# Transcendence of Various Infinite Series
## and
# Applications of Baker's Theorem

by

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# Abstract


We consider various infinite series and examine their arithmetic nature. Series of interest are of the form

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}, \quad \sum_{n\in\mathbb{Z}} \frac{f(n)A(n)}{B(n)}, \quad \sum_{n=0}^{\infty} \frac{z^n A(n)}{B(n)}$$

where $f$ is algebraic valued periodic function, $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ and $z$ is an algebraic number with $|z| \leq 1$. We also examine multivariable extensions

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}$$

and

$$\sum_{n_1,\ldots,n_k\in\mathbb{Z}} \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}.$$

These series are all very natural things to write down and we would like to understand them better. We calculate closed forms using various techniques. For example, we use relations between Hurwitz zeta functions, digamma functions, polygamma functions, Fourier analysis, discrete Fourier transforms, among other objects and techniques. Once closed forms are found, we make use of some of the well-known transcendental number theory including the

theorem of Baker regarding linear forms in logarithms of algebraic numbers to determine their arithmetic nature.

In one particular setting, we extend the work of Bundschuh [4] by proving the following series are all transcendental for positive $c \in \mathbb{Q} \setminus \mathbb{Z}$ and $k$ a positive integer:

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+c)^k}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^4-c^4)^{2k}}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^6-c^6)^{2k}}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^3 \pm c^3)^{2k}}$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 \pm c^3}, \quad \sum_{|n|\geq 2} \frac{1}{n^3-1}, \quad \sum_{|n|\geq 2} \frac{1}{n^4-1}, \quad \sum_{|n|\geq 2} \frac{1}{n^6-1}.$$

Bundschuh conjectured that the last three series are transcendental, but we offer the first unconditional proofs of transcendence.

We also show some conditional results under the assumption of some well-known conjectures. In particular, for $A_i(x), B_i(x) \in \overline{\mathbb{Q}}[x]$ with each $B_i(x)$ has only simple rational roots, if Schanuel's conjecture is true, the series (avoiding roots of the denominator)

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}$$

is either an effectively computable algebraic number or transcendental.

We also show that Schanuel's conjecture implies that the series

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

is either zero or transcendental, when $B(x)$ has non-integral roots.

We develop a general theory, analyzing various infinite series throughout.

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# Statement of Originality

The results of this thesis are original research, except of course where credit has been given. Within each chapter it has been spelled out clearly where methods are new and where methods are being adapted from previous work of others. Due credit has been given to those whose work came before this thesis.

# Contents

# Chapter 1

# Introduction

A formal Dirichlet series is defined as

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $s$ and $a_n, (n = 1, 2, 3, \ldots)$ are complex numbers. The most famous of these series is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for $\Re(s) > 1$. A basic question we can ask is "what do these series converge to?" in the case that they converge. Usually we can only say something specific about special cases. For instance, for $n$ a positive integer, we have

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where $B_{2n}$ is the $2n^{th}$ Bernoulli number. This equation was known to Euler. The formula shows that the values at even arguments of the Riemann zeta function have the property of being transcendental, which we define now.

**Definition 1.1.** A complex number is said to be **algebraic** if it is the root of a polynomial with integer coefficients.

**Definition 1.2.** A complex number is said to be **transcendental** if it is not the root of any polynomial with integer cofficients.

All complex numbers are either algebraic or transcendental, and this property is the central interest of this thesis.

Other series that are of of interest are Dirichlet $L-$series defined by

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for a periodic function $f$ and a complex number $s$. Much work on special values of these series has originated from a question of Chowla.

In 1969, during a conference on number theory at Stony Brook, S. Chowla [5] asked if there was a rational valued periodic function, that is not identically zero, with prime period $p$ such that the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges and vanishes. In 1973, Baker, Birch and Wirsing [3] used Baker's theory of linear forms in logarithms of algebraic numbers to answer this question with the following theorem:

**Theorem 1.3** *If $f$ is a non-vanishing function defined on the integers with algebraic values and period $q$ such that (i) $f(r) = 0$ if $1 < (r, q) < q$, (ii) the* *qth cyclotomic polynomial is irreducible over* $\mathbb{Q}(f(1), \ldots, f(q))$*, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In 2001, Adhikari, Saradha, Shorey, and Tijdeman [1] and then in 2007 Murty and Saradha [18], took things further and showed that for an algebraic valued periodic function, $f$, the sum

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

is either zero or transcendental when it converges. In [1] and [18], the authors also examined series of the form

$$\sum_{n=0}^{\infty} \frac{A(n)}{B(n)} \quad \text{and also} \quad \sum_{n=0}^{\infty} \frac{f(n)}{B(n)}$$

where $A(x)$ and $B(x)$ are polynomials with rational coefficients, $B(x)$ having only simple rational roots. More recently the authors of [10] examine similar series, introducing cases where $B(x)$ does not have simple roots. For this thesis, we extend the results on various infinite series, and offer our own proofs of some of their results when there is a simplification or alternate method. A common theme throughout the thesis is to make good use of the following theorem.

**Theorem 1.4** (A. Baker, [2]) *Let* $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$*. If* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$*, then* $1, \log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\overline{\mathbb{Q}}$*.*

Note that throughout this work, we take the principal value of the logarithm with argument in $(-\pi, \pi]$. It is easy to see that Theorem 1.4 (hereafter referred to as Baker's theorem) implies that any $\overline{\mathbb{Q}}$-linear form in logarithms of algebraic numbers is either zero or transcendental. This is the extent that Baker's theorem will be used.

In some cases we cannot conclude anything concrete with the state of the art being where it is. However, in some instances, we show that the well-known Schanuel's conjecture implies certain transcendence results. We state Schanuel's conjecture here.

**Conjecture 1.5** (S. Schanuel, [11]) *For any complex numbers $z_1, \ldots, z_n$ which are linearly independent over $\mathbb{Q}$, we have that the field*

$$\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$$

*has transcendence degree at least $n$ over $\mathbb{Q}$.*

In the above situation, Schanuel's conjecture says that there are at least $n$ algebraically independent numbers in the list $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$. There is no known proof of this conjecture, but if true, the conjecture does imply some nice results.

We now give an overview of the thesis clearly pointing out new contributions.

## 1.1 Overview and Contributions

Throughout the thesis we usually (unless otherwise specified) study series composed of a periodic function $f$, and algebraic number $|z| \leq 1$, and polynomials $A(x), B(x) \in \overline{\mathbb{Q}}[x]$. It is often the case that we compute a closed form for a series which involves what we call an effectively computable number. We point out that in [1] and [10], the authors call these numbers computable, however we wish to describe their nature a little more precisely. We define this here.

**Definition 1.6.** We say a number is **effectively computable** if there exists some terminating algorithm (or Turing machine) that can approximate it to arbitrary precision. In the usual setting, these numbers will lie in some algebraic number field generated by the values of $f$, $z$, the coefficients of $A(x)$ and $B(x)$, the roots of $B(x)$ and in some settings, some other obvious finite extension.

For our purposes, we will calculate these effectively computable numbers explicitly, or the algorithm to compute them will be clear. The isolation of these effectively computable numbers and our motivation will become clear when we first see them in Theorem 2.8.

Chapter 2 deals with series similar to those found in [18] and [1]. Namely we examine

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

where $f$ is a periodic function and $A(x), B(x)$ are polynomials. The methods found here are adapted from those found in [18] and used to extend results to a larger family of series. We calculate closed forms and build a general theory to analyze the arithmetic nature. We also give a concrete example of a family of series which are all transcendental. In particular we prove the following theorem.

**Theorem 2.10** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B) - 1$, *such that* $B(x)$ *has only simple roots,* $-p_1/q_1, \ldots, -p_k/q_k \in \mathbb{Q}$ *with* $(p_i, q_i) = 1$. *If* *there is a $q_j > 1$ which is coprime to each of the other $q_i$'s, then by omitting any integral roots of $B(x)$, the series*

$$\sum_{n=0}^{\infty} \frac{A(n)}{B(n)}$$

*is transcendental.*

In chapter 3 we study series similar to the series in chapter 2, however, we exchange periodic functions for exponentials. We analyze,

$$\sum_{n=0}^{\infty} \frac{z^n A(n)}{B(n)}$$

where $z$ is an algebraic number with $|z| \leq 1$. The results of this chapter are known, but the methods used are new and quite elegant in their simplicity. We show relatively elementary methods to finding closed forms for these series, and conclude transcendence results via similar methods used in chapter 2.

In chapter 4 we generalize the series studied in chapter 2 to include multivariable series of similar form. We give a full analysis and calculate closed forms. These results are completely new and have not appeared anywhere in the literature. We have:

**Corollary 4.10** *Let $f$ be an algebraic valued function, periodic in $k$ variables. Let $A_i(x), B_i(x) \in \overline{\mathbb{Q}}$ for $i = 1, \ldots, k$ such that each $B_i(x)$ has only simple rational roots. If Schanuel's conjecture is true, the series (avoiding* *roots of the denominator)*

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}$$

*is either equal to an effectively computable algebraic number or transcendental, when it converges.*

In chapter 5, we change things slightly by changing the summation from being over the natural numbers, to summation over the integers. This allows us to relax restrictions placed on $B(x)$, while still yielding nice results. We are able to use elementary techniques to obtain closed forms for various series, and then say something about the transcendental nature of some families of series. We develop the theory and show the following:

**Theorem 5.10** *Let $f$ be an algebraic valued periodic function with integer period. Let $A(x), B(x)$ be polynomials with algebraic coefficients, $B(x)$ with only rational roots, and $\deg(A) < \deg(B)$. The series, omitting the integral roots of $B(x)$,*

$$\sum_{n\in\mathbb{Z}} \frac{f(n)A(n)}{B(n)} = P(\pi),$$

*where $P(x)$ is a polynomial with algebraic coefficients and $\deg(P) \leq \deg(B)$.*

After the general theory is developed we examine specific cases. New results here include extending some work of Bundschuh [4].

**Theorem** *For positive $c \in \mathbb{Q}\setminus\mathbb{Z}$ and $k$ a positive integer, the following series are transcendental.*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+c)^k}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^4-c^4)^{2k}}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^6-c^6)^{2k}}, \quad \sum_{n\in\mathbb{Z}} \frac{1}{(n^3\pm c^3)^{2k}}$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 \pm c^3}, \quad \sum_{|n|\geq 2} \frac{1}{n^3-1}, \quad \sum_{|n|\geq 2} \frac{1}{n^4-1}, \quad \sum_{|n|\geq 2} \frac{1}{n^6-1}.$$

Bundschuh conjectured that the last three series are transcendental, but we offer the first unconditional proofs of transcendence.

We also closely analyze the case $\sum_{n\in\mathbb{Z}} 1/B(n)$ where $B(x)$ is a cubic polynomial. In some cases we obtain unconditional transcendence by an application of a theorem of Diaz regarding algebraic independence of exponentials of algebraic numbers. In particular we have the following theorem.

**Theorem 5.48** *Let* $A(x) \in \overline{\mathbb{Q}}[x] \setminus \{0\}$ *have degree less than 3. Also let* $f(x) := x^3 + ax^2 + bx + c \in \mathbb{Q}[x]$ *be irreducible with roots* $\alpha = \alpha_1, \alpha_2, \alpha_3$*. If* $f(x)$ *splits in* $\mathbb{Q}(\alpha)$ *such that*

$$\alpha_2 = r_1 + r_2\alpha + r_3\alpha^2 \quad \text{and} \quad \alpha_3 = s_1 + s_2\alpha + s_3\alpha^2$$

*and all of the ordered pairs* $(0,0), (1,0), (r_2, r_3), (s_2, s_3), (r_2+1, r_3), (s_2+1, s_3), (r_2+s_2, r_3+s_3), (r_2+s_2+1, r_3+s_3)$ *are distinct, then the series*

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{A(n)}{n^3+an^2+bn+c}$$

*is transcendental.*

In many other cases we cannot show certain series are transcendental without assuming either Schanuel's conjecture or a conjecture of Schneider. In the final section of the thesis, we begin studying what is needed to show that these series are transcendental. We begin by assuming Schanuel's conjecture, then lower the bar to examine the minimum assumption that is needed. In some cases, we alter our series very slightly to see what happens there. In particular we conclude the thesis with the following two results.

**Corollary 5.55** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B)$ *and* $B(x)$ *having algebraic roots that are not integers. If Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

*is either zero or transcendental.*

**Theorem 5.56** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with deg*$(A) <$ *deg*$(B)$ *and coefficients* $a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}$. *Suppose that* $B(X)$ *has only simple roots* $\alpha_1, \ldots, \alpha_k \notin \mathbb{Q}$ *and* $r_1, \ldots, r_l \in \mathbb{Q} \setminus \mathbb{Z}$. *If Schneider's conjecture (5.29 is true, the series*

$$S = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

*is either transcendental or belongs to the field*

$$K := \mathbb{Q}(i, \alpha_1, \ldots, \alpha_k, a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}, e^{2\pi i r_1}, \ldots, e^{2\pi i r_l}).$$

# Chapter 2

# Infinite sums and the Digamma and Polygamma functions

As mentioned in chapter 1, in this chapter we study series which are similar to those studied in [3], [1], [18], and [10]. We give our own proofs of various results and extend some results further. In particular we study series of the form

$$\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)}$$

where $f(x)$ is a periodic function and $A(x), B(x)$ are both polynomials. We assume throughout that $A(x)$ is not identically zero. The symbol $\sum'$ is to be interpreted as summation avoiding zeroes of $B(x)$. Later we will place conditions on each and conclude certain transcendence results. We will discover these conditions throughout the chapter. We take a slightly different approach, often simplifying the proofs, in the hope of gaining more insight. As a particular example which culminates from the theory we will build up, we prove the following:

**Theorem 2.10** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B) - 1$, *such that* $B(x)$ *has only simple roots,* $-p_1/q_1, \ldots, -p_k/q_k \in \mathbb{Q}$ *with* $(p_i, q_i) = 1$. *If there is a* $q_j > 1$ *which is coprime to each of the other* $q_i$*'s, then the sum*

$$\sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)}$$

*is transcendental.*

We begin by recalling some facts about certain special functions.

## 2.1 Preliminaries

We require some knowledge of the digamma function, $\psi(x)$, which is the logarithmic derivative of the gamma function. That is

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

We have the following well-known facts (see [18]). By logarithmically differentiating each of

$$\Gamma(x+1) = x\Gamma(x)$$

and also the Hadamard product

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} (1 + x/n)\, e^{-x/n},$$

we obtain

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{2.1}$$

and

$$-\psi(x) - \gamma = \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{n+x} - \frac{1}{n} \right) \tag{2.2}$$

where $\gamma$ is Euler's constant. We also recall Lemma 21 from [18] which states that for any integers $1 \leq a \leq q$,

$$-\psi(a/q) - \gamma = \log(q) - \sum_{b=1}^{q-1} \zeta_q^{-ba} \log(1 - \zeta_q^b) \tag{2.3}$$

where $\zeta_q$ is the primitive $qth$ root of unity, $e^{2\pi i/q}$.

As we see in equation (2.3), we can relate special values of the digamma function to logarithms of algebraic numbers. This is where Baker's theory enters into this work and allows us to conclude transcendence properties after we find closed forms for our series. Following this idea, Bundschuh [4] and later Murty and Saradha [18], using different techniques, showed that (2.3) is nonzero which, by Baker's theorem, implies the following theorem.

**Theorem 2.1** (P. Bundschuh [4], M. Ram Murty/N. Saradha [18]) *Let $q > 1$. Then, $\psi(a/q) + \gamma$ is transcendental for any $1 \leq a < q$.*

Murty and Saradha also showed [18] that at most one of the numbers

$$\gamma, \;\; \psi(a/q), \text{ with } (a,q) = 1, \;\; 1 \leq a \leq q,$$

is algebraic.

We now define the so called *polygamma function*. The polygamma function $\psi_k(x)$ is defined as the $k$th derivative of the the digamma function

$\psi(x) = \psi_0(x)$. Differentiating (2.2) $k-1$ times we obtain

$$\psi_{k-1}(x) = (-1)^k (k-1)! \sum_{n=0}^{\infty} \frac{1}{(n+x)^k}. \tag{2.4}$$

Note that for the last equality we have made the assumption that $x$ is a real variable that is not a negative integer. Equation (2.4) shows a connection between the polygamma function and the Hurwitz zeta function, which is defined as

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for $0 < x \leq 1$ and $\Re(s) > 1$. In this thesis we use only the fact that, $\zeta(s,x)$ extends meromorphically to the entire complex plane, with a simple pole at $s = 1$ with residue 1. Therefore, we refer the reader to [18] for a sufficient introduction to the Hurwitz zeta function.

In the sections ahead, we will use these tools to say something about infinite sums of rational functions.

## 2.2 Series with conditional convergence

Series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where $f(x)$ is a periodic function and $s \geq 1$ is an integer, are the starting point for the work done in this chapter. In [18], the authors obtain necessary and sufficient conditions for the convergence of such series at $s = 1$. For our purposes, we examine a slightly different series,

$$\sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s}$$

where $\alpha > 0$. We begin by examining convergence. These series converge absolutely for $\Re(s) > 1$, so we restrict ourselves to $s = 1$. The convergence conditions needed here are the same as those found in [18], but for completeness, we include the details.

**Proposition 2.2** *Let $f$ be a periodic function with period $q$. For $\alpha > 0$, the series*

$$\sum_{n=0}^{\infty} \frac{f(n)}{n+\alpha}$$

*converges if and only if*

$$\sum_{a=1}^{q} f(a) = 0.$$

**Proof.** We proceed by partial summation. Let

$$A(x) = \sum_{n \le x} f(n) = [x/q] \sum_{a=1}^{q} f(a) + \sum_{a=1}^{[x] \bmod q} f(a).$$

Let $S = \sum_{a=1}^{q} f(a)$. Since $[t] = t - \{t\}$ we have

$$A(x) = \frac{x}{q} S - \left\{ \frac{x}{q} \right\} S + \sum_{a=1}^{[x] \bmod q} f(a).$$

By partial summation we have that the series $\sum_{n\leq x} f(n)/(n+\alpha)$ is equal to

$$\frac{xS}{q(x+\alpha)} - \frac{\{x/q\}S}{x+\alpha} + \frac{1}{x+\alpha}\sum_{a=1}^{[x] \bmod q} f(a) + \int_1^x \frac{St/q - \{t/q\}S + \sum_{a=1}^{[t] \bmod q} f(a)}{(t+\alpha)^2}dt$$

As we send $x$ to infinity, the term

$$\frac{S}{q}\int_1^x \frac{t}{(t+\alpha)^2}dt$$

is the only term which does not converge unless we require that $S = 0$. When we have $S = 0$,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n+\alpha} = \int_1^{\infty} \frac{\sum_{a=1}^{[t] \bmod q} f(a)}{(t+\alpha)^2}dt.$$

∎

Now that convergence is understood, we relate various series to special values of the digamma function. We are now ready to prove a result similar to Theorem 16 in [18].

**Theorem 2.3** *Let $f$ be a periodic function with period $q \geq 1$ such that*

$$\sum_{a=0}^{q-1} f(a) = 0$$

*and take $\alpha > 0$. We have*

$$\sum_{n=0}^{\infty}\frac{f(n)}{n+\alpha} = \frac{-1}{q}\sum_{a=0}^{q-1} f(a)\psi(\frac{a+\alpha}{q}).$$

**Proof.** For $\Re(s) > 1$,

$$\sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^s} = \frac{1}{q^s} \sum_{a=0}^{q-1} f(a) \sum_{m=0}^{\infty} \frac{1}{(m + \frac{a+\alpha}{q})^s} = \frac{1}{q^s} \sum_{a=0}^{q-1} f(a) \zeta(s, \frac{a+\alpha}{q}).$$

This last series appears to have a simple pole at $s = 1$ since the Hurwitz zeta function has such a pole. Luckily the residue at $s = 1$ is $\sum_{a=0}^{q-1} f(a)/q = 0$ and we have analytic continuation of this series via analytic continuation of the Hurwitz zeta function. We insert the Riemann zeta function into this last equation since $\sum_{a=0}^{q-1} f(a)\zeta(s) = 0$. This yields

$$\frac{1}{q^s} \sum_{a=0}^{q-1} f(a) \left( \zeta(s, \frac{a+\alpha}{q}) - \zeta(s) \right).$$

We now proceed with limits. We have

$$\lim_{s \to 1^+} \zeta(s, x) - \zeta(s) = \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{n+x} - \frac{1}{n} \right)$$

which is equal to $-\psi(x) - \gamma$ by equation (2.2). Thus, taking the limit as $s$ goes to 1 we have,

$$\sum_{n=0}^{\infty} \frac{f(n)}{n+\alpha} = \frac{-1}{q} \sum_{a=0}^{q-1} f(a) \left( \psi(\frac{a+\alpha}{q}) + \gamma \right) = \frac{-1}{q} \sum_{a=0}^{q-1} f(a) \psi(\frac{a+\alpha}{q}).$$

∎

If we let $\alpha \in \mathbb{Q}$, we can now say something about the transcendence of such a series by equation (2.3) and then incorporating Baker's theorem.

Instead of stating a transcendence theorem for this previous series, we prove a more general result.

**Corollary 2.4** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *such that the* $\deg(A) < \deg(B)$ *and* $B(x)$ *has simple rational roots in* $[-1, 0)$. *Let* $f$ *be an algebraic valued periodic function as above, with* $\sum_{a=0}^{q-1} f(a) = 0$. *The series*

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

*is either zero or transcendental.*

**Proof.** Let $-\alpha_1, \ldots, -\alpha_k$ be the roots of $B(x)$, so that each $\alpha_i \in (0, 1]$. Since $B(x)$ has only simple roots, by partial fractions we can write

$$\frac{A(x)}{B(x)} = \sum_{i=1}^{k} \frac{c_i}{x + \alpha_i}$$

where each $c_i$ is some algebraic number. We can now rewrite our sum and use the previous theorem.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)} &= \sum_{i=1}^{k} c_i \sum_{n=0}^{\infty} \frac{f(n)}{n + \alpha_i} \\ &= \frac{-1}{q} \sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \left( \psi(\frac{a + \alpha_i}{q}) + \gamma \right) \end{aligned}$$

Writing $\alpha_i = p_i/q_i$, by (2.3) we have that our series is equal to,

$$\frac{-1}{q} \sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \left( -\log(q_i q) + \sum_{b=1}^{q_i q-1} \zeta_{q_i q}^{-b(aq_i + p_i)} \log(1 - \zeta_{q_i q}^{b}) \right).$$

The sum of the $f(a)$'s is zero, so we ignore the $\log(q_i q)$ terms. We next interchange the order of summation giving

$$-\sum_{i=1}^{k} c_i \sum_{b=1}^{q_i q-1} \zeta_{q_i q}^{-bp_i} \log(1-\zeta_{q_i q}^{b}) \left(\frac{1}{q}\right) \sum_{a=0}^{q-1} f(a)\zeta_q^{-ba}$$

which is equal to

$$-\sum_{i=1}^{k} c_i \sum_{b=1}^{q_i q-1} \widehat{f}(b)\zeta_{q_i q}^{-bp_i} \log(1-\zeta_{q_i q}^{b}).$$

where $\widehat{f}$ is the discrete Fourier transform of $f$ given by

$$\widehat{f}(n) := \frac{1}{q}\sum_{a=1}^{q} f(a)\zeta_q^{-an}.$$

We now have an algebraic linear combination of logarithms of algebraic numbers. By Baker's theorem, the sum is either zero or transcendental. ∎

Notice that in the previous theorem, we put certain restrictions on the roots of the denominator, $B(x)$. We assumed that the roots of $B(x)$ were simple and all within a short interval. In [1], the authors call such a $B(x)$ **reduced**. Later we will relax these restrictions.

At this point it doesn't seem possible to characterize exactly when the series is equal to zero. One reason is that for various integers $q_i q$, as in the theorem above, the values $\log(1-\zeta_{q_i q}^{b})$ are not linearly independent over $\mathbb{Q}$. For such values there are two ways that the series can converge to zero. One way is to have nonzero coefficients in front of the logarithms, but have cancellation since the logarithms are not linearly independent. The other way, for the series to vanish, is for each coefficient to be identically zero in some nontrivial way. In the case that the values $\log(1-\zeta_{q_i q}^b)$ are indeed linearly independent over $\mathbb{Q}$, the series $\sum_{n=0}^{\infty}\frac{f(n)A(n)}{B(n)}$ vanishes only when the coefficient for each logarithm is zero.

It is not obvious how to characterize either case. However, due to a formula of Gauss and a theorem of Murty and Saradha [16], we can give some conditions that will guarantee that our series does not vanish, which is a good starting point for understanding the whole story.

We have the following formula of Gauss which can be used to give an alternate closed form for the types of series we are interested in. Gauss first showed the formula in 1813, but Lehmer has a nice proof in [13]. For $1 \leq a < q$,

$$\psi(a/q) + \gamma = -\log 2q - \frac{\pi}{2}\cot\frac{\pi a}{q} + 2\sum_{0<j\leq q/2}\left(\cos\frac{2\pi a j}{q}\right)\log\sin\frac{\pi j}{q} \quad (2.5)$$

The formula relates the digamma function and $\gamma$ to a sum of logarithms of positive algebraic numbers and an algebraic multiple of $\pi$. The $\pi$ that appears is important due to the following lemma which we state without proof.

**Lemma 2.5** (Murty/Saradha, [16]) *Let $\alpha_1, \ldots, \alpha_n$ be positive algebraic numbers. If $c_0, c_1, \ldots, c_n$ are algebraic numbers with $c_0 \neq 0$, then*

$$c_0\pi + \sum_{j=1}^{n} c_j \log\alpha_j \neq 0$$

*and is a transcendental number.*

Apply this lemma to our series we obtain the following theorem.

**Theorem 2.6** *Let $A(x), B(x)$ and $f$ be as in Corollary 2.4. If*

$$\sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \cot\left(\frac{\pi(a+\alpha_i)}{q}\right) \neq 0$$

*then the series*

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

*is transcendental.*

**Proof.** As in the proof of Corollary 2.4, we rewrite our series as

$$\frac{-1}{q} \sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \left(\psi(\frac{a+\alpha_i}{q}) + \gamma\right).$$

Writing $\alpha_i = p_i/q_i$, by equation (2.5) each $\psi(\frac{a+\alpha_i}{q}) + \gamma$ is equal to

$$-\frac{\pi}{2} \cot\left(\frac{\pi(aq_i + p_i)}{qq_i}\right) + L_{i,a}$$

where $L_{i,a}$ is an algebraic combination of logs,

$$L_{i,a} = -\log(2qq_i) + 2 \sum_{0<j\leq qq_i/2} \cos\left(\frac{2\pi(aq_i+p_i)j}{qq_i}\right) \log\left(\sin\left(\frac{\pi j}{qq_i}\right)\right).$$

Grouping together the coefficients of $\pi$, by Lemma 2.5 we have the result. ∎

We will see the use of Gauss's formula again later in this chapter.

## 2.3 Series with absolute convergence

Next we change our setting ever so slightly. We no longer require that $\sum_{a=0}^{q-1} f(a) = 0$, but instead require that $\deg(A) < \deg(B) - 1$ to ensure we have convergence. Although this case does have overlap with the previous case, it is worth examining both for completeness. We begin immediately with a result very similar to Corollary 2.4.

**Theorem 2.7** *Let $f$ be an algebraically valued periodic function with period $q \geq 1$. Let $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ such that the* $\deg(A) < \deg(B) - 1$ *and $B(x)$ has simple rational roots in $[-1, 0)$. The series*

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

*is either zero or transcendental.*

**Proof.** Let $-\alpha_1, \ldots, -\alpha_k$ be the roots of $B(x)$. Since $B(x)$ has only simple roots, by partial fractions we can write

$$\frac{A(x)}{B(x)} = \sum_{i=1}^{k} \frac{c_i}{x + \alpha_i}$$

where each $c_i$ is some algebraic number. It turns out that $c_i = A(-\alpha_i)/B'(-\alpha_i)$ and considering the degrees of $A(x)$ and $B(x)$ we see that

$$\sum_{i=1}^{k} c_i = 0.$$

For reasons which will soon become clear, we separate our sum into two parts.

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)} = \sum_{n=q}^{\infty} \frac{f(n)A(n)}{B(n)} + \sum_{a=0}^{q-1} \frac{f(a)A(a)}{B(a)}.$$

We now analyze the first sum from the right hand side.

$$\sum_{n=q}^{\infty} \frac{f(n)A(n)}{B(n)} = \sum_{a=0}^{q-1} f(a) \sum_{n\equiv a(q), n\geq q} \sum_{i=1}^{k} \frac{c_i}{n+\alpha_i}$$

Since $\sum_{i=1}^{k} c_i = 0$, we insert $\sum_{i=1}^{k} \frac{-c_i}{n-a}$ into the innermost sum and obtain

$$\sum_{a=0}^{q-1} f(a) \sum_{n\equiv a(q), n\geq q} \sum_{i=1}^{k} c_i \left( \frac{1}{n+\alpha_i} - \frac{1}{n-a} \right)$$

which, after interchanging summation, is equal to

$$\sum_{a=0}^{q-1} f(a) \sum_{i=1}^{k} c_i \sum_{n\equiv a(q), n\geq q} \left( \frac{1}{n+\alpha_i} - \frac{1}{n-a} \right)$$

which equals

$$\frac{1}{q} \sum_{a=0}^{q-1} f(a) \sum_{i=1}^{k} c_i \sum_{m=1}^{\infty} \left( \frac{1}{m+(a+\alpha_i)/q} - \frac{1}{m} \right)$$

By equation (2.2), the innermost sum is equal to $-\psi(\frac{a+\alpha_i}{q}) - \gamma - \frac{q}{a+\alpha_i}$. Thus our sum is equal to

$$\frac{-1}{q}\sum_{a=0}^{q-1} f(a)\sum_{i=1}^{k} c_i\left(\psi\left(\frac{a+\alpha_i}{q}\right)+\gamma+\frac{q}{a+\alpha_i}\right)$$

which can be written as

$$\frac{-1}{q}\sum_{a=0}^{q-1} f(a)\sum_{i=1}^{k} c_i\left(\psi\left(\frac{a+\alpha_i}{q}\right)+\gamma\right)-\sum_{a=0}^{q-1}\frac{f(a)A(a)}{B(a)}.$$

Notice that the last sum is exactly the part which we omitted earlier. Putting it all together we have

$$\sum_{n=0}^{\infty}\frac{f(n)A(n)}{B(n)}=\frac{-1}{q}\sum_{a=0}^{q-1} f(a)\sum_{i=1}^{k} c_i\left(\psi\left(\frac{a+\alpha_i}{q}\right)+\gamma\right).$$

Similar to Corollary 2.4, we utilize equation (2.3) to relate values of the digamma function to a linear combination of logarithms. Writing $\alpha_i = p_i/q_i$ we obtain

$$\frac{-1}{q}\sum_{a=0}^{q-1} f(a)\sum_{i=1}^{k} c_i\left(\psi\left(\frac{aq_i+p_i}{q_iq}\right)+\gamma\right)$$

and by equation (2.3) this is equal to

$$\frac{-1}{q}\sum_{a=0}^{q-1} f(a)\sum_{i=1}^{k} c_i\left(-\log(q_iq)+\sum_{b=1}^{q_iq-1}\zeta_{q_iq}^{-b(aq_i+p_i)}\log(1-\zeta_{q_iq}^{b})\right).$$

At this point we simplify things by inserting the Fourier transform of $f$ to obtain,

$$\sum_{i=1}^{k} c_i\left(\widehat{f}(0)\log(q_iq)-\sum_{b=1}^{q_iq-1}\widehat{f}(b)\zeta_{q_iq}^{-bp_i}\log(1-\zeta_{q_iq}^{b})\right).$$

The end result is again a linear form in logarithms of algebraic numbers. By Baker's theorem, we have that the series $\sum_{n=0}^{\infty} f(n)A(n)/B(n)$ is either zero or transcendental. ∎

Note that in Corollary 2.4, we assumed that $\widehat{f}(0) = (1/q)\sum_{a=0}^{q-1} f(a) = 0$. The formula shown in this previous proposition reduces to the same form found in Corollary 2.4. Thus, in both cases we have the same closed form.

Ideally we would like to know exactly when series of these types converge to zero. However, as stated above, it is difficult to characterize when they vanish. For now, we leave that problem open and move onto a more general series.

We conclude this section by noting that we can also obtain formulae for various series where we do not restrict the roots of $B$ to be in $[-1, 0)$. We simply sum over all values of $n \geq 0$ where $B(n) \neq 0$. The end result is very similar, with the exception that instead of getting a linear form in logarithms of algebraic numbers, we obtain an algebraic number plus a linear form in logarithms. We show this result here.

**Theorem 2.8** *Let $f$ be an algebraically valued periodic function with integer period $q \geq 1$. Let $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ such that $B(x)$ has only simple roots in $\mathbb{Q}$. Omitting the roots of $B(x)$, the series*

$$\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)}$$

*is equal to an effectively computable algebraic number or is transcendental, when it converges.*

**Proof.** Let $-\alpha_1, \ldots, -\alpha_k$ be the roots of $B(x)$. Take the minimal natural number $r = qt$ such that $|\alpha_i| \leq r - 1$. We separate our series into two parts.

$$\begin{aligned}\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)} &= \sum_{n=0}^{r-1}{}' \frac{f(n)A(n)}{B(n)} + \sum_{n=r}^{\infty} \frac{f(n)A(n)}{B(n)} \\ &= \beta_1 + \sum_{n=0}^{\infty} \frac{f(n+r)A(n+r)}{B(n+r)}\end{aligned}$$

We took $r$ to be a multiple of $q$ so that $f(n+r) = f(n)$ for simplification. By partial fractions we can write

$$\frac{A(x+r)}{B(x+r)} = \sum_{i=1}^{k} \frac{c_i}{x + r + \alpha_i}$$

and using the same methods as above we have that our series is equal to

$$\beta_1 + \frac{-1}{q} \sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \left( \psi \left( \frac{a + r + \alpha_i}{q} \right) + \gamma \right).$$

For each pair $(a, i)$ we use (2.1) to write

$$\psi \left( \frac{a + r + \alpha_i}{q} \right) = \psi \left( \frac{p_{i,a}}{q_{i,a}} \right) + \beta_{i,a}$$

where $p_{i,a}/q_{i,a} \in (0, 1]$ and $\beta_{i,a}$ are rational numbers. Note that if we write $\alpha_i = p_i/q_i$, then $q_{i,a} = q_i q$. This allows us to once again use equation (2.3), and we have our sum is equal to

$$\beta + \frac{1}{q} \sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \left( \log(q_i q) - \sum_{b=1}^{q_i q - 1} \zeta_{q_i q}^{-b p_{i,a}} \log(1 - \zeta_{q_i q}^{b}) \right)$$

where $\beta = \left(\beta_1 - \frac{1}{q}\sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a)\beta_{i,a}\right)$. Notice that from the equality

$$\frac{x^q - 1}{x - 1} = 1 + x + \cdots + x^{q-1} = \prod_{b=1}^{q-1}(x - \zeta_q^b),$$

by letting $x = 1$ we see that $q = \prod_{b=1}^{q-1}(1 - \zeta_q^b)$ and we have that

$$\log(q) = \sum_{b=1}^{q-1} \log(1 - \zeta_q^b).$$

Using this identity we can simplify our sum to

$$\beta - \frac{1}{q}\sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \sum_{b=1}^{q_i q-1} (\zeta_{q_i q}^{-bp_{i,a}} - 1)\log(1 - \zeta_{q_i q}^b).$$

Our series is equal to a finite sum of algebraic numbers, depending only on $f, A$ and $B$, and an algebraic linear combination of logarithms of algebraic numbers. By Baker's theorem we are done. ∎

Similar to Theorem 2.6, we use Gauss's formula and Lemma 2.5 to show a certain conditional case of Theorem 2.8 is transcendental.

**Theorem 2.9** *Let $A(x), B(x)$ and $f$ be as in Theorem 2.8. For each pair $(a, i)$, take $s_{i,a} \equiv aq_i + p_i$ mod $(qq_i)$ such that $0 < s_{i,a} \leq qq_i$. If each $s_{i,a} < qq_i$ and*

$$\sum_{i=1}^{k} c_i \sum_{a=0}^{q-1} f(a) \cot\left(\frac{\pi s_{i,a}}{qq_i}\right) \neq 0$$

*then the series*

$$\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)}$$

*is transcendental.*

**Proof.** Starting from the line in the proof of Theorem 2.8 where we have that our series is equal to

$$\beta_1 - \frac{1}{q}\sum_{i-1}^{k} c_i \sum_{a=0}^{q-1} f(a)\left(\psi(\frac{a+r+\alpha_i}{q}) + \gamma\right)$$

we shift the digamma function the appropriate amount using 2.1 to obtain an algebraic number plus

$$-\frac{1}{q}\sum_{i-1}^{k} c_i \sum_{a=0}^{q-1} f(a)\left(\psi(\frac{s_{i,a}}{qq_i}) + \gamma\right).$$

Applying Gauss's formula (equation (2.5)) and collecting the coefficients of $\pi$, the result become clear. ∎

In the next section we give a family of series which are transcendental without assuming such conditions as in Theorems 2.6 and 2.9.

## 2.4 A family of transcendental series

As mentioned, it is not easy to characterize exactly when such series are algebraic. The following theorem illustrates at least one general case when the sum is equal to a transcendental number.

**Theorem 2.10** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B) - 1$*, such that* $B(x)$ *has only simple roots,* $-p_1/q_1, \ldots, -p_k/q_k \in \mathbb{Q}$ *with* $(p_i, q_i) = 1$*. If* *there is a $q_j > 1$ which is coprime to each of the other $q_i$'s, then the sum*

$$\sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)}$$

*is transcendental.*

**Proof.** Without loss of generality, we assume that $q_1 > 1$ and $(q_1, q_i) = 1$ for each $i \neq 1$. Similar to the proof of Theorem 2.8, let $r$ be the minimal natural number $r = qt$ such that $r > |p_i/q_i|$ for every $i = 1, \ldots, k$ so that our series

$$\sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)} = \beta - \sum_{i=1}^{k} c_i \left(\psi(r + p_i/q_i) + \gamma\right),$$

for some $\beta \in \overline{\mathbb{Q}}$. Using the identity $\psi(1 + z) = \psi(z) + 1/z$ we rewrite our sum as

$$\widetilde{\beta} - \sum_{i=1}^{k} c_i \left(\psi(\widetilde{p_i}/q_i) + \gamma\right),$$

where $0 < \widetilde{p_i} \leq q_i$ and $\widetilde{\beta} \in \overline{\mathbb{Q}}$. Note that $(\widetilde{p_i}, q_i) = (p_i, q_i) = 1$. From here we could apply Gauss's formula 2.5 and require the condition that

$$\sum_{i=1}^{k} c_i \cot\left(\frac{\pi \widetilde{p_i}}{q_i}\right) \neq 0$$

if each $q_i > 1$. We prove that the series is transcendental without assuming any of these conditions, hence we get the condition on sums of the cotangent function for free, and hence a stronger result in general.

Recall that from Theorem 2.1, we have $\psi(\widetilde{p_1}/q_1) + \gamma$ is transcendental, and therefore not equal to zero. By equation (2.3), we write our sum as a linear combination of logarithms,

$$\widetilde{\beta} - \sum_{i=1}^{k} c_i \left( -\log(q_i) + \sum_{b=1}^{q_i-1} \zeta_{q_i}^{-b\widetilde{p}_i} \log(1-\zeta_{q_i}^{b}) \right).$$

Using the identity $\log(q_i) = \sum_{b=1}^{q_i-1} \log(1-\zeta_{q_i}^{b})$ we simplify our sum to

$$\widetilde{\beta} - \sum_{i=1}^{k} c_i \sum_{b=1}^{q_i-1} (\zeta_{q_i}^{-b\widetilde{p}_i} - 1) \log(1-\zeta_{q_i}^{b}).$$

We need only show that the logarithmic part of the summation does not vanish. For this, we show that there is no linear relation, over $\mathbb{Q}$, between the $\log(1-\zeta_{q_1}^{b})$ terms and any other logarithms which appear. We can then use Baker's theorem to conclude linear independence over $\overline{\mathbb{Q}}$, and since the $q_1$ part of the summation is nonzero (Theorem 2.1), we can conclude that the entire series is transcendental.

Let $T_{q_1} \subseteq \{\log(1-\zeta_{q_1}), \ldots, \log(1-\zeta_{q_1}^{q_1-1})\}$ be a maximal set of linearly independent (over $\mathbb{Q}$) terms. Suppose that at least one other $q_i \geq 2$, or else we would be done. Let $S = \{q_i | q_i > 1, i = 2, \ldots, k\}$. Take $T_S \subseteq \{\log(1-\zeta_{q_i}^{j}) | q_i \in S, j = 1, \ldots, q_i - 1\}$ be any maximal linearly independent set. A linear relation between $T_{q_1}$ and $T_S$, with coefficients $x_i, y_i \in \mathbb{Z}$, can be written as

$$x_1 \log(1-\zeta_{q_1}^{a_1}) + \cdots + x_s \log(1-\zeta_{q_1}^{a_s}) = \sum_{q_i \in S} \sum_{j=1}^{t_i} y_{i,j} \log(1-\zeta_{q_i}^{b_{i,j}})$$

for some integers $a_l, t_i$ and $b_{i,j}$. This implies that

$$\prod_{i=1}^{s}(1-\zeta_{q_1}^{a_i})^{x_i} = \prod_{q_i\in S}\prod_{j=1}^{t_i}(1-\zeta_{q_i}^{b_{i,j}})^{y_{i,j}}. \tag{2.6}$$

We recall some facts from algebraic number theory. The ring of integers of any cyclotomic extension, $\mathbb{Q}(\zeta_m)$, is equal to $\mathbb{Z}[\zeta_m]$ ([15], Exercise 4.5.25). Also, $\prod_{i=1}^{m-1}(1-\zeta_m^i) = m$ since $\prod_{i=1}^{m-1}(x-\zeta_m^i) = 1+x+\cdots+x^{m-1}$. Thus the norm (in any field extension over $\mathbb{Q}(\zeta_m)$) of any factor $(1-\zeta_m^i)$ is an integer dividing some power of $m$.

Using these facts, we take the norm $N_K$ of both sides of equation (2.6) where $K = \mathbb{Q}(\zeta_{q_1},\ldots,\zeta_{q_k})$. Taking norms, $N_K(\prod_{i=1}^{s}(1-\zeta_{q_1}^{a_i})^{x_i})$ is a fraction of integers with numerator and denominator both dividing some power of $q_1$, while $N_K(\prod_{q_i\in S}\prod_{j=1}^{t_i}(1-\zeta_{q_i}^{b_{i,j}})^{y_{i,j}})$ is a fraction of integers whose numerator and denominator both divide some power of $q_2\cdots q_k$. Since $(q_1,q_i)=1$ for every $i=2,\ldots,k$, this implies that the norms are both equal to $\pm1$.

Now recall that if $(m,n)=1$, the intersection $\mathbb{Q}(\zeta_m)\cap\mathbb{Q}(\zeta_n)=\mathbb{Q}$. Since $\mathbb{Q}(\zeta_{q_2},\ldots,\zeta_{q_k})\subseteq\mathbb{Q}(\zeta_{q_1\cdots q_k})$, we have $\mathbb{Q}(\zeta_{q_1})\cap\mathbb{Q}(\zeta_{q_2},\ldots,\zeta_{q_k})=\mathbb{Q}$. Thus, the two products are in fact rational numbers with norm $\pm1$. Hence the two products are equal to $\pm1$. Squaring equation (2.6) and then taking log of both sides we have,

$$x_1\log(1-\zeta_{q_1}^{a_1})+\cdots+x_s\log(1-\zeta_{q_1}^{a_s}) = 0 = \sum_{q_i\in S}\sum_{j=1}^{t_i} y_{i,j}\log(1-\zeta_{q_i}^{b_{i,j}}).$$

Since the sets $T_{q_1}$ and $T_S$ are each linearly independent, we have a contradiction. Thus $T_{q_1}\cup T_S$ is a set of linearly independent values, over $\mathbb{Q}$. By Baker's theorem, the elements of $T_{q_1}\cup T_S\cup\{1\}$ are linearly independent over

$\overline{\mathbb{Q}}$. Finally, we conclude that the logarithms do not cancel, therefore our summation is transcendental. ∎

This last theorem really utilizes everything that we have built up thus far. For every result up until this point, the theorems read "... effectively computable algebraic (or zero) or transcendental..." or else we required some more complicated conditions to be satisfied. This example illustrates a general method. The trick is to somehow show that the transcendental part of such numbers does not vanish. With this, we end this section and begin relaxing conditions on $B(x)$ in the next section. We will see that there is a strong connection between the order of the roots of $B(x)$ and the degree of the digamma or polygamma function which appears.

## 2.5 Infinite sums and polygamma functions

We next relax the restriction that $B(x)$ has only simple roots. When $B(x)$ has simple roots, we obtained a linear form in special values of the digamma function, as we saw in the previous sections. Relating such series to the digamma function and using the connection to logarithms (equation (2.3)) is how we were able to say anything about the transcendental nature of these numbers. The next lemma shows a connection between the multiplicity of roots of $B(x)$ and special values of the polygamma function.

**Lemma 2.11** *Let $f$ be a periodic function with integer period $q$. Take $k \geq 1$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. Then*

$$\sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^k} = \frac{(-1)^k}{q^k(k-1)!} \sum_{a=1}^{q} f(a)\psi_{k-1}\left(\frac{a+\alpha}{q}\right)$$

*when the series converges.*

**Proof.** The case $k = 1$ is in the previous section(s), so assume that $k \geq 2$. From equation (2.4) above we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha)^k} &= \frac{1}{q^k} \sum_{a=0}^{q-1} f(a) \sum_{n=0}^{\infty} \frac{1}{(n+\frac{a+\alpha}{q})^k} \\ &= \frac{(-1)^k}{q^k(k-1)!} \sum_{a=0}^{q-1} f(a) \psi_{k-1}\left(\frac{a+\alpha}{q}\right)\end{aligned}$$

∎

We are now ready to calculate closed forms for series with very general $B(x)$.

**Theorem 2.12** *Let $f$ be an algebraically valued periodic function, with period $q \geq 1$. Take $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ with $-\alpha_1, \ldots, -\alpha_k \in \mathbb{Q} \backslash \mathbb{Z}_{\geq 0}$ the distinct roots of $B(x)$ with multiplicities $m_1, \ldots, m_k$ respectively. If the series converges, then*

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

*is an algebraic linear combination of values of various polygamma functions at rational points.*

**Proof.** For the series to converge, we must have at least that $\deg(A) < \deg(B)$ and so, by partial fractions can write

$$\frac{A(x)}{B(x)} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{c_{i,j}}{(x+\alpha_i)^j}.$$

Injecting this into our series we have

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{i,j} \sum_{n=0}^{\infty} \frac{f(n)}{(n+\alpha_i)^j}$$

By the previous lemma, this last sum is equal to

$$\sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{(-1)^j c_{i,j}}{q^j (j-1)!} \sum_{a=0}^{q-1} f(a) \psi_{j-1} \left( \frac{a+\alpha_i}{q} \right).$$

∎

Note that we could do the same trick of shifting the summation far enough to exclude all zeroes of $B(x)$, so that we could include the case that $B(x)$ has any rational zero. This method was demonstrated earlier in the proof of Theorem 2.8, and we simply state the result without proof.

**Theorem 2.13** *Let $f$ be an algebraically valued periodic function, with period $q \geq 1$. Take any $A(x), B(x) \in \overline{\mathbb{Q}}[x]$, $B(x)$ having only rational roots. If the series converges then*

$$\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)}$$

*is an effectively computable algebraic number plus an algebraic linear combination of values of various polygamma functions at rational points.*

In the previous sections of this chapter, we were able to state theorems regarding transcendence due to special values of the digamma function making an appearance. Here we can only illustrate the connection, in general, of series of this type to special values of the polygamma function. It is not known whether these special values are transcendental or even irrational.

Studying these special values is left for future work.

# Chapter 3

# Exponential Polynomial Sums

For this chapter we keep our summation over the natural numbers, but we remove the periodic function and insert an exponential. That is, we are interested in series of the form

$$\sum_{n=0}^{\infty} \frac{z^n A(n)}{B(n)}.$$

We will see that by examining series of this new form, we obtain results from the previous chapter for free. This chapter originates from pg 137 of [7]. Much of the theory has been shown in [1], however the proofs offered here are original and often more elegant.

## 3.1 Convergence of the basic case

We begin with a simpler series, where $A(x)$ is constant and $B(x)$ is a single linear polynomial. That is, examine

$$f_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{n+\alpha}.$$

For now, assume that the value of $\alpha$ is not an integer. Though it is clear that this series converges absolutely inside the unit circle and diverges at $z = 1$, it is unclear what happens on $|z| = 1$ with $z \neq 1$. We proceed with partial summation (see [15]). For any continuously differentiable function $f$ and any sequence $\{a_n\}_{n=1}^{\infty}$,

$$\sum_{0 \le n \le x} a_n f(n) = A(x)f(x) - \int_0^x A(t)f'(t)dt$$

where $A(x) = \sum_{0 \le n \le x} a_n$. Let $a_n = z^n$ so that $A(x) = A([x]) = (z^{[x]+1} - 1)/(z-1)$ and let $f(x) = 1/(x+\alpha)$. We have

$$\sum_{0 \le n \le x} \frac{z^n}{n+\alpha} = \frac{(z^{[x]+1} - 1)}{(z-1)(x+\alpha)} + \frac{1}{(z-1)} \int_0^x \frac{z^{[x]+1} - 1}{(t+\alpha)^2} dt$$

For $z \neq 1$ on the unit circle, sending $x$ to infinity we see that the first term disappears and the integral is $O(1/x)$ and therefore converges.

## 3.2 Exponential over a linear factor

We next utilize the absolute convergence inside the unit circle to obtain a pleasant closed form for this series when $\alpha$ is a rational number. We can then use Abel's convergence theorem to extend the closed form to the unit disc, except for the value $z = 1$. We state Abel's theorem without proof.

**Theorem 3.1** (Abel's Convergence Theorem) *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *inside the radius of convergence* $r$*, and that* $\sum_{n=0}^{\infty} a_n$ *is convergent. Then* $\lim_{x\to r^-} f(x) = \sum_{n=0}^{\infty} a_n r^n$.

**Proposition 3.2** *For* $p/q \in (0,1]$ *with* $(p,q) = 1$ *and any* $|z| \le 1, z \ne 0$ *or* $1$*, we have*

$$\sum_{n=0}^{\infty} \frac{z^n}{n + p/q} = -z^{-p/q} \sum_{t=0}^{q-1} \zeta_q^{-pt} \log(1 - \zeta_q^t z^{1/q}).$$

**Proof.** We start with the sum on the right hand side, inserting a series for $-\log(1-x)$.

$$-\sum_{t=0}^{q-1} \zeta_q^{-pt} \log(1 - \zeta_q^t z^{1/q}) = \sum_{t=0}^{q-1} \zeta_q^{-pt} \sum_{n=1}^{\infty} \frac{(\zeta_q^t z^{1/q})^n}{n}$$

For now we assume that $|z| < 1$ so that we have absolute convergence, and we interchange the order of summation and obtain,

$$\sum_{n=1}^{\infty} \frac{(z^{1/q})^n}{n} \sum_{t=0}^{q-1} \zeta_q^{t(n-p)}$$

The inner sum, being a sum of $q$-th roots of unity, is equal to $q$ if $n$ is congruent to $p$ mod $q$. Otherwise the sum is zero. So the only terms which survive are those with $n = mq + p$ for some $m \geq 0$. Our sum is equal to

$$q \sum_{m=0}^{\infty} \frac{z^{m+p/q}}{mq+p} = z^{p/q} \sum_{m=0}^{\infty} \frac{z^m}{m+p/q}.$$

Thus we have the result for $|z| < 1$. As mentioned above, we extend the equality via Abel's convergence theorem which extends equality of a function and its power series inside the unit circle to the boundary, as long as the series converges. We showed convergence in the previous section so we are done. ∎

In the case that $p/q$ is not in $(0, 1]$, we simply shift our summation enough, by a finite number of terms, so that we get a finite sum of terms and an infinite series similar to that which we analyzed above. We demonstrate this next. We also specify that $z$ be an algebraic number and prove a transcendence result.

**Proposition 3.3** *Let $z \neq 0$ or $1$ be an algebraic number with $|z| \leq 1$. For any $p/q \in \mathbb{Q}$ with $(p, q) = 1$ we have that the series*

$$\sum_{n=0}^{\infty}{}' \frac{z^n}{n+p/q}$$

*is an effectively computable algebraic number or is transcendental.*

**Proof.** Write $p/q = r + \widetilde{p}/\widetilde{q}$ where $\widetilde{p}/\widetilde{q} \in (0, 1]$. Rewrite the series

$$\sum_{n=0}^{\infty}{}' \frac{z^n}{n+p/q} = z^{-r} \sum_{n=0}^{\infty}{}' \frac{z^{n+r}}{n+r+\widetilde{p}/\widetilde{q}} = z^{-r} \sum_{n=r}^{\infty}{}' \frac{z^n}{n+\widetilde{p}/\widetilde{q}}.$$

We simplify this expression and obtain,

$$S + z^{-r} \sum_{n=0}^{\infty} \frac{z^n}{n + \widetilde{p}/\widetilde{q}}$$

where

$$S = -z^{-r} \sum_{n=0}^{r-1}{}' \frac{z^n}{n + \widetilde{p}/\widetilde{q}} \qquad \text{if } r > 0$$

and

$$S = z^{-r} \sum_{n=r}^{-1}{}' \frac{z^n}{n + \widetilde{p}/\widetilde{q}} \qquad \text{if } r < 0.$$

Note that $\sum'$ here is summation avoiding the original $-p/q$. If $r = 0$ then no shift is needed and $S = 0$. With $S$ explicitly given, the remainder of the sum is determined by Proposition 3.2. We have that our series is equal to

$$S - z^{-r-\widetilde{p}/\widetilde{q}} \sum_{t=0}^{\widetilde{q}-1} \zeta_{\widetilde{q}}^{-\widetilde{p}t} \log(1 - \zeta_{\widetilde{q}}^{t} z^{1/\widetilde{q}})$$

which simplifies to

$$S - z^{-p/q} \sum_{t=0}^{q-1} \zeta_q^{-pt} \log(1 - \zeta_q^t z^{1/q}).$$

We note that this closed form holds for general $|z| \leq 1$ with $z \neq 0$ or 1. Since $z$ is algebraic, the latter term is a linear form in logarithms of algebraic numbers. The logarithmic part is either zero or transcendental, by Baker's theorem. This completes the proof. ∎

## 3.3 Exponential times a polynomial

We next analyze the case when we have an exponential with a polynomial. Though it is easy to see that in this case the series equals an algebraic number, we compute the closed form relating the series to Stirling numbers of the second kind.

**Proposition 3.4** *For $P(x) \in \overline{\mathbb{Q}}[x]$ and $z$ algebraic with $|z| < 1$, we have that the series*

$$\sum_{n=0}^{\infty} z^n P(n)$$

*is algebraic.*

**Proof.** Assume that $z \neq 0$, or else we are done. By the ratio test we have that

$$\lim_{n\to\infty} \frac{|z^{n+1}P(n+1)|}{|z^n P(n)|} = |z| < 1$$

so we have absolute convergence. Writing $P(x) = \sum_{i=0}^{k} a_i x^i$ and separating the series by degree we have

$$\sum_{n=0}^{\infty} z^n P(n) = \sum_{i=0}^{k} a_i \sum_{n=0}^{\infty} z^n n^i$$

We relate the inner series to Stirling numbers of the second kind. Note that the Stirling number $S(n,k) \in \mathbb{N}$ is the number of ways of partitioning $n$ elements into $k$ nonempty sets. Also note that $S(i,0) = \delta_{i,0}$. There is a nice relationship between $x^i$ and these Stirling numbers. We have

$$x^i = \sum_{j=0}^{i} S(i,j)(x)_j$$

where $(x)_j$ is the falling factorial defined as $x(x-1)\cdots(x-j+1)$ for $j \geq 1$ and $(x)_0 = 1$. Inserting this into our series we have

$$\begin{aligned}\sum_{n=0}^{\infty} z^n P(n) &= \sum_{i=0}^{k} a_i \sum_{n=0}^{\infty} z^n \sum_{j=0}^{i} S(i,j)(n)_j \\ &= \sum_{i=0}^{k} a_i \sum_{j=0}^{i} S(i,j) \sum_{n=0}^{\infty} (n)_j z^n\end{aligned}$$

For the terms with $n < j$, the falling factorial $(n)_j = 0$. This leaves,

$$\sum_{i=0}^{k} a_i \sum_{j=0}^{i} S(i,j) \sum_{n=j}^{\infty} n(n-1)\cdots(n-j+1)z^n$$

which, by replacing $n$ by $n+j$, equals

$$\sum_{i=0}^{k} a_i \sum_{j=0}^{i} S(i,j) z^j \sum_{n=0}^{\infty} (n+1)\cdots(n+j)z^n.$$

The innermost sum is precisely the $j$-th derivative of a well-known series,

$$\left(\frac{z^j}{1-z}\right)^{(j)} = \left(\sum_{n=0}^{\infty} z^{n+j}\right)^{(j)} = \sum_{n=0}^{\infty} (n+1)\cdots(n+j)z^n.$$

Simple division yields, for any $j \geq 0$,

$$\frac{z^j}{1-z} = -(z^{j-1} + \cdots + z + 1) + \frac{1}{1-z}$$

and by taking $j$ consecutive derivative we obtain,

$$\left(\frac{z^j}{1-z}\right)^{(j)} = ((1-z)^{-1})^{(j)} = \frac{j!}{(1-z)^{j+1}}.$$

Finally we have,

$$\sum_{n=0}^{\infty} z^n P(n) = \sum_{i=0}^{k} a_i \sum_{j=0}^{i} \frac{S(i,j) j! z^j}{(1-z)^{j+1}}$$

which is clearly an algebraic number if $z$ is algebraic. ∎

## 3.4 Exponential times a rational function

From Propositions 3.3 and 3.4 we immediately get the following theorem.

**Theorem 3.5** *Let $z \neq 1$ be algebraic with $|z| \leq 1$. Let $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ such that $B(x)$ has simple rational roots. If it converges, the series*

$$\sum_{n=0}^{\infty}{}' \frac{z^n A(n)}{B(n)}$$

*is an effectively computable algebraic number or is transcendental.*

**Proof.** Write $A(x)/B(x) = Q(x) + R(x)/B(x)$ where $Q(x), R(x) \in \overline{\mathbb{Q}}[x]$

and $\deg(R) < \deg(B)$. By the ratio test we see that if $|z| < 1$ then we have absolute convergence, regardless of the degrees of $A(x)$ and $B(x)$. However, writing $A(x)/B(x) = Q(x) + R(x)/B(x)$, we see that the series converges on $|z| = 1$ (with $z \neq 1$) only if $Q(x) = 0$. For these cases, being the only cases of convergence, by partial fractions we write

$$\frac{R(x)}{B(x)} = \sum_{i=1}^{k} \frac{c_i}{x + p_i/q_i}$$

so that

$$\sum_{n=0}^{\infty}{}' \frac{z^n A(n)}{B(n)} = \sum_{n=0}^{\infty}{}' z^n Q(n) + \sum_{i=1}^{k} c_i \sum_{n=0}^{\infty}{}' \frac{z^n}{n + p_i/q_i}$$

and now Propositions 3.3 and 3.4 show the result. ∎

Theorem 3.5 encompasses previous work involving series of the form

$$\sum_{n=0}^{\infty} \frac{f(n)A(n)}{B(n)}$$

where $f(x)$ is an algebraic valued periodic function. Recall that by Fourier inversion on a finite group, $f(n)$ can be written

$$f(n) = \widehat{f}(0) + \widehat{f}(1)\zeta_q^n + \cdots + \widehat{f}(q-1)\zeta_q^{n(q-1)}.$$

We can write

$$\sum_{n=0}^{\infty}{}' \frac{f(n)A(n)}{B(n)} = \sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)} \sum_{a=0}^{q-1} \widehat{f}(a)\zeta_q^{an}$$
$$= \sum_{a=0}^{q-1} \widehat{f}(a) \sum_{n=0}^{\infty}{}' \frac{(\zeta_q^a)^n A(n)}{B(n)}$$

and by Theorem 3.5 we immediately get Theorem 2.8 as a corollary.

For the sake of completeness, we mention one final series with a general setting of exponentials with polynomials. The theorem is Theorem 4 of [1]. With the theory developed here it is possible to prove the general result, however, this author has no improvement on the proof of the authors of [1]. We state the theorem, but refer the reader to [1] for the complete proof. Note, that there is a small mistake in the proof of Theorem 4 from [1]. The authors relate their series to a sum of an effectively computable algebraic number and a linear form in logarithms of algebraic numbers. In the second last line of their proof, the authors simply forget the effectively computable algebraic number. With the statement of their theorem fixed, stated here, we can also drop a condition that was imposed on $Q(x)$. The condition that $Q(x)$ have all of its roots in $[-1, 0)$ is no longer needed. With the edited statement, the theorem is true with $Q(x)$ having any simple rational roots.

**Theorem 3.6** (Theorem 4 of [1]) *Let $P_1(x), \ldots, P_l(x) \in \overline{\mathbb{Q}}[x]$ and $\alpha_1, \ldots, \alpha_l \in \overline{\mathbb{Q}}$. Put $g(x) = \sum_{i=1}^{l} \alpha_i^x P_i(x)$. Let $Q(x) \in \overline{\mathbb{Q}}[x]$ have simple rational roots. If the series*

$$\sum_{n=0}^{\infty} \frac{g(n)}{Q(n)}$$

*converges, then the sum is either an effectively computable algebraic number or transcendental.*

Also note that with the theory developed in this chapter, we could simply add various convergent series and obtain a result similar to the statement of this theorem. It turns out, as shown in [1], the only series of this general form which converge are those which can be formed by adding convergent series of the type discussed in Theorem 3.5.

**Remark 3.7.** The reader may ask whether or not there are algebraic numbers which are not roots of unity and still satisfy $|\alpha| = 1$. The answer is yes. For an example, we introduce the so-called *Salem* numbers. A Salem number is a real algebraic number which has at least one conjugate on the unit circle. We do not discuss the Salem numbers in detail here, but note that they do exist and they give rise to algebraic numbers with absolute value one which are not roots of unity. The smallest known Salem number is

$$1.176280818\ldots$$

which is the largest real root of

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^3 - x^3 + x + 1.$$

This polynomial and value were found by Lehmer in 1933.

# Chapter 4

# Multivariable Zeta-type series

We now examine a generalization of the series which were studied in chapter 2. Using [18] as a model, we extend the results there to a multivariable version of the series $\sum_{n=1}^{\infty} \frac{f(n)}{n}$. We wish to analyze the series

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1 \cdots n_k}$$

which we interpret as

$$\lim_{N_1,\ldots,N_k\to\infty} \sum_{n_1=1}^{N_1} \cdots \sum_{n_k=1}^{N_k} \frac{f(n_1,\ldots,n_k)}{n_1 \cdots n_k}$$

where each $N_i$ can go to infinity independent of the other $N_j$'s. These series do not converge in all cases, but as we will see, we will require our function $f$ to satisfy certain conditions which are very similar to the required conditions of the one variable case from chapter 2. Once we discuss convergence, we will examine analytic continuation of these series which will lead to a general closed form.

Note that we make the following notational simplification. For a function $f$, periodic in $k$ variables with respective periods $q_1, \ldots, q_k$, we write $f$ as a function on $G := \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_k\mathbb{Z}$ and extend our definition of $f$ to $\mathbb{Z}^k$ by

$$f(n_1, \ldots, n_k) = f(n_1 \pmod{q_1}, \ldots, n_k \pmod{q_k}).$$

Let coset representatives for each factor in the group run from 1 to $q_i$. Throughout the chapter we will want to use the fact that $f$ is a function on this particular group.

## 4.1 Convergence of the multi-indexed series

We take convergence in this multi-indexed scenario to mean that the series converges to the same value regardless how $N_1$ through $N_k$ go to infinity. The next proposition shows exactly which conditions must be placed on $f$to get convergence of the series

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1 \cdots n_k}.$$

**Proposition 4.1** *Let $f(x_1, \ldots, x_k)$ be a function, periodic in $k$ variables with periods $q_1, \ldots, q_k$ respectively. The series*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1 \cdots n_k}$$

*converges if and only if we have that for every $i = 1, \dots, k$,*

$$\sum_{n_i=1}^{q_i} f(a_1, \dots, n_i, \dots, a_k) = 0 \qquad \forall\ a_j \in \{1, \dots, q_j\} \tag{4.1}$$

**Proof.** The convergence criteria says that if we fix $k-1$ of the variables, the sum over all possible values of the $i$th entry should be zero. We write

$$\sum_{n_1=1}^{N_1} \cdots \sum_{n_k=1}^{N_k} \frac{f(n_1, \dots, n_k)}{n_1 \cdots n_k} = \sum_{n_1=1}^{N_1} \frac{1}{n_1} \cdots \sum_{n_{k-1}=1}^{N_{k-1}} \frac{1}{n_{k-1}} \sum_{n_k=1}^{N_k} \frac{f(n_1, \dots, n_k)}{n_k}.$$

From our convergence results in chapter 2 for the one variable case, and using the fact that we need the series to converge irrespective of how $N_1$ through $N_k$ go to infinity, sending $N_k$ to infinity we know that the inner sum converges if and only if

$$\sum_{n_k=1}^{q_k} f(a_1, \dots, a_{k-1}, n_k) = 0$$

for each fixed $a_j \in \mathbb{Z}/q_j\mathbb{Z}$. Without loss of generality we could have changed the order of summation to isolate any of the $k$ entries of $f$, therefore we obtain that the criteria 4.1 are at the very least necessary.

We next show that the conditions 4.1 are sufficient as well. We induct on the number of indices in the summation. As stated, the case $\sum_{n=1}^{\infty} f(n)/n$ is in chapter 2. For $k \geq 2$ we write,

$$\sum_{n_1, \dots, n_k=1}^{\infty} \frac{f(n_1, \dots, n_k)}{n_1 \cdots n_k} = \sum_{n_1=1}^{\infty} \frac{1}{n_1} \sum_{n_2, \dots, n_k=1}^{\infty} \frac{f(n_1, n_2, \dots, n_k)}{n_2 \cdots n_k}.$$

By induction, the conditions 4.1 imply that the inner sum converges. Not only that, but the inner sum is periodic in $n_1$. Let

$$F(n_1) := \sum_{n_2,\ldots,n_k=1}^{\infty} \frac{f(n_1,n_2,\ldots,n_k)}{n_2\cdots n_k}.$$

Notice that $\sum_{n_1=1}^{q_1} f(n_1,a_2,\ldots,a_k) = 0$ for every $a_j \in \mathbb{Z}/q_j\mathbb{Z}$ implies that $\sum_{n_1=1}^{q_1} F(n_1) = 0$. Thus our series converges if and only if we have the criteria listed in equation (4.1) ∎

In the next section, we relate the series to values of the digamma function.

## 4.2 Polynomial forms of digamma values

Much like the one variable case of chapter 2, we inject the digamma function into these series. We do this in the same manner as before, via the Hurwitz zeta function.

**Theorem 4.2** *With $f$ periodic as above, satisfying the required convergence conditions (4.1), we have that*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1\cdots n_k} = \frac{1}{q_1\cdots q_k}\sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\psi\left(\frac{a_1}{q_1}\right)\cdots\psi\left(\frac{a_k}{q_k}\right).$$

**Proof.** We begin by examining

$$F(s_1,\ldots,s_k) := \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}}$$

as a function of $s_1,\ldots,s_k$, each $\Re(s_i) > 1$. We will show that this series is analytic for each $s_i = 1$, then take limits in the usual way. For $\Re(s_i) > 1$ we have the series is equal to,

$$\sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\left(\sum_{n_1\equiv a_1(q_1)} \frac{1}{n_1^{s_1}}\right)\cdots\left(\sum_{n_k\equiv a_k(q_k)} \frac{1}{n_k^{s_k}}\right)$$

which is equal to

$$\sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\left(\sum_{m_1=0}^{\infty} \frac{1}{(m_1q_1+a_1)^{s_1}}\right)\cdots\left(\sum_{m_k=0}^{\infty} \frac{1}{(m_kq_k+a_k)^{s_k}}\right).$$

Inserting the Hurwitz zeta function, we simplify to

$$\frac{1}{q_1^{s_1}\cdots q_k^{s_k}} \sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\zeta(s_1,a_1/q_1)\cdots\zeta(s_k,a_k/q_k).$$

The Hurwitz zeta function admits analytic continuation to the entire complex plane except for a simple pole with residue 1 at $s = 1$. In our case with $k$ variables, it becomes apparent that our convergence conditions (4.1) ensure analyticity of $F$ since any polar part of the series ($s_i = 1$ for any $i$) has residue 0 and is eliminated. Hence taking limits as all $s_i \to 1$ makes sense. Before taking limits, by the convergence conditions (4.1), we can rewrite $F$ as

$$\frac{1}{q_1^{s_1}\cdots q_k^{s_k}} \sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\left(\zeta(s_1,\frac{a_1}{q_1})-\zeta(s_1)\right)\cdots\left(\zeta(s_k,\frac{a_k}{q_k})-\zeta(s_k)\right)$$

Recall equation (2.2) which implies that

$$\lim_{s\to 1^+} \zeta(s,x) - \zeta(s) = -\psi(x) - \gamma.$$

Thus sending each $s_i$ to $1^+$ along the real axis, we obtain

$$\lim_{s_i\to 1^+} F(s_1,\ldots,s_k) = \frac{(-1)^k}{q_1\cdots q_k} \sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\psi(\frac{a_1}{q_1})\cdots\psi(\frac{a_k}{q_k}).$$

By analogy to Abel's theorem 3.1, we have a similar theorem for Dirichlet series (see [23] 9.12 on page 291). Since the series at $s_i = 1$ for each $i$ converges (see the previous section), we are able to take the limit inside the sum and conclude that

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1\cdots n_k} = \frac{(-1)^k}{q_1\cdots q_k} \sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\psi(\frac{a_1}{q_1})\cdots\psi(\frac{a_k}{q_k}).$$

∎

It is interesting that in [18], the authors found that the series $\sum_{n\geq 1} f(n)/n$ is equal to a linear combination of values of the digamma function. Analogously here, we find that the $k$ variable case is a polynomial form of degree $k$ in values of the digamma function. By degree $k$ we mean a homogeneous polynomial of degree $k$. Also analogous to what the authors of [18] found, the next corollary relates the series to generalized Euler constants. In [13], generalized Euler constants, $\gamma(a,q)$, are defined as

$$\lim_{N\to\infty} \left( \sum_{n>0,\ n\equiv a(q)} \frac{1}{n} - \frac{1}{q}\log N \right).$$

**Corollary 4.3** *With $f$ periodic as above, satisfying the required convergence conditions (4.1),*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1 \cdots n_k} = \sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\gamma(a_1,q_1)\cdots\gamma(a_k,q_k).$$

**Proof.** Making use of the identity

$$-\psi(a/q) = q\gamma(a,q) + \log q$$

(Theorem 7 from [13]) we have the result. ∎

In [16], the authors show that at most one of the infinite list of numbers $\gamma, \gamma(a,q)$ for $1 \leq a \leq q$ with $q \geq 2$ is algebraic. At this point in time, no transcendence results are known for polynomial forms of these generalized Euler constants, so we cannot say anything concrete regarding the transcendence of the previous series. However, as we will see in the next section, Schanuel's conjecture implies that the series in the previous corollary is transcendental when it does not vanish.

## 4.3 Fourier analysis of the series

We now take a different approach to analyzing the same family of series. We will Fourier analyze these series and obtain a different closed form. Recall the discrete Fourier transform on the finite Abelian group, $G = \mathbb{Z}/q_1\mathbb{Z} \times$

$\cdots \times \mathbb{Z}/q_k\mathbb{Z}$. For a function $f$ on $G$ we define in the usual way

$$\widehat{f}(x_1, \ldots, x_k) := \frac{1}{q_1 \cdots q_k} \sum_{(a_1,\ldots,a_k)\in G} f(a_1, \ldots, a_k)\zeta_{q_1}^{-a_1x_1} \cdots \zeta_{q_k}^{-a_kx_k}$$

and we have the inversion

$$f(n_1, \ldots, n_k) = \sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1, \ldots, a_k)\zeta_{q_1}^{a_1n_1} \cdots \zeta_{q_k}^{a_kn_k}.$$

We show an equivalence between the convergence conditions of equation (4.1), and conditions on the discrete Fourier transform of our $k$-periodic function. We point out that the following lemma is a special case of Frobenius reciprocity, but prove it directly for our case.

**Lemma 4.4** *Let $f$ be a function on the group $G = \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_k\mathbb{Z}$. We have the following equivalence:*

$$\widehat{f}(a_1, \ldots, 0, \ldots, a_k) = 0 \quad \forall\ a_j \in \mathbb{Z}/q_j\mathbb{Z},\ j \neq i$$

*where the zero is at the $i$-th position of $\widehat{f}$, if and only if*

$$\sum_{n_i=1}^{q_i} f(a_1, \ldots, n_i, \ldots, a_k) = 0 \quad \forall\ a_j \in \mathbb{Z}/q_j\mathbb{Z},\ j \neq i$$

.

**Proof.** We first assume the top conditions. Without loss of generality, we show the case that $i = k$. Examine,

$$\sum_{n_k=1}^{q_k} f(n_1, \ldots, n_k)$$

for fixed $n_1, \ldots, n_{k-1}$. By Fourier inversion this is equal to

$$\sum_{n_k=1}^{q_k} \sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1, \ldots, a_k)\zeta_{q_1}^{a_1 n_1} \cdots \zeta_{q_k}^{a_k n_k}$$

which equals

$$\sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1, \ldots, a_k)\zeta_{q_1}^{a_1 n_1} \cdots \zeta_{q_{k-1}}^{a_{k-1} n_{k-1}} \sum_{n_k=1}^{q_k} \zeta_{q_k}^{a_k n_k}$$

For all terms with $a_k \not\equiv q_k \pmod{q_k}$, the final sum is always 0. For $a_k \equiv q_k \pmod{q_k}$, the values $\widehat{f}(a_1, \ldots, a_k) = 0$ by assumption. So we have that the original sum is identically 0.

Now assume the second group of conditions and again examine the case that $i = k$ without loss of generality. By definition of the Fourier transform we have

$$\widehat{f}(n_1, \ldots, n_{k-1}, 0) = \frac{1}{q_1 \cdots q_k} \sum_{(a_1,\ldots,a_k)\in G} f(a_1, \ldots, a_k)\zeta_{q_1}^{-a_1 n_1} \cdots \zeta_{q_{k-1}}^{-a_{k-1} n_{k-1}}$$

which equals

$$\frac{1}{q_1 \cdots q_k} \sum_{a_1=1}^{q_1} \zeta_{q_1}^{-a_1 n_1} \cdots \sum_{a_{k-1}=1}^{q_{k-1}} \zeta_{q_{k-1}}^{-a_{k-1} n_{k-1}} \sum_{a_k=1}^{q_k} f(a_1, \ldots, a_k).$$

By assumption, the innermost sum is equal to 0 for every $a_1, \ldots, a_{k-1}$. Thus $\widehat{f}(n_1, \ldots, n_{k-1}, 0) = 0$ and we have the equivalence. ∎

Taking the convergence conditions in either form, it may not be obvious how to find such a function. The following lemma is a recipe for constructing functions (all of them) with such conditions.

**Lemma 4.5** *There are infinitely many functions $f$ on $G$ with*

$$\widehat{f}(a_1, \ldots, 0, \ldots, a_k) = 0 \quad \forall\, a_j \in \mathbb{Z}/q_j\mathbb{Z}$$

*for every $i = 1, \ldots, k$.*

**Proof.** Take any function $g$ on $G$ with such that $g(a_1, \ldots, a_k) = 0$ whenever at least one $a_i \equiv 0 (\mathrm{mod}\ q_i)$. We point out that choosing a function with these properties is quite easy since it is a function on a finite set of points. Put

$$f(n_1, \ldots, n_k) = \sum_{(a_1, \ldots, a_k) \in G} g(a_1, \ldots, a_k) \zeta_{q_1}^{a_1 n_1} \cdots \zeta_{q_k}^{a_k n_k}.$$

By Fourier inversion we have that $f$ is a function on $G$ and $\widehat{f} = g$, which satisfies the conditions. ∎

We next find an alternate closed form for the generalized series in question. Although we could have used the closed form in the previous section and then used equation (2.3) to relate the digamma function to logarithms of algebraic numbers, we derive the relation here.

**Theorem 4.6** *Let $f$ be a $k$-periodic function satisfying equations (4.1). In* *the case of convergence, the series*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1\cdots n_k}$$

*is equal to*

$$\sum_{(a_1,\ldots,a_k)\in G, a_i\not\equiv 0} \widehat{f}(a_1,\ldots,a_k)\log(1-\zeta_{q_1}^{a_1})\cdots\log(1-\zeta_{q_k}^{a_k}).$$

**Proof.** Assume $\Re(s_i) > 1$ for each $i$. By Fourier inversion we have

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \sum_{n_1,\ldots,n_k=1}^{\infty} \sum_{(a_1,\ldots,a_k)\in G} \frac{\widehat{f}(a_1,\ldots,a_k)\zeta_{q_1}^{a_1 n_1}\cdots\zeta_{q_k}^{a_k n_k}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

We interchange the order of summation and obtain,

$$\sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1,\ldots,a_k)\left(\sum_{n_1=1}^{\infty}\frac{\zeta_{q_1}^{a_1 n_1}}{n_1^{s_1}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{\zeta_{q_k}^{a_k n_k}}{n_k^{s_k}}\right)$$

The $k$ sums in parenthesis are examples of the Lerch zeta function. It is easy to see that when $a_i \equiv 0 \pmod{q_i}$ we have a simple pole at $s_i = 1$ with residue 1. Otherwise, there is no pole at $s_i = 1$. These poles are eliminated by our convergence conditions, so we have that our series equals

$$\sum_{(a_1,\ldots,a_k)\in G, a_i\not\equiv 0} \widehat{f}(a_1,\ldots,a_k)\left(\sum_{n_1=1}^{\infty}\frac{\zeta_{q_1}^{a_1 n_1}}{n_1^{s_1}}\right)\cdots\left(\sum_{n_k=1}^{\infty}\frac{\zeta_{q_k}^{a_k n_k}}{n_k^{s_k}}\right)$$

Sending each $s_i$ to 1 we obtain,

$$\sum_{(a_1,\ldots,a_k)\in G, a_i\not\equiv 0} \widehat{f}(a_1,\ldots,a_k)\log(1-\zeta_{q_1}^{a_1})\cdots\log(1-\zeta_{q_k}^{a_k}).$$

∎

For the sake of completeness, we now relate this method to the method used in the previous section.

**Corollary 4.7** *For a function $f$ satisfying equations (4.1),*

$$\sum_{(a_1,\ldots,a_k)\in G, a_i\not\equiv 0} \widehat{f}(a_1,\ldots,a_k)\log(1-\zeta_{q_1}^{a_1})\cdots\log(1-\zeta_{q_k}^{a_k})$$

*equals*

$$\frac{1}{q_1\cdots q_k}\sum_{(a_1,\ldots,a_k)\in G} f(a_1,\ldots,a_k)\psi(a_1/q_1)\cdots\psi(a_k/q_k).$$

**Proof.** Theorems 4.2 and 4.6 show the result. ∎

If we assume Schanuel's conjecture, 1.5, we can deduce a nice corollary from Theorem 4.6. Analogous to the one variable case of chapter 2, Schanuel's conjecture implies something about the transcendence of these multivariable series.

**Theorem 4.8** *Let $f$ be a $k$-periodic, algebraic valued function satisfying equations (4.1). If Schanuel's conjecture is true, then the sum*

$$\sum_{n_1,\ldots,n_k=1}^{\infty}\frac{f(n_1,\ldots,n_k)}{n_1\cdots n_k}$$

*is either zero or transcendental.*

**Proof.** By Theorem 4.6, the closed form for this series is a polynomial form, with algebraic coefficients, in logarithms of algebraic numbers. Schanuel implies that a set of linearly independent (over $\mathbb{Q}$) logarithms of algebraic numbers are in fact algebraically independent. Taking a maximal set of linearly independent logarithms that appear in the closed form for our series, and making appropriate substitutions of linear dependence relations, we obtain a polynomial form with algebraic coefficients in algebraically independent numbers. Thus the sum is either zero or transcendental. ∎

We next use this same technique in a very general setting.

## 4.4 Multivariable series of rational functions

The previous sections are interesting on their own, however we can conclude a lot more by simple analogy to chapter 2. We point out that we could use either method to find closed forms for the series of this section. We proceed with the method in the previous section (Fourier techniques), and use the theory that was developed in chapter 2. Similar to chapter 2, we assume that the polynomials analyzed here, $A_i(x)$ are never identically zero.

**Theorem 4.9** *Let $f$ be an algebraic valued function on $G$. Let $A_i(x), B_i(x) \in \overline{\mathbb{Q}}$ for $i = 1, \ldots, k$ such that each $B_i(x)$ has only simple rational roots. When it converges, the series*

$$\sum_{n_1,\ldots,n_k=0}^{\infty}{}' \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}$$

*is a $\overline{\mathbb{Q}}$ linear combination of polynomial forms of degree $j$ of logarithms of algebraic numbers for $j = 0, \ldots, k$.*

**Proof.** We use the Fourier transform of $f$ and interchange summation to obtain

$$\sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1,\ldots,a_k) \prod_{i=1}^{k} \left( {\sum_{n_i=0}^{\infty}}' \frac{\zeta_{q_i}^{a_i n_i} A_i(n_i)}{B_i(n_i)} \right).$$

Note that for each $i$, if $\deg(A_i) = \deg(B_i) - 1$, by Lemma 4.4 we require that each $\widehat{f}(a_1, \ldots, 0, \ldots, a_k) = 0$ where the zero is at the $i$th position. Otherwise $\deg(A_i) < \deg(B_i) - 1$ and we have convergence of each factor above. By Theorem 2.8, each factor is an effectively computable algebraic number plus a linear form in logarithms of algebraic numbers. The result is now clear. ∎

With the closed form calculated in the previous theorem, we immediately see an implication of Schanuel's conjecture.

**Corollary 4.10** *Let $f$ be an algebraic valued function on $G$. Let $A_i(x), B_i(x) \in \overline{\mathbb{Q}}$ for $i = 1, \ldots, k$ such that each $B_i(x)$ has only simple rational roots. If Schanuel's conjecture is true, the series*

$${\sum_{n_1,\ldots,n_k=0}^{\infty}}' \frac{f(n_1,\ldots,n_k) A_1(n_1) \cdots A_k(n_k)}{B_1(n_1) \cdots B_k(n_k)}$$

*is either equal to an effectively computable algebraic number or transcendental, when it converges.*

**Proof.** By Theorem 4.9, the series is a linear combination of polynomial forms of degree $j$ of logarithms of algebraic numbers for $j = 0, \ldots, k$. Similar to the proof of Theorem 4.8, Schanuel's conjecture implies that all of the logarithms which appear are linear combinations of some minimal set of algebraically independent logarithms. The result is now clear. ∎

Finally, we relax the condition placed on the $B_i(x)$'s that they have only simple roots. We use the one variable case from chapter 2 and relate the multivariable series to products of values of the polygamma function.

**Theorem 4.11** *Let $f$ be an algebraic valued function on $G$. Let $A_i(x), B_i(x) \in \overline{\mathbb{Q}}$ for $i = 1, \ldots, k$ such that each $B_i(x)$ has only rational roots. When it converges, the series*

$$\sum_{n_1,\ldots,n_k=0}^{\infty}{}' \frac{f(n_1,\ldots,n_k)A_1(n_1)\cdots A_k(n_k)}{B_1(n_1)\cdots B_k(n_k)}$$

*is a $\overline{\mathbb{Q}}-$linear combination of polynomial forms of degree $j$ of various polygamma functions at rational points for $j = 1, \ldots, k$.*

**Proof.** We rewrite our sum in the same manner as Theorem 4.9,

$$\sum_{(a_1,\ldots,a_k)\in G} \widehat{f}(a_1,\ldots,a_k) \prod_{i=1}^{k} \left( \sum_{n_i=0}^{\infty}{}' \frac{\zeta_{q_i}^{a_i n_i} A_i(n_i)}{B_i(n_i)} \right).$$

We now utilize Theorem 2.13 which implies that each series which appears as a factor here is an effectively computable algebraic number plus an algebraic linear combination of various polygamma functions at rational points. Products of these linear combinations yield the result. ∎

# Chapter 5

# Summation over $\mathbb{Z}$

In chapter 2 we were interested in series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)A(n)}{B(n)}.$$

In this chapter we examine similar series, but change the setting by taking the summation over all integers instead of restricting to the natural numbers. By taking the summation over all integers we obtain some nice results by using only elementary methods. After closed forms are found, some results from [1], [10], and [18] become corollaries of results found here.

Similar to chapter 4, we first examine the situation using very elementary techniques. We find closed forms and conclude transcendence results. Later, we examine our series with Fourier techniques. Summation over all integers has an obvious connection to Fourier series, and this will be explored.

## 5.1 Elementary methods of analysis

We begin by noting that by summation over $\mathbb{Z}$ we mean,

$$\sum_{n\in\mathbb{Z}} a_n = \lim_{N\to\infty} \sum_{|n|<N} a_n.$$

As mentioned above we wish to find closed forms for series similar to those studied thus far. We begin with an important example which will be used throughout this chapter.

We begin with the Hadamard product for $\sin(\pi z)$,

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Taking the logarithm we obtain,

$$\log(\sin(\pi z)) = \log(\pi) + \log(z) + \sum_{n=1}^{\infty} \left[\log\left(1 - \frac{z}{n}\right) + \log\left(1 + \frac{z}{n}\right)\right]$$

Differentiating this we obtain,

$$\pi \cot(\pi z) = \sum_{n\in\mathbb{Z}} \frac{1}{n+z} \tag{5.1}$$

It is important to note the $\pi$ in front. Recall that $\pi$ is transcendental. This quickly brings us to a transcendence result.

**Proposition 5.1** *Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$. The series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+\alpha}$$

*is transcendental (except in the case $\alpha \equiv \frac{1}{2}$ mod $\mathbb{Z}$ in which the series is zero).*

**Proof.** First, if $\alpha \equiv 1/2 \bmod \mathbb{Z}$, the series vanishes since for any $t \in \mathbb{Z}$,

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+\frac{1}{2}+t} = \sum_{n\in\mathbb{Z}} \frac{1}{-n+\frac{1}{2}+t} = \sum_{n\in\mathbb{Z}} \frac{1}{-n-\frac{1}{2}-t} = -\sum_{n\in\mathbb{Z}} \frac{1}{n+\frac{1}{2}+t}$$

each equality easy to see since we taking summation over all integers. Shifting the denominator by any integer does not change the sum.

Now suppose that $\alpha \neq 1/2$. Write $\alpha = p/q$. From equation (5.1) we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+p/q} = \pi\cot(\pi p/q) = \pi i \left(\frac{1+\zeta_q^{-p}}{1-\zeta_q^{-p}}\right).$$

Since $\zeta_q^{-p} \in \overline{\mathbb{Q}}$ we have $\pi$ times an algebraic number. Hence the series is transcendental. ∎

Using the closed form of equation (5.1) as a starting point, we extend this result using elementary techniques.

**Proposition 5.2** *For $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ and $k$ a positive integer we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^k} = \pi^k A_{\alpha,k}$$

*where $A_{\alpha,k}$ is an algebraic number.*

**Proof.** We have

$$\pi\cot(\pi z) = \sum_{n\in\mathbb{Z}} \frac{1}{(n+z)}.$$

Note that

$$\frac{d}{dz}(\cot(\pi z)) = -\pi \csc^2(\pi z),$$

$$\frac{d}{dz}(\csc(\pi z)) = -\pi \csc(\pi z)\cot(\pi z),$$

and for $m, n \geq 1$, the derivative of $\csc^m(\pi z)\cot^n(\pi z)$ is equal to

$$-\pi \left(m \csc^m(\pi z)\cot^{n+1}(\pi z) + n \csc^{m+2}(\pi z)\cot^{n-1}(\pi z)\right).$$

Using these rules it is easy to see that

$$\frac{d^{k-1}}{dz^{k-1}}(\pi \cot(\pi z)) = \pi^k P_k(\csc(\pi z), \cot(\pi z)),$$

where $P_k$ is a polynomial in two variables with integer coefficients. Since $\alpha$ is in $\mathbb{Q} \setminus \mathbb{Z}$, both of $\csc(\pi\alpha)$ and $\cot(\pi\alpha)$ are algebraic, and so we have that $P_k(\csc(\pi\alpha), \cot(\pi\alpha))$ is algebraic. Calculating the $(k-1)$st derivative of the right side of equation (5.1) (justified by uniform convergence of the series of derivatives) we have

$$\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{n\in\mathbb{Z}} \frac{1}{n+z}\right) = (-1)^{k-1}(k-1)! \sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^k},$$

hence,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^k} = \frac{(-1)^{k-1}(\pi\cot(\pi\alpha))^{(k-1)}}{(k-1)!}. \tag{5.2}$$

We now define the symbol $A_{\alpha,k}$ as

$$A_{\alpha,k} := \frac{(-1)^{k-1}(\pi\cot(\pi\alpha))^{(k-1)}}{\pi^k(k-1)!}$$

and the result is now clear. ∎

Similar to chapter 2, we now examine when the numerator is a periodic function and continue building up the types of series which we can analyze. Note that the closed form through this chapter hold for various values of $\alpha$, but we restrict to certain values. We do this because the series which arise when $\alpha \in \mathbb{Q}$ are very natural to write down, and also we can say something about the transcendence in these cases.

**Theorem 5.3** *Let $f$ be an algebraic valued periodic function with period $q$, $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, and $k$ be a positive integer. The series*

$$\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k}$$

*is in $\overline{\mathbb{Q}}\pi^k$.*

**Proof.** By direct calculation,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k} &= \sum_{a=1}^{q} f(a) \sum_{n\equiv a(q)} \frac{1}{(n+\alpha)^k} \\
&= \sum_{a=1}^{q} f(a) \sum_{n\in\mathbb{Z}} \frac{1}{(qn+a+\alpha)^k} \\
&= \frac{1}{q^k} \sum_{a=1}^{q} f(a) \sum_{n\in\mathbb{Z}} \frac{1}{(n+\frac{a+\alpha}{q})^k}
\end{aligned}$$

By Proposition 5.2 we have that this equals

$$\frac{\pi^k}{q^k} \sum_{a=1}^{q} f(a) A_{\frac{a+\alpha}{q},k}$$

Each $A_{\frac{a+\alpha}{q},k}$ is algebraic, so we have the result. ∎

From here we immediately obtain an interesting example.

**Example 5.4.** For $z \in \mathbb{R} \setminus \mathbb{Z}$, the series

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n+z} = \pi \csc(\pi z).$$

To see this we follow the proof of Theorem 5.3 by letting $q = 2$ with $f(1) = -1, f(2) = 1$. Calculating directly

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{f(n)}{n+z} &= \frac{\pi}{2}\left(-A_{\frac{1+z}{2},1} + A_{\frac{2+z}{2},1}\right) \\
&= \frac{\pi}{2}\left(-\cot\left(\frac{\pi}{2} + \frac{\pi z}{2}\right) + \cot\left(\pi + \frac{\pi z}{2}\right)\right) \\
&= \frac{\pi}{2}\left(\frac{\cos(\pi + \frac{\pi z}{2})}{\sin(\pi + \frac{\pi z}{2})} - \frac{\cos(\frac{\pi}{2} + \frac{\pi z}{2})}{\sin(\frac{\pi}{2} + \frac{\pi z}{2})}\right) \\
&= \frac{\pi}{2}\left(\frac{\cos(\frac{\pi z}{2})}{\sin(\frac{\pi z}{2})} + \frac{\sin(\frac{\pi z}{2})}{\cos(\frac{\pi z}{2})}\right) \\
&= \pi\left(\frac{\cos^2(\frac{\pi z}{2}) + \sin^2(\frac{\pi z}{2})}{2\cos(\frac{\pi z}{2})\sin(\frac{\pi z}{2})}\right) \\
&= \pi \csc(\pi z)
\end{aligned}$$

From this example, we obtain an alternate proof of a trigonometric identity. One can easily check the result by using a standard first year calculus trig table.

**Corollary 5.5** *For $z \in \mathbb{R} \setminus \mathbb{Z}$, we have* $\cot(\pi z) + \csc(\pi z) = \cot(\frac{\pi z}{2})$.

**Proof.** Adding the two series for $\pi\cot(\pi z)$ and $\pi\csc(\pi z)$ we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+z} + \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n+z} = \sum_{n\in\mathbb{Z}, n\equiv 0(2)} \frac{2}{n+z} = \sum_{n\in\mathbb{Z}} \frac{1}{n+\frac{z}{2}} = \pi\cot(\frac{\pi z}{2}).$$

∎

We can also use the same technique from Theorem 5.3 to examine the series $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{f(n)}{n^k}$. Note that this includes the case $\sum_{n\in\mathbb{Z}\setminus\{m\}} \frac{f(n)}{(n-m)^k}$ as well, simply by shifting $n \mapsto n+m$ and noticing that $f(n+m)$ is still periodic with period $q$.

**Theorem 5.6** *Let $f$ be an algebraically valued periodic function with period $q$ and let $k$ be a positive integer. The series*

$$\sum_{n\in\mathbb{Z}, n\neq 0} \frac{f(n)}{n^k}$$

*is in $\overline{\mathbb{Q}}\pi^k$.*

**Proof.** We calculate directly,

$$\begin{aligned}
\sum_{n\in\mathbb{Z},\, n\neq 0} \frac{f(n)}{n^k} &= \sum_{a=1}^{q-1} f(a) \sum_{n\equiv a\ (q)} \frac{1}{n^k} + f(q) \sum_{n\equiv 0\ (q),\, n\neq 0} \frac{1}{n^k} \\
&= \sum_{a=1}^{q-1} f(a) \sum_{m\in\mathbb{Z}} \frac{1}{(mq+a)^k} + f(q) \sum_{m\in\mathbb{Z},\, m\neq 0} \frac{1}{(mq)^k} \\
&= \sum_{a=1}^{q-1} \frac{f(a)}{q^k} \sum_{m\in\mathbb{Z}} \frac{1}{(m+\frac{a}{q})^k} + \frac{f(q)}{q^k} \sum_{m\in\mathbb{Z},\, m\neq 0} \frac{1}{(m)^k} \\
&= \pi^k \left( \sum_{a=1}^{q-1} \frac{f(a) A_{\frac{a}{q},k}}{q^k} + \frac{2f(q)Z(k)}{q^k} \right) \qquad (5.3)
\end{aligned}$$

where $Z(k) = 0$ if $k$ is odd and $Z(k) = \frac{\zeta(k)}{\pi^k} \in \mathbb{Q}$ if $k$ is even. In either case we see that the desired sum is an algebraic multiple of $\pi^k$ and we are done. ∎

As a special case, we immediately obtain a well-known theorem of Euler.

**Corollary 5.7** (Euler, 1735) *For* $\zeta(s)$*, the Riemann zeta function,* $\zeta(2) = \frac{\pi^2}{6}$.

**Proof.** Let $f$ have period $q = 2$ with $f(1) = 1$ and $f(2) = 0$. From the proof of Theorem 5.6 we have

$$\begin{aligned} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{f(n)}{n^2} &= \frac{\pi^2 A_{\frac{1}{2},2}}{4} \\ &= \frac{\pi^2(-1)^{2-1}(-\csc^2(\frac{\pi}{2}))}{(2-1)!4} \\ &= \frac{\pi^2}{4} \end{aligned}$$

Since $f(n)/n^2 = f(-n)/(-n)^2$, we also have that

$$\begin{aligned} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{f(n)}{n^2} &= 2\sum_{n=1}^{\infty} \frac{f(n)}{n^2} \\ &= 2\left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2^2}\sum_{m=1}^{\infty} \frac{1}{m^2}\right] \\ &= \frac{3}{2}\zeta(2) \end{aligned}$$

Equating the two, we have the well-known result. ∎

We generalize this idea of pairing up terms when $f$ has the same parity as the exponent of the denominator. This idea simplifies a result of Murty and Saradha from [17]. We first recall a definition from [17].

**Definition 5.8.** We say that a periodic function $f$ and an integer $k$ have the same parity if either $f$ is odd and $k$ is odd or $f$ is even and $k$ is even. That is $f(-n) = (-1)^k f(n)$ for every $n$.

In [17], the authors showed that for $f$ a periodic function and $k$ a positive integer, if $f$ and $k$ have the same parity, then the series

$$L(k,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k}$$

is an algebraic multiple of $\pi^k$. The theory developed here yields a new proof of this result.

**Proposition 5.9** *For an algebraic valued periodic function $f$ with period $q$ and a positive integer $k$, if $f$ and $k$ have the same parity then the series*

$$L(k,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k}$$

*is in $\overline{\mathbb{Q}}\pi^k$.*

**Proof.** Note that since $f$ and $k$ have the same parity that $\frac{f(-n)}{(-n)^k} = \frac{f(n)}{n^k}$. Thus we can write the desired series as

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \frac{1}{2} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{f(n)}{n^k} = \frac{\pi^k}{2q^k} \left( \sum_{a=1}^{q-1} f(a) A_{\frac{a}{q},k} \; + \; 2f(q)Z(k) \right).$$

∎

From Theorems 5.3 and 5.6 we are now ready to prove a result about the series

$$\sum_{n\in\mathbb{Z}}{}' \frac{f(n)A(n)}{B(n)}.$$

Similar to chapter 2, we require that the numerator and denominator have algebraic coefficients and the denominator has only rational roots, along with required restrictions on the degrees of the polynomials for convergence. We also take the summation over all integers, avoiding the zeros of $B(x)$.

**Theorem 5.10** *Let $f$ be an algebraic valued periodic function with integer period. Let $A(x), B(x)$ be polynomials with algebraic coefficients, $B(x)$ with only rational roots, and* $\deg(A) < \deg(B)$. *The series*

$$\sum_{n\in\mathbb{Z}}{}' \frac{f(n)A(n)}{B(n)} = P(\pi),$$

*where $P(x)$ is a polynomial with algebraic coefficients and* $\deg(P) \leq \deg(B)$.

**Proof.** Write

$$B(x) = \lambda(x - m_1)^{j_1} \cdots (x - m_r)^{j_r}(x + \alpha_1)^{k_1} \cdots (x + \alpha_s)^{k_s}$$

where the $m_1, \ldots, m_s$ are the distinct integer roots and $\alpha_1, \ldots, \alpha_r \in \mathbb{Q} \setminus \mathbb{Z}$ are distinct and $\lambda \in \overline{\mathbb{Q}}$. We can pull the $\lambda$ out front of the whole series so without loss of generality we may assume that $\lambda = 1$. By partial fractions, $A(n)/B(n)$ is equal to

$$\left(\frac{c_{1,1}}{(n-m_1)} + \cdots + \frac{c_{1,j_1}}{(n-m_1)^{j_1}}\right) + \cdots + \left(\frac{c_{r,1}}{(n-m_r)} + \cdots + \frac{c_{r,j_r}}{(n-m_r)^{j_r}}\right)$$
$$+ \left(\frac{d_{1,1}}{(n+\alpha_1)} + \cdots + \frac{d_{1,k_1}}{(n+\alpha_1)^{k_1}}\right) + \cdots + \left(\frac{d_{s,1}}{(n+\alpha_s)} + \cdots + \frac{d_{s,k_s}}{(n+\alpha_s)^{k_s}}\right)$$

where the $c_{i,j}$'s and $d_{i,j}$'s are algebraic numbers. Our summation becomes

$$\sum_{n\in\mathbb{Z}}{}' \frac{f(n)A(n)}{B(n)} = c_{1,1} \sum_{n\in\mathbb{Z}}{}' \frac{f(n)}{(n-m_1)} + \ldots + d_{s,k_s} \sum_{n\in\mathbb{Z}}{}' \frac{f(n)}{(n+\alpha_s)^{k_s}}$$

where each sum avoids $m_1, \ldots, m_r$. For some appropriate $i, j$ we have

$$c_{i,j} \sum_{n\in\mathbb{Z}}{}' \frac{f(n)}{(n-m_i)^j} = c_{i,j} \left[ \sum_{n\in\mathbb{Z}, n\neq m_i} \frac{f(n)}{(n-m_i)^j} - \sum_{t=1, t\neq i}^{r} \frac{f(m_t)}{(m_t-m_i)^j} \right].$$

From Theorem 5.6 above we have that this is equal to

$$\pi^j \Gamma_{i,j} + \gamma_{i,j}$$

for algebraic $\gamma_{i,j}, \Gamma_{i,j}$. Similarly we have

$$d_{i,j} \sum_{n\in\mathbb{Z}}{}' \frac{f(n)}{(n+\alpha_i)^j} = d_{i,j} \left[ \sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha_i)^j} - \sum_{t=1}^{r} \frac{f(m_t)}{(m_t+\alpha_i)^j} \right].$$

From Theorem 5.3 above we have that this is equal to

$$\pi^j \Delta_{i,j} + \delta_{i,j}$$

for algebraic $\delta_{i,j}, \Delta_{i,j}$. From here it is easy to see that

$$\sum_{n\in\mathbb{Z}}{}' f(n)A(n)/B(n) = P(\pi)$$

as desired. ∎

If we assume that $B(x)$ has no integer roots, we get a simpler result.

**Corollary 5.11** *In the same setting as the previous theorem, let $B(x)$ have roots only in $\mathbb{Q} \setminus \mathbb{Z}$. Then*

$$\sum_{n \in \mathbb{Z}} \frac{f(n)A(n)}{B(n)} = \pi P(\pi)$$

*is zero or transcendental, where $P(x)$ is a polynomial with algebraic coefficients.*

**Proof.** Note that since $B(x)$ has no integer roots, we can take summation is over all integers without a problem. Following the proof of the previous theorem, we no longer add and subtract terms for omitted integers and thus each individual summation from the previous proof is an algebraic multiple of a positive power of $\pi$. That is, each $\gamma_{i,j}$ and $\delta_{i,j}$ is zero. We obtain an algebraic polynomial in $\pi$ with zero constant term. Thus the series is either zero or transcendental. ∎

One would like to characterize exactly when series of the form from Theorem 5.10 are algebraic. That is, when does the transcendental part of the closed form disappear. This is difficult to answer due to the many variables which appear. Written as a polynomials in $\pi$, the coefficients could cancel out in some nontrivial way. We leave this problem open at this time.

Examining the closed forms we obtain, as in the previous corollary, we are able to enlarge our family of transcendental series via Nesterenko's theorem ([19]) that $\pi$ and $e^{\pi\sqrt{D}}$ are algebraically independent for any positive integer $D$. Nesterenko proved a more general result which we state now.

**Theorem 5.12** (Nesterenko, [19]) *For any imaginary quadratic field with discriminant $-D$ and character $\varepsilon$, the numbers*

$$\pi, e^{\pi\sqrt{D}}, \prod_{a=1}^{D-1} \Gamma(a/D)^{\varepsilon(a)}$$

*are algebraically independent.*

Before stating our theorem, we require a couple of lemmas. The first lemma allows us to take rational exponents of algebraically independent numbers, and still maintain algebraic independence.

**Lemma 5.13** *If $z_1, \ldots, z_l$ are algebraically independent over a field $K$ then so are $z_1^{p_1/q_1}, \ldots, z_l^{p_l/q_l}$ for any nonzero rational numbers $p_1/q_1, \ldots, p_l/q_l$.*

**Proof.** We proceed by examining transcendence degrees of field extensions. The transcendence degree of $K(z_1, \ldots, z_l)$ over $K$ is $l$, while $K(z_1^{1/q_1}, \ldots, z_l^{1/q_l})$ is algebraic over $K(z_1, \ldots, z_l)$, since $(z_j^{1/q_j})^{q_j} = z_j$. Thus, the transcendence degree of $K(z_1^{1/q_1}, \ldots, z_l^{1/q_l})$ over $K$ is also $l$. The numbers $\{z_1^{1/q_1}, \ldots, z_l^{1/q_l}\}$ are algebraically independent over $K$, so $\{z_1^{p_1/q_1}, \ldots, z_l^{p_l/q_l}\}$ are as well. ∎

Although (nontrivial) polynomial expressions in algebraically independent numbers are never zero, it is not clear whether or not rational functions evaluated at algebraically independent numbers can be algebraic, or zero. The following lemma examines this situation.

**Lemma 5.14** *Let $z_1, \ldots, z_l$ be algebraically independent over a field $K$. Also let $A(x_1, \ldots, x_l), B(x_1, \ldots, x_l) \in K[x_1, \ldots, x_l]$ be two nonzero polynomials whose quotient is not in $K$. The ratio*

$$\frac{A(z_1, \ldots, z_l)}{B(z_1, \ldots, z_l)}$$

*is not in $K$.*

**Proof.** Suppose the ratio is equal to some $\alpha \in K$. Clearing denominators, we have that $A(z_1, \ldots, z_l) = \alpha B(z_1, \ldots, z_l)$ which implies that

$$A(z_1, \ldots, z_l) - \alpha B(z_1, \ldots, z_l) = 0.$$

This contradicts the algebraic independence of the $z_i$'s. ∎

The previous lemmas allow us to work with rational functions in transcendental terms, with any rational exponents. These two lemmas are needed for the following result. Note that sums of ratios that appear in the previous lemma could be in $K$.

**Theorem 5.15** *Let $f$ be an algebraic valued periodic function with integer period, $q$. Let $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ be polynomials with $\deg(A) < \deg(B)$ and all the roots of $B(x)$ lie in $\mathbb{Q}(\sqrt{-D}) \setminus \mathbb{Z}$ for some positive integer $D$. The series*

$$\sum_{n \in \mathbb{Z}} \frac{f(n)A(n)}{B(n)}$$

*is either zero or transcendental.*

**Proof.** Without loss of generality, assume that

$$B(x) = (x + \alpha_1)^{k_1} \cdots (x + \alpha_s)^{k_s}$$

where each $\alpha_j = a_j + b_j\sqrt{-D}$ for $a_j, b_j \in \mathbb{Q}$. By partial fractions, we rewrite our series as

$$\sum_{n\in\mathbb{Z}} \frac{f(n)A(n)}{B(n)} = c_{1,1} \sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha_1)} + \ldots + c_{s,k_s} \sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha_s)^{k_s}}.$$

For a given pair $i, j$,

$$c_{i,j} \sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha_i)^j} = \frac{c_{i,j}\pi^j}{q^j} \sum_{a=1}^{q} f(a) A_{\frac{a+\alpha_i}{q},j}.$$

From the definition of $A_{x,k}$, being related to derivatives of the cotangent function,

$$\cot(\pi z) = i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1},$$

it is easy to see that $A_{\frac{a+\alpha_i}{q},j}$ is a rational expression in $e^{2\pi i\alpha_i/q}$ with algebraic coefficients. We write $e^{2\pi i\alpha_i/q}$ as $e^{2\pi ia_i/q}e^{-2\pi b_i\sqrt{D}/q}$. Writing the entire expression for the series as a polynomial in $\pi$, we see that each coefficient is an algebraic rational function evaluated at a transcendental number which is essentially $e^{\pi\sqrt{D}}$. Nesterenko's theorem 5.12 implies that each of the coefficients (if nonzero) and $\pi$ are algebraically independent, therefore the series either vanishes or is transcendental. ∎

We would like to also include cases where roots are in a real quadratic field, however it is not known if $\pi$ and $e^{\pi i\sqrt{D}}$ are algebraically independent for positive integer $D$.

We conclude this section with the following remark.

**Remark 5.16.** In [10], the authors state various theorems about infinite series (summation from 0 to $\infty$) of rational functions. They prove that certain series are related to derivatives of the cotangent function, similar to results found here. However, in their theorems, they always require some sort of symmetry condition. As an example the authors examine series of the form

$$\sum_{n=0}^{\infty} \frac{P_1(n)}{Q_1(n)} + \cdots + \frac{P_s(n)}{Q_s(n)}$$

where all ratios satisfy either

$$\frac{P_j(-x)}{Q_j(-x)} = \frac{P_j(r_j + x)}{Q_j(r_j + x)} \quad \text{or} \quad \frac{P_j(-x)}{Q_j(-x)} = (-1)^{r_j} \frac{P_j(r_j + x)}{Q_j(r_j + x)}$$

for some $r_j \in \mathbb{Z}$. It is easy to see that these symmetry conditions simply allow us to change from summation over $\mathbb{N}$ to summation over $\mathbb{Z}$ in a similar manner as was demonstrated in the proof of Proposition 5.9. Thus, many of the results in [10] are simple corollaries of the results found here. We note that our results show more because we do not require these symmetry conditions. We refer the reader to [10] for further examination.

## 5.2 A multivariable generalization

Using the results of the previous section, we easily obtain multivariable versions of each result. For notational simplification we write

$$G = \mathbb{Z}/q_1\mathbb{Z} \times \cdots \times \mathbb{Z}/q_r\mathbb{Z}$$

and take a function $f$ which is periodic in $r$ variables to mean a function on $G$ and then extend the definition of $f$ to a function on $\mathbb{Z}^r$.

**Theorem 5.17** *Take* $f : G \to \overline{\mathbb{Q}}$. *Let* $\alpha_1, \ldots, \alpha_r \in \mathbb{Q}$ *and* $k_1, \ldots, k_r$ *be* *positive integers. The series*

$$\sum_{n_1,\ldots,n_r\in\mathbb{Z}}{}' \frac{f(n_1,\ldots,n_r)}{(n_1+\alpha_1)^{k_1}\cdots(n_r+\alpha_r)^{k_r}}$$

*is in* $\overline{\mathbb{Q}}\pi^k$ *where* $k = k_1 + \cdots + k_r$.

**Proof.** We calculate directly. We rewrite our series according to congruence classes,

$$\sum_{(a_1,\ldots,a_r)\in G} f(a_1,\ldots,a_r)\prod_{i=1}^{r}\left(\sum_{n_i\equiv a_i(q_i)}{}' \frac{1}{(n_i+\alpha_i)^{k_i}}\right).$$

Writing $n_i = q_i n + a_i$ we obtain

$$\sum_{(a_1,\ldots,a_r)\in G} f(a_1,\ldots,a_r)\prod_{i=1}^{r}\left(\sum_{n\in\mathbb{Z}}{}' \frac{1}{(q_i n + a_i+\alpha_i)^{k_i}}\right)$$

which simplifies to

$$\frac{1}{(q_1^{k_1}\cdots q_r^{k_r})}\sum_{(a_1,\ldots,a_r)\in G} f(a_1,\ldots,a_r)\prod_{i=1}^{r}\left(\sum_{n\in\mathbb{Z}}{}' \frac{1}{(n+\frac{a_i+\alpha_i}{q_i})^{k_i}}\right).$$

By Proposition 5.2 and equation (5.3) of Theorem 5.6 this is equal to

$$\frac{\pi^k}{(q_1^{k_1}\cdots q_r^{k_r})}\sum_{(a_1,\ldots,a_r)\in G} f(a_1,\ldots,a_r)\prod_{i=1}^{r} F(q_i,a_i,\alpha_i,k_i)$$

where

$$F(q,a,\alpha,t) = \left\{ \begin{array}{lc} 2Z(t) \quad \text{if} \quad \alpha \in \mathbb{Z} \text{ and } a \equiv -\alpha \pmod{q} \\ A_{\frac{a+\alpha}{q},t} & \text{otherwise} \end{array} \right\}$$

and $Z(t) = 0$ if $t$ is odd and $\frac{\zeta(t)}{\pi^t}$ if $t$ is even. This is an algebraic multiple of $\pi^k$. ∎

From here it is not difficult to work with the case that there is a polynomial in the numerator and the denominator in each variable. As before we put certain conditions on these polynomials.

**Theorem 5.18** *Take* $f : G \to \overline{\mathbb{Q}}$. *Let* $A_1(x), \ldots, A_r(x) \in \overline{\mathbb{Q}}[x]$ *and let* $B_1(x), \ldots, B_r(x) \in \overline{\mathbb{Q}}[x]$ *with only rational roots. Suppose that* $\deg(A_i) < \deg(B_i)$, *then the series*

$$\sum_{n_1,\ldots,n_r \in \mathbb{Z}}{}' \frac{f(n_1,\ldots,n_r)A_1(n_1)\cdots A_r(n_r)}{B_1(n_1)\cdots B_r(n_r)} = P(\pi)$$

*where* $P(x)$ *is a polynomial with algebraic coefficients.*

**Proof.** This is similar to the previous theorem. We rewrite our series as

$$\sum_{(a_1,\ldots,a_r)\in G} f(a_1,\ldots,a_r) \prod_{i=1}^{r} \left( \sum_{n_i \equiv a_i(q_i)}{}' \frac{A_i(n_i)}{B_i(n_i)} \right).$$

Let $f_{a_i,q_i}(n)$ be the periodic function that is 1 when $n \equiv a_i (\bmod\ q_i)$ and 0 otherwise. By Theorem 5.10

$$\sum_{n_i \equiv a_i (q_i)}' \frac{A_i(n_i)}{B_i(n_i)} = \sum_{n_i \in \mathbb{Z}}' \frac{f_{a_i,q_i}(n_i) A_i(n_i)}{B_i(n_i)}$$

is a polynomial in $\pi$ with algebraic coefficients. The result follows. ∎

## 5.3 Fourier approach

We now examine the same types of series which were examined in section 5.1 using the theory of Fourier transforms. We obtain the same transcendence results via a different method. By utilizing Fourier techniques we obtain different closed forms for our series. Parts of this alternate technique can be applied in other settings as well. The main techniques used here are Fourier analysis in the classical sense as well as Fourier analysis of finite groups.

We begin by defining exactly what form we are taking for a Fourier series.

**Definition 5.19.** For a periodic function $f(x)$ defined on $(0, 1)$, which is Lebesgue integrable on $[0, 1]$ we define the $n$th Fourier coefficient of $f(x)$ by

$$f_n := \int_0^1 f(x) e^{-2\pi i n x} dx.$$

With this definition we have the Fourier series for $f(x)$ is

$$\sum_{n \in \mathbb{Z}} f_n e^{2\pi i n x}$$

We assume the reader is familiar with the properties of the Fourier transform and Fourier series, however we point out a theorem of Dirichlet regarding exactly what a given Fourier series converges to.

**Theorem 5.20** (Dirichlet's Theorem for Fourier series) *For a piecewise continuous periodic (with period one) function $f$ with a finite number of extrema and a finite number of discontinuities within any given interval that is absolutely integrable over a period and is bounded, for any $x$, we have that the Fourier series at $x$ is convergent and is given by*

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx} = \frac{f(x+)+f(x-)}{2}.$$

We next define convolution of periodic functions and proceed to analyze a specific Fourier series which will play an important role in the remainder of the chapter.

**Definition 5.21.** For two Lebesgue integrable, periodic functions $f, g$ on $\mathbb{R}$, both with period 1, we define the convolution $f * g$ as

$$(f*g)(t) := \int_0^1 f(x)g(t-x)dx.$$

It is easy to see from the definition of $*$ that for $f$ and $g$ both periodic with period 1, $f * g$ has period 1. Thus the next lemma is natural.

**Lemma 5.22** *Let $f, g$ be Lebesgue integrable, periodic functions with period 1 with Fourier coefficients $f_n, g_n$ respectively. For $h = f * g$ we have that $h_n = f_n g_n$.*

**Proof.** We calculate the coefficients $h_n$ directly,

$$h_n = \int_0^1 \int_0^1 f(x)g(t-x)e^{-2\pi int}dxdt$$

which, by Fubini, equals

$$\int_0^1 f(x) \int_0^1 g(t-x) e^{-2\pi i n t} dt dx.$$

If we let $z = t - x$ we obtain

$$\int_0^1 f(x) \int_{-x}^{1-x} g(z) e^{-2\pi i n (z+x)} dz dx$$

which is equal to

$$\int_0^1 f(x) e^{-2\pi i n x} dx \int_0^1 g(z) e^{-2\pi i n z} dz$$

since the second integrand has period 1. These last two integrals are equal to the product $f_n g_n$ and we are done. ∎

We next examine a special function, and its Fourier series. We first need a bit of notation in order to state our theorems more succinctly. For a periodic function $f$ with period 1, for the convolution of $f$ with itself $k$ times we write

$$f^{*k} := f * \cdots * f \qquad \text{(k times)}.$$

Using the previous lemma we examine a special function.

**Lemma 5.23** *For $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, let $g(t) = \frac{2\pi i}{1 - e^{-2\pi i \alpha}} e^{-2\pi i \alpha t}$ for $t \in (0, 1)$ and take copies on each unit interval $(n, n+1)$. Then for any $t \notin \mathbb{Z}$ and any positive* *integer $k$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{e^{2\pi int}}{(n+\alpha)^k} = g^{*k}(t) = \pi^k e^{-2\pi i\alpha\{t\}} B_{\alpha,k}(\{t\})$$

*where $B_{\alpha,k}(t)$ is a polynomial in $t$ with algebraic coefficients.*

**Proof.** We calculate the Fourier coefficients of $g$. We have

$$g_n = \frac{2\pi i}{1-e^{-2\pi i\alpha}} \int_0^1 e^{-2\pi i\alpha t} e^{-2\pi int} dt$$

which equals

$$\frac{2\pi i}{1-e^{-2\pi i\alpha}} \int_0^1 e^{-2\pi i(n+\alpha)t} dt.$$

Integrating we have

$$\frac{2\pi i}{1-e^{-2\pi i\alpha}} \left( \frac{e^{-2\pi i(n+\alpha)t}}{-2\pi i(n+\alpha)} \right)_0^1$$

which equals $1/(n+\alpha)$. Thus the first equality of the theorem comes from Lemma 5.22 above, since $g(t)$ is continuous and differentiable on $(0,1)$. Note that we are intentionally omitting the end points, and they will be analyzed later.

We prove the second equality by induction on $k$. By periodicity, assume that $t \in (0,1)$. For $k=1$ we have

$$g^{*1}(t) = g(t) = \pi^1 e^{-2\pi i\alpha t} \left( \frac{2i}{1-e^{-2\pi i\alpha}} \right).$$

Assume that the claim holds for all $1 \le n < k$. We calculate $g^{*k}$ directly. Splitting the convolution integral as follows, ensures that $0 < t - x < 1$ and

$0 < 1 + t - x < 1$ for the respective integrals. Write $g^{*k}(t) = (g^{*k-1} * g)(t)$ which splits as

$$\int_0^t g^{*k-1}(x)g(t-x)dx + \int_t^1 g^{*k-1}(x)g(1+t-x)dx.$$

By induction we make a substitution for $g^{*k-1}$

$$\pi^k \frac{2i}{(1-e^{-2\pi i\alpha})}\left[\int_0^t e^{-2\pi i\alpha x}B_{\alpha,k-1}(x)e^{-2\pi i\alpha(t-x)}dx + \int_t^1 e^{-2\pi i\alpha x}B_{\alpha,k-1}(x)e^{-2\pi i\alpha(1+t-x)}dx\right]$$

which equals

$$\pi^k e^{-2\pi i\alpha t}\left[\frac{2i}{(1-e^{-2\pi i\alpha})}\left(\int_0^t B_{\alpha,k-1}(x)dx \quad + \quad e^{-2\pi i\alpha}\int_t^1 B_{\alpha,k-1}(x)dx\right)\right].$$

Finally, the square brackets equate to a polynomial with algebraic coefficients thus our convolution equals $\pi^k e^{-2\pi i\alpha t}B_{\alpha,k}(t)$ as claimed and we are done. ∎

As mentioned, we were avoiding the end points 0 and 1 on purpose. Dirichlet's theorem 5.20 implies that, in our situation, the Fourier series converges to the average of the left and right hand limits of $g^{*k}$ at any given point. In $(0,1)$ this is the function itself since the function is continuous. At the end points we obtain an average. We find the closed form for the end points here.

**Corollary 5.24** *From the identity* $\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^k} = \pi^k A_{\alpha,k}$ *of section 5.1, we have*

$$\begin{aligned} i)\ A_{\alpha,1} &= \frac{i(1+e^{-2\pi i\alpha})}{(1-e^{-2\pi i\alpha})} \\ ii)\ A_{\alpha,k} &= e^{-2\pi i\alpha}B_{\alpha,k}(1) \quad \textit{for } k \geq 2. \end{aligned}$$

**Proof.** First note that from Dirichlet's theorem 5.20 we have,

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi ina} = \frac{f(a^-)+f(a^+)}{2}$$

where $c_n$ are the Fourier coefficients of $f(x)$ and $f(a^-), f(a^+)$ are the left and right limits respectively. In our case, $a$ is $t=1$ and $f$ is $g^{*k}$ as in Lemma 5.23. We have

$$\pi^k A_{\alpha,k} = \sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^k} = \pi^k \frac{e^{-2\pi i\alpha}B_{\alpha,k}(1)+B_{\alpha,k}(0)}{2}.$$

For $k=1$, we have that $B_{\alpha,1}(t) = 2i/(1-e^{-2\pi i\alpha})$ which proves $(i)$. Recall that the value of $A_{\alpha,1}$ has been calculated in section 5.1. For $(ii)$, from the analysis in Lemma 5.23, we have

$$B_{\alpha,k}(t) = \frac{2i}{1-e^{-2\pi i\alpha}}\left(\int_0^t B_{\alpha,k-1}(x)dx + e^{-2\pi i\alpha}\int_t^1 B_{\alpha,k-1}(x)dx\right).$$

Taking limits (note $B_{\alpha,k}$'s are polynomials) we have,

$$\begin{aligned} B_{\alpha,k}(0) &= \frac{2i}{1-e^{-2\pi i\alpha}}\left(0 + e^{-2\pi i\alpha}\int_0^1 B_{\alpha,k-1}(x)dx\right) \\ e^{-2\pi i\alpha}B_{\alpha,k}(1) &= e^{-2\pi i\alpha}\frac{2i}{1-e^{-2\pi i\alpha}}\left(\int_0^1 B_{\alpha,k-1}(x)dx + 0\right) \end{aligned}$$

Thus $e^{-2\pi i\alpha}B_{\alpha,k}(1) = B_{\alpha,k}(0)$ and so for $k \geq 2$ the equality

$$A_{\alpha,k} = e^{-2\pi i\alpha}B_{\alpha,k}(1).$$

What this amounts to is that $g(t)$ defined above is not continuous, but $g^{*k}(t)$

for $k \geq 2$ is continuous. ∎

With this preliminary Fourier analysis complete, we now wish to analyze the same type of series discussed in section 5.1. As mentioned, we will use Fourier analysis of finite groups to analyze the series $\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k}$ which in turn, by the same partial fractions trick, allows us to find alternate closed forms for all of the series in section 5.1.

**Theorem 5.25** *Let $f$ be an algebraic valued periodic function with integer period $q$. Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ and $k$ a positive integer. The series*

$$\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k}$$

*is in $\overline{\mathbb{Q}}\pi^k$.*

**Proof.** The discrete Fourier transform of $f$ is $\widehat{f}(n) = \frac{1}{q}\sum_{a=1}^{q} f(a)\zeta_q^{-an}$. Inversion yields $f(n) = \sum_{a=1}^{q} \widehat{f}(a)\zeta_q^{an}$. Making this substitution we have

$$\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k} = \sum_{a=1}^{q} \widehat{f}(a) \sum_{n\in\mathbb{Z}} \frac{\zeta_q^{an}}{(n+\alpha)^k}.$$

For each $0 < a < q$ we use the Fourier analysis from Lemma 5.23 to obtain

$$\sum_{n\in\mathbb{Z}} \frac{\zeta_q^{an}}{(n+\alpha)^k} = \pi^k \zeta_q^{-\alpha a} B_{\alpha,k}(a/q)$$

where each $B_{\alpha,k}(a/q)$ is algebraic. For $a = q$, from Corollary 5.2 we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^k} = \pi^k A_{\alpha,k}$$

where $A_{\alpha,k}$ is algebraic. Putting the two cases together we obtain,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k} &= \sum_{a=1}^{q} \widehat{f}(a) \sum_{n\in\mathbb{Z}} \frac{\zeta_q^{an}}{(n+\alpha)^k} \\ &= \pi^k \left( \sum_{a=1}^{q-1} \widehat{f}(a)\zeta_q^{-\alpha a} B_{\alpha,k}(a/q) + \widehat{f}(q) A_{\alpha,k} \right)\end{aligned}$$

From Corollary 5.24 above, for $k = 1$ we obtain,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{f(n)}{n+\alpha} &= \pi \left( \sum_{a=1}^{q-1} \widehat{f}(a)\zeta_q^{-\alpha a} B_{\alpha,1}(a/q) + \widehat{f}(q) A_{\alpha,1} \right) \\ &= \frac{\pi i}{(1-e^{-2\pi i\alpha})} \left( 2\sum_{a=1}^{q-1} \widehat{f}(a)\zeta_q^{-\alpha a} \quad + \widehat{f}(q)\left(1+e^{-2\pi i\alpha}\right) \right).\end{aligned}$$

For $k \geq 2$ we have

$$\sum_{n\in\mathbb{Z}} \frac{f(n)}{(n+\alpha)^k} = \pi^k \left( \sum_{a=1}^{q} \widehat{f}(a)\zeta_q^{-\alpha a} B_{\alpha,k}(a/q) \right).$$

In either case, the inner sum is algebraic which implies that the right hand side is zero or an algebraic multiple of $\pi^k$, which is what we desired. ∎

Next we examine the series where $\alpha = 0$. Of course we change our summation from all integers to all integers excluding 0. Before we state a theorem we need a lemma involving the Fourier series of Bernoulli functions. For completeness we define the Bernoulli polynomials here. They are defined recursively and have the following properties:

1. $b_0(x) = 1$

2. $b'_k(x) = kb_{k-1}(x)$

3. $\int_0^1 b_k(x)dx = 0$

The Bernoulli polynomials also make an appearance with a certain generating function.

$$\sum_{k=0}^{\infty} \frac{b_k(x)}{k!} t^k = \frac{te^{xt}}{e^t - 1}$$

The Bernoulli functions are defined as $B_k(x) = b_k(\{x\})$. Each $B_k(x)$ is periodic with period 1 so we can calculate the Fourier series for each.

**Lemma 5.26** (A. Hurwitz, 1890) *For any $x \notin \mathbb{Z}$ and $k \geq 1$ we have*

$$\frac{B_k(x)}{k!} = -\sum_{n \in \mathbb{Z}, n \neq 0} \frac{e^{2\pi i n x}}{(2\pi i n)^k}.$$

**Proof.** We calculate all the Fourier coefficients for every Bernoulli function at the same time. Recall the generating function above,

$$\sum_{k=0}^{\infty} \frac{b_k(x)}{k!} t^k = \frac{te^{xt}}{e^t - 1}.$$

Let $c_{k,n}$ denote the $n$th Fourier coefficient of $B_k(x)/k!$.

$$\begin{aligned} \sum_{k=0}^{\infty} c_{k,n} t^k &= \int_0^1 \sum_{k=1}^{\infty} \frac{B_k(x)}{k!} t^k e^{-2\pi i n x} dx \\ &= \int_0^1 \frac{te^{xt}}{e^t - 1} e^{-2\pi i n x} dx \\ &= \frac{t}{t - 2\pi i n} \end{aligned}$$

From the definition of the Bernoulli polynomials, for $n = 0$ we see that the only term that remains is for $k = 0$. That is $c_{k,0} = 0$ for every $k \geq 1$. For $n \neq 0$,

$$\begin{aligned}
\sum_{k=0}^{\infty} c_{k,n} t^k &= \frac{t}{t - 2\pi i n} \\
&= \frac{-t}{2\pi i n}\left(\frac{1}{1 - t/2\pi i n}\right) \\
&= \frac{-t}{2\pi i n}\sum_{k=0}^{\infty}\left(\frac{t}{2\pi i n}\right)^k \\
&= -\sum_{k=1}^{\infty}\frac{t^k}{(2\pi i n)^k}
\end{aligned}$$

This implies that $c_{0,n} = 0$ for $n \neq 0$, which we already knew since $B_0(x) = 1$. Also we get that $c_{k,n} = \frac{-1}{(2\pi i n)^k}$ for $n \neq 0$. By Dirichlet's theorem (see Appendix), we have the Fourier series for each Bernoulli function and the equality

$$\frac{B_k(x)}{k!} = -\sum_{n \in \mathbb{Z}, n \neq 0} \frac{e^{2\pi i n x}}{(2\pi i n)^k}$$

for every $k \geq 1$ and $x \notin \mathbb{Z}$. ∎

Note also that by Dirichlet's theorem, the above equality holds for all $x \in \mathbb{R}$ for $k \geq 2$ by the continuity of the Bernoulli functions for $k \geq 2$. We can immediately give an alternate proof of Theorem 5.6 using Lemma 5.26.

**Theorem 5.27** *Let $f$ be an algebraic valued periodic function with integer period $q$ and let $k$ be a positive integer. The series*

$$\sum_{n \in \mathbb{Z}, n \neq 0} \frac{f(n)}{n^k}$$

*is in* $\overline{\mathbb{Q}}\pi^k$.

**Proof.** We proceed as we did in Theorem 5.25.

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{f(n)}{n^k} \;=\; \sum_{a=1}^{q} \widehat{f}(a) \sum_{n\in\mathbb{Z},n\neq 0} \frac{\zeta_q^{na}}{n^k}.$$

From Lemma 5.26 we have that

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{\zeta_q^{an}}{n^k} = -\frac{(2\pi i)^k B_k(\frac{a}{q})}{k!}$$

for every $1 < a < q$. When $k \geq 2$ the equality holds for every $a = q$. For $k = 1$ the term for $a = q$ disappears, so we have

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{f(n)}{n} \;=\; -2\pi i \left( \sum_{a=1}^{q-1} \widehat{f}(a) B_1(\frac{a}{q}) \right)$$

For $k \geq 2$ we have

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{f(n)}{n^k} \;=\; -(2\pi i)^k \left( \sum_{a=1}^{q} \frac{\widehat{f}(a) B_k(\frac{a}{q})}{k!} \right)$$

The Bernoulli polynomials have rational coefficients so in both cases we have $\pi^k$ times an algebraic number and we are done. ∎

Using Theorems 5.25 and 5.27 we can prove the general result regarding series of the form

$$\sum_{n\in\mathbb{Z}}^{\infty}{}' \frac{f(n)A(n)}{B(n)},$$

namely Theorem 5.10. The proof that series of this form are polynomials in $\pi$ with algebraic coefficients rests solely on using partial fractions and applying Theorems 5.25 and 5.27 (or Theorems 5.3 and 5.6 in the previous section).

As we have seen, the two methods yield the same transcendence results. However, the methods demonstrated in this section give us a general recipe for finding closed forms. We need only recognize a series as the Fourier series of a particular function, then we will be able to conclude something about the convergent series. We end this section here simply by noting again that by passing from summation over the natural numbers to summation over the integers, we were able to conclude some nice transcendence results with relatively elementary methods.

## 5.4 Schanuel's conjecture and some examples

In this chapter, so far, our theorems have all involved finding generic closed forms for series, then saying something about the transcendence of the series. Our theorems have mostly been general statements that the series is either zero or transcendental, and due to the general nature of each series, we are not able to state a theorem which fully classifies these numbers as transcendental or not. Thus we examine some examples which lead us to study exactly what is needed for these series to be transcendental.

We will see that even in the early cases, the extent of our knowledge of transcendental number theory is quickly exhausted. In some cases we cannot say anything concrete, but we show implications of Schanuel's Conjecture. No proof of Schanuel's conjecture is known, but progress has been made on questions surrounding the conjecture. In particular, D. Roy [22] states the following conjecture. The notation $\mathcal{D}$ stands for the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

and *height* of a polynomial is taken to be the maximum of the absolute values of its coefficients.

**Conjecture 5.28** (D. Roy) *Let $l$ be a positive integer, $y_1, \ldots, y_l \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$ and let $\alpha_1, \ldots, \alpha_l \in \mathbb{C}^\times$. Moreover, let $s_0, s_1, t_0, t_1, u$ be positive numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}, \qquad \max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

*Assume that, for any sufficiently large positive integer $N$, there exists a nonzero polynomial $P_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in $X_0$, partial degree $\leq N^{t_1}$ in $X_1$ and height $\leq e^N$ which satisfies*

$$\left| \left(\mathcal{D}^k P_N\right) \left( \sum_{j=1}^{l} m_j y_j, \prod_{j=1}^{l} \alpha_j^{m_j} \right) \right| \leq exp(-N^u),$$

*for any integers $k, m_1, \ldots, m_l \in \mathbb{N}$ with $k \leq N^{s_0}$ and $max\{m_1, \ldots, m_l\} \leq N^{s_1}$. Then* $\operatorname{trdeg}(\mathbb{Q}(y_1, \ldots, y_l, \alpha_1, \ldots, \alpha_l)) \geq l$.

In [22], the author shows that Conjecture 5.28 is equivalent to Schanuel's conjecture. This reformulation offers a different approach to understanding Schanuel's conjecture, though we do not examine it here.

Similar to the exponential form in Schanuel's conjecture, Schneider (1952) [24] conjectured the following.

**Conjecture 5.29** (Schneider) *If $\alpha \neq 0, 1$ is algebraic and if $\beta$ is algebraic of degree $d \geq 2$, then the $d-1$ numbers*

$$\alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}$$

*are algebraically independent.*

Most recent progress on this conjecture is due to Diaz. Diaz's result is an improvement of a breakthrough result of Philippon [21]. Diaz improved the result by further analysis of Philippon's proof.

**Theorem 5.30** (G. Diaz, [6]) *If $\alpha$ is an algebraic number, not 0 or 1, and if $\beta$ is an irrational algebraic number of degree $d$, then*

$$\operatorname{trdeg}(\mathbb{Q}(\alpha^{\beta}, \alpha^{\beta^2} \ldots, \alpha^{\beta^{d-1}})) \geq \left[\frac{d+1}{2}\right].$$

Philippon's result is exactly the same in statement except the lower bound on transcendence degree is $[d/2]$. So far, the results of Philippon and Diaz are the best attack at proving Schneider's conjecture. In particular, Diaz shows the conjecture to be true for $d = 3$. We point out that in fact this $d = 3$ case was in fact first proved by Gel'fond [8].

**Theorem 5.31** (A. O. Gel'fond (1949), [8]) *For $\alpha$ algebraic of degree 3, and $a \neq 0, 1$ the numbers $a^{\alpha}$ and $a^{\alpha^2}$ are algebraically independent over $\mathbb{Q}$.*

We now study a few examples that use the theory developed throughout and also the theory mentioned here. Note that Bundschuh [4] examined the following types of series, but in a less general context. In particular, Bundschuh was interested in $\sum_{n\geq 2} 1/(n^s - 1)$ for integer $s \geq 3$. During the analysis, he briefly examined summation over $|n| \geq 2$ as well and showed that if Schanuel's conjecture is true, then these series are all transcendental.

We examine the case where the denominator is $n^s \pm c^s$ for various $c$, instead of Bundschuh's "$n^s - 1$". Unconditionally we prove that all of the series

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+c^2},\ \sum_{n\in\mathbb{Z}} \frac{1}{n^3 \pm c^3},\ \sum_{n\in\mathbb{Z}} \frac{1}{n^4-c^4},\ \sum_{n\in\mathbb{Z}} \frac{1}{n^6-c^6}$$

are transcendental for various values of $c$. We begin with some elementary observations.

**Theorem 5.32** *For any $c \in \mathbb{Q} \setminus \{0\}$ the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+c^2}$$

*is transcendental.*

**Proof.** We may assume that $c$ is positive. Let $F(x) = \frac{\pi}{c} e^{-2\pi c|x|}$. Applying the Fourier transform we obtain

$$\begin{aligned}
\widehat{F}(u) &= \int_{-\infty}^{\infty} F(x) e^{-2\pi iux} dx \\
&= \frac{\pi}{c} \int_{0}^{\infty} e^{-2\pi cx} e^{-2\pi iux} dx + \frac{\pi}{c} \int_{-\infty}^{0} e^{2\pi cx} e^{-2\pi iux} dx \\
&= \frac{1}{u^2+c^2}.
\end{aligned}$$

By the Poisson Summation formula we have the following equality,

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) e^{2\pi ivn}.$$

Setting $v = 0$ we obtain,

$$\frac{\pi}{c}\sum_{n\in\mathbb{Z}} e^{-2\pi c|n|} = \sum_{n\in\mathbb{Z}} \frac{1}{n^2+c^2}.$$

The sum on the left is simply two copies of a geometric series, which we simplify and obtain

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+c^2} = \frac{\pi(e^{2\pi c}+1)}{c(e^{2\pi c}-1)} \tag{5.4}$$

which is not zero.

By Nesterenko's theorem, $\pi$ and $e^{\pi}$ are algebraically independent, so our series is transcendental. ∎

Note that we could take powers of the denominator and by derivatives of the cotangent function, these series would all be transcendental as well. We give an even stronger result.

**Corollary 5.33** *Let $c_1, \ldots, c_k \in \mathbb{Q}\backslash\{0\}$ and let $s_1, \ldots, s_k$ be positive integers. The series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+c_1^2)^{s_1}\cdots(n^2+c_k^2)^{s_k}}$$

*is transcendental.*

**Proof.** First note that the series is not equal to zero since each term in the sum is positive. We write our summands as

$$\frac{1}{(n^2+c_1^2)^{s_1}\cdots(n^2+c_k^2)^{s_k}} = \frac{1}{(n+ic_1)^{s_1}(n-ic_1)^{s_1}\cdots(n+ic_k)^{s_k}(n-ic_k)^{s_k}}.$$

By partial fractions, we have that our series is equal to

$$\sum_{l=1}^{k}\sum_{m=1}^{s_l}\left[\alpha_{l,m}\sum_{n\in\mathbb{Z}}\frac{1}{(n+ic_l)^m}+\beta_{l,m}\sum_{n\in\mathbb{Z}}\frac{1}{(n-ic_l)^m}\right]$$

where each $\alpha_{l,m},\beta_{l,m}\in\overline{\mathbb{Q}}$. We obtain a closed form for our series by noting that each individual summation over $\mathbb{Z}$ is simply some number of derivative of $\pi\cot(\pi z)$ evaluated at $z=\pm ic_l$. Grouping powers of $\pi$ together, we have our series is equal to nonzero polynomial in $\pi$ with zero constant term. It is easy to see that each coefficient for a given power of $\pi$ is a rational expression in $e^{\pi}$, with rational exponents throughout. Since the series is nonzero, by Nesterenko's theorem we have transcendence. ∎

We next remove the square from $c$ from Theorem 5.32.

**Corollary 5.34** *For any rational $c>0$,*

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2+c}$$

*is transcendental.*

**Proof.** We substitute $\sqrt{c}$ for $c$ in equation (5.4) of Theorem 5.32 to obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2+c}=\frac{\pi(e^{2\pi\sqrt{c}}+1)}{\sqrt{c}(e^{2\pi\sqrt{c}}-1)}.$$

Recall that Nesterenko's theorem 5.12 shows that $\pi$ and $e^{\pi\sqrt{D}}$ are algebraically independent for any positive integer $D$. It is easy to see that $\sqrt{a/b}=\sqrt{ab}/b$ so we have that $\pi$ and $e^{\pi\sqrt{c}}$ are algebraically independent.

The series does not vanish and the result is now clear. ∎

Taking powers of the denominator in the previous corollary, we can also conclude transcendence of those series. This case includes the previous corollary, however we leave the former to show what's really going on when we remove the square on $c$.

**Corollary 5.35** *For any rational $c > 0$ and positive integer $k$,*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+c)^k}$$

*is transcendental.*

**Proof.** By partial fractions,

$$\frac{1}{(n^2+c)^k} = \sum_{j=1}^{k} \frac{\alpha_j}{(n+i\sqrt{c})^j} + \frac{\beta_j}{(n-i\sqrt{c})^j}$$

where each $\alpha_j, \beta_j \in \overline{\mathbb{Q}}$. Inserting this into our sum and relating the series to derivatives of the cotangent function, we obtain a polynomial in $\pi$ with algebraic coefficients and in particular, zero constant term. As before, each coefficient for a given power of $\pi$ is a rational expression in $e^{\pi\sqrt{c}}$. Since each term of the series is positive, our series is nonzero and by Nesterenko's theorem it is transcendental. ∎

The main thing that makes the previous corollary work is the fact that the roots of the denominator lie in an imaginary quadratic field. The general case of this is seen in Theorem 5.15.

From Theorem 5.32 we can also examine series with higher powers in the denominator.

**Theorem 5.36** *For any rational $c \in \mathbb{Q} \setminus \mathbb{Z}$, the series*

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^4 - c^4}$$

*is transcendental.*

**Proof.** By partial fractions, we can write

$$\frac{1}{x^4 - c^4} = \frac{1}{2c^2(x^2 + (ic)^2)} - \frac{1}{2c^2(x^2 + c^2)}.$$

From the closed form calculated in the proof of Theorem 5.32 our series,

$$\sum_{n \in \mathbb{Z}} \left( \frac{1}{2c^2(n^2 + (ic)^2)} - \frac{1}{2c^2(n^2 + c^2)} \right)$$

is equal to

$$\frac{\pi}{2c^3} \left( \frac{(e^{2\pi ic} + 1)}{i(e^{2\pi ic} - 1)} - \frac{(e^{2\pi c} + 1)}{(e^{2\pi c} - 1)} \right)$$

which simplifies to

$$\frac{\pi}{2c^3} \left( \frac{(e^{2\pi ic} + 1)(e^{2\pi c} - 1) - i(e^{2\pi c} + 1)(e^{2\pi ic} - 1)}{i(e^{2\pi ic} - 1)(e^{2\pi c} - 1)} \right). \tag{5.5}$$

This is equal to zero only if the numerator is zero, which implies that the algebraic coefficients of 1 and $e^{2\pi c}$ must both be zero. The coefficient of $e^{2\pi c}$ is

$$e^{2\pi ic} + 1 - ie^{2\pi ic} + i = 0$$

while the algebraic part of the numerator is

$$-e^{2\pi ic} - 1 - ie^{2\pi ic} + i = 0.$$

Putting these last two equations together we obtain that

$$i = ie^{2\pi ic}$$

which is a contradiction since $c \notin \mathbb{Z}$. By the theorem of Nesterenko, $\pi$ and $e^{\pi}$ are algebraically independent, so our series is transcendental. ∎

We give the following corollary without proof. The proof goes through similar to the previous corollary 5.35.

**Corollary 5.37** *For any rational $c \in \mathbb{Q} \setminus \mathbb{Z}$ and a positive integer $k$, the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^4 - c^4)^{2k}}$$

*is transcendental.*

In a specific case, Bundschuh conjectured the transcendence of the series

$$\sum_{|n|\geq 2} \frac{1}{n^4 - 1}.$$

We obtain this result unconditionally.

**Corollary 5.38** *The series*

$$\sum_{|n|\geq 2} \frac{1}{n^4 - 1}$$

*(and therefore $\sum_{n=2}^{\infty} 1/(n^4 - 1)$ as well) is transcendental.*

**Proof.** From equation (5.5) of Theorem 5.36, write

$$\sum_{|n|\geq 2} \frac{1}{n^4 - c^4} = \frac{1}{c^4} - \frac{2}{1-c^4} + \frac{\pi}{2c^3}\left(\frac{(e^{2\pi ic}+1)(e^{2\pi c}-1) - i(e^{2\pi c}+1)(e^{2\pi ic}-1)}{i(e^{2\pi ic}-1)(e^{2\pi c}-1)}\right).$$

We separate this sum as

$$\frac{1}{c^4} - \frac{\pi(e^{2\pi c}+1)}{2(e^{2\pi c}-1)} - \frac{2}{(1-c^4)} - \frac{\pi i(e^{2\pi ic}+1)}{2c^3(e^{2\pi ic}-1)}.$$

We wish to take the limit as $c$ goes to one, and we see that the last two terms in the sum cause us problems. Grouping these two terms together we have

$$\frac{1}{c^4} - \frac{\pi(e^{2\pi c}+1)}{2(e^{2\pi c}-1)} - \left[\frac{4c^3(e^{2\pi ic}-1) + \pi i(e^{2\pi ic}+1)(1-c^4)}{2c^3(e^{2\pi ic}-1)(1-c^4)}\right].$$

Applying L'Hôpital's rule twice to the troublesome terms, we send $c$ to 1 and obtain,

$$\sum_{|n|\geq 2} \frac{1}{n^4 - 1} = \frac{7e^{2\pi} - 7 - 2\pi - 2\pi e^{2\pi}}{4(e^{2\pi}-1)}.$$

The sum is now clearly positive (nonzero) and Nesterenko's theorem implies that the series is transcendental. ∎

In Theorem 5.36, we would like to place a "+" in front of $c^4$. In doing so we cannot conclude anything about the transcendence of the series. However, if we assume that Schanuel's conjecture is true, we get the following.

**Theorem 5.39** *For any rational $c \neq 0$, if Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^4 + c^4}$$

*is transcendental.*

**Proof.** First note that this series is nonzero. Rewrite

$$\frac{1}{n^4+c^4} = \frac{1}{n^4-(\xi c)^4}$$

where $\xi = e^{\pi i/4} = \sqrt{2}/2 + i\sqrt{2}/2$. Similar to the proof of Theorem 5.36, we have our series is equal to

$$\frac{\pi}{2\xi^3 c^3}\left(\frac{(e^{2\pi i\xi c}+1)(e^{2\pi\xi c}-1) - i(e^{2\pi\xi c}+1)(e^{2\pi i\xi c}-1)}{i(e^{2\pi i\xi c}-1)(e^{2\pi\xi c}-1)}\right). \qquad (5.6)$$

Notice that we can write $e^{2\pi ic\xi} = e^{\pi ic\sqrt{2}}e^{-\pi c\sqrt{2}}$.

The numbers 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$, so $\pi i$ and $\pi i\sqrt{2}$ are as well. Also, $\pi\sqrt{2} \in \mathbb{R}$, so the three numbers $\pi i, \pi i\sqrt{2}, \pi\sqrt{2}$ are linearly independent over $\mathbb{Q}$. Schanuel's conjecture implies that

$$\operatorname{trdeg}(\mathbb{Q}(\pi i, \pi i\sqrt{2}, \pi\sqrt{2}, e^{\pi i}, e^{\pi i\sqrt{2}}, e^{\pi\sqrt{2}})) \geq 3.$$

Thus $\pi, e^{\pi i\sqrt{2}}$, and $e^{\pi\sqrt{2}}$ must be algebraically independent. Since the factor inside the parenthesis must not vanish, and due to algebraic independence between the factor and $\pi$, we conclude that our series is transcendental. ∎

Note that Schanuel's conjecture is a very heavy assumption for the previous situation. The closed form for the series has a single power of $\pi$ times an element from $\overline{\mathbb{Q}}(e^{\pi i\xi c}, e^{\pi\xi c})$. We do not need algebraic independence, but some sort of multiplicative independence. If $\pi$ is irrational over $\overline{\mathbb{Q}}(e^{\pi i\xi c}, e^{\pi\xi c})$, then the series is transcendental. That brings us to a different problem regarding algebraic independence of $e^{\pi i\xi c}, e^{\pi\xi c}$. This leads us to the following corollary.

**Corollary 5.40** *For any rational $c \neq 0$, if Schneider's conjecture is true, the series*

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^4+c^4}$$

*is transcendental.*

**Proof.** Note that the series is positive and not zero. From equation (5.6), we obtain the closed form for our series,

$$\frac{1}{2i\xi^3c^3}\left(\frac{(e^{2\pi i\xi c}+1)(e^{2\pi\xi c}-1)-i(e^{2\pi\xi c}+1)(e^{2\pi i\xi c}-1)}{(e^{2\pi i\xi c}-1)(e^{2\pi\xi c}-1)}\right).$$

If this were equal to an algebraic number, the rational expression inside the parenthesis would be equal to an algebraic number $\alpha$. Clearing denominators, we obtain,

$$e^{2\pi i\xi c}e^{2\pi\xi c}(1-i-\alpha)+e^{2\pi i\xi c}(-1-i+\alpha)+e^{2\pi\xi c}(1+i+\alpha)+(-1+i-\alpha)=0.$$

Schneider's conjecture implies that $(e^{\pi i})^{\xi}$ and $(e^{\pi i})^{\xi^3}=e^{-\pi\xi}$ are algebraically independent, which implies that $e^{2\pi i\xi c}$ and $e^{2\pi\xi c}$ are algebraically independent. Thus, each coefficient above must be zero. This easily gives a contradiction and we have that the series is transcendental if Schneider's conjecture is true. ∎

We build one final series from previous series. As expected, a series built from other series which are not all unconditionally transcendental is not unconditionally transcendental.

**Corollary 5.41** *For any $c \in \mathbb{Q}\setminus\mathbb{Z}$, if Schanuel's conjecture is true, the* *series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^8 - c^8}$$

*is either zero or transcendental.*

**Proof.** We simply note that $\frac{1}{n^8-c^8} = \frac{1}{2c^4}\left(\frac{1}{n^4-c^4} - \frac{1}{n^4+c^4}\right)$. Separating the series by partial fractions, and examining the closed forms from the previous two theorems, our series will be equal to $\pi$ times an algebraic expression in $e^{\pi}, e^{\pi\sqrt{2}}$ and $e^{\pi i\sqrt{2}}$. Assuming Schanuel, the linear independence of $\pi i, \pi i\sqrt{2}, \pi, \pi\sqrt{2}$ implies $\pi, e^{\pi}, e^{\pi\sqrt{2}}$ and $e^{\pi i\sqrt{2}}$ are algebraically independent. The rational expression which will appear in the closed form for the series could be zero, algebraic, or transcendental, but since there is an algebraically independent $\pi$ in front we have that the series is either zero or transcendental. ∎

We could continue building new series in this fashion, but we are unable to say anything more unconditionally. Schanuel's conjecture implies that all of the series built up from previous series will be zero or transcendental. Near the end of this section we prove this statement in a general setting, so for now we move toward cases with odd exponents.

Before examining the cubic case, we point out that Theorem 5.32 gives us a nice form in which we can take limits and obtain $\zeta(2)$.

**Theorem 5.42** $\zeta(2) = \frac{\pi^2}{6}$.

**Proof.** We have that

$$\zeta(2) = \frac{1}{2} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}.$$

From equation (5.4) in the proof of Theorem 5.32, we have

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2+c^2} = \frac{\pi(e^{2\pi c}+1)}{c(e^{2\pi c}-1)} - \frac{1}{c^2}$$

which equals

$$\frac{\pi c e^{2\pi c} + \pi c - e^{2\pi c} + 1}{c^2(e^{2\pi c}-1)}.$$

Sending $c$ to zero, applying L'Hôpital's rule three times, we get $\zeta(2) = \frac{\pi^2}{6}$. ∎

We could perform the same type of calculation from the closed form of Theorem 5.36 to obtain $\zeta(4)$. In fact, we could use this technique to recover $\zeta(2k)$ from other closed forms as well. For instance, for any $k$ by taking limits in series of the form

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+c^2)^k}.$$

We do not examine this here and instead maintain focus on examining new series. We examine the following cubic case.

**Theorem 5.43** *For any* $c \in \mathbb{Q}\setminus\mathbb{Z}$ *the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3+c^3}$$

*is transcendental.*

**Proof.** Since $\sum_{n\in\mathbb{Z}} 1/(n^3-c^3) = -\sum_{n\in\mathbb{Z}} 1/(n^3+c^3)$ we may assume that $c > 0$. We factor the denominator $n^3+c^3 = (n+c)(n+c\rho)(n+c\rho^2)$ where $\rho = (-1-i\sqrt{3})/2$ is a primitive cube root of unity. By partial fractions we write our series as

$$\frac{1}{3c^2}\sum_{n\in\mathbb{Z}}\frac{1}{n+c}+\frac{\rho}{3c^2}\sum_{n\in\mathbb{Z}}\frac{1}{n+c\rho}+\frac{\rho^2}{3c^2}\sum_{n\in\mathbb{Z}}\frac{1}{n+c\rho^2}.$$

Using equation (5.1) and writing the exponential version of the cotangent function this is equal to

$$\frac{\pi i}{3c^2}\left[\frac{(e^{2\pi ic}+1)}{(e^{2\pi ic}-1)}+\rho\frac{(e^{2\pi ic\rho}+1)}{(e^{2\pi ic\rho}-1)}+\rho^2\frac{(e^{2\pi ic\rho^2}+1)}{(e^{2\pi ic\rho^2}-1)}\right]. \tag{5.7}$$

Since $\rho=-1/2-i\sqrt{3}/2$ we separate our exponential terms. After a little work we simplify our sum to

$$\frac{2\pi i\left[(e^{-2\pi ic}+e^{2\pi ic})+\rho(e^{\pi ic}e^{\pi c\sqrt{3}}+e^{-\pi ic}e^{-\pi c\sqrt{3}})+\rho^2(e^{\pi ic}e^{-\pi c\sqrt{3}}+e^{-\pi ic}e^{\pi c\sqrt{3}})\right]}{3c^2(e^{2\pi ic}-1)(e^{2\pi ic\rho}-1)(e^{2\pi ic\rho^2}-1)}.$$

By Nesterenko's theorem, $\pi$ and $e^{\pi c\sqrt{3}}$ are algebraically independent, so this sum is either zero or transcendental. If the numerator is zero, the algebraic coefficients of $e^{\pi c\sqrt{3}}$ and $e^{-\pi c\sqrt{3}}$ must both be zero. This implies that $\rho e^{\pi ic}+\rho^2e^{-\pi ic}=0$ and also $\rho^2e^{\pi ic}+\rho e^{-\pi ic}=0$. The first equation implies that $c=1/6+k_1$ for some integer $k_1$, while the second equation implies that $c=-1/6+k_2$ for some integer $k_2$. This is a contradiction, so the series is not zero and hence is transcendental. ∎

**Remark 5.44.** Note that unlike the case where we have even exponents, we cannot simply relate the series with summation over $\mathbb{Z}$ to summation over

$\mathbb{N}$. What we obtain instead is that at least one of

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + c^3} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^3 - c^3}$$

is transcendental. Related to this, note that the limit technique from above for obtaining $\zeta(2)$ fails here. By taking limits to obtain something that looks like $\zeta(3)$, the series over $\mathbb{Z}$ disappears. Having odd exponents puts us in a less desirable situation.

From Theorem 5.43, we obtain a nice result involving powers of this cubic case.

**Corollary 5.45** *For any $c \in \mathbb{Q} \setminus \mathbb{Z}$ and any positive integer $k$, the series*

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n^3 + c^3)^k}$$

*is zero or transcendental. In the case that $k$ is even, the series is transcendental.*

**Proof.** Our series

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+c)^k (n+\rho c)^k (n+\rho^2 c)^k}$$

can be rewritten using partial fractions in the usual way. Examining each series obtained from that, some algebraic number times

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+\rho^j c)^l}$$

is related to the $(l-1)$st derivative of the cotangent function. Simplifying everything, this is equal to a polynomial in $\pi$ with zero constant term. Each coefficient for a given power of $\pi$ is a rational expression in essentially $e^{\pi c\sqrt{3}}$. By Nesterenko's theorem, the numbers $\pi$ and $e^{\pi c\sqrt{3}}$ are algebraically independent which implies that the series is either zero or transcendental. Clearly if $k$ is even the series does not vanish and in that case we have transcendence.

∎

Similar to the quadratic case, we now remove the exponent on $c$. We cannot show that this series is transcendental, unconditionally, but if we assume Schanuel's conjecture to be true, this implies transcendence.

**Theorem 5.46** *For any $c \in \mathbb{Q}$, not a cube in $\mathbb{Z}$, if Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + c}$$

*is transcendental.*

**Proof.** We examine the three numbers $\pi i, \pi i\sqrt[3]{c}$ and $\pi i\sqrt[3]{c}\sqrt{-3} = -\pi\sqrt[3]{c}\sqrt{3}$. Note, we may assume that $c$ is not a cube, or else we would be in the case of Theorem 5.43. The third number is real while the other two are purely imaginary. The numbers 1 and $\sqrt[3]{c}$ are linearly independent over $\mathbb{Q}$, so $\pi i$ and $\pi i\sqrt[3]{c}$ are linearly independent as well. Thus the three numbers are linearly independent over $\mathbb{Q}$. If Schanuel's conjecture is true, we have that

$$\operatorname{trdeg}(\mathbb{Q}(\pi i, \pi i\sqrt[3]{c}, \pi\sqrt[3]{c}\sqrt{3}, e^{\pi i}, e^{\pi i\sqrt[3]{c}}, e^{\pi\sqrt[3]{c}\sqrt{3}})) \geq 3.$$

It is easy to see that this implies the algebraic independence of

$$\pi, e^{\pi i\sqrt[3]{c}}, \text{ and } e^{\pi\sqrt[3]{c}\sqrt{3}}.$$

Similar to the closed form calculated in the proof of Theorem 5.43, we have that our series is equal to

$$\frac{\pi i}{3\sqrt[3]{c^2}}\left[\frac{(e^{2\pi i\sqrt[3]{c}}+1)}{(e^{2\pi i\sqrt[3]{c}}-1)}+\rho\frac{(e^{2\pi i\sqrt[3]{c}\rho}+1)}{(e^{2\pi i\sqrt[3]{c}\rho}-1)}+\rho^2\frac{(e^{2\pi i\sqrt[3]{c}\rho^2}+1)}{(e^{2\pi i\sqrt[3]{c}\rho^2}-1)}\right].$$

After placing everything over a common denominator, the numerator is $2\pi i$ times

$$\begin{aligned}(e^{-2\pi i\sqrt[3]{c}}+e^{2\pi i\sqrt[3]{c}}) \quad &+ \quad \rho(e^{\pi i\sqrt[3]{c}}e^{\pi\sqrt[3]{c}\sqrt{3}}+e^{-\pi i\sqrt[3]{c}}e^{-\pi\sqrt[3]{c}\sqrt{3}})\\ &+ \quad \rho^2(e^{\pi i\sqrt[3]{c}}e^{-\pi\sqrt[3]{c}\sqrt{3}}+e^{-\pi i\sqrt[3]{c}}e^{\pi\sqrt[3]{c}\sqrt{3}})\end{aligned}$$

By Schanuel, our series is either 0 or transcendental. If the series is zero, this factor must be zero. Multiplying everything by $e^{2\pi i\sqrt[3]{c}}e^{\pi\sqrt[3]{c}\sqrt{3}}$ we have a nonzero polynomial expression of algebraically independent terms, which is equal to zero. This is a contradiction, thus our series is transcendental if Schanuel's conjecture is true. ∎

Similar to above, if we divide out the factor of $\pi$, that is examine the series

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^3+c},$$

we need something less than Schanuel's conjecture in order to conclude transcendence. In this case, the best reduction (so far) from using Schanuel, is to assume Schneider's conjecture.

**Theorem 5.47** *For any $c \in \mathbb{Q}$ which is not a cube, if Schneider's conjecture* *is true then the series*

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^3+c}$$

*is transcendental.*

**Proof.** From the proof of the previous theorem, we have that our series is equal to

$$\frac{i}{3\sqrt[3]{c^2}}\left[\frac{(e^{2\pi i\sqrt[3]{c}}+1)}{(e^{2\pi i\sqrt[3]{c}}-1)}+\rho\frac{(e^{2\pi i\sqrt[3]{c}\rho}+1)}{(e^{2\pi i\sqrt[3]{c}\rho}-1)}+\rho^2\frac{(e^{2\pi i\sqrt[3]{c}\rho^2}+1)}{(e^{2\pi i\sqrt[3]{c}\rho^2}-1)}\right].$$

Note that the number $\alpha=\sqrt[3]{c}\sqrt{-3}$ is algebraic of degree 6. If Schneider's conjecture is true, the 5 numbers $e^{\pi i\alpha},\ldots,e^{\pi i\alpha^5}$ are all algebraically independent. In particular, since $\alpha^4=9c\sqrt[3]{c}$, the algebraic independence of $e^{\pi i\alpha}$ and $e^{\pi i\alpha^4}$ implies that $e^{\pi i\sqrt[3]{c}}$ and $e^{\pi i\sqrt[3]{c}\sqrt{-3}}$ are algebraically independent. For simplicity, write $x$ and $y$ for $e^{\pi i\sqrt[3]{c}}$ and $e^{\pi i\sqrt[3]{c}\sqrt{-3}}$ respectively.

Returning to the series of interest, similar to the previous theorem where we isolated the numerator of the closed form, we simplify to $\frac{i}{3\sqrt[3]{c^2}}$ times

$$\frac{(x^2+x^{-2})+\rho(xy+x^{-1}y^{-1})+\rho^2(xy^{-1}+x^{-1}y)}{(x^2-1)(x^{-1}y^{-1}-1)(x^{-1}y-1)}.$$

If this were equal to an algebraic number, $\beta$, upon clearing the denominator we would have the equality

$$x^2+x^{-2}+\rho xy+\rho x^{-1}y^{-1}+\rho^2xy^{-1}+\rho^2x^{-1}y=\beta(x^2-1)(x^{-1}y^{-1}-1)(x^{-1}y-1).$$

The coefficient of $x^2$ implies that $\beta=1$, while the coefficient of $x^{-2}$ implies that $\beta=-1$. This contradiction implies that the series is nonzero and transcendental. ∎

In some cubic cases, for the series divided by $\pi$, we can say something unconditionally. We can even avoid the assumption that Schneider's conjecture is true. This is due to the fact that the theorems of Gel'fond and Diaz imply Schneider's conjecture for the case $d = 3$.

**Theorem 5.48** *Let $A(x) \in \overline{\mathbb{Q}}[x] \setminus \{0\}$ have degree less than 3. Also let $f(x) := x^3 + ax^2 + bx + c \in \mathbb{Q}[x]$ be irreducible with roots $\alpha = \alpha_1, \alpha_2, \alpha_3$. If $f(x)$ splits in $\mathbb{Q}(\alpha)$ such that*

$$\alpha_2 = r_1 + r_2\alpha + r_3\alpha^2 \quad \text{and} \quad \alpha_3 = s_1 + s_2\alpha + s_3\alpha^2$$

*and all of the ordered pairs $(0,0), (1,0), (r_2, r_3), (s_2, s_3), (r_2 + 1, r_3), (s_2 + 1, s_3), (r_2 + s_2, r_3 + s_3), (r_2 + s_2 + 1, r_3 + s_3)$ are distinct, then the series*

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{A(n)}{n^3 + an^2 + bn + c}$$

*is transcendental.*

**Proof.** We analyze our series by partial fractions. We have,

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{A(n)}{n^3 + bn^2 + cn + d} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{A(n)}{(n-\alpha)(n-\alpha_2)(n-\alpha_3)}$$

which equals

$$\beta_1 \frac{e^{2\pi i\alpha} + 1}{e^{2\pi i\alpha} - 1} + \beta_2 \frac{e^{2\pi i\alpha_2} + 1}{e^{2\pi i\alpha_2} - 1} + \beta_3 \frac{e^{2\pi i\alpha_3} + 1}{e^{2\pi i\alpha_3} - 1}$$

for some algebraic numbers $\beta_1, \beta_2, \beta_3$. We replace $\alpha_2$ and $\alpha_3$ with their linear representations in $1, \alpha$ and $\alpha^2$ as written above, so that

$$e^{2\pi i\alpha_2} = \gamma_1 e^{2\pi i r_2\alpha} e^{2\pi i r_3\alpha^2} \quad \text{and} \quad e^{2\pi i\alpha_3} = \gamma_2 e^{2\pi i s_2\alpha} e^{2\pi i s_3\alpha^2}$$

where $\gamma_1 = e^{2\pi i r_1}$ and $\gamma_2 = e^{2\pi i s_1}$ are algebraic. For simplicity, write $x := e^{2\pi i\alpha}$ and $y := e^{2\pi i\alpha^2}$ so that our series equals

$$\beta_1 \frac{(x+1)}{(x-1)} + \beta_2 \frac{(\gamma_1 x^{r_2} y^{r_3} + 1)}{(\gamma_1 x^{r_2} y^{r_3} - 1)} + \beta_3 \frac{(\gamma_2 x^{s_2} y^{s_3} + 1)}{(\gamma_2 x^{s_2} y^{s_3} - 1)}.$$

If this were an algebraic number, $-\delta$, after clearing denominators we have that

$$\beta_1(x+1)(\gamma_1 x^{r_2} y^{r_3} - 1)(\gamma_2 x^{s_2} y^{s_3} - 1) + \beta_2(x-1)(\gamma_1 x^{r_2} y^{r_3} + 1)(\gamma_2 x^{s_2} y^{s_3} - 1)$$
$$+\beta_3(x-1)(\gamma_1 x^{r_2} y^{r_3} - 1)(\gamma_2 x^{s_2} y^{s_3} + 1) + \delta(x-1)(\gamma_1 x^{r_2} y^{r_3} - 1)(\gamma_2 x^{s_2} y^{s_3} - 1)$$

equals zero.

Recall the theorem of Diaz, Theorem 5.30. Since $\alpha$ has degree 3, Diaz implies

$$\text{trdeg}(\mathbb{Q}(e^{\pi i\alpha}, e^{\pi i\alpha^2})) \geq \left[\frac{3+1}{2}\right] = 2.$$

Thus the two numbers $x$ and $y$ are algebraically independent. Again we point out that algebraic independence can also be obtained from an older result of Gel'fond, Theorem 5.31. The conditions placed on the orders pairs $(0,0), \ldots, (r_2+s_2+1, r_3+s_3)$ ensures that there are 8 distinct monomials in $x$ and $y$ (including the algebraic term), thus each of the coefficients must be zero. Examining the algebraic coefficients of $1, x, x^{r_2}y^{r_3}$, and $x^{s_2}y^{s_3}$ we have the following system of equations written in matrix form,

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system implies that $\beta_1 = 0 = \beta_2 = \beta_3 = \delta$. This is a contradiction, thus the series is transcendental. ∎

For completeness, we give a characterization of exactly when cubic polynomials split as described above.

**Lemma 5.49** *If $f(x) := x^3 + ax^2 + bx + c \in \mathbb{Q}[x]$ is irreducible then $f(x)$ splits in $\mathbb{Q}(\alpha)$ for one of its roots $\alpha$ if and only if the discriminant $18abc + a^2b^2 - 4b^3 - 4a^3c - 27c^2$ is a square in $\mathbb{Q}$.*

**Proof.** We recall some facts from abstract algebra. Let $\alpha = \alpha_1, \alpha_2$, and $\alpha_3$ be the distinct roots of $f(x)$. By definition, the discriminant

$$D := \prod_{i<j} (\alpha_i - \alpha_j)^2$$

which equals $18abc + a^2b^2 - 4b^3 - 4a^3c - 27c^2$. Let $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Since $f(x)$ splits in $K$, we examine the Galois group $G$ for $K$ over $\mathbb{Q}$. Note that the order of $G$ is either 3 or 6, corresponding to $C_3$ or $S_3$ respectively, since $[\mathbb{Q}(\alpha_1) : \mathbb{Q}] = 3$.

Suppose that $\sqrt{D}$ is a square. If $G = S_3$, for any $g \in G$ with order 2, $g$ simply interchanges two of the roots of $f$ leaving the other fixed. Suppose, without loss of generality, that $g$ leaves $\alpha_1$ fixed. We have

$$g(\sqrt{D}) = (\alpha_1 - \alpha_3)(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2) = -\sqrt{D}$$

which is a contradiction since $\sqrt{D} \in \mathbb{Q}$. Thus $G = C_3$ and by the fundamental theorem of Galois theory, we see that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ is a degree 3 extension, and therefore equal to $\mathbb{Q}(\alpha)$.

On the other hand, if $f(x)$ splits in $\mathbb{Q}(\alpha)$ then $|G| = 3$ which means $G = C_3$. The discriminant $\sqrt{D}$ is left fixed by $G$ so it must be in $\mathbb{Q}$ and $D$ is a square. ∎

The previous theorem does not characterize all cases where the denominator is cubic, but does cover many possibilities. The key point is that the denominator had splitting field $\mathbb{Q}(\alpha)$ for one of its roots $\alpha$ so that the other roots are linear combinations of powers of $\alpha$. With the current state of the theory, the theorems of Gel'fond and Diaz only takes us to the cubic case. For instance, if we take an irreducible quartic polynomial $B(x)$ with splitting field $\mathbb{Q}(\alpha)$ for one of its roots $\alpha$, then we could write the other three roots as $\mathbb{Q}$-linear combinations of $1, \alpha, \alpha^2$ and $\alpha^3$. We would then be able to write the series

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{B(n)}$$

as a rational expression in the numbers $e^{\pi i \alpha}, e^{\pi i \alpha^2}$, and $e^{\pi i \alpha^3}$. Theorem 5.31 of Gel'fond or Theorem 5.30 of Diaz implies that two of these numbers are algebraically independent, which does not allow us to conclude anything in our context.

The next attempt would be to assume Schneider's conjecture. The closed form would still include sums of rational expressions in algebraically independent terms which could be algebraic. Like in the cubic case, we might be able to find some special and very specific conditions to ensure transcendence, conditional upon Schneider's conjecture. We leave this open for now. We note that there is nothing special about a quartic polynomial in this situation. For any polynomial $B(x)$ with roots $\alpha_1, \ldots, \alpha_d$, at least one being irrational, we really want to analyze how $B(x)$ splits in $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ for a primitive element $\alpha$. We can express our series

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{B(n)}$$

as a sum of rational expressions in $e^{\pi i \alpha}, \ldots, e^{\pi i \alpha^{d-1}}$. If we assume Schneider's conjecture, these numbers are all algebraically independent. We cannot conclude that the series is transcendental in general, but as we will see later, we characterize exactly what possible algebraic values this sum can be.

Leaving Schneider's conjecture for now, we continue to build up concrete examples. We get a degree six series for free from Theorem 5.43. Simply note that

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^6 - c^6} = \frac{1}{2c^3} \sum_{n \in \mathbb{Z}} \left( \frac{1}{n^3 - c^3} - \frac{1}{n^3 + c^3} \right)$$

which simplifies to

$$\frac{-1}{c^3} \sum_{n \in \mathbb{Z}} \frac{1}{n^3 + c^3}$$

so there is nothing left to show. From the cubic cases, we immediately have the following theorem.

**Theorem 5.50** *For any rational $c \in \mathbb{Q}\backslash\mathbb{Z}$ and a positive integer $k$, the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^6 - c^6)^{2k}}$$

*is transcendental.*

**Proof.** We simply expand via partial fractions and then make use of our results from the Theorem 5.43. The series is nonzero, so Nesterenko's theorem implies transcendence. ∎

We point out the following improvement upon a conjecture of Bundschuh. Bundschuh conjectured that the following series was transcendental, and by the previous remarks, we show it unconditionally.

**Theorem 5.51** *The series*

$$\sum_{|n|\geq 2} \frac{1}{n^3 - 1}$$

*(and therefore $\sum_{|n|\geq 2} \frac{1}{n^6-1}$ and $\sum_{n=2}^{\infty} 1/(n^6 - 1)$ as well) is transcendental.*

**Proof.** We are interested in the cubic case $\sum_{|n|\geq 2} \frac{1}{n^3+c^3}$ which, from equation (5.7), equals

$$\frac{-1}{c^3} - \frac{1}{1+c^3} - \frac{1}{-1+c^3} + \frac{\pi i}{3c^2}\left[\frac{e^{2\pi i c}+1}{e^{2\pi i c}-1} + \rho\frac{e^{2\pi i c\rho}+1}{e^{2\pi i c\rho}-1} + \rho^2\frac{e^{2\pi i c\rho^2}+1}{e^{2\pi i c\rho^2}-1}\right].$$

Taking the limit as $c$ goes to $-1$ (via L'Hôpital's rule) we have

$$\sum_{|n|\geq 2} \frac{1}{n^6 - 1} = \frac{11 + 11e^{\pi\sqrt{3}} + 2\pi\sqrt{3} - 2\pi\sqrt{3}e^{\pi\sqrt{3}}}{6 + 6e^{\pi\sqrt{3}}}.$$

If this were equal to an algebraic number, we clear denominators and notice that $\pi$ appears by itself with coefficient $2\sqrt{3} \neq 0$ which contradicts Nesterenko's theorem that $\pi$ and $e^{\pi\sqrt{D}}$ are algebraically independent. Thus the series is transcendental. ∎

We are also interested in another degree six case. Unfortunately, similar to some cases shown above, we cannot conclude anything unconditionally. We have the following implication of Schanuel's conjecture.

**Theorem 5.52** *For any rational $c \neq 0$, if Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^6 + c^6}$$

*is transcendental.*

**Proof.** Following a previous trick, rewrite our series as

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^6 + c^6} = \sum_{n\in\mathbb{Z}} \frac{1}{n^6 - (\lambda c)^6}$$

where $\lambda = \sqrt{3}/2 + i/2$ is a primitive 12th root of unity. This series is equal to

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + (\lambda c)^3}.$$

Recall $\rho = -1/2 - i\sqrt{3}/2$. Equation (5.7) in the proof of Theorem 5.43 gives us the following closed form,

$$\frac{\pi i}{3\lambda^2 c^2}\left[\frac{(e^{2\pi ic\lambda}+1)}{(e^{2\pi ic\lambda}-1)} + \rho\frac{(e^{2\pi ic\lambda\rho}+1)}{(e^{2\pi ic\lambda\rho}-1)} + \rho^2\frac{(e^{2\pi ic\lambda\rho^2}+1)}{(e^{2\pi ic\lambda\rho^2}-1)}\right].$$

Note that $\rho = -\lambda^2$ and therefore $\rho^2 = \lambda^4$. We simplify our closed form to

$$\frac{\pi i}{3\lambda^2 c^2}\left[\frac{(e^{2\pi i c\lambda}+1)}{(e^{2\pi i c\lambda}-1)} + \lambda^2\frac{(e^{2\pi i c\lambda^3}+1)}{(e^{2\pi i c\lambda^3}-1)} + \lambda^4\frac{(e^{2\pi i c\lambda^5}+1)}{(e^{2\pi i c\lambda^5}-1)}\right].$$

The key numbers here are $e^{2\pi i c\lambda} = e^{\pi i c\sqrt{3}}e^{-\pi c}, e^{2\pi i c\lambda^3} = e^{-2\pi c}$, and $e^{2\pi i c\lambda^5} = e^{-\pi i c\sqrt{3}}e^{-\pi c}$. Nothing is known about the algebraic independence of $e^{\pi i\sqrt{3}}$ and $e^{\pi}$, however, the number $\lambda$ satisfies $\lambda^5 = \lambda^3 - \lambda$ and has degree 4, so as conjectured by Schneider [24], each term $e^{2\pi i\lambda^k}$ should be algebraically independent for $k \in \{1, 2, 3\}$. With the factor of $\pi$ in front, we are not able to conclude anything with Schneider's conjecture alone. Thus we need a stronger conjecture. If we assume Schanuel's conjecture to be true, noting that each term of the series is positive, then the series is transcendental. ∎

We now state a general case for series of this form. Examining the following series leads to a main idea for a more general result.

**Theorem 5.53** *For $p \geq 5$ any prime, and $c \in \mathbb{Q}\setminus\mathbb{Z}$, if Schanuel's conjecture is true, then each series*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^p + c^p}$$

*is either zero or transcendental.*

**Proof.** Let $\zeta$ be a primitive $p$th root of unity. We have that $1, \zeta, \ldots, \zeta^{p-2}$ are linearly independent over $\mathbb{Q}$. Thus $\pi i, \ldots, \pi i\zeta^{p-2}$ are linearly independent as well. Schanuel's conjecture implies that $\pi, e^{\pi i\zeta}, \ldots, e^{\pi i\zeta^{p-2}}$ are algebraically independent. Factoring the denominator of each term in our series we have

$$n^p + c^p = (n + c)\cdots(n + \zeta^{p-1}c)$$

and so we can rewrite our series as

$$\pi i \left( \alpha_0 \frac{e^{2\pi i c}+1}{e^{2\pi i c}-1} + \cdots \alpha_{p-1} \frac{e^{2\pi i c \zeta^{p-1}}+1}{e^{2\pi i c \zeta^{p-1}}-1} \right)$$

where each $\alpha_i$ is algebraic. Since there is a $\mathbb{Q}$-linear dependence

$$\zeta^{p-1} = -1 - \zeta - \cdots - \zeta^{p-2},$$

we can reduce the sum inside the parenthesis leaving only a rational function in algebraically independent terms which can be zero, algebraic or transcendental. Since there is a factor of $\pi$ out front, Schanuel implies that the series is either zero or transcendental. ∎

For most of this thesis we place restrictions on the individual terms of our series, restricting them to be more natural things to write down, like rational functions with rational roots. Schanuel's conjecture implies a general theorem about general series of rational functions. We generalize the main idea from the previous theorem to obtain an improvement on Theorem 4.4 of [9].

**Theorem 5.54** *Let* $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ *with* $\deg(A) < \deg(B)$ *and* $B(x)$ *having algebraic roots that are not integers. If Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

*is either zero or transcendental.*

**Proof.** Without loss of generality, write

$$B(x) = (x+\alpha_1)^{m_1} \cdots (x+\alpha_k)^{m_k}.$$

Let $T = \{\theta_1, \ldots, \theta_t\}$ be a maximal set of $\mathbb{Q}$-linearly independent numbers from the $\mathbb{Q}$ span of $\{\alpha_1, \ldots, \alpha_k\}$, such that 1 is not in the $\mathbb{Q}$ span of $T$. Note that in the case that $T$ is empty, $B(x)$ has only rational roots. This case was dealt with in sections 1 and 3 of this chapter and we already have the unconditional result that the sum is zero or transcendental. If $T$ is not empty, the numbers $1, \theta_1, \ldots, \theta_t$ are linearly independent over $\mathbb{Q}$ so $\pi i, \pi i\theta_1, \ldots, \pi i\theta_t$ are as well. Schanuel's conjecture implies that

$$\operatorname{trdeg}(\mathbb{Q}(\pi i, \pi i\theta_1, \ldots, \pi i\theta_t, e^{\pi i}, e^{\pi i\theta_1}, \ldots, e^{\pi i\theta_t})) \geq t + 1$$

which implies that $\pi, e^{\pi i\theta_1}, \ldots, e^{\pi i\theta_t}$ are algebraically independent over $\mathbb{Q}$.

By partial fractions and making use of equation (5.2) of Theorem 5.2, our series is equal to

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{i,j} \sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha_i)^j} = \pi \sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{i,j}(\cot(\pi\alpha_i))^{(j-1)}.$$

Each derivative of the cotangent function is a polynomial in cot and csc, therefore, each $(\cot(\pi\alpha_i))^{(j-1)}$ is a rational expression with algebraic coefficients evaluated at $e^{2\pi i\alpha_i}$. We rewrite each $\alpha_i$ is a linear combination

$$\alpha_i = \sum_{l=1}^{t} r_{i,l}\theta_l + R_i,$$

where each $r_{i,l}, R_i \in \mathbb{Q}$. Making this substitution and rearranging our sum as a polynomial in $\pi$, we have that each coefficient of a given power of $\pi$ is a rational expression in the numbers $e^{\pi i\theta_1}, \ldots, e^{\pi i\theta_t}$, each having rational exponents throughout. These coefficients can be zero, algebraic, or transcendental, but since $\pi, e^{\pi i\theta_1}, \ldots, e^{\pi i\theta_t}$ are algebraically independent, our series is either zero of transcendental. ∎

As mentioned a few times already, the full power of Schanuel's conjecture is not needed to imply transcendence results. Even in the previous theorem we need something much less than algebraic independence. Suppose $t$ is the largest multiplicity for any root of $B(x)$. If we examine the field $K$ over $\overline{\mathbb{Q}}$ generated by the $e^{\pi i\alpha_j}$'s, then in order to conclude transcendence, we need only have that either $\pi$ is transcendental over $K$ or $\pi$ has algebraic degree over $K$ greater than $t$. The highest power of $\pi$ that we see in any closed form from the above theorem will have power $t$, so we can conclude that the series is either zero or transcendental with this assumption on the relationship between $\pi$ and $K$.

We next give a special case of the previous theorem. If the roots of $B$ are all simple and all linearly independent, Schanuel implies the series is transcendental.

**Corollary 5.55** *Let $A(x), B(x)$ be as above with $B(x)$ having simple non-integral roots which are linearly independent over $\mathbb{Q}$. If Schanuel's conjecture is true, then the series*

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

*is transcendental.*

**Proof.** From the previous theorem, we know the series is either zero or transcendental, so we need only show it does not vanish. By partial fractions and then using equation (5.1) in the usual way, we write our series

$$\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)} = \pi \sum_{j=1}^{k} c_j \cot(\pi\alpha_j).$$

Using the exponential form of the cotangent functions this is equal to

$$\pi i \sum_{j=1}^{k} c_j \frac{e^{2\pi i\alpha_j}+1}{e^{2\pi i\alpha_j}-1}.$$

Finding a common denominator, this equals

$$\frac{\pi i}{\prod_{j=1}^{k}(e^{2\pi i\alpha_j}-1)} \sum_{j=1}^{k} c_j(e^{2\pi i\alpha_j}+1)\prod_{t\neq j}(e^{2\pi i\alpha_t}-1).$$

In order to show that our series does not vanish, we need only show that the polynomial

$$\sum_{j=1}^{k} c_j(X_j+1)\prod_{t\neq j}(X_t-1)$$

is not identically zero. If this polynomial is identically zero, by fixing $X_j = 1$ and $X_t = 2$ for every $t \neq j$, we see that $c_j = 0$, for each $j$. This is a contradiction since we discuss only cases where $A \neq 0$. Thus the series is not zero and Schanuel implies it is transcendental. ∎

We can eliminate the use of Schanuel's conjecture in at least one case, by dividing the series by $\pi$. For instance, if $B(x)$ has only simple roots $\alpha_1, \ldots, \alpha_k$, the series

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)} = \sum_{j=1}^{k} \beta_j\theta_j$$

where each $\beta_j \in \overline{\mathbb{Q}}$ and

$$\theta_j := \frac{e^{2\pi i\alpha_j}+1}{e^{2\pi i\alpha_j}-1}.$$

Note that for $\alpha_j \notin \mathbb{Q}$, $\theta_j$ is transcendental by Gel'fond's theorem. Thus as long as all the roots of $B(x)$ are not rational, we have a chance at showing that our series is transcendental. If $B(x)$ has only rational roots, then the series will be algebraic. We give the full analysis of this. This is illustrated in the following theorem.

**Theorem 5.56** *Let $A(x), B(x) \in \overline{\mathbb{Q}}[x]$ with deg$(A) <$ deg$(B)$ and coefficients $a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}$. Suppose that $B(X)$ has only simple roots $\alpha_1, \ldots, \alpha_k \notin \mathbb{Q}$ and $r_1, \ldots, r_l \in \mathbb{Q} \setminus \mathbb{Z}$. If Schneider's conjecture is true, the series*

$$S = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{A(n)}{B(n)}$$

*is either transcendental or belongs to the field*

$$K := \mathbb{Q}(i, \alpha_1, \ldots, \alpha_k, a_0, \ldots, a_{d_1}, b_0, \ldots, b_{d_2}, e^{2\pi i r_1}, \ldots, e^{2\pi i r_l}).$$

**Proof.** By direct calculation via partial fractions, in the usual way, our series is equal to

$$\sum_{j=1}^{k} c_j \frac{e^{2\pi i\alpha_j}+1}{e^{2\pi i\alpha_j}-1} + \sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m}+1}{e^{2\pi i r_m}-1}.$$

Each $c_j$ and $d_m$ is in $K$. If all of the roots are rational, the first sum is empty and result is clear.

Assume that $B(x)$ has at least one irrational root and suppose the sum,

$S \in \overline{\mathbb{Q}}$. We have

$$\begin{aligned} S - \sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1} &= \sum_{j=1}^{k} c_j \frac{e^{2\pi i \alpha_j} + 1}{e^{2\pi i \alpha_j} - 1} \\ &= \sum_{j=1}^{k} c_j + 2 \sum_{j=1}^{k} c_j \frac{1}{e^{2\pi i \alpha_j} - 1} \end{aligned}$$

so that

$$\sum_{j=1}^{k} c_j \frac{1}{e^{2\pi i \alpha_j} - 1} = \frac{1}{2} \left( S - \sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1} - \sum_{j=1}^{k} c_j \right) = \theta \in \overline{\mathbb{Q}}.$$

By assumption, $[\mathbb{Q}(\alpha_1, \ldots, \alpha_k) : \mathbb{Q}] := d > 1$, and so the primitive element theorem implies that there is a $\beta \in \overline{\mathbb{Q}}$ of degree $d$ such that $\mathbb{Q}(\alpha_1, \ldots, \alpha_k) = \mathbb{Q}(\beta)$. Thus, we have the equations

$$\alpha_j = \sum_{a=0}^{d-1} r_{a,j} \beta^a$$

where each $r_{a,j} \in \mathbb{Q}$. Take any integer $M \in \mathbb{Z}$ such that

$$\alpha_j = \frac{1}{M} \sum_{a=0}^{d-1} n_{a,j} \beta^a$$

where each $n_{a,j} \in \mathbb{Z}$. Let $\alpha = e^{\pi i/M}$. If Schneider's conjecture is true then the numbers

$$\alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent, which implies that

$$\alpha^{2\beta}, \ldots, \alpha^{2\beta^{d-1}}$$

are algebraically independent. Define $x_a := \alpha^{2\beta^a} = e^{2\pi i\beta^a/M}$ for $a = 1, \ldots, d-1$ so that

$$e^{2\pi i\alpha_j} = e^{\frac{2\pi i}{M}\sum_{a=0}^{d-1} n_{a,j}\beta^a} = \gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}}$$

where $\gamma_j$ is a root of unity.

Making this substitution we have that $\theta$ is equal to

$$\sum_{j=1}^{k} c_j \frac{1}{\gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}} - 1}.$$

We rewrite this expression to avoid any negative exponents, and obtain

$$\theta = \sum_{j=1}^{k} \frac{c_j \prod_{n_{a,j}<0} x_a^{-n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}.$$

Define $N_1 := \{j : n_{1,j} < 0\}$ and $P_1 := \{j : n_{1,j} > 0\}$. For any $j \in N_1$, the summand becomes

$$c_j \frac{(\prod_{n_{a,j}<0} x_a^{-n_{a,j}}) - (\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) + (\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}})}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}$$

which equals

$$-c_j + c_j \frac{\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}.$$

Separating our sum into parts with $j$ in $N_1$ or $P_1$ and then a sum over the $j$'s with $n_{1,j} = 0$ (which is essentially the part where $x_1$ is not present), we have

$$\theta + \sum_{j \in N_1} c_j \quad - \quad \sum_{j \notin N_1, P_1} c_j \frac{\prod_{n_{a,j}<0} x_a^{-n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}$$

equals

$$\sum_{j \in N_1} c_j \frac{\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})} + \sum_{j \in P_1} c_j \frac{\prod_{n_{a,j}<0} x_a^{-n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}.$$

Notice that the right hand side has $x_1$'s present while the left hand side does not. Also, the $x_1$'s present on the right are all on the denominator, and never in the numerator of any term. Thus, we can write the right hand side as $f(x_1)/g(x_1)$, a ratio of polynomials in $x_1$ with coefficients in the field generated by the remaining $x_a$'s, with $\deg(f) < \deg(g)$. $f(x_1)/g(x_1) \neq 0$, multiply both sides of the equation by $g(x_1)$ and we have a nonzero polynomial expression in $x_1$ with coefficients in $\overline{\mathbb{Q}}(x_2, \ldots, x_{d-1})$. Since $\deg(f) < \deg(g)$, this contradicts the algebraic independence of the $x_a$'s. So $f(x_1)/g(x_1) = 0$ which implies that

$$\theta + \sum_{j \in N_1} c_j = \sum_{j \notin N_1, P_1} c_j \frac{\prod_{n_{a,j}<0} x_a^{-n_{a,j}}}{(\gamma_j \prod_{n_{a,j}\geq 0} x_a^{n_{a,j}}) - (\prod_{n_{a,j}<0} x_a^{-n_{a,j}})}$$

both sides having nothing to do with $x_1$. Define

$$N_i := \{j : n_{i,j} < 0 \text{ and } n_{t,j} = 0 \text{ for } t < i\}$$
$$P_i := \{j : n_{i,j} > 0 \text{ and } n_{t,j} = 0 \text{ for } t < i\}.$$

Let $N$ be the union of the disjoint $N_a$'s. Following the same argument which was used to eliminate $x_1$, we eliminate all of the $x_a$'s and see that

$$\theta = -\sum_{j \in N} c_j$$

which implies

$$S = -2\sum_{j \in N} c_j + \sum_{j=1}^{k} c_j + \sum_{m=1}^{l} d_m \frac{e^{2\pi i r_m} + 1}{e^{2\pi i r_m} - 1}$$

which is in $K$. Hence $S$ is either transcendental or in $K$, and we have given the explicit formula for the algebraic case. ∎

**Remark 5.57.** We point out that, in the previous theorem, there are series which do vanish nontrivially, even when all of the roots of the denominator are irrational. As an example, examine the polynomial $x^2 - x - 1$ which has roots, $\phi$ and $1 - \phi$ where $\phi$ is the so-called golden ratio. It turns out that $\sum_{n \in \mathbb{Z}} (2n-1)/(n^2-n-1) = 0$. The key is that the roots of the denominator add to an integer. This gives a sort of telescoping series in which all terms of the sum cancel when we write the sum in a certain clever way.

We conclude by applying the previous theorem to a natural setting.

**Theorem 5.58** *Let $c \in \mathbb{Q} \backslash \mathbb{Z}$. For any prime $p \geq 3$, if Schneider's conjecture* *is true, then the series*

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{n^p - c^p}$$

*is transcendental.*

**Proof.** The series is equal to

$$\sum_{j=0}^{p-1}\alpha_j\frac{e^{2\pi i c\zeta^j}+1}{e^{2\pi i c\zeta^j}-1}$$

where $\zeta = e^{2\pi i/p}$ and each $\alpha_j$ is a nonzero algebraic number. By adding and subtracting one to each numerator we see that our series is algebraic if and only if

$$\sum_{j=1}^{p-1}\frac{\alpha_j}{e^{2\pi i c\zeta^j}-1} = \theta$$

is algebraic. If Schneider's conjecture is true, the numbers $x_1 := e^{2\pi i c\zeta}, \ldots, x_{p-2} := e^{2\pi i c\zeta^{p-2}}$ are algebraically independent. Since $\zeta^{p-1} = -1 - \cdots - \zeta^{p-2}$, we have

$$\theta = \sum_{j=1}^{p-2}\frac{\alpha_j}{x_j - 1} - \frac{\alpha_{p-1}}{x_1\cdots x_{p-2}-1}.$$

Following the elimination technique described in the previous theorem, we conclude that

$$\frac{\alpha_1}{x_1-1} - \frac{\alpha_{p-1}}{x_1\cdots x_{p-2}-1} = 0$$

so we cancel the terms. The only remaining term to contain $x_2$ is $\alpha_2/(x_2-1)$. By degree comparison, this must equal zero which implies that $\alpha_2 = 0$. This is a contradiction, so $\theta$ must be transcendental which yields the result. ∎

# Chapter 6

# Future Work

During this thesis there are some places which leave us with open questions. Throughout, we maintain our focus on studying infinite series and their arithmetic nature, and do not mention further areas of study that are of interest.

In chapter 2 we discussed special values of the digamma function. We examined linear relations between the digamma function evaluated at rational points $\psi(a/q)$. One can ask for more, and study whether or not certain values have some algebraic relation. For instance, are the values $\psi(a_1/q_1)$ and $\psi(a_2/q_2)$ algebraically dependent? Algebraically independent? It would also be interesting to examine $\psi(x)$ at irrational values of $x$.

We also discussed the polygamma function. Not much is known about these values. Special values of these functions can be connected to special values of the Hurwitz zeta function among other things. With a better understanding of these functions, we might be able to show that more values of the Hurwitz zeta function, or possibly the Riemann zeta function, are transcendental.

Also in chapter 2, we related our infinite series to a linear combination of logarithms of algebraic numbers. This allowed us to use Baker's famous theorem to conclude transcendence. The polylogarithm function is defined as

$$Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

for all values of $s \in \mathbb{C}$ and $|z| < 1$. It might be possible to find an analogue of Baker's theorem for the polylogarithm function. In particular, results from chapter 3 gives us a starting point.

The results of chapter 4 give us nice closed forms for infinite series with multiple indices. This chapter points toward the need to understand $k$-forms of logarithms of algebraic numbers. In general, studying any kind of algebraic relation between logarithms would be of interest. Schanuel's conjecture certainly implies algebraic independence in some instances, but Schanuel's conjecture is a very strong assumption and a proof of the conjecture is thought to be out of reach at this time.

Chapter 5 offers many avenues to keep pushing the theory toward some limit. The values $A_{x,k}$ that appear when analyzing the series

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+x)^k} = \pi^k A_{x,k}$$

are of interest. These values are simply derivatives of the cotangent function, however with a little more work, one might be able to recover some nice results. For instance, we might be able to give a new proof of Euler's theorem,

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where $B_{2n}$ is the $2n^{th}$ Bernoulli number.

In the final section of chapter 5, we examined many examples of series which exhaust the current state of the theory. Further study is needed to understand exactly what is needed to force various series to be algebraic or transcendental. As we saw, Schanuel's conjecture implies many transcendence results, however, this conjecture is really more powerful than what is needed. A reduction to Schneider's conjecture or possibly some other weaker theory is what we need. Waldschmidt [24] combines Schneider's conjecture and a conjecture of Gel'fond that $\log(\alpha)$ and $\alpha^{\beta}$ algebraically independent to state the following.

**Conjecture 6.1** (Gel'fond-Schneider) *The $d$ numbers*

$$\log(\alpha), \alpha^{\beta}, \ldots, \alpha^{\beta^{d-1}}$$

*are algebraically independent.*

Setting $\alpha = -1$ this conjecture implies that $\pi$ and certain exponentials are algebraically independent. This is what is needed for the many of the results in which we assume Schanuel to be true.

With the remark made after Theorem 5.56 about telescoping series in mind, one might think that the only possible way for these series to be algebraic is if there is some sort of telescoping cancellation. Waldschmidt [25] made the following conjecture.

**Conjecture 6.2** *For $A(x), B(x) \in \mathbb{Q}[x]$ the series*

$$\sum_{n=0}^{\infty}{}' \frac{A(n)}{B(n)}$$

*is either rational or transcendental. When it is rational it will be obvious,* *like in a telescoping series.*

Working toward extending the ideas in this thesis to possibly include this conjecture is of interest in the future. In general, this main theme of studying the values of somewhat natural series will be continued.

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