)_G = (\mathrm{SInd}_H^G \phi, \theta)_G,$$

which implies that

$$\phi^{(G)} = \mathrm{SInd}_H^G \phi$$

for all superclass functions ϕ on H. In other words, there is a unique superinduction which satisfies super Frobenius reciprocity.

Chapter 3

Galois and Automorphic Representations

3.1 Galois Representations and L-Functions

3.1.1 Artin L-Functions and Artin's Conjecture

In this section, we will set up the machinery and motivation for defining Artin L-functions and list their basic but important properties.

An *algebraic number field* k (or simply a *number field*) is a finite extension of the field of rational numbers, $\mathbb{Q}$. The *ring of integers* of k, denoted by $\mathcal{O}_k$, consists of all elements $x \in k$ such that x is a root of a non-zero monic polynomial with integral coefficients. This ring $\mathcal{O}_k$ is a *Dedekind domain*, i.e., an integral domain where every non-zero proper ideal can be factorised uniquely into a product of prime ideals (up to the order of the factors). The (absolute) *norm* of a non-zero ideal $\mathfrak{a}$ in $\mathcal{O}_k$ is defined by $N\mathfrak{a} = [\mathcal{O}_k : \mathfrak{a}] = |\mathcal{O}_k/\mathfrak{a}|$, i.e., the cardinality of the quotient ring $\mathcal{O}_k/\mathfrak{a}$. The *Dedekind zeta function* of k is defined by

$$\zeta_k(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s},$$

where the sum runs over all non-zero integral ideals of $\mathcal{O}_k$, and this series converges for $\Re(s) > 1$. Since $\mathcal{O}_k$ is a Dedekind domain and the norm is completely multiplicative, i.e., $N(\mathfrak{a}\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b})$ for all non-zero integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{O}_k$, one can deduce the *Euler product* for the Dedekind zeta function of k, namely,

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1},$$

where the product runs over all prime ideals in $\mathcal{O}_k$. Note that, if $k = \mathbb{Q}$, the Dedekind zeta function $\zeta_{\mathbb{Q}}(s)$ is exactly the Riemann zeta function, denoted by $\zeta(s)$. Like the Riemann zeta function, every Dedekind zeta function extends to a meromorphic function on $\mathbb{C}$ which has only a simple pole at $s = 1$. Moreover, every Dedekind zeta function is non-vanishing on $\Re(s) = 1$ and admits a functional equation relating values at s with values at $1 - s$. The famous *generalised Riemann hypothesis*, denoted GRH, asserts that every Dedekind zeta function is non-vanishing for s with $0 < \Re(s) < 1$ and $\Re(s) \neq \frac{1}{2}$.

By considering quadratic fields of the form $\mathbb{Q}(\sqrt{d})$ for some square-free integer d, one can write the Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{d})}(s)$ as a product of the Riemann zeta function and a Dirichlet L-function

$$\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s) L(s, \chi)$$

for some (non-trivial) Dirichlet character χ depending on d. (We recall that a *Dirichlet character* modulo m is a homomorphism

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$$

extended to $\mathbb{Z}$ by putting $\chi(n) = 0$ if $(n, m) \neq 1$. The *Dirichlet L-function* attached to χ is defined as

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

where the product runs over all primes.) Since this Dirichlet L-function can be extended to an entire function, one can deduce that the quotient $\zeta_{\mathbb{Q}(\sqrt{d})}(s)/\zeta(s)$ is entire. In general, replacing $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ by an *arbitrary* extension of number fields M/k, one may wonder whether the quotient $\zeta_M(s)/\zeta_k(s)$ of the Dedekind zeta functions is also entire. In fact, Dedekind conjectured that this quotient, $\zeta_M(s)/\zeta_k(s)$, should be entire, and proved in 1873 his conjecture for pure cubic extensions $M/\mathbb{Q}$, i.e., $M = \mathbb{Q}(\sqrt[3]{m})$ for some cube-free integer m. For the case of *Galois* extensions, Dedekind's conjecture was proved by Aramata and Brauer independently as the following.

Theorem 3.1 (Aramata-Brauer)**.** *Let M/k be a Galois extension of number fields. Then $\zeta_M(s)/\zeta_k(s)$ is entire.*

In the direction of Dedekind's conjecture for *non-normal* extensions, Uchida and van der Waall (independently) proved the following theorem which partially generalises the above theorem of Aramata and Brauer.

Theorem 3.2 (Uchida-van der Waall)**.** *Let M/k be an extension of number fields, and $\widetilde{M}$ a normal closure of M/k. If $\mathrm{Gal}(\widetilde{M}/k)$ is solvable, then $\zeta_M(s)/\zeta_k(s)$ is entire.*

To study Dedekind's conjecture, one needs to know how to *factorise* the Dedekind zeta functions. We shall start by recalling the theory about how primes *split* in a given Galois extension of number fields.

Let K/k be a Galois extension of number fields with Galois group G. Since $\mathcal{O}_K$

is also a Dedekind domain, for any prime ideal $\mathfrak{p}$ in $\mathcal{O}_k$, one has

$$\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_l^{e_l},$$

where $\mathfrak{P}_j$'s are distinct prime ideals in $\mathcal{O}_K$ and e_j's are positive integers. In this case, we say that the prime ideals $\mathfrak{P}_j$'s are *above* $\mathfrak{p}$ and denote this as $\mathfrak{P}_j|\mathfrak{p}$. In addition, if $e_j = 1$ for every j, then $\mathfrak{p}$ is called *unramified*. Otherwise, $\mathfrak{p}$ is *ramified*. Moreover, using the maximality of (non-zero) prime ideals in a Dedekind domain, it can be shown that for any prime ideals $\mathfrak{P}$ and $\mathfrak{P}'$ above $\mathfrak{p}$, there is a $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}'$. Therefore, one can conclude that G acts on $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_l\}$ *transitively*, i.e., there is exactly one G-orbit in $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_l\}$. Together with this transitivity, the unique factorisation implies that all e_j's are the same. Thus, one has

$$\mathfrak{p}\mathcal{O}_K = (\mathfrak{P}_1 \cdots \mathfrak{P}_l)^e,$$

where $e = e_j$ for any j. This e is called the *ramification index* of $\mathfrak{p}$.

On the other hand, since the Galois group G acts on the prime factors of $\mathfrak{p}\mathcal{O}_K$, it is natural to consider the stabiliser subgroup of each prime factor $\mathfrak{P}$ above $\mathfrak{p}$. Let $D_{\mathfrak{P}}$ denote the stabiliser subgroup of $\mathfrak{P}|\mathfrak{p}$, i.e.,

$$D_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\},$$

which is called the *decomposition group* at $\mathfrak{P}$. One also has the *inertia group* at $\mathfrak{P}$

$$I_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(x) \equiv x \pmod{\mathfrak{P}} \text{ for all } x \in \mathcal{O}_K\},$$

which is a normal subgroup of $D_{\mathfrak{P}}$. Since every (non-zero) prime ideal is maximal in a Dedekind domain, both $\mathcal{O}_K/\mathfrak{P}$ and $\mathcal{O}_k/\mathfrak{p}$ are fields; and, in fact, these fields are also *finite*. Moreover, it can be shown that the extension $(\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p})$ is *Galois* and $\mathrm{Gal}((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p}))$ is a cyclic group with a generator $x \mapsto x^{N\mathfrak{p}}$. Indeed, there is a canonical isomorphism

$$D_{\mathfrak{P}}/I_{\mathfrak{P}} \simeq \mathrm{Gal}((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p})).$$

Therefore, one can choose an element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ whose image in $\mathrm{Gal}((\mathcal{O}_K/\mathfrak{P})/(\mathcal{O}_k/\mathfrak{p}))$ is the generator described above. Such an element $\sigma_{\mathfrak{P}}$ is called a *Frobenius automorphism* at $\mathfrak{P}$ and it is only well-defined modulo $I_{\mathfrak{P}}$.

It can be shown that for any unramified $\mathfrak{p}$, $I_{\mathfrak{P}}$ is a trivial group for every $\mathfrak{P}|\mathfrak{p}$. In fact, combining the orbit-stabiliser theorem with the fact that there is exactly one G-orbit in $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_l\}$, and the canonical isomorphism stated above, one can show that the order of $I_{\mathfrak{P}}$ is equal to e, the ramification index of $\mathfrak{p}$, for any $\mathfrak{P}|\mathfrak{p}$. Thus, if $\mathfrak{p}$ is unramified, then $e = 1$ and so $I_{\mathfrak{P}}$ is a trivial group. Besides, since there are only *finitely many* ramified prime ideals in $\mathcal{O}_k$, one can deduce that all but finitely many $I_{\mathfrak{P}}$ are trivial for $\mathfrak{P}|\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_k$. For $\mathfrak{p}$ unramified, one can show that as $\mathfrak{P}$ ranges over the prime ideals above $\mathfrak{p}$, the $\sigma_{\mathfrak{P}}$ form a conjugacy class. This class is called the *Artin symbol* at $\mathfrak{p}$, denoted $\sigma_{\mathfrak{p}}$.

Using the above theory and representations of finite groups, Artin introduced his L-functions that generalise Dirichlet L-functions as follows. Let ρ be a complex finite-dimensional representation of $G = \mathrm{Gal}(K/k)$. The *Artin L-function* attached to ρ is defined by

$$L(s,\rho,K/k)=\prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\rho,K/k),$$

where the local L-function at $\mathfrak{p}$ is defined as

$$L_{\mathfrak{p}}(s,\rho,K/k)=\det(1-\rho|^{V^{I_{\mathfrak{P}}}}(\sigma_{\mathfrak{P}})N\mathfrak{p}^{-s})^{-1}$$

for $\Re(s)>1$. Here, the product runs over prime ideals in $\mathcal{O}_k$, $\mathfrak{P}$ denotes a prime ideal above $\mathfrak{p}$, and $V^{I_{\mathfrak{P}}}=\{v\in V \mid \rho(g)v=v \text{ for all } g\in I_{\mathfrak{P}}\}$. Sometimes we write $L(s,\chi,K/k)$ for $L(s,\rho,K/k)$, where $\chi=\operatorname{tr}\rho$ denotes the character of ρ. One can easily show that

$$L(s,1_G,K/k)=\zeta_k(s),$$

where 1_G denotes the trivial character of G, and that

$$L(s,\chi_1+\chi_2,K/k)=L(s,\chi_1,K/k)L(s,\chi_2,K/k)$$

for any characters χ_1 and χ_2 of G. Also, for any tower of Galois extensions $K/F/k$, any character ψ of $\mathrm{Gal}(F/k)$ defines a character $\mathrm{Inf}_{\mathrm{Gal}(F/k)}^{\mathrm{Gal}(K/k)}\psi$, called the *inflation* of ψ, of $\mathrm{Gal}(K/k)$ canonically through the quotient map $\mathrm{Gal}(K/k)\to\mathrm{Gal}(F/k)$, and

$$L(s,\mathrm{Inf}_{\mathrm{Gal}(F/k)}^{\mathrm{Gal}(K/k)}\psi,K/k)=L(s,\psi,F/k).$$

Moreover, for any character χ of $H\le G$, one has

$$L(s,\mathrm{Ind}_H^G\chi,K/k)=L(s,\chi,K/K^H),$$

where K^H is the fixed field of H. This property is called the *induction-invariance property of Artin L-functions.* Using these properties, one can deduce the following theorem of Artin and Takagi, generalising the decomposition of the Dedekind zeta functions of quadratic extensions of $\mathbb{Q}$ that we described previously.

Proposition 3.3 (Artin-Takagi Decomposition)**.**

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \neq 1_G} L(s, \chi, K/k)^{\chi(1)},$$

where the product runs over all non-trivial irreducible characters of G. In particular, one has

$$L(s, \mathrm{Reg}_G, K/k) = \zeta_K(s),$$

where Reg_G *is the regular representation of G.*

Artin conjectured that if ρ is irreducible and non-trivial, then $L(s, \rho, K/k)$ extends to an *entire* function and satisfies a functional equation. In fact, Artin showed that his conjecture is true if G is an *M-group.* In general, Artin's conjecture is still open and is viewed as a central problem in number theory.

It is worth noting that Artin seemed to be led to his L-functions and conjecture while trying to prove Dedekind's conjecture. Indeed, Dedekind's conjecture follows from Artin's conjecture. More precisely, for any intermediate field M of K/k, according to the fundamental theorem of Galois theory, there is a subgroup H of G such that M is the fixed field of H. Now, by Frobenius reciprocity, there are non-negative integers a_i's such that

$$\mathrm{Ind}_H^G 1_H = 1_G + \sum_i a_i \chi_i,$$

where 1_H and 1_G denote the trivial characters of H and G, respectively, and χ_i's are non-trivial irreducible characters of G. By the induction-invariance property of Artin L-functions and the above expression of $\mathrm{Ind}_H^G 1_H$, one can deduce that

$$L(s, 1_H, K/K^H) = L(s, 1_G, K/k) \prod_i L(s, \chi_i, K/k)^{a_i}.$$

Since $\zeta_M(s) = L(s, 1_H, K/K^H)$, $\zeta_k(s) = L(s, 1_G, K/k)$, and all a_i's are non-negative integers, Dedekind's conjecture follows from Artin's conjecture.

Now let us put our attention to infinite places of number fields. Firstly, let k be a number field. We recall that a *real embedding* of k is an injective field homomorphism from k to $\mathbb{R}$ and that a *complex embedding* of k is an injective field homomorphism from k to $\mathbb{C}$ whose image is not contained in $\mathbb{R}$. Dirichlet's unit theorem tells us that the rank of the group of units in $\mathcal{O}_k$ is $r = r_1 + r_2 - 1$, where

$$[k : \mathbb{Q}] = r_1 + 2r_2,$$

r_1 is the number of real embeddings k and $2r_2$ is the number of complex embeddings of k. When $r_2 = 0$, k is said to be *totally real*. For a real embedding (resp., a complex embedding) v of k, v is often called a *real infinite place* (resp., a *complex infinite place*) of k. We further recall that the *discriminant* of k is the square of the determinant of the n by n matrix whose (i, j)-entry is $\sigma_i(b_j)$, where n is the degree of k, $\{b_1, \cdots, b_n\}$ is an integral basis of $\mathcal{O}_k$, and $\sigma_1, \cdots, \sigma_n$ are embeddings of k.

To end this section, let us define the (global) Artin conductor (of χ with underlying space V). Let $\mathfrak{p}$ be a prime of k and $\mathfrak{P}$ be a prime of K above $\mathfrak{p}$. We let G_i denote the subgroup consisting of all σ of G acting trivially on $\mathcal{O}_K/\mathfrak{P}^{i+1}$. The group G_i is called the *i-th ramification group*. These higher ramification groups form a decreasing filtration

$$G \supseteq G_0 \supseteq G_1 \supseteq \cdots .$$

Furthermore, it can be shown that there exists N such that G_i is *trivial* for every $i \geq N$. Thus, we can define

$$n(\chi, \mathfrak{p}) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} \operatorname{codim} V^{G_i},$$

which is, in fact, a *finite* sum. Furthermore, Artin proved that $n(\chi, \mathfrak{p})$ is an *integer*. Moreover, for $\mathfrak{p}$ unramified, $n(\chi, \mathfrak{p}) = 0$. Thus, the following product

$$\mathfrak{f}(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{n(\chi, \mathfrak{p})},$$

where the product is over all primes of k, is a (well-defined) ideal of $\mathcal{O}_k$, called the *Artin conductor* of χ. From this, it is not hard to see that for any characters χ_1 and χ_2 of G, one has

$$\mathfrak{f}(\chi_1 + \chi_2) = \mathfrak{f}(\chi_1)\mathfrak{f}(\chi_2).$$

3.1.2 The Chebotarev Density Theorem

Throughout this section, we make use of some standard notations. We write $f \ll g$ or, equivalently, $f = O(g)$ if there is a constant M such that $|f(x)| \leq Mg(x)$ for all x sufficiently large. Also, we write $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to \infty$. We remark that all implied constants of estimates presented in this section are *absolute*.

As before, K/k denotes a Galois extension of number fields with Galois group G. For every unramified prime ideal $\mathfrak{p}$ in $\mathcal{O}_k$, $\sigma_{\mathfrak{p}}$ denotes the Artin symbol at $\mathfrak{p}$. The

Chebotarev density theorem essentially tells that the Artin symbols are *equidistributed* in the set of conjugacy classes of G. More precisely, the Chebotarev density theorem states the following.

Theorem 3.4 (Chebotarev Density Theorem). *Let C be a subset of G stable under conjugation and denote $\pi_C(x) = \#\{\mathfrak{p} \mid \mathfrak{p}$ is unramified with $N\mathfrak{p} \leq x$ and $\sigma_{\mathfrak{p}} \subseteq C\}$. Then*

$$\pi_C(x) \sim \frac{|C|}{|G|}\pi_k(x),$$

as $x \to \infty$, where $\pi_k(x) = \#\{\mathfrak{p} \mid \mathfrak{p}$ is unramified with $N\mathfrak{p} \leq x\}$.

Practically, one needs to know an *effective* version of the Chebotarev density theorem with *error terms* for studying problems from number theory. There are three versions: an unconditional version, a version assuming GRH, the generalised Riemann hypothesis asserting that the Dedekind zeta function is non-zero for $\Re(s) \neq \frac{1}{2}$ and $0 < \Re(s) < 1$, and a version assuming GRH and Artin's conjecture where the first two are covered in the fundamental paper [40] of Lagarias and Odlyzko, and the last one is due to M. R. Murty, V. K. Murty, and Saradha [48].

In the following theorems and corollaries, $n_k = [k\colon \mathbb{Q}]$ is the degree of k over $\mathbb{Q}$ and $n = [K\colon k]$ is the degree of K over k. Let d_k and d_K denote the absolute discriminants of $k/\mathbb{Q}$ and $K/\mathbb{Q}$, respectively. Let $P(K/k)$ denote the set of rational primes p for which there is $\mathfrak{p}$ of k with $\mathfrak{p}|p$ and $\mathfrak{p}$ is ramified in K. We then set

$$M(K/k) = nd_k^{\frac{1}{n_k}} \prod_{p\in P(K/k)} p.$$

Let $\mathfrak{f}(\chi)$ denote the Artin conductor of a character χ of $G = \mathrm{Gal}(K/k)$, and let $A_\chi = d_k^{\chi(1)} N\mathfrak{f}(\chi)$ denote the *conductor* of χ. The *offset logarithmic integral function* is defined as

$$\operatorname{Li} x = \int_2^x \frac{dt}{\log t}$$

for real variables $x > 2$.

To obtain a sharp error term for the Chebotarev density theorem, M. R. Murty, V. K. Murty, and Saradha [48] first derived the two estimates stated below.

Proposition 3.5. *For each unramified prime $\mathfrak{p}$ of k, let $\sigma_{\mathfrak{p}}$ denote the Artin symbol at $\mathfrak{p}$. Let χ be a character of G and let $\pi(x,\chi) = \sum_{N\mathfrak{p}\le x} \chi(\sigma_{\mathfrak{p}})$ where the sum is over unramified primes $\mathfrak{p}$ of k. Let $\delta(\chi)$ denote the multiplicity of the trivial character in χ. Suppose that the Artin L-function $L(s,\chi)$ is holomorphic for all $s \neq 1$ and is non-zero for $\Re(s) \neq \frac{1}{2}$ and $0 < \Re(s) < 1$. Then*

$$\pi(x,\chi) = \delta(\chi) \operatorname{Li} x + O\left(x^{\frac{1}{2}}(\log A_\chi + \chi(1) n_k \log x)\right) + O\left(\chi(1) n_k \log M(K/k)\right).$$

Lemma 3.6. *Let χ be an irreducible character of G. Then*

$$\log N\mathfrak{f}(\chi) \le 2\chi(1) n_k \left(\sum_{p \in P(K/k)} \log p + \log n \right).$$

From these estimates, M. R. Murty, V. K. Murty, and Saradha derived an effective version of the Chebotarev density theorem as follows.

Theorem 3.7. *Suppose that all Artin L-functions attached to all irreducible characters of $G = \mathrm{Gal}(K/k)$ are holomorphic at $s \neq 1$, and that GRH holds for $\zeta_K(s)$. Then*

$$\sum_C \frac{1}{|C|} \left| \pi_C(x) - \frac{|C|}{|G|} \operatorname{Li} x \right|^2 \ll x n_k^2 \log^2 \left(M(K/k)x\right),$$

where the sum on the left runs over conjugacy classes C of G.

We note that as mentioned above, effective versions of the Chebotarev density theorem with explicit error terms were first established by Lagarias and Odlyzko in [40]. If the generalised Riemann hypothesis for the Dedekind zeta function $\zeta_K(s)$ is assumed, Serre [58] further showed that

$$\pi_C(x) = \frac{|C|}{|G|} \operatorname{Li} x + O\left(\frac{|C|}{|G|} x^{\frac{1}{2}} (\log d_K + n_K \log x)\right), \tag{3.1}$$

where the big-O symbol is absolute. We also remark that there are unconditional versions, and refer the reader to [40] and [58].

Now by Theorem 3.7, one has

$$\pi_C(x) = \frac{|C|}{|G|} \operatorname{Li} x + O(|C|^{\frac{1}{2}} x^{\frac{1}{2}} n_k \log M(K/k)x). \tag{3.2}$$

On the other hand, if one writes the error term in (3.1) as

$$O\left(|C| x^{\frac{1}{2}} n_k \left(\frac{\log d_K}{n_K} + \log x\right)\right),$$

one can see that (3.2) is a better estimate as the factor $|C|$ in (3.1) is now replaced by $|C|^{\frac{1}{2}}$. These estimates are more versatile for many applications such as Artin's primitive root conjecture and the Lang-Trotter conjecture on Fourier coefficients of modular forms (cf. [48]).

3.1.3 Classical Heilbronn Characters

To study Artin's conjecture, Heilbronn introduced an innovative method. We now describe this.

As before, let K/k be a Galois extension of number fields with Galois group G, and fix $s_0 \in \mathbb{C}$. The *Heilbronn character* Θ_G (with respect to $s = s_0$) is defined by

$$\Theta_G = \sum_{\chi \in \mathrm{Irr}(G)} n(G, \chi)\chi,$$

where $n(G, \chi) = \mathrm{ord}_{s=s_0} L(s, \chi, K/k)$. One can see that the Heilbronn character might *not* be a character, and the Heilbronn character is a character or identically equal to zero if and only if Artin's conjecture is *locally* valid at $s = s_0$. By the works of Heilbronn-Stark (see Lemma 3.8 below), Foote-V. K. Murty [19], and M. R. Murty-Raghuram [49], one has the following collection of results connecting the zeros and poles of Artin L-functions and the Dedekind zeta functions.

Lemma 3.8 (Heilbronn-Stark Lemma)**.** *For any subgroup H of G,*

$$\Theta_G|_H = \Theta_H.$$

Theorem 3.9.

$$\sum_{\chi \in \mathrm{Irr}(G)} n(G, \chi)^2 \leq \left(\mathrm{ord}_{s=s_0} \zeta_K(s)\right)^2.$$

In addition, if G is solvable and χ' is a character of G of degree one, then

$$\sum_{\chi \neq \chi'} n(G, \chi)^2 \leq \left(\mathrm{ord}_{s=s_0} \frac{\zeta_K(s)}{L(s, \chi', K/k)}\right)^2.$$

In particular,

$$\sum_{\chi \neq 1_G} n(G, \chi)^2 \leq \left(\mathrm{ord}_{s=s_0} \frac{\zeta_K(s)}{\zeta_k(s)}\right)^2,$$

where 1_G is the trivial character of G.

Notice that these results imply that the zeros and poles (if any) of any Artin L-function are contained in the set of zeros of the Dedekind zeta function. In particular, applying the above theorem, one can easily see that all Artin L-functions are non-vanishing and holomorphic at $s = s_0$ if $\mathrm{ord}_{s=s_0} \zeta_K(s) = 0$. Moreover, in the case that the Dedekind zeta function has a simple zero at $s = s_0$, Stark [62] obtained the following holomorphy result.

Proposition 3.10. *If* $\mathrm{ord}_{s=s_0} \zeta_K(s) = 1$*, then all Artin L-functions attached to irreducible characters of* G *are holomorphic at* $s = s_0$*.*

3.1.4 Elliptic Curves and their L-Functions

Let k be a number field and E an elliptic curve defined over k. We recall that E is said to have *good reduction* at a finite place, i.e., a prime, v of k if $E \pmod v$ is still an elliptic curve. For every good reduction v of E, we let

$$Nv + 1 - a_v$$

represent the number of points of $E \pmod v$, where Nv stands for the absolute norm of v. The L-function $L(s, E, k)$ of E/k is defined as an Euler product:

$$L(s, E, k) = \prod_v L_v(s, E, k),$$

where the product is over all finite places of k. Moreover, for good reduction v of E,

$$L_v(s, E, k) = (1 - a_v Nv^{-s} + Nv^{1-2s})^{-1}.$$

For every finite extension F/k, E can be seen as an elliptic curve defined over F. Let $E_{/F}[n]$ denote the set of *n-torsion points* of E/F. By the work of Serre and Tate (cf. [55, 56]), one can associate a *compatible system of ℓ-adic representations* to E over F, i.e., for each prime ℓ,

$$\rho_F := \rho_{\ell,F} : \mathrm{Gal}(\overline{k}/F) \to \mathrm{Aut}(T_\ell(E,F)),$$

where $T_\ell(E,F)$ denotes the (ℓ-adic) *Tate module* of E/F, i.e., the inverse limit

$$T_\ell(E,F) = \varprojlim E_{/F}[\ell^n].$$

Furthermore, the L-function $L(s,E,F)$ of E/F is given by this family of ℓ-adic representations of E over F (see [55, 56] for details). Since $T_\ell(E,F) = T_\ell(E,k)$ as $\mathrm{Gal}(\overline{k}/F)$-modules, ρ_F is the restriction of ρ_k, which implies that

$$L(s,\rho_F) = L(s,\rho_k|_{\mathrm{Gal}(\overline{k}/F)}).$$

Now let us fix a Galois extension K/k and consider the m-th symmetric power of ρ_k. An analogous argument tells us that

$$(\mathrm{Sym}^m \rho_k)|_{\mathrm{Gal}(\overline{k}/F)} = \mathrm{Sym}^m \rho_F$$

for every intermediate field F of K/k. But

$$\mathrm{Ind}_{\mathrm{Gal}(\overline{k}/F)}^{\mathrm{Gal}(\overline{k}/k)} \left((\mathrm{Sym}^m \rho_k)|_{\mathrm{Gal}(\overline{k}/F)} \right) = \mathrm{Sym}^m \rho_k \otimes \mathrm{Ind}_{\mathrm{Gal}(\overline{k}/F)}^{\mathrm{Gal}(\overline{k}/k)} 1.$$

Putting everything together, we finally obtain

$$\begin{aligned} L(s, \operatorname{Sym}^m \rho_F) &= L(s, (\operatorname{Sym}^m \rho_k)|_{\mathrm{Gal}(\overline{k}/F)}) \\ &= L(s, \operatorname{Sym}^m \rho_k \otimes \operatorname{Ind}_{H_F}^G 1), \end{aligned} \tag{3.3}$$

where H_F is a subgroup of G such that $K^{H_F} = F$. We remark that if $F = K$, then H_F is the trivial group and the above formula, i.e., Equation (3.3), gives the Artin-Takagi decomposition for L-functions associated to elliptic curves.

3.1.5 Hecke L-Functions

Let k be a number field and $\mathfrak{m}$ a non-zero integral ideal of k. One defines the subgroup $I_{\mathfrak{m}}$ (resp., $P_{\mathfrak{m}}$) of the group I of fractional ideals in k (resp., the group P of principal ideals in k) by

$$I_{\mathfrak{m}} = \{\mathfrak{a} \in I \mid (\mathfrak{a}, \mathfrak{m}) = 1\},$$

$$P_{\mathfrak{m}} = \{\mathfrak{a} = (\alpha) \in P \cap I_{\mathfrak{m}} \mid \alpha \equiv 1 \pmod{\mathfrak{m}}\}.$$

One can show that $P_{\mathfrak{m}}$ is *normal* in $I_{\mathfrak{m}}$, and that the quotient group $H_{\mathfrak{m}} = I_{\mathfrak{m}}/P_{\mathfrak{m}}$ is a *finite abelian* group, called the *ray-class group* modulo $\mathfrak{m}$.

Let $\omega_\infty : \mathbb{Q}^\times \backslash k^\times \to \mathbb{C}^\times$ be a (unitary) character such that $U_{\mathfrak{m}} \subseteq \operatorname{Ker} \omega_\infty$, where $U_{\mathfrak{m}}$ denotes the group of units in $P_{\mathfrak{m}}$. Then, ω_∞ induces a homomorphism

$$\omega_\infty : P_{\mathfrak{m}} \to \mathbb{C}^\times.$$

From this, one can define a *Hecke character* of weight ω_∞ for $\mathfrak{m}$ as a homomorphism

$$\chi : I_{\mathfrak{m}} \to \mathbb{C}^\times,$$

which is (unitary) such that $\chi((\alpha)) = \omega_\infty(\alpha)$ if $\mathfrak{a} = (\alpha) \in P_\mathfrak{m}$, and is extended to I by setting $\chi(\mathfrak{a}) = 0$ if $(\mathfrak{a}, \mathfrak{m}) \neq 1$. We then come to the definition of Hecke L-functions as follows. The *Hecke L-function* of a Hecke character χ is defined as

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1},$$

where the sum is over all non-zero integral ideals and the product runs over all prime ideals.

As a simple example, for $k = \mathbb{Q}$, one can only have $\omega_\infty = 1$ and $\mathfrak{m} = (m)$ for some (unique) $m \geq 1$. Hence, the Hecke characters (resp., Hecke L-functions) modulo $\mathfrak{m}$ are exactly Dirichlet characters (resp., Dirichlet L-functions) modulo m.

3.1.6 Artin-Hecke L-Functions and CM-Elliptic Curves

We now recall the concept of *Artin-Hecke L-functions* developed by Weil [70].

Definition 3.1. *Let K/k be a Galois extension of number fields with Galois group G. Let ψ be a Hecke character of k and ρ be a complex representation of G with underlying vector space V. The Artin-Hecke L-function attached to ψ and ρ is defined by*

$$L(s, \psi \otimes \rho, K/k) = \prod_{\mathfrak{p}} \det(1 - \psi(\mathfrak{p}) \rho \mid^{V^{I_\mathfrak{P}}} (\sigma_\mathfrak{P}) N\mathfrak{p}^{-s})^{-1},$$

where the product runs over prime ideals in $\mathcal{O}_k$, $\mathfrak{P}$ denotes a prime ideal above $\mathfrak{p}$, $I_\mathfrak{P}$ is the inertia subgroup at $\mathfrak{P}$, and $V^{I_\mathfrak{P}} = \{v \in V \mid \rho(g)v = v$ for all $g \in I_\mathfrak{P}\}$. Usually we write $L(s, \psi \otimes \chi, K/k)$ for $L(s, \psi \otimes \rho, K/k)$ where $\chi = \operatorname{tr} \rho$.

We remark that for every 1-dimensional character χ of G, the Artin-Hecke L-function $L(s, \psi \otimes \chi, K/k)$ extends to a meromorphic function over $\mathbb{C}$ with only a possible pole at $s = 1$ since the corresponding L-function is a Hecke L-function. Moreover, Weil proved each of these L-functions $L(s, \psi \otimes \rho, K/k)$ extends to a *meromorphic* function on $\mathbb{C}$ by showing the following lemma and applying the Brauer induction theorem.

Lemma 3.11. *For any characters χ_1 and χ_2 of G and every character ϕ of H, we have*

1. $L(s, \psi \otimes (\chi_1 + \chi_2), K/k) = L(s, \psi \otimes \chi_1, K/k)L(s, \psi \otimes \chi_2, K/k)$, *and*

2. $L(s, \psi \otimes \mathrm{Ind}_H^G \phi, K/k) = L(s, \psi \circ N_{K^H/k} \otimes \phi, K/K^H)$, *where K^H is the subfield of K fixed by H and $N_{K^H/k}$ is the usual norm of K^H/k.*

We also recall two important facts from the theory of elliptic curves.

Theorem 3.12. *Let E be an elliptic curve defined over k. Suppose that E has CM by an order in an imaginary quadratic field F. If $F \subseteq k$, then the L-function $L(s, E, k)$ of E is the product of two Hecke L-functions of k. If $F \nsubseteq k$, then $L(s, E, k)$ is equal to a Hecke L-function of kF which is a quadratic extension of k.*

This result is due to Deuring [15]. From this theorem, M. R. Murty and V. K. Murty [46, Lemma 2] showed the following result, which was proved earlier by Shimura for CM-elliptic curves over $\mathbb{Q}$ by using Weil's converse theorem.

Theorem 3.13. *The generalised Taniyama-Shimura conjecture is valid for all CM-elliptic curves defined over k. In other words, every L-function of a CM-elliptic curve can be written in terms of Hecke L-functions.*

3.2 Automorphic Representations and the Langlands Reciprocity Conjecture

Let k be a number field. Denote the completion of k at finite v by k_v. Also, if v is real (resp., complex), then we set $k_v = \mathbb{R}$ (resp., $k_v = \mathbb{C}$). The *adèle ring* $\mathbb{A}_k$ of k is the restricted direct product $\prod'_v k_v$ over all places v of k with respect to $\{\mathcal{O}_{k_v}\}$, where $\mathcal{O}_{k_v}$ stands for the ring of v-adic integers. For any algebraic group G over k, it can be shown that $G(\mathbb{A}_k)$ is the restricted direct product $\prod'_v G(k_v)$ with respect to $\{G(\mathcal{O}_{k_v})\}$.

We begin by discussing L-functions attached to automorphic representations of GL_n. Our discussion is bound to be incomplete, so we refer the serious reader to [9] for details. When $G = GL_n$, one can show that the *L-group* of G is ${}^LG = {}^LG^0 \times W_k$ where ${}^LG^0$ is the connected component of ${}^LG$ and equal to $GL_n(\mathbb{C})$, and W_k is the *Weil group* of k. We recall that all upper-triangular matrices of G form a subgroup, which is called the *Borel subgroup* and often denoted by B. Also, if k_v is $\mathbb{R}$ (resp., $\mathbb{C}$), then the maximal compact subgroup K_v of $G(k_v)$ is $O(n)$ (resp., $U(n)$); otherwise, for any finite place v, $K_v = GL_n(\mathcal{O}_{k_v})$.

Now let us fix a character ω of $k^\times \backslash GL_1(\mathbb{A}_k)$, which is often called a *Grossen-character*, and consider the Hilbert space $L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$. For the right regular representation R of $G(\mathbb{A}_k)$ on $L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$, one has

$$(R(g)f)(x) = f(xg)$$

for any $f \in L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$ and $x, g \in G(\mathbb{A}_k)$. This is a *unitary* representation of $G(\mathbb{A}_k)$. From this, we define an *automorphic representation* to be an irreducible unitary subrepresentation of the right regular representation R of $G(\mathbb{A}_k)$ on $L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$. Similarly, a *cuspidal automorphic representation* is an irreducible unitary subrepresentation of the right regular representation of $G(\mathbb{A}_k)$ on $L_0^2(G(k)\backslash G(\mathbb{A}_k), \omega)$, where $L_0^2(G(k)\backslash G(\mathbb{A}_k), \omega)$ stands for the subspace of cusp forms of $L^2(G(k)\backslash G(\mathbb{A}_k), \omega)$. Moreover, a representation of $G(\mathbb{A}_k)$ is said to be *admissible* if its restriction to the maximal compact subgroup, $K = \prod_v K_v$, contains each irreducible representation of K with only finite multiplicity.

For any automorphic representation π of $G(\mathbb{A}_k)$, it has been shown that π can be written as a restricted tensor product $\otimes_v' \pi_v$, where for each place v, π_v is an irreducible admissible representation of $GL_n(k_v)$ such that for all but finitely many v, π_v is unramified, namely, the restriction of π_v to K_v contains the trivial representation. A place v will be called *unramified* (for π) if π_v is; otherwise, v is said to be *ramified*. It is known that for v unramified, π_v is induced from the Borel subgroup $B(k_v)$ of some tensor product $\mu_1 \otimes \cdots \otimes \mu_n$, where each μ_i is an unramified character of $k_v^\times$. For v finite, we let $\overline{\omega}$ be a generator for the maximal prime ideal of k_v (which is called the *uniformiser* for k_v). From this, we can further associate the semisimple conjugacy class $A(\pi_v)$ in ${}^LG^0$ to any unramified v, where

$$A(\pi_v) = \mathrm{diag}(\mu_1(\overline{\omega}), \cdots, \mu_n(\overline{\omega})).$$

We note that the eigenvalues of $A(\pi_v)$ are called the *Satake parameters* of π_v.

Now we define the (incomplete) *automorphic L-function* attached to π by

$$L(s,\pi) = \prod_v L(s,\pi_v),$$

where the product runs over all finite places of k and for v unramified,

$$L(s,\pi_v) = \det(I - A(\pi_v)Nv^{-s})^{-1}.$$

We remark that it is possible to define the complete automorphic L-function attached π and write down the precise description of $L(s,\pi_v)$ for ramified v (cf. [21, 30]). Moreover, by the work of Godement and Jacquet, we know that for every automorphic representation π of $GL_n(\mathbb{A}_k)$, $L(s,\pi)$ converges on some right half-plane and can be extended to a *meromorphic* function over $\mathbb{C}$. Moreover, if π is non-trivial and cuspidal, then $L(s,\pi)$ is *entire*. On the other hand, for any automorphic representations π_1 and π_2 of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively, by the theory of Rankin-Selberg convolutions developed by many authors, one can define the *Rankin-Selberg L-function* as

$$L(s,\pi_1\times\pi_2) = \prod_v L(s,\pi_{1,v}\times\pi_{2,v}),$$

where for v unramified, the local L-function is defined by

$$L(s,\pi_{1,v}\times\pi_{2,v}) = \det(I - A(\pi_{1,v})\otimes A(\pi_{2,v})Nv^{-s})^{-1}.$$

Via Rankin-Selberg convolutions, Jacquet and Shalika [33] showed that the L-function $L(s,\pi_1\times\pi_2)$ converges absolutely for $\Re(s) > 1$. Moreover, they proved the following.

Theorem 3.14. *Let π_1 and π_2 be cuspidal. Then the Rankin-Selberg L-function*

$L(s, \pi_1 \times \pi_2)$ has a simple pole at $s = 1$ if and only $\pi_2 \simeq \check{\pi}_1$, where $\check{\pi}_1$ denotes the contragredient of π_1.

Now we state the *Langlands reciprocity conjecture*, which sometimes is also called the *strong Artin conjecture.*

Conjecture 3.15. *For every Galois representation $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$, there exists an automorphic representation π of $GL_n(\mathbb{A}_k)$ such that*

$$L_v(s, \rho, K/k) = L(s, \pi_v)$$

for all but finitely many finite places v of k.

If such a π exists, then ρ is said to be of *automorphic type* (or *automorphic* for short), ρ is associated to π, or ρ corresponds to π; and we will write $\rho \leftrightsquigarrow \pi$.

We remark that as a consequence of the above theory, primarily, the result of Godement and Jacquet, Artin's conjecture follows from the Langlands reciprocity conjecture. Also, if Langlands reciprocity holds for characters χ_1 and χ_2 of G, then Artin's conjecture is valid for the Artin L-function $L(s, \chi_1 \otimes \chi_2, K/k)$. Furthermore, a result of Jacquet and Shalika (cf. [33, Theorem 4.7]) asserts that if $\chi \in \mathrm{Irr}(G)$ is associated to an automorphic representation π of $GL_n(\mathbb{A}_k)$, then π must be cuspidal.

3.3 Automorphy and Functoriality Results

3.3.1 Some Known Cases of Langlands Reciprocity

We first remark that by the works of Artin, Hecke, Iwasawa, and Tate, the Langlands conjecture for $GL(1)$ is precisely Artin reciprocity. The next big step was taken by

Langlands [37] and Tunnell [66] who proved the Langlands reciprocity conjecture for all irreducible 2-dimensional Galois representations with finite solvable image. Thus, by extracting the works of Artin and Langlands-Tunnell, one has

Theorem 3.16. *If a character χ of a solvable group G is of degree at most 2, then χ is of automorphic type.*

More recently, Khare and Wintenberger [34] proved Serre's modularity conjecture and then deduced Langlands reciprocity for any *odd* irreducible 2-dimensional representation over $\mathbb{Q}$ with non-solvable image.

We recall that a $\mathbb{C}$-representation (ρ, V) of a group G is said to be of $GO(n)$-type if $\dim V = n$ and it factors as

$$\rho : G \to GO_n(\mathbb{C}) \subset GL(V).$$

In his paper [54], Ramakrishnan derived the automorphy of solvable Artin representations of $GO(4)$*-type* as follows.

Theorem 3.17. *Let K/k be a Galois extension of number fields and ρ be a 4-dimensional representation of G whose image is solvable and lies in $GO_4(\mathbb{C})$. Then ρ is automorphic.*

One also has the following results concerning *symplectic* Galois representations and *hypertetrahedral* Galois representations due to Martin [41, 42].

Theorem 3.18. *Let K/k be a Galois extension of number fields and ρ be an irreducible 4-dimensional representation of $G = \mathrm{Gal}(K/k)$ into $GSp_4(\mathbb{C})$. If the projective image $\overline{G}$ of ρ (in $PGL_4(\mathbb{C})$) is isomorphic to $E_{2^4} \rtimes C_5$, then ρ is automorphic.*

Theorem 3.19. *Let K/k be a Galois extension of number fields and ρ be an irreducible 4-dimensional representation of $G = \mathrm{Gal}(K/k)$. Suppose $\overline{G}$ is an extension of A_4 by V_4. Then ρ is automorphic.*

As remarked by Martin, the case where $\overline{G} = V_4 \rtimes A_4$ yields examples of irreducible monomial 4-dimensional representations of $GO(4)$-type, which can also be shown to be automorphic by Theorem 3.17.

3.3.2 Base Change and Automorphic Induction

A key ingredient in the proof of the Langlands theorem on the automorphy of certain 2-dimensional Galois representations is the (normal) *cyclic base change* for GL_2, which has been generalised to GL_n by Arthur and Clozel [2] (à la Langlands) as follows.

Theorem 3.20. *Let K/k be a Galois extension of prime degree. Then for every (isobaric) representation π of $GL_n(\mathbb{A}_k)$, there exists a unique (isobaric) automorphic representation $\pi|_K$ of $GL_n(\mathbb{A}_K)$, called the base change of π to K, such that*

1. *a cuspidal representation Π of $GL_n(\mathbb{A}_K)$ is the base change π_K of π if and only if Π is Galois invariant (in particular, if Π is associated to $\rho|_K$ for some Galois representation ρ over k);*

2. *for any (isobaric) π' over k, $\pi'_K = \pi_K$ if and only if $\pi' = \pi \otimes \chi$ for some idèle class character χ of k;*

3. *for every Galois representation ρ over k associated to π, one has $\rho|_K \leftrightsquigarrow \pi|_K$; and*

4. *if χ is an idèle class character of k, then $(\pi \otimes \chi)|_K = \pi|_K \otimes \chi|_K$.*

Moreover, Arthur and Clozel [2] derived the adjoint map to base change, called *automorphic induction*, which corresponds to induction for Galois representations as stated in the following theorem.

Theorem 3.21. *Let K/k be a Galois extension of number fields of prime degree p, and Π denote an automorphic representation induced from cuspidal of $GL_n(\mathbb{A}_K)$ (or, in particular, a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$). Then there is an automorphic representation $I(\Pi)$ of $GL_{np}(\mathbb{A}_k)$, called the automorphic induction of Π, such that $L(s,\Pi) = L(s, I(\Pi))$; and $I(\Pi)$ is also induced from cuspidal. Moreover, if ρ is a Galois representation corresponding to Π, then $\mathrm{Ind}_{\mathrm{Gal}(\bar{k}/K)}^{\mathrm{Gal}(\bar{k}/k)} \rho \leftrightsquigarrow I(\Pi)$.*

Furthermore, one has a result of Jacquet [31].

Lemma 3.22. *Let K/k be a Galois extension of number fields of prime degree. Let π and σ be two cuspidal unitary automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_K)$, respectively. Then the Rankin-Selberg L-functions satisfy the following formal identity:*

$$L(s, B(\pi) \otimes \sigma) = L(s, \pi \otimes I(\sigma)).$$

For *non-normal* extensions, one has a theorem due to Jacquet, Piatetski-Shapiro, and Shalika [32] below.

Theorem 3.23. *Let K/k be a non-normal cubic extension of number fields. Let χ be an idèle class character of K and π an automorphic representation of $GL_2(\mathbb{A}_k)$. Then the automorphic induction $I(\chi)$ of χ and the base change $\pi|_K$ exist as automorphic representations of $GL_3(\mathbb{A}_k)$ and $GL_2(\mathbb{A}_K)$, respectively.*

Thus, by Theorems 3.21 and 3.23, all monomial characters of degree 3 are of automorphic type.

3.3.3 Tensor Products and Symmetric and Exterior Powers

In light of the Langlands reciprocity conjecture and the fact that the tensor product of any two Galois representations is still a Galois representation, the *principle of functoriality* asserts that the Rankin-Selberg convolutions $\pi_1 \times \pi_2$ of any cuspidal representations π_1 and π_2 of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively, is in fact an automorphic representation of $GL_{nm}(\mathbb{A}_k)$, denoted by $\pi_1 \otimes \pi_2$. In particular, if each Galois representation ρ_i is associated to π_i, then $\rho_1 \otimes \rho_2 \leftrightsquigarrow \pi_1 \otimes \pi_2$. When $m = 1$, this is known since for π_1 automorphic, any "twist" $\pi_1 \otimes \chi$ is also automorphic for any (unitary) character χ of $k^\times \backslash \mathbb{A}_k^\times$; and the functoriality was recently established for $GL(2) \times GL(2)$ by Ramakrishnan [53] and $GL(2) \times GL(3)$ by Kim-Shahidi [36].

In a slightly different vein, consider a representation $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$ and a *symmetric or exterior power lifting* $r : GL_n(\mathbb{C}) \to GL_m(\mathbb{C})$. For v unramified, the local L-function attached to $r(\rho)$ is defined as

$$L_v(s, r(\rho)) = \det(I - r(\rho(\sigma_v))Nv^{-s})^{-1}.$$

Similarly, for every automorphic representation π of $GL_n(\mathbb{A}_k)$, one can define the local automorphic L-function at (finite) unramified v as

$$L_v(s, \pi, r) = \det(I - r(A(\pi_v))Nv^{-s})^{-1}.$$

Again, inspired by the properties of Galois representations, the principle of functoriality conjectures that there should exist an automorphic representation $r(\pi)$ of $GL_m(\mathbb{A}_k)$ such that $L(s, r(\pi)_v) = L_v(s, \pi, r)$ for all unramified v. In particular, if ρ is a Galois representation corresponding to π, then $r(\rho) \leftrightsquigarrow r(\pi)$. For $n = 2$, Sym^2,

Sym^3, and Sym^4 have been shown to be functorial by Gelbart-Jacquet, Kim-Shahidi, and Kim, respectively. Also, Kim showed that $\wedge^2 : GL_4 \to GL_6$ is functorial. (In fact, Kim proved that $\wedge^2(\pi)$ equals an automorphic representation of $GL_6(\mathbb{A}_k)$ at all places, except possibly those above 2 and 3, and Henniart indicated how one can derive equality at the remaining places in a letter to Kim and Shahidi.)

3.3.4 Applications to the Langlands Reciprocity Conjecture

Applying the *functoriality* mentioned above and the works of Artin, Langlands, and many others, one knows that the Langlands reciprocity conjecture holds in the following cases.

1. the direct sum of (Galois) representations of automorphic type;

2. the induction of a representation of automorphic type from a subnormal subgroup of a solvable group;

3. the induction of a 1-dimensional representation from a subgroup of index 3;

4. $\mathrm{Sym}^m \rho$ for 2-dimensional automorphic ρ, where $m \leq 4$;

5. $\wedge^2\rho$ for 4-dimensional automorphic ρ;

6. the tensor product of two representations of automorphic type whose dimensions are 2 and 2, or 2 and 3;

7. any abelian twist of a representation of automorphic type;

8. representations of dimension at most 2 with (finite) solvable image;

9. the Asai lift of any 2-dimensional representation of automorphic type;

10. representations of $GO(4)$-type with solvable image;

11. 4-dimensional representations with projective image isomorphic to $E_{2^4} \rtimes C_5$ or an extension of A_4 by V_4;

12. representations with (finite) nilpotent images; and

13. odd 2-dimensional icosahedral representations over $\mathbb{Q}$.

We note that the first seven cases are straightforward applications of the functoriality results discussed in the preceding sections. On the other hand, although the second instance is well-known by experts, we still give a proof below as it will play a crucial role in helping us to study conjectures of Artin and Langlands later.

Proof of Case 2. We now consider a character χ of $G = \mathrm{Gal}(K/k)$ which is induced from an irreducible character ψ of a subnormal subgroup H of G. Assume, further, that G is solvable and that ψ is automorphic over the fixed field K^H, i.e., there is a cuspidal automorphic representation Π of $GL_{\psi(1)}(\mathbb{A}_{K^H})$ such that

$$L(s, \psi, K/K^H) = L(s, \Pi).$$

Since H is a subnormal subgroup of G, there is an invariant series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G,$$

where for each i, H_i is a normal subgroup of H_{i+1}. As G is finite, we may require each H_{i+1}/H_i is a finite simple group. Since G is solvable, each quotient group must be cyclic. Thus, each H_{i+1}/H_i is a cyclic group of prime order, and one has a tower of Galois extensions of prime degree

$$K \supset K^{H_1} \supset \cdots \supset K^{H_{m-1}} \supset k.$$

Now applying the Arthur-Clozel theorem of automorphic induction successively, one can derive that $\mathrm{Ind}_H^G \psi$ corresponds to an automorphic representation over k. In other words, Langlands reciprocity holds for χ. Moreover, if χ is irreducible, the earlier-mentioned result of Jacquet and Shalika asserts that π is necessarily cuspidal. □

The eighth case is the celebrated Artin reciprocity and the Langlands-Tunnell theorem (we will often refer to these celebrated results as the Artin-Langlands-Tunnell theorem), and the eleventh case is due to Martin. We remark that the proofs of the results of Langlands-Tunnell and Martin profoundly rely on the functoriality of base change and symmetric/exterior powers. The ninth and tenth cases are due to Ramakrishnan. The twelfth case is a theorem of Arthur-Clozel who utilised Artin reciprocity, their theory of automorphic induction, and the fact that all subgroups of a nilpotent group are subnormal. The last case follows from Khare-Wintenberger's proof of Serre's modularity conjecture.

3.3.5 Potential Automorphy

In his paper [65], Taylor proved the *potential automorphy* for certain symmetric power L-functions of non-CM elliptic curves and then deduced the *Sato-Tate conjecture* (over totally real fields). As remarked in [65], Taylor was building on his earlier work [13] and [22] with Clozel, Harris, and Shepherd-Barron (we note that [22] was cited as "Ihara's lemma and potential automorphy" in [65]). More recently, Barnet-Lamb, Geraghty, Harris, and Taylor [4] proved the potential automorphy for symmetric power L-functions in a more general setting.

We recall that the main theorem of Taylor et al. is: let k be a totally real field and E/k a non-CM elliptic curve. Then for any finite set S of natural numbers, there is a (finite) totally real Galois extension L/k such that for every $m \in S$, $\mathrm{Sym}^m \rho_k$ is automorphic over L, i.e., $(\mathrm{Sym}^m \rho_k)|_L$ is automorphic.

From now on, we fix a finite set S of natural numbers and let L be a totally real Galois extension L/k such that for every $m \in S$, $\mathrm{Sym}^m \rho_k$ is automorphic over L. We now recall two key steps of the proof of the Sato-Tate conjecture.

Theorem 3.24. *For any intermediate field F of L/k with L/F solvable,*

$$(\mathrm{Sym}^m \rho_k)|_F$$

is automorphic.

This is proved in [22] by Harris, Shepherd-Barron, and Taylor. The proof essentially applies the Arthur-Clozel theorem of base change and the fact that $(\mathrm{Sym}^m \rho_k)|_L$ is Galois-invariant. Moreover, from Theorem 3.24, Artin reciprocity, and the Brauer induction theorem, Taylor et al. showed the following.

Theorem 3.25. *$L(s, \mathrm{Sym}^m \rho_k)$ extends to a meromorphic function over $\mathbb{C}$.*

Chapter 4

Applications of Supercharacter Theory

4.1 Super Heilbronn Characters

Via the theory of supercharacters and superinduction discussed in Section 2.3, one can generalise the classical Heilbronn character as follows.

Definition 4.1 (Super Heilbronn Characters). *Let K/k be a Galois extension of number fields with Galois group G. Let H be a subgroup of G. Let G and H be compatible,* $\mathrm{Sup}(G)$ *be the set of all supercharacters of G, and* $\mathrm{Sup}(H)$ *be the set of all supercharacters of H. Assume that the restriction $\sigma|_H$ of any supercharacter σ of G to H is an integral combination of supercharacters of H. Then the super Heilbronn character Θ_H (with respect to $s = s_0$) is defined by*

$$\Theta_H = \sum_{\tau \in \mathrm{Sup}(H)} n(H,\tau) \frac{\tau}{\tau(1)},$$

where $n(H,\tau) = \frac{1}{m} \mathrm{ord}_{s=s_0} L(s, m \,\mathrm{SInd}_H^G \tau, K/k)$, *and* $m = lcm(\sigma(1) \colon \sigma \in \mathrm{Sup}(G))$.

One might ask why there are extra m and $\frac{1}{m}$ for each $n(H,\tau)$, and why one needs to *normalise* supercharacters appearing in Θ_H. First of all, since the superinduction

$\mathrm{SInd}_H^G \tau$ of a supercharacter τ of H might be a *rational* combination of supercharacters of G, $\mathrm{SInd}_H^G \tau$ might be a rational combination of irreducible characters of G. But it is more natural to consider Artin L-functions attached to *characters*. Thus, we use $m\,\mathrm{SInd}_H^G \tau$, which is actually a character, instead of $\mathrm{SInd}_H^G \tau$. However, if one considers the improper subgroup H of G, i.e., $H = G$, equipped with the same supercharacter theory, then the superinduction from H to G is the identity map, i.e, for any supercharacter σ of $H = G$, $\mathrm{SInd}_H^G \sigma = \sigma$. So

$$n(G,\sigma) = \frac{1}{m}\operatorname{ord}_{s=s_0} L(s, m\sigma, K/k) = \operatorname{ord}_{s=s_0} L(s,\sigma,K/k),$$

which coincides with the classical definition.

Secondly, when one regards the classical theory as a supercharacter theory, one is, in fact, considering $\mathrm{Sup}(G) = \{\sigma = \chi(1)\chi \mid \chi \in \mathrm{Irr}(G)\}$ instead of $\mathrm{Irr}(G)$. Therefore, from the definition of super Heilbronn characters, one has

$$\begin{aligned}
\Theta_G &= \sum_{\sigma\in\mathrm{Sup}(G)} n(G,\sigma)\frac{\sigma}{\sigma(1)} \\
&= \sum_{\sigma\in\mathrm{Sup}(G)} \operatorname{ord}_{s=s_0} L(s,\sigma,K/k)\frac{\sigma}{\sigma(1)} \\
&= \sum_{\chi\in\mathrm{Irr}(G)} \operatorname{ord}_{s=s_0} L(s,\chi(1)\chi,K/k)\frac{\chi(1)\chi}{\chi^2(1)} \\
&= \sum_{\chi\in\mathrm{Irr}(G)} n(G,\chi)\chi,
\end{aligned}$$

which gives the classical Heilbronn character.

To demonstrate that Artin L-functions attached to supercharacters enjoy similar properties of Artin L-functions attached to irreducible characters, we present the following result that generalises the previous works of Heilbronn and others in the context of supercharacters.

Proposition 4.1. *Let K/k be a Galois extension of number fields with Galois group G. One has*

$$\sum_{\sigma\in \mathrm{Sup}(G)} \frac{n(G,\sigma)^2}{\sigma(1)} \leq \left(\mathrm{ord}_{s=s_0}\, \zeta_K(s)\right)^2,$$

where $\zeta_K(s)$ is the Dedekind zeta function of K. In addition, if G is solvable and χ is a supercharacter of G of degree one, then

$$\sum_{\sigma\neq\chi} \frac{n(G,\sigma)^2}{\sigma(1)} \leq \left(\mathrm{ord}_{s=s_0} \frac{\zeta_K(s)}{L(s,\chi,K/k)}\right)^2.$$

In particular,

$$\sum_{\sigma\neq 1_G} \frac{n(G,\sigma)^2}{\sigma(1)} \leq \left(\mathrm{ord}_{s=s_0} \frac{\zeta_K(s)}{\zeta_k(s)}\right)^2,$$

where 1_G denotes the trivial character of G.

Proof. For every $\sigma \in \mathrm{Sup}(G)$, one can write σ as

$$\sigma = \sum_{\chi\in \mathrm{Irr}(G,\sigma)} \chi(1)\chi,$$

where $\mathrm{Irr}(G,\sigma)$ is the set of irreducible characters of G appearing in σ. Then

$$\begin{aligned}\mathrm{ord}_{s=s_0} L(s,\sigma,K/k) &= \mathrm{ord}_{s=s_0} L\left(s, \sum_{\chi\in \mathrm{Irr}(G,\sigma)} \chi(1)\chi, K/k\right)\\ &= \sum_{\chi\in \mathrm{Irr}(G,\sigma)} \chi(1)\mathrm{ord}_{s=s_0} L(s,\chi,K/k),\end{aligned}$$

which together with the Cauchy-Schwarz inequality implies that

$$\begin{aligned} n(G,\sigma)^2 = (\mathrm{ord}_{s=s_0} L(s,\sigma,K/k))^2 &= \left(\sum_{\chi\in \mathrm{Irr}(G,\sigma)} \chi(1)\mathrm{ord}_{s=s_0} L(s,\chi,K/k) \right)^2 \\ &\le \sum_{\chi\in \mathrm{Irr}(G,\sigma)} \chi^2(1) \sum_{\chi\in \mathrm{Irr}(G,\sigma)} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2 \\ &= \sigma(1) \sum_{\chi\in \mathrm{Irr}(G,\sigma)} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2. \end{aligned}$$

Since $\mathrm{Irr}(G) = \coprod_{\sigma\in \mathrm{Sup}(G)} \mathrm{Irr}(G,\sigma)$ is a disjoint union of all $\mathrm{Irr}(G,\sigma)$'s,

$$\begin{aligned} \sum_{\sigma\in \mathrm{Sup}(G)} \frac{n(G,\sigma)^2}{\sigma(1)} &\le \sum_{\sigma\in \mathrm{Sup}(G)} \frac{\sigma(1)\sum_{\chi\in \mathrm{Irr}(G,\sigma)} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2}{\sigma(1)} \\ &= \sum_{\chi\in \mathrm{Irr}(G)} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2 \\ &\le (\mathrm{ord}_{s=s_0} \zeta_K(s))^2, \end{aligned}$$

where the last inequality holds thanks to Theorem 3.9. Since any supercharacter χ' of G of degree one is exactly a 1-dimensional irreducible character of G, by an analogous argument, one has

$$\begin{aligned} \sum_{\sigma\neq\chi'} \frac{n(G,\sigma)^2}{\sigma(1)} &\le \sum_{\sigma\neq\chi'} \frac{\sigma(1)\sum_{\chi\in \mathrm{Irr}(G,\sigma)} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2}{\sigma(1)} \\ &= \sum_{\chi\in \mathrm{Irr}(G)\setminus\{\chi'\}} (\mathrm{ord}_{s=s_0} L(s,\chi,K/k))^2 \\ &\le \left(\mathrm{ord}_{s=s_0} \frac{\zeta_K(s)}{L(s,\chi',K/k)} \right)^2, \end{aligned}$$

where the last inequality is due to Theorem 3.9. The final part of the theorem can be obtained by taking $\chi' = 1_G$, the trivial character of G. $\square$

We also have the following *Heilbronn-Stark lemma for super Heilbronn characters.*

Lemma 4.2. *Under the same assumption as before, one has* $\Theta_G|_H = \Theta_H$ *for any subgroup* H *of* G.

Proof. By super Frobenius reciprocity, one has

$$\begin{aligned}
\Theta_G|_H &= \sum_{\sigma\in \mathrm{Sup}(G)} \frac{n(G,\sigma)}{\sigma(1)}\sigma|_H \\
&= \sum_{\sigma\in \mathrm{Sup}(G)} \frac{m}{m}\frac{n(G,\sigma)}{\sigma(1)}\sigma|_H \\
&= \frac{1}{m}\sum_{\sigma\in \mathrm{Sup}(G)} m\frac{n(G,\sigma)}{\sigma(1)} \sum_{\tau\in \mathrm{Sup}(H)} (\tau,\sigma|_H)_H \frac{\tau}{(\tau,\tau)_H} \\
&= \frac{1}{m}\sum_{\tau\in \mathrm{Sup}(H)} \left(\sum_{\sigma\in \mathrm{Sup}(G)} \frac{mn(G,\sigma)}{\sigma(1)} (\mathrm{SInd}_H^G\,\tau,\sigma)_G\right)\frac{\tau}{\tau(1)},
\end{aligned}$$

where $m = lcm\{\sigma(1)\colon \sigma\in \mathrm{Sup}(G)\}$. Since the restriction $\sigma|_H$ of any supercharacter σ of G to H is an integral combination of supercharacters of H, $(\mathrm{SInd}_H^G\,\tau,\sigma)$ is an integer for any supercharacter τ of H and any supercharacter σ of G. Now, we have

$$\begin{aligned}
\sum_{\sigma\in \mathrm{Sup}(G)} \frac{mn(G,\sigma)}{\sigma(1)} (\mathrm{SInd}_H^G\,\tau,\sigma)_G &= \mathrm{ord}_{s=s_0} L\left(s, \sum_{\sigma\in \mathrm{Sup}(G)} \frac{m(\mathrm{SInd}_H^G\,\tau,\sigma)_G}{\sigma(1)}\sigma, K/k\right) \\
&= \mathrm{ord}_{s=s_0} L\left(s, \sum_{\sigma\in \mathrm{Sup}(G)} \frac{m(\mathrm{SInd}_H^G\,\tau,\sigma)_G}{(\sigma,\sigma)}\sigma, K/k\right) \\
&= \mathrm{ord}_{s=s_0} L(s, m\,\mathrm{SInd}_H^G\,\tau, K/k)_G \\
&= mn(H,\tau).
\end{aligned}$$

Thus,

$$\begin{aligned}\Theta_G|_H &= \frac{1}{m}\sum_{\tau\in\mathrm{Sup}(H)}\left(\sum_{\sigma\in\mathrm{Sup}(G)}\frac{mn(G,\sigma)}{\sigma(1)}(\mathrm{SInd}_H^G\,\tau,\sigma)_G\right)\frac{\tau}{\tau(1)}\\ &= \frac{1}{m}\sum_{\tau\in\mathrm{Sup}(H)} mn(H,\tau)\frac{\tau}{\tau(1)}\\ &= \sum_{\tau\in\mathrm{Sup}(H)} n(H,\tau)\frac{\tau}{\tau(1)}\\ &= \Theta_H.\end{aligned}$$

□

Similar to the role played by the classical Heilbronn-Stark lemma in studying the relation between orders of the Dedekind zeta functions and Artin L-functions, one can obtain the following results by applying Lemma 4.2.

Proposition 4.3.

$$\frac{|H|}{|G|}\sum_{\tau\in\mathrm{Sup}(H)}\frac{n(H,\tau)^2}{\tau(1)} \le \left(\mathrm{ord}_{s=s_0}\,\zeta_K(s)\right)^2.$$

Proof. By the orthogonality property of supercharacters and Proposition 4.1,

$$\begin{aligned}(\Theta_G,\Theta_G)_G &= \left(\sum_{\sigma\in\mathrm{Sup}(G)} n(G,\sigma)\frac{\sigma}{\sigma(1)},\sum_{\sigma\in\mathrm{Sup}(G)} n(G,\sigma)\frac{\sigma}{\sigma(1)}\right)_G\\ &= \sum_{\sigma\in\mathrm{Sup}(G)}\frac{n(G,\sigma)^2}{\sigma(1)}\\ &\le \left(\mathrm{ord}_{s=s_0}\,\zeta_K(s)\right)^2.\end{aligned}$$

On the other hand, Lemma 4.2 gives

$$\begin{aligned}(\Theta_G, \Theta_G)_G &= \frac{1}{|G|}\sum_{g\in G}\Theta_G(g)\overline{\Theta_G(g)} \\ &\geq \frac{1}{|G|}\sum_{g\in H}\Theta_G(g)\overline{\Theta_G(g)} \\ &= \frac{1}{|G|}\frac{|H|}{|H|}\sum_{g\in H}\Theta_H(g)\overline{\Theta_H(g)} \\ &= \frac{|H|}{|G|}(\Theta_H, \Theta_H)_H \\ &= \frac{|H|}{|G|}\sum_{\tau\in \mathrm{Sup}(H)}\frac{n(H,\tau)^2}{\tau(1)},\end{aligned}$$

and thus the corollary follows. □

Corollary 4.4. *If* $\mathrm{ord}_{s=s_0}\,\zeta_K(s) = 0$*, then Artin L-functions* $L(s, m\,\mathrm{SInd}_H^G\,\tau, K/k)$ *attached to supercharacters* τ *of* H *are holomorphic and non-vanishing at* $s = s_0$*.*

Since the *first orthogonality property* states that the set of all supercharacters of G forms an orthogonal basis of the inner product space of superclass functions of G, one might expect that there should be a *second orthogonality property.* In fact, the expected second orthogonality property can be derived easily by using the first orthogonality property and linear algebra. However, for the sake of completeness and clarity, we shall state and prove the following lemma.

Lemma 4.5. *Let* $\mathrm{Sup}(G) = \{\sigma_1, \cdots, \sigma_n\}$ *and* $\{C_1, \cdots, C_n\}$ *be the sets of supercharacters and superclasses of* G*, respectively. Then*

$$\sum_{k=1}^{n}\frac{\sigma_k(C_i)\overline{\sigma_k(C_j)}}{\sigma_k(1)} = \begin{cases}\frac{|G|}{|C_i|} & \text{if } i = j, \\ 0 & \text{otherwise.}\end{cases}$$

In particular, for any superclass C,

$$\delta_C = \frac{|C|}{|G|} \sum_{\sigma \in \mathrm{Sup}(G)} \frac{\overline{\sigma(g_C)}\sigma}{\sigma(1)},$$

where δ_C denotes the characteristic function of C and g_C is an element of C.

Proof. For each k, let e_k be a representative of C_k. Then, for any i and j,

$$\begin{aligned}\delta_{ij}\sqrt{\sigma_i(1)\sigma_j(1)} &= (\sigma_i, \sigma_j) \\ &= \frac{1}{|G|}\sum_{g \in G} \overline{\sigma_i(g)}\sigma_j(g) \\ &= \frac{1}{|G|}\sum_{k=1}^{n} |C_k|\overline{\sigma_i(e_k)}\sigma_j(e_k),\end{aligned}$$

where δ_{ij} is the Kronecker delta. Setting $a_{ij} = \frac{\sigma_j(e_i)}{\sqrt{\sigma_j(1)}}$, one has

$$\begin{aligned}\delta_{ij} &= (\sigma_i, \sigma_j)_G \\ &= \frac{1}{|G|}\sum_{k=1}^{n} \overline{a_{ki}\sqrt{|C_k|}}a_{kj}\sqrt{|C_k|}.\end{aligned}$$

Considering a matrix $B = (b_{kj})$ where $b_{kj} = a_{kj}\frac{\sqrt{|C_k|}}{\sqrt{|G|}}$, the above equation implies that $B^*B = I = BB^*$. Hence,

$$\begin{aligned}\delta_{ij} &= \sum_{k=1}^{n} b_{ik}\overline{b_{jk}} \\ &= \frac{1}{|G|}\sum_{k=1}^{n} a_{ik}\sqrt{|C_i|}\overline{a_{jk}\sqrt{|C_j|}} \\ &= \frac{1}{|G|}\sum_{k=1}^{n} \frac{\sigma_k(e_i)}{\sqrt{\sigma_k(1)}}\sqrt{|C_i|}\overline{\frac{\sigma_k(e_j)}{\sqrt{\sigma_k(1)}}\sqrt{|C_j|}},\end{aligned}$$

as desired. □

4.2 Supercharacters and Artin's Conjecture

We remind the reader that our purpose of this chapter is applying supercharacter theory to study Artin L-functions. Thus, it is certainly desired to find a supercharacter theory of G satisfying the Artin conjecture, i.e., for any Galois extension K/k of number fields with Galois group G, the Artin conjecture holds for all Artin L-functions attached to supercharacters of such a supercharacter theory. To obtain such a theory, we shall invoke the Aramata-Brauer theorem.

First of all, for any Galois extension K/k with Galois group G, the Aramata-Brauer theorem asserts that the quotient $\zeta_K(s)/\zeta_k(s)$ is entire. In other words, Artin's conjecture holds for the Artin L-functions attached to supercharacters $\mathrm{Reg}_G - 1_G$ and 1_G. (We note that $\{\mathrm{Reg}_G - 1_G, 1_G\}$ gives the *maximal theory* of G.)

In [24], Hendrickson introduced the $*$-*product of supercharacter theories*, which produces a supercharacter theory of G from its normal subgroup N and the quotient group $H = G/N$ as follows.

Let G be a finite group and N be a normal subgroup of G. We equip N and G/N with supercharacter theories $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{J})$ respectively. Following [24], $(\mathcal{X}, \mathcal{K})$ is said to be G-*invariant* if for each $g \in G$ and $n \in N$, both n and $g^{-1}ng$ belong to the same superclass. Assuming that $(\mathcal{X}, \mathcal{K})$ is G-invariant, define

$$\mathcal{Z} = \{\mathrm{Ind}_N^G(\sigma_X) \mid X \in \mathcal{X} \setminus \{1_N\}\} \cup \{\mathrm{Inf}_{G/N}^G \sigma_Y \mid Y \in \mathcal{Y}\},$$

$$\mathcal{M} = \mathcal{K} \cup \{NJ \mid J \in \mathcal{J} \setminus \{e_H\}\}.$$

Hendrickson then proved the following (cf. [24, Theorem 4.3]):

Proposition 4.6. *The pair $(\mathcal{Z}, \mathcal{M})$ defines a supercharacter theory of G.*

This supercharacter theory is referred as the $*$-product of $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{J})$, and denoted by

$$(\mathcal{Z}, \mathcal{M}) = (\mathcal{X}, \mathcal{K}) * (\mathcal{Y}, \mathcal{J}).$$

Now we furthermore assume that N is equipped with the maximal theory and that $H = G/N$ is equipped with the classical theory. It is clear that the maximal theory of N is G-invariant. By the Aramata-Brauer theorem, the maximal theory of N satisfies the Artin conjecture. Thus, if Artin's conjecture is true for the classical theory of H, which is the case for H *nearly supersolvable* (see Section 6.1), then the $*$-product $(\mathcal{Z}, \mathcal{M})$, constructed as above, is a supercharacter theory of G, which satisfies Artin's conjecture as desired. We shall call it the *max-min theory.*

4.3 An Effective Chebotarev Density Theorem

In this section, we will make use of notations introduced in Section 3.1.2.

We now plan to extend the result of M. R. Murty, V. K. Murty, and Saradha, Theorem 3.7, to Artin L-functions attached to *supercharacters*. First of all, following the strategy developed in [48], one would need the following lemma which will play the main role in "counting primes".

Lemma 4.7. *Let π be a complex-valued linear function defined on the vector space of superclass functions of G. Then*

$$\sum_{C} \frac{1}{|C|} \left| \pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G) \right|^2 = \frac{1}{|G|} \sum_{\sigma \neq 1_G} \frac{|\pi(\sigma)|^2}{\sigma(1)},$$

where the sum on the left runs over superclasses C of G, and the sum on the right runs over the non-trivial supercharacters.

Proof. Since π is linear, by Lemma 4.5, one can write

$$\pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G) = \frac{|C|}{|G|}\sum_{\sigma\neq 1_G}\frac{\overline{\sigma(g_C)}\pi(\sigma)}{\sigma(1)},$$

where g_C is a representative of C. Therefore,

$$\begin{aligned}\left|\pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G)\right|^2 &= \frac{|C|}{|G|}\sum_{\sigma\neq 1_G}\frac{\overline{\sigma(g_C)}\pi(\sigma)}{\sigma(1)}\overline{\frac{|C|}{|G|}\sum_{\tau\neq 1_G}\frac{\overline{\tau(g_C)}\pi(\tau)}{\tau(1)}}\\ &= \frac{|C|^2}{|G|^2}\sum_{\sigma,\tau\neq 1_G}\pi(\sigma)\overline{\pi(\tau)}\frac{\overline{\sigma(g_C)}\tau(g_C)}{\sigma(1)\tau(1)}.\end{aligned}$$

Dividing both sides by $|C|$ and then taking summations running over all superclasses of G on both sides, one has

$$\begin{aligned}\sum_C\frac{1}{|C|}\left|\pi(\delta_C) - \frac{|C|}{|G|}\pi(1_G)\right|^2 &= \sum_C\frac{|C|}{|G|^2}\sum_{\sigma,\tau\neq 1_G}\pi(\sigma)\overline{\pi(\tau)}\frac{\overline{\sigma(g_C)}\tau(g_C)}{\sigma(1)\tau(1)}\\ &= \frac{1}{|G|}\sum_{\sigma,\tau\neq 1_G}\pi(\sigma)\overline{\pi(\tau)}\frac{1}{|G|}\sum_C|C|\frac{\overline{\sigma(g_C)}\tau(g_C)}{\sigma(1)\tau(1)}\\ &= \frac{1}{|G|}\sum_{\sigma,\tau\neq 1_G}\pi(\sigma)\overline{\pi(\tau)}\frac{1}{|G|}\sum_{g\in G}\frac{\overline{\sigma(g)}\tau(g)}{\sigma(1)\tau(1)}\\ &= \frac{1}{|G|}\sum_{\sigma\neq 1_G}\pi(\sigma)\overline{\pi(\sigma)}\frac{(\sigma,\sigma)}{\sigma(1)\sigma(1)}\\ &= \frac{1}{|G|}\sum_{\sigma\neq 1_G}\frac{|\pi(\sigma)|^2}{\sigma(1)},\end{aligned}$$

where the second last equality is due to the orthogonality property of $\mathrm{Sup}(G)$. □

For the purpose of counting primes, we also need to rewrite estimates described in Proposition 3.5 and Lemma 3.6 in the context of supercharacters as follows. As before, for each unramified prime $\mathfrak{p}$ of k, let $\sigma_{\mathfrak{p}}$ denote the Artin symbol at $\mathfrak{p}$. Let χ be a character of G and let $\pi(x,\chi) = \sum_{N\mathfrak{p}\leq x} \chi(\sigma_{\mathfrak{p}})$ where the sum is over unramified primes $\mathfrak{p}$ of k. Together with the definition of supercharacters, Proposition 3.5 gives:

Proposition 4.8. *Assuming GRH for the Dedekind zeta function of k, one has*

$$\pi(x, 1_G) = \operatorname{Li} x + O\left(x^{\frac{1}{2}}(\log d_k + n_k \log x)\right) + O(n_k \log M(K/k)),$$

where as before, $M(K/k)$ is defined as

$$M(K/k) = n d_k^{\frac{1}{n_k}} \prod_{p \in P(K/k)} p.$$

For any non-trivial supercharacter $\sigma \in \operatorname{Sup}(G)$, if the Artin L-function $L(s, \sigma, K/k)$ is entire and is non-zero for $\Re(s) \neq \frac{1}{2}$ and $0 < \Re(s) < 1$, then

$$\pi(x, \sigma) = O\left(x^{\frac{1}{2}}(\log A_\sigma + \sigma(1) n_k \log x)\right) + O(\sigma(1) n_k \log M(K/k)),$$

where $A_\sigma = d_k^{\sigma(1)} N\mathfrak{f}(\sigma)$ denotes the conductor of σ.

By the properties of Artin conductors, we also have below a generalisation of Lemma 3.6.

Lemma 4.9. *Let σ be a supercharacter of G. Then*

$$\log N\mathfrak{f}(\sigma) \leq 2\sigma(1) n_k \left(\sum_{p \in P(K/k)} \log p + \log n \right).$$

Proof. For any supercharacter σ, one can write σ as

$$\sigma = \sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi(1)\chi,$$

where $\mathrm{Irr}(G,\sigma)$ is the subset of $\mathrm{Irr}(G)$ consisting of all irreducible characters appearing in σ. Since, for any characters χ_1 and χ_2, $\mathfrak{f}(\chi_1 + \chi_2) = \mathfrak{f}(\chi_1)\mathfrak{f}(\chi_2)$, and the (absolute) norm N is completely multiplicative, one has

$$\begin{aligned}
\log N\mathfrak{f}(\sigma) &= \log N\mathfrak{f}\left(\sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi(1)\chi\right) \\
&= \log N\left(\prod_{\chi \in \mathrm{Irr}(G,\sigma)} \mathfrak{f}(\chi)^{\chi(1)}\right) \\
&= \sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi(1) \log N\mathfrak{f}(\chi).
\end{aligned}$$

Therefore, Lemma 3.6 implies that

$$\begin{aligned}
\log N\mathfrak{f}(\sigma) &= \sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi(1) \log N\mathfrak{f}(\chi) \\
&\leq \sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi(1) \left(2\chi(1)n_k\left(\sum_{p \in P(K/k)} \log p + \log n\right)\right) \\
&= 2\sum_{\chi \in \mathrm{Irr}(G,\sigma)} \chi^2(1) \left(n_k \sum_{p \in P(K/k)} \log p + \log n\right) \\
&= 2\sigma(1)n_k\left(\sum_{p \in P(K/k)} \log p + \log n\right).
\end{aligned}$$

□

Using the previous results, one can establish an *effective* version of the Chebotarev density theorem for *any* supercharacter theory as follows.

Theorem 4.10. *Suppose that all Artin L-functions attached to supercharacters of* $G = \mathrm{Gal}(K,k)$ *are holomorphic at* $s \neq 1$*, and that GRH holds for* $\zeta_K(s)$*. Then*

$$\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C) - \frac{|C|}{|G|}\operatorname{Li} x\right|^2 \ll x n_k^2 \log^2\left(M(K/k)x\right),$$

where the sum on the left runs over superclasses C *of* G*.*

Proof. First, observe that

$$\begin{aligned}\sum_C \frac{1}{|C|}\left|\frac{|C|}{|G|}\pi(x,1_G) - \frac{|C|}{|G|}\operatorname{Li} x\right|^2 &= \frac{1}{|G|^2}\sum_C |C|(\pi(x,1_G) - \operatorname{Li} x)^2 \\ &= \frac{1}{|G|}(\pi(x,1_G) - \operatorname{Li} x)^2.\end{aligned}$$

Applying Lemma 4.7, one has

$$\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C) - \frac{|C|}{|G|}\pi(x,1_G)\right|^2 = \frac{1}{|G|}\sum_{\sigma\neq 1_G}\frac{|\pi(x,\sigma)|^2}{\sigma(1)}.$$

On the other hand, for all non-trivial supercharacters σ of G, Proposition 4.8 gives

$$\pi(x,\sigma) = O\left(x^{\frac{1}{2}}(\log A_\sigma + \sigma(1)n_k\log x)\right) + O\left(\sigma(1)n_k\log M(K/k)\right),$$

and according to Lemma 4.9, this becomes

$$\pi(x,\sigma) \ll x^{\frac{1}{2}}\sigma(1)n_k\log\left(M(K/k)x\right).$$

Putting everything together and using the Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C)-\frac{|C|}{|G|}\operatorname{Li}x\right|^2\\
&=\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C)-\frac{|C|}{|G|}\pi(x,1_G)+\frac{|C|}{|G|}\pi(x,1_G)-\frac{|C|}{|G|}\operatorname{Li}x\right|^2\\
&\le\sum_C \frac{2}{|C|}\left|\pi(x,\delta_C)-\frac{|C|}{|G|}\pi(x,1_G)\right|^2+\sum_C \frac{2}{|C|}\left|\frac{|C|}{|G|}\pi(x,1_G)-\frac{|C|}{|G|}\operatorname{Li}x\right|^2\\
&=\frac{2}{|G|}\sum_{\sigma\neq 1_G}\frac{|\pi(x,\sigma)|^2}{\sigma(1)}+\frac{2}{|G|}(\pi(x,1_G)-\operatorname{Li}x)^2\\
&\ll\frac{1}{|G|}\sum_{\sigma\in\operatorname{Sup}(G)}x\sigma(1)n_k^2\log^2\left(M(K/k)x\right)\\
&=xn_k^2\log^2\left(M(K/k)x\right),
\end{aligned}$$

where the last equality holds since $|G|=\sum_{\chi\in\operatorname{Irr}(G)}\chi^2(1)=\sum_{\sigma\in\operatorname{Sup}(G)}\sigma(1)$. □

Corollary 4.11. *Under the same assumptions as the previous theorem,*

$$\pi(x,\delta_D)=\frac{|D|}{|G|}\operatorname{Li}x+O(|D|^{\frac{1}{2}}x^{\frac{1}{2}}n_k\log\left(M(K/k)x\right)),$$

where D is an arbitrary union of superclasses in G.

Proof. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\pi(x,\delta_D)-\frac{|D|}{|G|}\operatorname{Li}x\right|&=\left|\sum_{C\subseteq D}\left(\pi(x,\delta_C)-\frac{|C|}{|G|}\operatorname{Li}x\right)\frac{|C|^{\frac{1}{2}}}{|C|^{\frac{1}{2}}}\right|\\
&\le\left(\sum_{C\subseteq D}\frac{1}{|C|}\left|\pi(x,\delta_C)-\frac{|C|}{|G|}\operatorname{Li}x\right|^2\right)^{\frac{1}{2}}\left(\sum_{C\subseteq D}|C|\right)^{\frac{1}{2}}\\
&\ll\left(xn_k^2\log^2\left(M(K/k)x\right)\right)^{\frac{1}{2}}|D|^{\frac{1}{2}},
\end{aligned}$$

where the sums run over superclasses $C \subseteq D$. □

Remark 4.12. *As discussed in Section 4.2, the assumption of the Artin conjecture in our effective Chebotarev density theorem is automatically satisfied if one chooses the maximal theory or the max-min theory. We remark that these choices are sufficient for some arithmetic applications. For instance, to study the cyclicity problem of elliptic curves (modulo p), a key ingredient is the explicit formulae for*

$$\pi_1(x, \mathbb{Q}(E[m])/\mathbb{Q}) = \#\{p \leq x \mid p \text{ splits completely in } \mathbb{Q}(E[m])/\mathbb{Q}\},$$

where $\mathbb{Q}(E[m])$ *is the* m*-division field of an elliptic curve* $E/\mathbb{Q}$ *(cf. [57] and [12]). Since* $\pi_1(x, \mathbb{Q}(E[m])/\mathbb{Q}) = \pi(x, \delta_{\{1\}})$*, the number of primes* $p \leq x$ *with* $\sigma_p = \{e\}$*, and* $\{e\}$ *is a superclass, we can choose the maximal theory to get a desired estimate.*

Chapter 5

A Variant of Heilbronn Characters

5.1 Weak Arithmetic Heilbronn Characters

In this section, we will introduce *weak arithmetic Heilbronn characters* that generalise the classical Heilbronn characters, and we will discuss their properties.

From now on, G always denotes a finite group. For any subgroup H of G, we denote the trivial character and the regular representation of H by 1_H and Reg_H, respectively. In addition, $\langle h \rangle$ denotes the cyclic subgroup of H generated by an element $h \in H$, and e_H is the identity element of H.

Definition 5.1. *Let $I(G)$ be a set defined as*

$$I(G) = \{(H, \phi) \mid H \leq G \text{ is proper and cyclic or } H = G, \text{ and } \phi \text{ is a character of } H\},$$

and $n : I(G) \to \mathbb{Z}$ be a function satisfying the following three properties:

WAHC1. *$n(H, \phi_1 + \phi_2) = n(H, \phi_1) + n(H, \phi_2)$ for any characters ϕ_1 and ϕ_2 of H, where H is a cyclic subgroup or an improper subgroup of G;*

WAHC2. *$n(G, \mathrm{Ind}_H^G \phi) = n(H, \phi)$ for every cyclic subgroup H and every character*

ϕ of H; and

WAHC3. *$n(H,\phi) \geq 0$ for all cyclic subgroups H of G and all characters ϕ of H.*

Then the weak arithmetic Heilbronn character of a proper cyclic or improper subgroup H of G associated with such $n(H,\phi)$'s is defined by

$$\Theta_H = \sum_{\phi \in \mathrm{Irr}(H)} n(H,\phi)\phi,$$

which by condition WAHC2, is equal to $\sum_{\phi \in \mathrm{Irr}(H)} n(G, \mathrm{Ind}_H^G \phi)\phi$.

Such a formalism technique was used by Foote in [18] as well as by M. R. Murty and V. K. Murty in [46] to study certain L-functions. However, we will see such "abstract" Heilbronn characters are of interest in their own right. In fact, weak Heilbronn arithmetic characters and *arithmetic Heilbronn characters*, which will be discussed in the next section, inherit many properties of the classical Heilbronn characters. For instance, these Heilbronn characters also admit an *Artin-Takagi decomposition.*

Proposition 5.1 (Artin-Takagi Decomposition)**.**

$$n(G, \mathrm{Reg}_G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) n(G,\chi).$$

Proof. Since $\mathrm{Reg}_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$, the decomposition follows simply from condition WAHC1. □

By conditions WAHC2 and WAHC3, one can see $n(G,\chi) \geq 0$ for any character χ of G induced from a character of a cyclic subgroup of G. Also, condition WAHC2 implies a stronger condition: $n(\widetilde{H}, \mathrm{Ind}_H^{\widetilde{H}} \phi) = n(H,\phi)$ for any cyclic subgroup $\widetilde{H}$ of G

containing H, since

$$n(\widetilde{H}, \operatorname{Ind}_H^{\widetilde{H}} \phi) = n(G, \operatorname{Ind}_{\widetilde{H}}^G \operatorname{Ind}_H^{\widetilde{H}} \phi) = n(G, \operatorname{Ind}_H^G \phi) = n(H, \phi).$$

Now we shall state and prove several properties of weak arithmetic Heilbronn characters. Our methods are based on earlier works of Heilbronn, Stark, Foote, and V. K. Murty.

Lemma 5.2 (Heilbronn-Stark Lemma)**.** *Assume Θ_G is a weak arithmetic Heilbronn character. Then, for every cyclic subgroup H of G, one has*

$$\Theta_G|_H = \Theta_H.$$

Proof. By the definition, the first orthogonality property of irreducible characters, and Frobenius reciprocity, we have

$$\begin{aligned}
\Theta_G|_H &= \sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)\chi|_H \\
&= \sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi) \sum_{\phi \in \operatorname{Irr}(H)} (\chi|_H, \phi)\phi \\
&= \sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi) \sum_{\phi \in \operatorname{Irr}(H)} (\chi, \operatorname{Ind}_H^G \phi)\phi \\
&= \sum_{\phi \in \operatorname{Irr}(H)} \left(\sum_{\chi \in \operatorname{Irr}(G)} (\chi, \operatorname{Ind}_H^G \phi) n(G, \chi) \right) \phi.
\end{aligned}$$

Now we use conditions WAHC1 and WAHC2, and the first orthogonality property of irreducible characters again to get

$$\begin{aligned}\Theta_G|_H &= \sum_{\phi\in\mathrm{Irr}(H)} n\left(G, \sum_{\chi\in\mathrm{Irr}(G)} (\chi, \mathrm{Ind}_H^G \phi)\chi\right)\phi \\ &= \sum_{\phi\in\mathrm{Irr}(H)} n(G, \mathrm{Ind}_H^G \phi)\phi \\ &= \Theta_H.\end{aligned}$$

□

Like the classical Heilbronn-Stark lemma, the above lemma enables us to bound the coefficients of our Heilbronn characters.

Theorem 5.3.

$$\sum_{\chi\in\mathrm{Irr}(G)} n(G,\chi)^2 \le n(G, \mathrm{Reg}_G)^2.$$

Proof. We will give a proof based on the method developed in [19] and [46]. By the first orthogonality property and the definition of the (usual) inner product of class functions of G, one has

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(G)} n(G,\chi)^2 &= (\Theta_G, \Theta_G) \\ &= \frac{1}{|G|}\sum_{g\in G} |\Theta_G(g)|^2.\end{aligned}$$

Applying the Heilbronn-Stark lemma, for any $g \in G$, one has

$$\begin{aligned}\Theta_G(g) &= \Theta_{\langle g\rangle}(g) \\ &= \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} n(\langle g\rangle, \phi)\phi(g).\end{aligned}$$

Since $\langle g\rangle$ is cyclic, the triangle inequality and conditions WAHC2 and WAHC3 yield

$$\begin{aligned}|\Theta_G(g)| &= |\Theta_{\langle g\rangle}(g)| \\ &\leq \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} n(\langle g\rangle,\phi) \\ &= n\left(\langle g\rangle, \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)}\phi\right) \\ &= n(\langle g\rangle, \mathrm{Reg}_{\langle g\rangle}) \\ &= n(G,\mathrm{Reg}_G).\end{aligned}$$

Therefore, the theorem follows. □

Using this theorem and the fact that $n(G,\mathrm{Reg}_G) = n(G,\mathrm{Ind}_{\langle e_G\rangle}^G 1_{\langle e_G\rangle}) \geq 0$, one can immediately obtain the following analogues of famous theorems of Aramata-Brauer and Stark as mentioned earlier.

Corollary 5.4. $n(G,\mathrm{Reg}_G) \pm n(G,1_G) \geq 0$.

Corollary 5.5. *If $n(G,\mathrm{Reg}_G) \leq 1$, then $n(G,\chi) \geq 0$ for all irreducible characters χ of G.*

Proof. If $n(G,\mathrm{Reg}_G) = 0$, then the corollary follows from the above theorem immediately. Otherwise, for $n(G,\mathrm{Reg}_G) = 1$, by the Artin-Takagi decomposition, Proposition 5.1, one has

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)n(G,\chi) = n(G,\mathrm{Reg}_G) = 1.$$

In addition, Theorem 5.3 forces that all integers $n(G,\chi)$ are bounded by 1. Thus, we can conclude that there is exactly one character χ_0 of G such that $\chi_0(1) = 1$ and $n(G,\chi_0) = 1$. In other words, $n(G,\chi) = 0$ for any irreducible character $\chi \neq \chi_0$. □

In [46], M. R. Murty and V. K. Murty showed the following "twisting" result by using a formalism technique. We shall give a proof below by just checking that such twisting indeed defines a set of integers satisfying conditions WAHC1 to WAHC3.

Proposition 5.6. *Let $n(H,\phi)$'s be integers defining a weak Heilbronn character, i.e., these integers satisfy conditions WAHC1 to WAHC3. Let ρ be an arbitrary character of G. Suppose that for every cyclic subgroup H of G and irreducible character ϕ of H, we have $n(H, \rho|_H \otimes \phi) \geq 0$, then*

$$\sum_{\chi \in \mathrm{Irr}(G)} n(G, \rho \otimes \chi)^2 \leq n(G, \rho \otimes \mathrm{Reg}_G)^2.$$

Proof. For every cyclic subgroup H of G (or $H = G$) and every character ϕ of H, let $n'(H,\phi) = n(H, \rho|_H \otimes \phi)$. By the linearity of tensor product and the hypothesis of this theorem, it is easy to see that $n'(H,\phi)$'s satisfy conditions WAHC1 and WAHC3. On the other hand, since tensoring "commutes" with induction, we have

$$\begin{aligned} n'(H,\phi) &= n(H, \rho|_H \otimes \phi) \\ &= n(G, \mathrm{Ind}_H^G(\rho|_H \otimes \phi)) \\ &= n(G, \rho \otimes \mathrm{Ind}_H^G \phi) \\ &= n'(G, \mathrm{Ind}_H^G \phi). \end{aligned}$$

Therefore, this proposition follows from Theorem 5.3 immediately. □

5.2 Arithmetic Heilbronn Characters

In this section, we will put more conditions on $n(H,\phi)$'s, which make weak arithmetic Heilbronn characters capture almost all properties that we know for the classical

Heilbronn characters.

Definition 5.2. *Let $I(G)$ be a set defined as*

$$I(G) = \{(H, \phi) \mid H \text{ is a subgroup of } G, \text{ and } \phi \text{ is a character of } H\},$$

and $n : I(G) \to \mathbb{Z}$ be a function satisfying the following three properties:

AHC1. *$n(H, \phi_1 + \phi_2) = n(H, \phi_1) + n(H, \phi_2)$ for any subgroup H of G and any characters ϕ_1 and ϕ_2 of H;*

AHC2. *$n(G, \operatorname{Ind}_H^G \phi) = n(H, \phi)$ for every character ϕ of every subgroup H; and*

AHC3. *$n(H, \phi) \geq 0$ for all 1-dimensional characters ϕ of subgroups H of G.*

Then the arithmetic Heilbronn character of a subgroup H of G associated with such $n(H, \phi)$'s is defined as

$$\Theta_H = \sum_{\phi \in \operatorname{Irr}(H)} n(H, \phi)\phi,$$

which by condition AHC2, is equal to $\sum_{\phi \in \operatorname{Irr}(H)} n(G, \operatorname{Ind}_H^G \phi)\phi$.

It is clear that all arithmetic Heilbronn characters have properties discussed in the previous section. Moreover, since $n(H, \phi)$'s are now defined for all subgroups H of G, we have the following full-powered Heilbronn-Stark Lemma.

Lemma 5.7 (Heilbronn-Stark Lemma)**.** *For every subgroup H of G, one has*

$$\Theta_G|_H = \Theta_H.$$

Remark 5.8. *As pointed out by Professor Mike Roth (private communication), conditions AHC1-2 are equivalent to:*

AHC'. *Choose an integer q_i for each irreducible character χ_i of G.*

Proof (due to M. Roth). Let χ_i's be the irreducible characters of G. Given a function $n : I(G) \to \mathbb{Z}$ satisfying conditions AHC1-2, we set $q_i = n(G, \chi_i)$ for each i. Then for any character ϕ of G, as $\phi = \sum_i m_i \chi_i$ for some $m_i \geq 0$, condition AHC1 then gives that $n(G, \phi) = \sum_i m_i q_i$. Moreover, condition AHC2 says that $n(H, \phi) = n(G, \operatorname{Ind}_H^G \phi)$, which is already determined by the q_i's.

Thus, conditions AHC1-2 give q_i's, and conversely, from the above discussion, it is clear that any choice of q_i's gives a function $n : I(G) \to \mathbb{Z}$ satisfying conditions AHC1-2. □

As one can see now, conditions AHC1-2 are not really axioms, but rather a choice, the choice of a virtual character $\Theta_G = \sum_i q_i \chi_i$ with $q_i \in \mathbb{Z}$. Also, by the above discussion, one may replace conditions AHC1-2 by the formula

$$n(H, \phi) = (\operatorname{Ind}_H^G \phi, \Theta_G). \tag{5.1}$$

Therefore, one may form the function $n : I(G) \to \mathbb{Z}$ by the choice of a virtual character of G together with the formula (5.1). *Furthermore, with the same consideration, one may also define*

$$\Theta_H = \Theta_G|_H.$$

We further remark that this argument is also valid for weak arithmetic Heilbronn characters. The subtle difference between the two cases, which is really an axiom, is the condition WAHC3 or AHC3, requiring non-negativity on characters induced from 1-dimensional characters. Furthermore, considering the following two conditions:

WAHC. $(\mathrm{Ind}_H^G \phi, \cdot) \geq 0$ *for all cyclic subgroups* $H \leq G$ *and all 1-dimensional characters* ϕ *of* H*; and*

AHC. $(\mathrm{Ind}_H^G \phi, \cdot) \geq 0$ *for all subgroups* $H \leq G$ *and all 1-dimensional characters* ϕ *of* H*,*

each defines a cone in $\mathbf{C}(G)$*, the space of class functions of* G*. Moreover, axioms WAHC3 and AHC3 respectively are requiring* Θ_G *to be in this cone. It might be interesting to understand these cones in a few cases.*

Finally, as suggested by M. Roth, it is possible to simplify or shorten most of the proofs via "Θ_G*-perspective". For instance, formulas of type* $\sum n(G,\chi)^2$ *can be replaced by* (Θ_G, Θ_G)*, the "Artin-Takagi decomposition" is the linearity of the function* $(\cdot, \Theta_G)$*, etc.*

From now on, Θ_G always denotes an arithmetic Heilbronn character of G. Furthermore, we assume G is *solvable.* The following powerful lemma is essentially due to the work of Uchida and van der Waall, which is used by M. R. Murty and V. K. Murty [46] implicitly and is stated precisely in [49, Lemma 2.4].

Lemma 5.9. *Let* G *be a finite solvable group, and let* H *be a subgroup of* G*. Then*

$$\mathrm{Ind}_H^G 1_H = 1_G + \sum_i \mathrm{Ind}_{H_i}^G \phi_i,$$

where ϕ_i*'s are non-trivial 1-dimensional characters of some subgroups* H_i*'s of* G*.*

Following [49], we let $G^0 = G$, and define G^i to be $[G^{i-1}, G^{i-1}]$ for all $i \geq 1$. The series $\{G^i\}$ is called the *derived series* of G. Since G is solvable, such a series is eventually *trivia*l. Using this series, one may define the *level* of an irreducible character χ of G, denoted $l(\chi)$, as the least non-negative integer n such that χ is trivial on G^n. For instance, the level one characters are exactly the non-trivial 1-dimensional characters of G. In addition, M. R. Murty and Raghuram showed a stronger version of Lemma 5.9 (cf. [49, Lemma 2.5]).

Lemma 5.10. *Let G be a finite solvable group having more than one element, and let H be a subgroup of G. Let $\{G^i\}$ denote the derived series of G, and let m be the least non-negative integer such that $G^{m+1} = \langle e_G \rangle$. Then for all $i \geq 1$,*

$$\mathrm{Ind}_H^G 1_H = \mathrm{Ind}_{HG^i}^G 1_{HG^i} + \sum_j \mathrm{Ind}_{H_j}^G \phi_j,$$

where ϕ_j's are non-trivial 1-dimensional characters of some subgroups H_j's of G, and the sum might be empty.

Using these lemmas and the method developed in [49], we can prove the following sequence of properties for arithmetic Heilbronn characters.

Proposition 5.11. *Let H be a subgroup of G. Let χ and ϕ be 1-dimensional characters of G and H respectively. Then*

$$n(G, \mathrm{Ind}_H^G \phi) - (\chi|_H, \phi) n(G, \chi) \geq 0.$$

Proof. Note that if $(\chi|_H, \phi) = 0$, the theorem is clearly true by conditions AHC2 and AHC3. Suppose that $(\chi|_H, \phi) > 0$. Since both χ and ϕ are 1-dimensional, we obtain $\chi|_H = \phi$ and $(\chi|_H, \phi) = 1$. Following the proof of [49, Theorem 4.1], by Lemma 5.9, we first write

$$\mathrm{Ind}_H^G 1_H = 1_G + \sum \mathrm{Ind}_{H_i}^G \phi_i,$$

where ϕ_i's are non-trivial 1-dimensional characters of some subgroups H_i's of G. Since tensoring and induction "commute", by tensoring χ on the both sides of the above equation, we then get

$$\mathrm{Ind}_H^G \chi|_H = \chi + \sum \mathrm{Ind}_{H_i}^G (\chi|_{H_i}\phi_i).$$

As $\chi|_{H_i}\phi_i$'s are still 1-dimensional, by condition AHC3, $n(H, \chi|_{H_i}\phi_i) \geq 0$ for all i. Hence, the theorem follows from condition AHC2 and the fact that $(\chi|_H, \phi) = 1$ and $\chi|_H = \phi$. □

For any subgroup H of G, by taking $\chi = 1_G$ and $\phi = 1_H$, one can deduce an analogue of the Uchida-van der Waall theorem as below.

Corollary 5.12. *Let G be a solvable group, and H a subgroup. One has*

$$n(G, \mathrm{Ind}_H^G 1_H) - n(G, 1_G) \geq 0.$$

Moreover, by applying Lemma 5.10 and Proposition 5.11, it is possible to derive several analogues of M. R. Murty and Raghuram's results for arithmetic Heilbronn characters.

Theorem 5.13. *Let χ_0 be a 1-dimensional character of G. Then*

$$\sum_{\chi \in \mathrm{Irr}(G)\setminus\{\chi_0\}} n(G,\chi)^2 \leq (n(G, \mathrm{Reg}_G) - n(G,\chi_0))^2.$$

Proof. In light of the proof of [49, Theorem 4.4], we define a "truncated" (arithmetic)

Heilbronn character with respect to χ_0 as

$$\Theta_G^{\chi_0} = \sum_{\chi \in \mathrm{Irr}(G)\setminus\{\chi_0\}} n(G,\chi)\chi.$$

Taking norms on both sides of the above equation, one has

$$\begin{aligned} |\Theta_G^{\chi_0}|^2 &= \frac{1}{|G|}\sum_{g\in G}|\Theta_G^{\chi_0}(g)|^2 \\ &= \sum_{\chi \in \mathrm{Irr}(G)\setminus\{\chi_0\}} n(G,\chi)^2. \end{aligned}$$

On the other hand, by the Heilbronn-Stark lemma, Lemma 5.7, we have

$$\begin{aligned} \Theta_G^{\chi_0}(g) &= \Theta_G(g) - n(G,\chi_0)\chi_0(g) \\ &= \Theta_{\langle g\rangle}(g) - n(G,\chi_0)\chi_0(g) \\ &= \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} n(\langle g\rangle,\phi)\phi(g) - n(G,\chi_0)\sum_{\phi\in\mathrm{Irr}(\langle g\rangle)}(\chi_0|_{\langle g\rangle},\phi)\phi(g) \\ &= \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)}(n(\langle g\rangle,\phi) - n(G,\chi_0)(\chi_0|_{\langle g\rangle},\phi))\phi(g). \end{aligned}$$

Applying Proposition 5.11 with $H = \langle g\rangle$ and $\phi \in \mathrm{Irr}(\langle g\rangle)$, we get

$$n(\langle g\rangle,\phi) - n(G,\chi_0)(\chi_0|_{\langle g\rangle},\phi) \geq 0,$$

which combining with the triangle inequality gives

$$\begin{aligned}
|\Theta_G^{\chi_0}(g)| &\leq \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} (n(\langle g\rangle,\phi) - n(G,\chi_0)(\chi_0|_{\langle g\rangle},\phi)) \\
&= n\left(\langle g\rangle, \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} \phi\right) - n\left(G, \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)} (\chi_0|_{\langle g\rangle},\phi)\chi_0\right) \\
&= n(\langle g\rangle, \mathrm{Reg}_{\langle g\rangle}) - n(G, (\chi_0|_{\langle g\rangle}, \mathrm{Reg}_{\langle g\rangle})\chi_0) \\
&= n(G, \mathrm{Reg}_G) - n(G,\chi_0),
\end{aligned}$$

where the last equality holds provided that $(\chi_0|_{\langle g\rangle}, \mathrm{Reg}_{\langle g\rangle}) = \chi_0|_{\langle g\rangle}(1) = 1$. □

Proposition 5.14. *Let H be a subgroup of G, and let ϕ be any 1-dimensional character of H. Let S_ϕ denote the set of all 1-dimensional characters of G whose restrictions on H are ϕ. Then*

$$n(G, \mathrm{Ind}_H^G \phi) - \sum_{\chi\in S_\phi} n(G,\chi) \geq 0.$$

Proof. Note that if S_ϕ is empty, then the theorem is obviously true by conditions AHC2 and AHC3. Now we may assume S_ϕ is non-empty, and take $\chi_0 \in S_\phi$. Applying Lemma 5.10 with $i = 1$, we have

$$\mathrm{Ind}_H^G 1_H = \mathrm{Ind}_{HG^1}^G 1_{HG^1} + \sum \mathrm{Ind}_{H_j}^G \phi_j,$$

where for each j, ϕ_j is a non-trivial 1-dimensional character of a subgroup H_j of G, and the sum might be empty. Again, twisting the above equation by χ_0, we have

$$\mathrm{Ind}_H^G \phi = \mathrm{Ind}_{HG^1}^G \chi_0|_{HG^1} + \sum \mathrm{Ind}_{H_i}^G (\chi_0|_{H_i}\phi_i).$$

Since $\chi_0|_{H_i}\phi_i$'s are still 1-dimensional and $\mathrm{Ind}_{HG^1}^G \chi_0|_{HG^1}$ is exactly $\sum_{\chi\in S_\phi}\chi$, the proposition follows. $\square$

Theorem 5.15. *Let S be the set of all 1-dimensional characters of G. Then*

$$\sum_{\chi\in\mathrm{Irr}(G)\setminus S} n(G,\chi)^2 \le (n(G,\mathrm{Reg}_G) - n(G,\mathrm{Ind}_{G^1}^G 1_{G^1}))^2.$$

Proof. Following the proof of [49, Theorem 5.3], we define a truncated arithmetic Heilbronn character with respect to S as

$$\Theta_G^S = \sum_{\chi\in\mathrm{Irr}(G)\setminus S} n(G,\chi)\chi.$$

Taking norms on both sides of the above equation gives

$$\frac{1}{|G|}\sum_{g\in G}|\Theta_G^S(g)|^2 = \sum_{\chi\in\mathrm{Irr}(G)\setminus S} n(G,\chi)^2.$$

Thanks to the Heilbronn-Stark lemma, Lemma 5.7, we have

$$\begin{aligned}\Theta_G^S(g) &= \Theta_G(g) - \sum_{\chi\in S} n(G,\chi)\chi(g)\\ &= \Theta_{\langle g\rangle}(g) - \sum_{\chi\in S} n(G,\chi)\chi(g)\\ &= \sum_{\phi\in\mathrm{Irr}(\langle g\rangle)}\left(n(\langle g\rangle,\phi) - \sum_{\chi\in S} n(G,\chi)(\chi,\mathrm{Ind}_{\langle g\rangle}^G\phi)\right)\phi(g).\end{aligned}$$

Using Proposition 5.14 with $H = \langle g\rangle$ and $\phi\in\mathrm{Irr}(\langle g\rangle)$, we then obtain

$$n(G,\mathrm{Ind}_{\langle g\rangle}^G\phi) - \sum_{\chi\in S_\phi} n(G,\chi) \ge 0.$$

Observe that for every $\chi \in S$, $(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi)$ is either 0 or 1, and that $(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi) = 1$ if and only if $\chi \in S_\phi$. Thus, by condition AHC2, we may rewrite the above inequality as

$$n(\langle g\rangle, \phi) - \sum_{\chi \in S} n(G, \chi)(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi) \geq 0.$$

Finally, by the triangle inequality and the fact that for $\chi \in S$, $(\chi|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}) = 1$, and $\operatorname{Ind}_{G^1}^{G} 1_{G^1} = \sum_{\chi\in S} \chi$, one can deduce

$$\begin{aligned} |\Theta_G^S(g)| &\leq \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} \left(n(\langle g\rangle, \phi) - \sum_{\chi\in S} n(G, \chi)(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi) \right) \\ &= n(\langle g\rangle, \operatorname{Reg}_{\langle g\rangle}) - \sum_{\chi \in S} n(G,\chi)(\chi|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}) \\ &= n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^1}^{G} 1_{G^1}), \end{aligned}$$

which completes the proof. □

Corollary 5.16. *Let G be a solvable group. Then $n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^1}^{G} 1_{G^1})$ cannot be* 1.

Proof. Observe that $\operatorname{Reg}_G = \operatorname{Ind}_{G^1}^{G} 1_{G^1} + \sum_{\chi\notin S} \chi(1)\chi$ where S denotes the set of all 1-dimensional characters of G. If $n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^1}^{G} 1_{G^1})$ was equal to 1, then conditions AHC1 and AHC2 tell us that $\sum_{\chi\notin S} \chi(1) n(G, \chi) = 1$. However, Theorem 5.15 forces that there is at most one character $\chi' \notin S$ of G such that $n(G, \chi')$ is non-zero. In addition, the Artin-Takagi decomposition, Proposition 5.1, asserts that there should be a character $\chi' \notin S$ such that $n(G, \chi')$ is non-zero. But $\chi'(1) \geq 2$, which contradicts to the fact that $\chi'(1) n(G, \chi') = \sum_{\chi\notin S} \chi(1) n(G, \chi) = 1$. □

In [39, Lemma 3.2], Lansky and Wilson generalised results of M. R. Murty and Raghuram (cf. Lemma 5.10) by proving the following.

Lemma 5.17. *Let G be a finite solvable group, and let H be a subgroup of G. Let ϕ be a 1-dimensional character of H such that $\phi|_{H\cap G^i}$ is trivial, and let ϕ' be the unique extension of ϕ to a character of HG^i that is trivial on G^i. Then for any irreducible character χ of G, one has*

$$(\chi, \operatorname{Ind}_{HG^i}^G \phi') = \begin{cases} (\chi, \operatorname{Ind}_H^G \phi), & \text{if } l(\chi) \leq i, \\ 0, & \text{if } l(\chi) > i. \end{cases}$$

Adapting the method developed by Lansky and Wilson, it is possible now to obtain a generalisation of M. R. Murty and Raghuram's work in the setting of arithmetic Heilbronn characters as follows.

Proposition 5.18. *Let d be the greatest common divisor of the degrees of the characters in $\operatorname{Irr}(G)\backslash S^i$, where S^i denotes the set of irreducible characters of G of level less than or equal to i. Then $n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^i}^G 1_{G^i}) = kd$ for some non-negative integer k.*

Proof. By conditions AHC1 and AHC2, and Lemma 5.17 with $H = \langle e_G \rangle$, we have

$$\begin{aligned} n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^i}^G 1_{G^i}) &= n(G, \operatorname{Reg}_G) - \sum_{\chi \in S^i} \chi(1) n(G, \chi) \\ &= \sum_{\chi \in \operatorname{Irr}(G)\backslash S^i} \chi(1) n(G, \chi), \end{aligned}$$

which is a multiple of the greatest common divisor of the degrees of the characters χ of G with $l(\chi) > i$. Since the Aramata-Brauer theorem asserts that

$$n(G^i, \operatorname{Reg}_{G^i}) - n(G^i, 1_{G^i}) \geq 0,$$

by condition AHC2, we obtain $n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^i}^G 1_{G^i}) \geq 0$, which completes the proof. □

Proposition 5.19. *Let ϕ be a 1-dimensional character of a subgroup H of G. Then*

$$n(G, \operatorname{Ind}_H^G \phi) - \sum_{\chi \in S^i} (\chi, \operatorname{Ind}_H^G \phi) n(G, \chi) \geq 0,$$

where S^i denotes the set of irreducible characters of G of level less than or equal to i.

Proof. The proof is exactly the same as the proof in [39], but for the sake of completeness and clarity, we shall reproduce a proof in our setting. Firstly, we assume ϕ is trivial on $H \cap G^i$, then ϕ extends uniquely to a character ϕ' of $H \cdot G^i$. Now Lemma 5.17 implies that

$$\begin{aligned}\sum_{\chi \in S^i} (\chi, \operatorname{Ind}_H^G \phi) n(G, \chi) &= \sum_{\chi \in \operatorname{Irr}(G)} (\chi, \operatorname{Ind}_{HG^i}^G \phi') n(G, \chi) \\ &= n(G, \operatorname{Ind}_{HG^i}^G \phi').\end{aligned}$$

By Lemma 5.9, we have

$$\operatorname{Ind}_H^{HG^i} 1_H = 1_{H \cdot G^i} + \sum_j \operatorname{Ind}_{H_j}^{HG^i} \phi_j,$$

where ϕ_j's are non-trivial 1-dimensional characters of some subgroups H_j's of $H \cdot G^i$, and the sum might be empty. By twisting the above equation by ϕ', using the fact that tensoring and induction commute, and inducing everything to G, one has

$$\operatorname{Ind}_H^G \phi = \operatorname{Ind}_{HG^i}^G \phi' + \sum_j \operatorname{Ind}_{H_j}^G \phi'|_{H_j} \phi_j.$$

Thus, the theorem follows in this case that ϕ is trivial on $H \cap G^i$.

We remark that none of $\phi'|_{H_j}\phi_j$'s is trivial. If $\phi \neq 1_H$, then $(1_G, \operatorname{Ind}_H^G \phi) = 0$, and thus 1_G does not occur. On the other hand, if $\phi = 1_H$, then Lemma 5.17 and Frobenius reciprocity imply that $(1_G, \operatorname{Ind}_{HG^i}^G \phi') = (1_G, \operatorname{Ind}_H^G \phi) = 1$, and thus 1_G cannot occur in the summation in the above equation.

For the case that ϕ is non-trivial on $H \cap G^i$, Mackey's theorem (see, for example, [5, Sections 5.3 and 5.12]) and Frobenius reciprocity tell us that

$$\begin{aligned} ((\operatorname{Ind}_H^G \phi)|_{G^i}, 1_{G^i}) &= \sum_{G^i \backslash G/H} (\operatorname{Ind}_{xHx^{-1}\cap G^i}^{G^i} \phi^x, 1_{G^i}) \\ &= \sum_{G^i \backslash G/H} (\phi^x, 1_{xHx^{-1}\cap G^i}) \\ &= \sum_{G^i \backslash G/H} (\phi, 1_{H\cap G^i}) \\ &= 0, \end{aligned}$$

where for every $x \in G$, ϕ^x denotes the character of $xHx^{-1}\cap G^i$ given by $g \mapsto \phi(x^{-1}gx)$. Thus, $\operatorname{Ind}_H^G \phi$ contains no characters of level less than or equal to i, which means that $n(G, \operatorname{Ind}_H^G \phi) - \sum_{\chi\in S^i}(\chi, \operatorname{Ind}_H^G \phi)n(G,\chi) = n(G, \operatorname{Ind}_H^G \phi)$ in this case. Now the proposition follows from conditions AHC2 and AHC3. $\square$

Corollary 5.20. *Let ϕ_0 be a 1-dimensional character of a subgroup H of G, and $S^i_{\phi_0}$ the set of irreducible characters of level i occurring in $\operatorname{Ind}_H^G \phi_0$. Then*

$$\sum_{\chi\in S^i_{\phi_0}} (\chi, \operatorname{Ind}_H^G \phi_0) n(G,\chi) \geq 0.$$

Proof. If ϕ_0 is non-trivial on $H \cap G^i$, the last paragraph of the proof of Proposition

5.19 gives $(\chi, \operatorname{Ind}_H^G \phi_0) = 0$ for all $\chi \in S^i_{\phi_0}$, and the corollary follows immediately. Otherwise, ϕ_0 extends uniquely to a character ϕ of HG^i which is trivial on G^i. Then Proposition 5.19 (by replacing H and i by $H \cdot G^i$ and $i-1$, respectively) implies that

$$n(G, \operatorname{Ind}_{HG^i}^G \phi) - \sum_{\chi \in S^{i-1}} (\chi, \operatorname{Ind}_{HG^i}^G \phi) n(G, \chi) \geq 0.$$

By Lemma 5.17, the above difference is equal to

$$\sum_{\chi \in S^i} (\chi, \operatorname{Ind}_H^G \phi_0) n(G, \chi) - \sum_{\chi \in S^{i-1}} (\chi, \operatorname{Ind}_H^G \phi_0) n(G, \chi) = \sum_{\chi \in S^i_{\phi_0}} (\chi, \operatorname{Ind}_H^G \phi_0) n(G, \chi),$$

where S^j denotes the set of irreducible characters of G of level less than or equal to j. Hence, the corollary follows. □

Although we are not able to prove an analogue of Theorem 4.2 in [39], we can instead prove the following weaker result conjectured by M. R. Murty and Raghuram in [49].

Theorem 5.21. *For each $i \geq 1$,*

$$\sum_{\chi \in \operatorname{Irr}(G) \setminus S^i} n(G, \chi)^2 \leq (n(G, \operatorname{Reg}_G) - n(G, \operatorname{Ind}_{G^i}^G 1_{G^i}))^2,$$

where S^i denotes the set of irreducible characters of G of level less than or equal to i.

Proof. Again, we consider a truncated Heilbronn character with respect to S^i

$$\Theta_G^{S^i} = \sum_{\chi \in \operatorname{Irr}(G) \setminus S^i} n(G, \chi) \chi.$$

Taking norms on both sides of the above equation, we get

$$\frac{1}{|G|}\sum_{g\in G}|\Theta_G^{S^i}(g)|^2 = \sum_{\chi\in \mathrm{Irr}(G)\setminus S^i} n(G,\chi)^2.$$

Using the Heilbronn-Stark lemma, Lemma 5.7, one has

$$\begin{aligned}
\Theta_G^{S^i}(g) &= \Theta_G(g) - \sum_{\chi\in S^i} n(G,\chi)\chi(g)\\
&= \Theta_{\langle g\rangle}(g) - \sum_{\chi\in S^i} n(G,\chi)\chi(g)\\
&= \sum_{\phi\in \mathrm{Irr}(\langle g\rangle)}\left(n(\langle g\rangle,\phi) - \sum_{\chi\in S^i} n(G,\chi)(\chi, \mathrm{Ind}_{\langle g\rangle}^G \phi)\right)\phi(g).
\end{aligned}$$

Applying Proposition 5.19 with $H = \langle g\rangle$, we then obtain

$$n(\langle g\rangle,\phi) - \sum_{\chi\in S^i} n(G,\chi)(\chi, \mathrm{Ind}_{\langle g\rangle}^G \phi) \geq 0.$$

Therefore, the triangle inequality and Frobenius reciprocity yield

$$\begin{aligned}
|\Theta_G^{S^i}(g)| &\leq \sum_{\phi\in \mathrm{Irr}(\langle g\rangle)}\left(n(\langle g\rangle,\phi) - \sum_{\chi\in S^i} n(G,\chi)(\chi, \mathrm{Ind}_{\langle g\rangle}^G \phi)\right)\\
&= n(G, \mathrm{Reg}_G) - \sum_{\chi\in S^i} n(G,\chi)(\chi|_{\langle g\rangle}, \mathrm{Reg}_{\langle g\rangle})\\
&= n(G, \mathrm{Reg}_G) - n\left(G, \sum_{\chi\in S^i}\chi(1)\chi\right).
\end{aligned}$$

Using Lemma 5.10 with $H = \langle e_G \rangle$, we have

$$\mathrm{Reg}_G = \mathrm{Ind}_{G^i}^G 1_{G^i} + (*),$$

where (*) is a sum of monomial characters. Now $\mathrm{Ind}_{G^i}^G 1_{G^i}$ is exactly the sum of characters of G occurring in Reg_G which are trivial on G^i (or, equivalently, which have level less than or equal to i). This means that $\mathrm{Ind}_{G^i}^G 1_{G^i} = \sum_{\chi \in S^i} \chi(1)\chi$. Therefore, by conditions AHC1 to AHC3, we complete the proof. □

By an analogous argument of the proof of Corollary 5.16, one can deduce the following corollary.

Corollary 5.22. *Let G be a solvable group. Then $n(G, \mathrm{Reg}_G) - n(G, \mathrm{Ind}_{G^i}^G 1_{G^i})$ cannot be 1.*

At the end of this section, we give an application of our arithmetic Heilbronn characters to Artin L-functions.

Proposition 5.23. *Let Θ_G be an arithmetic Heilbronn character of a group G associated with integers $n(H, \phi)$. Let ρ be a character of G. Suppose that for every subgroup H of G, and 1-dimensional character ϕ of H, we have $n(H, \rho|_H \otimes \phi) \geq 0$. Then for any subgroup H of G, we have an arithmetic Heilbronn character defined by*

$$\Theta'_H = \sum_{\phi \in \mathrm{Irr}(H)} n'(H, \phi)\phi,$$

where $n'(H, \phi) = n(H, \rho|_H \otimes \phi)$. In particular, all properties we have shown for arithmetic Heilbronn characters also hold for Θ'_H.

Proof. The proof is similar to the proof of Proposition 5.6. By linearity of tensor product and the assumption of this theorem, it is easy to see that $n'(H,\phi)$'s satisfy conditions AHC1 and AHC3. Now since tensoring commutes with induction, by condition AHC2, we have

$$\begin{aligned} n'(H,\phi) &= n(H, \rho|_H \otimes \phi) \\ &= n(G, \operatorname{Ind}_H^G(\rho|_H \otimes \phi)) \\ &= n(G, \rho \otimes \operatorname{Ind}_H^G \phi) \\ &= n'(G, \operatorname{Ind}_H^G \phi). \end{aligned}$$

Therefore, the proposition follows. □

Let K/k be a solvable Galois extension of number fields with Galois group G. A deep theorem of Langlands-Tunnell asserts that all two dimensional representations of subgroups of G are automorphic. As a consequence, for any two dimensional representation ρ of G and any abelian character ϕ of a subgroup H of G, the Artin L-function $L(s, \rho|_H \otimes \phi, K/K^H)$ is holomorphic at $s \neq 1$. Fix $s_0 \neq 1$ and set

$$n'(H,\phi) = \operatorname{ord}_{s=s_0} L(s, \rho|_H \otimes \phi, K/K^H).$$

We recall that $n(H,\phi) = \operatorname{ord}_{s=s_0} L(s, \phi, K/K^H)$ defines the classical Heilbronn character. Hence, the above theorem assures that these $n'(H,\phi)$'s give a new arithmetic Heilbronn character. In particular, we have the following variant of the Uchida-van der Waall theorem (cf. Theorem 3.2) and M. R. Murty-Raghuram's inequality [49].

Theorem 5.24. *Let K/k be a solvable Galois extension of number fields with Galois group G, and let ρ be a two dimensional representation of G. Then for any subgroup*

H of G, the quotient

$$\frac{L(s, \operatorname{Ind}_H^G \rho|_H, K/k)}{L(s, \rho, K/k)}$$

is holomorphic at $s \neq 1$. Moreover, for every 1-dimensional character χ_0 of G, one has

$$\sum_{\chi \in \operatorname{Irr}(G) \setminus \{\chi_0\}} (\operatorname{ord}_{s=s_0} L(s, \rho \otimes \chi))^2 \leq \left(\operatorname{ord}_{s=s_0} \left(\frac{\zeta_K^2(s)}{L(s, \rho \otimes \chi_0, K/k)}\right)\right)^2 .$$

Proof. By Proposition 5.23, this theorem follows immediately from Corollary 5.12 and Theorem 5.13 and the identity

$$\rho \otimes \operatorname{Reg}_G = \rho \otimes \operatorname{Ind}_{\langle e_G \rangle}^G 1_{\langle e_G \rangle} = \operatorname{Ind}_{\langle e_G \rangle}^G \rho|_{\langle e_G \rangle} = 2 \operatorname{Ind}_{\langle e_G \rangle}^G 1_{\langle e_G \rangle} = 2 \operatorname{Reg}_G .$$

□

5.3 Applications to Artin-Hecke L-Functions and CM-Elliptic Curves

To avoid the situation that this chapter becomes a loyal servant of Nicolas Bourbaki, we shall apply our theory of arithmetic Heilbronn characters to study Artin-Hecke L-functions and L-functions of CM-elliptic curves. The central idea is due to M. R. Murty and V. K. Murty in [46] by setting $n(G, \chi)$ being equal to the orders of certain Artin-Hecke L-functions to establish an elliptic analogue of the Uchida-van der Waall theorem. As we will see, this brilliant idea will allow us to obtain several analytic properties of Artin-Hecke L-functions and L-functions of CM-elliptic curves. In particular, we derive the non-existence of simple zeros for the quotients of suitable L-functions of CM-elliptic curves.

First of all, we consider a (non-trivial) Hecke character ψ of infinite type of k, and fix a point $s_0 \in \mathbb{C}$. We may set $n^{\psi}(H, \phi) = \mathrm{ord}_{s=s_0} L(s, \psi \circ N_{K^H/k} \otimes \phi, K/K^H)$ for every character ϕ of any subgroup H of G. Using Lemma 3.11, it is easy to see that such $n^{\psi}(H, \phi)$'s define an arithmetic Heilbronn character. Moreover, by "linearity" of tensor product, for any Hecke characters ψ_1 and ψ_2 of infinite type of k, the integers $n^{\psi_1,\psi_2}(H, \phi) = n^{\psi_1}(H, \phi) + n^{\psi_2}(H, \phi)$ also give an arithmetic Heilbronn character.

We recall that, as discussed in Section 3.1.6, every L-function of a CM-elliptic curve can be written in terms of Hecke L-functions. Now fix $s_0 \in \mathbb{C}$ and suppose that K/k is a Galois extension of number fields with Galois group G. Let $L(s, E, K^H)$ be the L-function of E/K^H, which is either a single Hecke L-function or a product of two Hecke L-functions of K^H. Following the proof of Theorem 1 in [46], for each subgroup H of G and complex character ϕ of H, let $n(H, \phi)$ be the order of the L-function $L(s, \phi, E, K^H)$ at $s = s_0$, where $L(s, \phi, E, K^H)$ is the twist of $L(s, E, K^H)$ by ϕ (in particular, it is either a single Artin-Hecke L-function or a product of two Artin-Hecke L-functions). According to the conclusion of our previous discussion of Artin-Hecke L-functions, such integers $n(H, \phi)$ define an arithmetic Heilbronn character, and we hence can use the theory developed in the previous sections to these integers.

We do not intend to state all theorems and corollaries we can get but just mention two results. First of all, we have the following theorem that generalises M. R. Murty and V. K. Murty's elliptic analogue of the Uchida-van der Waall theorem (cf. [46]). Also, this theorem gives an elliptic analogue of M. R. Murty-Raghuram's inequality.

Theorem 5.25. *Suppose K/k is a solvable Galois extension with Galois group G, and let H be a subgroup of G. Let E be an elliptic curve over k and let χ and ϕ be*

1-dimensional characters of G and H, respectively. Then

$$\frac{L(s, \operatorname{Ind}_H^G \phi, E, k)}{L(s, \chi, E, k)^{(\chi|_H, \phi)}}$$

is entire. In addition, for every 1-dimensional character χ_0 of G, one has

$$\sum_{\chi \in \operatorname{Irr}(G)\setminus\{\chi_0\}} (\operatorname{ord}_{s=s_0} L(s, \chi, E, k))^2 \leq \left(\operatorname{ord}_{s=s_0}\left(\frac{L(s, E, K)}{L(s, \chi_0, E, k)}\right)\right)^2.$$

Moreover, we have an interesting result for L-functions of CM-elliptic curves below by applying Corollary 5.22.

Proposition 5.26. *Suppose K/k is a solvable Galois extension with Galois group G. Then for all $i \geq 1$,*

$$\frac{L(s, E, K)}{L(s, E, K^{G^i})}$$

cannot have any simple zero, where $G^0 = G$, $G^i = [G^{i-1}, G^{i-1}]$ for $i \geq 1$, K^{G^i} is the fixed field of G^i, and $L(s, E, K^{G^i})$ is the L-function of E/K^{G^i}.

Remark 5.27. *Note that as K^{G^i} is a subfield of K, it is clear that the group $E(K^{G^i})$ of K^{G^i}-rational points of E is a subgroup of $E(K)$. In other words, the algebraic rank of E/K^{G^i} is smaller than the algebraic rank of E/K. The above result then tells us that under the Birch-Swinnerton-Dyer conjecture, the difference between the algebraic ranks of E/K and E/K^{G^i} cannot be one, which is not obvious by only considering K^{G^i} as a subfield of K. It might be interesting to find a heuristic reason (or even a theoretic proof) to explain this phenomenon.*

5.4 Applications to Automorphic L-Functions and Elliptic Curves without CM

In this section, we will follow the path enlightened by [46] to demonstrate how arithmetic Heilbronn characters play a role in studying automorphic L-functions. First of all, in light of [46, Proof of Theorem 2], we prove the following lemma that allows us to construct arithmetic Heilbronn characters later.

Lemma 5.28. *Let K/k be a Galois extension of number fields with Galois group G, ρ a representation of G, and $n \geq 2$. Suppose that π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_k)$ such that for every intermediate field M of K/k with K/M solvable, $\pi|_M$ is automorphic (over M). Then the Rankin-Selberg L-function $L(s, \pi \otimes \rho)$ extends to a meromorphic function of s.*

Proof. By the Brauer induction theorem, one can write

$$\operatorname{tr} \rho = \sum_i m_i \operatorname{Ind}_{H_i}^G \chi_i,$$

where $m_i \in \mathbb{Z}$, χ_i is an abelian character of an elementary subgroup H_i of G, which is nilpotent. By Artin reciprocity, for each i, χ_i corresponds to a cuspidal automorphic representation of $GL_1(\mathbb{A}_{K^{H_i}})$. Since each H_i is nilpotent, H_i is solvable, and so $\pi|_{K^{H_i}}$ is automorphic. Now the Rankin-Selberg theory ensures that every $L(s, \pi|_{K^{H_i}} \otimes \chi_i)$ extends to an entire function. Thus, $L(s, \pi \otimes \rho)$ extends to a meromorphic function over $\mathbb{C}$. □

We first note that if the Langlands reciprocity conjecture holds for K/k, then the automorphy assumption on $\pi|_M$ can be easily removed by just applying the theory of

Rankin-Selberg L-functions. On the other hand, if one knows how to associate Galois representations to π and its "descents", then one can apply Arthur-Clozel's theory of base change to derive the desired automorphy result. In particular, if K/k is a totally real solvable extension and π is a "RAESDC" (regular algebraic essentially self-dual cuspidal) automorphic representation, then by the work of Taylor and his school, the extra automorphy assumption in the above lemma can be dropped (for more details and references, see Section 3.3.5).

Under the above assumption and notation, we now further assume that K/k is totally real and solvable. We let H be a subgroup of G and ϕ a character of H, and fix $s_0 \in \mathbb{C}$. We define $n(H, \phi)$ to be the order of the Rankin-Selberg L-function $L(s, \pi|_{K^H} \otimes \phi)$ at $s = s_0$. Since K/K^H is still a solvable Galois extension, by Lemmata 3.22 and 5.28, we know that $n(H, \phi)$'s define an arithmetic Heilbronn character. Again, we do not intend to restate all results established in the previous section but just mention two of them.

First of all, applying Proposition 5.11, we obtain the following theorem that can be seen as an analogue of M. R. Murty and Raghuram's variant of the Uchida-van der Waall theorem.

Theorem 5.29. *Under the assumption and notation as above. Let χ and ϕ be 1-dimensional characters of G and H respectively. Then the quotient*

$$\frac{L(s, \pi|_H \otimes \phi)}{L(s, \pi \otimes \chi)^{(\chi|_H, \phi)}}$$

is entire. Moreover, for every 1-dimensional character χ_0 *of* G*, one has*

$$\sum_{\chi \in \mathrm{Irr}(G)\setminus\{\chi_0\}} (\mathrm{ord}_{s=s_0} L(s, \pi \otimes \chi))^2 \leq \left(\mathrm{ord}_{s=s_0}\left(\frac{L(s, B(\pi))}{L(s, \pi \otimes \chi_0)}\right)\right)^2,$$

where $B(\pi)$ *is the base change of* π *to* K*.*

In fact, this also generalises [46, Theorem 4] that asserts that $L(s, \pi|_H)/L(s, \pi)$ is entire. On the other hand, one can use Corollary 5.22 to get the following.

Proposition 5.30. *Under the assumption and notation as above. Then for all* $i \geq 1$*,*

$$\frac{L(s, B(\pi))}{L(s, B^i(\pi))}$$

cannot have any simple zero where $G^0 = G$*,* G^i *denotes* $[G^{i-1}, G^{i-1}]$ *for all* $i \geq 1$*,* K^{G^i} *is the fixed field of* G^i*,* $B(\pi)$ *is the base change of* π *to* K*, and* $B^i(\pi)$ *is the base change of* π *to* K^{G^i}*, the fixed field of* G^i*.*

We note that the existence of $B^i(\pi)$ in the above theorem is due to the Arthur-Clozel theorem and the fact that each G^i is normal in G. We remark that our results also have other arithmetic applications. For instance, as mentioned in [46], the zeta function of any CM abelian variety over an arbitrary number field is given in terms of Hecke L-functions, and the Jacobian of a modular curve has the zeta function that is equal to a product of L-functions attached to modular forms by a theorem of Shimura. In both instances, one may obtain appropriate generalisation by setting integers equal to the orders of suitable L-functions (at $s = s_0 \in \mathbb{C}$) to define an arithmetic Heilbronn character.

At the end of this section, we shall apply the previous results to symmetric power

L-functions. Suppose that M/k is an extension of number fields contained in a totally real solvable Galois extension K/k with $G = \mathrm{Gal}(K/k)$. We denote H_M to be the subgroup of G such that $K^{H_M} = M$. Let E be a non-CM elliptic curve defined over k. As discussed in Section 3.1.4, for every intermediate field F of K/k, let $\rho_F = \rho_{E,F}$ denote a compatible system of ℓ-adic representations attached to E over F, i.e., for each prime ℓ,

$$\rho_F := \rho_{\ell,F} : \mathrm{Gal}(\overline{k}/F) \to \mathrm{Aut}(T_\ell(E,F)),$$

where $T_\ell(E,F)$ denotes (ℓ-adic) Tate module of E/F. Moreover, we have

$$L(s, \mathrm{Sym}^m \rho_F) = L(s, \mathrm{Sym}^m \rho_k \otimes \mathrm{Ind}_{H_F}^G 1), \tag{5.2}$$

where H_F is a subgroup of G such that $K^{H_F} = F$. Assuming the m-th symmetric power of ρ_k is automorphic, Lemma 5.28 implies that for every character χ of G, the Rankin-Selberg L-function

$$L(s, \mathrm{Sym}^m \rho_k \otimes \chi)$$

extends to a meromorphic function over $\mathbb{C}$.

Now fix $s = s_0 \in \mathbb{C}$, and for every character ϕ, define $n(H, \phi)$ to be the order of the L-function

$$L(s, (\mathrm{Sym}^m \rho_k)|_{K^H} \otimes \phi)$$

at $s = s_0$, where $(\mathrm{Sym}^m \rho_k)|_{K^H}$ is obtained in the same manner as in the proof of Theorem 3.24 (we note that Arthur-Clozel's theory of base change asserts that $(\mathrm{Sym}^m \rho_k)|_K$ is automorphic). Therefore, $n(H, \phi)$'s define an arithmetic Heilbronn character. As a consequence, we have the following elliptic analogue of the Uchida-van der Waall theorem that generalises [46, Theorem 2].

Proposition 5.31. *Under the assumption and notation as above. Let χ and ϕ be 1-dimensional representations of G and H, respectively. Then*

$$\frac{L(s, \operatorname{Sym}^m \rho_{K^H} \otimes \phi)}{L(s, \operatorname{Sym}^m \rho_k \otimes \chi)^{(\chi|_H, \phi)}}$$

is entire. Moreover, by equation (3.3)*, for every intermediate field F of K/k,*

$$\frac{L(s, \operatorname{Sym}^m \rho_F)}{L(s, \operatorname{Sym}^m \rho_k)}$$

is entire.

On the other hand, Proposition 5.30 and equation (3.3) give below an interesting result.

Proposition 5.32. *Under the assumption and notation as above. Then for all $i \geq 1$,*

$$\frac{L(s, \operatorname{Sym}^m \rho_K)}{L(s, \operatorname{Sym}^m \rho_{K^{G^i}})}$$

cannot admit any simple zero. In particular,

$$\frac{L(s, E, K)}{L(s, E, K^{G^i})}$$

has no simple zeros, where for any intermediate field F of K/k, $L(s, E, F)$ denotes the L-function of E/F.

Also, we have an elliptic analogue of M. R. Murty-Raghuram's inequality.

Theorem 5.33. *Under the assumption and notation as above. Suppose K/k is a totally real solvable Galois extension with Galois group G. Then for every 1-dimensional character χ_0 of G, one has*

$$\sum_{\chi \in \mathrm{Irr}(G)\setminus\{\chi_0\}} (\mathrm{ord}_{s=s_0} L(s, \mathrm{Sym}^m \rho_k \otimes \chi))^2 \leq \left(\mathrm{ord}_{s=s_0} \left(\frac{L(s, \mathrm{Sym}^m \rho_K)}{L(s, \mathrm{Sym}^m \rho_k \otimes \chi_0)}\right)\right)^2.$$

5.5 An Application of Weak Heilbronn Characters

As one can see, arithmetic Heilbronn characters indeed play a role which helps us to obtain analytic properties of L-functions. Meanwhile, one may wonder if we really need the notion of weak arithmetic Heilbronn characters, which seems impractical and unnecessary. Thanks to the recent groundbreaking work of Taylor and his school (cf. Section 3.3.5), this wonder may not be an issue. As we will demonstrate, it is possible to utilise all the results of *potential automorphy* and our weak arithmetic Heilbronn characters to study L-functions. However, for the sake of conceptual clarity, we shall only use Taylor's potential automorphy result here.

We again recall that Taylor's main theorem is: let k be a totally real field and E/k a non-CM elliptic curve. Then for any finite set S of natural numbers, there is a (finite) totally real Galois extension L/k such that for every $m \in S$, $\mathrm{Sym}^m \rho_k$ is automorphic over L, i.e., $(\mathrm{Sym}^m \rho_k)|_L$ is automorphic.

As before, we fix a finite set S of natural numbers and let L be a (finite) totally real Galois extension L/k such that for every $m \in S$, $\mathrm{Sym}^m \rho_k$ is automorphic over L, which is given by Taylor's theorem. We recall another key aspect in the proof of the Sato-Tate conjecture (cf. Theorem 3.24):

For any intermediate field F of L/k with L/F solvable, $(\mathrm{Sym}^m \rho_k)|_F$ is automorphic.

We first note that since every irreducible character ϕ of a cyclic subgroup H of $G = \mathrm{Gal}(L/k)$ can be identified as an automorphic representation of $GL_1(\mathbb{A}_{L^H})$ via Artin reciprocity, the above theorem and the Rankin-Selberg theory yield

$$L(s, (\mathrm{Sym}^m \rho_k)|_{L^H} \otimes \phi)$$

is entire.

In light of the method developed by Taylor et al., one can show the following.

Proposition 5.34. *For every character χ of $G = \mathrm{Gal}(L/k)$, $L(s, (\mathrm{Sym}^m \rho_k) \otimes \chi)$ extends to a meromorphic function over $\mathbb{C}$.*

Proof. As usual, the Brauer induction theorem asserts

$$\chi = \sum_i n_i \, \mathrm{Ind}_{H_i}^G \phi_i,$$

where for each i, n_i is an integer, and ϕ_i is a 1-dimensional character of a nilpotent subgroup H_i of G. According to Artin reciprocity, ϕ_i can be seen as a Hecke character over L^{H_i}. Putting everything together, one has

$$L(s, (\mathrm{Sym}^m \rho_k) \otimes \chi) = \prod_i L(s, (\mathrm{Sym}^m \rho_k)|_{L^{H_i}} \otimes \phi_i)^{n_i},$$

where $\phi_i \in \mathfrak{A}(GL_1(\mathbb{A}_{L^{H_i}}))$. By Theorem 3.24, $(\mathrm{Sym}^m \rho_k)|_{L^{H_i}}$ is automorphic over L^{H_i}. Now the Rankin-Selberg theory tells us that each $L(s, (\mathrm{Sym}^m \rho_k)|_{L^{H_i}} \otimes \phi_i)$ is entire, which completes the proof. $\square$

Therefore, for H cyclic or $H = G$, fixing $s_0 \in \mathbb{C}$ and setting

$$n(H, \phi) = \operatorname{ord}_{s=s_0} L(s, (\operatorname{Sym}^m \rho_k)|_{L^H} \otimes \phi),$$

the above discussion yields that $n(H, \phi)$'s define a weak arithmetic Heilbronn character. In particular, by Theorem 5.3, we then deduce:

Theorem 5.35.

$$\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)^2 \leq (\operatorname{ord}_{s=s_0} L(s, \operatorname{Sym}^m \rho_L))^2.$$

In particular, (if we choose S containing 1 in the very beginning)

$$|\operatorname{ord}_{s=s_0} L(s, \rho_k)| \leq \operatorname{ord}_{s=s_0} L(s, \rho_L).$$

We remark that the last inequality of analytic ranks is as predicted by the *Birch-Swinnerton-Dyer conjecture* for $s_0 = 1$ (cf. Remark 5.27).

Chapter 6

Conjectures of Artin and Langlands

6.1 Nearly Supersolvable Groups and Nearly Monomial Groups

As a consequence of *Artin reciprocity*, Artin's conjecture is true for any Galois extension of number fields whose Galois group is a *nilpotent group*, a *supersolvable group*, or an *M-group*. These classes of groups became an area of interest in their own right. For instance, Taketa's theorem [63] asserts that (finite) M-groups are necessarily *solvable* (cf. [5, Section 5.10]). Besides, it is also possible to generalise Taketa's theorem (see, for example, [5, Theorem 14.58]) to groups all of whose irreducible characters are induced from n-dimensional characters with $n \leq 2$. We will call these groups *nearly monomial groups* (or *NM-groups* for short).

Thanks to the works of Artin, Langlands, and Tunnell as well as the generalisation of Taketa's theorem as mentioned earlier, Artin's conjecture holds for every Galois extension of number fields whose Galois group is an NM-group. Undoubtedly, it is desired to classify the class of NM-groups not only for the purely group-theoretic interest but also for the purpose of studying L-functions.

By a result of Huppert (Proposition 2.4), if a group G admits an abelian normal subgroup N such that G/N is supersolvable, then G is an M-group. In light of this, we will introduce the notion of *nearly supersolvable groups* and discuss some properties of these groups. In fact, one goal of this section is showing that nearly supersolvable groups belong to the class of NM-groups. Now we shall start by defining nearly supersolvable groups.

Definition 6.1. *A finite group G is said to be nearly supersolvable (or NSS for short) if it has an invariant series of subgroups*

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G,$$

where each subgroup is normal in G, the quotient N_{i+1}/N_i is cyclic for every $i \geq 1$, and N_1 belongs to the class $\mathcal{C}$ consisting of groups whose irreducible representations are of dimension less than or equal to 2.

We note that the class $\mathcal{C}$ was classified by Amitsur (cf. [1, Theorem 3]).

Proposition 6.1. *Let G be a finite group. Then all irreducible characters of G are of degree 1 or 2 if and only if either*

1. *G is abelian,*

2. *G has an abelian subgroup of index 2, or*

3. *$G/\mathbf{Z}(G)$ is an abelian 2-group of order 8.*

This result has been generalised by Isaacs that if G is a group with $|\operatorname{cd}(G)| \leq 3$, where $\operatorname{cd}(G) = \{\chi(1) \mid \chi \in \operatorname{Irr}(G)\}$, then G must be solvable (cf. Theorem 2.11). Thus, by the results of Amitsur and Isaacs, all groups belonging to $\mathcal{C}$ are necessarily solvable. Therefore, nearly supersolvable groups are indeed *solvable*. On the other hand, it is clear that all supersolvable groups are nearly supersolvable. In fact, we will see that nearly supersolvable groups behave exactly like supersolvable groups. To state and prove this formally, we first recall below a lemma that assures that the class $\mathcal{C}$ is "closed" (cf. [51, Chapter 6, Lemma 1.3]).

Lemma 6.2. *Let G be a finite group. Suppose that all irreducible representations of G are of dimension 1 or 2. Assume that H is either a subgroup or a homomorphic image of G. Then every irreducible representation of H is of dimension 1 or 2.*

From this lemma, we can show that the class of NSS-groups is also "closed" as the following.

Proposition 6.3.

1. *Every subgroup of an NSS-group is NSS.*

2. *Every homomorphic image of an NSS-group is NSS. In particular, every quotient group of an NSS-group is NSS.*

Proof. Let G be a nearly supersolvable group with an invariant series of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G,$$

where each subgroup is normal in G, the quotient N_{i+1}/N_i is cyclic for every $i \geq 1$, and N_1 belongs to the class $\mathcal{C}$. Then for any subgroup H of G,

$$1 = H \cap N_0 \trianglelefteq H \cap N_1 \trianglelefteq \cdots \trianglelefteq H \cap N_{k-1} \trianglelefteq H \cap N_k = H$$

is an invariant series of H in which each quotient $(H \cap N_i)/(H \cap N_{i-1})$ is isomorphic to the subgroup $(H \cap N_i)N_{i-1}/N_{i-1}$ of N_i/N_{i-1}.

On the other hand, let $\phi : G \to H$ be a surjective homomorphism, then

$$1 = \phi(N_0) \trianglelefteq \phi(N_1) \trianglelefteq \cdots \trianglelefteq \phi(N_{k-1}) \trianglelefteq \phi(N_k) = H$$

is an invariant series of H. Moreover, for each i, $\phi(N_i)/\phi(N_{i-1})$ is a homomorphic image of the quotient group N_i/N_{i-1}. Now Lemma 6.2 implies that H is NSS whenever H is a subgroup or a homomorphic image of G. □

Like supersolvable groups, it is false in general that if both N and G/N are nearly supersolvable, then G is a nearly supersolvable group. However, we have the following weak substitute.

Lemma 6.4. *Let G be a group and N its normal subgroup. If N belongs to the class $\mathcal{C}$, and G/N is supersolvable, then G is nearly supersolvable.*

Proof. By lifting an invariant series of G/N to G, we have

$$N = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G,$$

where each subgroup is normal in G and every quotient N_{i+1}/N_i is cyclic. Since $N \in \mathcal{C}$, extending the above invariant series to the trivial subgroup completes the proof. □

Now we can state and prove our main theorem for this section.

Theorem 6.5. *All nearly supersolvable groups are NM-groups.*

Proof. According to the definition, for any nearly supersolvable group G, there is an invariant series of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G,$$

where each subgroup is normal in G, the quotient N_{i+1}/N_i is cyclic for every $i \geq 1$, and N_1 belongs to the class $\mathcal{C}$ consisting of groups whose irreducible representations are of dimension less than or equal to 2. Quotienting the above invariant series by N_1 then gives

$$\langle \overline{e} \rangle = \overline{N_1} \trianglelefteq \overline{N_2} \trianglelefteq \cdots \trianglelefteq \overline{N_k} = G/N_1,$$

where for each $i \geq 1$, $\overline{N_i} = N_i/N_1$. According to the third isomorphism theorem, each $\overline{N_{i+1}}/\overline{N_i}$ is isomorphic to N_{i+1}/N_i, which is cyclic. In other words, G/N_1 is supersolvable. Now by applying Theorem 2.3, G is a relative M-group with respect to N_1. As all irreducible characters of N_1 are of degree ≤ 2, we conclude that G is an NM-group. □

We recall below a result (cf. [71, pp. 6]) that gives sufficient conditions for groups being supersolvable, which will enable one to obtain some examples of NSS-groups.

Lemma 6.6.

1. *If $|G| = qp^n$ and $q|(p-1)$, then G is supersolvable. In particular, if $|G| = 2p^n$, then G is supersolvable.*

2. *Suppose that $q^2|(p-1)$ and $|G| = q^2p^n$. Then G is supersolvable. In particular, if $|G| = 4p^n$ with $4|(p-1)$, then G is supersolvable.*

By Proposition 6.1, a moment's reflection shows that all irreducible characters of any group of order $2p$ or $2p^2$ are of degree ≤ 2. Thus, one has

Corollary 6.7. *Let p be a prime. If $|G| = 4p^n$, and G admits a normal subgroup of order 2, 4, $2p$, or $2p^2$. Then G is NSS. If $|G| = 8p^n$, and G has a normal subgroup of order 4 or 8, then G is NSS. Moreover, if $|G| = 8p^n$ with $4|(p-1)$, and G admits a normal subgroup of order 2, $2p$, or $2p^2$, then G is NSS.*

To end this section, we give below a sufficient condition for groups of derived length ≤ 3 being NSS-groups.

Proposition 6.8. *Suppose that G has derived length ≤ 3. If G'/G'' is cyclic, then G is an NSS-group.*

Proof. Quotienting the derived series of G by G'' gives

$$1 \trianglelefteq G'/G'' \trianglelefteq G/G''.$$

Since G/G' is abelian, the third isomorphism theorem yields that the quotient

$$(G/G'')/(G'/G'')$$

is also abelian. As G'/G'' is cyclic, one can conclude that G/G'' is supersolvable. Moreover, since G'' is abelian, Lemma 6.4 asserts that G is NSS. □

6.2 Nearly Nilpotent Groups

As discussed in Section 3.3, Arthur and Clozel showed that all Galois representations with *nilpotent* image are automorphic via Artin reciprocity, their theory of automorphic induction, and the fact that all subgroups of a nilpotent group are subnormal. From this, one may ask for a classification of *subnormally monomial groups*, the groups all of whose irreducible characters are induced from 1-dimensional characters of subnormal subgroups.

We, however, note that due to the Langlands-Tunnell theorem, the theory of Arthur and Clozel indeed implies that all Artin L-functions attached to characters induced from 2-dimensional characters of subnormal groups are automorphic under a certain solvability condition. In light of this, we are interested in the classification of *subnormally NM-groups*, which leads us to consider *nearly nilpotent groups* as follows.

Definition 6.2. *A finite group G is called nearly nilpotent if it has a normal subgroup $N \in \mathcal{C}$ such that G/N is nilpotent, where $\mathcal{C}$ denotes the class consisting of groups whose irreducible representations are of dimension less than or equal to* 2*.*

Since all subgroups and homomorphic images of a nilpotent group are nilpotent, a moment's thought shows that all subgroups and homomorphic images of any nearly nilpotent group are nearly nilpotent. Also, as all nilpotent groups are supersolvable, all NN-groups form a "closed" subclass of the class of NSS-groups. In particular, all NN-groups are *solvable.*

Now let us consider a direct product $G = G_1 \times G_2$, where G_1 and G_2 are NM-groups. Note that for every irreducible character χ of G, there exist irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that $\chi = \chi_1 \times \chi_2$. Since both

G_1 and G_2 are NM-groups, for each i, there exists a subgroup H_i of G_i with an irreducible character $\psi_i \in \mathrm{Irr}(H_i)$ of degree ≤ 2 such that $\chi_i = \mathrm{Ind}_{H_i}^{G_i} \psi_i$. Thus,

$$\chi = \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (\psi_1 \times \psi_2).$$

However, now one can see that χ might not be induced from an irreducible character of degree ≤ 2. As a consequence, we cannot apply the Langlands-Tunnell theorem to deduce Artin's conjecture directly. But as each ψ_i is still of automorphic type (thanks to Artin reciprocity and the Langlands-Tunnell theorem), if we invoke the functoriality of $GL(n) \times GL(1)$ and $GL(2) \times GL(2)$ (cf. Section 3.3.3), then we are able to derive the automorphy of $\psi_1 \times \psi_2$. Thus, we have the following.

Proposition 6.9. *If K/k is a Galois extension of number fields whose Galois group is a direct product of two NM-groups, then Artin's conjecture is true for K/k.*

Moreover, by applying the Rankin-Selberg theory developed by Jacquet-Piatetski-Shapiro-Shalika, the above discussion then further yields:

Proposition 6.10. *If K/k is a Galois extension whose Galois group is a direct product of three (or four) NM-groups, then Artin's conjecture is true for K/k.*

In a slightly different vein, since any finite direct product of nilpotent groups is nilpotent, the Arthur-Clozel theory implies that the principle of functoriality is valid in this case. Naturally, one may want to find some "non-nilpotent" examples. Unfortunately, unlike nilpotent groups, the direct product of two nearly nilpotent groups might not be nearly nilpotent. In fact, by the previous discussion, one even cannot expect this would be an NM-group. Nevertheless, we have the following result.

Proposition 6.11. *If G_1 is a nearly nilpotent group and G_2 is an abelian-by-nilpotent group, i.e., G_2 admits an abelian normal subgroup N_2 with G_2/N_2 nilpotent, then $G_1 \times G_2$ is a nearly nilpotent group and so is of automorphic type.*

Proof. Since G_1 is a nearly nilpotent group, there is a normal subgroup N_1 of G_1 belonging to $\mathcal{C}$ such that G_1/N_1 is nilpotent. On the other hand, G_2 has an abelian normal subgroup N_2 such that G_2/N_2 is nilpotent. Thus, we have an invariant series

$$1 \trianglelefteq N_1 \times N_2 \trianglelefteq G_1 \times G_2.$$

Since $(G_1 \times G_2)/(N_1 \times N_2) \simeq (G_1/N_1) \times (G_2/N_2)$, which is a direct product of nilpotent groups, $(G_1 \times G_2)/(N_1 \times N_2)$ is nilpotent. Moreover, as all irreducible characters of $N_1 \times N_2$ are clearly of degree ≤ 2, $N_1 \times N_2 \in \mathcal{C}$. Thus, $G_1 \times G_2$ is nearly nilpotent. □

In addition, by invoking Ramakrishnan's functoriality of $GL(2) \times GL(2)$, one can show the direct product of two nearly nilpotent groups is still of automorphic type.

Theorem 6.12. *If G_1 and G_2 are nearly nilpotent, then $G_1 \times G_2$ is of automorphic type.*

Proof. Assume that K/k is a Galois extension of number fields with Galois group $G_1 \times G_2$. Since both G_1 and G_2 are nearly nilpotent, for each i, there exists $N_i \in \mathcal{C}$ such that G_i/N_i is nilpotent. Now Theorem 2.5 asserts that G_i is a relative SM-group with respect to N_i. As discussed above, for each irreducible character χ of $G_1 \times G_2$, there are irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that

$$\chi = \chi_1 \times \chi_2.$$

Also, for each i, there exist a subnormal subgroup H_i (containing N_i) of G_i and an irreducible character $\psi_i \in \mathrm{Irr}(H_i)$ such that $\chi_i = \mathrm{Ind}_{H_i}^{G_i} \psi_i$ and $\psi_i|_{N_i} \in \mathrm{Irr}(N_i)$. Thus,

$$\chi = \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (\psi_1 \times \psi_2).$$

On the one hand, as ψ_1 and ψ_2 are of degree ≤ 2, Artin reciprocity and Langlands-Tunnell's theorem assert that for each i, (by regrading ψ_i as an irreducible character of $H_1 \times H_2$) ψ_i corresponds to a cuspidal automorphic representation of dimension $\psi_i(1)$ over $K^{H_1 \times H_2}$. Thus, the functoriality of $GL(n) \times GL(1)$ and $GL(2) \times GL(2)$ implies that $\psi_1 \times \psi_2$ corresponds to a cuspidal automorphic representation (of dimension $\psi_1(1)\psi_2(1)$) over $K^{H_1 \times H_2}$. Note that as $H_1 \times H_2$ is subnormal in $G_1 \times H_2$, and $G_1 \times H_2$ is subnormal in $G_1 \times G_2$, we can conclude that $H_1 \times H_2$ is subnormal in $G_1 \times G_2$. Putting everything together, the above-mentioned theorems of Arthur-Clozel and Jacquet-Shalika yield that χ is cuspidal. $\square$

We now give some sufficient conditions for solvable groups to be of automorphic type. First of all, as any nilpotent group is isomorphic to a direct product of its Sylow subgroups and the derived subgroup of any supersolvable group is nilpotent, we have the following corollary.

Corollary 6.13. *If G is a supersolvable group of order $2^n p_1^{n_1} \cdots p_k^{n_k}$ with $n_i \leq 2$ and $n \leq 4$, then G is of automorphic type.*

Also, since all *Z-groups*, the groups whose all Sylow subgroups are cyclic, are supersolvable, a moment's reflection shows:

Corollary 6.14. *All Z-groups are of automorphic type. In particular, all groups of square-free order are of automorphic type.*

This section will close with below a semi-numerical result, which in particular, presents a simple proof for Cho and Kim's automorphy results of A_4, S_4, and $SL_2(\mathbb{F}_3)$.

Corollary 6.15. *Let p and q be distinct primes. If G is of order pq, p^2q, or p^2q^2, then G is of automorphic type.*

Proof. By the Sylow theorems, G must have a normal Sylow subgroup N (see, for example, [27, Theorems 1.30 and 1.31] and [61, 6.5.2]). Note that N is abelian, and that G/N is either a p-group or a q-group. Now the claim follows from Theorem 6.12 immediately. □

6.3 S-Accessible Characters

In light of the work of Arthur-Clozel on accessible characters, we prove the following.

Proposition 6.16 (à la Arthur et Clozel)**.** *Assume G is solvable and χ is irreducible. If χ is an integral sum of characters induced from irreducible characters, which are of automorphic type, of subnormal subgroups of G, then Langlands reciprocity holds for χ.*

Proof. As discussed in Section 3.3.4, all characters induced from irreducible characters, which are of automorphic type, of subnormal subgroups of G must be of automorphic type. Hence, χ corresponds to a (formal) integral sum of cuspidal automorphic representations. Thus, we can write

$$L(s,\chi,K/k) = \prod_i L(s,\pi_i)^{n_i},$$

where for each i, n_i is an integer, and $\{\pi_i\}_i$ is a finite set of distinct cuspidal automorphic representations (over k) such that $\pi_i \simeq \pi_j$ only if $i = j$.

As in [2], one can utilise Jacquet-Shalika's result, [33, Theorem 4.7], to complete the proof. However, for the sake of completeness, we sketch their argument as follows. By applying the theory of Rankin-Selberg convolutions and looking at the order of pole at $s = 1$ of $L(s, \chi \otimes \overline{\chi}, K/k)$, one has

$$1 = \sum_i n_i^2.$$

Since n_i's are integers, one can easily deduce that $|n_1| = 1$ (say) and $n_i = 0$ for any $i \neq 1$. Finally, if $n_1 = -1$, the Artin L-function would have "trivial poles" at some negative integers, which is impossible. □

Let S be a finite set of natural numbers. An irreducible character χ of G is called *S-accessible* if χ is an integral combination of characters induced from irreducible characters ψ_i of subnormal subgroups of G, where each $\psi_i(1)$ belongs to S. Moreover, a group is called S-accessible if all its irreducible characters are. For example, $\{1\}$-accessible characters (resp., groups) are exactly accessible characters (resp., groups) introduced by Arthur and Clozel, and nilpotent groups are $\{1\}$-accessible. Indeed, the author learned the above argument from Arthur and Clozel who showed all solvable accessible groups are of automorphic type and derived Langlands reciprocity for all nilpotent extensions. We now present below a generalisation of Arthur-Clozel's result.

Corollary 6.17. *Suppose G is solvable. If χ is a $\{1,2\}$-accessible character of G, then Langlands reciprocity holds for χ. Also, if $|G|$ is not divisible by 36 and χ is a $\{1,2,3\}$-accessible character of G, then Langlands reciprocity holds for χ.*

Proof. It suffices to show that any irreducible character ψ, with $\psi(1) \leq 3$, of any subgroup of G is of automorphic type. As all subgroups of G are solvable, if $\psi(1) \leq 2$, the assertion follows from the Artin-Langlands-Tunnell theorem. So we may assume $\psi(1) = 3$. Since 36 does not divide the order of any subgroup of G, Lemma 2.12 tells us that ψ must be monomial and hence of automorphic type. □

We note that all nearly nilpotent groups are solvable and $\{1, 2\}$-accessible. Thus, Langlands reciprocity holds for all nearly nilpotent extensions. However, all irreducible characters of a nearly nilpotent group are in fact induced from irreducible characters of degree at most 2 (cf. Theorem 2.3). As remarked in [2], Dade [14] has shown that if G is solvable, then $\{1\}$-accessible characters are monomial. It would be interesting to investigate whether a similar result holds or not. For example, are $\{1, 2\}$-accessible characters of a solvable group G all induced from irreducible characters of degree at most 2? We have no clue about this question; and instead of trying to answer this question, we will give a family of $\{1, 2, 3\}$-accessible groups in the next section.

6.4 Variants of Nearly Nilpotent and Nearly Supersolvable Groups

As before, K/k denotes a Galois extension of number fields with Galois group G. We first give below a result which presents a partial generalisation of the above-mentioned automorphy result of nearly nilpotent groups.

Theorem 6.18. *Suppose that* $36 \nmid |G|$*, and that* G *admits a normal subgroup* N *with* G/N *supersolvable and* $\mathrm{cd}(N) \subseteq \{1, 2, 3\}$*. Then the Artin conjecture is true for* K/k*. Moreover, if* G/N *is nilpotent, then* G *is of automorphic type.*

Proof. We first note that Theorem 2.11 asserts that N is solvable, and so is G. According to Theorem 2.3, every irreducible character χ of G is induced from an irreducible character ψ of degree at most 3 of a subgroup H of G. If $\psi(1) \leq 2$, ψ is automorphic by the Artin-Langlands-Tunnell theorem. On the other hand, for $\psi(1) = 3$, Lemma 2.12 tells us that ψ must be monomial as $|H|$ is not divisible by 36. Thus, ψ is automorphic (over K^H). From this and the induction invariance property of Artin L-functions, Artin's conjecture follows.

Assume, further, that G/N is nilpotent. Then Theorem 2.5 enables us to choose H being subnormal in G. As now χ is $\{1, 2, 3\}$-accessible, Corollary 6.17 yields that χ is of automorphic type. □

We give below a simple application of this theorem.

Corollary 6.19. *Let p be an odd prime. If $|G|$ is $8p$, then G is of automorphic type.*

Proof. Again the Sylow theorems asserts that G admits a normal Sylow subgroup N unless $G \simeq S_4$ (cf. [27, Theorems 1.32 and 1.33]). Assuming that G is not isomorphic to S_4, since all irreducible characters of N are of degree ≤ 2, and G/N is clearly nilpotent, Theorem 6.12 yields that G is of automorphic type.

Now suppose G is isomorphic to S_4. Then $\mathrm{cd}(G) = \{1, 2, 3\}$. Since $36 \nmid |S_4|$, Theorem 6.18 asserts that G is of automorphic type. □

Also, we have the following variant that generalises NSS-groups.

Proposition 6.20. *Suppose that $G = G_1 \times G_2$. For each i, assume that $160 \nmid |G_i|$, and that G_i admits a normal subgroup N_i with G_i/N_i supersolvable and $\mathrm{cd}(N_i) \subseteq \{1, 2, 4\}$. Then the Artin conjecture is true for K/k.*

Proof. Again, G is solvable as Theorem 2.11 ensures that each N_i is solvable. We observe that every irreducible character χ of G can be written as

$$\chi = \chi_1 \times \chi_2$$

for some $\chi_i \in \mathrm{Irr}(G_i)$. Moreover, Theorem 2.3 implies that each χ_i is induced from an irreducible character ψ_i of degree 1, 2, or 4 of a subgroup H_i of G_i. Also, the Artin-Langlands-Tunnell theorem yields ψ_i is automorphic if $\psi_i(1) \leq 2$.

For $\psi_i(1) = 4$, if ψ_i is imprimitive, then it must be induced from a character of degree at most 2, which can also be treated by the works of Artin and Langlands-Tunnell. So we may assume ψ_i is 4-dimensional and primitive. As $160 \nmid |G_i|$, Theorems 3.17 and 3.18 together with Lemma 2.13 assert that ψ_i is automorphic immediately.

Thus, one can conclude that χ is induced from a product of two irreducible characters of automorphic type. Now applying the theory of Rankin-Selberg L-functions, Artin's conjecture is valid for the Artin L-function attached to χ. □

By a similar argument, one can easily obtain a variant of Theorem 6.12.

Proposition 6.21. *Suppose that $G = G_1 \times G_2$ with $36 \nmid |G_i|$, and that each G_i admits a normal subgroup N_i with G_i/N_i supersolvable and* $\mathrm{cd}(N_i) \subseteq \{1, 2, 3\}$. *Then the Artin conjecture is true for K/k.*

We remark that one can, in fact, improve the above results via the classification of finite subgroups of linear groups. For instance, Lemma 2.13 tells us that the condition on indivisibility of $|G_i|$ by 160 can be weakened by only requiring that any subgroup of G_i has no quotient group isomorphic to $E_{2^4} \rtimes D_{10}$ or $E_{2^4} \rtimes F_{20}$. Similarly, the indivisibility of $|G|$ by 36 can be replaced by the condition that none of the subgroups of G has a quotient group isomorphic to the groups of order 36, 72, or

216 appearing in Lemma 2.12 whose precise description can be found, for example, in [43, Chapter 8].

We also have a variant of Theorem 6.18.

Proposition 6.22. *Suppose G is NSS. If G has a normal subgroup N with G/N nilpotent and* $\mathrm{cd}(N) \subseteq \{1,2,3\}$*, then G is of automorphic type.*

Proof. We induct on the order of $|G|$. By Theorem 2.5, G is a relative SM-group with respect to N. Thus, for every irreducible character χ of G, there exists a subnormal subgroup H with $N \le H \le G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in \mathrm{Irr}(N)$. If $H \neq G$, then the induction hypothesis assures that H is of automorphic type, and so applying Arthur-Clozel's theory completes the proof in this case.

Now assume that $H = G$. Since G is NSS, G is an NM-group, and χ must be induced from a character of degree 1 or 2. On the other hand, as $\chi|_N = \psi|_N$ is an irreducible character of N, χ is of degree ≤ 3. If $\chi(1) \le 2$, then Artin reciprocity and the Langlands-Tunnell theorem assert that χ is of automorphic type. Otherwise, for χ of degree 3, χ must be a monomial character. Now applying Arthur-Clozel's theory and Theorem 3.23 completes the proof. □

Corollary 6.23. *If G is a group of order* 54 *or* 162*, then G is of automorphic type.*

Proof. By [61, 7.2.15], G is a supersolvable group. Since any Sylow 3-subgroup P of G has index 2, P is a normal subgroup. As all non-trivial p-groups have non-trivial centre, $[P : \mathbf{Z}(P)] \le 27$. Thus, Lemma 2.9 yields that $\mathrm{cd}(P) \subseteq \{1,3\}$. Since G/P is cyclic, the corollary follows from Proposition 6.22 immediately. □

Now let us put our attention on groups of cube-free order. Firstly, we note that any Sylow subgroup of a group of cube-free order is abelian. Thus, by applying Proposition 2.4 with $N = G$, all solvable groups of cube-free order are M-groups. Thanks to the work of Qiao and Li, Proposition 2.1, we have the following refinement.

Theorem 6.24. *Assume $G = \mathrm{Gal}(K/k)$ is of cube-free order. If either $|G|$ is odd or G is a solvable group with a non-abelian Hall $\{2,3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$, then Langlands reciprocity holds for K/k.*

Proof. By the celebrated Feit-Thompson theorem, if $|G|$ is odd, then G is solvable. Thus, by Proposition 2.1, if $|G|$ is odd or G is solvable with a non-abelian Hall $\{2,3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$, then G is metabelian, which is $\{1\}$-accessible. Thus, the Langlands reciprocity conjecture follows. □

We recall that for a prime p with $3 \mid p+1$, Qiao and Li in [52] gave the following examples of groups which are not metabelian.

1. $C_p^2 \rtimes S_3$.

2. $C_p^2 \rtimes C_3 \rtimes C_4$.

Observe that these groups contain normal subgroups isomorphic to $C_p^2 \rtimes C_3$, and that $\mathrm{cd}(C_p^2 \rtimes C_3) \subseteq \{1,3\}$. Applying Theorem 6.18, we know that these groups are of automorphic type. Finally, we present the following result that gives another (non-nilpotent) example of the functoriality of the tensor product.

Proposition 6.25. *Assume that G_1 is a nearly nilpotent group and that G_2 is of order which is not divisible by 36. If G_2 has a normal subgroup N_2, whose irreducible characters are of dimension at most 3, such that G_2/N_2 is nilpotent, then $G_1 \times G_2$ is of automorphic type.*

Proof. Since G_1 is nearly nilpotent, there exists N_1, with $\mathrm{cd}(N_1) \subseteq \{1,2\}$, such that G_1/N_1 is nilpotent. Also, for each irreducible character χ of $G_1 \times G_2$, there are irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that $\chi = \chi_1 \times \chi_2$. Now Horváth's theorem tells us that for each i, there exist a subnormal subgroup H_i (containing N_i) of G_i and $\psi_i \in \mathrm{Irr}(H_i)$ such that $\chi_i = \mathrm{Ind}_{H_i}^{G_i} \psi_i$ and $\psi_i|_{N_i} \in \mathrm{Irr}(N_i)$. Thus, $\chi = \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\psi_1 \times \psi_2)$, where $\psi_1(1) \leq 2$ and $\psi_2(1) \leq 3$. Thus, $\psi_1 \times 1$ and $1 \times \psi_2$ are of degree less than or equal to 2 and 3, respectively.

By the assumption on the order of G_2, if $\psi_2(1) = 3$, then ψ_2 is a monomial character. Thus, Theorems 3.21 and 3.23 yield that $1 \times \psi_2$ is of automorphic type in this case. From the above discussion and the Artin-Langlands-Tunnell theorem, both $\psi_1 \times 1$ and $1 \times \psi_2$ must be of automorphic type. Observing that

$$\psi_1 \times \psi_2 = (\psi_1 \times 1) \otimes (1 \times \psi_2),$$

the functoriality of $GL(n) \times GL(1)$, $GL(2) \times GL(2)$, and $GL(2) \times GL(3)$ asserts that $\psi_1 \times \psi_2$ is also of automorphic type. Finally, as $H_1 \times H_2$ is subnormal in $G_1 \times G_2$, applying Arthur-Clozel's theorem completes the proof. □

6.5 Applications to Frobenius Groups

We recall that G is said to be Frobenius if there is a non-trivial proper subgroup H of G such that $g^{-1}Hg \cap H = 1$ whenever $g \in G \backslash H$. From the theory of Frobenius groups (cf. Chapter 2), we have the following lemma.

Lemma 6.26. *Suppose $G = N \rtimes H$ is a Frobenius group with Frobenius kernel N and solvable Frobenius complement H. If H is of automorphic type, then so is G.*

Proof. Let χ be an irreducible character of $G = \mathrm{Gal}(K/k)$. If $\mathrm{Ker}\,\chi$ contains N, then χ can be seen as an irreducible character of H. As H is of automorphic type, χ is automorphic over k. Otherwise, if $N \not\subseteq \mathrm{Ker}\,\chi$, then by Proposition 2.7, there is a $\psi \in \mathrm{Irr}(N)$ such that $\chi = \mathrm{Ind}_N^G \psi$. Since N is nilpotent, N is of automorphic type. In addition, K^N/k is a solvable Galois extension, Arthur-Clozel's theory yields that χ is automorphic over k. □

Now suppose that G is a Frobenius group, and H is a Frobenius complement of G. Assume, further, that the Fitting subgroup $\mathbf{F}(H)$ of H satisfies that $H/\mathbf{F}(H)$ is nilpotent. As every Sylow subgroup of H is either cyclic or a generalised quaternion group, all irreducible characters of $\mathbf{F}(H)$ are of degree ≤ 2. Thus, Theorem 6.12 and Lemma 6.26 assert that G is of automorphic type, which gives Zhang's result [75].

By a similar argument, one has a criterion below.

Lemma 6.27. *Let $G = \mathrm{Gal}(K/k)$ be a Frobenius group with Frobenius kernel N. If Artin's conjecture is true for K^N/k, then Artin's conjecture holds for K/k.*

Let us further borrow below a structure theorem of Frobenius complements (see, for example, [50, Lemmata 18.3 and 18.4] or [28, Theorems 6.14 and 6.15]). (We note that Frobenius complements are called Frobenius subgroups in [28].)

Proposition 6.28. *If H is a solvable Frobenius complement, then either:*

Type 1. *$H = SQ$, where S is a normal cyclic subgroup of H and Q is cyclic.*

Type 2. *$H = SQ$, where $S \trianglelefteq H$ is cyclic and Q is a generalised quaternion group.*

Type 3. *H is isomorphic to $SL_2(\mathbb{F}_3)$.*

Type 4. *$H/\mathbf{F}(H) \simeq S_3$, where $\mathbf{F}(H)$ is the Fitting subgroup of H.*

Now, by the results discussed previously, and the fact that every Sylow subgroup of a Frobenius complement is either cyclic or a generalised quaternion group, we have below a theorem.

Theorem 6.29. *Suppose that K/k is a solvable Frobenius Galois extension with Galois group G. Then the Artin conjecture holds for K/k. Moreover, if a Frobenius complement of G is of Type 1, 2, or 3, then Langlands reciprocity holds for K/k.*

Moreover, applying our method of low-dimensional groups, we still can say a little more for Frobenius complements of Type 4.

Proposition 6.30. *If G is a solvable Frobenius group G with Frobenius kernel N and Frobenius complement H, then any irreducible character χ of G is of automorphic type unless $N \subseteq \operatorname{Ker}\chi$, χ is of degree 6 and induced from a non-monomial character of degree 2, and H is of Type 4.*

Proof. As we have shown before, if $N \nsubseteq \operatorname{Ker}\chi$, χ is of automorphic type. Also Theorem 6.29 asserts that if H is not of Type 4, G is of automorphic type. Thus, we may assume $H/\mathbf{F}(H) \simeq S_3$ and $N \subseteq \operatorname{Ker}\chi$. In this case, χ can be seen as a character of H. Since $H/\mathbf{F}(H)$ is isomorphic to S_3, Theorem 2.3 implies that χ must be induced from an irreducible character ψ of degree ≤ 2 of a subgroup $\widetilde{H} \leq H$ of index 1, 2, 3, 6. Now by the Arthur-Clozel theory, Theorem 3.23, and the fact that the only non-subnormal subgroup of S_3 has index 3, the assertion follows. □

Corollary 6.31. *Assume that G is a solvable Frobenius group G with Frobenius complement H. If any Sylow 2-subgroup of the Fitting subgroup $\mathbf{F}(H)$ of H is abelian, then G is of automorphic type. In particular, if 16 does not divide $|G|$, then G is of automorphic type.*

Proof. By Theorem 6.29, we may assume $H/\mathbf{F}(H)$ is isomorphic to S_3. Observe that if 16 does not divide $|G|$, then 8 cannot divide $\mathbf{F}(H)$. In this case, any Sylow 2-subgroup of $\mathbf{F}(H)$ is abelian. Since for every $p > 2$, all Sylow p-subgroups of H are cyclic and $\mathbf{F}(H)$ is nilpotent, $\mathbf{F}(H)$ is abelian if any Sylow 2-subgroup of $\mathbf{F}(H)$ is.

Now assuming $\mathbf{F}(H)$ is abelian, the theory of relative M-groups tells us that all irreducible characters of H are monomial. Thus, Proposition 6.30 (together with its proof) implies H is of automorphic type. □

As shown in the proof of Proposition 6.30, we cannot derive the automorphy for irreducible characters of degree 6, induced from a character of degree 2. Nevertheless, if the existence of automorphic induction is assumed, one will have the following.

Theorem 6.32 (Conditional)**.** *If the non-normal cubic automorphic induction exists for all 2-dimensional cuspidal automorphic representations, then all solvable Frobenius groups are of automorphic type.*

6.6 Groups of Order at most 100

In [68], van der Waall applied group-theoretic methods together with a generalisation of Proposition 2.4 to show that all groups of order ≤ 100, twenty-four groups excepted, are monomial. Moreover, van der Waall described the 24 exceptional groups that are non-monomial. In light of the work of van der Waall, we will show that all groups, except A_5, of order at most 100, are of automorphic type.

Clearly, the trivial group is always of automorphic type. On the other hand, by the theorem of Arthur and Clozel, we know that all p-groups are of automorphic type.

Hence, if $|G|$ belongs to

$$\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$$

or

$$\{4, 8, 16, 32, 64, 9, 27, 81, 25, 49\},$$

then G is of automorphic type. There are 36 classes of groups.

According to Corollaries 6.15 and 6.19, any group of order pq, pq^2, p^2q^2, or $8p$ for some primes p and q is of automorphic type (thanks to Artin reciprocity, the Langlands-Tunnell theorem, and Arthur-Clozel's theory). Thus, if G has order 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 36, 38, 39, 40, 44, 45, 46, 50, 51, 52, 55, 56, 57, 58, 62, 63, 65, 68, 69, 74, 75, 76, 77, 82, 85, 86, 87, 88, 91, 92, 93, 94, 95, 98, 99, or 100, then G is of automorphic type. Here we have 51 classes of groups.

Now, there are only 13 remaining cases, namely, the groups of order 30, 42, 48, 54, 60, 66, 70, 72, 78, 80, 84, 90, or 96. If G is of order 30, 42, 54, 66, 70, or 78, G is of automorphic type by Corollaries 6.14 and 6.23. On the other hand, any group of order 90 has a normal subgroup of order 45, which is abelian. As a result, all groups of order 90 are metabelian and thus of automorphic type.

6.6.1 The Case $|G| = 48$

For G of order 48, G has a normal subgroup N of order 8 or 16. According to Lemma 2.9, all irreducible characters of N are of degree ≤ 2. Since G/N is either of order 3 or 6, G/N must be supersolvable. Thus, G is clearly NSS and NM. In addition, if $|G/N| = 3$, Theorem 6.12 asserts that G is of automorphic type.

Now assume $|N| = 8$. As G is an NM-group, Artin reciprocity, the Langlands-Tunnell theorem, and Theorem 3.23 ensure that every irreducible character of G of degree ≤ 3 is of automorphic type. On the other hand, we note that all irreducible representations of G are of dimension ≤ 4, which can easily be checked via GAP [20] for instance. By the fact that G is nearly supersolvable, and $[G : N] = 6$, we conclude that if χ is an irreducible character of degree 4, it must be induced from a 2-dimensional character of a subgroup H of G containing N. As $[G : H] = 2$, H is a normal subgroup, and thus χ is of automorphic type.

6.6.2 The Case $|G| = 60$

If G is of order 60, as a consequence of the Sylow theorems, G is either isomorphic to A_5, $A_4 \times C_5$, or $C_{15} \rtimes T$ where $T = C_4$ or $T = C_2^2$.

Since A_4 is of automorphic type, and C_5 is abelian, Artin reciprocity and the functoriality of $GL(n) \times GL(1)$ assert that $A_4 \times C_5$ is of automorphic type. On the other hand, for the third case, G is clearly of automorphic type thanks to Theorem 6.12. Therefore, we have the following.

Corollary 6.33. *Every non-simple group of order* 60 *is of automorphic type.*

6.6.3 The Case $|G| = 72$

Consider a group G of order 72. According to [68, Theorem II. 5.1], G is not monomial if and only if G' is the quaternion group of order 8. Moreover, a group G of order 72 with $|G'| = 24$ does not exist. Hence, any non-monomial group of order 72 must be nearly nilpotent and of automorphic type. For G monomial, G' must have order 1, 2, 4, 3, 6, 12, 9, 18, or 36. If G' is of order 1, 2, 4, 3, 6, 9, or 18, we know that

$\mathrm{cd}(G') \subseteq \{1, 2\}$, and so G is nearly nilpotent.

Now we assume G' is of order 12 or 36. By Lemma 2.10, $\mathrm{cd}(G') \subseteq \{1, 2, 3, 4\}$. On the other hand, Horváth's theorem tells us that for every $\chi \in \mathrm{Irr}(G)$, there exists a subnormal subgroup H with $G' \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_{G'} \in \mathrm{Irr}(G')$. Since every proper subgroup of a group of order 72 has been shown to be of automorphic type, if $H \neq G$, then Arthur-Clozel's theory of automorphic induction yields χ is of automorphic type. Thus, we may assume $H = G$. As G is monomial and solvable, if $\chi(1) \leq 3$, then χ is of automorphic type. Furthermore, if $4 \in \mathrm{cd}(G')$, Lemma 2.10 tells us that G' must be of order 36 and G'' is of order 9. Thus, G/G'' is a 2-group and G is nearly nilpotent.

6.6.4 The Case $|G| = 80$

Also a straightforward application of Sylow's theory yields that every group of order $16p$ has a normal Sylow subgroup unless $p = 3$. As a consequence, Lemma 2.9 and Theorem 6.12 assert every group of order $16p$ is of automorphic type unless $p = 3$. As shown above, for G of order 48, G is of automorphic type, and we hence have:

Corollary 6.34. *If G is of order $16p$, then G is of automorphic type. In particular, if $|G| = 80$, then G is of automorphic type.*

6.6.5 The Case $|G| = 84$

For G of order 84, by Proposition 2.1, G is either of the form $G = C_7 \rtimes C_3 \rtimes G_{\{2\}}$ or metabelian. By Theorem 2.3, it is easy to see that $\mathrm{cd}(C_7 \rtimes C_3) \subseteq \{1, 3\}$. As $36 \nmid |G|$, Theorem 6.18 asserts that G is of automorphic type. Thus, it remains to consider groups of order 96.

6.6.6 The Case $|G| = 96$

For the last case, $|G| = 96$, if $|G'|$ is 1, 2, 4, 8, 16, 3, 6, 12, or 24, then as above, we have $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$. Hence, by Theorem 6.18, G is of automorphic type.

Let $|G'|$ be 48. Then [68, Theorem II. 6.2] tells us that G'' is of order 16 and abelian. Since G/G'' is supersolvable, Theorem 2.3 asserts that for any $\chi \in \mathrm{Irr}(G)$, there exists a subgroup H with $G'' \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_{G''} \in \mathrm{Irr}(G'')$, and hence $\mathrm{cd}(G) \subseteq \{1, 2, 3, 6\}$. We note that if $\chi(1) = 6$, then it must be induced from a linear character of G'', which is normal in G. Thus, Arthur-Clozel's theory implies that χ is of automorphic type. Again, as G is monomial and solvable, if $\chi(1) \leq 3$, then χ is of automorphic type.

Now it remains to consider the case $|G'| = 32$. Let $\Phi(G')$ stand for the Frattini subgroup of G', i.e., the intersection of all maximal subgroups of G'. We recall that a p-group is termed extra-special if its centre, derived subgroup and Frattini subgroup all coincide. By the classification, [68, Theorem II. 6.5], we have either:

1. G is not monomial if and only if $\mathbf{Z}(G')$ is of order 8 and $\Phi(G')$ is of order 8 or 2 ([68, Cases (4-a) and (4-b)]); or

2. G is monomial if and only if $G' = Q * Q$, the extra-special group of order 32 of (+)-type ([68, Case (4-d-2)]).

For the first case, we note that if $|\mathbf{Z}(G')| = |\Phi(G')| = 8$, then van der Waall showed that $|G''| = 2$, which implies that $G' \in \Gamma_2$. (Here, Γ_2 is the Hall-Senior family of groups with the derived subgroups isomorphic to C_2 and the inner automorphism groups isomorphic to V_4.) It can be checked, by using the computer algebra package [20] (or even rather easily, but more tediously, by hand), that $\mathrm{cd}(G') = \{1, 2\}$ in this case. On the other hand, if $|\mathbf{Z}(G')| = 8$ and $|\Phi(G')| = 2$, then van der Waall proved that $G' \simeq C_2^2 \times Q$, which gives $\mathrm{cd}(G') = \{1, 2\}$. Thus, G is nearly nilpotent.

For the second case, van der Waall (see [68, pp. 125-126]) showed that for every irreducible representation ρ of G, either ρ can be regarded as a representation of $G/\mathbf{Z}(G')$, or ρ is faithful, monomial, and of dimension 4.

As remarked by van der Waall, $G/\mathbf{Z}(G')$ has the abelian derived subgroup and hence is monomial. We further note that this comment, in fact, tells us that $G/\mathbf{Z}(G')$ is metabelian and hence of automorphic type.

Finally, we assume ρ is faithful. Thus, $\rho(G)$ is a solvable subgroup of order 96 of $GL_4(\mathbb{C})$. As noted in [43, Chapter 4], since any scalar matrix in $\rho(G)$ lies in its centre $\mathbf{Z}(\rho(G))$ and Schur's lemma implies that $\mathbf{Z}(\rho(G))$ is contained inside the set of scalar matrices, the projective image of ρ in $PGL_4(\mathbb{C})$ is isomorphic to $\rho(G)/\mathbf{Z}(\rho(G)) \simeq G/\mathbf{Z}(G)$. Since $G \simeq (Q * Q) \rtimes C_3$, as may be checked in GAP [20] for example, one can deduce that $G/\mathbf{Z}(G)$ is isomorphic to $V_4 \rtimes A_4$. By a result of Martin, Theorem 3.19, ρ is of automorphic type which completes the proof.

Chapter 7

Concluding Remarks and Future Directions

Problems in arithmetic or, more generally, in mathematics enlighten the path for us to discover and understand new concepts. As discussed earlier, for any extension K/k of number fields, Dedekind conjectured that the quotient $\zeta_K(s)/\zeta_k(s)$ of the Dedekind zeta functions is entire. This indeed led Artin to his L-functions and holomorphy conjecture. It may be that we shall not see the complete resolution of either Dedekind's conjecture or Artin's conjecture shortly. However, they illuminate a deep relation among algebra, analysis, and arithmetic.

We recall that via works of Aramata-Brauer and Uchida-van der Waall, Dedekind's conjecture is valid whenever K/k is Galois, or K is contained in a solvable normal closure of k. Two key ingredients in their proofs are Artin reciprocity and the theory of monomial representations. Furthermore, they provide the background for the theory of Heilbronn characters. In light of these and the automorphy result for certain 2-dimensional (Galois) representations due to Langlands-Tunnell and Khare-Wintenberger, it may be interesting and possible to extend previous results via character theory as follows.

1. Investigate Dedekind's conjecture for non-solvable cases.

2. Study Heilbronn characters involving characters induced from characters of degree at most 2.

Also, via Arthur-Clozel's theory of automorphic induction, if K/k is a solvable Galois extension, then the quotient $\zeta_K(s)/\zeta_k(s)$ is equal to an automorphic L-function over k. Inspired by the result of Uchida and van der Waall, we further propose the following:

3. Study the strong Dedekind conjecture for K contained in a solvable normal closure of k. (That is, we want to show $\zeta_K(s)/\zeta_k(s)$ is automorphic over k.)

In a slightly different theme, we remark that the methods introduced in Chapter 6 allow one to study the Langlands reciprocity conjecture for solvable Galois extensions via elementary group theory (e.g. Sylow's theorems). Indeed, for solvable G, one can also argue using the derived subgroup G'. More precisely, as G/G' is abelian, our results obtained enable one to investigate the automorphy of G by simply considering $\mathrm{cd}(G')$, the set of character degrees of G', which can be easily computed via the computer algebra package [20]. From this, we have the following project in our minds.

4. Use a mix of theory and computation to investigate the automorphy of solvable groups of order greater than 100.

The Langlands program has provided us with an exuberant interplay of number theory and representation theory. Indeed, since the analytic theory of automorphic L-functions is well-developed, the "automorphy connection" allows us to study L-functions associated to arithmetic objects easier and resolve several famous conjectures including Fermat's last theorem (which follows from the modularity theorem of Wiles) and the Sato-Tate conjecture (which follows from the potential automorphy results of Taylor et al.). Thus, it is interesting and natural to seek arithmetic applications of the results presented in this thesis. For instance, in sieve theory, to study primes satisfying Chebotarev conditions, one of the main tools is a variant of the Bombieri-Vinogradov theorem due to M. Ram Murty and V. Kumar Murty [45], and Langlands reciprocity plays the crucial role in obtaining a better "level of distribution" in their theorem. From this, we propose our last project:

5. Find more applications of Langlands reciprocity in analytic number theory, especially, in sieve theory. We hope to apply this research to classical questions such as the Goldbach conjecture, the twin prime conjecture, the Artin primitive root conjecture, and the Lang-Trotter conjectures.

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