SPECIAL VALUES OF L-SERIES, PERIODIC COEFFICIENTS AND RELATED THEMES

by

Siddhi S. Pathak

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Abstract

This thesis is centered around the theme of special values of L-functions and other infinite series, which are often expected to be transcendental numbers. More specifically, we focus on the following two questions in various scenarios:

a) expressing the values in terms of certain special functions,

b) determining their arithmetic nature (i.e., whether they are rational or irrational, algebraic or transcendental).

Motivated by the conjectures of S. Chowla and P. Erdős, we first study the L-series $L(s, f)$ attached to a periodic arithmetical function f. Utilizing tools from transcendental number theory, we investigate the non-vanishing and the arithmetic nature of the values $L(1, f)$ and $L'(1, f)$. We introduce a probabilistic viewpoint towards the study of the values $L(k, f)$ for any integer $k \geq 1$, especially in the case when f is an Erdős function.

On a related note, we explore the irrationality of the values of Dedekind zeta-functions at positive integers using elementary means. We also initiate the study of elliptic analogs of the sum, $\sum_{n \in \mathbb{Z}} q(n)$, where $q(X)$ is a rational function. This opens new doors for future research.

Co-Authorship

The results in Chapters 5 and 6 were obtained jointly with my advisor, Prof. M. Ram Murty. The work discussed in Chapters 3 and 5 constitute my published research papers [77] and [65] respectively. The theorems in Chapter 4 will appear in my paper [76].

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Statement of Originality

I hereby certify that the work described in this thesis is the original work of the author. Collaboration with other authors is acknowledged and detailed in the section Co-Authorship. Results due to other authors are fully acknowledged and presented as such, in conformity with standard referencing practices in Mathematics.

Contents

Chapter 1

Introduction

The concept of a "number" lies at the heart of civilization. The earliest archaeological discoveries of human writing indicate a primordial awareness of the idea of natural numbers. This concept represents a significant step for higher thought and human evolution. The mathematician Richard Dedekind [30] wrote that "numbers are free creations of the human intellect; they serve as a means of grasping more easily and more sharply the diversity of things". Since numbers represent a gateway to our understanding of the world around us, their study has occupied some of the greatest minds since time immemorial.

For instance, the ideas of the number zero and the decimal system, which are the foundations of our advancement, were discovered in ancient India. The famous "zero of Gwalior" shows that the decimal system was in use well before 600 CE. In this context, Laplace [24] wrote

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position, as well as an absolute value;

a profound and important idea which appears so simple to us now that we ignore its true merit, but its very simplicity, the great ease which it has lent to all computations, puts our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

The school of Euclid discovered that all natural numbers are composed of prime numbers and so, prime numbers are the building blocks of all mathematics. A prime number is a natural number $n > 1$ which has no proper divisor d satisfying $1 < d < n$. For example, 2, 3, 5, 7, 11 are prime numbers. The number 9 is not prime since 3 is a proper divisor of 9. The study of prime numbers has been the focus of a galaxy of great mathematicians beginning with Euler, Gauss, Dirichlet, Riemann and Hurwitz.

In 1859, Riemann introduced the study of the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, with s being a complex variable. With his profound idea of analytic continuation, Riemann showed that $\zeta(s)$ extends analytically to the entire complex plane apart from a simple pole at $s = 1$. He emphasized that the study of prime numbers is intimately tied with the study of $\zeta(s)$ as a function of a complex variable. This function continues to be the focus of intense research to this day. The notorious Riemann hypothesis is the assertion that if $\zeta(s_0) = 0$ and $0 \le \Re(s_0) < 1$, then $\Re(s_0) = 1/2$. This is a major unresolved problem in number theory and if solved, would unveil many of the mysteries surrounding primes.

Euclid proved in his famous "Elements" that there are infinitely many prime numbers. In order to prove that there are infinitely many prime numbers in arithmetic progressions, Dirichlet introduced the L-series,

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\Re(s) > 1$, where $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ is a homomorphism and q is a natural number. Dirichlet proved that if $\chi \neq \chi_0$, the principal character, then $\lim_{s\to 1^+} L(s,\chi)$ exists, and the non-vanishing of $L(1,\chi)$ was shown to imply the infinitude of primes in an arithmetic progression. This non-vanishing could be achieved relatively easily when χ takes complex values. However, in order to show that $L(1,\chi) \neq 0$ for quadratic characters, Dirichlet had to discover the arithmetic significance of $L(1,\chi)$, by relating it to the class number of quadratic fields.

Motivated by Dirichlet's work, Chowla considered the general series

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where $f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$ and asked fundamental questions regarding the non-vanishing of $L(1,f)$. This thesis focuses on questions stemming from Chowla's consideration and related themes. Most notably among these is a conjecture of Erdős, which is centred around the non-vanishing of $L(1,f)$, where f is a non-multiplicative analog of a quadratic character, which we will call an Erdős function.

A complex number is said to be *algebraic* if it satisfies a polynomial equation with rational coefficients. The algebraic numbers form a field, and are denoted by $\overline{\mathbb{Q}}$. If a number is not algebraic, then it is said to be *transcendental*. An important problem in transcendental number theory is to identify 'new' numbers that are transcendental. If $L(1, f) \neq 0$, then Baker's theory of linear forms in logarithms implies that the value is transcendental. Therefore, the consideration of these questions lies on the interface between analytic and transcendental number theory.

Our analysis is a part of the general chapter of the study of special values of Dirichlet series. We will use a variety of methods from analytic number theory, probability theory and transcendental number theory to make some inroads into these questions. Our work is by no means exhaustive and future directions are indicated in the last chapter. However, the investigation has been fruitful and it seems to have advanced our understanding.

1.1 Organization of the thesis

We will use standard mathematical notation throughout the thesis. The symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are used to denote the natural numbers, integers and the rational numbers respectively. The field of algebraic numbers is denoted by $\overline{\mathbb{Q}}$. Further notation will be defined as required in the relevant discussion.

The results presented in this thesis are strung together by the common thread of special values of infinite series. In particular, the thesis is structured as follows.

Chapters 2, 3, 4 and 5 discuss progress related to the conjectures of S. Chowla and P. Erdős regarding the non-vanishing of the value $L(1, f)$, for periodic arithmetical functions f. More specifically, we introduce the background on L-series attached to periodic functions in Chapter 2. In Chapter 3, we study a conjecture of A. Livingston, regarding the linear independence of $\log \sin(a\pi/q)$ with $1 \leq a < q/2$ over the field of algebraic numbers. We prove this conjecture when $q = 4$ and $q \geq 5$ is prime (Theorem 3.2.3), and produce counterexamples in all other cases (Theorem 3.2.1). This settles Livingston's conjecture. The results from this chapter appeared in my paper [77].

We approach the Conjecture 3.0.1 (see Chapter 3) of Erdős from a probabilistic perspective in Chapter 4. In the first part, we utilize the method of moments to determine the existence and the characteristic function of the limiting distribution of the values $L(k, f)$ where $k \geq 1$ is a positive integer and f is an Erdős function with the same parity as k. A detailed statement of the result can be found in Theorem 4.1.2. The later part of this chapter is concerned with the proportion of Erdős functions for which Conjecture 3.0.1 is valid. In particular, we prove in Theorem 4.2.2 that the Erdős conjecture is true with "probability" one. These results will appear in my paper which has been accepted for publication [76].

In Chapter 5, we discuss the arithmetic nature of the value $L'(1, f)$, where f is a periodic arithmetical function. Under the assumption of a conjecture of S. Gun, M. R. Murty and P. Rath, we prove that in a family of linearly independent functions, periodic mod q, there can be *very few* functions f such that $L'(1, f)$ is algebraic. The cases when $L(1, f) \neq 0$ (Theorem 5.1.1) and when $L(1, f) = 0$ (Theorem 5.2.1) are treated separately. For a fixed prime $p \geq 7$, the above results can be applied to deduce that at least $(p-7)/2$ of the first Laurent series coefficients of $p^s\zeta(s, a/p)$ around $s = 1$, with $1 \leq a \leq p-1$, are transcendental (Corollary 5.3.2). The contents of this chapter have been published in my joint work [65].

Motivated by Euler's evaluation of $\zeta(2k)$ for $k \geq 1$, we discuss the special values of the Dedekind zeta-function in Chapter 6. Although much remains unknown regarding the nature of the values of the Dedekind zeta-function (see Chapter 6 for a definition) at positive integers, we demonstrate that more can be deduced from our current knowledge in the special case of CM-extensions. As a consequence of Theorem 6.3.1, we deduce in Corollary 6.3.2 that the values $\zeta_K(2m+1)$ are irrational as K varies over imaginary quadratic fields and $m \geq 1$ is fixed, except for at most one exception.

We initiate the study of elliptic analogues of the sums $\sum_{n\in\mathbb{Z}} q(n)$, where $q(X)$ is a rational function with algebraic coefficients, in Chapter 7. That is, we investigate the sum of the values of an appropriate rational function over the points of a two dimensional lattice in $\mathbb{C}$. We explicitly 'evaluate' these sums in terms of Weierstrass functions (Theorem 7.3.2) and study their transcendental nature in special cases (Theorems 7.4.1 and 7.5.1).

In the last chapter, Chapter 8, we record certain cognate ideas that arose during the course of our study, which we have relegated to future research.

Chapter 2

Dirichlet series with periodic coefficients

Let q be a positive integer. As noted earlier, in order to study the behaviour of primes in an arithmetic progression modulo q, Dirichlet defined and investigated appropriate analogues of the classical Riemann zeta-function. A Dirichlet character mod q is a group homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$. We identify the character χ with its extension to $\mathbb{Z}$, given by

$$\chi(n) := \begin{cases} \chi(n \bmod q) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, χ defines a completely multiplicative periodic function on the integers. A character χ is said to be odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. The identity homomorphism gives rise to the principal character χ_0, i.e.,

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet L-function attached to χ is defined by the product

$$L(s,\chi) := \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \;\; \Re(s) > 1.$$

Due to the multiplicativity of the character values, this can be expanded out into the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \;\; \Re(s) > 1.$$

It is easy to see that up to a holomorphic function, $L(s,\chi_0)$ is essentially the Riemann zeta-function $\zeta(s)$. More specifically,

$$L(s,\chi_0) = \zeta(s) \prod_{\substack{p \text{ prime}, \\ p|q}} \left(1 - \frac{1}{p^s}\right).$$

Thus, $L(s,\chi_0)$ has a simple pole at $s = 1$. However, using partial summation, it can be shown that when $\chi \neq \chi_0$, the Dirichlet series representation of $L(s,\chi)$ converges for $\Re(s) > 0$. In 1840s, Dirichlet took advantage of the pole of $L(s,\chi_0)$ at $s = 1$ and the fact that $L(1,\chi) \neq 0$ when $\chi \neq \chi_0$, to deduce that there are infinitely many primes, $p \equiv a \bmod q$ when $(a,q) = 1$.

The fact that $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$ is a vital step in Dirichlet's proof. It took Dirichlet almost four years to establish this result, during which he discovered the famous class number formula. Intrigued by the mystery surrounding Dirichlet's theorem, in the early 1960s, S. Chowla [17] asked if a similar non-vanishing result holds for general periodic functions on the integers that are not necessarily multiplicative. This initiated the study of general Dirichlet series attached to periodic arithmetical functions. In this chapter, we will focus on developing the background on Dirichlet series with periodic coefficients. These theorems will be useful in the later chapters.

2.1 Analytic continuation and functional equation

We will see that a Dirichlet series with periodic coefficients can be expressed as a linear combination of Hurwitz zeta-functions, $\zeta(s,x)$. In 1882, Hurwitz [46] recognized the central role of $\zeta(s,x)$ in the theory of Dirichlet L-functions and isolated it for independent study. For $\Re(s) > 1$ and $0 < x \leq 1$, the Hurwitz zeta function is defined as

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

Hurwitz proved that $\zeta(s,x)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$ with residue 1. In particular,

$$\zeta(s,x) = \frac{1}{s-1} - \psi(x) + O(s-1), \tag{2.1}$$

where ψ is the digamma function, which is defined as the logarithmic derivative of the gamma function.

Let q be a positive integer and f be an arithmetical function, periodic with period q. Define

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \Re(s) > 1.$$

Owing to the periodicity of f, we have

$$
\begin{aligned}
L(s,f) &= \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a+kq)^s} \\
&= \frac{1}{q^s} \sum_{a=1}^{q} f(a)\zeta\left(s, \frac{a}{q}\right),
\end{aligned} \tag{2.2}
$$

where $\zeta(s,x)$ is the Hurwitz zeta function. Thus, the analytic continuation of $\zeta(s,x)$ implies that $L(s,f)$ can be extended analytically to the entire complex plane except for a simple pole at $s=1$ with residue $\frac{1}{q}\sum_{a=1}^{q} f(a)$. Hence, $L(1,f) = \sum_{n=1}^{\infty} f(n)/n$ exists if and only if $\sum_{a=1}^{q} f(a) = 0$, which we will assume henceforth.

An important tool to develop the theory further is of the Fourier transform of f, as a function on $\mathbb{Z}/q\mathbb{Z}$. Given a function f, periodic with period q, define the Fourier transform of f as

$$
\widehat{f}(b) := \frac{1}{q} \sum_{a=1}^{q} f(a)\zeta_q^{-ab}, \tag{2.3}
$$

where $\zeta_q = e^{2\pi i/q}$. This can be inverted using the identity

$$
f(n) = \sum_{b=1}^{q} \widehat{f}(b)\zeta_q^{bn}. \tag{2.4}
$$

Thus, the condition for convergence of $L(1,f)$, i.e, $\sum_{a=1}^{q} f(a) = 0$ can be interpreted as $\widehat{f}(q) = 0$.

As an example, consider the function $e_{q,n}(a) := \zeta_q^{na}$, $n \in \mathbb{Z}$, defined on the integers.

Then the Fourier transform of $e_{q,n}$ is given by

$$\widehat{e_{q,n}}(b) = \frac{1}{q}\sum_{a=1}^{q} \zeta_q^{na}\,\zeta_q^{-ab} = \frac{1}{q}\sum_{a=1}^{q}\left(\zeta_q^{n-b}\right)^a = 0, \tag{2.5}$$

unless $n \equiv b \bmod q$, in which case, $\widehat{e_{q,n}}(n) = 1$. Another example is that of a primitive Dirichlet character χ modulo q. It can be shown that (see [77, Lemma 2.3])

$$\widehat{\chi}(n) = \frac{\tau(\chi)}{q}\,\overline{\chi(-n)}, \tag{2.6}$$

where $\tau(\chi) := \sum_{a=1}^{q} \chi(a)\zeta_q^a$ is the Gauss sum associated to χ.

In particular, since $\widehat{f}$ is another function periodic mod q, it is natural to ask if there exists a connection between $L(s, f)$ and $L(s, \widehat{f})$. In this direction, we prove the following lemma.

Lemma 2.1.1. *Let f be an arithmetical function periodic with period q. Then,*

$$L(1-s, f) = 2\,\Gamma(s)\left(\frac{q}{2\pi}\right)^s \cos\left(\frac{s\pi}{2}\right) L(s, \widehat{f}),$$

when f is even (i.e., f(-n) = f(n) for all n) and

$$L(1-s, f) = 2\,i\,\Gamma(s)\left(\frac{q}{2\pi}\right)^s \sin\left(\frac{s\pi}{2}\right) L(s, \widehat{f}),$$

when f is odd (i.e, f(-n) = -f(n) for all n).

Proof. The Hurwitz zeta function $\zeta(s, x)$ satisfies the following functional equation when the parameter x is rational.

$$\zeta\left(1-s,\frac{a}{q}\right)=\frac{2\,\Gamma(s)}{(2\pi q)^s}\sum_{b=1}^{q}\cos\left(\frac{\pi s}{2}-\frac{2\pi ab}{q}\right)\zeta\left(s,\frac{b}{q}\right), \tag{2.7}$$

where $0<a/q\leq 1$. For a proof of this fact, we refer the reader to [2, Chapter 12, Section 9]. Using this functional equation together with (2.2), we observe that

$$\begin{aligned}L(1-s,f)&=\frac{1}{q^{1-s}}\sum_{a=1}^{q}f(a)\zeta\left(1-s,\frac{a}{q}\right)\\&=\frac{2\,\Gamma(s)}{(2\pi q)^s\,q^{1-s}}\sum_{a=1}^{q}f(a)\sum_{b=1}^{q}\left[\cos\left(\frac{\pi s}{2}-\frac{2\pi ab}{q}\right)\zeta\left(s,\frac{b}{q}\right)\right]\\&=\frac{2\,\Gamma(s)}{q\,(2\pi)^s}\sum_{b=1}^{q}\zeta\left(s,\frac{b}{q}\right)\sum_{a=1}^{q}\left[f(a)\cos\left(\frac{\pi s}{2}-\frac{2\pi ab}{q}\right)\right].\end{aligned} \tag{2.8}$$

To simplify further, let

$$S_b:=\frac{1}{q}\sum_{a=1}^{q}f(a)\cos\left(\frac{\pi s}{2}-\frac{2\pi ab}{q}\right).$$

Since $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, we get

$$\begin{aligned}S_b&=\frac{1}{2q}\sum_{a=1}^{q}f(a)\left(e^{\frac{i\pi s}{2}-\frac{2\pi iab}{q}}+e^{\frac{-\pi is}{2}+\frac{2\pi iab}{q}}\right)\\&=\frac{e^{i\pi s/2}}{2}\left[\frac{1}{q}\sum_{a=1}^{q}f(a)\zeta_q^{-ab}\right]+\frac{e^{-i\pi s/2}}{2}\left[\frac{1}{q}\sum_{a=1}^{q}f(a)\zeta_q^{ab}\right]\\&=\frac{\widehat{f}(b)}{2}\left(e^{i\pi s/2}+(-1)^{\delta(f)}e^{-i\pi s/2}\right),\end{aligned}$$

by the definition (2.3) of Fourier transform and with

$$\delta(f) = \begin{cases} 0 & \text{if } f \text{ is even,} \\ 1 & \text{if } f \text{ is odd.} \end{cases}$$

Substituting this simplification of S_b in (2.8), we get

$$\begin{aligned} L(1-s,f) &= \frac{\Gamma(s)}{(2\pi)^s}\left(e^{i\pi s/2} + (-1)^{\delta(f)} e^{-i\pi s/2}\right)\left[\sum_{b=1}^{q} \zeta\left(s, \frac{b}{q}\right)\widehat{f}(b)\right] \\ &= \frac{\Gamma(s)\, q^s}{(2\pi)^s}\left(e^{i\pi s/2} + (-1)^{\delta(f)} e^{-i\pi s/2}\right) L(s, \widehat{f}), \end{aligned}$$

because of (2.3). This proves the lemma. □

This lemma is a generalization of the functional equation satisfied by the L-functions associated to primitive Dirichlet characters.

2.2 The value $L(1,f)$

In this section, we derive several ways of describing the value $L(1,f)$, for a periodic arithmetical function f. Recall that

$$L(s,f) = \frac{1}{q^s}\sum_{a=1}^{q} f(a)\,\zeta\left(s, \frac{a}{q}\right).$$

Thus, (2.1) gives us that

$$L(1,f) = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi\left(\frac{a}{q}\right), \tag{2.9}$$

provided that $\sum_{a=1}^{q} f(a) = 0$.

On the other hand, substituting (2.4) in the expression for $L(s, f)$ for $\Re(s) > 1$, we have,

$$\begin{aligned} L(s, f) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{x=1}^{q} \widehat{f}(x) \zeta_q^{xn} \\ &= \sum_{x=1}^{q} \widehat{f}(x) \sum_{n=1}^{\infty} \frac{\zeta_q^{xn}}{n^s}. \end{aligned}$$

Now, the partial summation formula, along with the condition $\widehat{f}(q) = 0$, gives

$$L(1, f) = -\sum_{b=1}^{q-1} \widehat{f}(b) \log(1 - \zeta_q^b), \tag{2.10}$$

where log denotes the principal branch.

This expression can also be derived in a more algebraic way as follows. Let

$$\mathbf{V}_q := \left\{ f \,\middle|\, f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}, \right\}.$$

Then $\mathbf{V}_q$ is clearly a $\mathbb{C}$-vector space. Moreover, it is equipped with the following inner product.

$$\langle f, g \rangle := \frac{1}{q} \sum_{a=1}^{q} f(a) \overline{g(a)}$$

for $f,\ g \in \mathbf{V}_q$. Thus, one can easily identify that for a periodic function f with $\sum_{a=1}^{q} f(a) = 0$,

$$L(1, f) = \langle f, \Psi_q \rangle, \tag{2.11}$$

where $\Psi_q(a) := \psi(a/q)$ for $1 \leq a \leq q$ is an arithmetical function, periodic modulo q. In this setting, another expression for $L(1, f)$ can be obtained using Parseval's theorem, i.e.,

$$\langle f, g \rangle = q \left\langle \widehat{f}, \widehat{g} \right\rangle .$$

Thus from (2.11), we get

$$L(1, f) = \left\langle \widehat{f}, \widehat{\Psi_q} \right\rangle .$$

Now, the Fourier transform of Ψ_q at $1 \leq b < q$ is

$$\widehat{\Psi_q}(b) = \frac{1}{q} \sum_{a=1}^{q} \psi\left(\frac{a}{q}\right) \zeta_q^{-ab} = \langle e_{q,-b}, \Psi_q \rangle = L(1, e_{q,-b}),$$

by (2.11) since $\widehat{e_{q,-b}}(q) = 0$ as seen in (2.5). But note that for $b \not\equiv 0 \bmod q$,

$$L(1, e_{q,-b}) = \sum_{n=1}^{\infty} \frac{e_{q,-b}(n)}{n} = \sum_{n=1}^{\infty} \frac{\zeta_q^{-bn}}{n} = -\log(1 - \zeta_q^{-b}).$$

Thus, the Fourier transform of Ψ_q is

$$\widehat{\Psi_q(b)} = \begin{cases} -\log(1 - \zeta_q^{-b}), & \text{if } b \not\equiv 0 \bmod q, \\ \frac{1}{q} \sum_{a=1}^{q} \psi\left(\frac{a}{q}\right), & \text{if } b \equiv 0 \bmod q. \end{cases}$$

Hence, the inner product of $\widehat{f}$ and $\widehat{\Psi_q}$ can be evaluated to

$$\left\langle \widehat{f}, \widehat{\Psi_q} \right\rangle = -\sum_{b=1}^{q-1} \widehat{f}(b) \log(1 - \zeta_q^{b}),$$

as $\widehat{f}(q) = 0$. Therefore, we obtain the identity that for an arithmetical function f, periodic with period q,

$$\frac{1}{q}\sum_{a=1}^{q} f(a)\psi\left(\frac{a}{q}\right) = -\sum_{b=1}^{q-1} \widehat{f}(b)\log(1-\zeta_q^b).$$

Thus, we have deduced that if f is an algebraic valued function periodic mod q, then $L(1, f)$ is a linear form in logarithm of algebraic numbers. This observation implies that $L(1, \chi)$ for $\chi \neq \chi_0$ is a linear form in logarithm of algebraic numbers. This observation opens the doors to the application of Baker's theory.

As a bi-product of our discussion above, notice that the computation of the Fourier transform of Ψ_q, together with Fourier inversion yield the famous Gauss's formula for the digamma function (as stated on [27, pg. 35-36]), for $1 \leq a < q$:

$$\psi\left(\frac{a}{q}\right) = -\gamma - \log q - \frac{\pi}{2}\cot\left(\frac{a\pi}{q}\right) + \sum_{b=1}^{r}\left\{\cos\left(\frac{2\pi ab}{q}\right)\log\left(4\sin^2\frac{\pi b}{q}\right)\right\} + (-1)^a \log 2\left(\frac{1+(-1)^q}{2}\right), \tag{2.12}$$

where $r := \lfloor (q-1)/2 \rfloor$. Substituting this in (2.9) and assuming $f(q) = 0$, we obtain

$$L(1, f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right) + \frac{2}{q}\sum_{b=1}^{r}\left\{\left[\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right)\right]\log\left(2\sin\frac{\pi b}{q}\right)\right\} - T_q, \tag{2.13}$$

where

$$T_q = \begin{cases} \frac{\log 2}{q}\left(\sum\limits_{k=1}^{q-1}(-1)^k f(k)\right) & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if f is an odd function, then the expression simplifies to

$$L(1,f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right). \tag{2.14}$$

On the other hand, simplifying (2.10) leads to another expression for $L(1,f)$ in the case when f is an odd function, as demonstrated in the argument below. Note that

$$1-\zeta_q^b = -(\zeta_q^{b/2}-\zeta_q^{-b/2})\zeta_q^{b/2} = -2i\biggl(\sin\left(\frac{b\pi}{q}\right)\biggr)e^{b\pi i/q}, \tag{2.15}$$

and so the principal value of the logarithm is

$$\log(1-\zeta_q^b) = \log\left(2\sin\frac{b\pi}{q}\right) + \left(\frac{b}{q}-\frac{1}{2}\right)\pi i \tag{2.16}$$

for $1 \le b < q$. Substituting (2.16) in (2.10), we get

$$\begin{aligned} L(1,f) &= -\sum_{b=1}^{q-1}\widehat{f}(b)\left[\log\left(2\sin\frac{b\pi}{q}\right) + \left(\frac{b}{q}-\frac{1}{2}\right)\pi i\right] \\ &= -\sum_{b=1}^{q-1}\widehat{f}(b)\log\left(2\sin\frac{b\pi}{q}\right) - \frac{i\pi}{q}\sum_{b=1}^{q-1} b\widehat{f}(b) + \frac{i\pi}{2}\sum_{b=1}^{q-1}\widehat{f}(b). \end{aligned} \tag{2.17}$$

Since f is an odd function, $\widehat{f}$ is also an odd function. Hence,

$$2\sum_{b=1}^{q-1}\widehat{f}(b) = \sum_{b=1}^{q-1}[\widehat{f}(b)+\widehat{f}(q-b)] = 0.$$

Therefore, the last term of (2.17) is zero. Now, note that $\sin(\pi-\theta)=\sin(\theta)$. Thus, $\sin(b\pi/q)$ is an even function of b. Hence, $\log(2\sin\frac{b\pi}{q})$ is even which implies that

$\widehat{f}(b)\log(2\sin\frac{b\pi}{q})$ is an odd function. Therefore, the first term of (2.17),

$$\sum_{b=1}^{q-1}\widehat{f}(b)\log\left(2\sin\frac{b\pi}{q}\right)=0.$$

This gives us that for odd periodic functions f,

$$L(1,f)=-\frac{i\pi}{q}\sum_{b=1}^{q-1}b\,\widehat{f}(b). \tag{2.18}$$

A similar simplification is possible when f is even, namely,

$$\begin{aligned}
L(1,f)&=-\sum_{b=1}^{q-1}\widehat{f}(b)\,\log(1-\zeta_q^b)\\
&=-\frac{1}{2}\sum_{b=1}^{q-1}\left(\widehat{f}(b)\,\log(1-\zeta_q^b)+\widehat{f}(q-b)\,\log(1-\zeta_q^{q-b})\right)\\
&=-\frac{1}{2}\sum_{b=1}^{q-1}\widehat{f}(b)\Big(\log(1-\zeta_q^b)+\overline{\log(1-\zeta_q^b)}\Big)\\
&=-\sum_{b=1}^{q-1}\widehat{f}(b)\,\log|1-\zeta_q^b|.
\end{aligned}$$

In the specific case of even primitive Dirichlet characters mod q, using (2.6), this gives

$$L(1,\chi)=\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\overline{\chi(a)}\,\log|1-\zeta_q^a|. \tag{2.19}$$

2.3 The value $L(k, f)$

Let $k \geq 1$ be an integer and f be an arithmetical function, periodic with period q and $\sum_{a=1}^{q} f(a) = 0$. Analogous to the Riemann zeta-function, the values of $L(s, f)$ at positive integers other than 1 are also of significant interest. From (2.2), we know that

$$L(k, f) = \frac{1}{q^k} \sum_{a=1}^{q} f(a) \sum_{n=0}^{\infty} \frac{1}{(n + (a/q))^k} = \frac{1}{q^k} \sum_{a=1}^{q} f(a) \zeta\left(k, \frac{a}{q}\right).$$

Thus, for $k \geq 2$, it suffices to study the value $\zeta(k, a/q)$.

The most important functions that act as pillars for our knowledge of the values $L(k, f)$ are the cotangent function, the digamma function and their derivatives. Their interaction is of foundational significance in our understanding. In particular, their series representations and in turn, their functional relation will play a central role in determining the arithmetic nature of $L(k, f)$.

Both these functions arise from the gamma function. The digamma function is the logarithmic derivative of $\Gamma(z)$, whereas the cotangent function can be obtained from the logarithmic differentiation of $\Gamma(z)\Gamma(1-z)$. More specifically, we have the infinite product expansion,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \quad z \neq 0, \, -1, \, -2, \, -3, \cdots,$$

where γ denotes the Euler-Mascheroni constant. Taking the logarithmic derivative of both sides gives the series representation of the digamma function,

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right), \tag{2.20}$$

for $z \neq 0, \, -1, \, -2, \, \cdots$. The higher derivatives of the digamma function are called the polygamma functions. Differentiating (2.20) gives

$$\psi_k(z) = (-1)^{k+1}\, k! \sum_{n=0}^{\infty} \frac{1}{(z+n)^{k+1}},$$

for $k \geq 1$. It is easy to see that for $k \geq 2$,

$$\zeta\left(k, \frac{a}{q}\right) = \frac{(-1)^{k-1}}{(k-1)!}\ \psi_{k-1}\left(\frac{a}{q}\right). \tag{2.21}$$

Thus, in order to study $L(k, f)$, an essential ingredient is the values of the polygamma function at rational arguments. However, these values are not well-understood. But one can exploit the relation of the polygamma functions with the derivatives of the cotangent function to derive more information in certain special cases. More details of this method can be found in [44].

In the same spirit as above, taking the logarithmic derivative of the infinite product expansion of $\Gamma(z)\Gamma(1-z)$ gives the series representation of the cotangent, by the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, for $z \notin \mathbb{Z}$. Therefore, we have

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right),$$

for $z \notin \mathbb{Z}$. As a result of the reflection formula, the two functions satisfy

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z.$$

On differentiating both sides with respect to z, we get

$$\psi_k(z) - (-1)^k \psi_k(1-z) = -\frac{d^k}{dz^k}\Big(\pi \cot \pi z\Big). \tag{2.22}$$

Since

$$\cot \pi x = i\,\frac{e^{i\pi x} + e^{-i\pi x}}{e^{i\pi x} - e^{-i\pi x}},$$

and $e^{i\pi a/q}$ is algebraic, one can conclude that $\cot(a\pi/q) \in \overline{\mathbb{Q}}$ for $1 \leq a < q$. Additionally, one can deduce that

$$\frac{d^k}{dx^k}\Big(\cot \pi x\Big)\Big|_{x=a/q} \in \pi^k \overline{\mathbb{Q}}.$$

Hence, we obtain that

$$\psi_k\left(\frac{a}{q}\right) - (-1)^k \psi_k\left(1 - \frac{a}{q}\right) \in \pi^k \overline{\mathbb{Q}}.$$

Therefore, one can study the arithmetic nature of the value $L(k, f)$ whenever we can pair the values of the polygamma function at a/q and $1-(a/q)$ to precisely get the above expression. In particular, *when k and f have the same parity*, one can express $L(k, f)$ using (2.22) as

$$L(k, f) = -\frac{(-1)^k}{(k-1)!\,q^k} \sum_{a=1}^{(q-1)/2} f(a)\left(\frac{d^{(k-1)}}{dz^{(k-1)}}(\pi \cot \pi z)\Big|_{z=a/q}\right), \tag{2.23}$$

and infer that $L(k, f)$ is an algebraic multiple of π^k in this case.

Analogous to (2.10), one can express the value $L(k, f)$ as a linear form in values of the k^{th} polylogarithm. For an integer $k \geq 1$ and $z \in \mathbb{C}$ with $|z| \leq 1$ if $k \geq 2$ and $|z| < 1$ if $k = 1$, the k^{th} polylogarithm is defined as

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Hence, when $k = 1$ and $|z| < 1$, $\mathrm{Li}_1(z) = -\log(1-z)$. Using the Fourier inversion formula (2.4), it can be seen that for $k > 1$,

$$\begin{aligned} L(k,f) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{b=1}^{q} \widehat{f}(b) \zeta_q^{bn} \\ &= \sum_{b=1}^{q} \widehat{f}(b) \sum_{n=1}^{\infty} \frac{(\zeta_q^b)^n}{n^k}. \end{aligned}$$

Therefore, we have for $k > 1$,

$$L(k,f) = \sum_{b=1}^{q} \widehat{f}(b)\, \mathrm{Li}_k(\zeta_q^b). \tag{2.24}$$

Hence, the vanishing and non-vanishing of $L(k,f)$ is linked with the linear relations among values of the polylogarithms at roots of unity.

2.4 The value $L'(1,f)$

In this section, we study the derivative of $L(s,f)$ at $s = 1$. In analogy with the notation of generalized Bernoulli numbers associated to Dirichlet characters, we define

$$B_{1,f} := \sum_{a=1}^{q} a\, f(a),$$

where f is an odd arithmetical function periodic with period q. Then,

Lemma 2.4.1. *For any arithmetical function f periodic with period q and $\sum_{a=1}^{q} f(a) = 0$,*

$$L'(0,f) = \frac{\log q}{q} B_{1,f} + \sum_{b=1}^{q} f(b) \log \Gamma\left(\frac{b}{q}\right).$$

Proof. By differentiating (2.2) with respect to s, we have

$$L'(s,f) = \frac{-\log q}{q^s}\left[\sum_{a=1}^{q} f(a)\,\zeta\left(s, \frac{a}{q}\right)\right] + \left[\frac{1}{q^s}\sum_{a=1}^{q} f(a)\,\zeta'\left(s, \frac{a}{q}\right)\right].$$

Substituting $s = 0$ gives

$$L'(0,f) = -\log q\left[\sum_{a=1}^{q} f(a)\,\zeta\left(0, \frac{a}{q}\right)\right] + \left[\sum_{a=1}^{q} f(a)\zeta'\left(0, \frac{a}{q}\right)\right].$$

Now, the values of the Hurwitz zeta-function and its derivative at $s = 0$ are given by

$$\zeta(0,x) = 1 + \zeta(0) - x,\ \zeta'(0,x) = \log\Gamma(x) + \zeta'(0),$$

where $\zeta(s)$ is the Riemann zeta function. A proof of the above fact can be found in [28]. Substituting these values in the expression obtained earlier, we get

$$\begin{aligned} L'(0,f) &= -(\log q)\big(1 + \zeta(0)\big)\left[\sum_{a=1}^{q} f(a)\right] + \frac{\log q}{q}\sum_{a=1}^{q} f(a)\,a \\ &\quad + \zeta'(0)\left[\sum_{a=1}^{q} f(a)\right] + \sum_{a=1}^{q} f(a)\log\Gamma\left(\frac{a}{q}\right) \\ &= \frac{\log q}{q}\sum_{a=1}^{q} f(a)\,a + \sum_{a=1}^{q} f(a)\log\Gamma\left(\frac{a}{q}\right), \end{aligned}$$

since $\sum_{a=1}^{q} f(a) = 0$. This proves the lemma. $\square$

One can now study the value $L'(1,f)$ using the functional equation obtained in

Lemma 2.1.1, when f is odd. However, when f is even, the gamma factors in the functional equation of $L(s,f)$ are not amenable to obtain the value at $s=1$.

Lemma 2.4.2. *Let f be an* odd *periodic arithmetical function with period q satisfying $f(q)=\widehat{f}(q)=0$. Then,*

$$L'(1,f)=\frac{i\pi}{q}\left\{\left(\left(1+\frac{1}{q}\right)(\log q)-\log 2\pi-\gamma\right)B_{1,\widehat{f}}+\sum_{b=1}^{q}\widehat{f}(b)\log\Gamma\left(\frac{b}{q}\right)\right\},$$

where $B_{1,g}:=\sum_{a=1}^{q}a\,g(a)$ for any odd arithmetical function periodic with period q.

Proof. Note that if f is an odd periodic arithmetical function, then so is $\widehat{f}$. Thus, differentiating the functional equation for $L(s,\widehat{f})$ from Lemma 2.1.1 gives

$$\begin{aligned}-L'(1-s,\widehat{f})&=2\,i\,\Gamma'(s)\left(\frac{q}{2\pi}\right)^{s}\sin\left(\frac{s\pi}{2}\right)L(s,f)\\&\quad+2\,i\,\Gamma(s)\left(\frac{q}{2\pi}\right)^{s}\log\left(\frac{q}{2\pi}\right)\sin\left(\frac{s\pi}{2}\right)L(s,f)\\&\quad+2\,i\,\Gamma(s)\left(\frac{q}{2\pi}\right)^{s}\frac{\pi}{2}\cos\left(\frac{s\pi}{2}\right)L(s,f)+2\,i\,\Gamma(s)\left(\frac{q}{2\pi}\right)^{s}\sin\left(\frac{s\pi}{2}\right)L'(s,f).\end{aligned}$$

Since $f(q)=\widehat{f}(q)=0$, both $L(s,f)$ and $L(s,\widehat{f})$ are entire. Thus, taking limit as s tends to 1 in the above expression, we have

$$\begin{aligned}-L'(0,\widehat{f})&=2\,i\,\Gamma(1)\frac{q}{2\pi}\sin\left(\frac{\pi}{2}\right)\left\{\left(\frac{\Gamma'}{\Gamma}(1)+\log\left(\frac{q}{2\pi}\right)\right)L(1,f)+L'(1,f)\right\}\\&=\frac{iq}{\pi}\left\{L'(1,f)+L(1,f)\left(\log\left(\frac{q}{2\pi}\right)-\gamma\right)\right\},\end{aligned}$$

as $\Gamma'(1)/\Gamma(1)=-\gamma$. By rearrangement, we get

$$L'(1,f)=\frac{i\pi}{q}L'(0,\widehat{f})-\left(\log\left(\frac{q}{2\pi}\right)-\gamma\right)L(1,f).$$

Using (2.18) derived in the last section, we have

$$L(1, f) = \frac{-i\pi}{q} B_{1,\widehat{f}},$$

for an odd periodic function f. This evaluation, together with Lemma 2.4.1 gives

$$\begin{aligned} L'(1, f) &= \frac{i\pi}{q}\left\{\frac{\log q}{q} B_{1,\widehat{f}} + \sum_{b=1}^{q} \widehat{f}(b) \log \Gamma\left(\frac{b}{q}\right) + \left(\log\left(\frac{q}{2\pi}\right) - \gamma\right) B_{1,\widehat{f}}\right\}, \\ &= \frac{i\pi}{q}\left\{\left(\frac{\log q}{q} + \log q - \log 2\pi - \gamma\right) B_{1,\widehat{f}} + \sum_{b=1}^{q} \widehat{f}(b) \log \Gamma\left(\frac{b}{q}\right)\right\}, \end{aligned}$$

from which the lemma is immediate. □

2.5 Laurent expansion around $s = 1$

Fundamental properties of a function are captured by its Laurent series coefficients. A case in point is Li's criterion for Riemann hypothesis obtained by X. J. Li [55] in 1997. More specifically, let

$$\lambda_n := \sum_{\rho}\left(1 - \left(1 - \frac{1}{\rho}\right)^n\right),$$

where the sum is over non-trivial zeros of the Riemann zeta-function. Then Li proved that the Riemann hypothesis is equivalent to the positivity of λ_n for all $n \in \mathbb{N}$. Furthermore, if

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} \eta_j (s-1)^j, \tag{2.25}$$

then it was shown in [15] that

$$\lambda_n = -\sum_{j=1}^{n}\left[\binom{n}{j}\eta_{j-1}\right] + 1 - (\log 4\pi + \gamma)\frac{n}{2} + \sum_{j=2}^{n} (-1)^j \binom{n}{j} (1 - 2^{-j})\zeta(j). \tag{2.26}$$

Thus, the Riemann hypothesis is intricately linked with the Laurent series coefficients of $-\zeta'/\zeta$. This inspires the study of the Laurent series coefficients of more general Dirichlet series. In this context, we prove the following proposition (see [64]).

Proposition 2.5.1. *Let*

$$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series, absolutely convergent on $\Re(s) > 1$. Suppose that for any $A > 0$,

$$S(x) := \sum_{n \leq x} a_n = \delta\, x + E(x), \tag{2.27}$$

for some $\delta \in \mathbb{R}$ and

$$E(x) = O\biggl(\frac{x}{(\log x)^A}\biggr). \tag{2.28}$$

Then, by partial summation, $f(s)$ can be analytically continued to $\Re(s) \geq 1$, with a possible simple pole at $s = 1$ and one can write its Laurent series expansion around $s = 1$ as

$$f(s) = \frac{\delta}{s-1} + \sum_{j=0}^{\infty} \eta_j(1, f)(s-1)^j.$$

Then,

$$\eta_0(1, f) = \delta + \int_1^{\infty} \frac{E(t)}{t^2}\, dt,$$

and for $j \geq 1$,

$$\eta_j(1, f) = \frac{(-1)^j}{j!} \int_1^{\infty} \frac{E(t)}{t^2} \Bigl((\log t)^j - j(\log t)^{j-1}\Bigr)\, dt. \tag{2.29}$$

Further, for $j \geq 1$,

$$\eta_j(1, f) = \frac{(-1)^j}{j!} \left\{ \lim_{x \to \infty} \left[\sum_{n \leq x} \frac{a_n (\log n)^j}{n} \right] - \delta \, \frac{(\log x)^{j+1}}{j+1} \right\}.$$

Proof. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Then the usual partial summation method gives,

$$\begin{aligned} f(s) &= s \int_1^{\infty} \frac{S(x)}{x^{s+1}} \, dx \\ &= \frac{\delta \, s}{s-1} + s \int_1^{\infty} \frac{S(x) - \delta \, x}{x^{s+1}} \, dx. \end{aligned}$$

By our hypothesis, the integral on the right hand side converges absolutely for $\Re(s) \geq 1$. Thus, we can derive the Laurent expansion at $s = 1$ using this integral. Writing $E(x) = S(x) - \delta x$, we find

$$\begin{aligned} s \int_1^{\infty} \frac{E(x)}{x^{s+1}} \, dx &= \big((s-1)+1\big) \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \, (\log x)^j}{j!} \, (s-1)^j \, dx \\ &= \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \, (\log x)^j}{j!} (s-1)^{j+1} \, dx + \\ &\qquad \int_1^{\infty} \frac{E(x)}{x^2} \sum_{j=0}^{\infty} \frac{(-1)^j \, (\log x)^j}{j!} \, (s-1)^j \, dx \\ &= \int_1^{\infty} \frac{E(x)}{x^2} \, dx + \\ &\qquad \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \, (s-1)^j \int_1^{\infty} \frac{E(x)}{x^2} \Big((\log x)^j - j (\log x)^{j-1} \Big) \, dx, \end{aligned}$$

the interchange of summation and integral being justified by the absolute convergence of the integral at $s = 1$. Thus, we see that an integral representation for the Laurent series coefficients at $s = 1$ of a general Dirichlet series can be obtained.

On the other hand, analysis with the help of partial summation gives

$$\sum_{n \le x} \frac{a_n (\log n)^j}{n} = \frac{S(x)\,(\log x)^j}{x} + \int_1^x \frac{S(t)}{t^2}\left((\log t)^j - j(\log t)^{j-1}\right) dt$$

$$= \frac{S(x)\,(\log x)^j}{x} + \delta \int_1^x \frac{(\log t)^j}{t}\, dt - j\,\delta \int_1^x \frac{(\log t)^{j-1}}{t}\, dt + \int_1^x \frac{E(t)}{t^2}\left((\log t)^j - j(\log t)^{j-1}\right) dt,$$

by (2.27). Hence, we deduce that

$$\sum_{n \le x} \frac{a_n (\log n)^j}{n} = \delta\, \frac{(\log x)^{j+1}}{j+1} + \int_1^\infty \frac{E(t)}{t^2}\left((\log t)^j - j(\log t)^{j-1}\right) dt + \left(\frac{S(x)\,(\log x)^j}{x} - \delta\,(\log x)^j\right) + \mathcal{E}(x),$$

where

$$\mathcal{E}(x) = \int_x^\infty \frac{E(t)}{t^2}\left((\log t)^j - j(\log t)^{j-1}\right) dt.$$

Note that $\mathcal{E}(x)$ is the tail of a convergent integral by (2.28) and since $j \ge 1$ and therefore, tends to zero as $x \to \infty$. Moreover, the third term on the right hand side also goes to zero as $x \to \infty$ by (2.27). On comparison with (2.29), the proposition is proved. □

The constants $\eta_j(1, \zeta)$ (known as Stieltjes constants) were first introduced by

Stieltjes (see [71, pg.161]), who proved that

$$\eta_j(1,\zeta) = \frac{(-1)^j}{j!}\left\{ \lim_{x\to\infty}\left[\sum_{n\leq x}\frac{(\log n)^j}{n}\right] - \frac{(\log x)^{j+1}}{j+1}\right\},$$

in a letter to Hermite in 1885. This formula seems to have been rediscovered by Briggs and Chowla in 1955 (see [71, pg. 163]). Clearly, this result can be stated in a more general setting, as is seen in Proposition 2.5.1.

As an arithmetic progression analog of the Stieltjes constants, we can consider the Laurent coefficients of the series

$$\zeta_{a,q}(s) := \sum_{\substack{n=1,\\ n\equiv a \bmod q}}^{\infty} \frac{1}{n^s}, \ \ \Re(s) > 1.$$

Although $\zeta_{a,q}(s) = 1/q^s\zeta(s, a/q)$, we will use the above notation to avoid the confusion arising from the factor q^{-s}. The standard notation for the eta-coefficients associated to the above series (which we continue to use henceforth) is $\eta_j(1,\zeta_{a,q}) = \gamma_j(a,q)$ for $j \geq 0$ and $\gamma_0(a,q) = \gamma(a,q)$. These coefficients were first studied by D. H. Lehmer [53] in the case $j = 0$ and by Knopfmacher [49] in general. We will refer to these constants as *generalized Stieltjes constants*.

A useful connection between value of derivatives of L-functions, attached to periodic functions at $s = 1$ and generalized Stieltjes constants is illustrated in the following lemma. We include its proof for the sake of completeness (see [49, Proposition 3.2]).

Lemma 2.5.2. *For an arithmetical function f, which is periodic with period q and satisfies $\widehat{f}(q) = 0$, we have*

$$L^{(k)}(1, f) = (-1)^k \sum_{a=1}^{q} f(a)\gamma_k(a, q),$$

where $\gamma_k(a, q)$ are generalized Stieltjes constants as defined earlier.

Proof. With a view to brevity, let

$$H_k(x, a, q) := \sum_{\substack{n \le x, \\ n \equiv a \bmod q}} \frac{\log^k n}{n},$$

for any positive real number x. Observe that

$$\begin{aligned} \sum_{n \le x} f(n) \frac{\log^k n}{n} &= \sum_{a=1}^{q} f(a) H_k(x, a, q) \\ &= \sum_{a=1}^{q} f(a) \left(H_k(x, a, q) - \frac{\log^{k+1} x}{q(k+1)} \right), \end{aligned}$$

since $q\,\widehat{f}(q) = \sum_{a=1}^{q} f(a) = 0$. Taking limit as x tends to infinity on both sides gives the result. $\square$

Chapter 3

The Erdős and Livingston conjectures

In a written communication with Livingston, Erdős [57] conjectured the following:

Conjecture 3.0.1. *(Erdős) Let q be a positive integer and f be an arithmetical function, periodic with period q. If $f(n) \in \{-1, 1\}$ when $q \nmid n$ and $f(n) = 0$ otherwise, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

whenever the series is convergent.

This conjecture will be discussed comprehensively in Chapter 4.

In 1965, Livingston [57] attempted to resolve the above conjecture. He predicted that to settle Conjecture 3.0.1, one would first have to prove:

Conjecture 3.0.2. *(Livingston) Let $q \geq 3$ be a positive integer. The numbers*

$$\left\{ \log\left(2 \sin \frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\} \text{ and } \pi$$

when q is odd, and

$$\left\{ \log\left(2\sin\frac{a\pi}{q}\right) : 1 \le a < \frac{q}{2} \right\},\ \pi \text{ and } \log 2$$

when q is even, are linearly independent over the field of algebraic numbers.

The above statement does not depend on the branch of log considered, as the values would only differ by an integer multiple of $2\pi i$.

In [77], we proved Livingston's conjecture when q is an odd prime or 4, and showed it to be false when $q \ge 6$ is composite. We also observed that Livingston's conjecture is not sufficient to imply Erdős's conjecture. We discuss these results in this chapter.

One of the main ingredients of the proofs is the following remarkable theorem of A. Baker [6, Theorem 2.1, pg. 10] on linear forms in logarithms of algebraic numbers.

Theorem 3.0.3 (A. Baker). *If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are non-zero algebraic numbers such that $\log\alpha_1$, $\log\alpha_2$, $\cdots$, $\log\alpha_n$ are linearly independent over the rationals, then 1, $\log\alpha_1$, $\log\alpha_2$, $\cdots$, $\log\alpha_n$ are linearly independent over the field of all algebraic numbers.*

3.1 The approach of Livingston

Let f be an Erdős function, i.e, $f(n) = \pm 1$ when $q \nmid n$ and $f(n) = 0$ whenever $q \mid n$. The condition for the existence of $L(1, f)$ implies that

$$\sum_{a=1}^{q} f(a) = \sum_{a=1}^{q-1} f(a) = 0. \tag{3.1}$$

As seen earlier in (2.13), $L(1,f)$ can be expressed as

$$L(1,f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right)+\frac{2}{q}\sum_{b=1}^{r}\left\{\left[\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right)\right]\log\left(2\sin\frac{\pi b}{q}\right)\right\}-T_q,$$

where

$$T_q = \begin{cases} \frac{\log 2}{q}\left(\sum\limits_{k=1}^{q-1}(-1)^k f(k)\right) & \text{if } q \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the numbers

$$\cot\left(\frac{a\pi}{q}\right) \text{ and } \cos\left(\frac{2\pi ab}{q}\right)$$

are algebraic for $1 \leq a < q$ and $1 \leq b < q$. Since $f(a) \in \overline{\mathbb{Q}}$ and $f(q) = 0$, we deduce that $L(1,f)$ is an algebraic linear combination of

$$\pi, \log\left(2\sin\frac{\pi}{q}\right), \log\left(2\sin\frac{2\pi}{q}\right), \cdots, \log\left(2\sin\frac{(q-1)\pi}{2q}\right)$$

together with $\log(2)$ when q is even. This led Livingston to predict that if Conjecture 3.0.1 were to be true, the above numbers should be linearly independent over $\overline{\mathbb{Q}}$. At this point, we make the following key observation - to conclude Conjecture 3.0.1 as an implication of Conjecture 3.0.2, one is still required to prove that the resulting relation is non-trivial. That is, if f is an Erdős function, not identically zero, then at least one of

$$\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right), \tag{3.2}$$

or

$$\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi ab}{q}\right), \quad 1 \leq b \leq r \tag{3.3}$$

or T_q is not zero. This question is not addressed by Conjecture 3.0.2 and hence, Livingston's conjecture is not sufficient to settle the conjecture of Erdős.

Remark. *If f is allowed to take values in $\overline{\mathbb{Q}}$ and q is odd, then there exist a plethora of examples of functions f that are not identically zero but for which* (3.2) *and* (3.3) *are both zero for all $1 \le b \le r$. These are given by the following theorem from [5]:*

Theorem 3.1.1. *Let $q \geq 3$ be a natural number. Then all odd, algebraically-valued functions f, periodic mod q, for which $L(1, f) = 0$ are given by the totality of linear combinations with algebraic coefficients of the following $\left\lfloor \frac{1}{2}(q-3) \right\rfloor$ functions:*

$$f_l(n) = (-1)^{n-1} \left(\frac{\sin n\pi/q}{\sin \pi/q} \right)^l, \quad \text{for } l = 3, 5, \cdots, (q-2) \tag{3.4}$$

when q is odd and

$$f_l(n) = (-1)^{n-1} \left(\frac{\cos n\pi/q}{\cos \pi/q} \right) \left(\frac{\sin n\pi/q}{\sin \pi/q} \right)^l \quad \text{for } l = 3, 5, \cdots, (q-1)$$

when q is even. The functions are linearly independent and take values in $\mathbb{Q}(\zeta_q)$, i.e, the q-th cyclotomic field.

Each f_l in the above theorem is an odd function. Since $\cos(2\pi ab/q)$ is an even function for $1 \le a < q$, (3.3) *is zero for all $1 \le b \le r$. $T_q = 0$ as q is odd. Thus,*

$$L(1, f) = \frac{-\pi}{2q} \sum_{a=1}^{q-1} f(a) \cot\left(\frac{a\pi}{q} \right),$$

which is zero by Theorem 3.1.1.

3.2 Main theorems

We note a useful observation before proceeding. If q is a positive integer and $1 \leq a < q/2$, then

$$2\sin\frac{a\pi}{q} = \frac{e^{ia\pi/q} - e^{-ia\pi/q}}{i} = ie^{-ia\pi/q}(1 - \zeta_q^a), \tag{3.5}$$

where $\zeta_q = e^{2\pi i/q}$. Since

$$\sin\frac{a\pi}{q} > 0,$$

for $1 \leq a < q/2$ and log denotes the principal branch,

$$\begin{aligned}\log\left(2\sin\frac{a\pi}{q}\right) &= \log\left(\left|1-\zeta_q^a\right|\right) + i0 = \log\left(\left|1-\zeta_q^a\right|\right)\\ &= \log\left(\left|1-\zeta_q^{-a}\right|\right) = \log\left(2\sin\frac{(q-a)\pi}{q}\right).\end{aligned} \tag{3.6}$$

3.2.1 The case $q \geq 6$, q composite

We will prove the following theorem in this section.

Theorem 3.2.1. *Conjecture 3.0.2 does not hold for $q \geq 6$ and q not prime. In fact, for a composite positive integer $q \geq 6$, the numbers*

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2}\right\}$$

are $\mathbb{Q}$-linearly dependent.

Proof. We prove the linear dependence of the numbers under consideration by giving an explicit $\mathbb{Q}$-relation among them. By (3.6), it suffices to exhibit a relation among logarithms of cyclotomic numbers. Now, since q is not prime, there is a divisor d of q such that $d \neq 1$, q. For such a divisor d, we have the following polynomial identity in $\mathbb{C}[X, Y]$:

$$X^{q/d} - Y^{q/d} = \prod_{j=1}^{q/d} \left(X - \zeta_{q/d}^{j} Y\right),$$

where $\zeta_{q/d} = e^{2\pi i d/q}$. Substituting $X = 1$ and $Y = \zeta_q^a$ for $(a, q) = 1$, we have

$$1 - e^{2\pi i a/d} = \prod_{j=1}^{q/d} \left(1 - e^{2\pi i (dj/q + a/q)}\right) = \prod_{j=1}^{q/d} \left(1 - e^{2\pi i (a+dj)/q}\right)$$

Thus, taking absolute values of both sides of the above equation gives us

$$\left(\left|1 - \zeta_q^{aq/d}\right|\right) = \prod_{j=1}^{q/d} \left(\left|1 - \zeta_q^{(a+dj)}\right|\right).$$

Taking logarithms of both sides, we obtain the following $\mathbb{Q}$-linear relation

$$\log\left(\left|1 - \zeta_q^{aq/d}\right|\right) - \sum_{j=1}^{q/d} \log\left(\left|1 - \zeta_q^{(a+dj)}\right|\right) = 0,$$

for all $1 \leq a < q$ and $(a, q) = 1$ and $d \mid q$, $d \neq 1, q$. Hence, using (3.6), we have

$$\log\left(2 \sin\left(\frac{aq}{d} \frac{\pi}{q}\right)\right) - \sum_{j=1}^{q/d} \log\left(2 \sin \frac{(a + dj)\pi}{q}\right) = 0. \tag{3.7}$$

Since we want a linear relation among

$$\left\{ \log\left(2 \sin \frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2} \right\},$$

we will replace $\log(2 \sin(b\pi/q))$ by $\log(2 \sin((q - b)\pi/q))$ whenever $b \geq q/2$. This is valid by (3.6). Now, we make the following observations. Suppose that there exists a

k such that $1 \leq k < q/2$ and

$$k \equiv a + dj \equiv a + dl \bmod q,$$

for some $1 \leq j, l \leq q/d$ and $j \neq l$. This implies that $q|d(j-l)$, which is impossible since $(j-l) < q/d$. Thus,

$$a + dj \not\equiv a + dl \bmod q, \tag{3.8}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Similarly,

$$-(a + dj) \not\equiv -(a + dl) \bmod q, \tag{3.9}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Suppose there exists a k such that $1 \leq k < q/2$ and

$$k \equiv a + dj \equiv -(a + dl) \bmod q,$$

for $1 \leq j, l \leq q/d$ and $j \neq l$. Thus, $q|(2a+d(j+l))$. Since $d|q$, we have $d|(2a+d(j-l))$, i.e, $d|2a$. But $(a, q) = 1$. Hence, $(a, d) = 1$, which implies that $d|2$. We assumed that $d \neq 1, q$. Therefore, $d = 2$. As a result, we have

$$a + dj \not\equiv -(a + dl) \bmod q, \tag{3.10}$$

for $1 \leq j, l \leq q/d$ and $j \neq l$ unless $d = 2$.

Thus, for $(a, q) = 1$, $d \mid q$ and $2 < d < q$, (3.7) along with (3.8), (3.9) and (3.10) give us a non-trivial $\mathbb{Q}$-relation, namely,

$$\mathfrak{R}_{a,d} := \sum_{1\leq k<q/2} \alpha_k \log\left(2\sin\frac{k\pi}{q}\right) = 0,$$

where α_k is determined as follows:

$$\alpha_k = -1 \text{ if } \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \not\equiv aq/d \bmod q \ \&\ k \equiv \pm(a+dj) \bmod q \\ \text{or } (aq/d \bmod q) \geq q/2,\ k \not\equiv -(aq/d) \bmod q \ \&\ k \equiv \pm(a+dj) \bmod q, \end{cases}$$

for some $1 \leq j \leq q/d$,

$$\alpha_k = 1 \text{ if } \begin{cases} \text{either } (aq/d \bmod q) < q/2,\ k \equiv aq/d \bmod q \ \&\ k \not\equiv \pm(a+dj) \bmod q \\ \text{or } (aq/d \bmod q) \geq q/2,\ k \equiv -(aq/d) \bmod q \ \&\ k \not\equiv \pm(a+dj) \bmod q, \end{cases}$$

for some $1 \leq j \leq q/d$ and

$$\alpha_k = 0, \text{ otherwise.}$$

To see that the above relation is non-trivial for q not prime and $q \geq 6$, note that at least one of the following scenarios happens- either $(aq/d \bmod q) < q/2$, in which case for $k \equiv aq/d \bmod q$, $\alpha_k = \pm 1$, or $(aq/d \bmod q) \geq q/2$, in which case for $k \equiv -(aq/d) \bmod q$, $\alpha_k = \pm 1$.

Hence, the numbers under consideration in Conjecture 3.0.2 are $\mathbb{Q}$-linearly dependent. As a result, Livingston's conjecture is false when q is not prime and $q \geq 6$. □

3.2.2 The case $q = 4$ or q an odd prime

Conjecture 3.0.2 holds for $q = 4$ because the set $\{1 \leq a < q/2\}$ is a singleton, namely, $a = 1$ and

$$\log\left(2\sin\frac{\pi}{4}\right) = \log\sqrt{2} \neq 0.$$

Hence, we concentrate on the case when $q = p$, an odd prime. In this case, we will show that Conjecture 3.0.2 is in fact true. The proof is based on the theory of matrices of Dedekind type.

Let $\mathfrak{M}$ be an $n \times n$ matrix with complex entries. Let $m_{i,j}$ denote the (i,j)-th entry of $\mathfrak{M}$. Then, $\mathfrak{M}$ is said to be of Dedekind type if there exists a finite abelian group, $G = \{x_1, x_2, \cdots, x_n\}$ and a complex valued function f on G such that

$$m_{i,j} = f(x_i^{-1}x_j),$$

for all $1 \leq i, j \leq n$. We will use the following well-known theorem regarding matrices of the Dedekind type:

Theorem 3.2.2. *Let $\mathfrak{M}$ be an $n \times n$ matrix of the Dedekind type. For a character χ on G (a homomorphism of G into $\mathbb{C}^*$), define*

$$S_\chi := \sum_{s \in G} f(s)\chi(s).$$

Then the determinant of $\mathfrak{M}$ is equal to

$$\prod_\chi S_\chi,$$

where the product runs over all characters of G. Thus, $\mathfrak{M}$ is invertible if and only if

$$S_\chi \neq 0,$$

for all characters χ of G.

For a proof of the above theorem and an exposition on properties of matrices of the Dedekind type, we refer the reader to [69]. The determinant of a matrix of the Dedekind type is often referred to as a Dedekind determinant.

With this background in place, we prove the following.

Theorem 3.2.3. *Let p be an odd prime. The numbers*

$$\left\{ \log\left(2\sin\frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\} \text{ and } \pi$$

are $\overline{\mathbb{Q}}$-linearly independent. Thus, Conjecture 3.0.2 is true when the modulus p is prime.

Proof. Let p be an odd prime. Our aim is to prove that the numbers

$$\left\{ \log\left(2\sin\frac{a\pi}{p}\right) : 1 \leq a \leq \frac{p-1}{2} \right\} \text{ and } \pi$$

are $\overline{\mathbb{Q}}$-linearly independent.

Suppose, to the contrary, that the above numbers have a $\overline{\mathbb{Q}}$-linear relation among them. Thus, there exist algebraic numbers $\beta_0, \beta_1, \cdots, \beta_r$, not all zero, such that

$$\beta_0 \pi + \sum_{a=1}^{r} \beta_a \log \left(2 \sin \frac{a\pi}{p} \right) = 0, \tag{3.11}$$

where $r = (p-1)/2$. If $\beta_0 \neq 0$, then (3.11) does not hold by the following Lemma from [68]:

Lemma 3.2.4. *If $c_0, c_1, \cdots, c_n$ are algebraic numbers and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are positive algebraic numbers with $c_0 \neq 0$, then*

$$c_0 \pi + \sum_{j=1}^{n} c_j \log \alpha_j \neq 0.$$

Thus, β_0 must be zero. Now, if the numbers

$$\left\{ \log \left(2 \sin \frac{a\pi}{p} \right) : 1 \leq a \leq \frac{p-1}{2} \right\}$$

are $\mathbb{Q}$-linearly independent, then by Baker's Theorem 3.0.3, the above numbers are also $\overline{\mathbb{Q}}$-linearly independent. This contradicts our assumption, and hence, the above numbers must satisfy a $\mathbb{Q}$-linear relation. Thus, there exist $b_1, b_2, \cdots, b_r$ such that

$$\sum_{a=1}^{r} b_a \log \left(2 \sin \frac{a\pi}{p} \right) = 0. \tag{3.12}$$

On clearing denominators, we can assume that

$$b_a \in \mathbb{Z}, \ 1 \leq a \leq \frac{(p-1)}{2}.$$

Since log denotes the principal branch and $\sin a\pi/p \in \mathbb{R}_{>0}$, (3.12) gives us the multiplicative relation -

$$\prod_{a=1}^{r}\left(2\sin\frac{a\pi}{p}\right)^{b_a} = 1.$$

Using (3.5), this relation can be interpreted as a relation among roots of unity and cyclotomic numbers, i.e,

$$\prod_{a=1}^{r}\left(ie^{-ia\pi/p}(1-\zeta_p^a)\right)^{b_a} = 1.$$

The above relation can be further simplified by raising both sides of the equation to the $4p$-th power. Since $\left(ie^{-ia\pi/p}\right)^{4p} = 1$, we are now left with the simpler multiplicative relation,

$$\prod_{a=1}^{r}\left(1-\zeta_p^a\right)^{B_a} = 1, \tag{3.13}$$

where $B_a := 4pb_a$ and each factor in the product belongs to the cyclotomic field, $\mathbb{Q}(\zeta_p)$.

Let G be the group $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$. Let $c \in G$ and σ_c be the unique automorphism of $\mathbb{Q}(\zeta_p)$ such that

$$\sigma_c(\zeta_p) = \zeta_p^c.$$

The action of $\sigma_{c^{-1}}$ on (3.13) gives us

$$\prod_{a=1}^{r}\left(1-\zeta_p^{ac^{-1}}\right)^{B_a} = 1.$$

On taking log of the above equation, we obtain the relation

$$\sum_{a=1}^{r} B_a \log\left(2\sin\frac{ac^{-1}\pi}{p}\right) = 0, \tag{3.14}$$

for all $1 \leq a \leq r$ and $1 \leq c \leq r$.

Define an $r \times r$ matrix $\mathfrak{M}$ whose $(a, c)^{\text{th}}$ entry is

$$\log\left(2\sin\frac{ac^{-1}\pi}{p}\right).$$

Thus, (3.14) can be rewritten as a matrix equation, i.e,

$$\mathfrak{M}v = 0,$$

where v the $r \times 1$ column vector with the a^{th}-entry being B_a. Since (3.12) was a non-trivial relation, $v \neq 0$. This is possible only if the determinant of $\mathfrak{M}$, $\det \mathfrak{M} = 0$.

Let $\mathfrak{M}^T$ denote the transpose of $\mathfrak{M}$. Notice that $\mathfrak{M}^T$ is a matrix of Dedekind type with $\mathfrak{f} : G \to \mathbb{C}$ given by

$$\mathfrak{f}(a) = \log\left(2\sin\frac{a\pi}{p}\right),$$

where G is as defined above. As mentioned in Theorem 3.2.2, $\mathfrak{M}^T$ is invertible if and only if

$$S_\chi := \sum_{a=1}^{r} \mathfrak{f}(a)\chi(a) \neq 0,$$

for all characters χ of the group G.

Observe that all characters of the group G are precisely the even Dirichlet characters modulo p. Thus, for a non-trivial even Dirichlet character χ, we can use (3.6) to express S_χ as:

$$\begin{aligned} S_\chi &= \sum_{a=1}^{r} \chi(a) \log\left(2\sin\frac{a\pi}{p}\right) = \sum_{a=1}^{r} \chi(a) \log\left(\left|1-\zeta_p^a\right|\right) \\ &= \frac{1}{2}\sum_{a=1}^{p-1} \chi(a) \log\left(\left|1-\zeta_p^a\right|\right) = \frac{p}{2\tau(\chi)} L(1,\overline{\chi}), \end{aligned}$$

where the last equality follows from (2.19) and the fact that

$$\tau(\chi) \neq 0.$$

For a proof of the above fact, we refer the reader to [60, Theorem 5.3.3, pg. 76]. By a famous theorem of Dirichlet,

$$L(1,\overline{\chi}) \neq 0.$$

Therefore, $S_\chi \neq 0$ when χ is a non-trivial character on G.

Let χ_0 be the trivial character on G, i.e, χ_0 is the trivial Dirichlet character modulo p. Then the factor S_{χ_0} is

$$\begin{aligned} S_{\chi_0} &= \sum_{a=1}^{r} \mathfrak{f}(a) = \sum_{a=1}^{r} \log\left(2\sin\frac{a\pi}{p}\right) = \sum_{a=1}^{r} \log\left(\left|1-\zeta_p^a\right|\right) \\ &= \frac{1}{2}\log\left(\prod_{a=1}^{p-1}\left|1-\zeta_p^a\right|\right) = \frac{1}{2}\log p \neq 0, \end{aligned}$$

where the last equality can be derived by noting that

$$\frac{1-X^p}{1-X} = \sum_{j=0}^{p-1} X^j = \prod_{a=1}^{p-1}(1-\zeta_p^a X),$$

substituting $X = 1$ and taking absolute values of both sides. Thus, $S_{\chi_0} \neq 0$. Hence, $\mathfrak{M}^T$, and in turn, $\mathfrak{M}$ is invertible. Therefore $v = 0$, which is a contradiction. This proves the theorem. □

It is not difficult to show that the numbers,

$$\frac{\sin(\pi a/p)}{\sin(\pi/p)}, \;\; 1 \leq a < p/2$$

are units in the cyclotomic field $\mathbb{Q}(\zeta_p)$. This observation leads to an alternate proof of Theorem 3.2.3. Although this proof is similar in flavour to our previous proof, we record a sketch of it here since it highlights the algebraic number theoretic aspect of Conjecture 3.0.2.

Proof. As before, we note that it suffices to prove that the numbers

$$\log\left(2\,\sin\frac{a\,\pi}{p}\right), \;\; 1 \leq a < p/2$$

are linearly independent over $\mathbb{Q}$. Suppose that

$$\sum_{a=1}^{r} b_a \log\left(2\,\sin\frac{a\,\pi}{p}\right) = 0, \;\; b_a \in \mathbb{Z}.$$

Rewriting this equation, we get

$$\sum_{a=1}^{r} b_a \log\left(\frac{\sin(\pi a/p)}{\sin(\pi/p)}\right) = \left(-\sum_{a=1}^{r} b_a\right) \log\left(2\,\sin\,\frac{\pi}{p}\right).$$

Since $b_a \in \mathbb{Z}$ and log denotes the principal branch, we have

$$\prod_{a=1}^{r} \left(\frac{\sin(\pi a/p)}{\sin(\pi/p)}\right)^{b_a} = \left(2\,\sin\,\frac{\pi}{p}\right)^{M},$$

where $M = -\sum_{a=1}^{r} b_a$. Now, note that $\sin z = (e^{iz} - e^{-iz})/(2i)$ and take the norm of both sides, to get

$$1 = \left(\prod_{a=1}^{p-1} (1 - \zeta_p^a)\right)^{M} = \Phi_p(1)^M,$$

where $\Phi_p(X)$ denotes the p^{th}-cyclotomic polynomial. Since $\Phi_p(X) = 1 + X + \cdots + X^{p-1}$, $\Phi_p(1) = p$. Therefore, the above equality reduces to $1 = p^M$, which is a contradiction is $M \neq 0$.

Hence, suppose that $M = 0$. We now obtain a multiplicative relation,

$$\prod_{a=1}^{r} \left(\frac{\sin(\pi a/p)}{\sin(\pi/p)}\right)^{b_a} = 1,$$

among the cyclotomic units

$$\xi_a := \frac{\sin(\pi a/p)}{\sin(\pi/p)}, \quad 1 \leq a < p/2.$$

But these are known to be multiplicatively independent (see [96, Lemma 8.1]). This completes the proof of Theorem 3.2.1. □

Chapter 4

Special values of L-series attached to Erdős functions

As noted earlier (see Conjecture 3.0.1), in a written correspondence with A. Livingston [57] in the early 1960s, Erdős conjectured the following.

Conjecture. *Let q be a positive integer. Let f be an arithmetical function, periodic with period q such that*

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{if } q \mid n. \end{cases}$$

Then the series $\sum_{n=1}^{\infty} f(n)/n \neq 0$, whenever it converges.

As seen in Chapter 2, the series $L(1, f) = \sum_{n=1}^{\infty} f(n)/n$ converges if and only if $\sum_{a=1}^{q} f(a) = 0$. Hence, for the sake of brevity, we say that a rational valued function f on the integers, periodic with period q is an *Erdős function* mod q if $f(n) \in \{-1, 1\}$ when $q \nmid n$ and $f(n) = 0$ otherwise and $\sum_{a=1}^{q} f(a) = 0$.

Erdős functions may be viewed as non-multiplicative analogues of quadratic Dirichlet characters. For a fundamental discriminant D, let χ_D be the quadratic character modulo $|D|$ given by the Kronecker symbol, i.e,

$$\chi_D(n) := \left(\frac{D}{n}\right).$$

In 1951, S. Chowla and P. Erdős [19] proved that the limit as N tends to infinity of the frequencies

$$\frac{\#\left\{D \,:\, |D| \leq N,\, L(1,\chi_D) \leq x\right\}}{N}$$

exists for all real x and is a continuous distribution function. In the 1960s, Barban [8, 9] calculated moments of $L(1,\chi_D)$ when $D < 0$, for all integer orders $k > 0$ and thus, showed that the characteristic function of the corresponding distribution has the form

$$\sum_{k=0}^{\infty} \frac{r(k)}{k!}\,(it)^k.$$

Here

$$r(k) = \sum_{\substack{n=1,\\ n \text{ odd}}}^{\infty} \frac{\phi(n)\,\tau_k(n^2)}{n^3},$$

where $\tau_k(n)$ is the k^{th} divisor function, i.e., number of ways of writing n as a product of k natural numbers.

Pursuing the analogy of Erdős functions as non-multiplicative analogs of quadratic Dirichlet characters, one may ask if certain properties of quadratic Dirichlet characters are also satisfied by Erdős functions. In this chapter, we follow the approach of Barban to understand the distribution of special values of L-series attached to Erdős functions.

For a positive integer q, let E_q be the set of Erdős functions mod q. Note that the convergence condition $\sum_{a=1}^{q} f(a) = 0$ implies that E_q is non-empty only when $q \geq 3$ is odd.

Although some progress has been made towards Conjecture 3.0.1, it remains open in the cases $q \equiv 1 \bmod 4$ or $q > 2\phi(q) + 1$. Conjecture 3.0.1 follows from a theorem of Baker, Birch and Wirsing [5] when q is prime. It was proved for $q < 2\phi(q) + 1$ by T. Okada [75] and for $q \equiv 3 \bmod 4$ by M. Ram Murty and N. Saradha [68]. In 2015, T. Chatterjee and M. Ram Murty [16] approached Conjecture 3.0.1 from a density theoretic perspective. They showed that if

$$S(x) := \#\big\{q \equiv 1 \bmod 4,\ q \leq x\ :\ \text{Erdős's conjecture holds for } q\big\},$$

then

$$\lim_{x \to \infty} \frac{S(x)}{x/4} \geq 0.82.$$

This can be interpreted as the Erdős conjecture being true for at least 82% of $q \equiv 1 \bmod 4$, which is the best possible lower bound using their methods. In the later section of this chapter, we use an alternate approach and improve on their result by proving that Erdős's conjecture is true with "probability" one.

Another aspect of this question is the non-vanishing of special values of $L(s, f)$ for an Erdős function f at positive integers greater than 1. In this direction, we ask the following question.

Question. *Let $q > 2$ and $k > 1$ be integers and f be an Erdős function mod q. Then is it true that $L(k, f) \neq 0$?*

In this context, observe the following (see [13, Section 9, 5.]). Let f be an Erdős function mod q and $k > 1$ be an integer. Suppose that $L(k, f) = 0$. Then observe that

$$|f(1)| = \left| \sum_{n=2}^{\infty} \frac{f(n)}{n^k} \right| \leq \zeta(k) - 1.$$

This implies that $2 \leq \zeta(k)$, i.e., $k < 2$. This establishes that $L(k, f) \neq 0$ for any Erdős function f if $k \geq 2$. Thus, Erdős's conjecture is interesting only when $k = 1$.

4.1 Distribution of $L(k, f)$

In this section, we will explicitly compute the characteristic function of the limiting distribution associated to the values $L(k, f)$ for an integer $k \geq 1$ and Erdős function f, with the same parity as k. We first quickly review the theory of higher dimensional Dedekind sums, a vital ingredient in our computations.

4.1.1 Higher dimensional Dedekind sums

In the course of evaluating moments of $L(k, f)$, we encounter a generalization of the higher dimensional Dedekind sums which were introduced by D. Zagier [101]. Classical Dedekind sums form a bridge between elementary number theory and topology. An excellent reference exposing this link is the book [45]. A generalization of the higher dimensional analog of these sums was studied by A. Bayad and A. Raouj [11], and are defined as follows. For $i = 0, \cdots, d$, let a_0 be a positive integer, $a_1, \cdots, a_d$ be positive integers co-prime to a_0 and $m_0, \cdots, m_d$ be non-negative integers. Define

$$C(a_i\,;\,a_0,\cdots,\widehat{a_i},\cdots,a_d\,|\,m_i\,;\,m_0,\cdots,\widehat{m_i},\cdots,m_d)$$
$$:= \begin{cases} \frac{1}{a_i^{m_i+1}} \sum_{k=1}^{a_i-1} \prod_{\substack{j=0,\\ j\neq i}}^{d} \left(\frac{d^{m_j}}{dz^{m_j}}(\cot z) \right)\Big|_{z=\pi k a_j/a_i}, & \text{if } a_i \geq 2, \\ 0 & \text{if } a_i = 1. \end{cases} \tag{4.1}$$

Here $\widehat{x_n}$ means that the term x_n is omitted. It can be shown that the number given by (4.1) is in fact rational. For a proof of this fact and further properties of the higher dimensional Dedekind sums, see [101] and [11]. Moreover, these sums satisfy a reciprocity law given by:

Theorem 4.1.1. *[11, Theorem 2.0.2] Let d be a positive integer, $a_0, \cdots, a_d$ be pairwise positive integers and $m_0, \cdots, m_d$ be non-negative integers. Let B_n denote the n^{th} Bernoulli number. Assume that the integer $M = d + m_0 + \cdots + m_d$ is even. Then we have*

$$\sum_{i=0}^{d} (-1)^{m_i}\, m_i! \sum_{\substack{l_0,\cdots,\widehat{l_i},\cdots,l_d\geq 0\\ l_0+\cdots+\widehat{l_i}+\cdots+l_d=m_i}} \left(\prod_{\substack{j=0\\ j\neq i}}^{d} \frac{a_j^{l_j}}{l_j!} \right)$$
$$\times\, C(a_i\,;\,a_0,\cdots,\widehat{a_i},\cdots,a_d\,|\,m_i\,;\,m_0+l_0,\cdots,m_{i-1}+l_{i-1},m_{i+1}+l_{i+1}\cdots,m_d+l_d)$$
$$= \begin{cases} -\left(R + (-1)^{d/2}\right) & \textit{if all } m_i \textit{ are zero,} \\ -R & \textit{otherwise,} \end{cases}$$

where [1]

$$R = \frac{(-1)^{M/2}\, 2^M}{\prod_{i=0}^{d} a_i^{m_i+1}} \sum_{\substack{j_0,\cdots,j_d \geq 0 \\ j_0+\cdots+j_d = M/2}} \prod_{i=0}^{d} a_i^{2j_i} A_{i,j_i}$$

and

$$A_{i,j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i-1-m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \geq (m_i+1)/2, \\ (-1)^{m_i}\, m_i! & \text{if } j_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4.1.2 Computation of moments

We obtain an expression for the characteristic function using the method of moments. More specifically, let B_m denote the m^{th} Bernoulli number defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}\, z^m.$$

For any positive integer n, let P_n denote the partially ordered set of partitions of n, i.e., for partitions $\underline{\lambda}$ and $\underline{\eta}$ of n, $\underline{\eta} \leq \underline{\lambda}$ if the parts of $\underline{\eta}$ can be obtained by merging the parts of $\underline{\lambda}$. The symbol $\binom{\underline{\lambda}}{\underline{\eta}}$ counts the number of ways in which parts of $\underline{\lambda}$ can be merged to obtain $\underline{\eta}$. For each $\underline{\lambda} \in P_n$, we inductively define $c(\underline{\lambda})$ as follows.

$$c((n)) := 2^{2nk}((k-1)!)^{2n}\left((-1)^{nk+1}\frac{B_{2nk}}{2nk!}\right)$$

[1] The minus sign in front of the right hand side is missing in the statement of [11, Theorem 2.0.2] but is evident from the proof.

and

$$c(\underline{\lambda}) = ((\lambda_1, \cdots, \lambda_m)) := \left[2^{2nk}((k-1)!)^{2n}\left(\prod_{i=1}^{m}(-1)^{\lambda_i k+1}\frac{B_{2\lambda_i k}}{2\lambda_i k!}\right)\right] - \sum_{\underline{\eta}<\underline{\lambda}}\binom{\underline{\lambda}}{\underline{\eta}}c(\underline{\eta}). \tag{4.2}$$

Then, using the method of moments, we show that

Theorem 4.1.2. *Fix a positive integer $k \geq 1$. For any integer $r \geq 1$ and real x, let E_{2r+1} be the set of Erdős functions mod $2r+1$ and*

$$E_{2r+1}^{(k)} := \{f \in E_{2r+1} \,:\, f \text{ is of the same parity as } k\}$$

and

$$F_r(x) := \frac{\#\left\{f \in E_{2r+1}^{(k)} \,:\, L(k,f) \leq x\right\}}{\#E_{2r+1}^{(k)}}.$$

Then, $F_r(x)$ converges to a distribution $F(x)$ at every point of continuity of the latter. Moreover, the corresponding characteristic function is entire and is given by

$$\phi(t) = \sum_{n=0}^{\infty}\frac{M(2n)}{(2n)!}(it)^{2n},$$

where

$$M(2n) := \frac{\pi^{2nk}}{((k-1)!)^{2n}\,2^{2n}}\left(\sum_{\underline{\lambda}\in P_n}c(\underline{\lambda})\right).$$

Proof. Our proof is based on the following lemma from [14, Theorem 30.2, pg. 390].

Lemma 4.1.3. *Suppose the distribution of X is determined by its moments, that the X_n have moments of all orders and that $\lim_n E[X_n^r] = E[X^r]$ for $r = 1, 2, \cdots$. Then $X_n \Rightarrow X$, i.e., the distribution of X_n converges to the distribution of X wherever the distribution of X is continuous.*

Thus, we first show that

$$M(n) := \lim_{r\to\infty} m_{2r+1}(n) = \lim_{r\to\infty} \frac{\sum_{f\in E_{2r+1}^{(k)}} L(k,f)^n}{\# E_{2r+1}^{(k)}}$$

is finite. Then to show that the limiting distribution is determined by its moments, we use a general theorem from [14, Theorem 30.1, pg. 388] stated below.

Theorem 4.1.4. *Let μ be a probability measure on the line having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k \mu(dx)$ of all orders. If the power series $\sum_{k=0}^{\infty} \alpha_k \, r^k/k!$ has a positive radius of convergence, then μ is determined by its moments.*

To begin with, we evaluate

$$m_q(n) := \frac{\sum_{f\in E_q^{(k)}} L(k,f)^n}{\# E_q^{(k)}},$$

for any non-negative integer n in terms of sums of the form (4.1). By (2.23),

$$\begin{aligned}
&\sum_{f\in E_q^{(k)}} L(k,f)^n \\
&= \frac{(-1)^{kn}}{\big((k-1)!\big)^n q^{kn}} \sum_{a_1,\cdots,a_n=1}^{(q-1)/2} \prod_{j=1}^{n} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\pi \cot \pi z) \Big|_{z=a_j/q} \right) \sum_{f\in E_q^{(k)}} f(a_1)\cdots f(a_n), \\
&= \frac{(-1)^{kn}\,\pi^k}{\big((k-1)!\big)^n q^{kn}} \sum_{a_1,\cdots,a_n=1}^{(q-1)/2} \prod_{j=1}^{n} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\cot z) \Big|_{z=a_j/q} \right) \sum_{f\in E_q^{(k)}} f(a_1)\cdots f(a_n).
\end{aligned}$$

Note that if $f \in E_q^{(k)}$ then $-f \in E_q^{(k)}$. Thus, if n is odd, the inner sum becomes zero by pairing terms corresponding to f and $-f$. Therefore, $m_q(n) = 0$ when n is odd.

Henceforth, let n be even and $p(n)$ denote the number of partitions of n. The above sum can be partitioned into $p(n)$ many sums according to the equality of the indices $a_1, \cdots, a_n$. In particular, for a partition $\underline{\lambda} = (\lambda_1, \cdots, \lambda_m)$ of n, the inner sum becomes

$$\sum_{f \in E_q^{(k)}} f(a_1)^{\lambda_1} \cdots f(a_m)^{\lambda_m}, \tag{4.3}$$

where $1 \leq a_1, \cdots, a_m \leq q-1$ are all distinct. Clearly, the above sum is $\#E_q^{(k)}$ when λ_l is even for all $1 \leq l \leq m$. Now, without loss of generality, suppose that λ_1 is odd. Since even and odd functions are determined by their values on $1 \leq a \leq r$, for any $f \in E_q^{(k)}$, there is a unique $f^- \in E_q^{(k)}$ such that

$$f^-(n) = \begin{cases} f(n) & \text{if } n \neq a_1, \\ -f(n) & \text{if } n = a_1, \end{cases}$$

for $1 \leq n \leq r$. Pairing up the terms corresponding to f and f^- in (4.3), we obtain that

$$\sum_{f \in E_q^{(k)}} f(a_1)^{\lambda_1} \cdots f(a_m)^{\lambda_m} = \begin{cases} \#E_q^{(k)}, & \text{if } \lambda_1, \cdots, \lambda_m \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, only those terms corresponding to partitions consisting of even parts survive and one can write

$$\begin{aligned} &\frac{1}{\#E_q^{(k)}} \sum_{f \in E_q^{(k)}} L(k, f)^n \\ &= \frac{\pi^k}{\big((k-1)!\big)^{2n} 2^{2n} q^{2kn}} \sum_{\substack{\underline{\lambda} = (\lambda_1, \cdots, \lambda_m), \\ \underline{\lambda} \in P_n}} \sideset{}{'}\sum_{a_1, \cdots, a_m = 1}^{q-1} \prod_{j=1}^{m} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\cot z) \Big|_{z = a_j/q} \right)^{2\lambda_j}, \end{aligned}$$

where $\sum'$ denotes that the sum is taken over *distinct* $1 \le a_1, \cdots, a_m \le q-1$. The inner sum can be expressed in terms of generalized higher dimensional Dedekind sums as follows. For any positive integer u, define

$$S_{q,k}^{(u)} := q\ C(q\,;\underbrace{1, 1, \cdots, 1}_{2u \text{ times}} \,|\, 0\,; \underbrace{k-1, \cdots, k-1}_{2u \text{ times}}).$$

With the notation as in the theorem, define $\mathfrak{S}_{q,k}^{(\underline{\eta})}$ inductively as follows. $\mathfrak{S}_{q,k}^{(n)} := S_{q,k}^{(n)}$ and for any $\underline{\lambda} = (\lambda_1, \cdots, \lambda_m)$,

$$\mathfrak{S}_{q,k}^{(\underline{\lambda})} := S_{q,k}^{(\lambda_1)} \cdots S_{q,k}^{(\lambda_m)} - \sum_{\underline{\eta} \le \underline{\lambda}} \binom{\underline{\lambda}}{\underline{\eta}} \mathfrak{S}_{q,k}^{(\underline{\eta})}. \tag{4.4}$$

Thus, we have that

$$m_q(2n) = \frac{\pi^k}{\big((k-1)!\big)^{2n}\, 2^{2n}\, q^{2kn}} \Big\{ \sum_{\underline{\lambda} \in P_n} \mathfrak{S}_{q,k}^{\underline{\lambda}} \Big\}.$$

To understand the asymptotic behaviour of $m_q(2n)$ as $q \to \infty$, we use the explicit evaluation of $S_{q,k}^{(u)}$ using Theorem 4.1.1. Thus,

$$\begin{aligned} S_{q,k}^{(u)} &= \sum_{t=1}^{q-1} \left(\frac{d^{k-1}}{dz^{k-1}} (\cot z) \right) \Big|_{z=\pi t/q}^{2u} \\ &= \begin{cases} -q\left(R + (-1)^u\right) & \text{if } k = 1, \\ -q\, R & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$R = \frac{(-1)^{uk}\, 2^{2uk}}{q} \sum_{\substack{j_0, \cdots, j_{2u} \\ j_0 + \cdots + j_{2u} = uk}} \alpha_{j_0, \cdots, j_{2u}}\, q^{2j_0},$$

for $\alpha_{j_0, \cdots, j_{2u}} \in \mathbb{Q}$ given by Theorem 4.1.1. In particular, $S_{q,k}^{(u)}$ is a polynomial in q of degree $2uk$ with leading coefficient $-(-4)^{uk} \alpha_{uk,0,\cdots,0}$, given explicitly by Theorem 4.1.1. Thus,

$$S_{q,k}^{(u)} \sim \left(2^{2uk} \left((k-1)!\right)^{2u} (-1)^{uk+1} \frac{B_{2uk}}{(2uk)!} \right) q^{2uk}, \quad \text{as } q \to \infty.$$

Using this, (4.4) and the definition of $c(\underline{\lambda})$ (4.2), we get that

$$\mathfrak{S}_{q,k}^{(\underline{\lambda})} \sim c(\underline{\lambda})\, q^{2nk},$$

as q tends to infinity. Hence,

$$M(2n) = \lim_{q \to \infty} m_q(2n) = \frac{\pi^{2nk}}{\left((k-1)!\right)^{2n} 2^{2n}} \left(\sum_{\underline{\lambda} \in P_n} c(\underline{\lambda}) \right).$$

Since the limit as q tends to infinity of $m_q(2n)$ exists, by Lemma 4.1.3, there exists a limiting distribution $F(x)$ whose odd moments are zero and even moments are given by $M(2n)$. Thus, the characteristic function of $F(x)$ is given by

$$\phi(t) := \sum_{n=0}^{\infty} \frac{M(2n)}{(2n)!} (it)^{2n}.$$

By Lemma 4.1.3 and Lemma 4.1.4, it suffices to show that $\phi(t)$ has positive radius of convergence. In fact, we prove that it is entire.

Let $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for $\Re(s) > 1$ be the Riemann zeta function. In 1737, Euler showed that

$$\zeta(2m) = (-1)^{m+1} \frac{B_{2m}\,(2\pi)^{2m}}{(2m)!\,2}.$$

Since $\zeta(2m+2) < \zeta(2m)$,

$$\left|\frac{B_{2m+2}}{(2m+2)!}\right| < \left|\frac{B_{2m+2}\,\pi^2}{(2m+2)!}\right| < \left|\frac{B_{2m}}{(2m)!}\right|.$$

Hence,

$$c(\underline{\lambda}) \leq \left| 2^{2nk}((k-1)!)^{2n} \left(\prod_{i=1}^{m} (-1)^{\lambda_i k+1} \frac{B_{2\lambda_i k}}{(2\lambda_i k)!} \right) \right| \leq 2^{2nk}((k-1)!)^{2n} \left| \frac{B_{2k}}{(2k)!} \right|^{m}.$$

and thus,

$$M(2n) = \frac{\pi^{2nk}}{((k-1)!)^{2n}\,2^{2n}} \left(\sum_{\underline{\lambda}\in P_n} c(\underline{\lambda}) \right) \leq \frac{(2\pi)^{2nk}}{2^{2n}}\, p(n) \left| \frac{B_{2k}}{(2k)!} \right|^{n} = \frac{p(n)}{2^n} \left(\frac{\zeta(2k)}{2} \right)^{n},$$

where $p(n)$ denotes the number of partitions of n. In 1918, Hardy and Ramanujan [41] showed that

$$p(n) \sim \frac{1}{(4\sqrt{3})\,n} \exp\left(\pi \sqrt{\frac{2n}{3}} \right) \text{ as } n \to \infty.$$

Now using Stirling's formula and the asymptotics for $p(n)$, we obtain that

$$\left(\frac{M(2n)}{(2n)!} \right)^{1/2n} \ll e^{-c \log n},$$

for a positive constant c. Therefore, applying the root test gives that the radius of convergence of $\phi(t)$ is infinite. This proves the theorem. □

4.2 Non-vanishing of $L(1, f)$

Let $q \geq 3$ be odd and E_q be the set of all Erdős functions mod q. Let $r := (q-1)/2$. We define a relation on this set as follows. For $f, g \in E_q$,

$$f \sim g \iff f(a) = g(a),\ \forall\, 1 \leq a \leq q,\ (a, q) \neq 1.$$

One can easily check that $\sim$ is an equivalence relation. Before proceeding, we prove the following proposition.

Proposition 4.2.1. *There exists at most one Erdős function f in every equivalence class of E_q under $\sim$, such that $L(1, f) = 0$.*

Proof. Suppose $f, g \in E_q$ are such that $L(1, f) = L(1, g) = 0$. Thus, by (2.9) and the convergence condition, we have

$$\begin{aligned}
\sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(a)\left[\psi\left(\frac{a}{q}\right) + \gamma\right] &= - \sum_{\substack{a=1 \\ (a,q)\neq 1}}^{q} f(a)\left[\psi\left(\frac{a}{q}\right) + \gamma\right] \\
&= - \sum_{\substack{a=1 \\ (a,q)\neq 1}}^{q} g(a)\left[\psi\left(\frac{a}{q}\right) + \gamma\right] \\
&= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} g(a)\left[\psi\left(\frac{a}{q}\right) + \gamma\right].
\end{aligned}$$

Therefore, we obtain

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big[f(a) - g(a)\big] \left[\psi\left(\frac{a}{q}\right) + \gamma\right],$$

which is a $\mathbb{Q}$-linear relation among the numbers

$$\psi(a/q) + \gamma, \;\; 1 \leq a \leq q, \; (a, q) = 1.$$

But these numbers are $\mathbb{Q}$-linearly independent as proven in [67, Theorem 4]. Hence, $f = g$. □

Using this proposition, we prove that almost all Erdős functions satisfy Erdős's conjecture.

Theorem 4.2.2. *Let $q \geq 3$ be an odd positive integer. Let E_q be the set of Erdős functions mod q and $V_q := \big\{ f \in E_q \, : \, L(1, f) = 0 \big\}$. Then,*

$$\lim_{x \to \infty} \Bigg\{ \Bigg(\sum_{\substack{3 \leq q \leq x, \\ q \text{ odd}}} \# V_q \Bigg) \Bigg/ \Bigg(\sum_{\substack{3 \leq q \leq x, \\ q \text{ odd}}} \# E_q \Bigg) \Bigg\} = 0.$$

Proof. In the light of Proposition 4.2.1, it suffices to count the number of equivalence classes of E_q under $\sim$. In order to count these, note that each equivalence class differs from the other based on the values of functions on $N_q := \{ \, a \, : \, 1 \leq a < q, \, (a, q) \neq 1 \}$. At each $a \in N_q$, an Erdős function f can take the value 1 or -1, with the only restriction that

$$\#\{1 \leq a < q \, : \, f(a) = 1\} = \#\{1 \leq a < q \, : \, f(a) = -1\} = \frac{q-1}{2}.$$

For simplicity of notation, let $n_q := (q - 1 - \phi(q))$ and recall that $r = (q-1)/2$.

(a) $r \geq n_q$: In this case, the number of $a \in N_q$ where a function takes the value 1 ranges from 0 to n_q. Thus, the total number of equivalence classes is

$$\left|E_q/\sim\right| = \sum_{k=0}^{n_q} \binom{n_q}{k} = 2^{n_q}.$$

(b) $r < n_q$: Let $j := n_q - r$. Then, the number of $a \in N_q$ where a function takes the value 1 has to be at least j. Hence, the number of equivalence classes is

$$\left|E_q/\sim\right| = \sum_{k=j}^{r} \binom{n_q}{k} < 2^{n_q}.$$

Therefore, in either case $|V_q| \leq 2^{(q-1-\phi(q))}$. Now, note that

$$\left|E_q\right| = \binom{2r}{r}.$$

Using the bounds by [84], one has

$$\sqrt{2\pi}\, n^{(n+\frac{1}{2})}\, e^{-n} < n! < \sqrt{2\pi}\, e\, n^{(n+\frac{1}{2})}\, e^{-n},$$

for all $n \in \mathbb{N}$, we get that

$$\binom{2r}{r} = \frac{(2r)\,!}{(r!)^2} \geq \frac{\sqrt{2\pi}\,(2r)^{(2r+1/2)}\, e^{-2r}}{\left(\sqrt{2\pi}\, e\, r^{(r+1/2)}\, e^{-r}\right)^2} = \frac{\sqrt{2}\, 2^{2r}\, r^{2r}\, \sqrt{r}}{\sqrt{2\pi}\, e^2\, r^{2r}\, r} = \frac{2^{2r}}{e^2 \sqrt{\pi}\, \sqrt{r}}.$$

Thus,

$$\Bigg(\sum_{\substack{3\leq q\leq x,\\ q \text{ odd}}} \#V_q\Bigg) \Bigg/ \Bigg(\sum_{\substack{3\leq q\leq x,\\ q \text{ odd}}} \#E_q\Bigg) \ll \frac{\sum_{r\leq x} 2^{2r-\phi(2r+1)}}{\sum_{r\leq x}(2^{2r}/\sqrt{r})} \ll \frac{\sum_{r\leq x} 2^{2r-(2r/(\log\log 2r))}}{\sum_{r\leq x}(2^{2r}/\sqrt{r})} \tag{4.5}$$

because by [50, pg. 217],

$$\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-C}.$$

The theorem now follows as the right hand side tends to zero as $x \to \infty$ by the following lemma [80, Problem 70, pg. 16],

Lemma 4.2.3. *Let the sequences a_n and b_n satisfy the conditions:*

$$b_n > 0, \quad n = 1, 2, \cdots ; \;\; b_1 + b_2 + b_3 + \cdots + b_n + \cdots \;\; \textit{diverges;}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = s.$$

Then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = s.$$

More specifically, one can take the sequence $a_n := 2^{2r-(2r/(\log\log 2r))}$ and the sequence $b_n := 2^{2r}/\sqrt{r}$ in the above Lemma and note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

This concludes the proof. □

Remark. *Theorem 4.2.2 can be made explicit by the computation of rates in equation* (4.5)*. Using Cauchy-Schwarz, it is easy to see that the limit in Theorem 4.2.2 goes to* 0 *as fast as* $4^{-x/\log\log x}$.

Chapter 5

Transcendental nature of $L'(1, f)$ and generalized Stieltjes constants

This chapter is devoted to the study of the arithmetic nature of the derivative of $L(s, f)$ at $s = 1$ for an arithmetical function f, periodic with period q satisfying $\sum_{a=1}^{q} f(a) = 0$. This problem has received scant attention in the literature. However, its significance can be illustrated by a curious result of Murty and Murty [63] which states that if there is some squarefree $D > 0$ and χ_D, the quadratic character attached to $\mathbb{Q}(\sqrt{-D})$, is such that $L'(1, \chi_D) = 0$, then, e^{γ} is transcendental. An analogous question of non-vanishing seems to occur in other contexts as well (see for example, [88]). Hence, the question of determining the arithmetic nature of $L'(1, f)$ is of utmost importance.

A conjecture put forth by S. Gun, M. Ram Murty and P. Rath [38] will play a fundamental role towards a partial solution to our question. The conjecture is the following.

Conjecture 5.0.1. *For any positive integer $q > 2$, let $\overline{V_{\Gamma}(q)}$ be the $\overline{\mathbb{Q}}$-vector space* *spanned by the real numbers*

$$\log \Gamma\left(\frac{a}{q}\right),\ 1 \le a \le q,\ (a,q) = 1.$$

Then the dimension of $\overline{V_\Gamma(q)}$ *is* $\phi(q)$.

This conjecture was inspired by a conjecture of Rohrlich[1] (see [95]) regarding the possible relations among the special values of the Γ-function. We note that Conjecture 5.0.1 is equivalent to the numbers $\{\log \Gamma(a/q) : 1 \le a \le q, (a,q) = 1\}$ being $\overline{\mathbb{Q}}$-linearly independent for $q > 2$. This is a major unsolved problem in number theory and is believed to be outside the scope of current mathematical tools.

Before proceeding, we state two notions that appear in our theorems. An arithmetical function periodic with period q is said to be of *Dirichlet type* if

$$f(n) = 0, \quad \text{whenever} \quad (n,q) > 1.$$

A set of arithmetical functions $\{f_1, f_2, \cdots, f_m\}$ is said to be *linearly independent over* $\overline{\mathbb{Q}}$ if

$$\sum_{j=1}^{m} \alpha_j f_j = 0, \text{ with } \alpha_j \in \overline{\mathbb{Q}} \Rightarrow \alpha_j = 0 \text{ for all } 1 \le j \le m.$$

We also observe that if $f_1, \cdots, f_r$ are arithmetical functions periodic with period q, then

$$\sum_{j=1}^{r} \alpha_j f_j = 0 \iff \sum_{j=1}^{r} \alpha_j \widehat{f_j} = 0, \tag{5.1}$$

[1]On communication with D. Rohrlich, he suggested that in fact, the conjecture referred to as Rohrlich's conjecture might be a folklore conjecture, as pointed out to him by J. Coates

for any complex numbers α_j, $1 \leq j \leq r$. This is immediate from the fact that the Fourier transform is a linear automorphism of the $\mathbb{C}$-vector space of arithmetical functions periodic with period q.

As seen in Chapter 2, the special value $L(1,f)$ has been extensively studied and is important in the context of our theorems. The following result of Baker, Birch and Wirsing [5] will be particularly useful in our scenario.

Theorem 5.0.2. *If f is a non-vanishing function defined on the integers with algebraic values and period q such that (i) $f(n) = 0$ whenever $1 < (n,q) < q$ and (ii) the q^{th} cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(1), f(2), \cdots, f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

5.1 Transcendence of $L'(1,f)$ when $L(1,f) \neq 0$

In this section, we prove the following theorem.

Theorem 5.1.1. *Let $q > 7$ be a positive integer. Define*

$$\mathfrak{F}_q := \Big\{ f : \mathbb{Z} \to \overline{\mathbb{Q}} \mid f \text{ is periodic with period } q,\ f \text{ is odd}, \\ \widehat{f} \text{ is of Dirichlet type},\ L(1,f) \neq 0 \Big\},$$

For $r > 2$, let $f_1, f_2, \cdots, f_r$ be $\overline{\mathbb{Q}}$-linearly independent elements of $\mathfrak{F}_q$. Then Conjecture 5.0.1 implies that at most three of the numbers

$$\Big\{ L'(1, f_j) \mid 1 \leq j \leq r \Big\}$$

are algebraic.

Proof. For convenience of notation, let

$$\mathscr{C} := \left(1 + \frac{1}{q}\right) \log q - \log 2\pi - \gamma.$$

Thus, Lemma 2.4.2 gives

$$L'(1, f_j) = \frac{i\pi}{q} \left\{ \mathscr{C} B_{1,\widehat{f_j}} + \sum_{b=1}^{q} \widehat{f_j}(b) \log \Gamma\left(\frac{b}{q}\right) \right\},$$

for all $1 \leq j \leq r$. By (2.18), the hypothesis $L(1, f_j) \neq 0$ implies that $B_{1,\widehat{f_j}} \neq 0$.

For $1 \leq k < l \leq r$, define

$$d_{k,l} := B_{1,\widehat{f_l}} L'(1, f_k) - B_{1,\widehat{f_k}} L'(1, f_l).$$

We claim that $d_{k,l} \neq 0$. Indeed, if $d_{k,l} = 0$, then we get that

$$\begin{aligned} 0 &= B_{1,\widehat{f_l}} L'(1, f_k) - B_{1,\widehat{f_k}} L'(1, f_l) \\ &= \frac{i\pi}{q} \left\{ \mathscr{C} \left(B_{1,\widehat{f_l}} B_{1,\widehat{f_k}} - B_{1,\widehat{f_k}} B_{1,\widehat{f_l}} \right) + \sum_{b=1}^{q} \left[B_{1,\widehat{f_l}} \widehat{f_k}(b) - B_{1,\widehat{f_K}} \widehat{f_l}(b) \right] \log \Gamma\left(\frac{b}{q}\right) \right\} \\ &= \sum_{b=1}^{q} \left[B_{1,\widehat{f_l}} \widehat{f_k}(b) - B_{1,\widehat{f_K}} \widehat{f_l}(b) \right] \log \Gamma\left(\frac{b}{q}\right), \end{aligned}$$

which is a $\overline{\mathbb{Q}}$-linear relation among the values of the log gamma function as $B_{1,\widehat{f_j}} \in \overline{\mathbb{Q}}$ for all $1 \leq j \leq r$. Therefore, Conjecture 5.0.1 gives that

$$B_{1,\widehat{f_l}} \widehat{f_k} - B_{1,\widehat{f_k}} \widehat{f_l} = 0$$

on all natural numbers. This implies $\overline{\mathbb{Q}}$-linear dependence of $\widehat{f_k}$ and $\widehat{f_l}$ and thus, contradicts the $\overline{\mathbb{Q}}$-linearly independence of f_k and f_l by (5.1). Hence, $d_{k,l}$ is not zero.

We now consider the ratio $d_{k,l}/d_{u,v}$ for $1 \leq k, u < l, v \leq r$ and $(k, l) \neq (u, v)$. If this ratio is algebraic, i.e.,

$$\frac{d_{k,l}}{d_{u,v}} = \eta \in \overline{\mathbb{Q}},$$

then we are led to argue that

$$\begin{aligned} 0 &= d_{k,l} - \eta d_{u,v} \\ &= \sum_{b=1}^{q} \left[B_{1,\widehat{f_l}} \widehat{f_k}(b) - B_{1,\widehat{f_k}} \widehat{f_l}(b) - \eta B_{1,\widehat{f_w}} \widehat{f_u}(b) + \eta B_{1,\widehat{f_u}} \widehat{f_w}(b) \right] \log \Gamma \left(\frac{b}{q} \right), \end{aligned}$$

which is a $\overline{\mathbb{Q}}$-linear relation among log gamma values. Hence, by Conjecture 5.0.1, we have

$$B_{1,\widehat{f_l}} \widehat{f_k} - B_{1,\widehat{f_k}} \widehat{f_l} - \eta B_{1,\widehat{f_w}} \widehat{f_u} + \eta B_{1,\widehat{f_u}} \widehat{f_w} = 0$$

on all natural numbers. Since $B_{1,\widehat{f_j}}$ are non-zero algebraic numbers, we obtain a non-trivial $\overline{\mathbb{Q}}$-linear relation among $\widehat{f_k}, \widehat{f_l}, \widehat{f_u}$ and $\widehat{f_w}$. The fact (5.1) transports this to $\overline{\mathbb{Q}}$-linear dependence of f_k, f_l, f_u and f_w, which contradicts our hypothesis. Thus, at most one of the $d_{k,l}$'s can be algebraic for $1 \leq k < l \leq r$.

As a result, if the four numbers, $L'(1, f_k), L'(1, f_l), L'(1, f_u)$ and $L'(1, f_w)$ are algebraic for $(k, l) \neq (u, w)$, then $d_{k,l}/d_{u,w}$ would be algebraic leading to a contradiction. Hence, the theorem follows. $\square$

5.2 Transcendence of $L'(1,f)$ when $L(1,f)=0$

We prove an analogue of Theorem 5.1.1 in the case when $L(1,f)=0$ in this section.

Theorem 5.2.1. *Let $q>5$ be a positive integer. Define*

$$\mathfrak{G}_q := \Big\{ f:\mathbb{Z}\to\overline{\mathbb{Q}} \mid f \text{ is periodic with period } q,\ f \text{ is odd}, \\ \widehat{f} \text{ is of Dirichlet type},\ L(1,f)=0 \Big\}.$$

For $r\geq 2$, let $f_1,\cdots,f_r$ be $\overline{\mathbb{Q}}$-linearly independent elements of $\mathfrak{G}_q$. Then under Conjecture 5.0.1, we conclude that at most one of the numbers

$$\Big\{ L'(1,f_j) \mid 1\leq j\leq r \Big\}$$

is algebraic.

Proof. Using the hypothesis that $L(1,f)=0$ for all $f\in\mathfrak{G}_{\mathfrak{q}}$ and (2.18), we know that

$$B_{1,\widehat{f}_j}=0,$$

for all $1\leq j\leq r$. Hence, Lemma 2.4.2 gives

$$L'(1,f_j)=\frac{i\pi}{q}\Bigg\{\sum_{b=1}^{q}\widehat{f}_j(b)\log\Gamma\Big(\frac{b}{q}\Big)\Bigg\},$$

for all $1\leq j\leq r$. Suppose that for $1\leq k<l\leq r$,

$$\frac{L'(1,f_k)}{L'(1,f_l)}=\xi\in\overline{\mathbb{Q}}.$$

Then simplifying the above expression gives

$$\sum_{b=1}^{q}\left[\widehat{f_k}(b)-\xi\widehat{f_l}(b)\right]\log\Gamma\left(\frac{b}{q}\right)=0,$$

which is an algebraic linear relation among the log gamma values. Therefore, by Conjecture 5.0.1, we get that

$$\widehat{f_k}-\xi\widehat{f_l}=0$$

on all natural numbers. This implies the $\overline{\mathbb{Q}}$-linear dependence of the functions $\widehat{f_k}$ and $\widehat{f_l}$ and thus, contradicts the $\overline{\mathbb{Q}}$-linear independence of f_k and f_l by (5.1). Hence, the quotient $L'(1,f_k)/L'(1,f_l)$ is transcendental for all $1\leq k<l\leq r$, which in turn leads us to conclude that at most one of the numbers under consideration is algebraic. □

5.3 Transcendence of generalized Stieltjes constants

The connection between derivatives of $L(s,f)$ for periodic arithmetical functions f at $s=1$ and generalized Stieltjes constants has been noted in Chapter 2. In this section, we utilize this link to throw light on the arithmetic nature of certain generalized Stieltjes constants and their linear combinations.

Corollary 5.3.1. *Let $q>7$ be a positive integer. Then assuming Conjecture 5.0.1, we deduce that at most three of the following numbers are algebraic:*

$$\left\{L'(1,\chi)=-\sum_{a=1}^{q}\chi(a)\gamma_1(a,q)\mid\chi\text{ is an odd primitive Dirichlet character mod }q\right\}.$$

Proof. Let q be any natural number greater than 7 and χ be an odd primitive Dirichlet character modulo q. It suffices to show that $\chi\in\mathfrak{F}_{\mathfrak{q}}$ i.e, that $\widehat{\chi}$ is of Dirichlet type and that $L(1,\chi) \neq 0$. The latter follows from the famous theorem of Dirichlet [2, Theorem 6.20 and Section 7.3]. The former was seen earlier in (2.6). This completes the proof of the corollary. $\square$

Corollary 5.3.2. *For an odd prime p greater than 7, Conjecture 5.0.1 implies that at least $(p-7)/2$ of the numbers*

$$\{\gamma_1(a,p) \mid 1 \leq a \leq p-1\}$$

are transcendental.

Proof. We begin the proof by observing that the functions f_j defined below are in $\mathfrak{F}_{\mathfrak{p}}$. For $1 \leq j \leq (p-1)/2$,

$$f_j(n) := \begin{cases} 1 & \text{if } n \equiv j \bmod p, \\ -1 & \text{if } n \equiv -j \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, each f_j is periodic with odd prime period p, f_j is odd and $f_j(p) = 0$. Moreover, $\sum_{a=1}^{p} f_j(a) = 0$ and by Theorem 5.0.2, $L(1, f_j) \neq 0$ for all $1 \leq j \leq r$. Thus, $f_j \in \mathfrak{F}_{\mathfrak{p}}$ for all $1 \leq j \leq r$. Also note that the functions $\{f_j : 1 \leq j \leq (p-1)/2\}$ are $\overline{\mathbb{Q}}$-linearly independent. Therefore, Theorem 5.1.1 implies that at least $\frac{(p-1)}{2} - 3$ of the numbers

$$\{\gamma_1(a,p) - \gamma_1(p-a,p) : 1 \leq a \leq (p-1)/2\}$$

are transcendental. Since the difference of two numbers being transcendental implies that at least one of them is transcendental, the result follows. $\square$

Chapter 6

Values of Dedekind zeta-function at odd positive integers

Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Then the Dedekind zeta-function attached to K is defined as

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K,\\ \mathfrak{p}\neq 0}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1}, \quad \Re(s) > 1,$$

where the product is over non-zero prime ideals in $\mathcal{O}_K$ and N denotes the absolute norm. Since $\mathcal{O}_K$ is a Dedekind domain, the non-zero integral ideals can be expressed uniquely as a product of prime ideals. Thus, the infinite product above can be expanded into the Dirichlet series

$$\zeta_K(s) = \sum_{\substack{\mathfrak{A}\subseteq\mathcal{O}_K,\\ \mathfrak{A}\neq 0}} \frac{1}{N\mathfrak{A}^s}, \quad \Re(s) > 1,$$

where the sum is over non-zero integral ideals in $\mathcal{O}_K$. The function $\zeta_K(s)$ was introduced by R. Dedekind, who also conjectured its analytic continuation, which was proved later by Hecke [42]. His proof used a multi-dimensional version of one of the techniques from Riemann's 1859 paper, namely the use of the transformation formula of theta series. Thus, $\zeta_K(s)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$. The residue at $s = 1$ is given by the analytic class number formula,

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,h\,R}{\omega\,\sqrt{|d_K|}},$$

where r_1 is the number of real embeddings of K , $2r_2$ is the number of complex embeddings of K, h denotes the class number, R is the regulator, ω is the number of roots of unity in K and d_K is the discriminant of K (see [62, Chapter 1]).

Analogous to the Riemann zeta-function, the Dedekind zeta-function captures crucial information about the distribution of prime ideals in $\mathcal{O}_K$. For example, the non-vanishing of $\zeta_K(s)$ on the line $\Re(s) = 1$ along with its simple pole at $s = 1$, implies the prime ideal theorem. The prime ideal theorem asserts that if $\pi_K(x) := \#\{\mathfrak{p} \in \mathcal{O}_K \,:\, \mathfrak{p} \text{ is prime }, N\mathfrak{p} \leq x\}$, then

$$\pi_K(x) \sim \frac{x}{\log x},$$

as $x \to \infty$. For a proof of this theorem, we refer the reader to the exposition by M. R. Murty and V. K. Murty in [62, Theorem 3.2].

Moreover, the Dedekind zeta function satisfies a functional equation,

$$\xi_K(s) = \xi_K(1-s),$$

where

$$\xi_K(s) := \left(\frac{\sqrt{|d_K|}}{2^{r_2}\,\pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\,\zeta_K(s)s(s-1)$$

is entire. Since the gamma function has poles at negative integers, from the functional equation one can deduce that $\zeta_K(s)$ is always zero at all non-zero negative even integers. Additionally, if K is not totally real (i.e., $r_2 > 0$), then $\zeta_K(s)$ is zero at all odd negative integers as well. Thus, the only non-zero values of $\zeta_K(s)$, at negative integers $-m$, arise when K is totally real and $m > 0$ is odd. Via the functional equation, these values correspond to $\zeta_K(2n)$ for any integer $n > 0$. In 1940, Hecke [43] proved that $\zeta_K(2n)$ is an algebraic multiple of π^{4n} for a real quadratic field K. This led him to conjecture similar phenomena when K is any totally real field. Indeed, it was shown by C. L. Siegel and H. Klingen [48] independently, that when F is totally real, $\zeta_F(1-2n)$ is rational. This translates to $\zeta_F(2n)$ being an algebraic multiple of $\pi^{2n[F:\mathbb{Q}]}$, generalizing Euler's 1737 theorem for the Riemann zeta-function. The method utilized by them relied on the theory of Hilbert modular forms. An accessible exposition of the proof can be found in the appendix of Siegel's TIFR lecture notes [91].

When K is not totally real, nothing is known regarding the irrationality or transcendence of $\zeta_K(n)$. In 1990, D. Zagier [102] put forth a conjecture connecting these values to the polylogarithm function,

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad |z| < 1.$$

He conjectured that $\zeta_K(n)$ is a simple multiple of the determinant of a matrix whose entries are linear combinations of polylogarithms evaluated at a certain number in K. The case $n = 3$ of Zagier's conjecture was settled by A. Goncharov [36]. However, we are still far from understanding the nature of these numbers. A number field E is said to be *CM* if there exists a subfield F of E, such that F is totally real, and E is a totally imaginary quadratic extension of F. The aim of this chapter is to highlight that an irrationality result for the values of the Dedekind zeta function of CM-number fields can be deduced from our current knowledge, using "elementary" means.

6.1 Artin L-functions

In this section, we summarize certain relevant facts regarding Artin L-functions. A gentle introduction to Artin L-functions can be found in N. Snyder's senior thesis, titled "Artin L-functions: A Historical Approach" [93]. A more concise account is included in the excellent monograph by M. R. Murty and V. K. Murty [62, Chapter 2].

Algebraic number theoretic preliminaries

Let E/F be a Galois extension of number fields, with Galois group G. Let $\mathcal{O}_E$ and $\mathcal{O}_F$ denote the ring of integers of E and F respectively. Fix a non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}_F$, and let $\mathfrak{p}\mathcal{O}_E = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$, for $\mathfrak{P}_i \subseteq \mathcal{O}_E$, prime. The prime ideals $\mathfrak{P}_i$ are said to lie above $\mathfrak{p}$. Since $\mathcal{O}_E$ and $\mathcal{O}_F$ are Dedekind domains, the non-zero prime ideals are maximal. This gives us the corresponding residue field extension, $\overline{E_i}/\overline{F}$, where $\overline{E_i} = \mathcal{O}_E/\mathfrak{P}_i$ and $\overline{F} = \mathcal{O}_F/\mathfrak{p}$. The group G acts transitively on the primes above $\mathfrak{p}$. Therefore, we obtain the identity

$$n = efg, \tag{6.1}$$

where n is the degree of the extension $[E:F]$, $e = e_1 = \cdots = e_g$ is the ramification index, $f = f_1 = \cdots = f_g$ is the degree of the residue field extension and g is the number of primes above $\mathfrak{p}$.

Let $\mathfrak{P}$ be a prime lying above $\mathfrak{p}$. The stabilizer of $\mathfrak{P}$ for the action of G is called the decomposition group at $\mathfrak{P}$ and is denoted by

$$G_{\mathfrak{P}} = \{\sigma \in G \,:\, \sigma(\mathfrak{P}) = \mathfrak{P}\}\,.$$

Let $\overline{E} = \mathcal{O}_E/\mathfrak{P}$. Then $\overline{E}/\overline{F}$ is a finite extension of finite fields. Hence, $\overline{E}/\overline{F}$ is Galois with a cyclic Galois group. There exists a canonical map

$$\phi_{\mathfrak{P}} : G_{\mathfrak{P}} \to \mathrm{Gal}\big(\overline{E}/\overline{F}\big),$$

via the Frobenius automorphism, defined below. The surjectivity of $\phi_{\mathfrak{P}}$ follows from the primitive element theorem. If $I_{\mathfrak{P}} := \ker(\phi_{\mathfrak{P}})$, then we have

$$G_{\mathfrak{P}}/I_{\mathfrak{P}} \simeq \mathrm{Gal}\big(\overline{E}/\overline{F}\big).$$

The subgroup $I_{\mathfrak{P}}$ is called the inertia group at $\mathfrak{P}$. In other words,

$$I_{\mathfrak{P}} = \{\sigma \in G_{\mathfrak{P}} \,:\, \sigma(x) \equiv x \bmod \mathfrak{P},\ \forall x \in \mathcal{O}_E\}\,.$$

Now, the orbit-stabilizer theorem implies that

$$|G_{\mathfrak{P}}| = \frac{|G|}{g} = \frac{n}{g}.$$

When $\mathfrak{p}$ is unramified, $e = 1$ and

$$f = \left|\mathrm{Gal}\left(\overline{E}/\overline{F}\right)\right| = \frac{|G_{\mathfrak{P}}|}{|I_{\mathfrak{P}}|} = \frac{n}{g\,|I_{\mathfrak{P}}|}.$$

Therefore, the identity (6.1) gives

$$n = \frac{n}{g\,|I_{\mathfrak{P}}|} \times g.$$

Hence, $|I_{\mathfrak{P}}| = 1$ when $\mathfrak{p}$ is unramified in E.

As mentioned earlier, $\mathrm{Gal}\left(\overline{E}/\overline{F}\right)$ is cyclic. A unique element of the Galois group, $\sigma_{\mathfrak{P}}$, having the effect

$$\sigma_{\mathfrak{P}}(x) \equiv x^{N\mathfrak{p}} \bmod \mathfrak{P},$$

generates $\mathrm{Gal}\left(\overline{E}/\overline{F}\right)$. Any element of $G_{\mathfrak{P}}$, in the coset of $I_{\mathfrak{P}}$ which corresponds to $\sigma_{\mathfrak{P}}$ will be called a *Frobenius element* at $\mathfrak{P}$ and will continue to be denoted as $\sigma_{\mathfrak{P}}$. In the case when $\mathfrak{p}$ is unramified (which is for all but finitely many primes), $I_{\mathfrak{P}}$ is trivial and thus, the Frobenius element $\sigma_{\mathfrak{P}}$ is well-defined. Suppose $\sigma \in G \backslash G_{\mathfrak{P}}$. Then, $\sigma(\mathfrak{P})$ is also a prime above $\mathfrak{p}$. It can be easily checked that

$$G_{\sigma(\mathfrak{P})} = \sigma G_{\mathfrak{P}} \sigma^{-1},\ I_{\sigma(\mathfrak{P})} = \sigma I_{\mathfrak{P}} \sigma^{-1}.$$

Thus, if $\mathfrak{p}$ is unramified, then the Frobenius elements corresponding to all the primes above $\mathfrak{p}$ form a conjugacy class in G. This conjugacy class is denoted by $\sigma_{\mathfrak{p}}$. Note that if G is abelian, $\sigma_{\mathfrak{p}}$ is a genuine element of G, called the *Artin symbol* at $\mathfrak{p}$.

Representations of finite groups

We will now recall certain crucial facts regarding representations of finite groups. An excellent reference for this topic is Serre's book, Linear Representations of Finite Groups [90].

Let G be a finite group and V be a finite dimensional $\mathbb{C}$-vector space. A (complex) linear representation of G is a group homomorphism,

$$\rho : G \to GL(V),$$

that is, for every $g, h \in G$, $\rho(g)$ defines an automorphism of V with $\rho(g) \,.\, \rho(h) = \rho(g.h)$. Thus, G acts on the vector space V via ρ. The space V is called the representation space of G. To avoid cumbersome notation, we will refer to the representation (V, ρ) as either ρ or V depending upon the context.

The degree of ρ, sometimes denoted as $\dim(\rho)$ is the dimension of the vector space V. Suppose $W \subseteq V$ is a G-invariant subspace. Then W is said to be a subrepresentation of V. Every representation has itself and the 0-space as subrepresentations. If these are the only subrepresentations, then the representation is said to be *irreducible*. As a consequence of Maschke's theorem, every representation of G can be expressed as a direct sum of irreducible representations.

The character χ of the representation ρ is defined as $\chi(g) = \mathrm{Tr}(\rho(g))$, the trace of the linear map $\rho(g)$. Characters of irreducible representations are called irreducible characters. Note that $\chi(1) = \dim(\rho)$.

Functions that are constant on conjugacy classes are called *class functions*. Characters of a group are important examples of class functions. An inner product can be defined on the space of all class functions on G by

$$\langle f_1 \,|\, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

The set $\widehat{G}$ consisting of irreducible characters of G forms an orthonormal basis for the space of class functions on G. Hence, the number of irreducible characters of G is equal to the number of conjugacy classes in G.

Let V be a representation of G with character ψ and suppose that

$$V = \oplus_{j=1}^{r} m_j \, W_j,$$

where W_j's are non-isomorphic irreducible representations of G with characters χ_j and $m_j \in \mathbb{N}$. Then, $m_j = \langle \psi \,|\, \chi_j \rangle$. Thus, representations with the same character are isomorphic, i.e., characters determine a representation.

If H is subgroup of G, then characters on H can be extended to characters on G in the following way. Let f be a class function on H. Let $r = [G : H]$ and $g_1, \cdots, g_r$

be the coset representatives such that $G = \cup_{i=1}^{r} g_i\, H$. Let

$$\widetilde{f}(g) := \begin{cases} f(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Define a class function $\mathrm{Ind}_H^G(f)$ on G as

$$\mathrm{Ind}_H^G(f)(g) = \sum_{i=1}^{r} \widetilde{f}(g_i^{-1}\, g\, g_i).$$

Thus, for a character χ of H, $\mathrm{Ind}_H^G(\chi)$ is a character of G. Let χ be a character of G and ψ be a character of H. Then *Frobenius reciprocity* states that

$$\left\langle \chi \,|\, \mathrm{Ind}_H^G \psi \right\rangle_G = \left\langle \mathrm{Res}_H^G \chi \,|\, \psi \right\rangle_H ,$$

where $\mathrm{Res}_H^G \chi$ denotes the restriction of χ to G.

A standard example of a group representation is the regular representation. For every $g \in G$, define the symbol x_g and let $V_G := \oplus\, x_g\, \mathbb{C}$. Then, the regular representation of G is defined as

$$\Phi_G : G \to GL(V_G),$$

where $\Phi_G(g') \cdot x_g := x_{g'g}$, extended $\mathbb{C}$-linearly. We will denote the character of the regular representation as reg_G. It is easy to see that

$$\mathrm{reg}_G(g) = \begin{cases} |G|, & \text{if } g = e_G, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, from the definition of induction of characters, one can observe that $\mathrm{Ind}_{\{e_G\}}^G(1) = \mathrm{reg}_G$, where e_G is the identity of G and 1 denotes the trivial character.

Thus, if χ be an irreducible character of G, then

$$\langle \mathrm{reg}_G \,|\, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \mathrm{reg}_G(g)\, \overline{\chi(g)} = \frac{1}{|G|} \,.\, |G| \,.\, \chi(1) = \chi(1).$$

Therefore, from the theory recalled earlier, we obtain

$$\mathrm{reg}_G = \sum_{\chi \in \widehat{G}} \chi(1)\, \chi, \tag{6.2}$$

where $\widehat{G}$ denotes the group of all irreducible characters of G and $\chi(1)$ is the dimension of the irreducible representation corresponding to χ.

Definition of Artin L-functions

Let E/F be a Galois extension of number fields with Galois group G. Let $\rho : G \to GL(V)$ be a representation of G with character χ. Then the Artin L-function associated to the extension E/F and the representation ρ is defined as

$$L(s, \chi, E/F) = \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_F, \\ \mathfrak{p} \text{ prime}}} L_{\mathfrak{p}}(s, \chi, E/F),$$

where the local factors at each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ are as follows. Suppose first that $\mathfrak{p}$ is unramified in E. As seen earlier, $\sigma_{\mathfrak{p}}$ denotes the conjugacy class corresponding to the Frobenius at $\mathfrak{p}$. The local factor at $\mathfrak{p}$ is defined as the inverse of the characteristic polynomial of any element in $\sigma_{\mathfrak{p}}$ (well-defined) evaluated at $N\mathfrak{p}^{-s}$, i.e.,

$$L_{\mathfrak{p}}(s, \chi, E/F) = \det\left(I - \rho(\sigma_{\mathfrak{p}})N\mathfrak{p}^{-s}\right)^{-1}.$$

Now suppose that $\mathfrak{p}$ is ramified in E and fix a prime $\mathfrak{P}$ above $\mathfrak{p}$. Let $V^{I_{\mathfrak{P}}}$ be the subspace of vectors fixed by the inertia group $I_{\mathfrak{P}}$, pointwise. That is,

$$V^{I_{\mathfrak{P}}} = \left\{v \in V \ : \ \rho(\iota) \cdot v = v, \ \text{ for all } \iota \in I_{\mathfrak{P}}\right\}.$$

Since $I_{\mathfrak{P}}$ is a normal subgroup of $G_{\mathfrak{P}}$, one can see that $V^{I_{\mathfrak{P}}}$ is $G_{\mathfrak{P}}$-invariant. Let $\sigma_{\mathfrak{P}}$ be any Frobenius element at $\mathfrak{P}$. Then,

$$L_{\mathfrak{p}}(s, \chi, E/F) = \det\left(I - \rho(\sigma_{\mathfrak{P}})\big|_{V^{I_{\mathfrak{P}}}} N\mathfrak{p}^{-s}\right)^{-1},$$

where $\sigma|_{V^{I_{\mathfrak{P}}}}$ denotes σ restricted to the invariant subspace $V^{I_{\mathfrak{P}}}$ for $\sigma \in G_{\mathfrak{P}}$. Note that the above definition is independent of the choice of the Frobenius element. The infinite product consisting of all these local factors converges absolutely for $\Re(s) > 1$ and defines the Artin L-function associated to ρ and the extension E/F.

The Artin L-function satisfies a functional equation in the same spirit as the Riemann zeta-function. At the infinite primes, i.e., the Archimedean places, the corresponding Euler factors are defined as follows. Let ν be an Archimedean place of

F. Then,

$$L_\nu(s,\chi,E/F) = \begin{cases} \left((2\pi)^{-s}\,\Gamma(s)\right)^{\dim(\rho)}, & \text{if } \nu \text{ is complex,} \\ \left(\pi^{-s/2}\,\Gamma(s/2)\right)^{a}\left(\pi^{-(s+1)/2}\,\Gamma((s+1)/2)\right)^{b} & \text{if } \nu \text{ is real.} \end{cases}$$

Here a is the dimension of the $+1$ eigenspace of complex conjugation and b is the dimension of -1 eigenspace of complex conjugation. Hence,

$$a + b = \dim(\rho).$$

Therefore, the gamma factors for $L(s,\rho,E/F)$ are

$$\gamma(s,\chi,E/F) = \prod_{\nu \text{ - Archimedean place of } F} L_\nu(s,\chi,E/F).$$

An important invariant that makes an appearance in the functional equation is the Artin conductor, $\mathfrak{f}_\chi$. The Artin conductor is an ideal in the ring $\mathcal{O}_F$ and is defined by the restriction of χ to the inertia group and its various subgroups. We refrain from giving the technical definition here and refer the reader to [62] for the precise version. However, we note one of the useful connections of the Artin conductor to the relative discriminants of number fields. In 1931, E. Artin [3] proved the conductor-discriminant formula for any Galois extension of number fields E/F. This formula states that

$$\mathfrak{D}_{E/F} = \prod_{\chi\in\widehat{G}} \mathfrak{f}_\chi^{\chi(1)}, \tag{6.3}$$

where $\mathfrak{D}_{E/F}$ denotes the relative discriminant of E/F.

Let

$$A_\chi = d_F^{\chi(1)} N_{F/\mathbb{Q}}\, \mathfrak{f}_\chi \in \mathbb{Q},$$

where d_F denotes the absolute discriminant of the field F. The completed Artin L-function can then be defined as

$$\Lambda(s, \chi, E/F) := A_\chi^{s/2}\, \gamma(s, \chi, E/F)\, L(s, \chi, E/F).$$

This completed Artin L-function satisfies the functional equation

$$\Lambda(s, \chi, E/F) = W(\chi)\Lambda(1 - s, \overline{\chi}, E/F), \tag{6.4}$$

for all $s \in \mathbb{C}$. The number $W(\chi)$ is called the Artin root number and is a complex number of absolute value 1, carrying deep arithmetic meaning. One important observation here is that if χ is real-valued, then $W(\chi) = \pm 1$. This can be seen by comparing the above functional equation with its complex conjugate.

These L-functions take more familiar shape in certain scenarios. For example, if the extension considered is the trivial extension $\mathbb{Q}/\mathbb{Q}$, then the corresponding Artin L-function reduces to the well-known Riemann zeta-function. Suppose E/F is Galois with Galois group G. Then, the Artin L-function obtained by considering the trivial representation of G is nothing but the Dedekind zeta-function attached to the ground field, $\zeta_F(s)$. On the other hand, the Artin L-functions associated to the irreducible representations of $(\mathbb{Z}/n\mathbb{Z})^*$ and the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ are precisely the Dirichlet L-functions mod n. Thus, we see that Artin L-functions are general L-functions encompassing most of the zeta and L-functions corresponding to classical settings.

One of the important functorial properties of Artin L-functions is that if ρ_1 and ρ_2 are two representations of $G = \mathrm{Gal}(E/F)$ with characters χ_1 and χ_2, then

$$L(s, \chi_1 + \chi_2, E/F) = L(s, \chi_1, E/F)L(s, \chi_2, E/F).$$

Also, if $H \leq G$ and χ is a character of H, then

$$L(s, \mathrm{Ind}_H^G \chi, E/F) = L(s, \chi, E/E^H),$$

where E^H is the fixed field of H. Using these two properties, together with (6.2) and the fact that the regular representation of G can be induced from the trivial representation, we obtain the factorization

$$\zeta_E(s) = \zeta_F(s) \prod_{\substack{\chi \in \widehat{G}, \\ \chi \neq 1}} L(s, \chi, E/F)^{\chi(1)}. \tag{6.5}$$

Artin conjectured that any Artin L-function $L(s, \chi, E/F)$ associated to a character χ of $\mathrm{Gal}(\overline{F}/F)$ extends to an analytic function to the entire complex plane except for a possible pole at $s = 1$, of order equal to the multiplicity of the trivial representation in the representation determined by χ. This is one of the classical conjectures in number theory and remains unresolved in general. It is known in the special case when $\mathrm{Gal}(E/F)$ is abelian. In this case, by Artin's reciprocity law, the Artin L-function of an irreducible character corresponds to a Hecke L-series, which is known to be entire (see [51, Chapter 9] for further details). There are also some recent results in the 2-dimensional case due to Langlands [52], Tunnell [94], Khare and Wintenberger [47].

6.2 Values of L-functions at negative integers

Euler's earlier proof of the evaluation of $\zeta(2k)$ for any positive integer k relied on the theory of Weierstrass products, which was developed only later. So his proof was not rigorous, although the idea was sound. After Riemann's 1859 paper, an alternate way to understand $\zeta(2k)$ emerged, thanks to the functional equation of $\zeta(s)$. One of Riemann's methods of proving the analytic continuation and functional equation was using contour integration along the Hankel contour. As a bi-product of this method, Riemann succeeded in evaluating $\zeta(-n)$ for a positive integer n in terms of Bernoulli numbers. The n^{th} Bernoulli number can be defined using the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \ \ |z| < 2\pi.$$

Thus, the Bernoulli numbers are rational numbers. Riemann showed that when n is a positive integer,

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \in \mathbb{Q}.$$

Now using the functional equation of $\zeta(s)$, one can readily evaluate $\zeta(2k)$.

The analogous idea goes through when determining the values of the Hurwitz zeta-function $\zeta(s, x)$ at negative integers. Recall that for $0 < x \leq 1$, it is defined as

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}, \ \ \Re(s) > 1.$$

Again, the method of integrating an appropriate kernel along the Hankel contour leads to the analytic continuation of $\zeta(s,x)$ to the entire complex plane, except for a simple pole at $s=1$ with residue 1. This also gives that

$$\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1},\ n \in \mathbb{Z}_{\geq 0}, \tag{6.6}$$

where $B_m(X)$ denotes the m^{th} Bernoulli polynomial, defined by an exponential twist of the generating function for the Bernoulli numbers, that is,

$$\frac{z\, e^{Xz}}{e^z-1} = \sum_{m=0}^{\infty} \frac{B_m(X)}{m!}\, z^m, \quad |z| < 2\pi.$$

Therefore, it is apparent that the Bernoulli polynomials have rational coefficients. The proofs of analytic continuation alluded to in the above exposition can be found in [2].

Fix a positive integer q and let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character mod q. The Dirichlet L-series attached to χ is given by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1.$$

On collecting terms of the same residue class modulo q, we get

$$L(s,\chi) = \sum_{a=1}^{q} \chi(a) \sum_{n=0}^{\infty} \frac{1}{(a+nq)^s} = \frac{1}{q^s} \sum_{a=1}^{q} \chi(a) \zeta\left(s, \frac{a}{q}\right),\ \Re(s) > 1.$$

The analytic continuation of the Hurwitz zeta-function and the fact that $\sum_{a=1}^{q} \chi(a) = 0$ when $\chi \neq \chi_0$ imply that $L(s,\chi)$ is entire when $\chi \neq \chi_0$. Furthermore, for any integer

$n \geq 0$, by (6.6),

$$L(-n,\chi) = \frac{q^n}{n+1}\sum_{a=1}^{q}\chi(a)B_{n+1}\left(\frac{a}{q}\right).$$

Therefore, if χ is a quadratic character, then $L(-n,\chi) \in \mathbb{Q}$.

Similarly, the Siegel-Klingen theorem determines the values of Dedekind zeta-functions attached to totally real fields at odd negative integers and proves that these values are rational. These results were generalized to Artin L-functions in special cases by Coates and Lichtenbaum [22]. In particular, they show the following (as stated in P. J. Wong [99]). Let E/F be a Galois extension of number fields with Galois group G. Let ρ be a representation of G with character χ and $L(s,\chi,E/F)$ be the associated Artin L-function. Let $\mathbb{Q}(\chi) = \mathbb{Q}(\{\chi(g) : g \in G\})$ and m be a fixed positive integer. Then, $L(-m,\chi,E/F) \in \mathbb{Q}(\chi)$ provided that

a) m is odd and the fixed field $E^{\ker(\rho)}$ of $\ker(\rho)$ is totally real, or

b) m is even, the fixed field $E^{\ker(\rho)}$ is totally imaginary, complex conjugation is central in $\mathrm{Gal}(E^{\ker(\rho)}/F)$ and $\chi(c) = -\dim(\rho)$, where $c \in G$ denotes complex conjugation.

This also generalizes the similar observation in case of Dirichlet L-functions mentioned above.

6.3 The Main theorem and Corollaries

In this section, we focus on Dedekind zeta-functions attached to CM-number fields. Some naturally occurring examples of CM-extensions include imaginary quadratic extensions of $\mathbb{Q}$ and cyclotomic extensions, i.e., $\mathbb{Q}(\zeta_q)/\mathbb{Q}(\zeta_q+\zeta_q^{-1})$ for a positive integer q.

In order to state the main theorem of this chapter, we define the notion of rational equivalence. Two complex numbers α and β are said to be rationally equivalent, i.e., $\alpha \sim_{\mathbb{Q}} \beta$ if $\beta = u\,\alpha$ for some $u \in \mathbb{Q}$. With this definition, we show that

Theorem 6.3.1. *Fix a totally real number field F. Let E_1 and E_2 be two CM-extensions of F and d_{E_1} and d_{E_2} be their respective discriminants. Then, for a fixed integer $m > 0$,*

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{d_{E_2}}{d_{E_1}}\right)^{1/2}.$$

Proof. Let $G_j := \mathrm{Gal}(E_j/F)$ for $j = 1, 2$. Then we have, $G_j = \{1, c_j\}$, where c_j denote complex conjugation. Let the characters corresponding to c_j be $\chi_j : G_j \to \{\pm 1\}$ where $\chi_j(c_j) = -1$. By the factorization (6.5),

$$\zeta_{E_j}(s) = \zeta_F(s) L(s, \chi_j, E_j/F), \ \ j = 1, 2.$$

Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} = \frac{L(2m+1, \chi_1, E_1/F)}{L(2m+1, \chi_2, E_2/F)}.$$

The functional equation of Artin L-functions (6.4) relate the value at $2m+1$ with the value at $-2m$. Since F is totally real, all Archimedean places of F are real and hence, none of the gamma factors appearing in the functional equation have poles at these integers. Thus, for $j = 1, 2$, we have

$$A_{\chi_j}^{(2m+1)/2}\, \gamma(2m+1, \chi_j, E_j/F)\, L(2m+1, \chi_j, E_j/F)$$
$$= W(\chi_j)\, A_{\chi_j}^{-m}\, \gamma(-2m, \chi_j, E_j/F)\, L(-2m, \chi_j, E_j/F).$$

This implies that

$$L(2m+1,\chi_j,E_j/F) = W(\chi_j)\left(A_{\chi_j}\right)^{-2m-1/2} \frac{\gamma(-2m,\chi_j,E_j/F)}{\gamma(2m+1,\chi_j,E_j/F)} L(-2m,\chi_j,E_j/F).$$

Note that $\ker(\chi_j) = 1$. Thus, it can be easily checked that the hypotheses for the Coates-Lichtenbaum theorem are true. Hence, we deduce that

$$L(-2m,\chi_j,E_j/F) \in \mathbb{Q}.$$

It can also be seen that these values are non-zero. Indeed, the zeros at negative integers appear as a result of poles of the Euler factors at infinity. But F is totally real and the extension is totally imaginary. Hence, the gamma factors appearing in the functional equation are of the form $\Gamma((s+1)/2)$ and $\Gamma(1-(s/2))$, which do not have poles at $s = -2m$. Moreover, since χ_j are real valued for $j = 1$, 2, $W(\chi_j) = \pm 1$ as seen earlier. Hence,

$$L(2m+1,\chi_j,E_j/F) \sim_{\mathbb{Q}} A_{\chi_j}^{-1/2} \frac{\gamma(-2m,\chi_j,E_j/F)}{\gamma(2m+1,\chi_j,E_j/F)}.$$

On taking the ratio of $L(2m+1,\chi_j,E_j/F)$ for $j = 1$, 2, the gamma factors cancel as they are same for both the Artin L-functions under consideration. Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{A_{\chi_2}}{A_{\chi_1}}\right)^{1/2}.$$

The factor of d_F will be common to both the values, and disappears in the ratio.

Thus, the contributing factor reduces to

$$\frac{N_{F/\mathbb{Q}}\mathfrak{f}_{\chi_1}}{N_{F/\mathbb{Q}}\mathfrak{f}_{\chi_2}}.$$

Since the conductor of the trivial representation is the unit ideal, the conductor-discriminant formula (6.3) implies

$$\mathfrak{f}_{\chi_j} = \mathfrak{D}_{E_j/F}.$$

The relative discriminant of E_j/F is related to the absolute discriminant of E_j by the formula

$$d_{E_j} = d_F^2 \ . \ N_{F/\mathbb{Q}}\mathfrak{D}_{E_j/F}.$$

The statement of the theorem now follows. □

Specializing to the case of imaginary quadratic fields leads to the following interesting corollary.

Corollary 6.3.2. *Let m be a fixed integer, $m \geq 1$. Then the numbers*

$$\left\{ \zeta_K(2m+1) \ : \ K/\mathbb{Q} \text{ is an imaginary quadratic extension} \right\}$$

are irrational with at most one exception.

Proof. By Theorem 6.3.1, we know that if K_1 and K_2 are two imaginary quadratic extensions of $\mathbb{Q}$, then

$$\frac{\zeta_{K_1}(2m+1)}{\zeta_{K_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{d_{K_2}}{d_{K_1}}\right)^{1/2}.$$

Since K_1 and K_2 are distinct quadratic extensions of $\mathbb{Q}$, their will be atleast one odd prime p such that $p|d_{K_j}$, $p^2 \nmid d_{K_j}$ and $p \nmid d_{K_l}$ for j, $l = 1, 2$ and $j \neq l$. Thus, the factor of $\sqrt{p}$ will contribute towards the irrationality of the quotient. □

Corollary 6.3.3. *Let m be a fixed integer, $m \geq 1$. Fix a totally real field F. Consider the family $\mathfrak{F}$ of number fields E such that E/F is a CM-extension, d_E is not a perfect square and if E_1, $E_2 \in \mathfrak{F}$, then $(d_{E_1}, d_{E_2}) = 1$. Then the numbers*

$$\left\{ \zeta_E(2m+1) \, : \, E \in \mathfrak{F} \right\}$$

are irrational with at most one exception.

Proof. The conditions on the family $\mathfrak{F}$ ensure that for any E_1 and E_2 in $\mathfrak{F}$, d_{E_1}/d_{E_2} is not a perfect square in $\mathbb{Q}$. The corollary is now immediate from Theorem 6.3.1. □

Remark. *It is not immediately clear if an infinite family as in Corollary 6.3.3 exists for any given totally real number field F. We relegate finding such examples to future research. Another direction of investigation is to explore if the above theorem gives any information regarding the Lichtenbaum conjectures [56] regarding the first non-vanishing coefficient in the Taylor series expansion of $\zeta_F(s)$ around a negative integer. For an introduction and further reading in this context, see [81].*

Chapter 7

Transcendental sums associated to elliptic functions

7.1 Introduction

Motivated by the intricate questions of non-vanishing and transcendence surrounding the sums $\sum_{n=1}^{\infty} A(n)/B(n)$ and $\sum_{n\in\mathbb{Z}} A(n)/B(n)$ for $A(X)$, $B(X) \in \overline{\mathbb{Q}}[X]$, we devote this chapter to the study of their 2-dimensional analog. Let Λ be the lattice spanned by the non-zero complex numbers ω_1 and ω_2, that is, $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. We assume that ω_1 and ω_2 are $\mathbb{R}$-linearly independent so that Λ is a two-dimensional lattice. Then for suitable co-prime polynomials, $A(X)$ and $B(X)$, we study the sum

$$\mathcal{S}(A,B) := \sum_{\omega\in\Lambda} \frac{A(\omega)}{B(\omega)} = \lim_{N\to\infty} \lim_{M\to\infty} \sum_{\substack{m,n\in\mathbb{Z},\\ |m|\le M,\\ |n|\le N}} \frac{A(m\omega_1+n\omega_2)}{B(m\omega_1+n\omega_2)}. \tag{7.1}$$

In [44], the authors highlight and exploit the connection of the digamma function and its derivatives (i.e., the polygamma functions) to sums of the form $\sum_{n=1}^{\infty} A(n)/B(n)$. In the same spirit, sums of the form $\sum_{n\in\mathbb{Z}} A(n)/B(n)$ are fundamentally related to the cotangent function and its derivatives (see [70] for more details). Similarly, we will show that sums of the form (7.1) are associated to linear combinations of the Weierstrass functions. Thus, the study of the arithmetic nature of (7.1) is in essence an attempt to understand linear relations among values of the Weierstrass functions and their derivatives.

7.2 Elliptic functions

In this section, we overview relevant properties of elliptic functions and results concerning the arithmetic nature of their values. We also prove certain lemmas that play a vital role in the later sections.

7.2.1 The Weierstrass functions: Review and Lemmas

A detailed introduction to elliptic functions can be found in the classic book of Whittaker and Watson [98, Chapter XX]. However, for a brief review, we refer the reader to [66, Chapter 10].

One of the most important properties governing the behaviour of trigonometric functions $\sin(z)$ and $\cos(z)$ is that they are periodic with period 2π. A natural extension of this theory lies in doubly periodic functions. Given two non-zero complex numbers, ω_1 and ω_2, such that ω_1/ω_2 is not purely real, a function f is said to be *doubly periodic* if

$$f(z+\omega_1) = f(z), \quad f(z+\omega_2) = f(z),$$

for all $z \in \mathbb{C}$ where f is defined. An *elliptic function* with respect to the lattice Λ is a meromorphic function on $\mathbb{C}$ that is doubly periodic. The elements of Λ are called *periods* of the lattice. The numbers ω_1 and ω_2 that generate the lattice Λ are called *fundamental periods*. The values of an elliptic function f can be determined by its values on the *fundamental parallelogram*,

$$D = \left\{ r\omega_1 + s\omega_2 \,:\, r,\, s \in \mathbb{R},\ 0 \leq r,\, s < 1 \right\},$$

which is bounded. Thus, Liouville's theorem implies that any entire elliptic function is constant.

An example of a non-constant elliptic function is provided by the Weierstrass $\wp$ function associated to Λ, defined as

$$\wp(z;\Lambda) := \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \ \ z \in \mathbb{C}\backslash\Lambda, \tag{7.2}$$

where Λ^* denotes the set of non-zero periods. Whenever the lattice is fixed in our discussion, we will write the Weierstrass $\wp$ function as $\wp(z)$. It can be shown that the Weierstrass $\wp$ function converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$, thus defining an analytic function on $\mathbb{C}\backslash\Lambda$, with poles of order 2 at every period in Λ. On differentiating the series term-by-term for $z \in \mathbb{C}\backslash\Lambda$, it follows that the derivatives of $\wp$,

$$\wp^{(j)}(z) = (-1)^j (j+1)! \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^{j+2}}, \ \ j \geq 1,$$

are elliptic functions on $\mathbb{C}$. Moreover, if

$$\wp(z+\omega) = \wp(z) + c(\omega),$$

for some constant $c(\omega)$, then evaluating both sides at $z = -\omega/2$ and noting that $\wp(z)$ is an even function, one deduces that $\wp(z)$ defines an elliptic function on the complex plane with respect to the lattice Λ. Although it may seem that the Weierstrass $\wp$ function is only one specific example of an elliptic function, it turns out that $\wp(z)$ and its derivative $\wp'(z)$ are in a sense, universal. That is, any elliptic function with respect to Λ is a rational function in $\wp$ and $\wp'$.

Given a lattice Λ, the associated Eisenstein series of weight $2k$ for an integer $k \geq 2$ is given by the absolutely convergent sum

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda^*} \frac{1}{\omega^{2k}}.$$

For $k = 1$, the above sum does not converge absolutely and is dependent on the order of summation. However,

$$G_2(z) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2}, \quad \Im(z) > 0,$$

where $\sum'$ means that the term $(m, n) = (0, 0)$ is omitted in the summation, can be shown to converge using a technique of Hurwitz (see[Chapter 5] [61]). Since the fundamental periods ω_1 and ω_2 can be chosen such that $\Im(\omega_1/\omega_2) > 0$, the above series would define an Eisenstein series of weight 2 for Λ.

Eisenstein series play an important role in the theory of the Weierstrass $\wp$ function. For instance, they appear as coefficients in the Laurent series expansion of $\wp(z)$ around

$z = 0$. Another example of their appearance is the following. Let

$$g_2(\Lambda) := 60\, G_4(\Lambda), \quad g_3(\Lambda) := 140\, G_6(\Lambda).$$

If the lattice is fixed in our discussion, we will drop the reference to Λ for convenience.

The Weierstrass functions $\wp(z)$ and $\wp'(z)$ satisfy a differential equation with coefficients g_2 and g_3, i.e.,

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{7.3}$$

Thus, the points $(\wp(z), \wp'(z))$ for $z \in \mathbb{C}\backslash\Lambda$ lie on the curve defined by the equation

$$y^2 = 4x^3 - g_2 x - g_3. \tag{7.4}$$

In fact, the points $(\wp(z), \wp'(z))$ parametrize all complex points on the curve (7.4). Let ω_1 and ω_2 be fundamental periods of Λ. Observe that

$$\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0,$$

as $\wp'(z)$ is an odd function. Hence, the numbers

$$\wp\left(\frac{\omega_1}{2}\right), \ \wp\left(\frac{\omega_2}{2}\right) \text{ and } \wp\left(\frac{\omega_1 + \omega_2}{2}\right) \tag{7.5}$$

are roots of the equation $4x^3 - g_2 x - g_3 = 0$. Therefore, if g_2, g_3 are algebraic, then the above numbers are also algebraic.

From (7.3), one deduces an addition law for the Weierstrass $\wp$ function. Let z_1, $z_2 \in \mathbb{C}\backslash\Lambda$, with $z_1 \pm z_2 \notin \Lambda$. Then

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2. \tag{7.6}$$

Taking the limit as $z_1 \to z_2$, one obtains the duplication formula

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp(z)}\right)^2.$$

Thus, when the invariants g_2, g_3 are algebraic, the addition law together with the observation that the numbers (7.5) are algebraic implies that

$$\wp\left(\frac{\omega_1}{n}\right) \text{ and } \wp\left(\frac{\omega_2}{n}\right)$$

are algebraic numbers. Furthermore, the addition and duplication formula give that

$$\wp\left(\frac{m}{n}\,\omega_1\right) \text{ and } \wp\left(\frac{m}{n}\,\omega_2\right),$$

with m, $n \in \mathbb{Z}$ and $\gcd(m, n) = 1$, are also algebraic.

Since the derivatives of the Weierstrass $\wp$ function will be essential to our understanding of the nature of $\mathcal{S}(A, B)$, we prove the following lemma regarding their representation as a polynomial in $\wp$ and $\wp'$.

Lemma 7.2.1. *Let $\wp(z)$ be the Weierstrass $\wp$ function associated to a lattice Λ, with*

$g_2(\Lambda)$, $g_3(\Lambda) \in \overline{\mathbb{Q}}$. Then there exist polynomials $F_l(X)$, $G_l(X) \in \overline{\mathbb{Q}}[X]$ such that

$$\wp^{(l)}(z) = F_l\left(\wp(z)\right)\, \wp'(z) + G_l\left(\wp(z)\right).$$

Moreover, $F_{2l}(X) = 0$, $G_{2l+1}(X) = 0$ and $\deg F_{2l+1}(X) = l$, $\deg G_{2l}(X) = l + 1$.

Proof. The existence of such polynomials can be seen from the algebraic differential equation,

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\,\wp(z) - g_3.$$

On differentiating both sides with respect to z, we get

$$\wp'(z)\,\wp''(z) = \left(6\,\wp^2(z) - \frac{g_2}{2}\right)\wp'(z),$$

that is,

$$\wp^{(2)}(z) = 6\,\wp^2(z) - \frac{g_2}{2}. \tag{7.7}$$

Therefore, $\wp^{(2)}(z)$ can be expressed as a polynomial in $\wp(z)$. Thus, differentiation of (7.7) with respect to z implies that the higher derivatives of $\wp(z)$ can also be expressed as polynomials in $\wp$ and $\wp'$, with the degree of $\wp'$ being one, owing to (7.3). Note that $\phi_l(z) := F_l(\wp(z))\,\wp'(z) + G_l(\wp(z))$ defines an elliptic function with respect to the lattice Λ. Since $\wp'(z)$ has poles of order 3 at the lattice points, the order of poles of $\phi_l(z)$ at these points would be odd. Suppose l is even. Then the function $\wp^{(l)}(z)$ has poles of an even order at the periods of Λ. Thus, $F_{2l}(X) = 0$ and $\wp^{2l}(z) = G_{2l}(\wp(z))$. Differentiating this with respect to z gives $G_{2l+1}(X) = 0$ for $l \geq 0$.

That the degrees of the polynomials F_l and G_l are as stated in the lemma, can be seen by induction on l. The statement of the lemma clearly holds for $l = 0,\ 1$. Suppose that the lemma holds for all $k \leq l$, $1 < l$. If $l = 2m$ for some $m \in \mathbb{N}$. Then

$$\wp^{(2m)}(z) = G_{2m}\left(\wp(z)\right),$$

with $\deg G_{2m} = m + 1$. Differentiating both sides with respect to z gives

$$\wp^{(2m+1)}(z) = G'_{2m}\left(\wp(z)\right)\ \wp'(z),$$

where $G'_{2m}(X)$ is the derivative of the polynomial $G_{2m}(X)$. Therefore, $F_{2m+1}(X) = G'_{2m}(X)$, $\deg F_{2m+1}(X) = m$ and $G_{2m+1} = 0$. Similarly, suppose $l = 2m + 1$ for some $m \in \mathbb{N}$. Then

$$\wp^{(2m+1)}\left(\wp(z)\right) = F_{2m+1}\left(\wp(z)\right)\ \wp'(z),$$

where $\deg F_{2m+1} = m$. Differentiating this expression gives

$$\wp^{(2m+2)}\left(\wp(z)\right) = F'_{2m+1}\left(\wp(z)\right)\left(\wp'(z)\right)^2 + F_{2m+1}\left(\wp(z)\right)\ \wp''(z).$$

Hence, using (7.3) and (7.7), we obtain

$$\wp^{(2m+2)}\left(\wp(z)\right) = F'_{2m+1}\left(\wp(z)\right)\left(4\wp^3(z) - g_2\wp(z) - g_3\right) + F_{2m+1}\left(\wp(z)\right)\left(6\,\wp^2(z) - \frac{g_2}{2}\right).$$

Thus, $\wp^{(2m+2)}(z) = G_{2m+2}(\wp(z))$, for a polynomial $G_{2m+2}(X) \in \overline{\mathbb{Q}}[X]$. Suppose that

$$F_{2m+1}(X) = \sum_{k=0}^{m} f(k)\,X^k,$$

then the leading term in $G_{2m+2}(X)$ is evidently

$$\Big(m\, f(m) + 6\, f(m)\Big)\, X^{m+2}.$$

Since $f(m) \neq 0$ and $m > 0$, the coefficient of X^m in $G_{2m+2}(X)$ is not zero. Thus, the degree of $G_{2(m+1)}(X) = m+2$, as claimed in the lemma. $\square$

Another related function that makes an appearance in our study is the Weierstrass zeta-function, defined as

$$\zeta^*(z;\Lambda) = \frac{1}{z} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right),$$

which converges absolutely and uniformly on compact subsets of $\mathbb{C}\backslash\Lambda$, and hence, defines an analytic function there. Once again, we will drop the reference to the lattice in the notation if it is fixed in our discussion. One can check that ${\zeta^*}'(z) = -\wp(z)$, for $z \in \mathbb{C}\backslash\Lambda$.

Note that $\zeta^*(z)$ is *not* doubly periodic. In particular,

$$\zeta^*(z+\omega) = \zeta^*(z) + \eta(\omega), \tag{7.8}$$

for some constant $\eta(\omega)$ independent of z. Also, note that the Weierstrass zeta-function is an odd function. Thus, evaluating (7.8) at $z = -\omega_1/2$ and $\omega = \omega_1$ gives

$$\eta_1 = 2\,\zeta^*\left(\frac{\omega_1}{2}\right),\;\; \eta_2 = 2\,\zeta^*\left(\frac{\omega_2}{2}\right). \tag{7.9}$$

If ω_1 and ω_2 are the fundamental periods of Λ, then the corresponding $\eta_1 := \eta(\omega_1)$ and $\eta_2 := \eta(w_2)$ are called *quasi-periods* of the Weierstrass zeta-function. One can observe that $\eta(\omega)$ is $\mathbb{Z}$-linear in ω and hence, other quasi-periods of ζ^* are $\mathbb{Z}$-linear combinations of η_1 and η_2.

The fundamental periods and quasi-periods are related by the Legendre relation, namely,

$$\omega_1 \, \eta_2 - \omega_2 \, \eta_1 = 2\pi i, \quad \text{if } \Im(\omega_1/\omega_2) > 0.$$

The Weierstrass zeta-function also satisfies an addition law given by

$$\zeta^*(z_1 + z_2) = \zeta^*(z_1) + \zeta^*(z_2) + \frac{1}{2}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right), \tag{7.10}$$

for $z_1,\, z_2 \in \mathbb{C}\backslash\Lambda$ such that $z_1 \pm z_2 \notin \Lambda$. As $z_1 \to z_2$, we obtain a duplication formula analogous to the one for the Weierstrass $\wp$ function. More generally, on repeated application of the addition law, we have for $m \in \mathbb{N}$,

$$\zeta^*(mz) = m\, \zeta^*(z) + \frac{1}{2}\sum_{j=2}^{m-1} \mathcal{F}_j(z) + \frac{1}{2}\,\frac{\wp''(z)}{\wp'(z)}, \tag{7.11}$$

where

$$\mathcal{F}_j(z) = \frac{\wp'(jz) - \wp'(z)}{\wp(jz) - \wp(z)},$$

provided that $jz \notin \Lambda$ for all $0 \le j \le m-1$.

With the background established so far, we prove the following lemmas which will be particularly useful in the following sections.

Lemma 7.2.2. *Let Λ be a lattice with algebraic invariants g_2, g_3 and fundamental periods ω_1 and ω_2. Let $n,\, l \in \mathbb{Z}$ with $n > 1$ and $\gcd(l, n) = 1$. Then there exists an algebraic number $\epsilon(l,\, n)$ such that*

$$\zeta^*\left(\frac{l}{n}\,\omega_1\right) = \frac{2l}{n}\,\zeta^*\left(\frac{\omega_1}{2}\right) + \epsilon(l,\, n).$$

The analogous statement holds when ω_1 is replaced by ω_2.

Proof. It suffices to prove the lemma when $1 \le l \le n-1$. Indeed, suppose the lemma holds in this case. Suppose that $l \ge n$. Then one can write

$$\frac{l}{n} = q + \frac{l'}{n},$$

with $q \in \mathbb{N}$ and $0 \le l' \le n-1$. Thus,

$$\zeta^*\left(\frac{l}{n}\,\omega_1\right) = \zeta^*\left(q\,\omega_1 + \frac{l'}{n}\,\omega_1\right) = \zeta^*\left(\frac{l'}{n}\,\omega_1\right) + \eta(q\,\omega_1) = \frac{2l'}{n}\,\zeta^*\left(\frac{\omega_1}{2}\right) + \epsilon(l',\, n) + 2q\,\zeta^*\left(\frac{\omega_1}{2}\right),$$

which proves the lemma in this case. Moreover, if $l < 0$, then the lemma follows from the fact that $\zeta^*(z)$ is an odd function.

Now suppose that $1 \le l \le n-1$. We would like to take $m = n$ and $z = \omega_1/2n$ in (7.11). Since ω_1 is a fundamental period, we see $(j/2n)\,\omega_1 \notin \Lambda$ for $0 < j < n$. Therefore, we have

$$\zeta^*\left(\frac{\omega_1}{2}\right) = n\,\zeta^*\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\sum_{j=2}^{n-1} \mathcal{F}_j\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\,\frac{\wp''}{\wp'}\left(\frac{\omega_1}{2n}\right).$$

As seen earlier, since g_2 and g_3 are algebraic, the values $\wp(\omega_1/2n)$, $\wp'(\omega_1/2n)$ and $\wp''(\omega_1/2n)$ are all algebraic. Therefore, we get

$$\zeta^*\left(\frac{\omega_1}{2n}\right) = \frac{1}{n}\,\zeta^*\left(\frac{\omega_1}{2}\right) + \delta(l,\, n), \tag{7.12}$$

for some algebraic number $\delta(l,\, n)$, which can be explicitly written down from the expression above. Once again, we aim to utilize (7.11) with $m = 2l$ and $z = \omega_1/2n$. Since $0 < l < n$ and ω_1 is a fundamental period, $(k/2n)\,\omega_1 \notin \Lambda$ for $0 < k < 2l$. Hence,

$$\zeta^*\left(\frac{l}{n}\,\omega_1\right) = 2l\,\zeta^*\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\sum_{k=2}^{2l-1}\mathcal{F}_k\left(\frac{\omega_1}{2n}\right) + \frac{1}{2}\,\frac{\wp''}{\wp'}\left(\frac{\omega_1}{2n}\right).$$

The lemma is now immediate from (7.12). □

We remark here that above proof will not go through in the most general case of rational multiples of any non-zero period in Λ, since (7.11) will not be valid when one of the intermediate values $jz \in \Lambda$ for some $0 < j < m$. In particular, if $\omega_0 = a_0\,\omega_1 + b_0\,\omega_2$ and n are such that $j\omega_0/2n \in \Lambda$, then the above proof will fail. For example, consider $\omega_0 = 4\omega_1 + 4\omega_2$, $n = 6$ and $j = 3$. Then $j\omega_0/2n \in \Lambda$ while $\omega_0/n \notin \Lambda$ and $j < n$. However, using the above lemma, one can extend the result to the value of $\zeta^*(z)$ at non-integral rational multiples of any non-zero period in Λ.

Lemma 7.2.3. *Let Λ be a lattice with algebraic invariants g_2, g_3 and fundamental periods ω_1 and ω_2. Let $n,\, l \in \mathbb{Z}$ with $n > 1$ and $\gcd(l, n) = 1$. Let $\omega_0 = a_0\,\omega_1 + b_0\,\omega_2 \in \Lambda^*$. Suppose that $\omega_0/n \notin \Lambda$. Then there exists an algebraic number $\epsilon(l,\, n,\, a_0,\, b_0)$ such that*

$$\zeta^*\left(\frac{l}{n}\,\omega_0\right) = \frac{2l}{n}\left(\zeta^*\left(\frac{\omega_0}{2}\right)\right) + \epsilon(l,\, n,\, a_0,\, b_0).$$

Proof. As earlier, one can assume that $0 < l < n$. We consider two cases.

a) $n \mid a_0$, $n \nmid b_0$ (or symmetrically, $n \mid b_0$, $n \nmid a_0$): Let $a_0 = qn$. Note that $lb_0/n \notin \mathbb{Z}$ as $(l, n) = 1$ and $n \nmid b_0$. Thus,

$$\zeta^*\left(\frac{l}{n}\,\omega_0\right) = \zeta^*\left(\frac{l\,a_0}{n}\,\omega_1 + \frac{l\,b_0}{n}\,\omega_2\right) = \zeta^*\left(lq\,\omega_1 + \frac{l\,b_0}{n}\,\omega_2\right)$$
$$= lq\,\eta_1 + \zeta^*\left(\frac{l\,b_0}{n}\,\omega_2\right) = 2lq\,\zeta^*\left(\frac{\omega_1}{2}\right) + \frac{2lb_0}{n}\,\zeta^*\left(\frac{\omega_2}{2}\right) + \delta(l,\, n,\, a_0,\, b_0),$$

as $\eta_1 = a\zeta^*(\omega_1/2)$ and by Lemma 7.2.2. This proves the result.

b) $n \nmid a_0$, $n \nmid b_0$: Using the relation $\zeta^*(z+\omega) = \zeta^*(z) + \eta(\omega)$ for $z \notin \Lambda$ and $\omega \in \Lambda$, one can reduce to the situation where $0 < la_0 < n$ and $0 < lb_0 < n$. By (7.10), we get

$$\zeta^*\left(\frac{la_0}{n}\,\omega_1 + \frac{lb_0}{n}\,\omega_2\right) = \zeta^*\left(\frac{la_0}{n}\,\omega_1\right) + \zeta^*\left(\frac{lb_0}{n}\,\omega_2\right) + \frac{1}{2}\left(\frac{\wp'(la_0\omega_1/n) - \wp'(lb_0\omega_2/n)}{\wp(la_0\omega_1/n) - \wp(lb_0\omega_2/n)}\right).$$

Since g_2 and g_3 are algebraic, as seen earlier, the Weierstrass $\wp$ function and its derivatives take algebraic values at non-integral rational multiples of the fundamental periods. Thus, the last term is an algebraic number. The lemma now follows from Lemma 7.2.2.

□

7.2.2 Transcendence of values of Weierstrass functions

An excellent reference for the compilation of results regarding algebraic independence of values of elliptic functions is the book "Number Theory IV", by N. I. Fel'dman and Yu. V. Nesterenko [32]. Proofs for some of the results can also be found in [66].

Throughout this subsection, we consider the lattice Λ, with fundamental periods ω_1 and ω_2 such that $\Im(\omega_1/\omega_2) > 0$. Furthermore, we also suppose that *the associated invariants g_2 and g_3 are algebraic.*

Recall that the ring of endomorphisms (as a $\mathbb{Z}$-module) of the lattice Λ, say $E(\Lambda)$, consists of complex numbers λ such that $\lambda\Lambda \subseteq \Lambda$. It can be shown that $E(\Lambda)$ is either $\mathbb{Z}$ or an order in an imaginary quadratic field. The second case occurs if and only if $\tau := \omega_1/\omega_2$ is a quadratic imaginary irrational number. In this case, the lattice Λ (or the associated Weierstrass $\wp$ function) is said to have *complex multiplication* and the field $K = \mathbb{Q}(\tau)$ will be called the *field of complex multiplication.* Since complex multiplication adds more structure to the lattice, several conjectures are often proven in the CM case and are open otherwise.

It is well known that the exponential function e^z takes transcendental values at algebraic arguments, by the Hermite-Lindemann theorem. Analogously, one may enquire regarding the arithmetic nature of $\wp(\alpha)$ for an algebraic number α. In the 1930s, Th. Schneider [85], [86], [87] proved several important results regarding the values of $\wp(z)$ and $\zeta^*(z)$ as well as their periods and quasi-periods. We state a few relevant theorems below (as appeared in [32]).

Theorem 7.2.4 (Schneider)**.** *Suppose $\alpha \in \mathbb{C}\backslash\Lambda$ is algebraic. Then $\wp(\alpha)$ is transcendental.*

Theorem 7.2.5 (Schneider)**.** *Suppose that the invariants of $\wp(z)$ and $\zeta^*(z)$, namely, $g_2,\ g_3 \in \overline{\mathbb{Q}}$ and let $\phi(z) := az + b\,\zeta^*(z)$ for $a,\ b \in \overline{\mathbb{Q}}$ with $|a| + |b| > 0$. If β is an algebraic number with $\beta \notin \Lambda$, then at least one of the numbers $\phi(\beta)$ and $\wp(\beta)$ is* *transcendental.*

The above two theorems together with the Schneider-Lang theorem imply that

Theorem 7.2.6 (Schneider)**.** *Any non-zero period or quasi-period of* Λ *is transcendental.*

However, more can be said in case Λ has complex multiplication. A crucial lemma of D. Masser [58, Lemma 3.1] in this context is the following.

Lemma 7.2.7 (Masser)**.** *Let* $\wp(z)$ *be a Weierstrass* $\wp$ *function with algebraic invariants* g_2, g_3 *and complex multiplication. Let* ω_1, ω_2 *and* η_1, η_2 *be certain periods and quasi-periods respectively. Then* ω_2 *and* η_2 *are algebraic over the field* $\mathbb{Q}(\omega_1, \eta_1)$*.*

As a consequence of this lemma and an important theorem of Yu. V. Nesterenko [72], the following result can be obtained when Λ has complex multiplication (see [66, Chapter 17] for details).

Theorem 7.2.8. *Let* $\wp(z)$ *be a Weierstrass* $\wp$ *function for a lattice with algebraic invariants* g_2, g_3 *and complex multiplication by an order of the imaginary quadratic field* K*. Let* ω *be a non-zero period and* η *the corresponding quasi-period. Then for any* $\tau \in K$ *with* $\Im(\tau) \neq 0$*, each of the sets*

$$\{\pi, \omega, e^{2\pi i\tau}\} \text{ and } \{\omega, \eta, e^{2\pi i\tau}\}$$

are algebraically independent over $\mathbb{Q}$*.*

In case of the exponential function, we have the Lindemann-Weierstrass theorem, which states that

Theorem 7.2.9 (Lindemann-Weierstrass). *If $\alpha_1, \cdots, \alpha_r$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then*

$$e^{\alpha_1}, \cdots, e^{\alpha_r}$$

are algebraically independent over $\mathbb{Q}$.

This theorem includes the special case mentioned earlier, that e^α is transcendental for non-zero algebraic α. Thus, it is natural to wonder if an elliptic analog of the Lindemann-Weierstrass theorem exists. The following version of the elliptic Lindemann-Weierstrass was conjectured by G. V. Chudnovsky in 1980 [21].

Conjecture 7.2.10 (Chudnovsky). *Suppose that $n \geq 1$, the Weierstrass $\wp(z)$ has algebraic invariants and does not have complex multiplication and $\alpha_1, \cdots, \alpha_n$ are algebraic numbers that are $\mathbb{Q}$-linearly independent. Then the numbers*

$$\wp(\alpha_1), \cdots, \wp(\alpha_n)$$

are algebraically independent over $\mathbb{Q}$.

This conjecture has only been proved when $n = 1$, which is Theorem 7.2.4.

In 1983, Wüstholz [100] and P. Philippon [78] independently proved that the elliptic analog of the Lindemann-Weierstrass theorem is true in the CM-case. In particular, we have

Theorem 7.2.11 (G. Wüstholz, P. Philippon). *Suppose that $n \geq 1$, the Weierstrass $\wp(z)$ has algebraic invariants and complex multiplication, and $\alpha_1, \cdots, \alpha_n$ are linearly* *independent over the field of complex multiplication. Then the numbers*

$$\wp(\alpha_1), \cdots, \wp(\alpha_n)$$

are algebraically independent over $\mathbb{Q}$.

In the 1960s, S. Schanuel proposed a conjecture about values of the exponential function while attending a course given by S. Lang. The conjecture reads as follows.

Conjecture 7.2.12 (Schanuel)**.** *Let* $x_1, \cdots, x_n \in \mathbb{C}$ *be such that they are linearly independent over* $\mathbb{Q}$. *Then the transcendence degree of the field*

$$\mathbb{Q}\big(x_1, x_2, \cdots, x_n, e^{x_1}, e^{x_2}, \cdots, e^{x_n}\big)$$

over $\mathbb{Q}$ *is at least* n.

This conjecture encompasses most of the known results regarding the arithmetic nature of values of the exponential function. Many more interesting corollaries can be deduced conditional upon this conjecture. Schanuel's conjecture is known when $n = 1$ (the Hermite-Lindemann theorem) and when $x_1, \cdots, x_n \in \overline{\mathbb{Q}}$ by Theorem 7.2.9 (the Lindemann-Weierstrass theorem). That Schanuel's conjecture can be regarded as a special case of the generalized Grothendieck period conjecture, was shown by C. Bertolin [12]. Another specialization of the Grothendieck conjecture is an elliptic analog of the classical Schanuel conjecture. Following the authors in [79], we call this the *elliptic Schanuel conjecture* and state it below.

Conjecture 7.2.13 (Elliptic Schanuel)**.** *Let* Λ *be a lattice and* $\wp$, ζ^* *denote the associated Weierstrass functions. Let* K *be the field of endomorphisms of* Λ *(i.e.,*

$K = \mathbb{Q}(\tau)$ if Λ has complex multiplication and $K = \mathbb{Q}$ otherwise). Let $x_1, \cdots,$ $x_n \in \mathbb{C}\backslash\Lambda$ such that they are linearly independent over K. Then

$$\text{tr deg } \mathbb{Q}\bigg(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, x_1, \cdots, x_n, \wp(x_1), \cdots, \wp(x_n), \zeta^*(x_1), \cdots, \zeta^*(x_n)\bigg) \geq 2n + \frac{4}{[K : \mathbb{Q}]}.$$

Interestingly, there is another conjecture which is proposed as an elliptic analog of the Schanuel conjecture by M. R. Murty and V. K. Murty in [63]. Although a special case of the general Grothendieck conjecture, this conjecture provides a different avenue for exploration.

Conjecture 7.2.14 (Murty-Murty)**.** *Let $x_1, x_2, \cdots, x_n$ be $\overline{\mathbb{Q}}$-linearly independent numbers. Let $\wp_2, \cdots, \wp_n$ be Weierstrass $\wp$ functions associated to non-isogenous CM elliptic curves $E_2, \cdots, E_n$ defined over $\overline{\mathbb{Q}}$. If $x_2, \cdots, x_n$ are not contained in the poles of $\wp_i$, $2 \leq i \leq n$, then the transcendence degree of the field*

$$\mathbb{Q}\bigg(x_1, \cdots, x_n, e^{x_1}, \wp_2(x_2), \cdots, \wp_n(x_n)\bigg)$$

is at least n.

7.3 Evaluation of $\mathcal{S}(A, B)$

In this section, we establish the connection between sums of the form (7.1) and Weierstrass functions. Before we proceed, we prove the following lemma which will be crucial towards our goal.

Lemma 7.3.1. *Let $A(X)$, $B(X) \in \mathbb{C}[X]$ be co-prime polynomials with* $\deg A \leq$

$\deg B - 3$. *Let the distinct roots of* $B(X)$ *be* $\alpha_1, \cdots, \alpha_r$ *with multiplicities* $\mu_1, \cdots, \mu_r$. *Suppose that the partial fraction decomposition of* $A(X)/B(X)$ *is*

$$\frac{A(X)}{B(X)} = \sum_{i=1}^{r} \sum_{j=1}^{\mu_r} \frac{\lambda_{i,j}}{(X-\alpha_i)^j}.$$

Let $\mathcal{M} := \max_{1\le i\le r} \mu_i$ *and define* $\lambda_{i,j} = 0$ *for* $\mu_i < j \le \mathcal{M}$, $1 \le i \le r$. *Then*

$$(a) \sum_{i=1}^{r} \lambda_{i,1} = 0, \text{ and } (b) \sum_{i=1}^{r} (\lambda_{i,2} + \lambda_{i,1}\,\alpha_i) = 0.$$

Proof. From the partial fraction decomposition, we obtain

$$A(X) = \sum_{i=1}^{r} \sum_{j=1}^{\mu_r} \lambda_{i,j} \frac{B(X)}{(X-\alpha_i)^j}. \tag{7.13}$$

Since $j \ge 1$, $\deg(B(X)/(X-\alpha_j)^j) \le \deg B(X) - 1$. However, $\deg A(X) \le \deg B(X) - 3$. Hence, the coefficients of $X^{\deg B - 1}$ and $X^{\deg B - 2}$ must be zero on the right hand side. Since $X^{\deg B - 1}$ is the highest degree term in (7.13), the contribution to this term is solely from the polynomials $B(X)/(X-\alpha_i)$ for $1 \le i \le r$. This implies part (a) of the lemma.

Similarly, towards part (b), the coefficient of $X^{\deg B - 2}$ is comprised of the coefficient of the highest degree term in the polynomials $B(X)/(X-\alpha_i)^2$ for $1 \le i \le r$ as well as the coefficient of the second highest degree term in the polynomials $B(X)/(X-\alpha_i)$. Since the coefficient of the second highest degree term in a polynomial is given by the negative of the sum of its roots, the coefficient of $X^{\deg B - 2}$ in

$B(X)/(X-\alpha_{i_0})$ equals

$$-\Big[\sum_{i=i}^{r}(\alpha_i\mu_i)-\alpha_{i_0}\Big].$$

Let $S := \sum_{i=1}^{r}\alpha_i\mu_i$. Therefore, the coefficient of $X^{\deg B-2}$ on the right hand side of (7.13) becomes

$$\sum_{i=1}^{r}\lambda_{i,2}+\sum_{i=1}^{r}\lambda_{i,1}\big(\alpha_i-S\big)=\sum_{i=1}^{r}\lambda_{i,2}+\sum_{i=1}^{r}\lambda_i\alpha_i-S\sum_{i=1}^{r}\lambda_{i,1}.$$

By part (a) of this lemma, the last term in the above expression vanishes and part(b) is proved. □

We now prove the main result connecting sum of rational function over lattices to Weierstrass functions.

Theorem 7.3.2. *Let Λ be a two-dimensional lattice in $\mathbb{C}$. Let $A(X)$, $B(X)\in\mathbb{C}[X]$ be as in Lemma 7.3.1. Further suppose that none of the roots of $B(X)$ lie in Λ. Then*

$$\mathcal{S}(A,B)=-\sum_{i=1}^{r}\lambda_{i,1}\,\zeta^*(\alpha_i)+\sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}}\frac{\lambda_{i,j}}{(j-1)!}\,\wp^{(j-2)}(\alpha_i),$$

where $\zeta^*(z)$ is the Weierstrass zeta-function and $\wp(z)$ is the Weierstrass $\wp$ function.

Proof. Suppose $\Lambda=\omega_1\,\mathbb{Z}\oplus\omega_2\,\mathbb{Z}$. Let $\sum'$ mean that the term $(m,n)=(0,0)$ is omitted from the summation below.

$$\sum_{\substack{m,n\in\mathbb{Z},\\|m|\le M,\\|n|\le N}}{}' \frac{A(m\,\omega_1+n\,\omega_2)}{B(m\,\omega_1+n\,\omega_2)}=\sum_{i=1}^{r}\sum_{j=1}^{\mathcal{M}}\sum_{\substack{|m|\le M,\\|n|\le N}}{}' \frac{\lambda_{i,j}}{((m\,\omega_1+n\,\omega_2)-\alpha_i)^j}.$$

The only terms for which convergence needs to be checked are $j = 1$, 2 as

$$\sum_{\omega \in \Lambda^*} 1/|\omega|^{2+\epsilon} < \infty,$$

for any $\epsilon > 0$. For $j = 1$, we add the necessary factors for convergence and subtract the extra terms. This gives

$$\begin{aligned}
&\sum_{i=1}^{r} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{\lambda_{i,1}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)} \\
&= \left(\sum_{i=1}^{r} \lambda_{i,1}\right) \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)} + \left(\sum_{i=1}^{r} \lambda_{i,1}\alpha_i\right) \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \\
&- \sum_{i=1}^{r} \lambda_{i,1} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \left\{ \frac{1}{(\alpha_i - (m\,\omega_1 + n\,\omega_2))} + \frac{1}{(m\,\omega_1 + n\,\omega_2)} + \frac{\alpha_i}{(m\,\omega_1 + n\,\omega_2)^2} \right\}.
\end{aligned}$$

The above expression can be further simplified using Lemma 7.3.1. The term corresponding to

$$\sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)}$$

cancels and one can replace $\sum_{i=1}^{r} \lambda_{i,1}\alpha_i = -\sum_{i=1}^{r} \lambda_{i,2}$. Similar computations for $j = 2$

give

$$\sum_{i=1}^{r} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{\lambda_{i,2}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^2}$$
$$= \sum_{i=1}^{r} \lambda_{i,2} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \left\{ \frac{1}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^2} - \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \right\}$$
$$+ \left(\sum_{i=1}^{r} \lambda_{i,2} \right) \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}.$$

The coefficients corresponding to the sum

$$\sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}$$

in the terms $j = 1$ and $j = 2$ cancel. Thus, we obtain

$$\sum_{i=1}^{r} \sum_{j=1}^{\mathcal{M}} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{\lambda_{i,j}}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^j}$$
$$= \left(\sum_{i=1}^{r} \lambda_{i,1} \right) \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)} + \left(\sum_{i=1}^{r} \lambda_{i,1}\alpha_i \right) \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{(m\,\omega_1 + n\,\omega_2)^2}$$
$$+ \sum_{i=1}^{r} \lambda_{i,2} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \left\{ \frac{1}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^2} - \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \right\}$$
$$+ \sum_{i=1}^{r} \sum_{j=3}^{\mathcal{M}} \lambda_{i,j} \sideset{}{'}\sum_{\substack{|m|\le M,\\ |n|\le N}} \frac{1}{((m\,\omega_1 + n\,\omega_2) - \alpha_i)^j}.$$

Hence, the theorem is proved by taking the limits as $M,\ N \to \infty$ and the term corresponding to $m = n = 0$ is added to both sides. □

Remark. *This theorem shows that $\mathcal{S}(A, B)$ can be expressed as a linear combination of the Weierstrass zeta-function and its derivatives. This is analogous to the evaluation of sums of the form $\sum_{n\in\mathbb{Z}}^{\infty} A(n)/B(n)$ in terms of the cotangent function and its derivatives. In this sense, the Weierstrass zeta-function $\zeta^*(z)$ can be thought of as an elliptic analogue of the cotangent function, $\pi\cot(\pi z)$.*

7.4 Roots of $B(X)$: Rational multiples of periods

In the next two sections, we will concentrate on determining the arithmetic nature of the sum $\mathcal{S}(A, B)$. The values of $\cot z$ when z is a rational multiple of the period π are well understood. Similarly, our earlier discussion throws light on the values of the Weierstrass functions at rational multiples of non-zero periods in the lattice. Now, we put the analysis of Section 7.2.1 into context by investigating the nature of $\mathcal{S}(A, B)$ when the roots of $B(X)$ are non-integral rational multiples of non-zero periods. We prove

Theorem 7.4.1. *Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ with $\Im(w_1/w_2) > 0$ and $g_2(\Lambda)$, $g_3(\Lambda) \in \overline{\mathbb{Q}}$. Let $A(X) \in \overline{\mathbb{Q}}[X]$ and $B(X) \in \mathbb{C}[X]$ be as in Theorem 7.3.2. Fix a non-zero period in Λ, say ω_0. Assume that*

$$\alpha_i = \frac{a_i}{b_i}\omega_0 \;\notin \Lambda,$$

where a_i, $b_i \in \mathbb{Z}$, $a_i \neq 0$, $b_i > 1$ and $\gcd(a_i, b_i) = 1$.

a) If all roots of $B(X)$ are simple, then $\mathcal{S}(A, B)$ is either zero or transcendental.

b) If $B(X)$ has at least one repeated root and $\sum_{i=1}^{r}\lambda_{i,1}\,\alpha_i \neq 0$, then $\mathcal{S}(A, B)$ is either zero or transcendental provided that Λ has complex multiplication.

Proof. Recall that by Theorem 7.3.2, we have

$$\mathcal{S}(A,B) = -\sum_{i=1}^{r} \lambda_{i,1}\,\zeta^*(\alpha_i) + \sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\,\wp^{(j-2)}(\alpha_i).$$

If $\alpha_i = (a_i/b_i)\,\omega_1$ as in the hypothesis of the above theorem, then as seen in Section 7.2.1,

$$\wp^{(j-2)}\left(\frac{a_i}{b_i}\,\omega_0\right) \in \overline{\mathbb{Q}}, \ \text{ for all } 1 \le i \le r,\ 2 \le j \le \mathcal{M}.$$

Moreover, Lemma 7.2.3 implies that

$$\zeta^*\left(\frac{a_i}{b_i}\,\omega_0\right) = \frac{a_i}{b_i}\left(2\,\zeta^*\left(\frac{\omega_0}{2}\right)\right) + \epsilon(a_i,\, b_i),$$

for some algebraic number $\epsilon(a_i,\, b_i)$. Therefore, we get that

$$\begin{aligned}\mathcal{S}(A,B) = -\left(\sum_{i=1}^{r} \lambda_{i,1}\,\frac{a_i}{b_i}\right)\left(2\,\zeta^*\left(\frac{\omega_0}{2}\right)\right) + \sum_{i=1}^{r} \lambda_{i,1}\,\epsilon(a_i,\, b_i) \\ + \sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\,\wp^{(j-2)}(\alpha_i).\end{aligned} \tag{7.14}$$

Now suppose that $B(X)$ has simple roots. Then part (b) of Lemma 7.3.1 implies that

$$\sum_{i=1}^{r} \lambda_{i,1}\,\frac{a_i}{b_i} = 0,$$

and there are no terms involving the Weierstrass $\wp$ function and its derivatives. Therefore, (7.14) reduces to

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} \lambda_{i,1}\,\epsilon(a_i, b_i). \tag{7.15}$$

Thus, the algebraicity of the values of $\wp(z)$ and its derivatives and the nature of

$\lambda_{i,j}$ enable us to express $\mathcal{S}(A, B)$ as a rational function with algebraic coefficients evaluated at ω_0. More specifically, we know that for $i \leq i_0 \leq r$,

$$\lambda_{i_0,1} = \frac{A(\alpha_{i_0})}{B'(\alpha_{i_0})} = \frac{A(\alpha_{i_0})}{\prod_{\substack{1\leq i\leq r,\\ i\neq i_0}}(\alpha_{i_0} - \alpha_i)}.$$

Since $\deg(A) \leq \deg(B) - 3$ and $\alpha_i = (a_i/b_i)\,\omega_1$,

$$\lambda_{i_0,1}\,\epsilon(a_{i_0},\, b_{i_0}) = \frac{F_{i_0}(\omega_0)}{\omega_0^{r-1}},$$

where $F_{i_0}(X) \in \overline{\mathbb{Q}}[X]$ of degree less than or equal to $\deg B - 3 = r - 3$. If $\mathcal{S}(A, B)$ is a non-zero algebraic number, then clearing the denominators in

$$\frac{\left(\sum_{i=1}^{r} F_i(\omega_0)\right)}{\omega_0^{r-1}}$$

gives a polynomial relation of ω_0 over $\overline{\mathbb{Q}}$. This polynomial is non-trivial by degree considerations. This implies that ω_0 is algebraic and contradicts Schneider's Theorem 7.2.6. Thus, $\mathcal{S}(A, B)$ is either zero or transcendental. This proves part a).

Now assume that $B(X)$ has multiple roots. By the hypothesis in part (b), the coefficient of $\eta_0 = 2\,\zeta^*(\omega_0/2)$ in (7.14) is not zero. Hence, the right hand side of (7.14) is a non-trivial polynomial in η_0 over $\overline{\mathbb{Q}}(\omega_0)$. However, Theorem 7.2.8 proves the algebraic independence of ω_0 and η_0, provided that Λ has complex multiplication. Therefore, $\mathcal{S}(A, B)$ cannot be a non-zero algebraic number, and is transcendental. □

Remark. *It is possible that the sum $\mathcal{S}(A, B) = 0$ in certain cases. For example, let*

$A(X) = 1$ *and*

$$B(X) = \left(X - \frac{m_1}{2}\,\omega_1\right)\left(X - \frac{m_2}{2}\,\omega_1\right)\left(X - \frac{m_3}{2}\,\omega_1\right),$$

where m_1*,* m_2 *and* m_3 *are distinct odd positive integers. Then the linearity of the* η*-function implies that*

$$\zeta\left(\frac{m_j}{2}\,\omega_1\right) = m_j\,\zeta\left(\frac{\omega_1}{2}\right), \quad j = 1,\, 2,\, 3.$$

Therefore, $\epsilon(m_j, 2) = 0$ *for all* $j = 1,\ 2$ *and* 3*. Thus, equation* (7.15) *implies that* $\mathcal{S}(A, B) = 0$.

Remark. *If* $B(X)$ *has multiple roots and the coefficient of* η_0 *is zero, then* (7.14) *gives*

$$\mathcal{S}(A, B) = \sum_{i=1}^{r}\sum_{j=1}^{\mu_i} \mathcal{A}_{i,j}\,\lambda_{i,j},$$

for some algebraic numbers $\mathcal{A}_{i,j}$*. Since*

$$\lambda_{i,j} = \frac{1}{(\mu_i - j)!}\left[\frac{d^{(\mu_i - j)}}{dX^{(\mu_i - j)}}\left\{(X - \alpha_i)^{\mu_i}\,\frac{A(X)}{B(X)}\right\}\right]\Bigg|_{X=\alpha_i},$$

it follows that $\lambda_{i,j}$ *are rational functions in* ω_0*. If we show that* $\sum_{i=1}^{r}\sum_{j=1}^{\mu_i} \mathcal{A}_{i,j}\,\lambda_{i,j}$ *is a non-trivial rational function, then it follows that* $\mathcal{S}(A, B)$ *is either zero or transcendental. However, this is more subtle and does not follow from mere degree considerations.*

7.5 Roots of $B(X)$: Algebraic numbers

Inspired by the conjectures and theorems regarding the transcendental nature of values of the Weierstrass functions at algebraic arguments (Section 7.2.2), we study the nature of $\mathcal{S}(A,B)$ when $B(X)$ has algebraic roots. In particular, we prove the following.

Theorem 7.5.1. *Let Λ be a lattice with $g_2(\Lambda)$, $g_3(\Lambda) \in \overline{\mathbb{Q}}$. Let $A(X)$, $B(X) \in \overline{\mathbb{Q}}[X]$ be co-prime polynomials as in Theorem 7.3.2. Suppose the roots of $B(X)$, namely, $\alpha_1, \alpha_2, \cdots, \alpha_r$ are $\mathbb{Q}$-linearly independent. Then we have the following.*

a) If $\lambda_{i,1} \neq 0$ for at least one i, $1 \leq i \leq r$, then Conjecture 7.2.13 implies that $\mathcal{S}(A,B)$ is transcendental.

b) If $\lambda_{i,1} = 0$ for all $1 \leq i \leq r$ and Λ does not have complex multiplication, then $\mathcal{S}(A,B)$ is transcendental conditional on Conjecture 7.2.10.

c) If $\lambda_{i,1} = 0$ for all $1 \leq i \leq r$ and Λ has complex multiplication, then $\mathcal{S}(A,B)$ is transcendental.

Proof. We recall that by Theorem 7.3.2,

$$\mathcal{S}(A,B) = -\sum_{i=1}^{r} \lambda_{i,1}\, \zeta^*(\alpha_i) + \sum_{i=1}^{r}\sum_{j=2}^{\mathcal{M}} \frac{\lambda_{i,j}}{(j-1)!}\, \wp^{(j-1)}(\alpha_i).$$

Since the roots of $A(X)$, $B(X) \in \overline{\mathbb{Q}}[X]$, the partial fraction coefficients $\lambda_{i,j} \in \overline{\mathbb{Q}}$. Therefore, $\mathcal{S}(A,B)$ is an algebraic linear combination of $\zeta^*(\alpha_i)$, $\wp(\alpha_i)$ and special values of the derivatives of $\wp(z)$ at α_i. As seen in Lemma 7.2.1, the derivatives of $\wp(z)$ can be expressed as polynomials in $\wp$ an $\wp'$ with algebraic coefficients. Thus,

$\mathcal{S}(A,B)$ is, in fact, a polynomial with algebraic coefficients evaluated at $\zeta^*(\alpha_i)$, $\wp(\alpha_i)$ and $\wp'(\alpha_i)$ for $1 \le i \le r$. Suppose $\lambda_{i_0,1} \neq 0$ for some i_0, $1 \le i_0 \le r$, then $\mathcal{S}(A,B)$ is a non-trivial polynomial in $\zeta^*(\alpha_{i_0})$ with coefficients in

$$\overline{\mathbb{Q}\Big(\{\wp(\alpha_i)\,:\,1 \le i \le r\} \Big)} \cup \overline{\mathbb{Q}}\Big(\{\zeta^*(\alpha_i)\,:\,1 \le i \le r,\, i \neq i_0\} \Big).$$

If $\mathcal{S}(A,B)$ is algebraic, then one would obtain that $\zeta^*(\alpha_{i_0})$ is algebraic over the above field, implying that the transcendence degree of

$$\mathbb{Q}\Big(\omega_1, \omega_2, \eta_1, \eta_2, \wp(\alpha_1), \cdots, \wp(\alpha_r), \zeta^*(\alpha_1), \cdots, \zeta^*(\alpha_r)\Big)$$

is less than $2r+4$ if Λ does not have CM, and the transcendence degree of

$$\mathbb{Q}\Big(\omega_1, \eta_1, \wp(\alpha_1), \cdots, \wp(\alpha_r), \zeta^*(\alpha_1), \cdots, \zeta^*(\alpha_r)\Big)$$

is less than $2r+2$ if Λ has CM, by Lemma 7.2.7. This contradicts Conjecture 7.2.13, thus proving part (a).

Now suppose that $\lambda_{i,1} = 0$ for all $1 \le i \le r$. Thus,

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} \sum_{j=2}^{\mathcal{M}} \widetilde{\lambda}_{i,j}\, \wp^{(j-2)}(\alpha_i),$$

where $\widetilde{\lambda}_{i,j} := \lambda_{i,j}/(j-1)! \in \overline{\mathbb{Q}}$. Let

$$\mathcal{M}_e := \begin{cases} \frac{\mathcal{M}-1}{2}, & \text{if } \mathcal{M} \text{ is odd,} \\ \frac{\mathcal{M}}{2}, & \text{if } \mathcal{M} \text{ is even,} \end{cases}$$

and

$$\mathcal{M}_o := \begin{cases} \frac{\mathcal{M}-1}{2}, & \text{if } \mathcal{M} \text{ is odd,} \\ \frac{\mathcal{M}-2}{2}, & \text{if } \mathcal{M} \text{ is even.} \end{cases}$$

Thus, separating the terms corresponding to j-even and j-odd in the expression for $\mathcal{S}(A, B)$ gives

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} \sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, \wp^{(2u-1)}(\alpha_i) + \sum_{i=1}^{r} \sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, \wp^{(2v-2)}(\alpha_i).$$

Now using Lemma 7.2.1, we express the derivatives of the Weierstrass $\wp$ function as polynomials in $\wp$ and $\wp'$ to get

$$\begin{aligned} \mathcal{S}(A,B) &= \sum_{i=1}^{r} \left\{ \left[\sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, F_{2u-1}\Big(\wp(\alpha_i)\Big) \right] \wp'(\alpha_i) \right\} + \sum_{i=1}^{r} \sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i)\Big) \\ &= \sum_{i=1}^{r} \Big(\mathcal{C}_i\, \wp'(\alpha_i) \Big) + \mathcal{D}, \end{aligned}$$

where $\mathcal{C}_i$, $\mathcal{D} \in \overline{\mathbb{Q}}(\wp(\alpha_1), \cdots, \wp(\alpha_r))$ are defined as

$$\mathcal{C}_i = \sum_{u=1}^{\mathcal{M}_o} \widetilde{\lambda}_{i,2u+1}\, F_{2u-1}\Big(\wp(\alpha_i)\Big), \quad \mathcal{D} = \sum_{i=1}^{r} \sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i)\Big).$$

Suppose that $\mathcal{S}(A, B)$ is algebraic and that $\mathcal{C}_{i_0} \neq 0$ for some i_0 with $1 \leq i_0 \leq r$.

This implies that $\wp'(\alpha_i)$ satisfies a linear equation over the field

$$\overline{\mathbb{Q}}\bigg(\{\wp(\alpha_i) \,:\, 1 \leq i \leq r\} \bigg) \cup \overline{\mathbb{Q}}\bigg(\{\wp'(\alpha_i) \,:\, 1 \leq i \leq r,\, i \neq i_0\} \bigg).$$

By Conjecture 7.2.10, if Λ does not have CM or by Theorem 7.2.11 if Λ has CM, we know that the numbers $\wp(\alpha_i)$ are algebraically independent for $1 \leq i \leq r$. Thus, the above conclusion implies that $\wp'(\alpha_{i_0})$ satisfies a non-trivial linear relation over the field $\overline{\mathbb{Q}}(\wp(\alpha_{i_0}))$. However, we know by (7.3) and Theorem 7.2.4 that $\wp'(\alpha_{i_0})$ is quadratic over $\overline{\mathbb{Q}}(\wp(\alpha_{i_0}))$. Hence, the theorem is proved in this case.

Now assume that $\mathcal{C}_i = 0$ for all $1 \leq i \leq r$. We will show that this can only happen if the polynomial $A(X)$ is the identically zero polynomial to begin with. Let

$$G_{2l}(X) := \sum_{m=0}^{l+1} g_l(m)\, X^m,$$

where $G_{2l}(X)$ is the polynomial that we encountered in Lemma 7.2.1. Therefore, we have

$$\begin{aligned}
\mathcal{S}(A,B) &= \sum_{i=1}^{r} \sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v}\, G_{2v-2}\Big(\wp(\alpha_i) \Big) \\
&= \sum_{i=1}^{r} \sum_{v=1}^{\mathcal{M}_e} \widetilde{\lambda}_{i,2v} \sum_{m=0}^{v} g_{v-1}(m)\, \big(\wp(\alpha_i)\big)^m \\
&= \sum_{i=1}^{r} \sum_{m=0}^{\mathcal{M}_e} \delta_{m,i}\, \big(\wp(\alpha_i)\big)^m,
\end{aligned}$$

where $\delta_{m,i} = \sum_{v=m}^{\mathcal{M}_e} g_{v-1}(m)\, \widetilde{\lambda}_{i,2v}$ and the last step is obtained by interchanging the order of summation. Therefore, if

$$P_i(X) := \sum_{m=0}^{\mathcal{M}_e} \delta_{m,i}\, X^m \ \in \overline{\mathbb{Q}}[X],$$

then we get

$$\mathcal{S}(A,B) = \sum_{i=1}^{r} P_i\Big(\wp(\alpha_i)\Big).$$

If $P_i(X)$ is a non-constant polynomial for even a single i, $1 \le i \le r$, then by Conjecture 7.2.10 in the non-CM case and by Theorem 7.2.11 in the CM case, we deduce that $\mathcal{S}(A,B)$ must be transcendental.

Thus, we can assume that $\deg P_i(X) = 0$ for all $1 \le i \le r$. Therefore, the coefficients $\delta_{m,i} = 0$ for $1 \le i \le r$, $1 \le m \le \mathcal{M}_e$, that is,

$$\sum_{v=0}^{\mathcal{M}_e} g_{v-1}(m)\, \widetilde{\lambda}_{i,2v} = 0, \tag{7.16}$$

since Lemma 7.2.1 proves that $g_{v-1}(m) = 0$ for $m > v$. The above relation can be interpreted as a matrix equation as follows. Let $\mathcal{G}$ be the $\mathcal{M}_e \times \mathcal{M}_e$ matrix whose (m,v)-th entry is $g_{v-1}(m)$ for $1 \le v,\ m \le \mathcal{M}_e$. Let $\mathcal{N}$ be the $\mathcal{M}_e \times r$ matrix whose (v,i)-th entry is $\widetilde{\lambda}_{i,2v}$, for $1 \le i \le r$, $1 \le v \le \mathcal{M}_e$. Hence, equation (7.16) is equivalent to the matrix identity

$$\mathcal{G}\,\mathcal{N} = \Big[0\Big],$$

where the right hand side is the 0-matrix. Observe that $\mathcal{G}(m,v) = 0$ for all $m > v$, i.e., the matrix $\mathcal{G}$ is an upper triangular matrix. Thus,

$$\det(\mathcal{G}) = \prod_{m=1}^{\mathcal{M}_e} g_{m-1}(m).$$

By Lemma 7.2.1, $\deg G_{2m-2}(X) = m$, which implies that $g_{m-1}(m) \neq 0$ for all $m \in \mathbb{N}$. Hence, $\mathcal{G}$ is an invertible matrix and $\mathcal{N}$ is the zero matrix. This concludes the proof of parts (b) and (c). $\square$

Owing to the involved nature of the addition formula of $\zeta^*(z)$ and $\wp(z)$, it is not a priori clear if $\mathcal{S}(A, B)$ is transcendental or even non-zero, in the case when $\alpha_1, \cdots,$ α_r are not $\mathbb{Q}$-linearly independent. We relegate this to future study.

Chapter 8

Avenues for future research

In the thesis so far, we have made progress towards understanding the nature of special values of L-series and other infinite series in various scenarios. However, many more paths remain unexplored. We record a few ideas in this chapter, that we came across during the course of our study. This will help organize our thought in order to further our understanding and chart a course for further research.

8.1 Consequences of a result of S. Fischler

A major advancement in the field of arithmetic nature of special values, was the breakthrough result of Apéry [1] in 1978, proving the irrationality of $\zeta(3)$. Further progress was achieved by T. Rivoal [83] and K. Ball and T. Rivoal [7], when they showed the following:

Theorem 8.1.1 (K. Ball, T. Rivoal)**.** *Given an $\epsilon > 0$, there exists an integer $N(\epsilon) > 0$ such that for all $n \geq N(\epsilon)$,*

$$\dim_{\mathbb{Q}} \, \mathbb{Q}\Big(\zeta(3), \; \zeta(5), \; \zeta(7), \; \cdots, \zeta(2n+1)\Big) \geq \frac{(1-\epsilon)\, \log n}{1 + \log 2},$$

as n tends to infinity.

As a corollary of the above theorem, they deduce that infinitely many of the values $\zeta(2k+1)$ for an integer $k \geq 1$ are irrational. These results were further refined by W. Zudilin [104] and he proved that

Theorem (W. Zudilin)**.** *At least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.*

The analogue of the Ball-Rivoal theorem for special values of Dirichlet L-functions was first investigated by M. Nishimoto [73]. An alternate proof for Nishimoto's theorem was provided by S. Fischler [33] in 2018. In particular, Fischler proved the following.

Theorem 8.1.2 (S. Fischler)**.** *Let $N \geq 1$ and $f : \mathbb{N} \to \mathbb{C}$ such that $f(n+N) = f(n)$ for all $n \in \mathbb{N}$. Let $p \in \{0, 1\}$, $a \geq 2$, $z_0 \in \{1, e^{i\pi/N}\}$; put*

$$\xi_j := \sum_{n=1}^{\infty} \frac{f(n)\, z_0^n}{n^j}, \quad \textit{for any } j \in \{1,\, 2, \cdots, a\},$$

except that $\xi_1 := 0$ if $z_0 = 1$. Then as $a \to \infty$,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\left(\left\{\xi_j \,:\, 1 \leq j \leq a,\, j \equiv p \bmod 2\right\}\right) \geq \frac{(1+o(1))}{N + \log 2} \log a.$$

As a corollary, Fischler proves that infinitely many of the values $L(k, f)$, for a periodic function f and k varying over the positive integers with a fixed parity, are irrational. Thus, taking f to be a Dirichlet character modulo N and $z_0 = 1$, one obtains the analogue of the Ball-Rivoal theorem for Dirichlet L-functions. Since $L(k, \chi)$ when k and χ have the same parity is an algebraic multiple of π^k (see Section 2.3), the above theorem is more interesting when k and χ have opposite parity.

In this section, we derive an interesting implication of Fischler's theorem on the Chowla-Milnor conjecture, which we explain below. In 1982, inspired by S. Chowla's earlier work, S. Chowla and his daughter, P. Chowla put forth the following conjecture.

Conjecture (Chowla-Chowla). *For an odd prime $p > 3$, the $\lfloor (p-1)/2 \rfloor$ numbers*

$$\zeta\left(2, \frac{1}{p}\right), \ \zeta\left(2, \frac{2}{p}\right), \cdots, \zeta\left(2, \frac{p-1}{2p}\right)$$

are linearly independent over $\mathbb{Q}$.

This motivated Milnor [59] to assert a more general statement, namely,

Conjecture 8.1.3 (Chowla-Milnor). *Let $q > 3$ and $k \geq 2$ be positive integers. Then the numbers*

$$\left\{\zeta\left(k, \frac{a}{q}\right) \ : \ 1 \leq a < q, \ (a, q) = 1\right\}$$

are linearly independent over $\mathbb{Q}$.

From Chapter 2, it can be seen that the Chowla-Milnor conjecture can be reinterpreted as a statement about the non-vanishing of $L(k, f)$ for a rational-valued periodic function f of Dirichlet type. Following the method outlined by Baker, Birch and Wirsing in [5], S. Gun, M. R. Murty and P. Rath observed in [39] that the Chowla-Milnor conjecture would follow provided that we have an analog of Baker's theory for linear forms in *polylogarithms* evaluated at algebraic numbers. Recall that for $z \in \mathbb{C}$, $|z| \leq 1$ and any integer $k \geq 2$ or for $|z| < 1$ when $k = 1$, the k-th polylogarithm is defined as

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

In [39], the following conjecture was proposed as an analog of Baker's theorem (Theorem 3.0.3).

Conjecture 8.1.4 (Polylog Conjecture)**.** *Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be non-zero algebraic numbers with $|\alpha_i| \leq 1$ for $1 \leq i \leq r$. Assume that $\mathrm{Li}_k(\alpha_1), \cdots, \mathrm{Li}_k(\alpha_r)$ are $\mathbb{Q}$-linearly independent. Then $\mathrm{Li}_1(\alpha_1), \cdots, \mathrm{Li}_k(\alpha_r)$ are $\overline{\mathbb{Q}}$-linearly independent.*

Moroever, the authors proved that the Polylog conjecture implies the Chowla-Milnor conjecture in [39]. However, the Polylog conjecture seems out of reach at the current stage.

Remark. *Another approach towards the Chowla-Milnor conjecture is suggested by the expression* (2.24),

$$L(k, f) = \sum_{b=1}^{q} \widehat{f}(b) \, \mathrm{Li}_k(\zeta_q^b).$$

Thus, any information regarding the linear independence of values of $\mathrm{Li}_k(z)$ at roots of unity would be helpful. Indeed, such a conjecture (see [82, Conjecture 7.1.2, Pg. 261]) exists in the literature, albeit in cohomological language. However, in 1989, D. Ramakrishnan [82, Proposition 7.2.5, pg. 263] proved that this conjecture has an equivalent formulation in terms of values of the polylogarithm. More specifically, the statement of the conjecture that is relevant to us is as follows.

Conjecture 8.1.5. *Let $m, n \geq 1$ be integers and $\mathrm{Li}_m(z)$ denote the m^{th} polylogarithm.*

Let

$$D_m(z) := \begin{cases} \Re(\mathrm{Li}_m(z)) & \text{if } m \text{ is odd,} \\ \mathrm{Im}(\mathrm{Li}_m(z)) & \text{if } m \text{ is even.} \end{cases}$$

Further let

$$J_m := \begin{cases} \{j \in \mathbb{Z} \,:\, 1 < j < n/2, (j,n) = 1\} & \text{if } m = 1, \\ \{j \in \mathbb{Z} \,:\, 0 < j < n/2, (j,n) = 1\} & \text{if } m > 1. \end{cases}$$

Then, the numbers

$$\left\{ D_m(\zeta_n^j) \,:\, j \in J_m \right\}$$

are linearly independent over $\mathbb{Q}$*.*

Although the above conjecture seems relevant, we cannot directly apply it to the Chowla-Milnor scenario. As noted before, the Chowla-Milnor conjecture would follow is we show that for a rational-valued periodic function f *of Dirichlet type,* $L(k, f) \neq 0$*. For such an* f*,* $\widehat{f}(b) \in \mathbb{Q}(\zeta_q)$ *for* $1 \leq b \leq q$ *and it is not necessary that* $\widehat{f}(b) = 0$ *if* $(b, q) > 1$*. Since* (2.24) *expresses* $L(k, f)$ *as a linear combination over* $\mathbb{Q}(\zeta_q)$ *(and not* $\mathbb{Q}$*) of values of the* k^{th}*-polylogarithm at roots of unity, Conjecture 8.1.5 is not sufficient to conclude the Chowla-Milnor conjecture. We refer the reader to [82] for further details regarding Conjecture 8.1.5.*

For positive integers $k \geq 2$ and $q > 3$, define $V_{k,q}$ to be the $\mathbb{Q}$-vector space generated by

$$\left\{ \zeta\left(k, \frac{a}{q}\right) \,:\, 1 \leq a < q,\, (a, q) = 1 \right\}.$$

Then the Chowla-Milnor conjecture is equivalent to the statement that

$$\dim_{\mathbb{Q}} V_{k,q} = \phi(q),$$

where ϕ denotes Euler's totient function.

Motivated by (2.21) and (2.23), we define the following two subspaces of $V_{k,q}$,

$$V_{k,q}^{+} := \mathrm{Span}_{\mathbb{Q}}\left(\left\{\zeta\left(k, \frac{a}{q}\right) + (-1)^k \zeta\left(k, 1 - \frac{a}{q}\right) \,:\, 1 \le a < q/2,\ (a,q) = 1\right\}\right),$$
$$V_{k,q}^{-} := \mathrm{Span}_{\mathbb{Q}}\left(\left\{\zeta\left(k, \frac{a}{q}\right) - (-1)^k \zeta\left(k, 1 - \frac{a}{q}\right) \,:\, 1 \le a < q/2,\ (a,q) = 1\right\}\right).$$

It is easy to see that $V_{k,q} = V_{k,q}^{+} + V_{k,q}^{-}$. Using a result of T. Okada [74] on the $\mathbb{Q}$-linear independence of values of the cotangent and the fact that

$$\zeta\left(k, \frac{a}{q}\right) + (-1)^k \zeta\left(k, 1 - \frac{a}{q}\right) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \big(\pi \cot \pi z\big)\Big|_{z=a/q},$$

(see Section 2.3 for more details), S. Gun, M. R. Murty and P. Rath [39] proved that

$$\dim_{\mathbb{Q}} V_{k,q}^{+} = \phi(q)/2.$$

Thus, the Chowla-Milnor conjecture implies that

$$\dim_{\mathbb{Q}} V_{k,q}^{-} = \phi(q)/2.$$

However, nothing is known about these complementary spaces $V_{k,q}^{-}$. In fact, even the weaker statement of $\dim_{\mathbb{Q}} V_{k,q}^{-} > 1$ remains unproven.

For any $x > 0$ and $\mathcal{P} \in \{0, 1\}$, let

$$V_{\mathcal{P},q}(x) := \operatorname{Span}_{\mathbb{Q}}\left(\left\{\zeta\left(k, \frac{a}{q}\right) \,:\, 1 \leq a < q,\, (a,q) = 1,\, 2 \leq k \leq x,\, k \equiv \mathcal{P} \bmod 2\right\}\right).$$

In the light of the Ball-Rivoal theorem, one may ask if for a fixed positive integer $q > 3$, then does there exists a non-zero lower bound for $\dim_{\mathbb{Q}} V_{\mathcal{P},q}(x)$ as $x \to \infty$? An answer to this question lies in the observation that for $\Re(s) > 1$,

$$\frac{1}{q^s} \sum_{\substack{a=1,\\(a,q)=1}}^{q} \zeta\left(s, \frac{a}{q}\right) = \sum_{\substack{n=1,\\(n,q)=1}}^{\infty} \frac{1}{n^s} = \left(\prod_{p|q}(1 - p^{-s})\right) \zeta(s).$$

Hence, for any integer $k \geq 2$,

$$\sum_{\substack{a=1,\\(a,q)=1}}^{q} \zeta\left(k, \frac{a}{q}\right) = \left(q^k \prod_{p|q}(1 - p^{-k})\right) \zeta(k),$$

i.e., $\zeta(k)$ is a rational multiple of the sum of $\zeta(k, a/q)$ for $1 \leq a < q$, $(a,q) = 1$. Thus,

$$\dim_{\mathbb{Q}} V_{\mathcal{P},q}(x) \geq \dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\left(\left\{\zeta(k) \,:\, 2 \leq k \leq x,\, k \equiv \mathcal{P} \bmod 2\right\}\right).$$

If $\mathcal{P} = 0$, then $\zeta(2n)$ is a rational multiple of π^{2n} by Euler's theorem and by Lindemann's theorem on transcendence of π, the dimension of the space on the right hand side is $[x] - 1$. On the other hand, if $\mathcal{P} = 1$, then the lower bound for the right hand side is precisely Theorem 8.1.1. Thus, the number

$$\zeta\left(k, \frac{a}{q}\right)$$

is irrational for infinitely many $2 \leq k$ and $1 \leq a < q$, $(a, q) = 1$.

However, more can be said in this context from Theorem 8.1.2. Fix a positive integer $q > 3$ *and* a residue class $b \bmod q$ with $1 \leq b < q$ and $(b, q) = 1$. Define the function

$$f_b(n) := \begin{cases} 1 & \text{if } n \equiv b \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $z_0 = 1$. Thus, Theorem 8.1.2 implies that

Corollary 8.1.6.

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\left(\left\{\zeta\left(j, \frac{b}{q}\right) \,:\, 2 \leq j < a,\, j \equiv \mathcal{P} \bmod 2\right\}\right) \geq \frac{(1 + o(1))}{q + \log 2} \log a,$$

as $a \to \infty$.

Hence, given a positive integer $q > 3$ *and* a residue class b with $1 \leq b < q$, $(b, q) = 1$, there are infinitely many positive integers $k \geq 2$ such that $\zeta(k, b/q)$ is irrational.

Moreover, for $\dim_{\mathbb{Q}} V_{k,q}^{-}$, where it is not known if even one number in $V_{k,q}^{-}$ is irrational, Theorem 8.1.2 gives a non-trivial answer. Once again, fix a positive integer $q > 3$ and a co-prime residue class $b \bmod q$. Let $\mathcal{P} \in \{0, 1\}$ and $z_0 = 1$. Define the function

$$f_b^{-}(n) := \begin{cases} 1 & \text{if } n \equiv b \bmod q, \\ (-1)^{\mathcal{P}+1} & \text{if } n \equiv -b \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Then Theorem 8.1.2 implies that

Corollary 8.1.7. *For any* $\mathcal{P} \in \{0,1\}$,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\left(\left\{\zeta\left(j, \frac{b}{q}\right) - (-1)^j \zeta\left(j, 1-\frac{b}{q}\right) \; : \; 2 \le j < a,\; j \equiv \mathcal{P} \bmod 2\right\}\right) \ge \frac{(1+o(1))}{q+\log 2} \log a,$$

as $a \to \infty$.

In other words, there are infinitely many positive integers j with $j \ge 2$ and $j \equiv \mathcal{P} \bmod 2$ such that the number

$$\zeta\left(j, \frac{b}{q}\right) - (-1)^j \zeta\left(j, 1-\frac{b}{q}\right)$$

is irrational. Since the above number belongs to $V_{j,q}^-$, we deduce that there are infinitely many j such that

$$\dim_{\mathbb{Q}} V_{j,q}^- \ge 2.$$

This conclusion also follows from a theorem of S. Gun, M. R. Murty and P. Rath [39, Theorem 2]. However, the current approach also gives us a lower bound on the smallest j such that $\dim_{\mathbb{Q}} V_{j,q} > 1$. Indeed, we can choose a in Theorem 8.1.2 such that $\log a/(q + \log 2) > 1$. Thus, we get that

Corollary 8.1.8. *For a fixed* $\mathcal{P} = 0$ *or* 1, *at least one of*

$$\zeta\left(j, \frac{b}{q}\right) - (-1)^j \zeta\left(j, 1-\frac{b}{q}\right) \quad \textit{for } 2 \le j \le 2e^q,\; j \equiv \mathcal{P} \bmod 2,$$

is irrational.

Remark. *By* (2.21)*, one can translate the above conclusions to values of polygamma functions at rational arguments.*

8.2 Classifying $\mathbb{Q}$-linear relations among Galois conjugates

The first half of this thesis focused on the problem of non-vanishing of $L(1, f)$, for suitable periodic functions f. In particular, when f is odd, this question reduces to linear relations among the numbers

$$\cot\left(\frac{a\pi}{q}\right), \quad 1 \leq a < q,$$

due to (2.14). Moreover, if f is of Dirichlet type and rational valued, then one is led to study all $\mathbb{Q}$-linear relations among

$$\cot\left(\frac{a\pi}{q}\right), \quad 1 \leq a < q.$$

Observing that $\cot(a\pi/q) \in \mathbb{Q}(\zeta_q)$ and for $1 \leq a < q$ and $(a, q) = 1$,

$$\cot\left(\frac{a\pi}{q}\right) = \sigma_a\left(\cot\left(\frac{\pi}{q}\right)\right),$$

where $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$, with $\sigma_a(\zeta_q) = \zeta_q^a$. In this section, following a paper of K. Girstmair [34], we present this problem in a general algebraic framework.

Group Rings of finite abelian groups

Before discussing the problem at hand, we review some facts from the general study of group rings of finite abelian groups. An accessible survey of this theory can be found in [4].

Let G be a finite abelian group. The group ring $\mathbb{Q}[G]$ is the set of formal $\mathbb{Q}$-linear combinations,

$$\mathbb{Q}[G] = \left\{ \sum_{\sigma \in G} a_\sigma \, \sigma \ : \ a_\sigma \in \mathbb{Q} \right\},$$

together with the canonical addition, and multiplication given by

$$\left(\sum_{\sigma \in G} a_\sigma \, \sigma \right) \left(\sum_{\tau \in G} b_\tau \, \tau \right) = \sum_{\eta \in G} \left(\sum_{\substack{\sigma, \tau \in G, \\ \sigma\tau = \eta}} a_\sigma \, b_\tau \right) \eta.$$

This endows $\mathbb{Q}[G]$ with a ring structure. Furthermore,

Proposition. *The ring $\mathbb{Q}[G]$ can be written as a direct sum of cyclotomic fields.*

Proof. By the structure theorem for finite abelian groups, it suffices to show that the group ring of a cyclic group is a direct sum of cyclotomic fields. Suppose C_n denotes the cyclic group of order n. Then it can be easily seen that

$$\mathbb{Q}[C_n] \simeq \frac{\mathbb{Q}[X]}{(X^n - 1)}.$$

Now, we know the factorization

$$X^n - 1 = \prod_{d \mid n} \Phi_d(X),$$

where Φ_d denotes the d^{th}-cyclotomic polynomial, which is irreducible. Therefore,

$$\mathbb{Q}[C_n] \simeq \bigoplus_{d \mid n} \frac{\mathbb{Q}[X]}{(\Phi_d(X))} \simeq \bigoplus_{d \mid n} \mathbb{Q}(\zeta_d).$$

□

Since $\mathbb{Q}[G]$ is a direct sum of fields, we get that

Proposition 8.2.1. *Every ideal in* $\mathbb{Q}[G]$ *is principal.*

Additionally this implies that $\mathbb{Q}[G]$ has only finitely many ideals, and hence, is Artinian. Since $\mathbb{Q}[G]$ is a finite dimensional $\mathbb{Q}$-vector space, one can see that $\mathbb{Q}[G]$ is semi-simple. Every ideal of $\mathbb{Q}[G]$ is a $\mathbb{Q}$-vector subspace of $\mathbb{Q}[G]$. Therefore, every ideal of $\mathbb{Q}[G]$ is non-nilpotent. Thus, by [23, Theorem 24.2], we deduce that:

Proposition. *Every ideal in* $\mathbb{Q}[G]$ *is generated by an idempotent element. Moreover, if*

$$L := \mathbb{Q}[G]\, e, \;\; L' := \mathbb{Q}[G]\,(1-e),$$

for an idempotent e*, then*

$$\mathbb{Q}[G] = L \,\oplus\, L',$$

and L' *is said to be the complementary ideal of* L*.*

Given an ideal of $\mathbb{Q}[G]$, it would be helpful to have a method of obtaining its generator from the intrinsic structure of the ideal. This can be carried out as follows. Let $\widehat{G}$ denote the group of characters of G, i.e.,

$$\widehat{G} = \Big\{\chi : G \to \mathbb{C}^*,\, \chi \text{ is a group homomorphism}\Big\},$$

together with the group law given by multiplication. Let

$$K := \mathbb{Q}\Big(\big\{\chi(\sigma) \,:\, \chi \in \widehat{G},\, \sigma \in G\big\}\Big).$$

Then $K \subseteq \mathbb{Q}(\zeta_{|G|})$, which has an abelian Galois group, and hence, $K/\mathbb{Q}$ is Galois.

The group ring $K[G]$ will also play an important role in our analysis. Note that the Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ extends naturally to $K[G]$ by

$$\tau \cdot \left(\sum_{\sigma \in G} a_\sigma \, \sigma \right) := \sum_{\sigma \in G} \tau(a_\sigma) \, \sigma,$$

for $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ and $a_\sigma \in K$, for all $\sigma \in G$. There is also a natural action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\widehat{G}$ given by

$$\chi^\tau := (\tau \, \cdot \, \chi) = \tau \circ \chi,$$

for $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ and $\chi \in \widehat{G}$. We will say that a subset $X \subseteq \widehat{G}$ is *closed under conjugation* if for all $\chi \in X$ and $\tau \in \mathrm{Gal}(K/\mathbb{Q})$, $\chi^\tau \in X$.

To each $\chi \in \widehat{G}$, we attach a 'special' idempotent element in $K[G]$, namely,

$$\epsilon_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1}) \, \sigma.$$

Let $X \subseteq \widehat{G}$ and $\epsilon_X := \sum_{\chi \in X} \epsilon_\chi$. Suppose that X is closed under conjugation. Then for any $\tau \in \mathrm{Gal}(K/\mathbb{Q})$,

$$\tau \, \cdot \, \epsilon_X = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\chi \in X} \chi^\tau(\sigma^{-1}) \, \sigma = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\chi \in X} \chi(\sigma^{-1}) \sigma = \epsilon_X,$$

as X is closed under the action of $\mathrm{Gal}(K/\mathbb{Q})$. Therefore, for $X \subseteq \widehat{G}$, that are closed under conjugation,

$$\epsilon_X \in \mathbb{Q}[G].$$

In fact, these elements *intrinsically* define all ideals of $\mathbb{Q}[G]$.

Indeed, using the theory of 'central idempotents' (see [34]), one can show that there exists an order preserving bijection between $\mathcal{A} := \{\mathfrak{A} : \text{ ideal in } \mathbb{Q}[G]\}$ and $\mathcal{X} := \{X : X \subseteq \widehat{G}, \text{ closed under conjugation}\}$. This bijection is explicitly given as follows.

$$\begin{aligned} \mathcal{A} &\xleftrightarrow{\;\sim\;} \mathcal{X} \\ \mathfrak{A} &\longmapsto X_{\mathfrak{A}} := \left\{\chi \in \widehat{G} : \chi\big|_{\mathfrak{A}} \neq 0\right\} \\ \mathbb{Q}[G]\epsilon_X &\longleftarrow\!\shortmid X \end{aligned}$$

Note that in the above bijection, $\chi \in \widehat{G}$ is to be treated as a $\mathbb{Q}$-linear map on $\mathbb{Q}[G]$, extended by

$$\chi\left(\sum_{\sigma \in G} a_\sigma\, \sigma\right) = \sum_{\sigma \in G} a_\sigma\, \chi(\sigma).$$

Thus, given an ideal $\mathfrak{A}$ of $\mathbb{Q}[G]$, it suffices to identify the set of characters $\chi \in \widehat{G}$ such that

$$\chi\big|_{\mathfrak{A}} \neq 0,$$

in order to find a generator of $\mathfrak{A}$. Alternatively, one can determine the set of all characters $\chi \in \widehat{G}$ such that

$$\chi\big|_{\mathfrak{A}'} = 0,$$

where $\mathfrak{A}'$ denotes the complementary ideal of $\mathfrak{A}$. This is because $\mathfrak{A} = \mathbb{Q}[G]\epsilon_{X_{\mathfrak{A}}}$,

$$\mathfrak{A}' = \mathbb{Q}[G]\epsilon_{X_{\mathfrak{A}'}} = \mathbb{Q}[G](1 - \epsilon_{X_{\mathfrak{A}}}),$$

as $\mathfrak{A} \oplus \mathfrak{A}' = \mathbb{Q}[G]$. Note that by orthogonality of characters,

$$\sum_{\chi \in \widehat{G}} \epsilon_\chi = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \sum_{\sigma \in G} \chi(\sigma^{-1})\, \sigma = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \sum_{\chi \in \widehat{G}} \chi(\sigma^{-1}) = 1.$$

Therefore, we obtain

$$1 - \epsilon_{X_{\mathfrak{A}}} = \frac{1}{|G|} \sum_{\chi \notin X_{\mathfrak{A}}} \epsilon_\chi = \epsilon_{\widehat{G} \setminus X_{\mathfrak{A}}}.$$

Hence, the bijection stated above implies that $X_{\mathfrak{A}'} = \widehat{G} \setminus X_{\mathfrak{A}}$ and we get,

$$\chi \in X_{\mathfrak{A}} \iff \chi \notin X_{\mathfrak{A}'} \iff \chi\big|_{\mathfrak{A}'} = 0.$$

Thus, a generator of an ideal $\mathfrak{A}$ can be identified by determining its complementary ideal $\mathfrak{A}'$ and the characters $\chi \in \widehat{G}$ such that $\chi\big|_{\mathfrak{A}'} = 0$.

Application to linear relations among Galois conjugates of cyclotomic numbers

Fix a positive integer $q > 3$ and let $\mathbb{Q}(\zeta_q)$ be the q^{th} cyclotomic field. Let $G := \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \simeq (\mathbb{Z}/q\mathbb{Z})^*$. Suppose that $b \in \mathbb{Q}(\zeta_q)$. Our aim is to classify all the relations

$$\sum_{\sigma \in G} a_\sigma\, \sigma(b) = 0, \;\; a_\sigma \in \mathbb{Q}.$$

This motivates the definition of the following action. For $\sigma \in G$ and $b \in \mathbb{Q}(\zeta_q)$, define

$$\sigma \cdot b := \sigma(b). \tag{8.1}$$

Since we want to study $\mathbb{Q}$-linear relations, we extend the above action $\mathbb{Q}$-linearly to the group ring $\mathbb{Q}[G]$. Thus, (8.1) defines a $\mathbb{Q}[G]$-module structure on $\mathbb{Q}(\zeta_q)$ given by

$$\left(\sum_{\sigma \in G} a_\sigma \, \sigma\right) \cdot b := \sum_{\sigma \in G} a_\sigma \, \sigma(b).$$

Thus, it is easy to see that for a fixed $b \in \mathbb{Q}(\zeta_q)$, the set

$$\mathfrak{A}_b := \{\alpha \in G \, : \, \alpha \cdot b = 0\}$$

forms an ideal of the ring $\mathbb{Q}[G]$. Thanks to Proposition 8.2.1, we know that every ideal is principal. Therefore, it suffices to find a generator of $\mathfrak{A}_b$ and we can achieve that by the technique outlined earlier.

In order to recognize the complementary ideal of $\mathfrak{A}_b$, we realize that $\mathfrak{A}_b = \ker(E_b)$, where $E_b : \mathbb{Q}[G] \to \mathbb{Q}(\zeta_q)$ is such that

$$E_b(\gamma) := \gamma \cdot b.$$

Therefore, the complementary ideal $\mathfrak{A}'_b$ is isomorphic to the image of E_b,

$$\mathrm{Im}(E_b) = \{\gamma \cdot b \, : \, \gamma \in \mathbb{Q}[G]\}.$$

By the normal basis theorem, $\mathbb{Q}(\zeta_q)/\mathbb{Q}$ has a normal basis, i.e., a basis generated by an element and its Galois conjugates. Let $x \in \mathbb{Q}(\zeta_q)$ generate a normal basis for $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. Then

$$b = \sum_{\sigma \in G} b_\sigma \, \sigma(x), \;\; b_\sigma \in \mathbb{Q}.$$

That is, if we denote $\beta = \sum_{\sigma \in G} b_\sigma \sigma \in \mathbb{Q}[G]$, then $b = \beta \cdot x$. Thus,

$$\operatorname{Im}(E_b) = \Big\{ (\gamma \, \beta) \cdot x \, : \, \gamma \in \mathbb{Q}[G] \Big\} \simeq_{\mathbb{Q}-\text{v.s.}} \mathbb{Q}[G]\beta.$$

Hence, the complementary ideal $\mathfrak{A}'_b = \mathbb{Q}[G]\beta$, and the relevant characters will be

$$X_b := \Big\{ \chi \in \widehat{G} \, : \, \chi(\beta) = 0 \Big\}.$$

However, an answer to the classification question is more favourable in terms of b, rather than β. Thus, our next step would be to eliminate the occurrence of x in our study.

We define "character coordinates" for numbers in $\mathbb{Q}(\zeta_q)$ following [34]. Suppose that χ is a character modulo q of conductor f, and is induced from the primitive character χ_f. Then for any $c \in \mathbb{Q}(\zeta_q)$, let

$$Y\left(\chi \mid c\right) := \frac{1}{\tau(\overline{\chi_f})} \sum_{\sigma \in G} \chi(\sigma^{-1}) \, \sigma(c).$$

It is easy to see using the orthogonality of characters that

$$\sum_{\chi \in \widehat{G}} \tau(\overline{\chi_f}) Y\left(\chi \mid c\right) = |G|\, c,$$

and hence the name, 'character coordinates'.

Returning back to our application, we compute the character coordinates of the element b. Recall that $b = \beta \cdot x$, with $\beta = \sum_{\gamma \in G} b_\gamma\, \gamma$. Thus,

$$\begin{aligned} Y\left(\chi \mid b\right) &= \frac{1}{\tau(\overline{\chi_f})} \sum_{\sigma \in G} \chi(\sigma^{-1}) \sigma \Bigg(\sum_{\gamma \in G} b_\gamma\, \gamma(x) \Bigg) \\ &= \frac{1}{\tau(\overline{\chi_f})} \sum_{\gamma \in G} b_\gamma \sum_{\sigma \in G} \chi(\sigma^{-1})\, \sigma\big(\gamma(x)\big) \\ &= \frac{1}{\tau(\overline{\chi_f})} \sum_{\gamma \in G} b_\gamma\, \chi(\gamma) \sum_{\tau \in G} \chi(\tau^{-1}) \tau(x) \\ &= \chi(\beta)\, Y\left(\chi \mid x\right). \end{aligned}$$

Now, by a theorem of Leopoldt [54, Satz 2], $Y(\chi \mid x) \neq 0$. Therefore, we get that

$$\chi(\beta) = 0 \iff Y\left(\chi \mid b\right) = 0.$$

Thus, given $b \in \mathbb{Q}(\zeta_q)$, to classify all $\mathbb{Q}$-linear relations among the Galois conjugates of b, it suffices to determine the set X_b of characters χ such that $Y\left(\chi \mid b\right) = 0$, as

$$\mathfrak{A}_b = \mathbb{Q}[G] \epsilon_{X_b}.$$

In the particular case of $b = \cot(\pi/q)$, it can be seen from (2.14) that

$$Y\big(\chi \mid \cot(\pi/q)\big) = \begin{cases} -\dfrac{2\,q}{\pi\,\tau(\overline{\chi_f})}\,L(1,\chi), & \text{if } \chi(-1) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Dirichlet's theorem that $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$ implies that (see [18], [74])

Theorem 8.2.2 (T. Okada)**.** *For a positive integer $q > 3$, the numbers*

$$\cot\left(\frac{a\pi}{q}\right),\ 1 \leq a < q/2,\ (a,q) = 1$$

are linearly independent over $\mathbb{Q}$.

Indeed, the above method is more general and can also be applied to derive the analogue of the above theorem for derivatives of the cotangent at rational arguments, since $L(m,\chi) \neq 0$ for $m > 1$. We aim to generalize this method in order to understand *multiplicative* relations among cyclotomic numbers.

8.3 Special values of the Epstein zeta and related functions

Let a, b and $c \in \mathbb{Z}$ and

$$Q(X,Y) := aX^2 + bXY + cY^2$$

be a binary quadratic form. Alternatively, this may be viewed as

$$Q(X,Y) = \begin{bmatrix} X \\ Y \end{bmatrix}^t \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

The quadratic form $Q(X,Y)$ is said to be *positive definite* if the associated matrix is positive definite, that is,

$$a > 0, \;\; 4ac - b^2 > 0.$$

This is equivalent to $Q(m,n) > 0$ for $(m,n) \in \mathbb{Z} \times \mathbb{Z} \backslash \{(0,0)\}$. A Dirichlet series can be then attached to Q, namely,

$$Z_Q(s) := \sum_{\substack{m,n \in \mathbb{Z}, \\ (m,n) \neq (0,0)}} \frac{1}{Q(m,n)^s} = \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^s},$$

where $r_Q(n)$ denotes the number of ways of representing n in terms of Q, i.e.,

$$r_Q(n_0) = \# \left\{ (m,n) \in \mathbb{Z}^2 \, : \, Q(m,n) = n_0 \right\}.$$

We show below that the above series is absolutely convergent in $\Re(s) > 1$.

Lemma 8.3.1. *Let $Q(X,Y) = aX^2 + bXY + cY^2$ be a positive definite binary quadratic form with a, b, $c \in \mathbb{Z}$. Then the series $Z_Q(s)$ converges absolutely for $\Re(s) > 1$.*

Proof. Since $Q(X,Y)$ is positive definite, the associated matrix A, is a positive symmetric matrix. Thus, there exists an orthogonal matrix U such that $UAU^t = D$, where D is a diagonal matrix. Moreover, if

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

with λ_1, λ_2 being the eigenvalues of A, then $\lambda_1, \lambda_2 > 0$. Suppose that $\lambda_2 \geq \lambda_1$. Thus,

$$\begin{aligned} Q(m,n) &= (m\,n)A(m\,n)^t \\ &= (m\,n)U^t\big(UAU^t\big)U(m\,n)^t \\ &= \lambda_1 u_1^2 + \lambda_2 u_2^2, \end{aligned}$$

where we let

$$U \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $\lambda_2 \geq \lambda_1$, $Q(m,n) \geq \lambda_1(u_1^2 + u_2^2) = \lambda_1(m^2+n^2)$ as transformation by U is an isometry. Therefore,

$$\sum_{(m,n)\neq(0,0)} Q(m,n)^{-s} \leq \lambda_1^{-s} \sum_{(m,n)\neq(0,0)} \left(m^2+n^2\right)^{-s} = \lambda_1^{-s} \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s},$$

where $r_2(n)$ is the number of ways of representing n as a sum of two squares. Since $r_2(n) < d(n)$, where $d(n)$ denotes the number of divisors of n (see [61, Exercise 1.3.4]), and $d(n) = O(n^\epsilon)$, we see that the series $Z_Q(s)$ converges absolutely in the region $\Re(s) > 1$. □

Remark. *In 1903, P. Epstein [31] introduced and studied the generalization of $Z_Q(s)$ when Q is a positive definite quadratic form in n variables. He proved its analytic continuation, functional equation as well as analogues of Kronecker's limit formula for these general zeta-functions. Hence, the functions $Z_Q(s)$ are called Epstein zeta-functions in his honour.*

Following the multi-dimensional analogue of Riemann's method of analytic continuation using theta series, it can be shown that $Z_Q(s)$ can be analytically continued to the entire complex plane except for a simple pole at $s = 1$. Moreover, the function $Z_Q(s)$ satisfies a functional equation akin to that of the Riemann zeta function, relating $Z_Q(s)$ to $Z_Q(1-s)$. Contrary to the Riemann zeta-function, $Z_Q(s)$ does not necessarily satisfy the Riemann hypothesis. In fact, Davenport and Heilbronn [25], [26] proved in 1936 that if $Q(X,Y) = aX^2+bXY+cY^2$ is a positive definite quadratic form with $D = b^2-4ac$ being a fundamental discriminant such that $\mathbb{Q}(\sqrt{D})$ has class number $h(D) > 1$, then $Z_Q(s)$ has infinitely many zeros in the region $\sigma > 1$.

Thus, it is natural to inquire if the analogue of Euler's theorem for $\zeta(2k)$ holds for these zeta-functions. In particular, is it possible to 'evaluate' $Z_Q(n)$ for any positive integer $n > 1$? This question was addressed in a paper of J. R. Smart [92] in 1971. He obtained an expression for the value $Z_Q(k)$ in terms of $\zeta(2k)$, $\zeta(2k-1)$ and certain Fourier series expansions when $ac - b^2 = 1$. This was later generalized to the case of any $D = 4ac - b^2 > 0$ by K. Williams and Zhang N. [103]. The argument relies on the Fourier series expansion of the Epstein zeta-function, and we include it here for the sake of completeness.

S. Chowla and A. Selberg announced certain results concerning the Epstein zeta-function of a binary quadratic form in 1949 (see [20]), the proofs of which were published much later in [89]. They derived the analytic continuation and functional equation of $Z_Q(s)$ using the following transformation formula. We give the proof of this transformation formula as we believe it to be a general identity, and the authors omit the proof in [89].

Lemma 8.3.2. *For $\alpha \in \mathbb{R}$ and $x > 0$,*

$$\sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^2 \pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x} \cos(2\pi n\alpha).$$

Proof. For a fixed $\alpha \in \mathbb{R}$ and $x > 0$, let $f(y) = e^{-(y+\alpha)^2\pi x}$. Since f is continuous and absolutely integrable on $\mathbb{R}$,

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e^{-(y+\alpha)^2\pi x}\, e^{-2\pi i t y}\, dy.$$

Making the change of variable $y + \alpha \mapsto y$, we obtain

$$\widehat{f}(t) = e^{2\pi i t\alpha} \int_{-\infty}^{\infty} e^{-y^2\pi x - 2\pi i t y}\, dy.$$

Completing square of the exponent of the integrand gives

$$\widehat{f}(t) = e^{-\pi t^2/x}\, e^{2\pi i t\alpha} \int_{-\infty}^{\infty} e^{-\pi\left(\sqrt{x}y + \frac{it}{\sqrt{x}}\right)^2}\, dy.$$

Now substitute $\sqrt{x}y + (it/\sqrt{x}) = z$. Thus, we obtain that

$$\widehat{f}(t) = \frac{e^{-\pi t^2/x}\, e^{2\pi i t\alpha}}{\sqrt{x}} \int_L e^{-\pi z^2}\, dz,$$

where $L = \{z \in \mathbb{C} : \mathrm{Im}(z) = t/\sqrt{x}\}$. This can be reduced to the integral along the real axis by contour integration. Thus, for $T > 0$, consider the positively oriented rectangular contour $\mathcal{C}_T$ from $(T, 0)$ to $(T, t/\sqrt{x})$ to $(-T, t/\sqrt{x})$ to $(-T, 0)$ and back to $(T,0)$. Let

$$I_1 := \int_{[(-T,0),(T,0)]} e^{-\pi z^2}\, dz,\ I_2 := \int_{[(T,0),(T,t/\sqrt{x})]} e^{-\pi z^2}\, dz,$$
$$I_3 := \int_{[(T,t/\sqrt{x}),(-T,\sqrt{x})]} e^{-\pi z^2}\, dz,\ I_4 := \int_{[(-T,t/\sqrt{x}),(-T,0)]} e^{-\pi z^2}\, dz,$$

where $[(r,s),(u,v)]$ denotes the straight line from (r,s) to (u,v). Thus, $\int_{\mathcal{C}_T} e^{-\pi z^2}\, dz = I_1 + I_2 + I_3 + I_4$. It is not difficult to show that

$$\lim_{T\to\infty} |I_2 + I_4| = 0.$$

Hence, $\lim_{T\to\infty} I_1 = -\lim_{T\to\infty} I_3$ as $e^{-\pi z^2}$ is entire. Therefore,

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1 \Rightarrow \widehat{f}(t) = \frac{e^{-\pi t^2/x}\, e^{2\pi i t\alpha}}{\sqrt{x}}.$$

The identity is then a consequence of the Poisson summation formula. □

We briefly outline the following theorem of S. Chowla and A. Selberg, which was also proved by Bateman and Grosswald [10] and is related to an earlier work of M. Deuring [29, Equation (9)].

Theorem 8.3.3 (S. Chowla, A. Selberg)**.** *Given a positive definite quadratic form $Q(X,Y) = aX^2 + bXY + cY^2$, with discriminant $D = 4ac - b^2 > 0$,*

$$Z_Q(s) = 2a^{-s}\zeta(2s) + \frac{2^{2s}\, a^{s-1}\sqrt{\pi}}{\Gamma(s)\, D^{s-\frac{1}{2}}}\,\zeta(2s-1)\Gamma\left(s-\frac{1}{2}\right) + R_Q(s), \tag{8.2}$$

where

$$R_Q(s) = \frac{4\,\pi^s\, 2^{s-\frac{1}{2}}}{\sqrt{a}\,\Gamma(s)\, D^{\frac{s}{2}-\frac{1}{4}}} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\,\sigma_{1-2s}(n)\,\cos\left(\frac{n\pi b}{a}\right) I_n(s),$$

with

$$I_n(s) = \int_0^\infty t^{s-\frac{3}{2}} e^{-\frac{-\pi n\sqrt{D}}{2a}\left(t+\frac{1}{t}\right)}\, dt, \tag{8.3}$$

and $\sigma_z(n) := \sum_{d|n} d^z$ *for any* $z \in \mathbb{C}$.

Proof. Suppose that $\Re(s) > 1$. The first term in the formula is obtained from the term corresponding to $n = 0$ in

$$Z_Q(s) = \sum_{(m,n)\neq(0,0)} \left(am^2 + bmn + cn^2\right)^{-s}.$$

Let

$$\mathcal{Z}(s) := \sum_{n=-\infty, n\neq 0}^{\infty} \sum_{m\in\mathbb{Z}} \left(am^2 + bmn + cn^2\right)^{-s}.$$

Taking a common and completing the square of the resulting summand gives

$$a^s \mathcal{Z}(s) = \sum_{n=-\infty, n\neq 0}^{\infty} \sum_{m\in\mathbb{Z}} \left[\left(m + \frac{b}{2a}\, n\right)^2 + \frac{D}{4a^2}\, n^2\right]^{-s}. \tag{8.4}$$

We now proceed as in Riemann's proof of analytic continuation of $\zeta(s)$. Thus, using the integral representation of the gamma function and a suitable change of variables, we get

$$\pi^{-s}\, a^s\, \Gamma(s)\, \mathcal{Z}(s) = \int_0^\infty t^{s-1} \left(\sum_{n=-\infty, n\neq 0}^{\infty} e^{-\frac{D}{4a^2} n^2 \pi t} \sum_{m\in\mathbb{Z}} e^{-\left(m+\frac{b}{2a}n\right)^2 \pi t} \right) dt.$$

Thus, applying Lemma 8.3.2 and separating the term corresponding to $m = 0$ implies

$$\pi^{-s}\, a^s\, \Gamma(s)\, \mathcal{Z}(s) = 2\int_0^\infty t^{s-\frac{3}{2}} \left(\sum_{n=1}^{\infty} e^{-\frac{D}{4a^2}n^2\pi t}\right) dt + 4\int_0^\infty t^{s-\frac{3}{2}} \left(\sum_{m,n=1}^{\infty} e^{-\frac{D}{4a^2}n^2\pi t - \frac{\pi m^2}{t}} \cos\left(\frac{2\pi mnb}{a}\right)\right) dt.$$

Interchanging integral and summation in the second term and a suitable change of variables transforms it into the series $R_Q(s)$ up to the constant factors. For the first term, substitute $u = D\pi t/(4a^2)$ and get

$$\int_0^\infty t^{s-\frac{3}{2}} \left(\sum_{n=1}^{\infty} e^{-\frac{D}{4a^2}n^2\pi t}\right) dt = \left(\frac{D\pi}{4a^2}\right)^{\frac{3}{2}-s-1} \int_0^\infty u^{s-\frac{3}{2}} \sum_{n=1}^{\infty} e^{-un^2}\, du = \left(\frac{4a^2}{D\pi}\right)^{s-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{2(s-1/2)}} \int_0^\infty w^{s-\frac{3}{2}} e^{-w}\, dw,$$

by the change of variable $n^2 u \mapsto w$. This corresponds to the second term in the formula (8.2) and completes the proof. □

In [10], Bateman and Grosswald identified the integral $I_n(s)$ as a specialization of the Bessel function,

$$K_\nu(z) := \frac{1}{2}\int_0^\infty e^{-\frac{z}{2}\left(u+\frac{1}{u}\right)}\, u^{\nu-1}\, du.$$

This observation, together with Theorem 8.3.3 implies that

$$Z_Q(s) = 2a^{-s}\zeta(2s) + \frac{2^{2s}\, a^{s-1}\sqrt{\pi}}{\Gamma(s)\, D^{s-\frac{1}{2}}}\, \zeta(2s-1)\, \Gamma\left(s - \frac{1}{2}\right) + \frac{4\,\pi^s\, 2^{s+\frac{1}{2}}}{\sqrt{a}\,\Gamma(s)\, D^{\frac{s}{2}-\frac{1}{4}}} H(s), \quad (8.5)$$

where the function $H(s)$ is given by the infinite series

$$H(s) = \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\, \sigma_{1-2s}(n)\, \cos\left(\frac{n\pi b}{a}\right)\, K_{s-\frac{1}{2}}\left(\frac{\pi n\sqrt{D}}{a}\right).$$

Since we are interested in values of $Z_Q(s)$ at positive integers, we set $s = k > 1$ in (8.5). The following formula of the Bessel function (see [97, pg. 80 (12)]) is particularly useful in our setting.

$$K_{k-\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{r=0}^{k-1} \frac{(k-1+r)!}{r!(k-1-r)!(2z)^r}.$$

Evaluating (8.5) at $s = k$, together with the above formula at $z = \pi n\sqrt{D}/a$, we have

$$Z_Q(k) = 2a^{-k}\zeta(2k) + \frac{2^{2k}\, a^{k-1}\sqrt{\pi}}{(k-1)!\, D^{k-\frac{1}{2}}}\, \zeta(2k-1)\, \Gamma\left(k - \frac{1}{2}\right)$$
$$+ 2\left(\frac{2\pi}{\sqrt{D}}^{k}\right) \sum_{r=0}^{k-1} \binom{k-1+r}{k-1} \frac{1}{(k-1-r)!} \left(\frac{a}{2\pi\sqrt{D}}\right)^r \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{k+r}} \left(e^{2\pi i n\tau} + e^{-2\pi i n\bar{\tau}}\right), \tag{8.6}$$

using the fact that $\sigma_{1-2k}(n) = n^{1-2k}\sigma_{2k-1}(n)$, and

$$\tau := \frac{b}{2a} + i\frac{\sqrt{D}}{2a} \in \mathbb{H}.$$

Here $\mathbb{H}$ denotes the upper half plane, $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. This proves [92, Equation (2.4)] for all $D = 4ac - b^2 > 0$. However, the main aim of Smart in [92] is to connect these values to certain functions with modular behaviour. On the other hand, our motivation is to investigate (8.6) from the point of view of transcendence.

In this light, we note a connection of the series appearing in (8.6) to certain integrals of classical Eisenstein series, called Eichler integrals. In this regard, we refer the reader to the paper of E. Grosswald [37] and S. Gun, M. R. Murty and P. Rath [40]. In 1972, E. Grosswald [37] considered the function

$$F_k(z) := \sum_{n=1}^{\infty} \sigma_{-k}(n)\, e^{2\pi i n z} = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k}\, e^{2\pi i n z},$$

for any $k \in \mathbb{Z}$ and $z \in \mathbb{H}$. When $k \leq -3$, the function $F_k(z)$ is essentially the classical Eisenstein series $E_{1-k}(z)$. On the other hand, for odd positive values of k, $F_k(z)$ can be obtained by successive integration of the corresponding Eisenstein series minus the constant term. In particular, for an *even* positive integer $k > 2$, let

$$E_k(z) = \gamma_k + \sum_{n=1}^{\infty} \sigma_k(n)\, e^{2\pi i n z}, \quad z \in \mathbb{H}, \quad \gamma_k = \frac{-B_k}{2k},$$

where B_n denotes the n^{th} Bernoulli number. Then, for any *odd* $k > 1$,

$$F_k(z) = \frac{(2\pi i)^k}{(k-1)!} \int_{i\infty}^{z} \left(E_{k+1}(\tau) - \gamma_{k+1}\right) (\tau - z)^{k-1}\, d\tau,$$

which is an Eichler integral in the sense of Goldstein [35].

In the context of (8.6), we note that for a positive integer $k > 1$, $0 \leq r \leq k-1$ and $z \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{k+r}}\, e^{2\pi i n z} = \frac{F_{2k-1}^{(k-1-r)}(z)}{(2\pi i)^{k-1-r}}.$$

Hence, the value $Z_Q(k)$ can be expressed as

$$Z_Q(k) = 2a^{-k}\zeta(2k) + \frac{2^{2k}\, a^{k-1}\sqrt{\pi}}{(k-1)!\, D^{k-\frac{1}{2}}}\,\zeta(2k-1)\,\Gamma\left(k-\frac{1}{2}\right)$$
$$+\,2\left(\frac{2\pi}{\sqrt{D}}^{\,k}\right)\sum_{r=0}^{k-1}\binom{k-1+r}{k-1}\frac{1}{(k-1-r)!}\left(\frac{a}{2\pi\sqrt{D}}\right)^{r}\left(\frac{F_{2k-1}^{(k-1-r)}(\tau)+F_{2k-1}^{(k-1-r)}(-\overline{\tau})}{(2\pi i)^{k-1-r}}\right).$$

Thus, in order to understand the arithmetic nature of $Z_Q(k)$, it is imperative to study the transcendental or algebraic nature of the special values of $F_k(z)$ and its derivatives for $z \in \mathbb{H}$. This was carried out by S. Gun, M. R. Murty and P. Rath [40] wherein they investigate the values of the function

$$F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha), \quad \alpha \in \mathbb{H}.$$

Therefore, in the special case of $\tau = i$, it may be possible to derive explicit formulae for the special values of the associated Epstein zeta-function. We relegate this to future research. The Epstein zeta function is also related to the Dedekind zeta function attached to imaginary quadratic fields, ideal class zeta-functions as well as non-holomorphic Eisenstein series. Thus, the above observation might provide some new insight into the special values of related zeta-functions as well.

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