

**Bidders' Behaviour and Theory of Share Auctions with
Applications to the Colombian Primary Bond Market**

by

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Abstract

Although most governments sell their bonds through a share auction, little is known about behaviour of bidders in these auctions. This thesis analyzes the literature on government securities auctions, focusing primarily on structural empirical estimation. Additionally, it examines bidders behaviour in Colombian government bond auctions during 2007, including the additional sale done after the auction.

The thesis summarizes the different structural methodologies that have been developed to determine what the best auction for a particular case is. It discusses the advantages and disadvantages of each methodology and explores assumptions and robustness when confronted with data. To make these comparisons more straightforward, a unified notation is introduced and several methods are applied to the same auctions, uniform price auctions conducted by the government of Colombia.

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Chapter 1

General Introduction

Government securities¹ have been sold through auctions for many years. The US started using the auction mechanism to sell short term securities (known as Treasury bills) in 1929 and longer term securities (known as Treasury notes and bonds) in 1970.² Despite the long and broad use of government securities auctions (studied in the literature as share auctions) our understanding of bidders' behaviour is quite limited (Hortaçsu [11]).

Chapter 2 contributes to the literature by analyzing the behaviour of bidders in the Colombian government bond auctions. Three main findings are presented. First, in contrast with other treasury auctions (Castellanos [6]), the market clearing price in the Colombian auctions tends to be above the price in the secondary market. I explore this phenomenon and illustrate that a key institutional detail involving a secondary sale contingent on the primary auction may explain this difference with other auctions. Second,

¹ Although government securities include bonds and other types of investments, government bonds and securities are used interchangeably throughout this document as synonyms of government debt in general.

² Source: U.S. Treasury: Bureau of The Public Debt.

using identifiers that allow me to follow individual bidders across auctions I analyze the determinants of the stepwise demands. I find that predetermined variables explain the number of steps (bid-points) and the quantities. However, bid prices exhibit significant unexplained variation. Third, for demands that have 3 or more bid-points, 93% of the variability is captured by a linear regression. This result is similar to what Hortaçsu finds for Turkish auctions. Theoretically there is no reason for bid-points to be nearly co-linear. At the same time, game-theoretic models of share auctions are quite difficult to implement. This pattern in demands may play an important role in developing feasible and robust estimation methods.

Little as we know about bidders' behaviour, we know no more theoretically about the auctions themselves. The literature about this type of auctions is still developing and several important questions remain unanswered. From the theoretical side only three equilibrium strategies for uniform price auctions are known. Chapter 3 compares these few known equilibrium strategies and shows that, even when a closed form for an optimal strategy can be found, estimating the structural parameters is not an easy task. From both the theoretical and empirical sides, one of the questions that remains unanswered is what the most appropriate auction format is from the point of view of the seller. Ausubel and Cramton [4] concluded that the superiority of an auction mechanism can only be determined empirically on a case-by-case basis. Researchers who have tried to address this problem empirically have ended up with different results.

Against this backdrop, empirical researchers confronting new problems in go-

vernment bond auctions need to be familiar with several methodologies. Chapter 4 addresses this need: it summarizes the different methodologies recently developed to determine what the best auction for a particular case is. It discusses the advantages and disadvantages of each methodology and explores their robustness when confronted with the real world. To make these comparisons more straightforward, a unified notation is introduced and several methods are applied to the same auctions, uniform price auctions conducted by the government of Colombia.

Closing up the ideas of the previous three chapters, Chapter 5 concludes.

Chapter 2

Bidders' Behaviour in Government Securities Auctions: A case study for Colombia

2.1 Introduction

This chapter examines the bidders' behaviour in the Colombian government bond auctions during 2007 for the period in which there is no uncertainty in the supply. The approach taken here is different from the one followed by the other descriptive studies that have contributed to the knowledge of bidders' behaviour, including the analyses by Elsinger and Zulehner [8], Keloharju et al. [17], and Nyborg et al. [20] for Austria, Finland, and Sweden, respectively.¹

All these works estimate similar regressions to analyze the effect that exogenous variables have on the behaviour of bidders, concluding that the volatility of the secondary market price of the bond being auctioned has a significant impact.² In my case, instead of concentrating on the exogenous variables that correlate with the bidder's behaviour, I focus on the bidder decision problem within the auction. Additionally, I

¹ See Appendix C for the main characteristics of the auction in these and other countries.

² When volatility increases, bidders reduce the price levels at which they bid, reduce quantity demanded, and increase the dispersion of their bids.

analyze how an additional sale done by the government, three days after the auction, affects the behaviour of bidders.

2.2 Government Securities Auctions

This section describes institutional details of how governments sell their securities through auctions. Additionally, it explains two important concepts commonly used in papers that study government bond auctions: share auction and residual supply.

Before the auction begins, the government announces the amount of securities it intends to sell (the supply). When the auction opens, bidders submit a sealed demand consisting of price and quantity pairs (bid-points). The number of pairs allowed differs by country, e.g, Korea: 5 (Kang and Puller [13]), Czechoslovakia: 10 (Kastl [14]), Turkey (Hortaçsu [12]) and Colombia: unlimited (see Appendix A). In some countries, e.g. Colombia, instead of price-quantity pairs, bidders submit interest rate-quantity pairs. In those cases the bond price can be recovered from the interest rate.

When the auction closes the government sorts the bid-points from highest to lowest price and adds the quantities demanded until the supply is met—at the market clearing price. The market clearing price is thus the price where the demanded quantity equals the supply. All the submitted pairs above or at the market clearing price are winning pairs. Bidders who submitted winning pairs are allocated securities from the pool and pay these to the government.

Once the equilibrium price is found governments differ in what price winners

pay. In a discriminatory auction (sometimes called a pay-as-you-bid auction) bidders pay the price they offered for their winning bids, i.e. each bidder pays a different price according to their bid. In a uniform price auction, all winners pay the market clearing price. According to Brenner et al. [5], from a survey delivered to 48 countries in 2005, 24 used a discriminatory auction while 9 were using a uniform price auction. Nine countries were using both mechanisms, depending on the security being auctioned. The remaining six were using pricing rules which are neither uniform nor discriminatory.

For purposes of illustration, assume a government issues securities worth US\$10,000 under an auction and there are three bidders who submit the following bidding pairs (top panel):

Example setting: Supply: \$10,000. Number of bidders: 3.

Table 2.1: Bids (price,quantity) and Auction Outcome

Bidder 1	Bidder 2	Bidder 3
(102,1000)	(103,5000)	(102,3000)
(101,1000)		(98,3000)
(99,1000)		
Discriminatory Auction (Price, Securities Won)		
(102,10)	(103,50)	(102,30)
(101,10)		
Uniform Auction (Price, Securities Won)		
(101,20)	(101,50)	(101,30)

In government securities auctions, the price submitted usually corresponds to the price the bidder is willing to pay for an imaginary bond with a face value of 100.

The market clearing price for this auction is 101. Bidder 1 has two winning pairs while bidder 2 and 3 have one winning pair each. Under the discriminatory format (middle

panel), bidder 1 will have to pay 102 for each of the 10 bonds (1000/100) in the first pair³, and 101 for each of the additional 10 bonds of the second pair. Bidder 2 and bidder 3 will have to pay 103 and 102 for each of the 50 and 30 bonds they demanded, respectively. Under the uniform price format (lower panel) the three bidders will have to pay 101 for each bond they won: 20, 50 and 30 each bidder respectively. In this example, the government earns \$10,240 under the discriminatory price and \$10,100 with a uniform price.

Ex-post analysis will always show greater revenue for the seller under the discriminatory format. But some countries continue to use the uniform one since bidders behave strategically and alter their bids according to the mechanism used. Therefore, using the discriminatory format will not necessarily make the government richer.

Without loss of generality we can normalize the quantity supplied to 1 so that bids become shares. Table 2.2 shows the step demand function under the share framework for the example previously shown.

Table 2.2: Step Demand (price,share)

Bidder 1	Bidder 2	Bidder 3
(102,0.1)	(103,0.5)	(102,0.3)
(101,0.2)		(98,0.6)
(99,0.3)		

Another important concept in the study of government securities auctions—which is used in Section 4.3—is residual supply. The residual supply of a bidder is calculated

³ Another way of seeing this, is that for the first pair bidder 1 receives a bond with a face value of 1000 and has to pay 102% of that face value.

by subtracting the aggregate bid function of all other bidders⁴ at each price from the total supply. Figure 2.1 shows that the point (the black dot in the figure) where the individual step demand function (in this case Bidder 1's function) intersects the residual supply determines the market clearing price and the total quantity (the share) won by the bidder.⁵

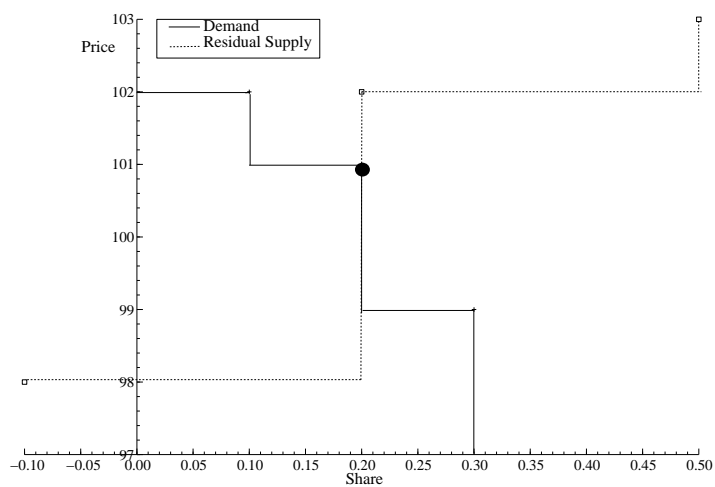


Figure 2.1: Demand and Residual Supply of Bidder 1

⁴ In an auction with N bidders, the aggregate bid function of all other bidders is obtained by “horizontally adding” the $N - 1$ individual demand functions.

⁵ The intersection between the residual supply and the bidder's demand determines the total quantity won by the bidder, except, when there is more than one bidder that submitted a pair at the market clearing price.

2.3 The Data and the Colombian Government's Auction

2.3.1 The Data

The data analyzed in this thesis covers the auctions of long term bonds—bonds with maturity of at least one year—denominated in Colombian pesos (COP)⁶ held in 2007, specifically the ones without uncertainty in the supply. The sample consists of 48 auctions held between March and August 2007.⁷ Auctions were held every two weeks (on Wednesdays)⁸, and each time the government ran 4 simultaneous auctions. The bonds matured at the following dates with corresponding coupons in parenthesis: May 14, 2009 (8.75%), November 24, 2010 (7.5%), October 28, 2015 (8%), and July 24, 2020 (11%). All the auctions were reopenings of an existing security, therefore identical securities were traded in the secondary market both before and after the auction. A total of US\$1.03 billion⁹ was raised with the issue of these bonds during the sample period.

Table 2.3 presents summary statistics for each of the bonds. The mean coverage,

⁶ The Colombian government also auctions bonds denominated in Real Value Units (RVU: account unit which reflects the purchasing power based exclusively on the Consumer Price Index (CPI) variation), in US dollars (they have not be done since November 2001), and in COP with the coupon linked to the Colombian CPI.

⁷ The auction held on March 14 2007, for the bond with maturity on June 12 2008, is not taken into account because this is the only auction for this bond in the sample—actually, it was the last time the bond was auctioned. In Chapter 4, this auction is taken into account in the estimations. After August 2007 there were no more auctions, because by then, the government had satisfied already its financing needs for that year. For the auctions held between January and February the government did not announce the specific supply for each bond; instead it announced a global supply to be divided by its own discretion between the group of bonds.

⁸ According to the regulation [21] auctions can be held on the second and fourth Wednesday of the monetary weeks of the month. In the sample we have 12 different auction dates; for one of them the time space between auctions is 21 days.

⁹ The exchange rate of August 31,2007 (2173.17 pesos/US) is used throughout the paper.

ratio of total bids in an auction to the amount sold, for each bond is close to 4, meaning that on average the maximum demand is 4 times the supply. From an individual perspective, the government issued 33% in the bond with longer maturity. The longer term auctions attracted more bidders and had a higher market clearing price (MCP). Figure 2.2 shows that this is also the bond with the highest transaction volume during the sample period. The scarce liquidity for the bond with maturity in 2015 might explain its lower price among the 4 bonds. To be able to compare prices over time, this chapter presents and does all the analysis with clean prices. Clean price is the price of a bond excluding any interest that has accrued since issue or the most recent coupon payment whereas the dirty price is the price of a bond including the accrued interest. When clean prices change it is for an economic reason, therefore clean prices are more stable over time than dirty prices. Dirty prices change day to day depending on where the current date is in relation to the coupon dates, in addition to any economic reasons.

Table 2.3: Summary Statistics per Bond (I)

Bond	Amount Issued (US millions) ^a	MCP Mean	Coverage			# of Bidders		
			Mean	Min	Max	Mean	Min	Max
5/14/2009	209.5	98.5	4.1	1.9	7.2	8.3	6	11
11/24/2010	251.0	93.2	3.8	2.6	6.1	10.0	8	13
10/28/2015	205.0	88.7	4.1	2.3	6.1	8.7	6	12
7/24/2020	333.5	107.8	4.1	2.7	6.3	11.6	10	13

^a The total amount issued does not have to be equal to the money raised by the government, since the last is obtained by multiplying the amount issued and its corresponding market clearing price in percentage terms.

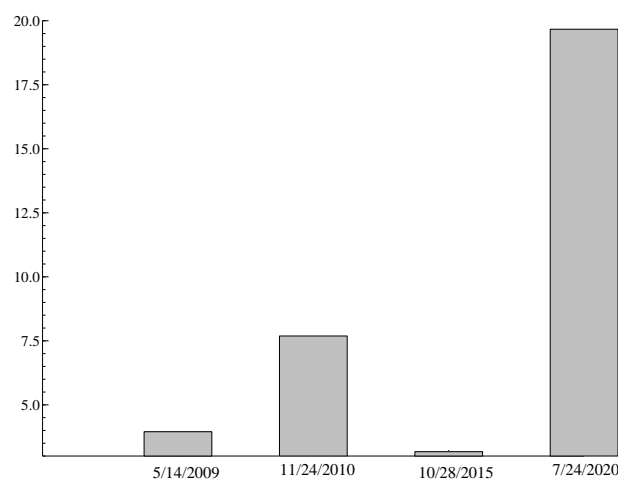


Figure 2.2: Volume Traded per Bond March-August 2007 (US billions).

Source: Colombian Electronic Negotiation System (known in Colombia as SEN)

2.3.2 The Auction of the Colombian Government

This subsection presents how the auction was conducted, its rules during the period of study, and some descriptive statistics. Appendix B presents a table with the most relevant modifications to the auction's rules from 2002 to 2007.

The day before the auction, the Central Bank of Colombia, on behalf of the government,¹⁰ announces through an electronic system the different securities it intends to sell and their corresponding supply. The auctions for the securities are done simultaneously but from the government's perspective each auction is independent.¹¹

The day of the auction bidders submit their bids electronically from 9:30 am to

¹⁰ Everything related to the auction is decided by the government. The Central Bank just executes the auction.

¹¹ From the bidders' perspective, if they have budget constraints the simultaneous auctions might be not independent.

10 am; bids are private, and bidders do not know how many bidders are sending bids. As discussed later, bidders do know the number of participants allowed to submit bids. The government sets the maximum quantity per bid-point equal to US\$100 million¹² and the maximum difference between the highest and lowest yield per bidder in an auction to be equal to 75 basis points (b.p).¹³ In the data these limits are not binding. The maximum quantity a bidder submitted was US\$23 million, and the maximum dispersion in yields was 36 b.p. The minimum quantity per bid-point is COP\$500 million (US\$0.23 million) and is not binding either: none of the 1210 bid-points in the data set hit this lower bound.

As an example of an actual auction, Table 2.4 presents the bids for the six bidders that participated in the auction held on April 25, 2007 for the bond with maturity in 2015. In this auction bidders presented quantities at eleven different prices. Bidder 1 submitted four bid-points, bidder 9 three bid-points, bidder 7 two bid-points, and the other two bidders single pairs.

Table 2.5 summarizes the 463 bids observed in the 48 auctions in the data set. The types of bids are quite varied. While single bid pairs make up 24% of the bids, nearly the same percentage have more than 3 bid-points. The maximum number of bid-points submitted by a bidder in the data set is 9. On average the number of bid-points submitted by a bidder in an auction is 2.4.

¹² In the regulation this quantity is set in US dollars. To obtain the maximum quantity in Colombian Pesos for each auction, the exchange rate of the auction's date has to be used.

¹³ Recall that in Colombia a bid is a pair of a yield and a quantity.

When the auction closes at 10:00 am, the government organizes the bid-points from highest to lowest price, adds the quantities demanded, and finds the price at which the supply is met. All the submitted bid-points above or at the market clearing price are winning bid-points. When there is a tie, two or more bid-points at the market clearing price, the quantities received by bidders are rationed proportionately.¹⁴ As can be seen in Table 2.6, 28% of submitted bid-points were winners (at or above the market clearing price). Of these, 23% were rationed. Bidders who submitted winning bid-points are allocated securities from the pool and pay these to the government at the market clearing price (i.e. this is a uniform price auction). In the auction presented in Table 2.4, the supply was COP\$21,835.4 million,¹⁵ implying a market clearing price of COP\$87.671. Bidder 1 set the market clearing price, and their pair at the market clearing price was rationed to COP\$335.4 million to satisfy supply. In this case there was only one bidder at the market clearing price, hence the pro-rata rule was not used.

The government releases the auction's results the same day the auction is held. The public announcement includes the market clearing price, the total quantity demanded and the total quantity sold. Three working days after the auction, the government might hold the non-competitive round (NCR), which the next section discusses in de-

¹⁴ The aggregate marginal quantity demanded at the market clearing price is computed. The marginal quantity demanded at the market clearing price by a given bidder is divided by the aggregate marginal quantity demanded at the market clearing price to determine the proportion of the rationed quantity that bidder is to receive. The rationed quantity is determined by subtracting the aggregate quantity demanded at all prices strictly above the market clearing price from the final supply.

¹⁵ In reality, the government announces the supply in cost value. For clarity, here it is presented in nominal value. In cost value the supply announced for this auction was COP\$20,000 million. The cost value of each submitted pair is obtained by multiplying the price in percentage terms (dirty price; this paper presents everything in terms of clean prices) and the nominal value. The dirty market clearing price for this auction was 91.594.

tail.

2.3.3 The Non-Competitive Round

The NCR is an additional sale of the security previously auctioned. This sale is done by the government if the coverage of the auction was at least 1.2. Only bidders that obtained bonds in the auction have the right to acquire more bonds in the NCR. The supply for the NCR is determined according to:¹⁶

- If the coverage of the auction was at least 2, the supply for the NCR is 80% of the auction's supply.
- If the coverage of the auction was between 1.2 and 2, the supply for the NCR is 55% of the auction's supply.

Bidders who bought bonds in the auction can buy an additional amount in the NCR in proportion to the quantity won in the auction. Roughly speaking, a bidder who acquired 30% of the bonds auctioned can buy 30% of the supply in the NCR.¹⁷ The price that bidders have to pay for the bonds demanded in the NCR is the average secondary market price of the bond on the auction date, which is calculated and published by the Colombian Securities and Stocks Exchange (CSSE, known in Colombia as BVC).¹⁸

Between 12:30 pm and 1:00 pm of the NCR date, bidders with the right to participate

¹⁶ See Appendix B for its modifications from 2002 to 2007.

¹⁷ The actual rules for the amount each bidder can buy in the NCR are more complicated. See Appendix B.

¹⁸ This is the price that Colombian agents have to use to value their assets at market value. It is calculated daily.

submit the amount they want to purchase.¹⁹

From the bidder's perspective, the NCR is a sale without quantity or price uncertainty. From the government's perspective, it is a sale with quantity uncertainty since bidders can demand less than the maximum they are allowed to. Given the characteristics of the NCR, when bidders submit their bids for the original auction, they are implicitly submitting a bid to buy an European call option. This option can be exercised on the NCR date to buy the bond at the strike price—the average secondary market price for the bond on the auction date. In a market with no liquidity problems, we would only expect bidders to exercise their option if the market price of the bond on the NCR date is higher than the strike price. In this way, bidders can make profits by buying the bonds in the NCR and selling them in the secondary market at a higher price.

For the auction presented in Table 2.4, the coverage was 3.05, hence the government held the NCR on April 30 with a supply of COP\$17468.3 million. Neither of the two bidders that had the right to attend the NCR participated. A plausible explanation is that on April 30, the bond was trading at a lower price than the one at which bidders could have obtained it in the NCR. Additionally, as shown in Figure 2.2, the bond with maturity in 2015 had low liquidity during the sample period: in the Colombian Electronic Negotiation System (ENS)²⁰, one of the two trading systems²¹, there were no transactions between April 21 and May 7, 2007.

¹⁹ This quantity is required to be less than or equal to the maximum amount each bidder can obtain.

²⁰ Known in Colombia as SEN.

²¹ In Colombia there are two systems of negotiations: Electronic Negotiation System (ENS), managed by the Central Bank, and Electronic Negotiation Market (ENM), managed by the CSSE.

According to Table 2.3 the Colombian government held the NCR for all the auctions—the minimum coverage was always higher than 1.2. In fact, the supply for the NCR was equal to 80% of the amount sold in the primary auction for all except one of the auctions where the coverage was 1.9. During the sample period the government issued 73.8% of the bonds through the auction and 26.2% in the NCR. From the bonds issued through the NCR, 9.6%, 17.9%, 24.5%, and 47.9% correspond to the bond with maturity in 2009, 2010, 2015 and 2020, respectively.

The NCR, with the characteristics explained, is a unique feature of the government of Colombia's auction. In other countries (e.g. USA, Canada, France) bidders can submit a non-competitive bid, but this is an offer sent at the same time as the competitive bids that consists of a limited amount that will be served at a price equal to the average price of the awarded competitive bids. From a bidder's perspective these types of non-competitive bids have price but not quantity uncertainty²² and introduce uncertainty in the auction's supply. The closest case to Colombia's NCR is the one done by the Mexican government, called the buy-option round in Castellanos and Oviedo [7]. In the Mexican case bidders can buy securities at the weighted allocation price resulting from the primary auction after it takes place.²³ Further, the round is held the same day of the auction, and the non-competitive bid has quantity uncertainty because there is no guarantee that sufficient supply will be available to meet all demands.

²² Recall that competitive bids are price-quantity pairs. A non-competitive bid is only a quantity.

²³ In Colombia the NCR was established in 2003; before the government held a second round which had the same characteristics as the buy option in Mexico (see Appendix B).

2.3.4 The Bidders

In 1998 the Colombian government introduced the Market Makers program to improve government securities' liquidity in the secondary market and to promote investment in those securities. Its members, known as market makers, are the only ones that can participate in the auction.

At the beginning of each year the members of the market makers program are chosen and ranked by the government according to their participation in the auction and in the secondary market in the previous year. In the secondary market, market makers are required to quote bid and ask prices in the trading systems. These quotes must be within a specified bid ask spread. In the primary market, to be able to continue being a member of the program for the next year, the entity has to acquire a minimum of 4.25% of the securities auctioned during the current year. The NCR can be seen as a benefit that market makers receive to compensate for the risk they have to assume to fulfil their obligations.

The list of markets makers is publicly available and the entities that can participate in the program are: banks, brokers, and financial corporations. In 2007, the program had a total of 14 members: 9 banks, 3 brokers, and 2 financial corporations.

2.4 Analysis and Findings

2.4.1 The NCR

How does the NCR affect the bidding behaviour in the auctions? As mentioned, because of the NCR, bidders are bidding to buy bonds and implicitly an European call option. This should be reflected in prices and it is indeed supported by the data as shown in Figure 2.3, which illustrates that in most of the auctions, the government sells the bonds at a higher price than the one observed in the secondary market. This behaviour would not make sense without the NCR. In fact, it has not been reported in other countries where typically the government sells the bonds at a lower price than the one present in the secondary market—known in the literature as underpricing (see Table 1 of Castellanos [6]). For example, in Finland according to Keloharju et al., on average government securities were underpriced by 0.041% of their face value from 1992 to 1999. On the other hand, the average overpricing in Colombia is 0.13% of face value.

The difference between the market clearing price (MCP) and the secondary market price can be seen as a lower limit for the price of the option. Table 2.7 shows that this difference between prices is higher for the bonds in which bidders exercise a higher percentage of the option. This ex-post analysis illustrates that bidders incorporate the price of the option in their bidding strategy and pay more for the options which have a higher probability of being exercised.

Having mentioned that the NCR might be influencing the bidders' behaviour, another question of interest is whether the government makes or loses money by offering the NCR, in comparison to selling the bonds at the secondary market price. To consider this the following was done for each auction: i) The amount issued in the auction was multiplied by the difference between the market clearing price and the secondary market price on the day of the auction. A positive quantity indicates the amount of money the government is earning with respect to the market. According to Figure 2.3 we expect most of these quantities to be positive. ii) The amount issued in the NCR was multiplied by the difference between the price bidders have to pay in the NCR and the secondary market price on the day of the NCR. Recall that we expect bidders to exercise the option when the secondary market price is higher than the NCR's price; therefore, we expect this quantity to be negative.

Summing across auctions the government earned COP\$3.36 billion. Across NCRs the government lost COP\$3.60 billion with respect to the market. In net, the government's loss is COP\$0.24 billion, which is 0.08% of the nominal amount issued.

2.4.2 The Bidders

The minimum amount that market makers have to buy in the primary market to maintain their status is typically mentioned by the market analysts and bidders as one of the aspects affecting the bidding behaviour (see Appendix C). Their argument is that bidders submit higher demands and prices in the last auctions of the year in order to

satisfy this requirement. As Figure 2.4 and 2.5 illustrate, this is not what happened in 2007, as there is no marked increase in coverage or in prices in the last auctions. Additionally, Figure 2.6 shows that with the exception of bidder 11 (dashed pattern), bidders maintain fairly constant the percentage of bonds obtained in the auction through time.

2.4.2.1 The Bidders' Decisions

Every time there is an auction, bidders face the following decisions if they decide to attend: i) how many bid-points to submit, ii) how much quantity to demand at each point, and iii) how much to bid at each point. This subsection analyzes the data with respect to these decisions variables. According to interviews with some participants, their decisions are influenced by the market conditions, their businesses and their performance in the market makers program (see Appendix C). As is expected, they do not explicitly solve an optimization problem to determine their bids. However, the factors they mention as input to their decision are consistent with strategic models of bidding.

Table 2.8 illustrates how the entry and number of bid-points decisions vary among bidders. For the entry decision, we see that only 2 of the 14 bidders participated in all the auctions. For the number of bid points, the table shows that on average bidder 7 submits the highest number of bid-points (4.6), whereas bidder 3 submits the lowest (1.4). Additionally, the table indicates that the bidder who demands the most is not the one who wins the most and vice versa: the ratio between share demanded and share

won ranges from 1.7 for bidder 6 to 180 for bidder 2. Specifically, bidder 6 demands on average almost 50% of the pool of securities and wins close to 30% of the pool, whereas bidder 2 demands 20% and ends up with 0.11%.

In most theoretical models and related structural estimation of share auctions, with Kastl's work as an exception, researchers have assumed that bidders submit a continuous downward sloping demand from which the researcher sees only some points (see Chapter 4). Under this view, bidders do not choose how many points to submit, neither do they explicitly decide how much to bid and for how much. Their decision is the functional form of the demand curve. Kastl [14], gets closer to reality by modeling demand as a step function where bidders choose how many points to submit.

According to Kastl, if bidders behave strategically and do not face a cost for the submission of bid-points, they should submit as many points as they are allowed to. The data for Colombia shows that bidders reduce the dimensionality of the strategic space by choosing to submit a small number of bid-points—on average bidders submit 2.4 bid-points per auction. This is the same behaviour documented by Kastl for Czechoslovakia.²⁴

Regarding the quantity and price decisions, for estimation purposes, Kastl assumes that bidders choose optimally how much to demand at each step (quantity-bid) but not how much to bid (price-bid). In Colombia the data shows precisely the opposite given that the variability of the bid quantities is much lower than the variability of the

²⁴ To rationalize this behaviour, Kastl assumes that bidders face a submission cost. Chapter 4 presents Kastl's model and estimation in more detail.

bid price. With respect to the decision variable quantity, the data shows that almost 80% of the bid-points are either for COP\$5,000 million or for COP\$10,000 million,²⁵ while with respect to price, there are just 8 cases in which the bidder submitted the same price twice—maximum number of times a price is repeated.²⁶ Figure 2.7 presents the frequency histograms for the bid prices per bond and for the bid quantity over all the bonds.

The behaviour of bidders 4 and 6 suggests that the choice variable is the price, whereas, the quantity and also the number of bid-points seem to be predefined. Bidder 4 maintained the same strategy in terms of number of bid-points and bid quantity for 3 consecutive auction dates (12 auctions), then changed it and maintained the new strategy constant—changing only the price—for 7 consecutive auction dates. For the first 3 consecutive auction dates the bidder submitted two bid-points for the bond 5/14/2009, a single bid-point for the bonds 11/24/2010 and 10/28/2015, and two for the bond 7/24/2020. For the next 7 consecutive auction dates,²⁷ this bidder submitted two bid-points for each of the four bonds. Figure 2.8 illustrates that though neither the quantities nor the number of bid-points submitted change, the prices and the slope of the line between the two bid-points do change across time. In the 10 auction dates described, this bidder submitted 74 bid-points, each of them for a quantity of COP\$10,000 million.

²⁵ 47% of the bid quantities are for COP\$5,000 million and 29% for COP\$10,000 million.

²⁶ The comparison might be overstated because when bidders submit more than one bid-point the bid prices should be different, whereas the quantities not.

²⁷ From May 23, 2007 to August 22, 2007.

For the other bidder, for the 10 auction dates in which they participated, the bidder maintained a constant strategy for 6 consecutive auctions by submitting two bid-points for the bonds with shorter maturity, each bid-point for COP\$10,000 million. In the sample set, this bidder submitted 39 bid-points, 38 of which were for COP\$10,000 million.

2.4.2.2 Reducing the Dimensionality of the Bidder's Demand

Even though the previous analysis has shown that there is little variation in the quantities that bidders submit per bid-point, the study of bidders' demands is still a complex task because bidders submit step demands with different number of bid-points at different prices and shares. This was illustrated across auctions by Table 2.8; as Figure 2.9 shows it also happens within an auction. This figure shows the step demand²⁸ for each of the 6 bidders that attended the auction on April 25, 2007 for the bond that expires on 2015.

To reduce the dimensionality of the bidders' demand I estimate the following linear regression for each bidder that submitted at least 3 bid-points:

$$p_{i,j,l} = b_0 + b_1 q_{i,j,l} + \epsilon_{i,j,l} \quad (2.1)$$

where $p_{i,j,l}$ is the j -th price submitted by bidder i in auction l , and $q_{i,j,l}$ is the correspon-

²⁸ To obtain the step demand, the bid-points submitted by the bidder are sorted from highest to lowest price and the quantities are summed up. See Table (2.4). To present the data under the share auction framework quantities were normalized by the supply.

ding share quantity demanded at this price—not the marginal share quantity submitted. In this way, a bidder’s demand is represented by an intercept and a slope, where the slope indicates how elastic is the demand with respect to the price.²⁹ Table 2.9 presents the estimation results for each of the bonds, showing that the slope of the bidders’ demand (\hat{b}_1) differs across bonds.³⁰ On average bidders present more inelastic demands the longer the maturity of the bond. But, as seen in Table 2.3, this does not necessarily translate into higher market clearing prices.

Over all the bonds, 93% of the variability in the bidders’ demand can be explained by the linear specification. This is similar to what Hortaçsu finds for Turkey, where the variability explained is 92%.³¹

As a further exercise, I estimate a linear fit for the residual supply (an intercept and a slope) for each bidder that submitted at least 3 bid-points, and intersect it with the estimated linear demand to find the market clearing price under the linearity assumption. Figure 2.10 displays the distribution of the difference between estimated market clearing prices and actual ones. It has a mean of 0.036 and a variance of 0.073. For this figure, given the distribution of estimated coefficients of residual supplies and demands, 100 random market clearing prices were estimated for each bidder. Though there is no apparent reason why bidders should submit a linear demand function as assumed by most of the theoretical work (see Chapter 3), the data seems to support this assumption.

²⁹ To be able to compare results among the different types of bonds, prices in each auction were normalized by their corresponding market clearing price.

³⁰ The results show that it also differs across bidders within auctions for a given bond.

³¹ In Hortaçsu’s case, he analyzes one type of security (the 3 month bill) and his estimates are $\hat{b}_0=0.8494$ and $\hat{b}_1=-0.0004$.

2.4.3 Summary of Findings

Section 2.4 has analyzed the bidders bidding behaviour using a previously unexplored data set of the Colombian government bond auctions. Three main findings were presented and discussed: i) in contrast to results from other countries, auctions in Colombia exhibit overpricing with respect to the secondary market. The NCR was analyzed empirically and presented as a possible explanation for this, ii) bidders seem to reduce the dimensionality of the bidding space by choosing a small number of bid-points, 2.4 on average, and by choosing quantities that do not vary much—the variability of the quantities chosen is much lower than the one for the prices, and iii) for the bidders who choose 3 or more bid-points, even though their demand is a step function, the points do not deviate far from a linear function.

Table 2.4: Bids Submitted by Bidders

Price ^b	Quantities ^a					
	Bidder1	Bidder4	Bidder5	Bidder7	Bidder9	Bidder12
87.823					5000	
87.772	5000					
87.721					5000	
87.681						
87.676	5000				1500	
87.671	5000					
87.620				5000		
87.570		10000		5000		
87.524	5000					
87.519			10000			
86.271						5000

^a COP million. Auction Date: April 25,2007. Bond Auctioned: October 28 2015 (8%). The bidder's number corresponds to the id in the data set.

^b Corresponds to the clean price the bidder is willing to pay for an imaginary bond with a face value of 100.

Table 2.5: Number of Bid-points Submitted by Bidders

# Bid-points	%
1	23.76
2	32.61
3	20.52
>3	23.11

Table 2.6: Summary Statistics per Bond(II)

Bond	# of Bid-points			# of Winning Bid-points		
	Mean	Min	Max	Mean	Min	Max
5/14/2009	15.6	9	26	5.3	3	7
11/24/2010	19.0	13	25	7.3	4	12
10/28/2015	15.8	11	25	5.9	5	9
7/24/2020	25.5	20	33	9.9	6	15

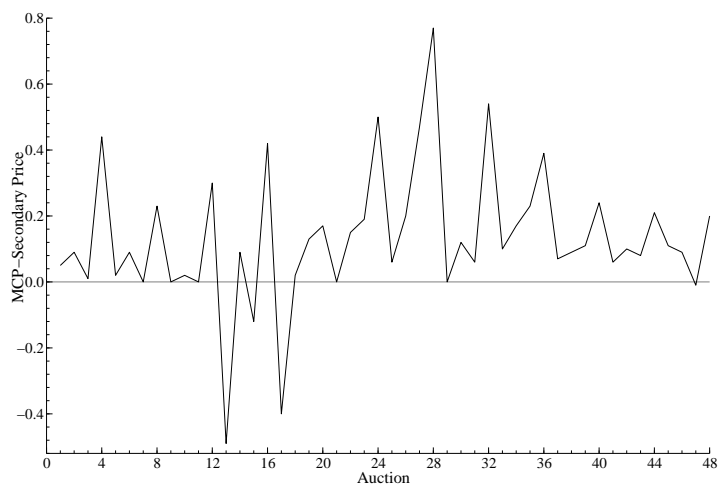


Figure 2.3: Difference between Market Clearing Price(MCP) and the Secondary Market Price.

Auction 1, 2, 3, and 4 corresponds to bond 5/14/2009, 11/24/2010, 10/28/2015, and 7/24/2020, respectively. The same for all the other subsequent sets of 4 auctions. The secondary market price is the mean price of the security on the day of the auction reported by ENS. For auctions 1, 7, 9, 11, 15, 21, 29, and 31, there were not transactions on ENS; the price published by CSSE for mark to market was used instead.

Table 2.7: Difference in Prices and % Issued in the NCR

Maturity	MCP-Secondary Price	% NCR
5/14/2009	0.08	9.6
11/24/2010	0.14	17.9
10/28/2015	0.16	24.5
7/24/2020	0.37	47.9

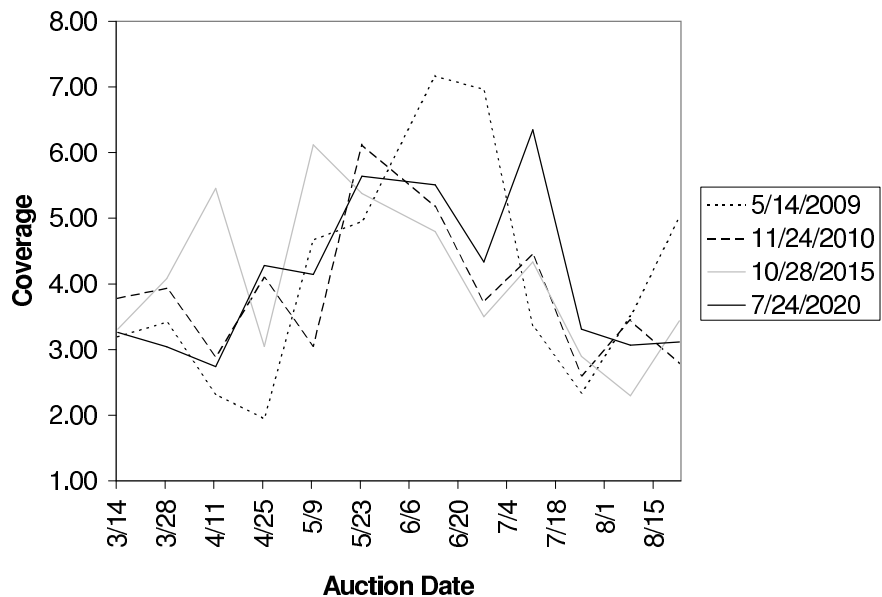


Figure 2.4: Coverage per Bond per Auction

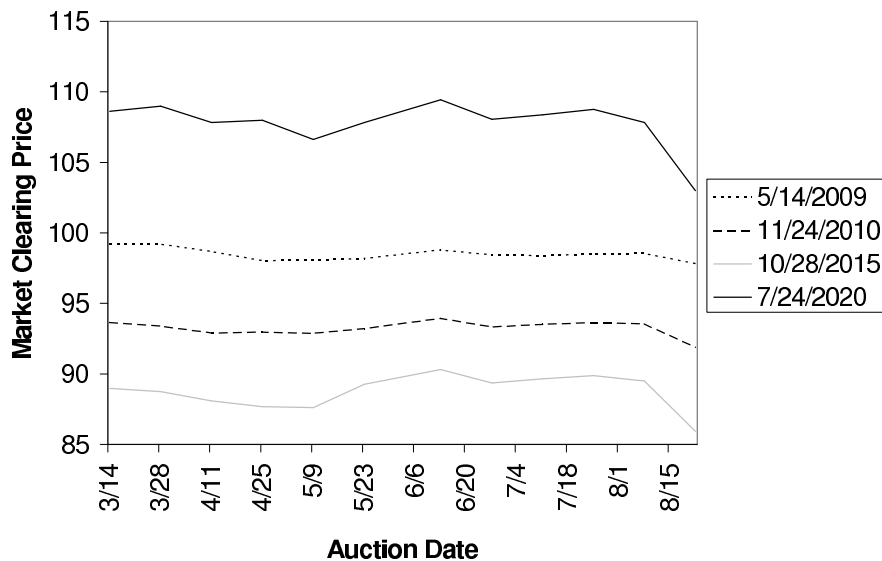


Figure 2.5: Market Clearing Price (clean price) per Bond per Auction

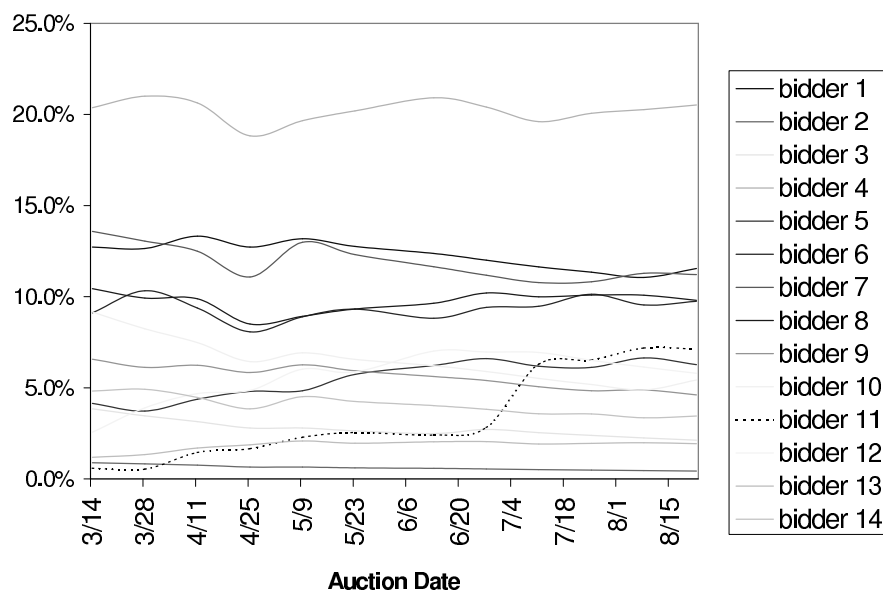


Figure 2.6: Percentage Obtained through the Auctions in 2007

The first observation corresponds to the percentage obtained from January through March 14.

Table 2.8: Descriptive Statistics per Bidder

Bidder Id	# of Auctions Attended	Share Demanded ^a	Share Won ^a	# of Bid-points Submitted ^a
1	44	36.91%	12.38%	2.4
2	17	19.15%	0.11%	2.2
3	17	30.06%	1.89%	1.4
4	48	44.13%	21.80%	1.8
5	46	41.19%	8.70%	2.0
6	20	47.51%	28.54%	2.0
7	48	63.76%	9.56%	4.6
8	43	51.93%	9.97%	2.8
9	44	32.65%	3.34%	3.9
10	20	31.98%	3.21%	2.1
11	32	42.93%	16.21%	2.1
12	44	37.34%	9.11%	3.0
13	26	25.26%	3.58%	1.8
14	14	54.96%	6.62%	1.7

^a Average per Auction.

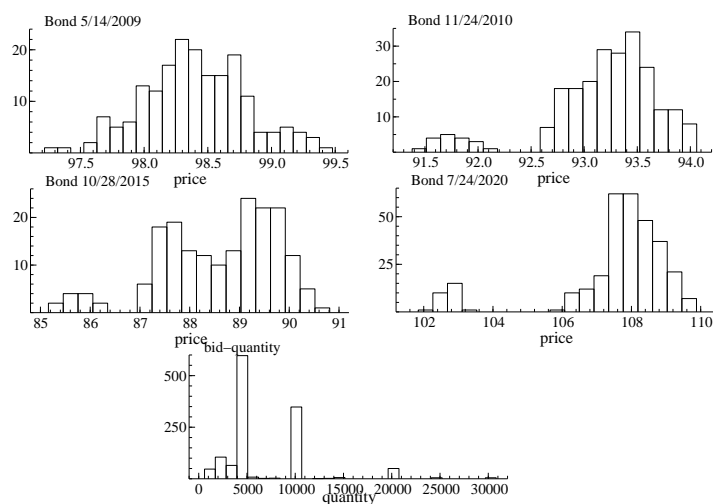


Figure 2.7: Frequency Histograms

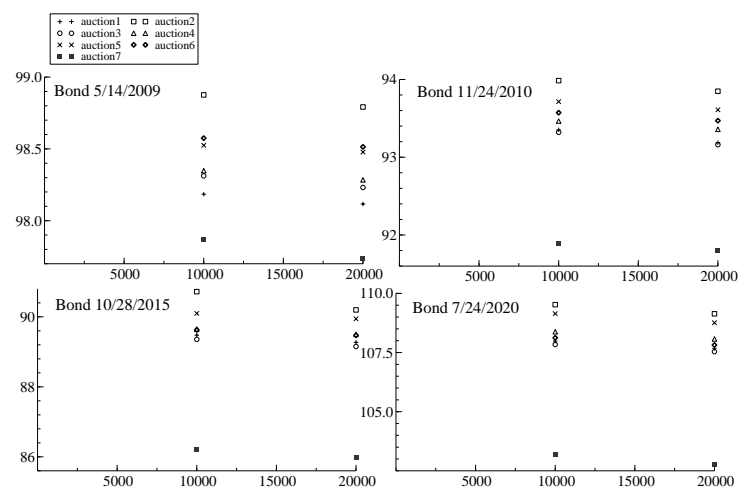


Figure 2.8: Bidder 4's Demands per Bond for 7 Consecutive Auction Dates

x-axis: Quantity (COP million), y-axis: Clean Price

Table 2.9: Linear Fit to Bidders' Demands (mean values)

Maturity	\hat{b}_0	St.Error \hat{b}_0	\hat{b}_1	St.Error \hat{b}_1	R^2
5/14/2009	1.00037	0.00023	-0.00364	0.00051	0.94160
11/24/2010	1.00100	0.00037	-0.00705	0.00119	0.93692
10/28/2015	1.00251	0.00087	-0.01167	0.00225	0.92528
7/24/2020	1.00263	0.00091	-0.02535	0.00436	0.93370
All bonds	1.00177	0.00064	-0.01193	0.00240	0.93419

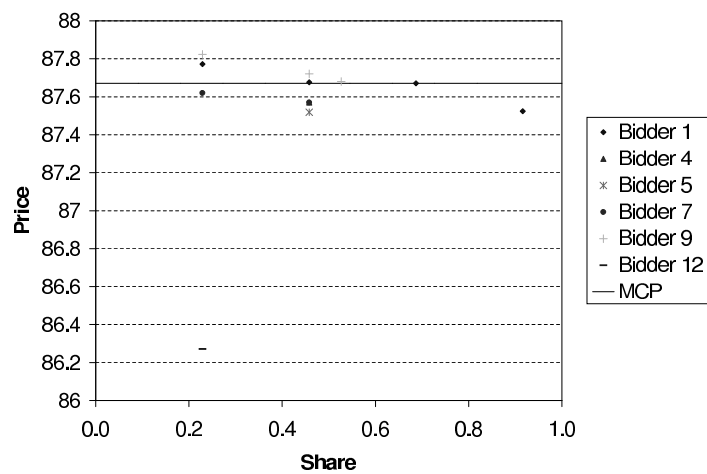


Figure 2.9: Bidders' Demands. (Auction held on April 25,2007)

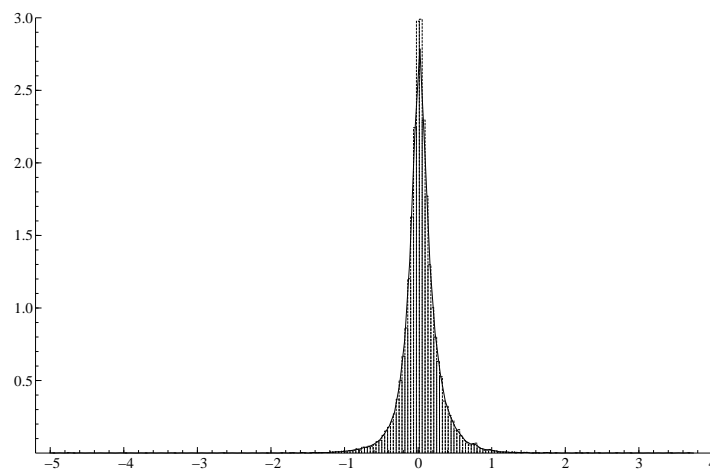


Figure 2.10: Density of the Difference between Estimated MCP and Actual MCP

Observations: 20,500. 4 outliers are not included.

Chapter 3

Known Strategies for Share Auctions

3.1 Introduction

This chapter begins with a brief discussion of the two information structures generally adopted in auctions, followed by a comparison of the three known closed-form strategies for share auctions with symmetric bidders. Until now, there is no known closed strategy with asymmetric bidders. The chapter closes up with an attempt to estimate the structural parameters of a uniform price auction using simulated data under one of the known strategies.

3.2 Information Structures

Auctions are typically studied as noncooperative games of incomplete information where the seller is unsure about the values that bidders attach to the object(s) being sold—the maximum amount each bidder is willing to pay. Thus, to model the strategic behavior of the game we need to assume a specific structure for those values—an information structure. The two most commonly investigated information structures (or

paradigms) in the auction context are the Independent Private Values Paradigm (IPVP) and the Common Value Paradigm (CVP).

Most of the theoretical and econometric research within auctions of one indivisible object has been undertaken using the IPVP (Paarsch, Harry J. and Hong, Han [23]). This is not the case in research on structural estimation of government securities auctions, however, where both paradigms have been equally used. Armantier and Sbaï and Fevrier et al. use CVP. The other 3 papers presented in Chapter (4) use IPVP. On the theoretical side, most research has been done under the CVP.

The IPVP assumes that each bidder values the object independently from other bidders, whereas the CVP assumes that the object is worth the same for all bidders: the IPVP recognizes differences in individuals' preferences; the CVP does not. Thus, besides determining the best auction from the seller's perspective, when imposing the IPVP researchers are also interested in evaluating the efficiency of the auction in terms of the buyers. That is, does the auction format allocate the object to the bidder with the highest valuation? This is not a concern if the CVP is assumed since the object is worth the same ex ante unknown value for all bidders. A third paradigm is the Affiliated-Values Paradigm (AVP). In this case, the utility of each bidder is a function of their own private valuation and unknown common value. To my knowledge no work has used the AVP to estimate models of share auctions.

Under the IPVP, bidder i assigns a value V_i to the object¹, which is independently

¹ Throughout, random variables are denoted with capital letters, and their realizations with lower case letters.

and identically distributed according to a common knowledge distribution $F_V(v)$.² Bidder i knows the realization v_i of V_i and that other bidders' values are independently distributed according to F_V .

Under the CVP, the value of the object, V , is unknown at the time of the auction but is the same for all bidders. Bidder i is assumed to receive a signal, S_i , which has a conditional density $f_{S|V}(s|v)$ known to all bidders, and which is assumed to be an unbiased estimator of the true value. The sole explanation for heterogeneity in bids across auction participants is the difference in opinions (signals) concerning the object's true value.

Real-life auctions probably display an information structure that incorporates elements belonging to both the IPVP and the CVP, making the AVP an attractive information structure to impose. As stated by Paarsh and Hong [23], however, it is difficult to do empirical work within this paradigm because models are unidentified in the econometric sense.³

3.2.1 IPVP vs CVP

Given the difficulty in estimating a model within the more general AVP, the work on government securities auctions has been confined to either the IPVP or the CVP.

Which of these paradigms fits the characteristics of these types of auctions better? The

² This is the general setting used for independent private value auctions. In share auctions the setting is a little bit different as Section 4.3 shows.

³ A model is identified if, given the implications of equilibrium behavior in a particular auction game, the joint distribution of bidders valuations and signals is uniquely determined by the joint distribution of observables (the bids).

CVP can be justified by the existence of a secondary market for government securities. Bonds bought in the auction can be sold at the secondary market price, which would be the common but unknown valuation of the object. Nonetheless, not all bidders sell the securities after the auction and some of them acquire them for private reasons: to hedge positions, to satisfy certain government requirements,⁴ or to sell them to private customers. Thus, both private and common values can be present in a government bond auction.

Hortacsu and Kastl [16] test the two information structures for share auctions⁵ using a method that can only be applied to a few specific countries, e.g. Canada, where the researcher can observe the modifications that bidders make to the submitted bids upon observing the bids of their customers. The major distinction between customers and bidders, is that customers cannot bid on their own account in the auction, but have to route their bids through one of the bidders. The important characteristic of the Canadian data is that bidders are required to identify bids submitted by customers in the electronic bidding system. Applying their methodology to Canadian data delivered mixed results: they could not reject the null hypothesis that private values exist for short term bills (3 months), but they found evidence to reject it for longer term bills (12 months).⁶

⁴ Two examples: i) in Turkey banks are required to maintain 35% of their portfolios in government securities, and ii) in most countries the institutions that can participate in the auction must be members of the market makers program, which requires them to acquire a minimum amount of the securities being auctioned.

⁵ Parsch [22] does it for single unit auctions.

⁶ The null hypothesis of private values is rejected at a 5% level.

3.3 The Uniform Auction under the CVP: Equilibrium Strategies

So far, efforts to come up with an optimal strategy for the discriminatory share auction under either the CVP or IPVP have been unsuccessful.⁷ Similarly, no closed expression for a uniform price auction under the IPVP is found in the literature. This reduces the field of known equilibrium strategies to uniform price auctions within the CVP: in this case, three optimal strategies have been identified for auctions with N symmetric bidders. In this document, each of them is named according to its author(s) as follows: Fevrier et al.'s strategy, Kyle's strategy, and Wang and Zender's strategy.

Fevrier et al.'s strategy: Fevrier et al. show that the unique equilibrium strategy in the class of continuous demand functions that are linear in the signals is:

$$\phi^*(p, s_i, N; \alpha, \beta, \gamma) = \frac{1 - \left\{ \frac{\beta}{N} + s_i \right\} \left\{ \frac{\Gamma(N+\alpha)}{\Gamma(N+\alpha+1/\gamma)} \frac{1+\gamma}{\gamma} p \right\}^\gamma}{N-1}. \quad (3.1)$$

The assumptions behind this result are the following: the amount supplied is known, the signal S_i given $V = v$ follows an exponential distribution with parameter v^γ , and the value V is assumed to have the following distribution function with parameters α , β and γ :

$$F_V(v|\alpha, \beta, \gamma) = \int_0^v \gamma u^{\gamma-1} \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\gamma(\alpha-1)} e^{-\beta u^\gamma} du \quad (3.2)$$

Kyle's strategy: assumes the above setting but with uncertainty in the supply.⁸

That is, bidders know that the supply is $1 - \Omega$, where Ω follows a normal distribution

⁷ Hortaçsu [12] provides an analytic characterization of the equilibrium in the restrictive case of two bidders with private values in which bidders have linear demand functions and possess exponentially distributed private signals regarding the value of the security.

⁸ Three possible reasons for uncertainty in the supply: i) in some countries bidders are allowed to sub-

with mean 0 and standard deviation σ_Ω . With $V \sim N(\mu_V, \sigma_V^2)$ and $S_i|(V = v) \sim N(v, \sigma_S^2)$ the optimal strategy is:

$$\begin{aligned} \phi^*(p, s_i, N; \mu_V, \sigma_V, \sigma_S, \sigma_\Omega) &= \frac{N-2}{N(N-1)} + \frac{2N-2}{N(N-1)} \left(\frac{N-2}{N(N-1)} \right)^{1/2} \frac{\sigma_\Omega \sigma_S}{\sigma_V^2} \mu_V \\ &+ \left(\frac{N-2}{N(N-1)} \right)^{1/2} \frac{\sigma_\Omega}{\sigma_S} s_i - \left(\frac{N-2}{N(N-1)} \right)^{1/2} \left[\frac{2N-2}{N(N-1)} \frac{\sigma_\Omega \sigma_S}{\sigma_V^2} + \frac{\sigma_\Omega}{\sigma_S} \right] p. \end{aligned} \quad (3.3)$$

This result follows Kyle [18]⁹ and Fevrier et al. [9]¹⁰. The latter mention that under Kyle's assumptions but without supply uncertainty there does not exist an equilibrium strategy linear in the signal and price. This result holds true even if the IPVP is assumed (this is shown in Appendix D).

Wang and Zender's strategy: Wang and Zender [26] show that under the same setting as in Kyle but when all bidders have Constant Absolute Risk Aversion (CARA) utility, with coefficient ρ , the unique symmetric equilibrium in linear strategies is:

$$\phi^*(p, s_i, N; \mu_V, \sigma_V, \sigma_S, \sigma_\Omega) = A + Bs_i - Cp$$

where B is the unique positive root of the cubic equation:

$$\rho \sigma_S^2 B^3 + \frac{N}{N-1} B^2 + \frac{\rho \sigma_\Omega^2}{N-1} B - \frac{(N-2) \sigma_\Omega^2}{(N-1)^2 \sigma_S^2} = 0,$$

mit non-competitive bids which are served before competitive bids, therefore the supply for the auction is the supply announced minus the amount served to non-competitive bids, ii) sometimes governments announce a bracket for the supply, and iii) in some cases the government has the discretion to modify the supply after the auction.

⁹ A paper not exactly related to share auctions but to the market microstructure literature in finance.

¹⁰ In Fevrier et al., $\Omega \sim N(\mu_\Omega, \sigma_\Omega^2)$ and $(1 - \mu_\Omega)$ multiplies the first term in (3.3). Wang and Zender's strategy and Fevrier et al.'s can be related to each other by setting $\mu_\Omega = 0$.

and

$$\begin{aligned}
 A &= (C - B)\mu_V + \frac{\Lambda}{1 + (N - 1)\Lambda} \\
 C &= B \left[1 + \frac{\sigma_S^2}{(1 + (N - 1)\Lambda)\sigma_V^2} \right] \\
 \Lambda &= \frac{(N - 1)B^2}{(N - 1)B^2 + \frac{\sigma_Q^2}{\sigma_S^2}} \in \left(0, \frac{N - 2}{2(N - 1)} \right)
 \end{aligned}$$

3.3.1 The Strategies Compared

In all three strategies presented the bid function is decreasing in price. This holds true notwithstanding different distributions of the valuation and signals, with or without the existence of uncertainty in supply, and assuming that bidders are risk averse. The resulting strategies also imply that as the number of bidders increase the demand schedules always become more inelastic.

There are two main differences among the strategies. The first is that when the value and the signals are assumed to be normally distributed, the bidding function is increasing in the signal. In Fevrier et al.'s setup the lower the signal the higher the demand for a given price.¹¹ The other main difference is that under the normality assumption bidders will submit demand schedules with the same slope. Fevrier et al.'s strategy, in contrast, predicts that on a typical demand diagram the intercept with the x-axis (quantity) will be the same for all bidders, but the slope and intercept with the y-axis (price) will differ by bidder.

¹¹ In Fevrier et al. each bidder's signal follows an exponential distribution with parameter v^γ , which implies a mean of $1/v^\gamma$ that is decreasing in v . Therefore, if the bidder gets a higher signal they infer that the value of the object is lower and submits a lower bid.

The main difference between Kyle and Wang and Zender is risk aversion. Figure 3.1 presents the two strategies under the same set of parameters ($\mu_v = 100, \sigma_V = 0.5, \sigma_S = 0.5, \sigma_\Omega = 0.1, N = 3$, for Wang and Zender's strategy $\rho = 0.99$). When bidders are risk averse the demand schedules they submit are more inelastic.

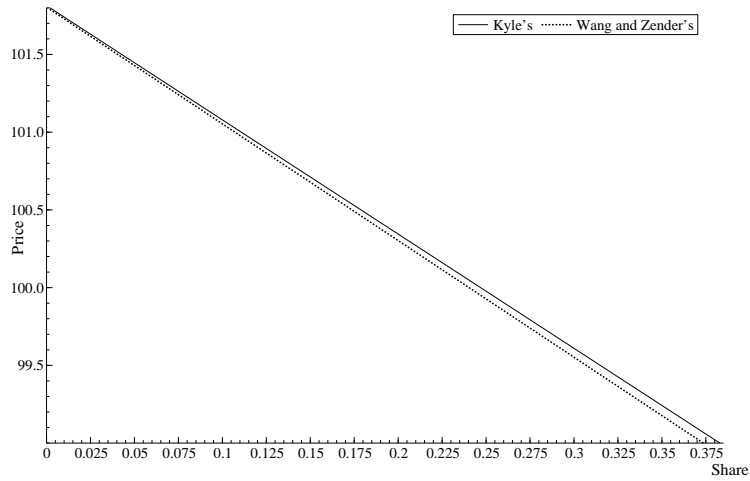


Figure 3.1: Kyle's and Wang and Zender's Bid Functions

Table 3.1 shows that when the coefficient of risk aversion is very low the strategies are practically the same. As bidders become more risk averse, the demand schedules become more inelastic (the price coefficient, C , decreases).

Kyle's and Wang and Zender's bid functions predict that: i) higher uncertainty in the supply is followed by more elastic bid schedules, ii) $N \geq 3$ for the bid schedule to make sense, iii) the higher the number of bidders, the more inelastic the bid schedules, iv) the higher the uncertainty in the object's value, the lower the elasticity of the bid schedule, and v) if $\sigma_S < \sigma_V$, the higher the uncertainty in the signal, the more inelastic

Table 3.1: Kyle's and Wang and Zender's coefficients

	A	B	C
R.neutral	5.609977	0.081650	0.136083
$\rho = 0.0001$	5.609969	0.081649	0.136082
$\rho = 0.99$	5.533163	0.079507	0.133216

The first row corresponds to Kyle's strategy, the second and third row to Wang and Zender's strategy. As in Wang and Zender's strategy, for Kyle's strategy, A, B, and C, correspond to the independent coefficient, the coefficient that multiplies the signal, and the coefficient that multiplies the price, respectively.

the bid schedule will be.¹² Figure 3.2 illustrates that the behaviour of the price coefficient for both strategies is very similar—visually there is no difference—and that as the uncertainty in the signal increases the change in the price coefficient decreases.

Regarding the market clearing price, Kyle's and Wang and Zender's equilibrium strategies predict that it will be higher as: i) the uncertainty in the supply is higher, ii) the expected value of the object is higher, iii) the uncertainty in the value of the object is lower, and iv) the uncertainty in the bidder's signal is lower.

Finally, Fevrier et al.'s strategy is a generalization of an equilibrium strategy derived in Wilson [27]. Wilson assumes that V follows a gamma distribution with parameters α and β , and that the signals are exponentially distributed with parameter v (when $V = v$). His setup corresponds to the special case where the parameter γ appearing in the distribution function (3.2) is equal to 1. When $\gamma = 1$, (3.1) reduces

¹² In Kyle's strategy this can be seen by taking the derivative of the coefficient that multiplies the price with respect to σ_S . In Wang and Zender's case there is not such an exact condition but numerically it can be seen that the behaviour is very similar. See Figure 3.2.

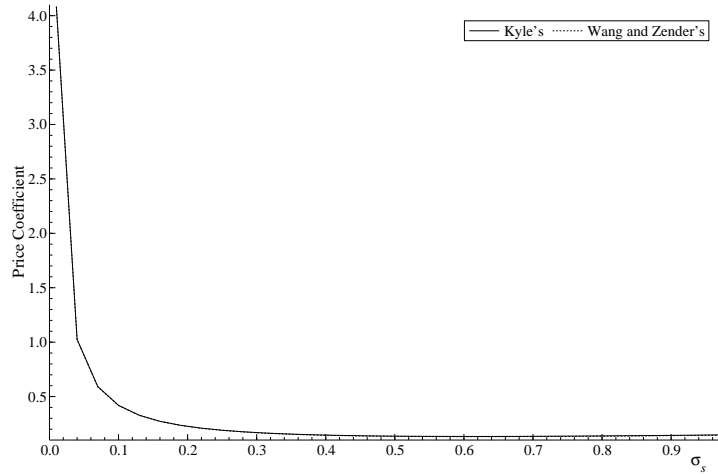


Figure 3.2: Signal Uncertainty and Price Coefficient ($\mu_v = 100, \sigma_V = 0.5, \sigma_\Omega = 0.1, N = 3, \rho = 0.6$)

to—Wilson’s strategy—:¹³

$$\phi^*(p, s_i, N; \alpha, \beta) = \frac{1}{N-1} - \frac{2(\beta + N s_i)}{N(N + \alpha)(N-1)} p. \quad (3.4)$$

Comparing Wilson’s strategy—instead of Fevrier et al.’s—with the other two strategies previously depicted is more straightforward. For this comparison, the mean of the value of the object, α/β , is set equal to μ_V , and its variance, α/β^2 , equal to σ_V^2 (the mean and the variance assumed in the previous setups). When $\mu_V = 100$ and $\sigma_V = 0.5$, α has to be equal to 40,000 and β to 400. Since the signals are exponentially distributed with parameter v , if the object’s value is 100 (which is the mean of the distribution of the valuation)¹⁴, the mean of the signals is $1/100$ and the variance is $1/(100^2)$. Figure 3.3 shows that strategies differ according to the parametric framework

¹³ Recall that $\Gamma(z+1) = z\Gamma(z)$.

¹⁴ In Kyle’s and Wang and Zender’s settings when the object’s value was 100, signals were assumed to be normally distributed with mean 100.

adopted,¹⁵ even though in both frameworks the object's value has the same mean and variance.¹⁶

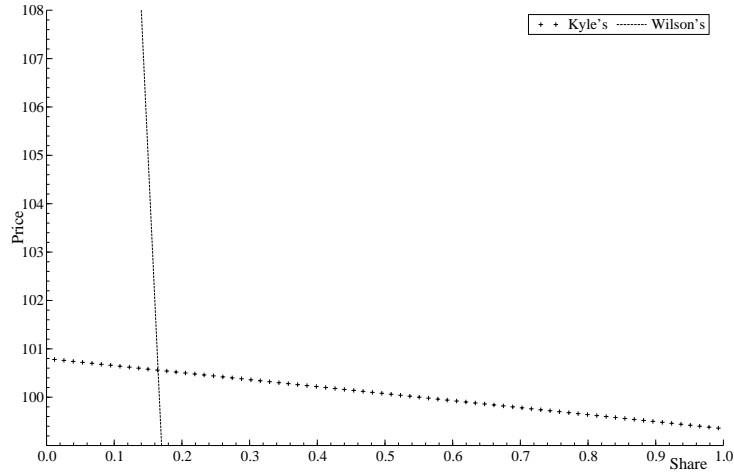


Figure 3.3: Kyle's and Wilson's Bid Functions (N=3)

According to (3.4), the optimal demand function is:

$$p = \frac{N(N + \alpha)}{2(\beta + N s_i)} - \frac{N(N - 1)(N + \alpha)}{2(\beta + N s_i)} \phi_i, \quad (3.5)$$

where the demand's slope is $N - 1$ times the intercept. Subsection 2.4.2.2 estimates a linear fit for the bidders' demands in the Colombian government bond auctions, uniform price auctions without uncertainty in the supply, finding that this ratio is 0.01190, which is not consistent with Wilson's optimal demand function.

¹⁵ My conjecture is that the difference in the strategies is due more to the different parametric frameworks adopted than to the fact that in one setting there is supply uncertainty whereas in the other one there is not.

¹⁶ If the signals are set to have a mean equal to 100, which is similar to the mean used in Kyle's and Wang and Zender's setups (I assumed signals were normally distributed with mean v , and $V \sim N(100, 0.5^2)$), then $\alpha = 0.0004$ and $\beta = 0.04$. In this case for the price range of 99 to 108 the bids did not make sense; they ranged from -3349 to -3653.

3.3.2 Simulated Method of Moments for Wilson's Strategy

Having gone through the main characteristics of the optimal strategies for a uniform auction under the CVP, a related question of interest concerns the estimation of the structural parameters. Assuming there is no uncertainty in the supply, this can be illustrated using Wilson's strategy, i.e. trying to estimate the primitives of the model, the parameters α and β , by simulating a set of bidding functions. The details of this exercise follow.

Bidding functions were simulated for 50 identical auctions ($l = 1, \dots, L$; $L = 50$) with eight bidders each ($N_l = 8 \ \forall l$). For each auction, the valuation of the object follows a $\Gamma(3, 2)$ distribution and each bidder's signal is exponentially distributed with decay rate v_l . Thus, for each auction the following was done:

- (1) Draw a random valuation, $v_l \sim \Gamma(3, 2)$.
- (2) Draw $N = 8$ signals, $s_i \sim \text{Exponential}(v_l)$.
- (3) Compute bid functions following (3.4). These simulated bidding functions correspond to the data that a researcher would have available in a real-world scenario.

Next, recover the structural parameters, $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta})$,¹⁷ and compare them with their true counterparts, $\boldsymbol{\theta}_0 = (3, 2)$. A common way of doing this is using simulated method of moments (SMM), where the goal is to minimize the sum of squared deviations between

¹⁷ Throughout the thesis vectors are denoted in bold.

the empirical moments and their theoretical counterparts. Because bidders in share auctions bid a demand function not a single price, the moment matched for this exercise was the mean of the area under the bidding function, i.e, the bidder's total willingness to pay.¹⁸ For bidder i , the area under their bidding function, Δ_i , is given by:

$$\Delta_i = \int_0^{p_i^{max}} \left(\frac{1}{N-1} - \frac{2(\beta + Ns_i)}{N(N+\alpha)(N-1)} p \right) dp$$

where $p_i^{max} = \frac{N(N+\alpha)}{2(\beta+Ns_i)}$ and corresponds to the price at which bidder i 's demand equals zero. The integral evaluates to:

$$\Delta_i = \frac{N(N+\alpha)}{4(N-1)(\beta+Ns_i)},$$

and the SMM parameter estimates satisfy

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{L} \sum_{l=1}^L \sum_{i=1}^{N_l} (\Delta_i^l - m_i^l(\theta))^2$$

where L is the number of auctions in the data set, N_l is the number of bidders in auction l , Δ_i^l is the area under the bidding function submitted by bidder i in auction l , and $m_i^l(\theta)$ is its corresponding expectation (or first moment). For each parameter vector, θ , the procedure to simulate $m_i^l(\theta)$ takes the following steps for draws $mc = 1, \dots, MC$ (see Hong and Shum [10]):

- (1) Draw a random valuation, $v_{mc} \sim \Gamma(\theta)$ and draw $N = 8$ signals, $s_i \sim \text{Exponential}(v_{mc})$ holding the seed constant for random number generation across different values of θ .¹⁹

¹⁸ Typically in an auction of an indivisible object the first moment to match is the mean of the single bids. See Hong and Shum [10].

¹⁹ This is the same as the steps 1 and 2 used to generate the data, except that there $\theta = \theta_0$.

(2) Evaluate the area under the bidding function which correspond to the drawn

$$\text{signals, } \Delta_{i,mc}^l = \Delta_i^l(s_{i,mc}; \boldsymbol{\theta}).$$

(3) Compute the simulated moment

$$m_i^l(\boldsymbol{\theta}) = \frac{1}{MC} \sum_{mc=1}^{MC} \Delta_{i,mc}^l.$$

3.3.2.1 Results

To solve for the parameter vector $\boldsymbol{\theta}$, several different starting values (1600) and two optimization routines (BFGS and Simplex) in Oxmetrics were used. None of the tries reached convergence, which can be explained by Figure 3.4. As shown in the graph, the objective function has very slight curvature, i.e, it has long ridges in the parameter space,²⁰ which makes the two parameters not separately identifiable.²¹

Further analysis was done by changing the value of the parameters to investigate the effect on the area under the bidding function, the moment that forms the basis of the estimation procedure. The top panel of Figure 3.5 illustrates that for higher signals the area under the bidding function becomes similar irrespective of the number of bidders. So, even variation among the number of bidders does not help to identify the model. Since Wilson's strategy is decreasing in the signal, the plots are negative-sloping. Additionally, for lower signals the area under the bidding function is higher as

²⁰ While not detailed here, the exercise was also done using $\alpha = 3$ and $\beta = 0.003$. The results were very similar.

²¹ Optimizing over β , and setting α equal to the benchmark parameter ($\alpha = 3$), the routine converged to the true β value ($\beta = 2$). The same exercise was done for α given different starting values (α was increased by 0.5 from 0.5 until 20); no convergence was reached even though the final values are not that far from the benchmark (after 1000 iterations the range of the final α was 2.59-3.46).

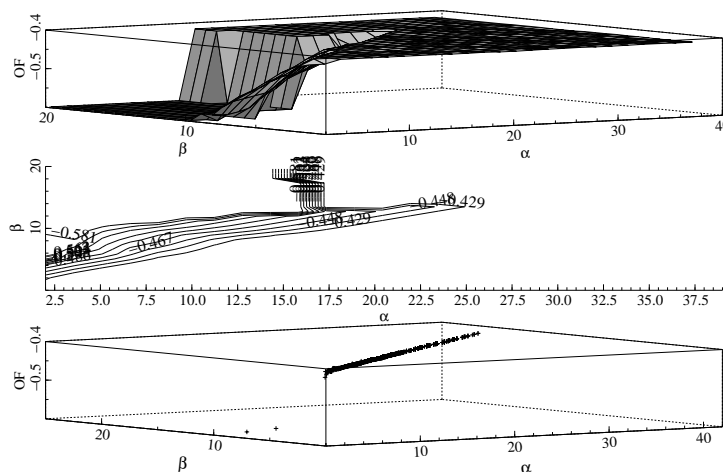


Figure 3.4: Objective function given α and β (Values are negative because the optimization routines are set for maximization).

N increases, i.e. as there is a larger number of participants, whereas this phenomenon reverses for higher signals. This implies that for a given bidder, as the signal increases, the winner's curse (which encourages more cautious bidding as N increases) dominates over the competition effect (which encourages more aggressive bidding).

The lower panel of Figure 3.5 also shows that as the signal increases the similarity of the area under the bidding function for different parameters increases. In conclusion, for high values of the signal, perturbations in the parameters of the model do not lead to significant changes in the bidder's total willingness to pay, explaining why from a computational point of view the parameters are not identified and estimable.

As a further exercise, using the data from the auctions of the Colombian government, I maximized the signal's log likelihood function to estimate the structural parameters. The result of the exercise is the same: Wilson's strategy (3.4) is not iden-

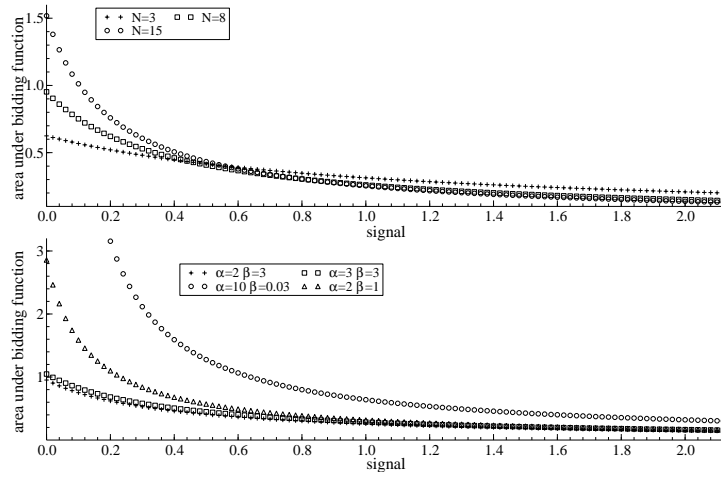


Figure 3.5: Variation of the bidder's total willingness to pay in α , β and N (For upper panel $\alpha = 3$, $\beta = 2$, for lower panel $N = 8$)

tified when allowing both α and β to vary. Signals (for each bidder that submitted two or more bid-points) were recovered from the estimated slope of a linear regression with submitted price as the dependent variable and demanded share as the independent variable. In this case the non-identification arises because the data transforms two parameters into a one dimensional value that is linearly dependent on the two parameters.

In conclusion, even though closed-form optimal strategies exist, it does not necessarily imply that structural parameters can be estimated.

Chapter 4

Empirical Structural Estimation

4.1 Introduction

Structural empirical research on government bond auctions is even younger than the theoretical analysis of multi-share auctions. Currently, the main contributions include two published and three unpublished papers: Armantier and Sbaï [2], Kang and Puller [13], Fevrier et al. [9], Hortaçsu [11], and Kastl [14]. This chapter categorizes and discusses this literature according to the information structure assumed. From each paper, rather than focusing on the theoretical parts, the relevant points for empirical estimation are highlighted and the difficulties that each methodology presents are discussed. Furthermore, when the proposed methodology is applicable to uniform price auctions, the primitives of the model are estimated using real data from auctions done by the Colombian government. The methods are applied to a single data set to compare their actual performance.

All methods and estimates are presented under the same framework: a perfectly divisible unit of a good is auctioned to N ($N \geq 2$) symmetric risk neutral bidders who

maximize their own expected utility. Adding the CVP assumption results in the framework used by Fevrier et al.. In Armantier and Sbaï's model, in contrast, bidders are asymmetric and risk averse and the supply is unknown; the decision of using a common framework seeks to improve clarity and comparability. Under the IPVP previous work has assumed risk neutral bidders and the optimality condition holds with asymmetric and/or uncertainty in the supply; it is the estimation that has to be modified if the researcher wants to take these features into account.

Up to now all the papers on structural estimation within the CVP have adopted a parametric framework. Hence, their objective is to recover the parameters of the distributions assumed in the model. In contrast, the papers assuming the IPVP have used non-parametric estimation and their goal has been recovering the marginal valuation of each bidder.

4.2 Empirical Work under the CVP

Armantier and Sbaï [2], and Fevrier et al. [9]¹ have the following aspects in common: i) parametric estimation, ii) bidders submit a continuous differentiable downward sloping demand function², and iii) a discriminatory auction.

In the CVP setting, the actual value $V \in \Theta_V$ of the good is the same to each bidder, but it is unknown at the time of the auction. The c.d.f of the true value, denoted

¹ Armantier and Lafhel [1] applies Armantier and Sbaï's methodology to Canadian data. Castellanos and Oviedo [7] apply Fevrier et al.'s method to Mexican data.

² As the example setting shows, in reality, bidders submit a step demand.

$F_V(v|\boldsymbol{\delta}_V)$, depends on a parameter vector $\boldsymbol{\delta}_V$ to be estimated. Prior to the auction, each bidder $i = 1, \dots, N$ receives a private signal $s_i \in \Theta_S$ about the value of the good. This signal is generated from a conditional distribution with c.d.f, $F_{S|V}(s_i|V, \boldsymbol{\delta}_S)$, where $\boldsymbol{\delta}_S$ is another parameter vector to be estimated. Therefore, the whole vector of unknown parameters to be estimated is $\boldsymbol{\theta} = (\boldsymbol{\delta}_V, \boldsymbol{\delta}_S)$. From a player's perspective $F_V(\cdot|\boldsymbol{\delta}_V)$, N , and $F_{S|V}(\cdot|\cdot, \boldsymbol{\delta}_S)$, are all common knowledge.

Each bidder submits a sealed bid after receiving the private signal. This bid consists of a schedule that specifies the share of the good demanded, $\phi_i(p, s_i)$, for any price $p > 0$. The auctioneer aggregates the individual bids to determine the market clearing price, p^0 , i.e., the price for which aggregate demand equals one. To characterize an optimal strategy³ —a Bayesian Nash Equilibrium (BNE)— researchers typically restrict their attention to symmetric strategies, so $\phi_i(\cdot, \cdot) = \phi(\cdot, \cdot) \forall i$. If all bidders except i use the optimal strategy $\phi^*(\cdot, \cdot)$, and i uses the strategy $\phi_i(\cdot, \cdot)$ then, the market clearing price is determined by:

$$\sum_{j \neq i} \phi^*(p^0, s_j) + \phi_i(p^0, s_i) = 1.$$

The market clearing price depends on the private signals received by the rivals of bidder i . Since bidder i knows the distribution function from which the signals, S_j $j \neq i$, are drawn and also the function $\phi^*(\cdot, \cdot)$, she can determine the (conditional) distribution

³ An optimal strategy is one that is optimal for any one bidder when each other bidder is using it.

function of the random variable P^0 : That is:

$$\begin{aligned} H(p; v, \phi_i) &\equiv Pr(P^0 \leq p | V = v, \phi_i(p, s_i) = \phi_i) \\ &= Pr\left(\sum_{j \neq i} \phi^*(p, S_j) \leq 1 - \phi_i | V = v\right). \end{aligned}$$

The profit of bidder i is:

$$(v - p^0)\phi_i(p^0, s_i) - \tau \int_{p^0}^{p_i^{max}} \phi_i(p, s_i) dp \quad (4.1)$$

where p_i^{max} is the highest price for which bidder i 's demand is strictly positive. The uniform and discriminant pricing formats are the special cases of (4.1) when $\tau = 0$ and 1, respectively. Bidder's i expected utility can be written in two ways. One, using the equilibrium price distribution $H(p; v, \phi_i)$,

$$E \left\{ \left(\int_0^\infty \left((V - p)\phi_i(p, s_i) - \tau \int_p^{p_i^{max}} \phi_i(p, s_i) dp \right) \frac{\partial H(p; v, \phi_i)}{\partial p} \right) | S_i = s_i \right\}. \quad (4.2)$$

The other one, in unconditional terms—not derived for a given signal but rather across all signals:

$$\int_{\Theta_V} \int_{\Theta_S^N} \left[(V - p^0)\phi_i(p^0, s_i) - \tau \int_{p^0}^{p_i^{max}} \phi_i(p, s_i) dp \right] d\Upsilon(V, \mathbf{s}) \quad (4.3)$$

where \mathbf{s} is the vector of signals, $\mathbf{s} = (s_1, \dots, s_N)$, and $\Upsilon(V, \mathbf{s})$ is the joint distribution of the true value V and the vector of private signals \mathbf{s} .

The optimal strategy $\phi^*(\cdot, \cdot)$ satisfies the following first order condition (Euler-Lagrange equation) of the conditional expected utility maximization problem (4.2) (see Appendix E for details):

$$E \left\{ (V - p)H_p(\cdot) - \tau H(\cdot) - \phi_i(p, s_i)(\tau - 1)H_{\phi_i}(\cdot) \middle| S_i = s_i \right\} = 0 \quad (4.4)$$

where the market clearing price distribution and its partial derivatives are evaluated at

$$\phi_i(p, s_i) = \phi^*(p, s_i).$$

4.2.1 Constrained Strategic Equilibrium

Armantier and Sbaï adopt the constrained strategy equilibrium technique (hereafter CSE) developed by Armantier et al. [3] to approximate intractable BNEs. The idea of the CSE approach is to calculate an equilibrium within a family of expanding constrained sets of amenable strategies. This sequence of CSEs can be shown to converge towards a BNE under standard conditions.

To keep all the methodologies under the same setting I present the CSE approximation method when bidders are risk neutral and symmetric and there is no uncertainty in the supply. The CSE approach is still relevant in this case because there are not analytic solutions for the discriminatory auction.

Each bidder i chooses the optimal polynomial bidding function of the form:

$$\phi_i^{(k)}(p, s_i, \mathbf{d}_i) = d_{0,i} + \sum_{j=1}^k \sum_{j'=0}^j d_{m,i} s_i^{(j-j')} p^{j'} \quad (4.5)$$

$m = j(j+1)/2 + j'$. Where the constrained strategy of order 1 (CS(1)), $k = 1$, is:

$$\phi_i^{(1)}(p, s_i, \mathbf{d}_i) = d_{0,i} + d_{1,i} s_i + d_{2,i} p. \quad (4.6)$$

Since polynomials can approximate any continuous function, the sequence of CSEs, CS(1),... CS(k), as $k \rightarrow \infty$ converges to a BNE (Armantier et al. [3]). For purposes of illustration, I show the CSE approach for $k = 1$.

In a share auction with constrained strategies given by (4.6), the market clearing price is defined as:

$$\sum_{i=1}^N d_{0,i} + d_{1,i}s_i + d_{2,i}p = 1. \quad (4.7)$$

We can write p^0 as a function, $p^0(\mathbf{s}, \mathbf{d})$, where $\mathbf{d} = (d_{0,1}, d_{1,1}, d_{2,1}, \dots, d_{0,N}, d_{1,N}, d_{2,N})$.

Under CS(1), the market clearing price is given by:

$$p^0(\mathbf{s}, \mathbf{d}) = \frac{1 - \sum_{i=1}^N d_{0,i} - \sum_{i=1}^N d_{1,i}s_i}{\sum_{i=1}^N d_{2,i}}.$$

Within the CSE approach, bidder i maximizes their expected unconditional utility with respect to $\mathbf{d}_i = (d_{0,i}, d_{1,i}, d_{2,i})$ to find the optimal constrained strategy of order 1. Given that we are analyzing a discriminatory auction format within the constrained strategy approach, in (4.3), τ is set equal to 1 and $\phi_i(p, s_i) = \phi_i^{(1)}(p, s_i, \mathbf{d}_i)$. Therefore, the maximization problem of bidder i becomes:

$$\begin{aligned} \max_{\mathbf{d}_i} & \int_{\Theta_V} \int_{\Theta_S^N} [(V - p^0(\mathbf{s}, \mathbf{d})) (d_{0,i} + d_{1,i}s_i + d_{2,i}p^0(\mathbf{s}, \mathbf{d})) \\ & - \int_{p^0(\mathbf{s}, \mathbf{d})}^{p_i^{max}} (d_{0,i} + d_{1,i}s_i + d_{2,i}p) dp] d\Upsilon(V, \mathbf{s}). \end{aligned}$$

The parametrization of the constrained strategies provides a major computational advantage as the determination of the CSE reduces to finding the \mathbf{d}_i that solves the following system of non-linear equations

$$\begin{aligned} \int_{\Theta_V} \int_{\Theta_S^N} \left[(V - p^0(\cdot)) \left(\frac{\partial \phi_i^{(1)}(\cdot)}{\partial p^0} \frac{\partial p^0(\cdot)}{\partial \mathbf{d}_i} + \frac{\partial \phi_i^{(1)}(\cdot)}{\partial \mathbf{d}_i} \right) - \right. \\ \left. \frac{\partial \Phi_i^{(1)}(p, s_i, \mathbf{d}_i)|_{p=p_i^{max}}}{\partial \mathbf{d}_i} + \frac{\partial \Phi_i^{(1)}(p, s_i, \mathbf{d}_i)|_{p=p^0}}{\partial \mathbf{d}_i} \right] d\Upsilon(V, \mathbf{s}) = 0, \quad (4.8) \end{aligned}$$

$\forall i \in N$, where $\Phi_i^{(1)}(\cdot) = d_{0,i}p + d_{1,i}s_i p + d_{2,i}p^2/2$, is the primitive of $\phi_i^{(1)}(\cdot)$ with respect to p . Because \mathbf{d}_i is a 3 by 1 vector, (4.8) represents a system of 3 equations. In the case of symmetric bidders only one representative bidder needs to be considered, therefore (4.8) reduces to a system of 3 non-linear equations which must be solved numerically. Given that the integral in (4.8) is $N + 1$ (the number of bidders plus one) and N is typically greater than 8 (see Appendix A), Armantier and Sbaï use a quasi Monte Carlo method. Suppose that all bidders except i use the optimal strategy $\phi^{(1)*}(p, s_j, \mathbf{d}^*) = d_0^* + d_1^*s_j + d_2^*p \forall j \neq i$, and that i uses the strategy $\phi_i^{(1)}(p, s_i, \mathbf{d}_i) = d_{0,i} + d_{1,i}s_i + d_{2,i}p$.

Then the system of 3 non-linear equations that needs to be solved is:

$$\begin{aligned} \frac{1}{MC} & \sum_{mc=1}^{MC} \left[(v_{mc} - p^0(\mathbf{s}_{mc}, \mathbf{d}^*)) \left(\frac{\partial \phi_i^{(1)}(\cdot)}{\partial p^0} \Big|_{(p_{mc}^0, s_{i,mc}, \mathbf{d}^*)} \frac{\partial p^0(\cdot)}{\partial \mathbf{d}_i} \Big|_{(\mathbf{s}_{mc}, \mathbf{d}^*)} \right. \right. \\ & + \left. \left. \frac{\partial \phi_i^{(1)}(\cdot)}{\partial \mathbf{d}_i} \Big|_{(p_{mc}^0, s_{i,mc}, \mathbf{d}^*)} \right) - \frac{\partial \Phi_i^{(1)}(\cdot)}{\partial \mathbf{d}_i} \Big|_{(p_{mc}^{max}, s_{i,mc}, \mathbf{d}^*)} \right. \\ & \left. + \frac{\partial \Phi_i^{(1)}(\cdot)}{\partial \mathbf{d}_i} \Big|_{(p_{mc}^0, s_{i,mc}, \mathbf{d}^*)} \right] = 0 \quad \mathbf{d}_i = d_{0,i}, d_{1,i}, d_{2,i} \end{aligned} \quad (4.9)$$

where v_{mc} denotes the mc random valuation generated from F_V , \mathbf{s}_{mc} a vector of N random signals generated from $F_{S|V}(s|v_{mc})$, and $\frac{\partial p^0(\cdot)}{\partial d_{0,i}} \Big|_{(\mathbf{s}_{mc}, \mathbf{d}^*)}$ the derivative of the market clearing price with respect to $d_{0,i}$ evaluated at the mc random vector of signals and at the optimal vector of parameters.⁴ We follow Armantier and Sbaï and assume that V_l , the true value of the bond at auction l , $l = 1, \dots, L$, is normally distributed with mean $\mu_{V_l} = \delta_0 + \delta_1 Yield_l + \delta_2 Maturity_l$ and variance σ_V^2 , and that the conditional distribution of $s_{i,l}$, the private signal of bidder i at auction l , follows a normal distribution

⁴ First the derivative is taken with respect to $d_{0,i}$, then symmetry, $\mathbf{d}_i = \mathbf{d}^*$, is imposed.

with mean v_l (given $V_l = v_l$) and variance σ_S^2 , then the vector of parameters to estimate in this simpler model is⁵ $\boldsymbol{\theta} = (\delta_0, \delta_1, \delta_2, \sigma_V, \sigma_S)$.

Up to this point, we can find the CSE(1) ($d_{0,i} = d_0^*, d_{1,i} = d_1^*, d_{2,i} = d_2^*$) given some initial $\boldsymbol{\theta}$. To find the structural vector $\hat{\boldsymbol{\theta}}$ that fits the model with the data Armantier and Sbaï apply the simulated method of moments (SMM):

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} [\mathbf{A}'\mathbf{W}\mathbf{A}] \quad (4.10)$$

where \mathbf{W} is the weighting matrix, and

$$\mathbf{A} = \sum_{l,k,i} \mathbf{Z}_l \left(q_{i,k,l} - \frac{1}{MC_1} \sum_{m=1}^{MC_1} \phi^{(1)*}(p_k, s_{i,m}, \mathbf{d}^*) \right).$$

Here $q_{i,k,l}$ is the observed quantity in the data submitted by bidder i at price k in auction l , MC_1 is the size of the Monte Carlo approximation, $s_{i,m}$ is a simulated private signal, and \mathbf{Z}_l is a vector of exogenous variables (e.g. maturity, nominal yield) for auction l . This estimation technique minimizes the weighted distance between the quantity actually demanded by each bidder in the sample at each possible price and the equilibrium expected quantity demanded at that price, as determined by the CSE.

4.2.1.1 Armantier and Sbaï's Empirical Findings

Armantier and Sbaï estimate a structural model with data from 118 French government discriminatory bond auctions held between May 1998 and December 2000. Their model allows for asymmetric and risk averse bidders. Under these assumptions

⁵ In this case $\boldsymbol{\delta}_V = (\delta_0, \delta_1, \delta_2, \sigma_V)$ and $\boldsymbol{\delta}_S = (\sigma_S)$.

they cannot solve (4.4) for ϕ^* analytically, therefore they adopt the CSE. Their counterfactual analysis reports that a shift from the discriminatory to the uniform price format would benefit the government.

4.2.1.2 An Implementation of the Constrained Strategic Equilibrium

Armantier et al. [3] provide an example that shows the good performance of the CSE methodology for the case of a first price auction with symmetric risk neutral bidders. They show numerically that the CSE approximates very well the known BNE (see figure 1 of their paper). This subsection evaluates the performance of the CSE relative to a known BNE for the common value uniform price share auction with N symmetric risk neutral bidders.

In particular, consider again Wilson's strategy previously given in (3.4):

$$\phi^*(p, s_i, N; \alpha, \beta) = \frac{1}{N-1} - \frac{2\beta}{N(N-1)(N+\alpha)}p - \frac{2N}{N(N-1)(N+\alpha)}ps_i. \quad (4.11)$$

Fevrier et al. show that (4.11) is the unique equilibrium strategy in the class of demand functions that are linear in the signals.⁶

Compare that strategy to a CS(2) strategy of the form:

$$\phi_i^{(2)}(p, s_i, \mathbf{d}_i) = d_{0,i} + d_{1,i}s_i + d_{2,i}p + d_{3,i}s_i^2 + d_{4,i}s_ip + d_{5,i}p^2. \quad (4.12)$$

Because (4.12) is a special case of (4.11) we should expect the CSE approach to do

⁶ When $\gamma = 1$ the distribution (3.2) corresponds to the gamma distribution with parameters α and β . Using the fact that $\Gamma(z+1) = z\Gamma(z)$, when $\gamma = 1$ the optimal strategy (3.1) reduces to (4.11).

quite well and result in the following constrained optimal parameters:

$$\begin{aligned} d_0^* &\approx \frac{1}{N-1}, & d_1^* &\approx 0, & d_2^* &\approx -\frac{2\beta}{N(N-1)(N+\alpha)}, & d_3^* &\approx 0, \\ d_4^* &\approx -\frac{2N}{N(N-1)(N+\alpha)}, & d_5^* &\approx 0. \end{aligned} \quad (4.13)$$

The simpler CSE(1) approximation might also do well. In a uniform price auction with two symmetric risk neutral bidders ($N = 2$), under the constrained strategy of order 1, $\phi_i^{(1)}(p, s_i, \mathbf{d}_i) = d_{0,i} + d_{1,i}s_i + d_{2,i}p$, the maximization problem for the representative bidder $i = 1$ becomes⁷ :

$$\max_{d_{0,1}, d_{1,1}, d_{2,1}} \int_{\Theta_V} \int_{\Theta_{S_1}} \int_{\Theta_{S_2}} [(V - p^0(\mathbf{s}, \mathbf{d})) (d_{0,1} + d_{1,1}s_1 + d_{2,1}p^0(\mathbf{s}, \mathbf{d}))] d\Upsilon(V, \mathbf{s}) \quad (4.14)$$

where the market clearing price is given by

$$p^0(\mathbf{s}, \mathbf{d}) = p^0 = \frac{1 - d_{0,1} - d_{0,2} - d_{1,1}s_1 - d_{1,2}s_2}{d_{2,1} + d_{2,2}}.$$

Therefore, the first order conditions are:

$$\begin{aligned} \int_{\Theta_V} \int_{\Theta_{S_1}} \int_{\Theta_{S_2}} (V - p^0(\mathbf{s}, \mathbf{d})) \left(1 - \frac{d_{2,1}}{d_{2,1} + d_{2,2}} \right) \\ + \phi_1^{(1)}(p^0, s_1, \mathbf{d}_1) \left(\frac{1}{d_{2,1} + d_{2,2}} \right) d\Upsilon(V, \mathbf{s}) &= 0 \\ \int_{\Theta_V} \int_{\Theta_{S_1}} \int_{\Theta_{S_2}} (V - p^0(\mathbf{s}, \mathbf{d})) \left(s_1 - \frac{d_{2,1}s_1}{d_{2,1} + d_{2,2}} \right) \\ + \phi_1^{(1)}(p^0, s_1, \mathbf{d}_1) \left(\frac{s_1}{d_{2,1} + d_{2,2}} \right) d\Upsilon(V, \mathbf{s}) &= 0 \end{aligned}$$

⁷ In equation (4.3) τ was set equal to 0 and $\phi_i(p, s_i) = \phi_i^{(1)}(p^0, s_i, \mathbf{d}_i)$.

$$\begin{aligned}
& \int_{\Theta_V} \int_{\Theta_{S_1}} \int_{\Theta_{S_2}} (V - p^0(\cdot)) \left(\frac{-d_{2,1}(1 - d_{0,1} - d_{0,2} - d_{1,1}s_1 - d_{1,2}s_2)}{(d_{2,1} + d_{2,2})^2} + p^0(\cdot) \right) \\
& + \phi_1^{(1)}(p^0, s_1, \mathbf{d}_1) \left(\frac{1 - d_{0,1} - d_{0,2} - d_{1,1}s_1 - d_{1,2}s_2}{(d_{2,1} + d_{2,2})^2} \right) d\Upsilon(V, \mathbf{s}) = 0. \quad (4.15)
\end{aligned}$$

The first equation corresponds to the first order condition of the parameter $d_{1,0}$, the second one of the parameter $d_{1,1}$ and the third one of the parameter $d_{1,2}$. When we impose symmetry among the bidders, $d_{0,1} = d_{0,2} \equiv d_0$, $d_{1,1} = d_{1,2} \equiv d_1$, $d_{2,1} = d_{2,2} \equiv d_2$, and approximate the integrals using Monte Carlo we arrive at the non-linear system to solve:

$$\begin{aligned}
& \frac{1}{MC} \sum_{mc=1}^{MC} (v_{mc} - p^0(\mathbf{s}_{mc}, \mathbf{d})) \left(1 - \frac{d_2}{2d_2} \right) + \phi^{(1)}(p_{mc}^0, s_{1,mc}, \mathbf{d}) \left(\frac{1}{2d_2} \right) = 0 \\
& \frac{1}{MC} \sum_{mc=1}^{MC} (v_{mc} - p^0(\cdot)) \left(s_{1,mc} - \frac{d_2 s_{1,mc}}{2d_2} \right) + \phi^{(1)}(p_{mc}^0, s_{1,mc}, \mathbf{d}) \left(\frac{s_{1,mc}}{2d_2} \right) = 0 \\
& \frac{1}{MC} \sum_{mc=1}^{MC} (v_{mc} - p^0(\mathbf{s}_{mc}, \mathbf{d})) \left(\frac{-d_2(1 - 2d_0 - d_1 \sum_{j=1}^2 s_{j,mc})}{(2d_2)^2} + p^0(\mathbf{s}_{mc}, \mathbf{d}) \right) \\
& + \phi^{(1)}(p_{mc}^0, s_{1,mc}, \mathbf{d}) \left(\frac{1 - 2d_0 - d_1 \sum_{j=1}^2 s_{j,mc}}{(2d_2)^2} \right) = 0 \quad (4.16)
\end{aligned}$$

where

$$p^0(\mathbf{s}_{mc}, \mathbf{d}) = p_{mc}^0 = \frac{1 - 2d_0 - d_1 \sum_{j=1}^2 s_{j,mc}}{2d_2},$$

$\mathbf{s}_{mc} = (s_{1,mc}, s_{2,mc})$ and $\mathbf{d} = (d_0, d_1, d_2)$. To implement the CSE technique numerically, MC random points need to be drawn. Each random point, $mc = 1, \dots, MC$, consist of a random valuation, v_{mc} , drawn from $\Gamma(\alpha, \beta)$ and, N ($N = 2$) random signals generated from the exponential distribution with parameter v_{mc} . Results are reported for $(\alpha, \beta) = (3, 2)$ which corresponds to a gamma distribution with a mean of 1.5 and a variance of 0.75, and $MC = 10^4$.

To solve for the CSE(1) of a uniform price share auction with two bidders where $V \sim \Gamma(3, 2)$, I attempted to solve the system (4.16) in Ox using the routine SolveNLE. I tried 1331 different starting values, but none reached strong convergence. In 501 cases the program reached weak convergence.⁸ In these cases the range of each parameter, d_0, d_1, d_2 , was $(-0.04047, 0.05280)$, $(-0.00571, 0.00440)$, $(-0.03135, 0.02395)$, respectively. Figure (4.1) depicts the frequency histogram for each parameter.

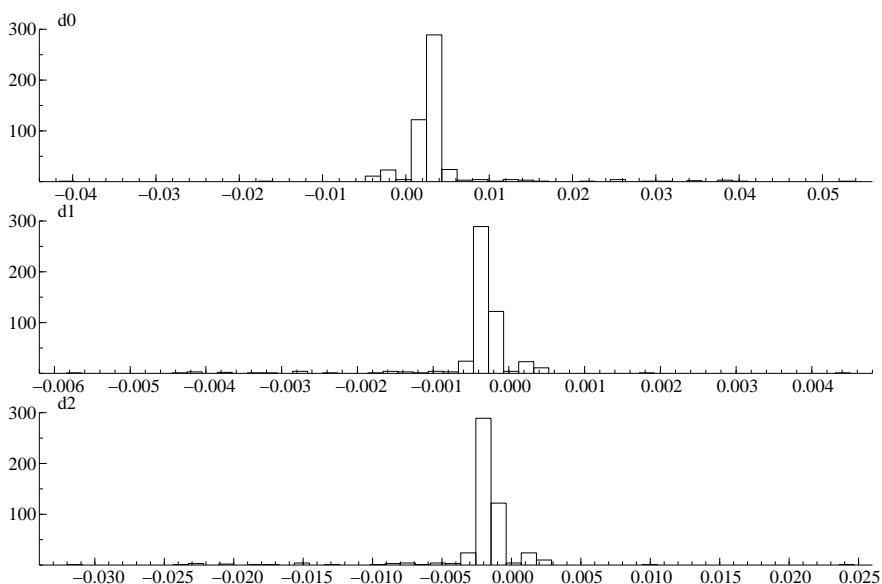


Figure 4.1: Frequency histogram for the estimated parameters when the program obtained weak convergence

To estimate the CSE(2) we have to set the maximization problem for bidder i as we did in equation (4.14) for the CSE(1), where now the bid is constrained to be as

⁸ According to Ox, weak convergence is obtained when the step length has become too small and the norm of the non-linear system of equations is less than 0.005. Strong convergence is obtained when norm of the non-linear system of equations is less than 0.0001.

(4.12), and where the market clearing price, p^0 , is defined by

$$\sum_{j=1}^2 \phi_j^{(2)}(p^0, s_j, \mathbf{d}_j) = 1.$$

Then we have to derive first order conditions⁹ for bidder i . For the CSE(1) this was system (4.15). Finally we impose symmetry across the two bidders and compute the three dimensional integral using Monte Carlo. We end up with a system of 6 non-linear non-redundant equations with 6 unknowns (for the CSE(1) this was system (4.16)). Table (4.1) shows that at least when the starting values are set to be equal (left panel) or very close (right panel) to the parameters of the BNE, the constrained strategy procedure performed well. It returned strong convergence and final values close to the BNE parameters. When the starting values are not close (for example when all the starting parameters are set equal to 1) the procedure does not converge. This is based on 30 different starting values. These results suggest the CSE is not a robust procedure due to sensitivity to starting values for the implied non-linear system of equations to be solved.

Table 4.1: Optimum Parameters for the Constrained Strategy of Order 2 ($MC = 10^6$)

	Starting Values	Optimum Parameters	Starting Values	Optimum Parameters
d_0	1	1.000700	0.9	1.000700
d_1	0	0.003911	0.00001	0.003912
d_2	-0.4	-0.384880	-0.3	-0.384880
d_3	0	0.000048	0.00001	0.000048
d_4	-0.4	-0.411390	-0.3	-0.411390
d_5	0	-0.008153	0.00001	-0.008155

⁹ Without loss of generality we can choose bidder 1 to be the representative bidder. Therefore we have to take derivatives with respect to $d_{0,1}, d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}, d_{5,1}$.

4.2.1.3 Evaluation of Armantier and Sbaï's Methodology

The advantage of the CSE technique is that it provides a controlled approximation to the intractable BNE. The method can be easily adapted to handle asymmetries, risk aversion, and uncertainty in the supply. Additionally, given the estimated structural parameters, it allows to do revenue comparisons among other formats without needing to know their optimal strategy.

For a practitioner who wants to find the approximate equilibrium of a share auction the constrained strategy approach may not be suitable since it requires as starting values almost the optimum parameters of the BNE: the ones that the practitioner wants to find. Indeed, when applied to the data from the Colombian government uniform price auctions, the method failed to converge for different starting values.

4.2.2 Two Step Estimator

Fevrier et al. propose a two-step estimator to find the vector of structural parameters, θ , of a discriminatory auction. In the first step the distribution of the observed bids, $G(\cdot)$, is estimated using a non-parametric kernel. In the second step, $\hat{\theta}$ minimizes an empirical counter part of the Euler condition:

$$\hat{\theta} = \arg \min_{\theta} \sum_{t=1}^L m_t^2(p_t^0; \theta).$$

Here L is the number of auctions in the data set, p_t^0 is the observed market clearing

price of auction t , and

$$m_t(p_t^0; \boldsymbol{\theta}) \equiv \frac{1}{L} \sum_{l=1}^L K \left(\frac{N_t - N_l}{h_N} \right) * \left(\prod_{z=1}^{N_Z} K \left(\frac{Z_{z,t} - Z_{z,l}}{h_{Z_z}} \right) \right) * \mathbf{1}(p_t^0 \leq p_t^0) * ((N_l - 1)(\Theta - p_t^0) - (p_t^0 - p_l^0)). \quad (4.17)$$

Where

$$\Theta = \frac{\Gamma(N_l + \alpha_l + 1/\gamma)}{\Gamma(N_l + \alpha_l) \left(\beta_l + \sum_{i=1}^{N_l} \beta_l \left[\frac{1}{\hat{G}^{1/\alpha_l}(q_{i,p_t^0,l}|N_l, \mathbf{Z}_l)} - 1 \right] \right)^{1/\gamma}},$$

$K(\cdot)$ is a kernel, N_Z is the number of exogenous variables, h_N, h_{Z_z} are bandwidth parameters, $\mathbf{Z}_l = (Z_{1,l}, \dots, Z_{N_z,l})$ is the vector of exogenous variables for auction l , and $q_{i,p_t^0,l}$ is the observed quantity submitted by bidder i at price p_t^0 in auction l . Under the assumption that V_l is distributed as (3.2) with parameters $\alpha_l = (1, \mathbf{Z}_l) \cdot \boldsymbol{\alpha}$, $\beta_l = (1, \mathbf{Z}_l) \cdot \boldsymbol{\beta}$ and γ , and that the signals follow an exponential distribution with parameter v_l^γ , Θ corresponds to $E\{V_l | S_1^l = s_1^l, \dots, S_{N_l}^l = s_{N_l}^l\}$.

Setting $\tau = 1$ in equation (4.4) we find that

$$E \left\{ (V - p) H_p(p; V, \phi_i(p, s_i)) - H(p; V, \phi_i(p, s_i)) \middle| S_i = s_i \right\} = 0 \quad (4.18)$$

is the Euler equation for the discriminatory auction. Fevrier et al. show that (4.18) can be rewritten as

$$E\{(N-1)[E\{V | S_1 = s_1, \dots, S_N = s_N\} - p] \mathbf{1}(P^0 \leq p)\} - E\{(p - P^0) \mathbf{1}(P^0 \leq p)\} = 0. \quad (4.19)$$

The first expectation is with respect to S_1, \dots, S_N (the random variable P^0 only depends on the signals); the second expectation is with respect to V given the signals; and

the third is with respect to P^0 . A comparison of (4.19) and (4.17) shows the latter is indeed an empirical counterpart of the Euler condition.

4.2.2.1 Fevrier et al.'s Empirical Findings

Fevrier et al. estimate a structural model with data from 45 French government discriminatory bond auctions held during January 2005 and December 2005. The secondary market price, the nominal yield, and the maturity of the bond are the exogenous variables included in the estimation. Evaluating Fevrier et al.'s estimates of α and β at their characteristics' sample mean, I find that $V^{12.28} \sim \Gamma(3022, 8.4 * 10^{-22})$. At a first glance their estimates might seem surprising because they imply that $V^{12.28}$ has an incredible huge variance $(3022/(8.4 * 10^{-22})^2)$ and a mean of $3.6 * 10^{24}$. However, the implied mean of V is 111, which is close to the bonds' prices that we see in the secondary market.

Given the estimated structural parameter vector, $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$, using the known optimal strategy for the uniform price auction, equation (3.1), and the fact that the signals can be recover from the non-parametric distribution of bids, $\hat{s}_{i,t} = F_{S|Z}^{-1}(1 - \hat{G}(q_{i,p_t^0,t}|N_t, \mathbf{Z}_t; p_t^0)|\mathbf{Z}_t; \hat{\theta})$, Fevrier et al. calculate a hypothetical market clearing price for each auction under the uniform price format. From this they obtain the hypothetical revenue for the government if it had used this format. They find that the French government would had received 5% less under the uniform price format.

4.2.2.2 Evaluation of Fevrier et al.'s Methodology

The main advantage of this methodology is that it does not require an explicit solution for the equilibrium to estimate parameters of the model. One drawback is that by combining non-parametric (step 1) and parametric (step 2) estimation it may be seen as suffering from the worst of both approaches. From the non-parametric side the quality of the estimates depends crucially on the precision of the non-parametric estimation. From the parametric side, the estimates are not distribution free. If the researcher wants to compare revenue across formats the parametric framework has to provide an optimal known strategy for the hypothetical formats. Further, to recover the private signal from the observed demand step function using the estimated distribution of bids, may not result in a unique signal that rationalizes the multiple pairs submitted by the bidder, as assumed by the model. Fevrier et al. downplay this issue by recovering the signals from just one point: the market clearing price of each discriminatory auction. Signals derived from other prices would differ. In effect rather than deriving implied signals from the estimates it is necessary to formulate an estimate of the signals based on the distribution implied by different price points.

Fevrier et al.'s methodology was developed for discriminatory auctions. To apply it to data from uniform price auctions new Euler conditions must be derived. Additionally, to compare actual revenue under the uniform price auction with hypothetical revenue under a discriminatory auction, a strategic equilibrium for the discriminatory auction is required and no such solution is currently known.

4.3 Empirical Work under the IPVP

The starting point for estimation of share auctions within the IPVP is Hortaçsu[11]. Models estimated by Kang and Puller, and Kasl extend Hortaçsu's non-parametric methodology. This section states the general framework of the IPVP for share auctions assuming continuous demand function. The next subsection presents Hortaçsu's resampling method. Subsection 4.3.2 illustrates how Kang and Puller estimate the marginal valuations for a uniform price auction using the resampling method. Subsection 4.3.3 presents Kasl's work. He estimates the bidders' valuations in a uniform price auction using the resampling method also but takes into account that bidders submit a step demand function. For these last two subsections, marginal valuations for the bidders that participated in the Colombian government auction, between March and August 2007, are estimated.

In the IPVP setting with symmetric bidders each player receives a private signal, s_i , that determines his marginal valuation, $v(q, s_i)$, from winning a given share, q , of the object. As in the CVP, bidder i 's demand function is modeled as a function of the price and the private signal, $\phi(p, s_i)$. Assuming continuously differentiable downward sloping demand functions, bidder i 's maximization problem is:

$$\max_{\phi_i} \left\{ \left(\int_0^\infty \left(\int_0^{\phi_i(p, s_i)} v(q, s_i) dq - p\phi_i(p, s_i) - \tau \int_p^{p_i^{max}} \phi_i(p, s_i) dp \right) \frac{\partial H(\cdot)}{\partial p} \right) \Big| s_i \right\}, \quad (4.20)$$

where $H(p; \phi_i(p, s_i)) = Pr(\sum_{j \neq i} \phi_j(p, S_j) \leq 1 - \phi_i(p, s_i))$.

Comparing (4.20) with (4.2) two differences emerge. First, under the CVP it has

been typically assumed that the unknown marginal value of the good is constant across quantity. This is not the case for the private marginal valuation within IPVP. Second, within the IPVP the expectation with respect to the valuation (the outer expected value in (4.2)) is not needed because the valuation is private and known by bidder i .

Following a similar procedure as the one outlined in Appendix E for the CVP, the Euler-Lagrange necessary condition for the above optimization problem is:¹⁰

$$(v(\phi_i(p, s_i)) - p)H_p(p; \phi_i(p, s_i)) - \tau H(p; \phi_i(p, s_i)) - \phi_i(p, s_i)(\tau - 1)H_{\phi_i}(p; \phi_i(p, s_i)) = 0.$$

For the uniform price auction ($\tau = 0$) this reduces to

$$v(\phi_i(p, s_i)) = p - \frac{H_{\phi_i}(p; \phi_i(p, s_i))}{H_p(p; \phi_i(p, s_i))} \phi_i(p, s_i), \quad (4.21)$$

and for the discriminatory price auction ($\tau = 1$),

$$v(\phi_i(p, s_i)) = p + \frac{H(p; \phi_i(p, s_i))}{H_p(p; \phi_i(p, s_i))}. \quad (4.22)$$

These two equations state that in equilibrium bidders will never bid a price that is higher than their valuation, $v(\phi_i(p, s_i)) \geq p$. In the uniform price auction bidders shade their bid by an amount $\frac{H_{\phi_i}(\cdot)}{H_p(\cdot)} \phi_i(p, s_i)$. The denominator can be interpreted as the “density” of the market clearing price when bidder i bids $\phi_i(p, s_i)$. The numerator is the shift in the probability distribution of the market clearing price due to a change in bidder i 's bid. This latter effect is always negative since increasing a bid increases the market clearing price and therefore decreases the probability that the market clearing price is lower than a given p . For the discriminatory auction the shading factor is $\frac{H(\cdot)}{H_p(\cdot)}$.

¹⁰ Because bidders are symmetric the Euler-Lagrange condition has to be evaluated at $\phi_i(p, s_i) = \phi(p, s_i)$.

4.3.1 The Resampling Method

The optimality conditions (4.21) and (4.22) show that, to recover the bidder's valuation, the market clearing price distribution, $H(p; \phi(p, s_i))$, needs to be estimated. To do so Hortaçsu proposes the following resampling method for each bidder in each auction:

- (1) Fix bidder i among the total N_l bidders in auction l .
- (2) From the sample of N_l step demand functions in the auction, draw a random sample of $N_l - 1$ step demand functions, with replacement, giving equal probability of $1/N_l$ to each of them in the original sample.
- (3) Construct the residual supply function generated by these $N_l - 1$ resampled step demand functions.
- (4) Intersect the residual supply function with bidder i 's bid function to find the market clearing price.
- (5) Repeat steps 2, 3, and 4 a large number of times, B .

This resampling procedure generates B market clearing prices conditional on bidder i 's step demand function. One can then estimate the market clearing price distribution by counting the frequency with which a given p remained above the resampled market clearing prices.

Hortaçsu recovers the marginal valuation for each submitted pair for each bidder in a discriminatory auction, using the following discretized version of the optimality condition (4.22):¹¹

$$v_i(q_{i,k,l}, s_i) = p_{i,k,l} + \frac{H(p_{i,k+1,l}; \mathbf{q}_{i,l})(p_{i,k,l} - p_{i,k+1,l})}{H(p_{i,k,l}; \mathbf{q}_{i,l}) - H(p_{i,k+1,l}; \mathbf{q}_{i,l})}, \quad (4.23)$$

where $\mathbf{q}_{i,l} = (q_{i,1,l} < \dots < q_{i,K,l})$ is the submitted vector of quantities by bidder i in auction l and $q_{i,k,l}$ is the quantity submitted by bidder i at the corresponding price $p_{i,k,l}$ ($\mathbf{p}_{i,l} = (p_{i,1,l} > \dots > p_{i,K,l})$) in auction l . It is important to note that $k = 1, \dots, K$ might differ by bidders in an auction, as it represents the number of pairs submitted by a bidder. In terms of the example setting presented in Chapter 2, for bidder 1, $K = 3$, $\mathbf{q}_1 = (0.1, 0.2, 0.3)$, $\mathbf{p}_1 = (102, 101, 99)$, whereas for bidder 3, $K = 2$, $\mathbf{q}_3 = (0.3, 0.6)$, $\mathbf{p}_3 = (102, 98)$.

4.3.1.1 Hortaçsu's Empirical Findings

With the recovered marginal valuations from 27 discriminatory auctions held by the Turkish government, Hortaçsu compares the actual revenue received by the government with the hypothetical revenue of a uniform price auction in which the bids are the estimated marginal valuations (truthful bidding). He finds that the discriminatory format used generated a higher revenue than this hypothetical best case for the uniform auction.¹² Therefore, he concludes that the discriminatory auction is better for the

¹¹ Recall that equation (4.22) was obtained assuming that bidders submit a continuous demand function and not a step demand function as in reality.

¹² Remember that under the assumption of continuously differentiable downward sloping demand functions, bidders in equilibrium will never bid a price higher than their marginal valuation. Therefore the highest revenue the seller can receive is when bidders bid their valuation.

government in terms of revenue. It is important to note that if his result had been the opposite, that the hypothetical best case gives a higher revenue, he would not have been able to rank the two mechanisms in terms of expected revenue.

4.3.1.2 Implementation of the Resampling Method

Figure 4.2 illustrates the resampling procedure with $B = 2$ for a bidder (bidder #3) who participated in the Colombian government bond auction held on March 14, 2007 for the bond that expired on December 12, 2008. The actual market clearing price in this auction was 105.997. This is the point where the actual residual supply intersects the bidder's step demand function. Figure 4.3 depicts the histogram of resampled market clearing prices for the same bidder using $B = 5000$. Here we can see that the probability of a market clearing price below 105.7 is very low as it is the probability of a market clearing price higher than the first bid price submitted by bidder 3 (106.157).

Table (4.2) shows the market clearing price distribution function, obtained with the resampling procedure ($B = 5000$), that bidder #3 faces for each price submitted in the auction.¹³ It also illustrates one of the potential difficulties of applying Hortaçsu's resampling methodology. Suppose that bidder 3 demands quantities at the two lowest prices. Using equation (4.23) to recover the marginal valuation for the quantity demanded at the price 105.543 is not possible because the denominator is equal to zero. The researcher can assume that the valuation is equal to the price since in a discriminatory

¹³ In this auction there were 18 different prices. Bidder 3 submitted three pairs, therefore $K = 3$ for their case.

auction the incentive to shade bids is low for values well below the expected market clearing price.

4.3.1.3 Evaluation of Hortaçsu's Methodology

The biggest advantage of Hortaçsu's methodology is that it is easy to implement. Additionally, it can easily accommodate asymmetries, uncertainty in the supply and stochastic participation.

Its main drawback is the lack of precise estimates in the case where the number of bidders is small. However, Hortaçsu does not quantify what this means; he only states that the resampling method should not be performed if for example $N = 3$. In Hortaçsu's data set the mean number of bidders per auction is 69, which is a big number compared with other data sets that have been studied (See Appendix A). Another drawback is that it can not be extended to the CVP.

4.3.2 The Resampling Method for the Uniform Price Auction

Kang and Puller extend Hortaçsu's methodology to the uniform price auction. Assuming that the marginal valuation function is a step function which has a constant value of $v(q_k)$ on (q_{k-1}, q_k) , Kang and Puller show that bidder i 's valuations can be

recovered from the first-order condition:

$$v_i(q_{i,k,l}, s_i) = \frac{p_{i,k,l}(H(p_{i,k,l}; \mathbf{q}_{i,l}) - H(p_{i,k+1,l}; \mathbf{q}_{i,l})) - H_{q_{i,k,l}}(\cdot)(p_{i,k-1,l}q_{i,k-1,l} - p_{i,k,l}q_{i,k,l})}{H(p_{i,k,l}; \mathbf{q}_{i,l}) - H(p_{i,k+1,l}; \mathbf{q}_{i,l}) + H_{q_{i,k,l}}(\cdot)(q_{i,k,l} - q_{i,k-1,l})} \quad \forall k. \quad (4.24)$$

The empirical procedure assumes: i) $p_{i,0,l} = p_{i,1,l}$, ii) $q_{i,0,l} = 0$, and iii) $H(p_{i,K+1,l}) = 0$.

This last assumption means that the bidder assigns a probability equal to zero to the event that the market clearing price is lower than the lowest price that she submitted, which also explains why the bidder did not submit more bidding quantities at lower prices.

To estimate the market clearing price distribution and its derivative, Kang and Puller use Hortaçsu's resampling method. Specifically, to estimate $H_{q_{i,k,l}}(p_{i,k,l}; \mathbf{q}_{i,l}) = \frac{\partial H(p_{i,k,l}; \mathbf{q}_{i,l})}{\partial q_{i,k,l}}$, they use the fact that the distribution of the market clearing price can be represented as a function of the sum of $N - 1$ rivals' bid quantities which are i.i.d. random variables from bidders i 's perspective. That is:¹⁴

$$H(p_k; \mathbf{q}_i) = Pr(P^0 \leq p_k | \mathbf{q}_i) = Pr(q_{i,k} \leq 1 - \sum_{j \neq i} q_j(p_k)).$$

In words, given a price p_k , the probability that the market clearing price is less than or equal to this price is the same as the probability that bidder i 's demanded quantity at p_k , $q_{i,k}$, is less than their residual supply at p_k , $RS_i(p_k)$. Let $R(q_{i,k})$ and $r(q_{i,k})$ denote

¹⁴ Omitting the auction index, l .

the c.d.f and p.d.f of $RS_i(p_k)$ conditional on \mathbf{q}_i , then $H(p_k; \mathbf{q}_i) = 1 - R(q_{i,k})$. Hence,

$$\begin{aligned} H_{q_{i,k}}(p_k; \mathbf{q}_i) &= \frac{\partial}{\partial q_{i,k}}(1 - Pr(RS_i(p_k) \leq q_{i,k})) \\ &= \frac{\partial}{\partial q_{i,k}}(1 - R(q_{i,k})) = -r(q_{i,k}). \end{aligned}$$

Because the resampling procedure provides B resampled residual supply functions, we can estimate $r(q_{i,k})$ at any price using kernel estimation. Therefore,

$$\hat{H}_{q_{i,k}}(p_k; \mathbf{q}_i) = -\frac{1}{Bh} \sum_{i=1}^B K\left(\frac{RS_i(p_k) - q_{i,k}}{h}\right).$$

4.3.2.1 Kang and Puller's Empirical Findings

Kang and Puller recover the marginal valuations of 30 Korean government securities auctions for the three year bond. These auctions were conducted between September 1999 and April 2002. In 20 of them the uniform price format was used while in 10 the discriminatory format was employed. For the discriminatory auctions Kang and Puller follow the same procedure as Hortaçsu. The only difference is that instead of using equation (4.23) to recover the valuations, they use the subtle modification proposed by Hortaçsu in Appendix 8.2.

Because Kang and Puller have data from both formats, they are able to conclude that in terms of revenue and efficiency the discriminatory format outperforms the uniform by an amount less than 1%.

4.3.2.2 Implementation of Kang and Puller's Methodology

Applying Kang and Puller's methodology, I estimate the marginal valuation function of each bidder in each of the 49 uniform price auctions that the Colombian government ran between March and August, 2007. For the kernel estimation I use a normal kernel with bandwidth equal to the standard deviation of the resampled residual supplies multiplied by $1.06 * B^{-1/5}$.

Figure 4.4 shows the estimated marginal valuation and the bidding function for bidder #3, who attended the Colombian government auction held on March 14, 2007 for the bond that expired on December 12, 2008 (the same auction and bidder I used to show Hortaçsu's resampling method). The difference between the bid price and the marginal valuation measures by how much the bidder shaded the bid.

In uniform price auctions, the bidder has an incentive to bid below their valuation for the x^{th} unit of a good if that bid has some probability of lowering the market clearing price that is paid for all other $x - 1$ units. In accordance, the figure shows that the bidder did not shade her bid at the highest price. This characteristic is present in the estimation of all the 473 bidders' demands in the data set.

From 1231 bidding pairs I am able to recover 1230 valuations; the only valuation that I cannot recover is one for which $\hat{H}(p_{i,k,l}; \mathbf{q}_{i,l}) = \hat{H}(p_{i,k+1,l}; \mathbf{q}_{i,l})$ and $\frac{\partial \hat{H}(p_{i,k,l}; \mathbf{q}_{i,l})}{\partial q_{i,k,l}} = 0$. From the 1230 recovered valuations, 272 are lower than the corresponding bid price. This does not make sense since the valuation at each bid-point should be above the corresponding price. This is violated in 266 cases when

$\hat{H}(p_{i,k,l}; \mathbf{q}_{i,l}) = \hat{H}(p_{i,k+1,l}; \mathbf{q}_{i,l})$. Marginal valuations are estimated using equation (4.24), which is the optimality condition of the bidder's expected profit:¹⁵

$$\begin{aligned} & \sum_{k=1}^K [Pr(MPC = p_{i,k}) * (\text{payoff if } MCP = p_{i,k})] \\ &= \sum_{k=1}^K [H(p_{i,k}; \vec{q}_i) - H(p_{i,k+1}; \vec{q}_i)] \left(\int_0^{q_{i,k}} v(q, s_i) dq - p_{i,k} q_{i,k} \right). \end{aligned}$$

When $H(p_{i,k}; \mathbf{q}_i) = H(p_{i,k+1}; \mathbf{q}_i)$ the submitted bid at step k does not affect the expected payoff. Without any other explanation, for the other 6 cases the inconsistency comes from employing the resampling method with a small number of bidders. When $\hat{v}_i(q_{i,k,l}) < p_{i,k,l}$, the marginal valuation can be reset equal to price. Kang and Puller faced this problem for 33% of their bid-points.

4.3.2.3 Evaluation of Kang and Puller's Methodology

As Kang and Puller's methodology is a straight forward extension of Hortaçsu's methodology to the uniform price auction, it inherits the advantages and disadvantages of the latter.

If the practitioner has data from a uniform price auction, even though the marginal valuations can be recovered, it is not possible to do a revenue comparison with the discriminatory auction because truthful bidding under a discriminatory format will always give more revenue than any bidding with a uniform format payment. Fortunately, Kaslt [14] comes up with a solution to this problem, which I explain and implement in the next subsection.

¹⁵ In this equation the subscript l is omitted.

4.3.3 The Resampling Method with Step Bidding Functions

Kastl extends Hortaçsu's procedure by explicitly taking into account that bidders submit a step demand function. His analysis focuses on strategic decisions of the bidders on where to locate each step; the decision implicitly depends on the location of other steps.

To rationalize the fact that bidders use fewer steps than what they are allowed to and that the number of steps differs across auctions and bidders, Kaslt assumes that for each bid-point bidders face a cost of submission that might differ across bidders and/or time. Kaslt develops a model that incorporates these features and characterizes its equilibrium.

Under the assumption that bidder i 's valuation is strictly increasing in s_i and weakly decreasing and continuous in q ,¹⁶ Kaslt characterizes a set of conditions for the uniform price auction¹⁷ that any BNE has to satisfy for every step k in the K_i -step demand function.¹⁸ The conditions account for ties among the bidders and ensure that a local perturbation of the bid-price at step k is not optimal. However, his empirical exercise is just based on the condition that rules out profitable local perturbations of the

¹⁶ This allows for a constant (flat) marginal valuation schedule as the one assumed under the CVP.

¹⁷ Kaslt [15] characterizes the set of optimal conditions for the discriminatory auction. A practitioner that has data from a discriminatory auction can estimate the marginal valuations using Hortaçsu's resampling method and equation (3) of that paper.

¹⁸ For ease of notation the auction index l is dropped.

bid-quantity at step k and ignores ties. This condition is:

$$v(q_k, s_i) = E(P^0 | p_{i,k} > P^0 > p_{i,k+1}) + \frac{q_{i,k}}{Pr(p_{i,k} > P^0 > p_{i,k+1})} \frac{\partial E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1})}{\partial q_{i,k}}. \quad (4.25)$$

The empirical estimation taking ties into account is possible but an additional assumption on the marginal valuation curve has to be imposed. In (4.25) the cost that bidder i incurs for submitting K_i steps does not appear because it does not affect the location of the steps (conditional on K_i).

To estimate the different pieces of (4.25), one employs Hortaçsu's resampling method to obtain B resampled market clearing prices. To estimate $E(P^0 | p_{i,k} > P^0 > p_{i,k+1})$ take the mean of the resampled market clearing prices that are greater than $p_{i,k+1}$ and smaller than $p_{i,k}$. To obtain $Pr(p_{i,k} > P^0 > p_{i,k+1})$ count the number of market clearing prices that are in the range and divide it by B . To construct the estimator of the derivative, Kaslt suggests the following:

$$\frac{\partial E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1})}{\partial q_{i,k}} = \frac{E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1}, q_{i,k} - \epsilon) - E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1})}{\epsilon} \quad (4.26)$$

where $E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1}) = E(P^0 | p_{i,k} \geq P^0 \geq p_{i,k+1}) Pr(p_{i,k} \geq P^0 \geq p_{i,k+1})$, and the two factors on the right hand side are estimated as before.¹⁹

$E(P^0; p_{i,k} \geq P^0 \geq p_{i,k+1}, q_{i,k} - \epsilon)$ is the expected value of the market clearing price being between $p_{i,k}$ and $p_{i,k+1}$ when the quantity bid by bidder i at step k is perturbed

¹⁹ Note that here we do not have the strict sign that we had before.

to be $q_{i,k} - \epsilon$. To estimate this joint expected value we have to obtain a new set of resampled market clearing prices taking into account that bidder i 's demand vector is the perturbed one.

Kastl also proposes an alternative method to evaluate the performance of the employed auction without having to obtain counterfactual strategies. To evaluate the performance in terms of revenue he proposes to estimate the bidder's expected utility²⁰ and compare it with the bidder's actual utility. Under the equilibrium assumption, the observed bid function of each bidder should be a best response to the equilibrium strategies of other bidders; therefore, the estimated expected utility should deliver the highest utility the bidder can obtain. The expected utility can be obtained using the distribution of the market clearing price and the assumption that the marginal valuation is a step function. Thus, the expected utility of bidder i is estimated by:

$$\frac{1}{B} \sum_{b=1}^B \left(\sum_{k=1}^{K_i} (\hat{v}(q_k, s_i) - p_b^0)(q_k - q_{k-1}) \mathbf{1}(p_k \geq p_b^0) \right), \quad (4.27)$$

where p_b^0 is the b^{th} market clearing price obtained with the resampling procedure.²¹

4.3.3.1 Kastl's Empirical Findings

Kastl recovers the valuations from 28 uniform price auctions held by the Czechoslovakian government for the 3-month bill. To circumvent the problem of lack of precision when employing the resampling method with a small number of bidders,

²⁰ In equation (4.20) the bidder is maximizing the expected utility. For the uniform price auction set $\tau = 0$.

²¹ Recall that $p_0 = p_1$ and $q_0 = 0$.

Kastl pools 4 auctions for each resampling set.

Under the assumption of a constant marginal valuation step function, Kastl concludes that bidders do not have enough (local) market power around the expected market clearing price to adversely affect the auction's revenue. That is, the actual market prices are not far from those obtained under truthful bidding. The mean difference between the prices is 0.5 basis points.

In terms of revenue and efficiency Kastl reports that the uniform price auction performs well. He estimates that the auction failed to extract at most 0.03% of the expected surplus while implementing an allocation resulting in almost all of the efficient surplus—the ratio between actual utility of each bidder and the utility obtained under an efficient allocation is close to 1.

4.3.3.2 Implementation of Kastl's Methodology

Figure 4.5 illustrates the estimated marginal valuation using Kastl's methodology, for each point that bidder #3 submitted in the auction held by the Colombian government on March 14, 2007. To make this figure comparable with Figure (4.4), I assume that the number of bidders that participate in the auction is common knowledge.²² Aside from this figure all the other results that I present in this subsection are done assuming that the number of potential bidders is common knowledge among the bidders. In government securities auctions, this assumption is more realistic, given

²² In this auction from 14 potential bidders (financial institutions pre-approved to participate in the government auctions for 2007) 10 attended it.

that bidders do not meet for the auction (bids are sent electronically); that usually (e.g. Canada, Colombia, Czechoslovakia, Korea) all participants have to be registered with the auctioneer; and the list of registered participants is publicly available. Following Kaslt and Kang and Puller,²³ I assume that any bidder for whom I do not observe a bid function in a given auction submitted a bid of zero for any quantity, and I include such a bid in the sample from which I resample.

Figure 4.5 shows that for the last step bidder #3 submitted a price that is above their marginal valuation. As (4.21) shows this is not equilibrium behaviour under the assumption that bidders submit a differentiable downward sloping bid. But, when we take into account that bidders submit a finite number of points as their bids, the need to “bundle” bids for several units together introduces new trade-offs. A higher bid increases the probability of winning the bundle, but might imply a loss on the last unit of the bundle. If the gain on the other units of the bundle is higher than this loss, then a rational bidder may submit a bid higher than their marginal valuation for that last unit.

Following Kaslt’s estimation²⁴ I recover the valuation for each bid-point submitted by each bidder in each auction in the data set that I am using. I find that due to the bundling effect in 658 out of 1231 bid-points, the bidders bid higher than the marginal valuation. Additionally, 49 estimated marginal valuations violate the assumption that they are weakly decreasing in the quantity, $v(q_k, s_i) > v(q_{k-1}, s_i)$; for those cases I set

²³ In Kang and Puller estimation they assume that the number of bidders that participate in the auction is common knowledge among the bidders. But, as a robust check they redo their estimation assuming that the number of potential bidders is the one that is common knowledge. They find that their revenue and efficiency results are very similar under both settings.

²⁴ In equation (4.26) I set $\epsilon = 0.01$.

the marginal valuation to be equal to the previous estimated marginal valuation, that is $v(q_k, s_i) = v(q_{k-1}, s_i)$.

Finally, when a bidder submits bid-points at p_k and p_{k+1} , and no other bidders submit bid-points between these two prices, the valuation for step k , $v(q_k, s_i)$, cannot be recovered using (4.25). In those cases the conditional expected value is not defined and the probability of the event that the market clearing price is between these two prices is zero. To circumvent this problem the researcher can set the valuation equal to the price.²⁵ If a bid step can never set the market clearing price, then the bidder may not shade the bid and bid at the marginal value. In my estimation I had to set 467 marginal valuations to be equal to the price.²⁶

Given the marginal valuation point estimates for each auction, $\hat{v}(q_k, s_i)$, and assuming that the marginal valuation is a step function which has a constant value of $\hat{v}(q_k, s_i)$ on (q_{k-1}, q_k) , I estimate the market clearing price under truthful bidding. In 40 of 49 auctions, the market clearing price under truthful bidding is lower than the actual one. On average the absolute difference between these two values is 0.1. In comparison with Kastl's result I estimate that in Colombia bidders have more market power.

²⁵ In an email communication, Kastl stated that another way to avoid the problem is to define a price grid that is finer than the actual one (for example 1/4 of a basis point) and then estimate the distribution of the market clearing price on that grid non-parametrically using kernels. In this way there will be some mass strictly between p_k and p_{k+1} .

²⁶ Note that the precision of my estimates for the marginal valuation might be low, given that the numbers of bidders in the Colombian auctions is small and that I am not pooling data across auctions to avoid potential measurement biases due to the presence of unobserved (to the econometrician) factors that vary from auction to auction.

In terms of revenue, on average, the Colombian government failed to extract at most 0.02% of expected surplus²⁷. For 203 of 473 bid demands, the expected utility is negative, with an average of -0.005. Ignoring the lack of precision in my estimates, this indicates the data are not consistent with the optimally conditions associated with the assumed functional form for the marginal valuation.

To evaluate the performance of the auction in terms of efficiency we should compare the actual utility of each bidder with the utility obtained under an efficient allocation. In contrast to Kaslt's results, this ratio is not close to one, which indicates that the actual allocation is inefficient. Since actual utility is negative in 11 auctions, the ratio (actual utility/utility under the efficient allocation) ranges from -1.47 to 3.20, with a mean of 0.35 and a standard deviation of 0.73. Once again, ignoring the lack of precision of my estimates, these results indicate that the assumption of the valuation function as a step function is inadequate.

4.3.3.3 Evaluation of Kastl's Methodology

The advantage of Kastl's work is that it gets closer to reality as it takes into account that bidders submit a step demand function. Additionally, it proposes an alternative mechanism to evaluate the performance of the auction without having to obtain counterfactual strategies. This evaluation can be implemented for data from both uniform price and discriminatory auctions.

²⁷ This average does not take into account the negative expected utilities.

While in Kang and Puller the assumption that the marginal valuation is a step function which has a constant value of $\hat{v}(q_k, s_i)$ on (q_{k-1}, q_k) is essential to estimate the marginal valuation, here is not; it is necessary to do counterfactuals.

Given that it employs Hortaçsu's resampling method, the estimation suffers from lack of precision when a small number of bidders attend the auction.

4.4 The Methodologies Compared

Most countries sell their public debt through a discriminatory or a uniform price auction, in which bidders submit a step demand function. In either type of auction, the market clearing price is established as the highest price at which demand equals or exceeds the supply. When the demand at the market clearing price is higher than the supply, the bid pairs at the market clearing price are rationed. Aside from the pricing rule used in the auction, another difference among countries is related to the supply. In some countries the supply is fixed while in others it has some uncertainty. This uncertainty arises because non-competitive bids are allowed, because the government has some discretion to modify it, or because the government announces a bracket for the supply.

- Bidders: which is the bidders' attitude towards risk? Are bidders symmetric or asymmetric? Do bidders know the number of other bidders attending the auction? These are the three key questions that the researcher needs to answer to set up and estimate an auction model.

First of all, Armantier and Sbaï's work is the only one in which bidders are risk averse; as shown, however, their methodology is also applicable to risk neutral bidders. In all the other works bidders are risk neutral and the methodologies cannot be easily adapted to account for risk aversion.

Second, all the models and the methodologies were presented and implemented assuming bidders are symmetric. Nonetheless, all methodologies can be easily adapted to account for asymmetric groups of bidders²⁸ —where bidders are asymmetric among groups but symmetric within the group. When modeling asymmetry the practitioner faces two problems: i) the asymmetry across bidders has to be defined exogenously by the researcher; in other words, the researcher has to establish how to divide the bidders, and ii) if the researcher is using the resampling method for estimation, the division of bidders in asymmetric groups reduces the number of bidders in each resampling set, hence reducing the precision of the marginal valuation estimates.

Finally, Armantier and Sbaï and Fevrier et al. assume that the numbers of bidders that attend the auction is common knowledge among the participants.

Given that bidders usually submit their demands electronically, that the list of admitted participants in government securities auctions is publicly available,

and that not all of them attend all the auctions, a more suitable assumption is

²⁸ In fact all methodologies except Fevrier et al.'s estimate an asymmetric share auction model either as their objective estimation or as a robustness check. In Fevrier et al.'s case they mention that their estimation method can easily be adapted to the case of asymmetric bidders. This is because the Euler condition (4.19) is still valid in the asymmetric setup. Under asymmetry, the distribution function of the bids would have to be estimated separately for each subgroup of identical bidders.

that the number of potential bidders is the one that is common knowledge. As shown in Section 4.3.2, Hortaçsu's resampling method can easily be adjusted to either assumption.

- **Step Demand:** In real-world government securities auctions bidders submit a step demand function. However, Kaslt is the only one that considers this in the model. In Kaslt's model each bid pair is costly to submit, and hence, in equilibrium, bidders submit finite number of points as their bids. Kaslt states that the discrete bidding can have important consequences for empirical analysis. In contrast, all the other works assume a bid function that is continuously differentiable and from which the econometrician observes few points.
- **Rationing:** In real-life government securities auctions, rationing is done in a pro-rata fashion. With the exception of Kaslt's work, the other papers do not consider rationing. Kaslt incorporates rationing in his model by including the pro-rata rule, but his estimation is done assuming that at most one bidder is rationed. To allow for more bidders being rationed additional assumptions on the marginal valuation function are required. In the case of Armantier and Sbaï, and Fevrier et al., on the other hand, the probability of rationing is zero because demands are assumed to be continuous. Hortaçsu, and Kang and Puller avoided the complication posed by rationing by defining the market clearing price as the price at which the total demand falls just short of the total supply.
- **Supply:** All methodologies can be implemented with or without uncertainty in

the supply. In the cases where there is uncertainty, the supply becomes a random variable, therefore the expected utility has to account for this new dimension of uncertainty.

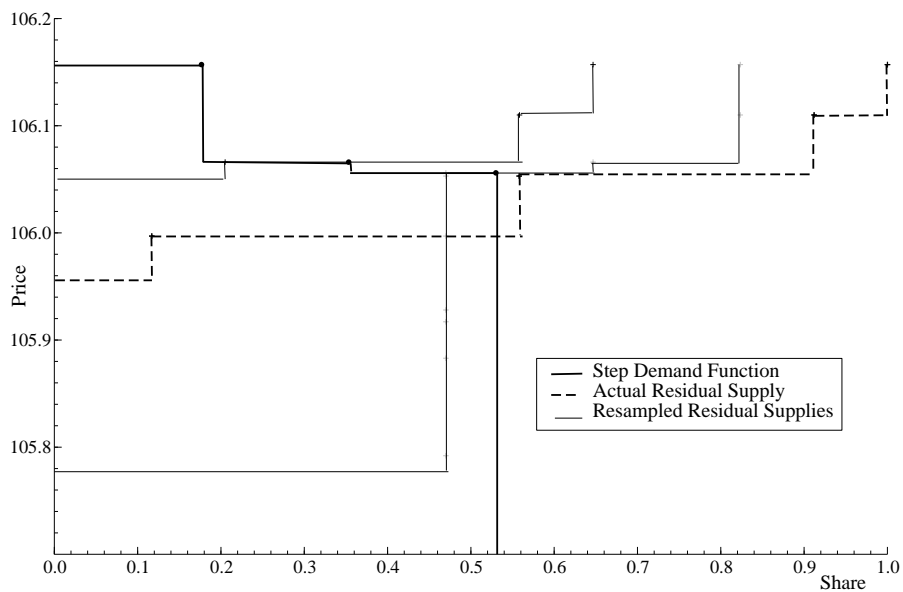


Figure 4.2: Resampling Procedure. ($B=2$)

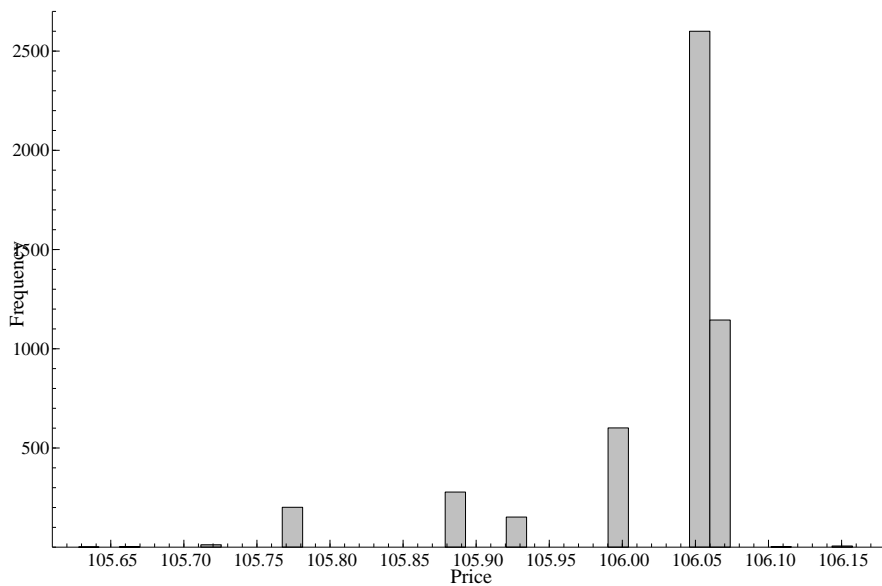


Figure 4.3: Frequency histogram of the resampled market clearing price ($B=5000$)

Table 4.2: Estimated Distribution for Bidder #3

Price	$H(p, \vec{q}_3)$
106.157	1
106.11	0.9988
106.066	0.9984
106.056	0.7694
106.053	0.3902
105.997	0.2494
105.928	0.1292
105.917	0.0988
105.883	0.0988
105.792	0.0432
105.781	0.0432
105.769	0.022
105.713	0.003
105.656	0.0006
105.633	0.0002
105.599	0
105.543	0
104.814	0

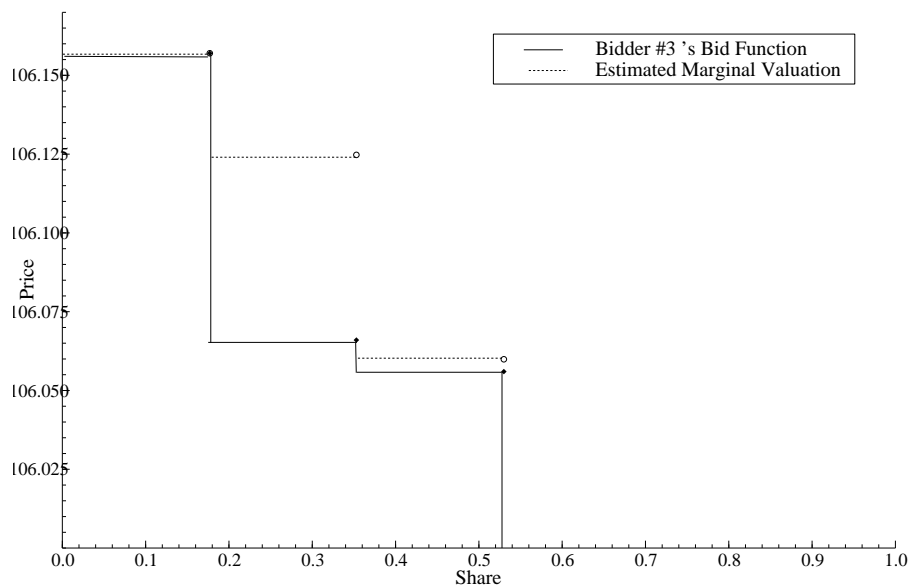


Figure 4.4: Marginal Valuation Estimation based on Kang and Puller

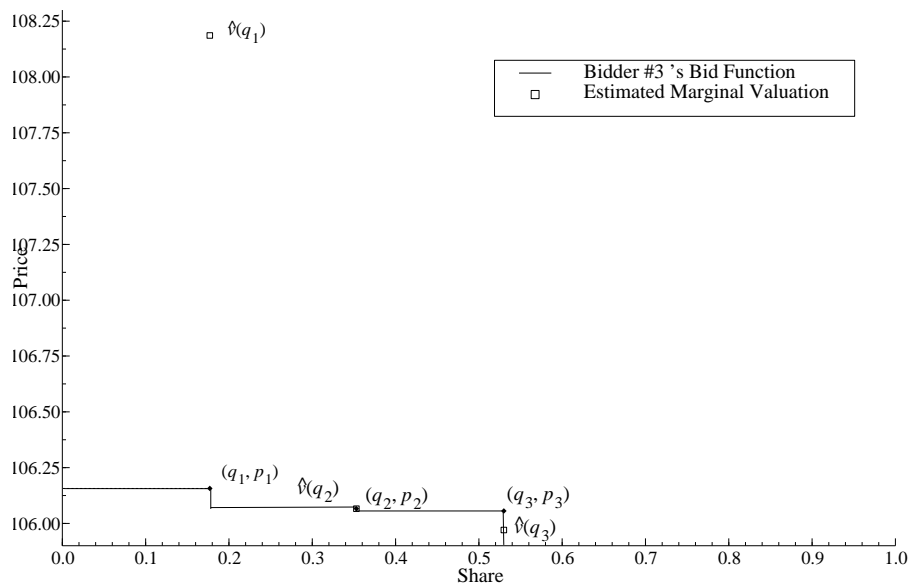


Figure 4.5: Marginal Valuation Estimation based on Kaslt

Chapter 5

Conclusion

In an auction of a single divisible object, bidders have to decide the price to bid. In government bond auctions, in contrast, bidders have to choose how many bid-points to submit, and the quantity and the price to bid for each of them. This higher dimensionality of the strategic space complicates the analysis of bidders' behaviour and explains why our understanding of it is far from complete.

Moreover, the analysis of government bond auctions is still developing. Few equilibrium strategies are known and the estimation of its parameters is complicated. This thesis has explored these issues and as a contribution to the still developing literature on government bond auctions and tool for future practitioners, the last chapter reviewed the different methodologies that have done structural estimation of share auctions. For a practitioner needing to estimate a structural model for government securities, I propose the following steps which are anchored in the topics that this thesis has addressed.

- Find a data set that identifies all the bid pairs submitted by each bidder in each

auction.

- Decide the information structure that best fits the data's auction environment.

Under the CVP the researcher is confined to use Armantier and Sbaï's or Fevrier et al.'s methodology. The first one seems to be very promising for its flexibility but is not easy to implement in the case of share auctions. The second one, so far, is only developed for discriminatory auctions. Under the IPVP, Kaslt's methodology is advantageous since it incorporates the fact that bidders submit finite step demand functions.

- Estimate the model.
- Evaluate the performance of the employed mechanism. Here, if the researcher has data from a discriminatory auction and is using Fevrier et al.'s methodology, he or she is constrained to assume one of the parametric specifications that provides an analytical solution of the equilibrium strategy for the uniform auction. Within the IPVP, Kaslt proposes a method to assess the performance of the auction without having to obtain counterfactual strategies and which can be implemented for data from both uniform price and discriminatory auctions.

By comparing and contrasting all structural approaches for estimating government securities auctions this thesis has clarified some aspects of this complex institution. Further research needs to be encouraged, and both governments and other public institutions should seek ways to achieve this. The former are direct beneficiaries and the latter could

benefit from multiple spill over effects; for example electricity auctions are similar to government securities auctions.

One of the directions of future research could be to explore auctions as a repeated game and/or analyze the implications of the interactions between the auctions and the secondary market; until now, all methods have treated each auction as an independent and isolated game. Contributions in the analytical analysis of auctions are also needed. Recall that until now, only some optimal strategies under specific parametric frameworks are known. Finally, all structural estimation is based on the assumption that bidders behave strategically. But do they? According to Sade et al.'s experimental approach, bidding strategies under both auction mechanisms are generally inconsistent with the equilibrium strategies identified by the theory. Following the lines of McAdams [19], future research can explore how far bidders are from playing the best response in real-life auctions.

Having acknowledge the difficulties that studying government bond auctions entails, and the vast opportunities for further research within the field, as government issuance of debt increases, research in the area must probably do likewise. I hope this thesis contributes to these efforts.

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Appendix A: Main Characteristics of Government Securities Auctions

Table (1) presents the main characteristics of government securities auctions across some countries, as reported by recent studies.

Table 1: Main Characteristics of Government Securities Auctions

Reference	Data Set	Sample Period ^b	# Auctions ^c	Format ^a	Bonds Maturity ^d	Coverage	# bidders ^e	# bid-points allowed	# bid-points by bidder ^f	Uncertain Supply
[7]	Mexico	01/01-04/02	180	D	short	3.24	19	∞	3.8	no
[8]	Austria	91-2006	137	D	NM ^g	2.63	NM	∞	5	no
[9]	France	01/05-12/05	49 ^h	D	long	2.25	20	∞	2.9	yes ⁱ
[11]	Turkey	10/91-09/93	25	D	3 m.	3.57	69	∞	6.9	yes/ ^j
[13]	Korea	09/99-04/02	30	20 D, 10 U	3 y.	NM	25.8	5	3.1	no
[14]	Czech.	11/99-12/00	28	U	3 m.	NM	13	10	2.3	yes ^k
[17]	Finland	91-99	232	U	long	NM	8	∞	2.7	yes/ ^l
[20]	Sweden	90-94	400	D	short, long	2.41	12	∞	2.9	yes ^m
[24]	Germany	98-02	93	D	long	NM	42	∞	NM	yes ⁿ

^a D: discriminatory, U: uniform.

^b MM/YY

^c Analyzed in the paper.

^d Short term: maturity < 1 year. Long term: maturity \geq 1 year.

^e Mean.

^f Mean # of bid-points submitted by bidder per auction.

^g Not mentioned.

^h 24 different auction's dates.

ⁱ Non-competitive bids are allowed & the government announces a bracket for the global supply.

^j For the first 17 auctions the government did not preannounce the total quantity for sale.

^k Non-competitive bids are allowed.

^l The supply is not preannounced. In 1998 the maximum supply starts to be announced.

^m The government has the right to withdraw securities from the auction after bids have been submitted.

ⁿ Non-competitive bids are allowed & the government can set a % of the supply to be sold in the secondary market.

Appendix B: Modifications to the Regulation

Table (2) illustrates the main modifications to the regulation of the auction and the non-competitive round (NCR) since 2002 until 2007. The first panel, presents the main characteristics of the auction in 2002. The other panels point out the modifications. In 2002 instead of the NCR there was a second round. Their difference is that the second round was held the same day of the auction and that its demand could be higher than its supply, therefore bidders' bid for the second round had quantity uncertainty.

Table 2: Main Modifications to the Auction and the NCR, 2002-2007

Year	Regulation
January 2002	
Auction ^a (10-11 a.m)	
Maximum Bid Supply	US\$ 175 million In November, instead of a supply for each bond, the government starts to announce a global supply for all the bonds that are auctioned simultaneously but independently. The only connection between the auctions is the global supply.
Settlement Date	2 working days after the auction
Second Round	
Date	Auction's date (3-4 p.m)
Supply (supply2 ^{SR})	- Auction's coverage $\geq 2 \rightarrow \text{supply2}^{SR} = 0.5 * \text{supply1}$ - $1.2 \leq A. \text{ coverage} < 2 \rightarrow \text{supply2}^{SR} = 0.25 * \text{supply1}$ - A. coverage $< 1.2 \rightarrow \text{supply2}^{SR} = 0$
Price	Auction's market clearing price
Participants	Market makers ^b who were able to buy in the auction
Maximum bid	supply2 ^{SR}
Allocation Rule	If demand $> \text{supply2}^{SR}$, the market marker who won the greatest amount in the auction will get his bid, then the second market market maker and so on.
Settlement Date	Same as auction's settlement date
January 2003	
Second Round: last year it was held	
Non-Competitive Round	
Date	Auction's settlement date (12:30-1:00 p.m)
Supply ^c (supply2)	- A. coverage $\geq 2 \rightarrow \text{supply2} = 0.5 * \text{supply1}$ - $1.2 \leq A. \text{ coverage} < 2 \rightarrow \text{supply2} = 0.25 * \text{supply1}$ - A. coverage $< 1.2 \rightarrow \text{supply2} = 0$
Price	Auction's market clearing price
Participants	Top 9 ranked market makers who won in the auction and did not go to the second round
Maximum bid	% of supply2. The % is the same as the % won in the auction
Settlement Date	Same as auction's settlement date

^a The auction's supply per bond—not the global supply—is denoted by supply1.

^b The members of the market makers program are divided between market makers and aspirants to market makers. In the thesis this distinction was not made—all members of the program were called market makers.

^c Because of the transition between the second round and the NCR, the supply for the NCR is supply2 minus the allocation in the second round.

Year	Regulation
August 2005	
Auction (9:30-10:00 a.m)	
Maximum Bid	US\$ 100 million
Settlement Date	Same day as the auction
Non-Competitive Round	
Participants	Market makers who won in the auction
Maximum bid	% of supply2. The % is the same as the % won in the auction (for this the aspirants to market makers are not taken into account).
Settlement Date	The day of the NCR, which is the third working day after the auction
January 2006	
Non-Competitive Round	
Supply	<ul style="list-style-type: none"> - A. coverage $\geq 2 \rightarrow \text{supply2} = 0.65 * \text{supply1}$ - $1.2 \leq \text{A. coverage} < 2 \rightarrow \text{supply2} = 0.4 * \text{supply1}$ - A. coverage $< 1.2 \rightarrow \text{supply2} = 0$
Participants	Market makers (MM) and aspirants to market makers (AMM) who won in the auction
Maximum bid (allocation)	<ul style="list-style-type: none"> - A first amount to MM: $0.5 * \text{supply1}$ if A. coverage ≥ 2, or $0.25 * \text{supply1}$ if $1.2 \leq \text{A. coverage} < 2$. Once this amount is defined, the amount that corresponds to each MM is proportional to what they won in the auction (excluding AMM). - The second amount is $0.1 * \text{supply1}$ and is distributed between the MM and AMM that are in the top 5 positions of the MM program for the current year, according to the amount that was awarded to them in the auction. - The third amount is $0.05 * \text{supply1}$ and is distributed between the MM and AMM that are in the positions 6 to 10 of the program, according to the amount that was awarded to them in the auction.
Price	Average market price of the auction's date
January 2007	
Auction	
Supply	In March, instead of the global supply the government comes back to announce the supply per bond.

Non-Competitive Round	
Supply	<ul style="list-style-type: none"> - A. coverage $\geq 2 \rightarrow$ supply2 = 0.8*supply1 - $1.2 \leq$ A. coverage $< 2 \rightarrow$ supply2 = 0.55*supply1 - A. coverage $< 1.2 \rightarrow$ supply2 = 0
Maximum bid (allocation)	<ul style="list-style-type: none"> - A first amount to MM: 0.5*supply1 if A. coverage ≥ 2, or 0.25*supply1 if $1.2 \leq$ A. coverage < 2. Once this amount is defined the amount that corresponds to each MM is proportional to what they won in the auction (excluding AMM). - The second amount is 0.2*supply1 and is distributed between the MM and AMM that are in the top 5 positions of the MM program for the current year, according to the amount that was awarded to them in the auction - The third amount is 0.1*supply1 and is distributed between the MM and AMM that are in the positions 6 to 10 of the program, according to the amount that was awarded to them in the auction

Appendix C: Oral Survey

This appendix summarizes the main points of an oral survey conducted in July 2008 to three market makers. The questions asked were:

- (1) Do you follow any type of strategy for the bidding? Which factors affect your bid?
- (2) When you participate in the auction do you take into account the possibility of it being followed by a NCR? Would your bids change if the NCR did not exist?
- (3) Before the auction, do you have a rough idea of who is going to participate in it? do you communicate with the other potential bidders to know if they are going to participate in the auction?
- (4) Do you consider that bidders are asymmetric?

The answers were:

- (1) None of the bidders follow any type of estimation. Their bids depend on: Bidder A: i) their score in the market makers program, ii) their stock of bonds, iii) the market conditions, and iv) how do they think the auction is going to be. Bidder B: i) the market conditions, ii) their score in the market makers program, iii) their utilities, depending on them they can make a risky bid that can give high returns, and iv) if they took a short position before the auction, which depends on the market's conditions. Bidder C: i) their clients, and ii) the existence of the NCR.
- (2) Bidder A and Bidder B: the participation decision and the bids would change if there was not NCR. Bidder C: the aggressiveness of the bids would change if there was not NCR; the NCR option can give them high profits. Their participation decision probably would not be affected, because they feel that as market makers their duty is to go the auction.

- (3) Bidder A: They do not communicate with the other potential bidders and do not know who is going to participate in the auction. Bidder B: They do not communicate with the other potential bidders before the auction. Sometimes, through their clients, they get some idea of who is going to participate in the auction. Bidder C: They assume that all the potential bidders are going because that is their duty as members of the program of market makers.
- (4) Bidder A: Yes. Bidder B: Yes. They will divide the bidders in three groups. One for the bidders that go always with high demands. Another one for the bidders that take a moderate position—like them—sending constant bids through all the auctions. They try to get 15% of the supply in each auction. And a last group for the bidders that go just when they see a clear opportunity for earning profits. They consider that the bidders in this group are playing a risky strategy because at the end of the year they might have to buy bonds at any price to satisfy the requirement of the market makers program in which members have to buy at least 4.25% of the primary auction. Bidder C: Yes. There are 2 aggressive bidders. The others can be grouped in one group.

Appendix D: Optimal Strategy for the Uniform Format Within the IPVP

Here I show that under the IPVP for the uniform price auction with $N \geq 2$ risk neutral symmetric bidders and known supply, there is no optimal strategy that is linear in the private valuation and in the price when the valuations are normally distributed with mean μ_V and standard deviation σ_V .

Following a similar procedure as the one in Appendix E, we can find that the Euler condition for the uniform price auction ($\tau = 0$) under the IPVP when each bidder i has a constant private marginal valuation, v_i , is:

$$v_i - p = -\frac{H_{\phi_i}(p; \phi_i)}{H_p(p; \phi_i)} \phi_i(p, v_i) \quad (1)$$

Since we want to find an optimal strategy that is linear in the valuation and in the price we “guess” that the optimal strategy is $\phi^*(p, v_i) = A + Bv_i + Cp$. The objective is to find the parameters A, B, C that satisfy the Euler equation (1). Assuming that all bidders except bidder i play this strategy and that bidder i plays $\phi_i(p, v_i)$ we have that the c.d.f for the market clearing price, $H(p, \phi_i)$ becomes

$$\begin{aligned} H(p; \phi_i) &= Pr(1 \geq \phi_i + (N-1)A + B \sum_{j \neq i} V_j + (N-1)Cp) \\ &= Pr(1 - \phi_i - (N-1)A - (N-1)Cp \geq B \sum_{j \neq i} V_j) \\ &= \Psi(\Delta) \end{aligned}$$

where

$$\Delta = \frac{1 - \phi_i - (N-1)A - (N-1)Cp - B(N-1)\mu_V}{\sqrt{(N-1)B^2\sigma_V^2}},$$

Ψ denotes the Normal Standard c.d.f, and the third line follows from the fact that $B \sum_{j \neq i} V_j \sim N(B(N-1)\mu_V, B^2(N-1)\sigma_V^2)$. Evaluating the derivatives $H_{\phi_i}(p; \phi_i)$ and $H_p(p; \phi_i)$ we have that:

$$\begin{aligned} H_{\phi_i}(p; \phi_i) &= -\frac{1}{\sqrt{(N-1)B^2\sigma_V^2}} \psi(\Delta) \\ H_p(p; \phi_i) &= -\frac{(N-1)C}{\sqrt{(N-1)B^2\sigma_V^2}} \psi(\Delta) \end{aligned}$$

where ψ denotes the p.d.f of the normal standard distribution.

Given that bidders are symmetric, for $\phi^*(\cdot, \cdot)$ to be the optimal strategy, the Euler condition should be satisfied for $\phi_i(\cdot, \cdot) = \phi^*(\cdot, \cdot)$. Plugging the derivatives and bidder i 's strategy, $\phi_i(p, v_i) = A + Bv_i + Cp$, in the Euler equation (1) we get:

$$(N - 1)C(v_i - p) = A + Bv_i + Cp.$$

Comparing coefficients we can see that $A = 0$, $B = (N - 1)C$, and $C = -(N - 1)C$, which contradicts the assumption that $N \geq 2$.

Appendix E: Euler-Lagrange Condition for the Discriminatory and Uniform Format Within the CVP

Wilson [27] derives the Euler-Lagrange condition for the uniform auction, and Fevrier et al. [9] for the discriminatory format. Here I present the derivation under a general framework.

To find the optimal strategy, the representative bidder, bidder i , maximizes his expected utility with respect to $\phi_i(p, s_i)$. The following functional optimization problem

$$\max_{\phi_i} E \left\{ \left(\int_0^\infty \left((V - p)\phi_i(p, s_i) - \tau \int_p^{p_i^{max}} \phi_i(p, s_i) dp \right) \frac{\partial H(p; V, \phi_i(\cdot))}{\partial p} \right) \Big| S_i = s_i \right\} \quad (2)$$

is going to be solved using calculus of variations. The expected value is with respect to the value of the object given $S_i = s_i$. Denoting

$$f(p) \equiv (V - p)\phi_i(p, s_i) - \tau \int_p^{p_i^{max}} \phi_i(p, s_i) dp$$

and

$$q'(p) \equiv \frac{\partial H(p; V, \phi_i(p, s_i))}{\partial p} \equiv H_p(p; V, \phi_i(p, s_i))$$

then the inner integral of (2), by integration by parts, is equal to:

$$f(p)q(p) \Big|_0^\infty - \int_0^\infty f'(p)q(p)$$

Where $q(p) = H(p; V, \phi_i(p, s_i))$ and $f'(p)$ is equal to:

$$f'(p) = ((V - p)\phi_{i_p}(p, s_i) - \phi_i(p, s_i) - \tau[\phi_i(p_i^{max}, s_i) - \phi_i(p, s_i)])dp.$$

As a result we have that the inner integral of (2) is equal to:

$$\left(\left[(V - p)\phi_i(p, s_i) - \tau \int_p^{p_i^{max}} \phi_i(p, s_i) dp \right] H(p; V, \phi_i(p, s_i)) \right) \Big|_0^\infty - \int_0^\infty ((V - p)\phi_{i_p}(p, s_i) - \phi_i(p, s_i) - \tau[\phi_i(p_i^{max}, s_i) - \phi_i(p, s_i)])H(p; V, \phi_i(p, s_i)) dp$$

where $\phi_{i_p}(p, s_i)$ denotes the derivative of the strategy with respect to price. Given that $H(p; V, \phi_i)$ is the c.d.f of the market clearing price we have that: $H(\infty; V, \phi_i(p, s_i)) = 1$, $H(0; V, \phi_i(p, s_i)) = 0$. Additionally, $\phi_i(p_i^{max}, s_i) = 0$, $\phi_i(\infty, s_i) = 0$ and $\int_{-\infty}^{p_i^{max}} \phi_i(p, s_i) dp = 0$. Hence, the optimization problem becomes:

$$\max_{\phi_i} E \left\{ - \int_0^{\infty} [(V - p)\phi_{i_p}(p, s_i) + \phi_i(p, s_i)(\tau - 1)] H(p; V, \phi_i(p, s_i)) dp \middle| S_i = s_i \right\}.$$

The expression in brackets can be written as a function $g(p, \phi_i(p, s_i), \phi_{i_p}(p, s_i), V)$. Using this notation, the maximization problem can be rewritten as

$$\max_{\phi_i} E \left\{ - \int_0^{\infty} g(p, \phi_i(p, s_i), \phi_{i_p}(p, s_i), V) dp \middle| S_i = s_i \right\}.$$

The Euler equation (which is a necessary condition for optimality) is given by

$$E \left\{ - \frac{\partial g}{\partial \phi_i} + \frac{\partial}{\partial p} \frac{\partial g}{\partial \phi_{i_p}} \middle| S_i = s_i \right\} = 0.$$

Evaluating the derivatives we have:

$$\begin{aligned} \frac{\partial g}{\partial \phi_i} &= [(V - p)\phi_{i_p}(p, s_i) + \phi_i(p, s_i)(\tau - 1)] H_{\phi_i}(p; V, \phi_i(p, s_i)) \\ &\quad + (\tau - 1)H(p; V, \phi_i(p, s_i)) \\ \frac{\partial g}{\partial \phi_{i_p}} &= (V - p)H(p; V, \phi_i(p, s_i)) \\ \frac{\partial}{\partial p} \frac{\partial g}{\partial \phi_{i_p}} &= (V - p) \{ H_p(p; V, \phi_i(p, s_i)) + H_{\phi_i}(p; V, \phi_i(p, s_i))\phi_{i_p}(p, s_i) \} \\ &\quad - H(p; V, \phi_i(p, s_i)), \end{aligned}$$

where the subscripts denote the partial derivatives. As a result, the Euler condition is

$$E \left\{ (V - p)H_p(\cdot) - \tau H(\cdot) - \phi_i(p, s_i)(\tau - 1)H_{\phi_i}(\cdot) \middle| S_i = s_i \right\} = 0.$$

The strategy $\phi^*(p, s_i)$ is optimal if the above condition is satisfied for $\phi_i(p, s_i) = \phi^*(p, s_i)$.