Topics in Combinatorics and Random Matrix Theory

by

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Abstract

Motivated by the longest increasing subsequence problem, we examine sundry topics at the interface of enumerative/algebraic combinatorics and random matrix theory.

We begin with an expository account of the increasing subsequence problem, contextualizing it as an "exactly solvable" Ramsey-type problem and introducing the RSK correspondence. New proofs and generalizations of some of the key results in increasing subsequence theory are given. These include Regev's single scaling limit, Gessel's Toeplitz determinant identity, and Rains' integral representation. The double scaling limit (Baik-Deift-Johansson theorem) is briefly described, although we have no new results in that direction.

Following up on the appearance of determinantal generating functions in increasing subsequence type problems, we are led to a connection between combinatorics and the ensemble of truncated random unitary matrices, which we describe in terms of Fisher's random-turns vicious walker model from statistical mechanics. We prove that the moment generating function of the trace of a truncated random unitary matrix is the grand canonical partition function for Fisher's random-turns model with reunions.

Finally, we consider unitary matrix integrals of a very general type, namely the "correlation functions" of entries of Haar-distributed random matrices. We show that these expand perturbatively as generating functions for class multiplicities in symmetric functions of Jucys-Murphy elements, thus addressing a problem originally raised by De Wit and t'Hooft and recently resurrected by Collins. We argue that this expansion is the CUE counterpart of genus expansion.

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Chapter 1

Introduction

1.1 Motivation

Consider a library of N books, labelled bijectively with the numbers $\{1, \dots, N\}$, and arranged on a single long bookshelf. The configuration of the books corresponds to a permutation σ from the symmetric group $S(N)$. How many operations are required to sort the books into canonical ascending order $1, 2, \dots, N$ from left to right on the shelf?

One could proceed as follows: locate book 1. If it is located on the left end of the shelf, do nothing. If not, remove it, and replace it at the left and of the shelf. Now locate book 2. If it is located immediately to the right of book 1, do nothing. If not, remove it, and replace it immediately to the right of book 1. Continuing this process will sort the books, but this algorithm is not optimal.

A better strategy uses the fact that some of the books may already be in the correct relative order. In fact, the minimum number of operations required to sort the books is

$$N - \operatorname{lis}(\sigma),$$

where $\operatorname{lis}(\sigma)$ is the length of the longest *increasing subsequence* in σ (defined below). It is easy to describe an algorithm that sorts the books in this number of operations (the structure of this simple algorithm will be clear from Example 15 in Chapter 2). The permutation statistic $\operatorname{lis}(\sigma)$ thus emerges as a natural measure of "order;" when $\operatorname{lis}(\sigma)$ is large, the books are already close to the desired ascending configuration and can be sorted quickly, whereas many operations are required if $\operatorname{lis}(\sigma)$ is small.

Now imagine that you are a librarian responsible for sorting the shelf at the end of each day. Since this is a repetitive chore, you would be primarily interested in its statistical properties. On average, how many operations will be required to sort the books? What is the probability that more than a given number of operations will be required? If N is large, smooth approximations may be more useful than exact answers.

These are very basic questions which turn out to be rather difficult to answer. Mathematically, the problem is to determine the distribution of the length of the longest increasing subsequence in a random permutation. This fundamentally important and ostensibly elementary question was raised by Ulam [Ula61] in the 1960's. The problem was fully solved a mere ten years ago. The solution, a very advanced analytical argument presented by Baik, Deift, and Johansson in [BDJ99], marks the culmination of a half-century of research. Alternative proofs (and generalizations) were presented soon afterward by Okounkov [Oko00], Borodin, Okounkov, and Olshanski [BOO00], and Johansson [Joh01]. Indeed, far from being a simple combinatorial curiosity, the *longest increasing subsequence problem* is deeply intertwined with several of the most active areas of modern mathematics, such as asymptotic representation theory and properties of large random matrices. Interest in the increasing subsequence problem has greatly intensified in the wake of [BDJ99], and several very useful and accessible survey articles have been written [AD99, Dei00, Sta06].

In the initial chapters of this thesis, we will describe a small part of the vast body of research which has been conducted in and around the longest increasing subsequence problem, adding some new results, and new proofs of known results, along the way. This portion of the thesis represents a newcomer's efforts to understand and catch up to the dramatic recent developments in this area. It is hoped that these chapters are not too much of a sprint, and might serve as a mostly self-contained presentation of the combinatorial foundation of the subject. From a didactic perspective, part of the appeal of the increasing subsequence problem is that it provides a combinatorially motivated point of entry into the daunting edifice of random matrix theory. This perspective will be emphasized, and our principal goal is to bring into focus the underlying structures which lead to a relationship between increasing subsequences and random matrices.

In later chapters, we consider problems which are less obviously related to increasing subsequences. This departure occurs in two stages. First, we consider the relationship between the increasing subsequence problem and unitary matrix integrals first observed by Rains [Rai98], and consider various ways in which this can be generalized. This leads to combinatorial interpretations of averages over the ensemble of truncated random unitary matrices. This parameter-dependent ensemble emerged a decade ago in work of Sommers and Zyczkowski [SZ00] on scattering theory. Previously, this ensemble lacked any connection to combinatorics. We show that various averages over these ensembles of random contractions enumerate configurations of random-turns vicious walkers, which are themselves combinatorial generalizations of increasing subsequences. We thus obtain a one-parameter family of extensions of the relationship between the Circular Unitary Ensemble and the increasing subsequence problem.

In the final chapter of this thesis, which consists of work undertaken in collaboration with Sho Matsumoto, we consider a problem which has no *a priori* connection to either increasing subsequences or random matrix theory: the class basis expansion of symmetric functions evaluated on the alphabet of Jucys-Murphy elements. The Jucys-Murphy elements [Juc74, Mur81] are a distinguished family of generators of the Gelfand-Tsetlin subalgebra which have an unexpected symmetry property — although they are typically non-central, symmetric functions of them are. The problem of computing the multiplicity of a given conjugacy class in a given symmetric function of the JM-elements is an intriguing combinatorial problem with connections to various branches of mathematics, such as random walk problems [13], intrinsic representation theory of the symmetric groups [38], vertex operators [29], and Hurwitz numbers [19].

In Chapter 8, we give a new application of the Jucys-Murphy elements. We prove that perturbative series of a large class of unitary matrix integrals, which have been intensively studied by both physicists and mathematicians, expand as generating functions for conjugacy class multiplicities in symmetric functions of Jucys-Murphy elements. The study of such integrals thus becomes a part of JM-element theory. We obtain new results on class expansions, and reconstruct the entire theory of unitary matrix integrals using these results in an elegant and unified way. This approach, which is a consequence of the *Weingarten calculus* developed by Collins [Col03], Collins-Śniady [11], Banica-Collins [BC07], and Banica-Speicher [6], parallels the well-established coupling between Gaussian integrals over the space of Hermitian matrices and the map enumeration problem.

1.2 Guide to main results

This section is a guide to the main original results obtained in this thesis. Notation and terminology not explained here will be presented below, where the lack of context will be remedied and proper references given.

1.2.1 The single scaling limit

Let $u(d,N)$ denote the number of permutations in $S(N)$ with increasing subsequence length bounded by d. Let $t(d,N)$ denote the number of involutions in $S(N)$ with longest decreasing subsequence of length exactly d and longest increasing subsequence of length exactly N.

Theorem 1 (Asymptotic Knuth Theorem, Novak 2009)**.** *For any fixed $d \geq 1$,*

$$u(d,dn) \sim t(d,2n)$$

as $n \to \infty$.

Corollary 2 (Regev 1981)**.** *For any fixed $d \geq 1$,*

$$u(d,N) \sim (2\pi)^{\frac{1-d}{2}} \left(\prod_{i=0}^{d-1} i! \right) d^{2N+\frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}}$$

as $N \to \infty$.

1.2.2 Determinantal identities

Let $\mathbb{W}_d$ denote the "Weyl graph" obtained by intersecting the type A Weyl chamber in $\mathbb{R}^d$ with the integer lattice $\mathbb{Z}^d$. This is a graded graph with lowering and raising operators L, R.

Theorem 3 (Novak 2008)**.** *The lowering and raising operators on $\mathbb{W}_d$ commute: $LR = RL$.*

Corollary 4 (Gessel 1990)**.** *Let $I_k(t)$ denote the modified Bessel function of the first kind of order k. For any $d \geq 1$,*

$$\sum_{n \geq 0} u(d,n) \frac{t^{2n}}{n!n!} = \det(I_{j-i}(2t))_{1 \leq i,j \leq d}.$$

Corollary 5 (Forrester 2001)**.** *The refined partition function $Z_d(L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}; \mu, \lambda)$ of Fisher's random-turns vicious walker model depends only on the number of L's and R's in the word $L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}$, and not on their order.*

1.2.3 Random contraction matrices

Let $\mathcal{U}(d+q)$ denote the group of $(d+q) \times (d+q)$ complex unitary matrices, and $\mathcal{B}(d)$ the semigroup of $d \times d$ matrices with operator norm ≤ 1. Let $T^{(q)} : \mathcal{U}(d+q) \to \mathcal{B}(d)$ be the map sending a unitary matrix to its $d \times d$ principal submatrix, and $\tau_d^{(q)}$ the pushforward of normalized Haar measure under $T^{(q)}$.

Let $Z_d(N;q)$ denote the number of ways in which d random-turns vicious walkers can depart adjacent initial positions $d, d-1, \dots, 1$ on the integer lattice $\mathbb{Z}$ and arrive at new adjacent positions $d+q, d-1+q, \dots, 1+q$ by taking N steps.

Theorem 6 (Novak 2008). *For any* $d \geq 1, q \geq 0$

$$\frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP) = \sum_{N \geq 0} Z_d(N;q) \frac{t^N}{N!},$$

were $H_{R(d,q)} = \prod_{i=0}^{d-1} \frac{(q+i)!}{i!}$.

Corollary 7 (Rains 1998). *For any* $d, n \geq 1$

$$u(d,n) = \int_{\mathcal{U}(d)} |\,\mathrm{Tr}\, U|^{2n} dU.$$

1.2.4 Symmetric functions of Jucys-Murphy elements

Let Ξ_n denote the multiset $\{\{J_1, \ldots, J_n, 0, 0, \ldots\}\}$ of Jucys-Murphy elements in the symmetric group algebra $\mathbb{C}[S(n)]$. Let Λ denote the algebra of symmetric functions and consider the specialization $\Lambda \to Z(n)$ defined by $f \mapsto f(\Xi_n)$, where $Z(n)$ is the center of $\mathbb{C}[S(n)]$. Denote by $G_\mu(f,n)$ the multiplicity of the conjugacy class $\mathfrak{c}_\mu(n)$ of permutations of reduced-cycle type μ in the symmetric function $f(\Xi_n)$.

Theorem 8 (Matsumoto and Novak 2009). *Let* μ *be a fixed partition. Then:*

1. *The map* $f \mapsto G_\mu(f,n)$ *is a linear function on* Λ *taking values in the polynomials* $\mathbb{C}[n]$.
2. $G_\mu(f,n)$ *vanishes when* $\deg f < |\mu|$.
3. *When* $\deg f = |\mu|$, *the coefficient* $G_\mu(f) = G_\mu(f,n)$ *is independent of* n.

Corollary 9 (Stanley 2009, Olshanski 2009). *Let* λ *be a random partition of* n *distributed according to the Plancherel measure* $\mathfrak{P}_n$. *Let* A_λ *denote the alphabet of contents of* λ. *Then, for any symmetric function* $f \in \Lambda$, *the expected value* $\langle f(A_\lambda) \rangle_{\mathfrak{P}_n}$ *is a polynomial function of* n.

Let $m_\lambda \in \Lambda$ be the monomial symmetric function of type λ, and introduce the notation $L^\lambda_\mu(n) = G_\mu(m_\lambda, n)$.

Theorem 10 (Matsumoto and Novak 2009)**.** *With $L^\lambda_\mu(n)$ as above, we have the following:*

1. *$L^\lambda_\mu(n)$ is a polynomial function of n.*

2. *$L^\lambda_\mu(n)$ vanishes unless $|\mu| \leq |\lambda|$.*

3. *$L^\lambda_\mu(n)$ vanishes unless $|\mu| \equiv |\lambda| \mod 2$.*

4. *When $|\mu| = |\lambda|$, the n-independent coefficients $L^\lambda_\mu = L^\lambda_\mu(n)$ are given explicitly by*

$$L^\lambda_\mu = \sum_{(\lambda^{(1)},\ldots,\lambda^{(\ell(\mu))}) \in \mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} \mathrm{RC}(\lambda^{(i)}), \tag{1.1}$$

where the sum runs over refinement sequences of partitions

$$\mathfrak{R}(\lambda, \mu) = \{(\lambda^{(1)}, \ldots, \lambda^{(\ell(\mu))}) : \lambda^{(i)} \vdash \mu_i \text{ and } \lambda = \lambda^{(1)} \cup \cdots \cup \lambda^{(\ell(\mu))}\} \tag{1.2}$$

and $\mathrm{RC}(\lambda)$ denotes the refined Catalan number of type λ.

Let $h_k \in \Lambda$ be the complete homogeneous symmetric function of degree k, and introduce the notation $F^k_\mu(n) = G_\mu(h_k, n)$. By definition $h_k = \sum_{\lambda \vdash k} m_\lambda$, and thus $F^k_\mu(n) = \sum_{\lambda \vdash k} L^\lambda_\mu(n)$. By the previous theorem, non-zero coefficients are of the form $F^{|\mu|+2g}_\mu(n)$ for $g \geq 0$.

Let Cat_m denote the Catalan number

$$\mathrm{Cat}_m = \frac{1}{m+1}\binom{2m}{m},$$

and let $T(m, n)$ denote the Carlitz-Riordan central factorial number

$$T(m, n) = 2\sum_{k=1}^{n} \frac{k^{2m}(-1)^{n-k}}{(n-k)!(n+k)!}.$$

Theorem 11 (Matsumoto and Novak 2009). *The following exact formulas hold:*

1. *For any partition* μ *with* $\mathrm{wt}(\mu) \leq n$,

$$F_{\mu}^{|\mu|}(n) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i}.$$

2. *For any non-negative integer* $g \geq 0$,

$$F_{(n-1)}^{n-1+2g}(n) = \mathrm{Cat}_{n-1} T(n-1+g, n-1).$$

1.2.5 Correlation functions of the CUE and JM elements

Recall the above notation: $F_{\mu}^{k}(n) = G_{\mu}(h_k, n)$ and non-zero coefficients are of the form $F_{\mu}^{|\mu|+2g}(n)$ for $g \geq 0$.

Let $\mathcal{U}(N)$ be the compact group of $N \times N$ complex unitary matrices equipped with normalized Haar measure dU. We consider integrals of the form

$$\left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \int_{\mathcal{U}(N)} \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} dU,$$

where the u_{ij} are matrix elements, $1 \leq n \leq N$, and $\pi \in S(n)$ is a permutation. These integrals define Collins' *Weingarten function.*

Theorem 12 (Matsumoto and Novak 2009). *Let* $1 \leq n \leq N$ *and let* $\pi \in S(n)$ *be a permutation of reduced cycle-type* μ. *Then*

$$(-1)^{|\mu|} N^{n+|\mu|} \left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \sum_{g \geq 0} \frac{F_{\mu}^{|\mu|+2g(n)}}{N^{2g}}.$$

Corollary 13 (Collins 2003). *Let* $\mathrm{Cat}_k = \frac{1}{k+1}\binom{2k}{k}$ *be the Catalan number. With hypotheses as above,*

$$(-1)^{|\mu|} N^{n+|\mu|} \left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i} + O(\frac{1}{N^2}).$$

Corollary 14 (Collins 2003)**.** *Let $\gamma \in S(n)$ be the full cycle $\gamma = (1, 2, \ldots, n)$. With hypotheses as above,*

$$\left\langle \prod_{k=1}^{n} u_{kk}\overline{u_{k\gamma(k)}} \right\rangle_N = \frac{(-1)^{n-1}\,\mathrm{Cat}_{n-1}}{N(N^2-1^2)\ldots(N^2-(n-1)^2)}.$$

Chapter 2

Increasing Subsequences

2.1 Monotone subsequences in permutations

The study of monotone subsequences in permutations was initiated by Erdős and Szekeres in their now classic 1935 paper [ES35], wherein the following definition was formulated.

Definition. *A permutation σ from the symmetric group $S(N)$ is said to have an* increasing subsequence *of length k if there exist indices*

$$1 \leq i_1 < i_2 < \cdots < i_k \leq N$$

such that

$$\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k).$$

Similarly, σ has a decreasing subsequence *of length k if the above holds with the second set of inequalities reversed.*

If σ has either an increasing or decreasing subsequence of length k, it is said to have a monotone subsequence *of length k.*

Example 15. Consider the following permutation from $S(9)$:

$$4\ \boxed{1}\ 9\ 3\ \boxed{2}\ 7\ \boxed{6}\ \boxed{8}\ 5.$$

The elements of an increasing subsequence of length four are boxed, and there is no longer increasing subsequence. This permutation can be sorted in $9 - 4 = 5$ operations, shown below. The element which has been moved is underlined in each step.

$$4\ \boxed{1}\ 9\ \boxed{2}\ \underline{3}\ 7\ \boxed{6}\ \boxed{8}\ 5$$

$$\boxed{1}\ 9\ \boxed{2}\ 3\ \underline{4}\ 7\ \boxed{6}\ \boxed{8}\ 5$$

$$\boxed{1}\ 9\ \boxed{2}\ 3\ 4\ \underline{5}\ 7\ \boxed{6}\ \boxed{8}$$

$$\boxed{1}\ 9\ \boxed{2}\ 3\ 4\ 5\ \boxed{6}\ \underline{7}\ \boxed{8}$$

$$\boxed{1}\ \boxed{2}\ 3\ 4\ 5\ \boxed{6}\ 7\ \boxed{8}\ \underline{9}.$$

2.2 Ramsey Theory

Looking for monotone subsequences in a large random permutation is a particular instance of looking for order in a large random structure. The problem of detecting the existence of ordered substructures in large random structures is of fundamental importance in mathematics and the natural sciences. It has many incarnations and arises in a wide variety of contexts. In mathematics, the body of results which treat this issue is known collectively as *Ramsey theory*. While there is no precise definition of what constitutes a Ramsey-type theorem, the subject may be characterized as the set of those results which reinforce the mantra, coined by Motzkin, that "complete disorder is impossible."

The field was initiated in 1930 by Ramsey [Ram30], who proved the following graph-theoretical result (Ramsey's theorem was independently rediscovered in a slightly different form in [ES35]).

Theorem 16 (Ramsey's Theorem)**.** *Let (p, q) be an arbitrary pair of positive integers. There exists a least positive integer $R(p, q)$ such that any edge bicolouring of the complete graph $K_{R(p,q)}$ necessarily contains either a monochromatic copy of K_p or a monochromatic copy of K_q.*

The numbers $R(p, q)$ whose existence is guaranteed by Theorem 16 are called *Ramsey numbers*. Borrowing a phrase from statistical physics (see e.g. [Bax82]), one imagines the Ramsey number $R(p, q)$ as delineating a "phase transition" in the edge colouring problem. For example, it is possible to construct edge bicolourings of K_2, K_3, K_4, K_5 which have no monochromatic K_3-subgraph, see figure (2.1). However, any arbitrary edge bicolouring of $K_6, K_7, K_8, \ldots$ necessarily contains a monochromatic copy of K_3. Thus $R(3, 3) = 6$, marking the point at which the possibility of a monochromatic triangle crystallizes into a certainty.

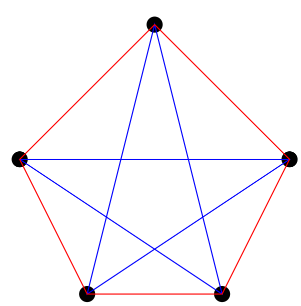


Figure 2.1: An edge bicolouring of K_5 which contains no monochromatic K_3.

Ramsey numbers for small (p, q) are tabulated in Figure 2.2. The computation

	1	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	9	14	18
4	1	4	9	18	25	?
5	1	5	14	25	?	?
6	1	6	18	?	?	?

Figure 2.2: Ramsey numbers, known and unknown

of Ramsey numbers is a notoriously difficult problem; the question marks in Figure 2.2 indicate Ramsey numbers whose value is presently unknown (data excerpted from [Rad06]).

Another famous Ramsey-type result is Szemerédi's Theorem. For a random subset of the natural numbers, the appropriate notion of a structured subset is an arithmetic progression. Erdős and Turan conjectured, and Szemerédi proved [Sze75], that every sufficiently dense set of positive integers contains a relatively long arithmetic progression.

Theorem 17 (Szemerédi's Theorem)**.** *For any positive integer k and any positive real number $\delta > 0$, there exists a least positive integer $S = S(k, \delta)$ such that any subset of $\{1, 2, \ldots, S\}$ of size $\lceil \delta S \rceil$ contains an arithmetic progression of length k.*

As with Ramsey numbers, the problem of determining the Szemerédi numbers $S(k, \delta)$ appears to be very difficult, see [Gow01, GT08].

An anologue of Szemerédi's theorem in which the set of all positive integers is replaced by the set of prime numbers was recently obtained by Green and Tao [GT08], thus settling (in the affirmative) the famous folklore conjecture that there exist arbitrarily long arithmetic progressions of prime numbers. The ultimate problem of this type seems to be Erdős' conjecture that if X is a set of natural numbers such that $\sum_{x \in X} x^{-1}$ diverges, then X contains arbitrarily long arithmetic progressions.

2.3 The Erdős-Szekeres Theorem

The Erdős-Szekeres Theorem is a Ramsey-type result for permutations.

Theorem 18 (Erdős-Szekeres Theorem)**.** *Let (p, q) be an arbitrary pair of positive integers. Every permutation of the numbers $1, 2, \ldots, (p-1)(q-1)+1$ necessarily contains either a decreasing subsequence of length p or an increasing subsequence of length q.*

Proof. Following [AZ04], we present the original proof due to Erdős and Szekeres. This argument makes use of the Pigeonhole Principle: if A and B are two finite sets of cardinalities a and b, respectively, and $a > b$, then for any function $f : A \to B$ there exists $y \in B$ such that

$$|f^{-1}(y)| \geq \left\lceil \frac{a}{b} \right\rceil.$$

A second, more transparent proof which replaces the Pigeonhole Principle with the Robinson-Schensted-Knuth correspondence is given in Chapter 3.

First note that if either p or q is equal to 1, then the claim is trivial. We therefore assume $\min(p, q) \geq 2$.

In order to apply the Pigeonhole Principle to the problem at hand, let σ be an arbitrary permutation of order $(p-1)(q-1)+1$ represented in one-line notation

$$\sigma = a_1 a_2 \ldots a_{(p-1)(q-1)+1},$$

i.e. $a_i = \sigma(i)$. Associate to each a_i the number d_i, which is the length of the longest decreasing subsequence starting at a_i (note that a decreasing subsequence of maximal length need not be unique, so really we should say "*a* longest decreasing subsequence"). If $d_i \geq p$ for some $1 \leq i \leq (p-1)(q-1)+1$ we are done, so suppose that $d_i < p$ for all $1 \leq i \leq (p-1)(q-1)+1$. Then the function $f : a_i \mapsto d_i$ maps

$$\{1, \ldots, (p-1)(q-1)+1\} \to \{1, \ldots, p-1\}.$$

Hence, by the Pigeonhole Principle, there exists $s \in \{1, \ldots, p-1\}$ such that $a_i \mapsto s$ for some

$$\left\lceil \frac{(p-1)(q-1)+1}{p-1} \right\rceil = q$$

numbers $a_{i_1}, \ldots, a_{i_q}$, $i_1 < \cdots < i_q$. Now consider a_{i_1} and a_{i_2}. If $a_{i_1} > a_{i_2}$, then we would obtain a decreasing subsequence of length $s+1$ by prepending a_{i_1} to the decreasing subsequence of length s starting at a_{i_2}. This cannot be, since by definition of the function f the length of the longest decreasing subsequence starting at a_{i_2} is s. Hence $a_{i_1} < a_{i_2}$. Applying this argument to each consecutive pair $(a_{i_2}, a_{i_3}), (a_{i_3}, a_{i_4}), \ldots$ we see that

$$a_{i_1} < a_{i_2} < \cdots < a_{i_q},$$

which is an increasing subsequence of length q in σ. □

The numbers $ES(p,q) = (p-1)(q-1)+1$ are the permutation analogue of the Ramsey numbers $R(p,q)$ and the Szemerédi numbers $S(k,\delta)$. In contrast to the Ramsey and Szemerédi numbers, they are given by a simple quadratic function of p and q. Indeed, the Erdős-Szekeres theorem is significantly easier than its counterparts for graphs and sets of integers. Once again borrowing from the vocabulary of statistical mechanics, we conclude that our chosen measure of order in permutations has a certain "exactly solvable" quality not present in other contexts. Emboldened by this, we pursue finer questions regarding the structure of the monotone subsequences in a random permutation.

Chapter 3

The RSK Correspondence

3.1 Introduction

The proof of the Erdős-Szekeres theorem given in Chapter 2 required a fair bit of the ingenuity for which its progenitors are famous. It is not clear if or how such an argument can be adjusted to tackle finer probabalistic questions which go beyond existence statements. What is needed is an encoding of permutations which elucidates the structure of their monotone subsequences.

This need is met by the Robinson-Schensted-Knuth (RSK) correspondence [Rob38, Sch61, Knu73], which links the longest increasing subsequence problem to Young diagrams and tableaux. These latter combinatorial objects are ubiquitous in mathematics, occurring in such diverse fields as representation theory, algebraic geometry, random matrix theory, and mathematical physics (see e.g. the short survey [Yon07]). In our setting, the importance of Young tableaux stems from the fact that they furnish a flexible reformulation of the increasing subsequence problem. This reformulation indicates that the problem lies at the confluence of several streams of mathematics.

3.2 Partitions

Given a natural number N, a *partition* of N is a decomposition of N into a sum of smaller natural numbers (called the "parts" of the partition), without regard to order to the order of the parts.

Example 19. The partitions of 5 are:

$$\begin{aligned}5 &= 5\\ &= 4+1\\ &= 3+2\\ &= 3+1+1\\ &= 2+2+1\\ &= 2+1+1+1\\ &= 1+1+1+1+1.\end{aligned}$$

Note that the representation of a partition is not necessarily unique; for instance $4+1$ and $1+4$ are the same partition of 5. We therefore fix the convention that partitions will always be written with their parts listed in weakly decreasing order. It is also convenient to consider the existence of an "empty partition" $\emptyset$, which is thought of as the unique partition of $N=0$. The set of partitions of N is denoted $\mathbb{Y}(N)$. A generic partition is denoted λ, with $\ell(\lambda)$ being the number of parts. Depending on context, we will use any of the usual notations $\lambda \in \mathbb{Y}(N)$, $|\lambda| = N$, $\lambda \vdash N$ to indicate that λ is a partition of N.

The study of partitions has a long history. In the eighteenth century, Euler conducted extensive investigations of the *partition function* $p(N) := |\mathbb{Y}(N)|$ which counts the number of partitions of N. The term "partition function" occurs in various other

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$p(N)$	1	2	3	5	7	11	15	22	30

Figure 3.1: Initial values of the partition function.

contexts in mathematics and mathematical physics. All of these are elaborations of the classical partition function $p(N)$ in the sense that they refer to a (possibly weighted) sum or integral over all configurations.

There is no known closed form expression for $p(N)$. However, its values may be computed from Euler's generating function

$$\sum_{N\geq 0} p(N)x^N = \prod_{i\geq 1} \frac{1}{1-x^i}. \tag{3.1}$$

The initial values of the partition function are tabulated in Figure 3.2.

Despite its classical roots, the study of partitions remains an active area of research. In number theory, the modern study of partitions focuses on arithmetic properties of Euler's partition function. The partition function was tabulated by MacMahon up to $N = 200$ around 1914. MacMahon's tables were studied by Ramanujan, who noticed the curious congruence phenomena

$$p(5k+4) \equiv 0 \mod 5$$

$$p(7k+5) \equiv 0 \mod 7$$

$$p(11k+6) \equiv 0 \mod 11.$$

Ramanujan [Ram21] proved the first two of these. The deeper meaning of Ramanujan's congruences is a concern of number theory. A satisfactory explanation was found as recently as 2000, when Ono [Ono00] used the theory of mock theta functions to prove the existence of similar congruences for every modulus coprime to six.

A different strand of partition theory studies the statistical properties of large random partitions. This area of asymptotic combinatorics was initiated by the Russian school around Vershik and Kerov in the late 1970's, and has since drawn the interest of many other researchers. The basic philosophy [Ver96] involves adopting an approach inspired by statistical physics: roughly, one considers the set $\mathbb{Y}(N)$ of partitions of N as the canonical ensemble, equipped with a Boltzmann factor derived from an appropriately chosen "energy" functional on partitions. In Chapter 4, the analogy with statistical mechanics will naturally emerge from our efforts to determine the asymptotic number of permutations with decreasing subsequence length bounded by a fixed number d (the "single scaling limit").

In statistical mechanics, the first task is always to compute the partition function. This was done for partitions (asymptotically) by Hardy and Ramanujan [HR17] in 1917. They proved that

$$p(N) \sim \frac{e^{\pi\sqrt{\frac{2N}{3}}}}{\sqrt{48}N},\ N \to \infty, \tag{3.2}$$

which precisely means that

$$\lim_{N\to\infty} \frac{N}{e^{\pi\sqrt{\frac{2N}{3}}}} p(N) = \sqrt{48}.$$

Hardy and Ramanujan extracted the asymptotics (3.2) from Euler's generating function (3.1) via a complex-analytic argument, see [New91]. The techniques developed in their calculation evolved into the Hardy-Littlewood *circle method*, which until today remains one of the most frequently used techniques in analytic number theory. Interestingly, a precursor to Szeméredi's theorem which deals with arithmetic progressions of length three was obtained by Roth [Rot53] using the circle method.

Hardy and Ramanujan's derivation of the $p(N)$ asymptotics is reminiscent of Hadamard and de la Valée Poussin's 1896 analytic proof of the Prime Number Theorem, and was to some extent inspired by this earlier result. Let $\pi(N)$ denote the number of primes $\leq N$. As with $p(N)$, no exact expression for $\pi(N)$ is known. The Prime Number Theorem asserts that

$$\lim_{N\to\infty} \frac{\log N}{N}\pi(N) = 1. \tag{3.3}$$

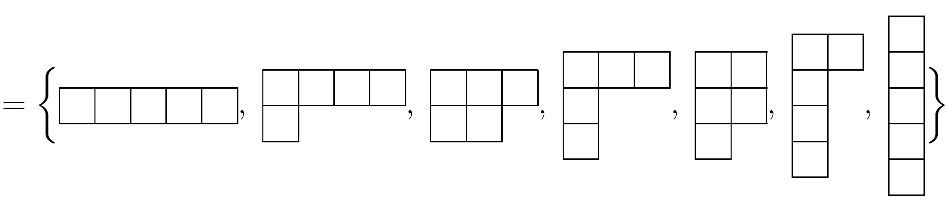
Hademard and de la Vallée Poussin extracted this from the Euler product identity[1]

$$\sum_{N\geq 0} \frac{1}{N^x} = \prod_{p\geq 2} \left(\frac{1}{1-p^{-x}}\right)^{[p \text{ prime}]} \tag{3.4}$$

via a complex-analytic argument, see [New91].

In many situations it is useful to represent partitions diagrammatically. This technique goes back to Young, and Sylvester and Ferrers. Given a partition $N = \lambda_1 + \lambda_2 + \dots$, the corresponding *Young diagram* $\lambda = (\lambda_1, \lambda_2, \dots)$ is a top-left justified array of unit squares, called "cells," with λ_1 cells in the first row, λ_2 squares in the second row, etc. We identify a partition with its diagram, and make no distinction between the two.

Example 20. For $N = 5$, we have

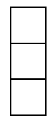
$\mathbb{Y}(5) =$ 

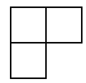
There is an obvious relationship between permutations and partitions coming from cycle structure. Recall that every permutation σ in $S(N)$ can be factored uniquely as a product of disjoint cycles (disregarding the order of the cycles). The lengths of the cycles in the factorization of σ define a partition of N, called the *cycle-type* of σ.

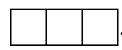
[1] We are using Iverson's bracket notation: $[P]$ returns the value 1 if P is true, and 0 if P is false. For example, the Kronecker delta δ_{ij} becomes $[i = j]$ in Iverson's notation. The eponymous bracket notation was introduced by Kenneth E. Iverson [Ive62] in his development of the APL programming language, and later popularized by Knuth [Knu92, GKP94]. The use of Iverson's notation seems particularly appropriate in this thesis, since Iverson was a graduate of Queen's University.

Since two permutations are conjugate if and only if they have the same cycle type, the conjugacy classes of $S(N)$ are naturally labelled by the partitions of N.

Example 21. For $N = 3$, the labeling of conjugacy classes of $S(3)$ by Young diagrams is the following:

$\{(1)(2)(3)\} \longleftrightarrow$ 

$\{(12)(3), (13)(2), (23)(1)\} \longleftrightarrow$ 

$\{(123), (132)\} \longleftrightarrow$ .

3.3 Young's lattice

A much subtler relationship between permutations and partitions was discovered by Robinson [Rob38] and Schensted [Sch61], and later clarified and extended by Knuth [Knu73]. In order to describe their result, we need to change our perspective slightly.

Instead of considering the set $\mathbb{Y}(N)$ of partitions of a fixed number N, let us group all partitions together into a single set

$$\mathbb{Y} = \bigcup_{N \geq 0} \mathbb{Y}(N).$$

This is similar to the passage from canonical ensemble to grand canonical ensemble in statistical mechanics. The diagrammatic representation of partitions suggests a natural partial order $\subseteq$ on $\mathbb{Y}$, obtained by declaring $\mu \subseteq \lambda$ if and only if the diagram of μ is contained in the diagram of λ when we align their top left corners as in Figure 3.2.

The containment order makes $\mathbb{Y}$ into a lattice known as *Young's lattice*. The first few levels of the Hasse graph of Young's lattice, which is known as the *Young graph*

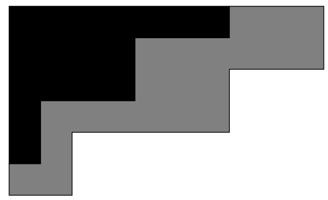


Figure 3.2: Ordering Young diagrams by inclusion.

and also denoted $\mathbb{Y}$, are shown in Figure 3.3 below.

Figure 3.3: The initial levels of the Young graph.

The Young graph is the prototypical example of a *graded graph*.

Definition. *A* graded graph *is a pair* (G, r) *consisting of a locally finite graph* G *together with a function* $r : G \to \mathbb{Z}$ *defined on the vertices of* G*, called a* rank function*. The incidence relation and rank function on* G *are required to be compatible in the sense that if* u, v *are adjacent vertices of* G*, then either* $r(v) = r(u) + 1$ *(denoted* $u \nearrow v$*) or* $r(v) = r(u) - 1$ *(denoted* $u \searrow v$*).*

The Young graph is a simple, connected graded graph. The rank function on the

Young graph is defined by

$$|\lambda| = \text{no. of cells in } \lambda,$$

and evidently all vertices of $\mathbb{Y}$ have non-negative rank.

Definition. *Given a partition $\lambda \in \mathbb{Y}$, a* standard Young tableau *of shape λ is a walk $\emptyset \to \lambda$ on $\mathbb{Y}$ in which each step consists of the addition of a single cell, viz. a walk of the form*

$$\emptyset \nearrow \cdots \nearrow \lambda.$$

The the set of all standard Young tableaux of shape λ is denoted $\operatorname{Tab}\lambda$. One may consider λ as a simple multi-cellular organism, in which case $\operatorname{Tab}\lambda$ represents the set of all possible growth histories of λ. It is often convenient to represent a standard Young tableau of shape $\lambda \in \mathbb{Y}(N)$ as a labelling of the cells of λ with the numbers $1, \ldots, N$ which is increasing along rows and columns.

Example 22. The walk

$$\emptyset \nearrow \ydiagram{1} \nearrow \ydiagram{1,1} \nearrow \ydiagram{2,1}$$

is a standard Young tableau of shape $\ydiagram{2,1}$ which corresponds to the labelling

$$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}.$$

There is one more standard Young tableau of shape $\ydiagram{2,1}$, namely

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}.$$

Thus

$$\operatorname{Tab}\ydiagram{2,1} = \left\{ \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \right\}.$$

Graded graphs occur in diverse situations, most notably representation theory, where they arise as *branching graphs* (a.k.a *Bratteli diagrams*). Consider a tower of finite groups

$$\{1\} = G(0) \subset G(1) \subset \cdots \subset G(N) \subset \ldots. \tag{3.5}$$

Let $G(N)^\wedge$ denote the set of (equivalence classes of) irreducible complex finite-dimensional representations of $G(N)$. The branching graph $\mathbb{B}$ has vertex set

$$\mathbb{B} = \bigcup_{N \geq 0} G(N)^\wedge. \tag{3.6}$$

Two vertices μ, λ of $\mathbb{B}$ are joined by k edges if and only if $\mu \in G(N-1)^\wedge$, $\lambda \in G(N)^\wedge$, and the multiplicity of μ in the restriction of λ to $G(N-1)$ is k. If the tower (3.5) has simple branching (i.e. the restriction of an irreducible representation of $G(N)$ to $G(N-1)$ splits as a direct sum of irreducibles with multiplicities either 0 or 1), then the branching graph $\mathbb{B}$ is a simple graded graph.

The tower of symmetric groups

$$S(0) \subset S(1) \subset \cdots \subset S(N) \subset \ldots, \tag{3.7}$$

where $S(N-1) \subset S(N)$ means that $S(N-1)$ is canonically embedded in $S(N)$ as the subgroup of permutations which fix the element N, has simple branching. Since the irreducible representations of any finite group are in bijection with its conjugacy classes, and since the conjugacy classes of $S(N)$ are labelled by $\mathbb{Y}(N)$, we have $S(N)^\wedge = \mathbb{Y}(N)$. Indeed, the branching graph of the symmetric group tower is precisely the Young graph $\mathbb{Y}$.

A classical result in the representation theory of the symmetric groups is that the dimension of an irreducible $S(N)$-module V^λ labelled by the Young diagram λ is precisely equal to $|\operatorname{Tab}\lambda|$, the number of standard Young tableaux of shape λ. Hence we denote $\dim\lambda := |\operatorname{Tab}\lambda|$. Thus the dimension of the irrep of $S(N)$ labelled by λ has a nice combinatorial description, and consequently several nice formulas for $\dim\lambda$ are available. The *Frobenius formula* computes $\dim\lambda$ in terms of the row-lengths of λ :

$$\dim\lambda = \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)}(\lambda_i+\ell(\lambda)-i)!}\prod_{1\leq i<j\leq\ell(\lambda)}(\lambda_i-\lambda_j+j-i). \tag{3.8}$$

Another useful formula, due to Frame, Robinson, and Thrall [FRT54], computes $\dim\lambda$ in terms of a combinatorial parameter on partitions known as *hook-length.* The hook-length of a cell $\Box\in\lambda$ is by definition the number of cells to the right of $\Box$, plus the number of cells beneath $\Box$, plus one, see Figure 3.4. The *hook-length formula* is

$$\dim\lambda = \frac{|\lambda|!}{\prod_{\Box\in\lambda}h(\Box)}. \tag{3.9}$$

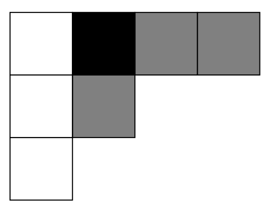


Figure 3.4: A cell of hook-length four.

A classical theorem of Burnside [Bur11] asserts that the sum of the squares of the dimensions of the irreducible representations of any finite group is equal to the cardinality of the group. Thus,

$$\sum_{\lambda\in\mathbb{Y}(N)}(\dim\lambda)^2 = N!. \tag{3.10}$$

Given the very concrete combinatorial interpretation of $\dim\lambda$ just discussed, Burnside's theorem suggests the existence of an underlying bijection

$$S(N)\longleftrightarrow\bigcup_{\lambda\in\mathbb{Y}(N)}(\operatorname{Tab}\lambda)\times(\operatorname{Tab}\lambda)$$

between permutations and pairs of standard Young tableaux of the same shape. This bijection is precisely the Robinson-Schensted-Knuth (RSK) correspondence, which will be described presently.

3.4 $\mathbb{Y}$ as a differential poset

Before moving on to the description of the RSK correspondence, let us give a purely combinatorial proof of Burnside's identity in which representation theory is bypassed in favour of an argument which relies on the intrinsic structure of the Young graph. This argument is due to Stanley [Sta88].

Associated to any simple graded graph (G, r) is a pair of linear operators $L, R \in \operatorname{End} \mathbb{C}[G]$, where $\mathbb{C}[G]$ is the free $\mathbb{C}$-vector space spanned by the vertices of G. These operators are called the *lowering* and *raising* operators, and are defined by

$$L(v) := \sum_{v \searrow u} u \tag{3.11}$$

$$R(v) := \sum_{w \nearrow v} v \tag{3.12}$$

and linear extension. The number of walks of the form

$$u \underbrace{\nearrow \nearrow \cdots \nearrow}_{a_1} \underbrace{\searrow \searrow \cdots \searrow}_{b_1} \cdots \underbrace{\nearrow \nearrow \cdots \nearrow}_{a_k} \underbrace{\searrow \searrow \cdots \searrow}_{b_k} v$$

from u to v on G which take a_1 $\nearrow$-steps, followed by b_1 $\searrow$-steps, …, followed by a_k $\nearrow$-steps, followed by b_k $\searrow$-steps is equal to

$$[v]L^{b_k}R^{a_k}\dots L^{b_1}R^{a_1}(v),$$

where $[\cdot]$ is the "coefficient of" functional on $\mathbb{C}[G]$.

A pair of standard Young tableaux of shape λ is an ordered pair

$$(\emptyset \nearrow \cdots \nearrow \lambda, \emptyset \nearrow \cdots \nearrow \lambda)$$

of walks on $\mathbb{Y}$. By reversing the second walk we can instead view a pair of standard Young tableaux of shape λ as a closed walk on $\mathbb{Y}$ of the form

$$\emptyset \nearrow \cdots \nearrow \lambda \searrow \cdots \searrow \emptyset,$$

see Figure (3.5). Thus in order to prove the Burnside identity (3.10), it suffices to

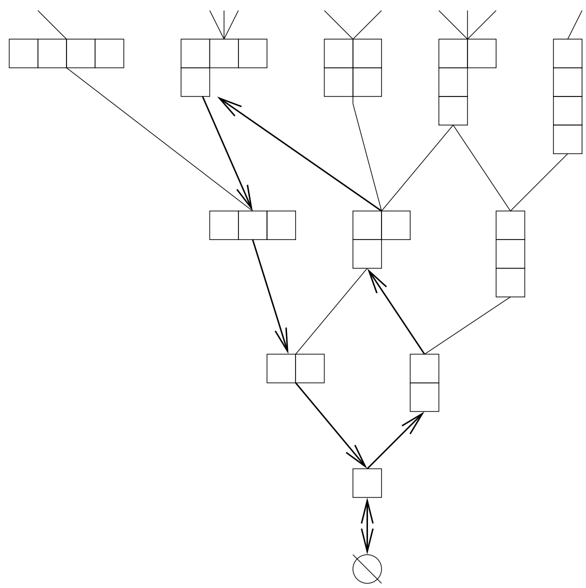


Figure 3.5: A closed walk equivalent to a pair of standard Young tableaux.

verify that

$$[\emptyset]L^N R^N(\emptyset) = N!. \tag{3.13}$$

This can be done using the fact that the raising and lowering operators L, R on $\mathbb{Y}$ satisfy the Heisenberg commutation relation.

Theorem 23 (Stanley, [Sta88])**.** *The raising and lowering operators on the Young graph satisfy the commutation relation $LR - RL = I$.*

Proof. This proof is due to Stanley.

Let μ be an arbitrary Young diagram. We compute the coefficients $[\lambda]LR(\mu)$ and $[\lambda]RL(\mu)$ for general λ.

Suppose first that $\lambda \neq \mu$. If $[\lambda]LR(\mu) \neq 0$, then λ can be obtained by in a unique way by adding a cell s to μ and deleting a (necessarily different) cell t from μ. Thus $[\lambda]LR(\mu) = 1$. But then λ can be obtained from μ in a unique way by deleting and then adding a cell, namely first delete t and then add s. Thus $[\lambda]RL(\mu) = 1$ also. On the other hand, if $[\lambda]LR(\mu) = 0$, then λ cannot be obtained from μ by first adding a cell and then deleting a cell, and hence it cannot be obtained from μ by first deleting a cell and then adding a cell, whence $[\lambda]RL(\mu) = 0$ also. Thus $[\lambda](LR - RL)(\mu) = 0$ for any $\lambda \neq \mu$.

Now suppose that $\lambda = \mu$. Let r be the number distinct row-lengths occurring in μ. Then there are $r + 1$ ways to obtain μ from μ by first adding a cell and then deleting that same cell, and only r ways to obtain μ from μ by first deleting and then adding a cell, as shown schematically in Figure 3.6. Thus $[\mu]LR(\mu) = r + 1$ and $[\mu]RL(\mu) = r$, whence $[\mu](LR - RL)(\mu) = 1$.

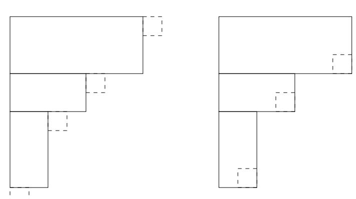


Figure 3.6: The four ways to add a cell to and three ways to delete a cell from a multirectangular diagram with $r = 3$.

It follows that $(LR - RL)(\mu) = \mu$ for any $\mu \in \mathbb{Y}$. □

The unital $\mathbb{C}$-algebra generated by two non-commuting indeterminates u, v subject only to the relation $uv - vu = 1$ is known as the *Weyl algebra*. For interest's sake, we mention that the automorphism group of the Weyl algebra is the subject of a famous conjecture of Dixmier. Dixmier's conjecture asserts that every endomorphism of the Weyl algebra is actually an automorphism. The Dixmier conjecture is closely related to an even more famous open problem, the Jacobian conjecture, which claims that every polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ with non-vanishing Jacobian has a polynomial inverse. Further details and references can be found in [BKK05].

Theorem 23 constructs a Fock space representation of the Weyl algebra on $\mathbb{C}[\mathbb{Y}]$ by sending u to L and v to R. One obtains a more traditional Fock space representation of the Weyl algebra on the polynomial ring $\mathbb{C}[x]$ by mapping u to the derivative operator D_x and v to the "multiplication by x" operator M_x. Because of this, Young's lattice is referred to as a *differential poset*. This terminology was introduced in [Sta88], wherein other examples of differential posets are constructed. Note that the vacuum space in the first representation is $\mathbb{C}\emptyset$, while the vacuum space in the second representation is $\mathbb{C}1$, the constant polynomials. Thus

$$[\emptyset]L^N R^N(\emptyset) = [1]D_x^N M_x^N(1) = [1]D_x^N(x^N) = N!,$$

which proves the Burnside formula for $S(N)$.

3.5 The RSK correspondence

We now construct the claimed bijection

$$S(N) \xrightarrow{\text{RSK}} \bigcup_{\lambda \in \mathbb{Y}(N)} (\operatorname{Tab}\lambda) \times (\operatorname{Tab}\lambda) \tag{3.14}$$

between permutations and pairs of standard Young tableaux. We will follow the poset-theoretic development due to Greene and Fomin (as surveyed in [BF01]), which has some advantages over the more traditional “bumping and sliding” algorithmic description (see [Knu73]).

Let X be a poset. A *chain* in X is a totally ordered subset of X, and an *antichain* in X is a subset in which no two elements are comparable. Let $\mathcal{P}$ denote the collection of all finite posets. Greene and Fomin (independently) discovered a remarkable map

$$\lambda : \mathcal{P} \to \mathbb{Y}$$

which assigns to each $X \in \mathcal{P}$ a Young diagram $\lambda(X) \in \mathbb{Y}$ using the chain-antichain structure of X.

Suppose $|X| = N$. Let c_k, $k \geq 0$ denote the maximal cardinality of a union of k (not necessarily distinct) chains in X. By definition, $c_0 = 0$. Similarly, let a_k, $k \geq 0$ denote the maximal cardinality of a union of k antichains in X. Define

$$\lambda_k := c_k - c_{k-1},\ k \geq 1 \tag{3.15}$$

and

$$\mu_k := a_k - a_{k-1},\ k \geq 1. \tag{3.16}$$

Theorem 24 (Duality theorem for finite posets)**.** *The sequences* $\lambda(X) = (\lambda_1, \lambda_2, \dots)$ *and* $\mu(X) = (\mu_1, \mu_2, \dots)$ *are weakly decreasing and form conjugate partitions of* N*.*

The term “conjugate” means that the diagrams of μ and λ are the transpose of one another, see Figure 3.7.

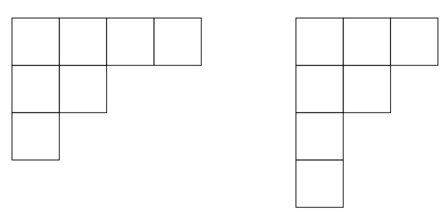


Figure 3.7: Conjugate Young diagrams.

The map $X \mapsto \lambda(X)$ is called the Greene-Fomin map. It is a many-to-one map; indeed, the number $|\mathcal{P}(N)|$ of posets with N elements grows like [S2]

$$|\mathcal{P}(N)| \sim Ce^{\frac{\log 2}{4}N^2+(\frac{3}{2}+\log 2)N-N\log N-\log N} \tag{3.17}$$

as $N \to \infty$, where C is the constant

$$C = \frac{2}{\pi}\sum_{i\geq 0}\frac{1}{2^{i(i+1)}},$$

which is much faster than the sub-exponential growth of $p(N)$ given by the Hardy-Ramanujan formula.

The Greene-Fomin map has an important "functorial" property: if x is an extremal (i.e. maximal or minimal) element of X, then

$$\lambda(X - \{x\}) \subseteq \lambda(X).$$

This functoriality allows one to compute $\lambda(X)$ recursively by "growing" X according to an arbitrary linear extension (recall that a *linear extension* of a poset X of cardinality N is an order-preserving map $X \to [N]$, where $[N]$ is the N-chain $1 \leq \cdots \leq N$. Thus a linear extension of X is an extension of the partial order on X to a total order). Because of the functoriality of the Greene-Fomin map, each linear extension of X produces a standard Young tableau of shape $\lambda(X)$ (but not in a unique way — distinct linear extensions may produce the same tableau).

Example 25. Let us compute the shape $\lambda(X)$ of the poset X whose Hasse graph is given in Figure 3.8 using the linear extension depicted in Figure 3.9.

Functoriality of λ tells us that we can determine $\lambda(X)$ by growing X according to the given linear extension, a process which is depicted in Figure 3.10.

Let $\sigma \in S(N)$ be a permutation, and consider the poset X_σ whose points form

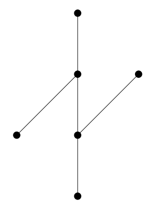


Figure 3.8: The Hasse graph of a finite poset.

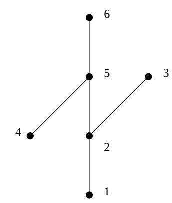


Figure 3.9: A linear extension.

the set

$$\{(i, \sigma(i)) : 1 \leq i \leq N\},$$

with partial order defined by

$$(i, \sigma(i)) \leq (j, \sigma(j)) \iff i \leq j \text{ and } \sigma(i) \leq \sigma(j).$$

Then, by construction, each chain in X_σ corresponds to an increasing subsequence in σ and similarly each antichain in X_σ corresponds to a decreasing subsequence in σ. Thus the length $\text{lis}(\sigma)$ of the longest increasing subsequence in σ is equal to the length of the first row of the Greene-Fomin image $\lambda(X_\sigma)$, while the length $\text{lds}(\sigma)$ of the longest decreasing subsequence in σ is equal to the number of rows in $\lambda(X_\sigma)$.

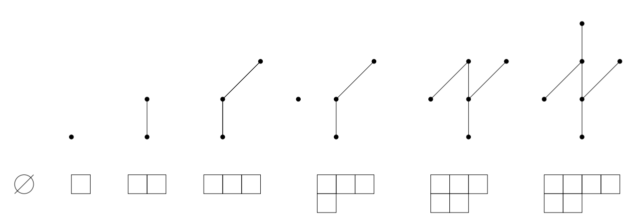


Figure 3.10: Growth of a poset.

Note that the Greene-Fomin image of X_σ has nothing to do with the cycle structure of σ, see Figure 3.11.

There are two canonical linear extensions of the permutation poset X_σ. First, we can extend the partial order on X_σ to a total order by declaring

$$(i, \sigma(i)) \leq (j, \sigma(j)) \iff i \leq j.$$

This is the WE-extension (west-to-east) of X_σ. Second, there is the SN-extension (south-to-north) of X_σ defined by

$$(i, \sigma(i)) \leq (j, \sigma(j)) \iff \sigma(i) \leq \sigma(j).$$

Let P_σ be the standard Young tableau obtained by growing X_σ according to the WE-extension, and let Q_σ be the standard Young tableau obtained by growing X_σ according to the SN-extension. The map

$$\mathrm{RSK} : S(N) \longrightarrow \bigcup_{N \in \mathbb{Y}(N)} (\mathrm{Tab}\,\lambda) \times (\mathrm{Tab}\,\lambda)$$

defined by

$$\mathrm{RSK}(\sigma) = (P_\sigma, Q_\sigma)$$

is the Robinson-Schensted-Knuth correspondence. It is in fact a bijection, though we have not actually proved this (though we have shown that the domain and range of RSK have the same cardinality). By construction, the RSK correspondence has the following three fundamental properties:

RSK 1. The length of the longest increasing subsequence in σ is equal to the length of the first row of $\lambda(X_\sigma)$.

RSK 2. The length of the longest decreasing subsequence in σ is equal to the number of rows in $\lambda(X_\sigma)$.

RSK 3. If $\mathrm{RSK}(\sigma) = (P_\sigma, Q_\sigma)$, then $\mathrm{RSK}(\sigma^{-1}) = (Q_\sigma, P_\sigma)$.

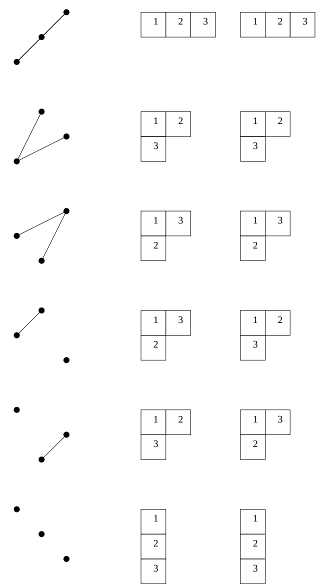


Figure 3.11: The RSK correspondence for $S(3)$. Hasse graphs of permutation posets are on the left.

The RSK correspondence for $S(3)$ is shown in Figure 3.11. Note that property (RSK 3), which is a direct consequence of the fact that the WE-extension of X_σ is the SN-extension of $X_{\sigma^{-1}}$ and *vice versa*, implies that $\mathrm{RSK}(\sigma) = (P_\sigma, P_\sigma)$ if and only if σ is an involution. In other words, the RSK correspondence also entails a bijection

$$\{\sigma \in S(N) : \sigma^2 = \mathrm{id}\} \longleftrightarrow \bigcup_{\lambda \in \mathbb{Y}(N)} \mathrm{Tab}\,\lambda$$

which inherits properties (RSK 1) and (RSK 2).

In summary: RSK operates by assigning to a permutation σ a pair of standard Young tableaux determined by the two canonical linear extensions of the permutation poset X_σ. For our purposes, the inner workings of RSK are not really important. What is important is that RSK gives us a formula for the number of permutations $\sigma \in S(N)$ with longest increasing subsequence of length exactly a and longest decreasing subsequence of length exactly b, namely

$$\sum_{\substack{\lambda \in \mathbb{Y}(N) \\ \lambda_1 = a \\ \ell(\lambda) = b}} (\dim \lambda)^\beta, \tag{3.18}$$

where $\beta = 2$ if we want to count unrestricted permutations and $\beta = 1$ if we want to count involutions.

3.6 Another proof of the Erdős-Szekeres theorem

As an illustration of the power of the RSK method, we give another (much simpler) proof of the the Erdős-Szekeres Theorem from Chapter 2.

Theorem 26. *If $\sigma \in S((p-1)(q-1)+1)$, then σ either has a decreasing subsequence of length p or an increasing subsequence of length q (or both).*

Proof. The Greene-Fomin image of the permutation poset X_σ is a Young diagram with $(p-1)(q-1)+1$ cells. Suppose that σ has neither a decreasing subsequence of length p nor an increasing subsequence of length q. Then the number of rows in $\lambda(X_\sigma)$ is at most $p-1$, and the length of the first row of $\lambda(X_\sigma)$ is at most $q-1$. Hence $\lambda(X_\sigma) \subseteq R(p-1, q-1)$, where $R(p-1, q-1)$ is the $(p-1) \times (q-1)$ rectangular Young diagram. This is a contradiction, since $|\lambda(X_\sigma)| = (p-1)(q-1)+1$ whereas $|R(p-1, q-1)| = (p-1)(q-1)$.

This argument also shows that the Erdős-Szekeres number $(p-1)(q-1)+1$ is sharp, since any pair of tableaux (P, Q) of common shape $R(p-1, q-1)$ corresponds to a permutation in $S((p-1)(q-1))$ with longest increasing subsequence of length $p-1$ and longest decreasing subsequence of length $q-1$. $\square$

Chapter 4

The Single Scaling Limit

4.1 Introduction

Recall that our stated reason for introducing RSK was to go beyond the existence theorems of Ramsey theory by investigating the finer probabalistic structure of monotone subsequences in random permutations. In this chapter we will give the first such application of RSK, and simultaneously observe the first tangible connection (both in this thesis and historically) between random permutations and random matrices.

Let

$$\mathsf{P}(\sigma) = \frac{1}{N!}$$

be the uniform probability measure on the symmetric group $S(N)$, and let lis_N (respectively lds_N) denote the length of the longest increasing (respectively decreasing) subsequence in a uniformly random permutation from $S(N)$. Let

$$F_N(x) := \mathsf{P}(\mathrm{lis}_N \leq x) = \mathsf{P}(\mathrm{lds}_N \leq x)$$

be the cumulative distribution function of lis_N (equivalently lds_N). As an immediate consequence of the RSK formula (3.18), we have

$$\boxed{F_N(x) = \frac{1}{N!} \sum_{\substack{\lambda \in \mathbb{Y}(N) \\ \lambda_1 \leq x}} (\dim \lambda)^2 = \frac{1}{N!} \sum_{\substack{\lambda \in \mathbb{Y}(N) \\ \ell(\lambda) \leq x}} (\dim \lambda)^2.} \tag{4.1}$$

The power of the representation (4.1) may not be readily apparent. However, it ultimately leads to a very refined understanding of the asymptotics of $F_N(x)$.

Theorem 27 (Johansson, [Joh98])**.** *For any fixed* $s \in \mathbb{R}$

$$\lim_{N\to\infty} F_N(s\sqrt{N}) = c[s = 2] + [s > 2],$$

where $c \approx 0.9694$ *is a constant.*

Theorem 27 implies that, for N large, $F_N(x)$ rises sharply from close to 0 to close to 1 as one crosses the critical point $x \approx 2\sqrt{N}$. This fact was first observed computationally by Ulam [Ula61] in the 1960's, and led him to conjecture the asymptotic behaviour

$$\lim_{N\to\infty} \frac{\mathbb{E}(\mathrm{lis}_N)}{\sqrt{N}} = 2 \tag{4.2}$$

for the average value of lis_N. Ulam's asymptotics (4.2) can indeed be deduced from Theorem 27, see [Joh98]. This answers our very first question about the statistics of the book-sorting problem: when N is large, the expected number of operations required to sort the books is $N - 2\sqrt{N}$.

A vast refinement of Theorem 27, due to Baik, Deift, and Johansson, describes the fluctuations of lis_N around its mean.

Theorem 28 (Baik-Deift-Johansson, [BDJ99])**.** *For any fixed* $t \in \mathbb{R}$,

$$\lim_{N\to\infty} F_N(2\sqrt{N} + tN^{1/6}) = F(t),$$

where $F(t)$ *is the Tracy-Widom distribution function.*

The point here is that the Tracy-Widom distribution $F(t)$ is a known distribution (it will be described in Chapter 5). According to the Baik-Deift-Johansson theorem, the constant which appears in Johansson's theorem is $c = F(0)$.

Theorems 27 and 28 can be compared to their counterparts for the number cyc_N of cycles in a random permutation:

$$\lim_{N\to\infty} \mathsf{P}(\mathrm{cyc}_N \leq s\log N) = \frac{1}{2}[s=1] + [s>1] \tag{4.3}$$

$$\lim_{N\to\infty} \mathsf{P}(\mathrm{cyc}_N \leq \log N + t\sqrt{\log N}) = \Phi(t). \tag{4.4}$$

Here

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx \tag{4.5}$$

is the cumulative distribution function of a standard normal random variable. These results are due to Goncharov (1944, in Russian), see [FS06] for a description of Goncharov's work in English.

The passage from the boxed formula (4.1) to Theorems 27 and 28 is rather involved, and will be described in more detail in Chapters 5 and 6. For now, suffice to say that both theorems involve the asymptotics of $F_N(x)$ in a delicate *double scaling limit* where $x, N \to \infty$ simultaneously, with specified scaling ($x \sim s\sqrt{N}$ in Theorem 27 and $x \sim 2\sqrt{N} + tN^{1/6}$ in Theorem 28). In order to facilitate such an analysis, (4.1) must be recast in an alternative form which is well-poised for a subtle asymptotic analysis. In Chapter 5, we will see how to do this by "Poissonizing" the parameter N.

In this chapter, we investigate the asymptotic behaviour of $F_N(x)$ in the *single scaling limit* where $N \to \infty$ with x fixed. The asymptotics of $F_N(x)$ in this regime can be deduced directly from the formula (4.1), without the need for auxiliary constructions. As shown by Regev [Reg81] in 1981, the asymptotics of $F_N(x)$ in the single scaling limit lead to a very natural and direct connection with random matrices.

Of course, we have

$$\lim_{N\to\infty} F_N(x) = 0$$

for any fixed $x \in \mathbb{R}$. This is intuitively obvious: the probability $F_N(4)$ that a random permutation from $S(N)$ has no increasing subsequence of length 5 is significant for $N = 7$, but very small when $N = 10^7$. One wants to know precisely the rate of decay of the probability $F_N(x)$ as $N \to \infty$ with x fixed.

Theorem 29 (Regev, [Reg81]). *Let $d \geq 1$ be a fixed positive integer. Then*

$$\lim_{N\to\infty} \frac{N^{N+\frac{d^2}{2}}}{(d^2 e)^N} F_N(d) = \gamma_d,$$

where γ_d is a constant depending only on d. The value of γ_d is given explicitly by

$$\gamma_d = (2\pi)^{-\frac{d}{2}} \left(\prod_{i=0}^{d-1} i! \right) d^{\frac{d^2}{2}} 2^{\frac{1-d^2}{2}}.$$

Theorem 29 says that the asymptotic rate of decay of the probability $F_N(d)$ is

$$F_N(d) \sim \gamma_d \frac{(d^2 e)^N}{N^{N+\frac{d^2}{2}}},\ N \to \infty,$$

viz. faster than exponential. This can be compared with the corresponding decay rate for cycles,

$$\mathsf{P}(\mathrm{cyc}_N \leq d) \sim \frac{1}{(d-1)!} \frac{(\log N)^{d-1}}{N},\ N \to \infty, \tag{4.6}$$

which can be obtained from standard estimates of Stirling numbers [Wil] and is evidently much slower. In effect, this shows that for any fixed but arbitary number d, a large random permutation of size $N \gg d$ is much more likely to have an increasing subsequence of length $> d$ than it is to have more than d cycles.

Let us call the constant γ_d which occurs in Theorem 29 *Regev's constant.* We will see that the value of Regev's constant is closely related to the value of the d-dimensional integral

$$\Psi_d(2) = \int_{\mathbb{R}^d} e^{-\sum_{i=1}^d x_i^2} \prod_{1\le i<j\le d} (x_i - x_j)^2 d\mathbf{x},$$

which was first evaluated by Dyson and Mehta [DM63] in the 1960's as the partition function of the Gaussian Unitary Ensemble (defined below). In fact, we will obtain a combinatorial evaluation of this integral.

In this chapter, we begin with a simple heuristic derivation of Regev's result which is inspired by an exact formula of Knuth for the probability $F_N(2)$, namely

$$F_N(2) = \frac{(2N)!}{N!N!(N+1)!}. \tag{4.7}$$

The proof of Knuth's result, which is based on the RSK correspondence, provides sufficient intuition to guess an asymptotic formula for $F_N(d)$ *including the value of* γ_d using nothing more than Stirling's formula. Moreover, it suggests a new rigorous derivation of Theorem 29 which circumvents the computation of the GUE partition function, a calculation which is required in Regev's original proof [Reg81]. An additional benefit of the argument presented here is that it reveals a surprising asymptotic "duality" between pattern avoidance/containment in permutations/involutions.

4.2 The Asymptotic Knuth Theorem

Our goal in this chapter is to determine the asymptotics of the number $u(d, N)$ of permutations in $S(N)$ with no increasing subsequence of length $d+1$ in the limit where $N \to \infty$ and $d \ge 1$ is fixed but arbitrary. Actually, it will be more convenient to think of $u(d, n)$ as the number of permutations in $S(N)$ with no *decreasing* subsequence of length $d+1$. Undoubtedly,

> "the best way to solve an asymptotic problem is to have a nice exact formula valid before the limit, which would make passing to the limit conceptually clear, if not completely straightforward" [Oko201].

Unfortunately, no nice exact formula is known for $u(d, N)$.

Our strategy is to compare $u(d, N)$ with the number $t(d, N)$ of *involutions* in $S(N)$ with longest decreasing subsequence of length *exactly* d and longest increasing subsequence of length *exactly* N. The number $t(d, N)$ can be computed exactly, and its asymptotics can be computed from this exact expression using Stirling's formula. Thus by establishing an asymptotic correspondence between $u(d, N)$ and $t(d, N)$ (in a certain scaling), the asymptotics of $u(d, N)$ become accessible.

Theorem 30 (Asymptotic Knuth Theorem)**.** *For any fixed* $d \geq 1$,

$$u(d, dn) \sim t(d, 2n)$$

as $n \to \infty$.

We obtain the following remarkable result of Regev [Reg81] as a corollary of the Asymptotic Knuth Theorem (AKT).

Corollary 31 (Regev [Reg81])**.** *For any fixed* $d \geq 1$,

$$u(d, N) \sim (2\pi)^{\frac{1-d}{2}} \left(\prod_{i=0}^{d-1} \right) d^{2N+\frac{d^2}{2}} (2N)^{\frac{1-d^2}{2}}$$

as $N \to \infty$.

Proof. Assume that the AKT (Theorem 30) holds. By the RSK correspondence, we have

$$t(d, q) = \dim R(d, q) = (dq)! \prod_{i=0}^{d-1} \frac{i!}{(q+i)!},$$

where $R(d, q)$ is the $d \times q$ rectangular Young diagram and the second equality is a direct consequence of either Frobenius' formula or the hook-length formula for the dimension of a Young diagram. Applying Stirling's formula, which says that

$$q! \sim \sqrt{2\pi} q^{q+\frac{1}{2}} e^{-q}$$

as $q \to \infty$, we obtain

$$\dim R(d, q) \sim (2\pi)^{\frac{1-d}{2}} \left(\prod_{i=0}^{d-1} \right) d^{dq+\frac{1}{2}} q^{\frac{1-d^2}{2}}$$

as $q \to \infty$ with d fixed. Making the substitution $q = 2N/d$ in this asymptotic and applying the AKT yields the statement of the theorem. □

The rest of this Chapter is devoted to a proof of the AKT. We will see that the first hint of a connection between the increasing subsequence problem and random matrix theory appears in this proof.

4.3 Knuth's Theorem

Theorem 30 is called the Asymptotic Knuth Theorem because it is an asymptotic version of the following well-known result of Knuth [Knu73].

Theorem 32 (Knuth [Knu73]). *For any* $N \geq 1$,

$$u(2, N) = t(2, N) = \mathrm{Cat}_N,$$

where $\mathrm{Cat}_N = \frac{1}{N+1}\binom{2N}{N}$ *is the Catalan number.*

Proof. Let us give a bijective proof of Knuth's theorem using RSK. From RSK, we know that

$$u(2, N) = \sum_{\lambda \in \mathbb{Y}_2(N)} (\dim \lambda)^2,$$

where $\mathbb{Y}_2(N)$ is the set of Young diagrams with N cells and at most 2 rows. For example, $u(2,3) = 5$, corresponding to the five pairs

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}.$$

Also by RSK, we know that $t(2,N)$ is equal to $\dim R(2,N)$, the number of standard Young tableaux on the $2 \times N$ rectangular Young diagram $R(2,N)$. For example, $t(2,3) = 5$ corresponding to the five tableaux

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5&6\\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&6\\ \hline\end{array}.$$

The proof of Knuth's theorem therefore reduces to finding a bijection

$$\operatorname{Tab} R(2,N) \longleftrightarrow \bigcup_{\lambda \in \mathbb{Y}_2(N)} (\dim \lambda) \times (\dim \lambda).$$

The required bijection is easy to describe. Given a rectangular tableau $R \in \operatorname{Tab} R(2,N)$, let P be the standard tableau delineated by the first N entries $1, \ldots, N$ of R, and let Q be the standard tableau delineated by the remaining N entries $N+1, \ldots, 2N$ of R. Now rotate Q through $180°$, reverse the order of its entries, and subtract N from each entry to obtain a standard tableau Q'. The bijection we seek is

$$R \longleftrightarrow (P, Q').$$

When $N = 3$, this bijection looks like

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}\ \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}\ \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}\ \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline 3&\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}\ \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline 2&\\\cline{1-1}\end{array}$$

□

4.4 Failure of Knuth's theorem for $d \geq 3$

Let us try the "rectangle splitting" procedure from the proof of Knuth's theorem on a general rectangle $R(d, N)$. First, we need to assume that the rectangle has an even number of cells, so set $N = 2n$.

As an example, take $d = 3$ and $n = 2$, so that we're looking at standard Young tableaux on the 3×4 rectangle. One such tableau is

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&7&8&9\\\hline 6&10&11&12\\\hline\end{array},$$

and this splits into the tableaux

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&\multicolumn{3}{c}{}\\\cline{1-1} 6&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array},$$

which evidently do not have the same shape. Moreover,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

and

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline\end{array}$$

are perfectly valid RSK pairs which do not occur as the splitting of any rectangular tableau $R \in \operatorname{Tab} R(3,4)$. Thus the proof of Knuth's theorem breaks down for $d \geq 3$, but that does not render the underlying idea useless. Let us quantify how badly it breaks down, by finding an explicit expression for the error term $E(d,n)$ defined implicitly by

$$u(d, dn) = t(d, 2n) + E(d, n).$$

Theorem 33.

$$E(d,n) = \frac{1}{2} \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\dim \mu - \dim \mu^*)^2 + \sum_{\substack{\nu \in \mathbb{Y}_d(dn) \\ \nu_1 > 2n}} (\dim \nu)^2,$$

where μ^* *is the* complement *of* μ *relative to* $R(d,2n)$, *i.e. the diagram with row-lengths*

$$(2n - \mu_d, \ldots, 2n - \mu_1).$$

Proof. The algorithm

1. Input: A rectangular Young tableau $R \in \operatorname{Tab} R(d,2n)$.

2. Let P be the standard Young tableau delineated by the entries $1, \ldots, dn$ of R. Let Q be the skew tableau delineated by the entries $dn+1, \ldots, 2dn$ of R.

3. Let Q' be the tableau obtained by rotating Q through $180°$.

4. Let Q'' be the tableau obtained from Q' by making the substitutions $2dn \mapsto 1, 2dn-1 \mapsto 2, \ldots, dn+1 \mapsto dn$.

5. Output: A pair of standard Young tableaux (P, Q''), whose shapes are complementary relative to $R(d, 2n)$.

implements a bijection

$$\operatorname{Tab} R(d,2n) \longleftrightarrow \bigcup_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\operatorname{Tab}\mu) \times (\operatorname{Tab}\mu^*).$$

Thus,

$$\boxed{\dim R(d,2n) = \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\dim\mu)(\dim\mu^*).} \tag{4.8}$$

This decomposition, although very simple, is crucial to our arguments. Grouping terms according to whether they are self-complementary or not, this is refined to

$$\dim R(d,2n) = \sum_{\substack{\lambda \in \mathbb{Y}_d(dn) \\ \lambda \subseteq R(d,2n) \\ \lambda=\lambda^*}} (\dim\lambda)^2 + \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n) \\ \mu\neq\mu^*}} (\dim\mu)(\dim\mu^*). \tag{4.9}$$

On the other hand, by the definition of $u(d,N)$ we have

$$u(d,dn) = \sum_{\substack{\lambda \in \mathbb{Y}_d(dn) \\ \lambda \subseteq R(d,2n) \\ \lambda=\lambda^*}} (\dim\lambda)^2 + \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n) \\ \mu\neq\mu^*}} (\dim\mu)^2 + \sum_{\substack{\nu \in \mathbb{Y}_d(dn) \\ \nu_1>2n}} (\dim\nu)^2. \tag{4.10}$$

Substituting for the first (self-complementary) group of terms in (4.10) using (4.9) and completing the square yields the identity

$$u(d,dn) = \dim R(d,2n) + \underbrace{\frac{1}{2}\sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\dim\mu - \dim\mu^*)^2 + \sum_{\substack{\nu \in \mathbb{Y}_d(dn) \\ \nu_1>2n}} (\dim\nu)^2}_{E_d(n)}.$$

□

Note that when $d = 2$, the error term $E(2, n)$ is identically zero, since *all* diagrams in $\mathbb{Y}_2(2n)$ fit inside the rectangle $R(2, 2n)$, and are self-complementary relative to $R(2, 2n)$. Thus the proof of Theorem 33 reduces to the proof of Knuth's theorem when $d = 2$. When $d \geq 3$, diagrams which are not contained in $R(d, 2n)$, or are contained in $R(d, 2n)$ but are not self-complementary, are always present. Thus $E(d, n) > 0$ as soon as $d > 2$. However, even though $E(d, n)$ is always present for $d \geq 3$, we might hope that it is *asymptotically negligible*. A little numerical evidence encourages that hope. For $d = 3$ and $n = 100$, we compute

$$t(3, 200) \approxeq 6.32277 \times 10^{276}.$$

Comparing this with

$$u(3, 300) \approxeq 6.34023 \times 10^{276},$$

a value which was computed from Gessel's generating function (discussed in Chapter 5), we find that the latter is only slightly larger.

4.5 The Maximal Shape

The form of the error term $E(d, n)$ strongly indicates that the overwhelming contribution to the partition sum defining $u(d, dn)$ comes from self-complementary Young diagrams contained in the rectangle $R(d, 2n)$. In particular, diagrams Λ which maximize the dimension function over $\mathbb{Y}_d(dn)$ should be self-complementary relative to $R(d, 2n)$. Let us now show that there is in fact a *unique* $\Lambda \in \mathbb{Y}_d(dn)$ which maximizes the dimension function.

Note that the canonical diagram from $\mathbb{Y}_d(dn)$ which is self-complementary relative to $R(d, 2n)$ is of course the $d \times n$ rectangle $R(d, n)$. In fact, we will see that the maximal diagram Λ is asymptotically equal to $R(d,n)$, up to a first approximation. In order to formalize this, we parameterize deviation from $R(d,n)$ by introducing the *normalized coordinates* of a diagram $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{Y}_d(dn)$ defined by

$$\tilde{\lambda}_i := \frac{\lambda_i - n}{\sqrt{n}},\ 1 \le i \le d, \tag{4.11}$$

see Figure 4.1.

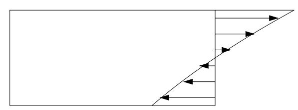


Figure 4.1: Normalized coordinates measure deviation from $R(d,n)$ on the scale $\sqrt{n}$.

The normalized coordinates of $R(d,n)$ are $(0,0,\ldots,0)$. More generally, observe that for an arbitrary diagram $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{Y}_d(dn)$ the normalized coordinates satisfy

$$\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_d \tag{4.12}$$

and

$$\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_d = 0, \tag{4.13}$$

and thus $(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_d) \in \overline{\Omega_d}$, where Ω_d is the $(d-1)$-dimensional open region

$$\Omega_d = \{(y_1,\ldots,y_d) \in \mathbb{R}^d : y_1 > \cdots > y_d,\ y_1 + \cdots + y_d = 0\}. \tag{4.14}$$

It is also clear that the dimension of λ is expressed in terms of normalized coordinates by the formula

$$\dim(n + \tilde{\lambda}_1\sqrt{n},\ldots,n + \tilde{\lambda}_d\sqrt{n}). \tag{4.15}$$

Finally, if $\mu = (\mu_1,\ldots,\mu_d) \subset R(d,2n)$, then the normalized coordinates of μ^* are

$$(-\tilde{\mu}_d,\ldots,-\tilde{\mu}_1), \tag{4.16}$$

where $\tilde{\mu}_1, \ldots, \tilde{\mu}_d$ are the normalized coordinates of μ.

In order to do asymptotics, we want the dimension function

$$\dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n}) \tag{4.17}$$

to be well-defined for all $(y_1, \ldots, y_d) \in \Omega_d$ and smooth for n sufficiently large. More generally, we want to extend dim to all of $\overline{\mathfrak{W}_d}$, where

$$\mathfrak{W}_d = \{(t_1, \ldots, t_d) \in \mathbb{R}^d : t_1 > \cdots > t_d\} \tag{4.18}$$

is the d-dimensional (open, type A) *Weyl chamber*, which will appear many times in this thesis. This can be done using the Frobenius formula and the Gamma function. Because the Gamma function is meromorphic with simple poles at the non-positive integers, we will need to include a cut-off in our extended dimension function. This cut-off plays no role in asymptotic calculations.

Definition. *The extended dimension function is defined on all of* $\overline{\mathfrak{W}_d}$ *by*

$$\dim(t_1, \ldots, t_d) := \frac{\Gamma(t_1 + \cdots + t_d + 1)}{\prod_{i=1}^d \Gamma(t_i + d - i + 1)} \prod_{1 \le i < j \le d} (t_i - t_j + j - i)[t_d \ge 0]. \tag{4.19}$$

The following is a key technical result.

Theorem 34. *For any fixed* $(y_1, \ldots, y_d) \in \Omega_d$,

$$\lim_{n \to \infty} C_{d,n} \dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n}) = e^{-W(y_1, \ldots, y_d)},$$

where

$$C_{d,n} = (2\pi)^{\frac{d}{2}} \frac{n^{dn + \frac{d(d+1)}{4}}}{(dn)! e^{dn}} \sim (2\pi)^{\frac{d-1}{2}} \frac{n^{\frac{(d-1)(d+2)}{4}}}{d^{dn + \frac{1}{2}}}$$

and

$$W(y_1, \ldots, y_d) = \frac{1}{2} \sum_{i=1}^{d} y_i^2 - \sum_{1 \le i < j \le d} \log(y_i - y_j).$$

Proof. This is a computation using Stirling's formula.

From the extended Frobenius formula, we have

$$\dim(n+y_1\sqrt{n},\dots,n+y_d\sqrt{n}) = \frac{(dn)!}{\prod_{i=1}^{d}\Gamma(n+y_i\sqrt{n}+d-i+1)}\prod_{1\le i<j\le d}((y_i-y_j)\sqrt{n}+j-i)$$

for n sufficiently large.

Let us first analyze the asymptotics of the product

$$\frac{1}{\prod_{i=1}^{d}\Gamma(n+y_i\sqrt{n}+d-i+1)}.$$

We begin by noting that $\Gamma(n+y_i\sqrt{n}+d-i+1)\sim n^{d-i+1}\Gamma(n+y_i\sqrt{n})$, which is a simple consequence of the usual functional equation $\Gamma(z+1)=z\Gamma(z)$. Thus

$$\frac{1}{\prod_{i=1}^{d}\Gamma(n+y_i\sqrt{n}+d-i+1)}\sim\frac{1}{n^{\frac{d(d+1)}{2}}\prod_{i=1}^{d}\Gamma(n+y_i\sqrt{n})}.$$

Taking logarithms yields

$$\log\left(\frac{1}{\prod_{i=1}^{d}\Gamma(n+y_i\sqrt{n}+d-i+1)}\right)\sim-\frac{d(d+1)}{2}\log n-\sum_{i=1}^{d}\log\Gamma(n+y_i\sqrt{n}).$$

By Stirling's formula,

$$\log\Gamma(N)\sim\frac{1}{2}\log 2\pi+(N-\frac{1}{2})\log N-N$$

for N large. Thus

$$\sum_{i=1}^{d}\log\Gamma(n+y_i\sqrt{n})\sim\frac{d}{2}\log 2\pi-dn+\sum_{i=1}^{d}(n+y_i\sqrt{n}-\frac{1}{2})\log(n+y_i\sqrt{n}).$$

Note that, even though we are performing a summation, there is no need to keep track of the error terms coming from Stirling's formula. This is because the number of terms in the sum is constant, namely d. Now since

$$\log(n+y_i\sqrt{n})=\log n+\log(1+\frac{y_i}{\sqrt{n}}),$$

we have

$$\sum_{i=1}^{d}(n+y_i\sqrt{n}-\frac{1}{2})\log(n+y_i\sqrt{n})=dn\log n-\frac{d}{2}\log n+\sum_{i=1}^{d}(n+y_i\sqrt{n}-\frac{1}{2})\log(1+\frac{y_i}{\sqrt{n}}).$$

Using the expansion

$$\log(1 + \frac{y_i}{\sqrt{n}}) = \frac{y_i}{\sqrt{n}} - \frac{y_i^2}{2n} + O(n^{-3/2})$$

for n sufficiently large, we find that

$$\sum_{i=1}^{d}(n + y_i\sqrt{n} - \frac{1}{2}) \log(1 + \frac{y_i}{\sqrt{n}}) \sim \frac{1}{2}\sum_{i=1}^{d} y_i^2.$$

Putting this all together, we find that

$$\frac{1}{\prod_{i=1}^{d} \Gamma(n + y_i\sqrt{n} + d - i + 1)} \sim \frac{e^{dn}}{(2\pi)^{\frac{d}{2}} n^{dn+\frac{d^2}{2}}} e^{-\frac{1}{2}\sum_{i=1}^{d} y_i^2}$$

as $n \to \infty$.

The second group of factors is much easier to handle:

$$\prod_{1\leq i<j\leq d} ((y_i - y_j)\sqrt{n} + j - i) \sim n^{\frac{d(d-1)}{4}} \prod_{1\leq i<j\leq d} (y_i - y_j).$$

Thus

$$\dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n}) \sim \frac{(dn)!e^{dn}}{(2\pi)^{\frac{d}{2}} n^{dn+\frac{d(d+1)}{4}}} e^{-W(y_1,\ldots,y_d)}$$

as $n \to \infty$, as claimed.

□

4.6 The maximal shape and the log gas

We want to use Theorem 34 to locate the diagram $\Lambda \in \mathbb{Y}_d(dn)$ which asymptotically maximizes the dimension function. Theorem 34 tells us that to locate Λ we must locate the maximum of e^{-W}, or equivalently the minimum of W, where W is the mixed quadratic/logarithmic form which emerged in the proof of Theorem 34.

Note that W is well-defined on the open Weyl chamber $\mathfrak{W}_d$. It will be convenient to extend the domain of W by inserting absolute values in the logarithmic terms, viz.

$$W(x_1, \ldots, x_d) = \frac{1}{2}\sum_{i=1}^{d} x_i^2 - \sum_{1\le i<j\le d} \log|x_i - x_j|. \tag{4.20}$$

W is now well-defined on all of $\mathbb{R}^d$, with the exception of the hyperplanes

$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = x_j\},\ 1 \le i < j \le d$$

which make up the walls of the Weyl chamber, where W hits $+\infty$. Thus e^{-W} vanishes on the boundary of $\mathfrak{W}_d$.

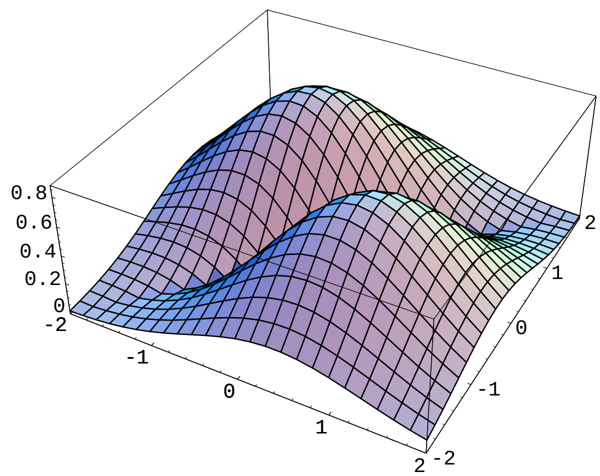


Figure 4.2: Plot of the surface $e^{-W(x_1,x_2)}$ for $-2 \le x_1, x_2 \le 2$.

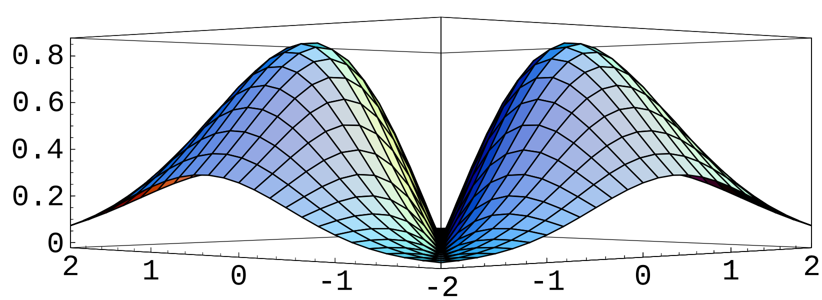


Figure 4.3: The same surface, viewed from the point $(-3, -3, 0)$.

The bivariate function $W(x_1, x_2)$ is plotted as a surface in $\mathbb{R}^3$ in Figures 4.2 and

4.3. Inspection of these plots makes it clear that W attains a strict maximum on $\mathfrak{W}_d$, at least in the case $d = 2$. This was proved by Stieltjes, who located the maximum (see [AR, AAR99]).

Theorem 35 (Stieltjes)**.** *W is strictly minimized over $\mathfrak{W}_d$ by the roots $z_1 > \cdots > z_d$ of the d^{th} Hermite polynomial $H_d(x)$.*

Stieltjes' result together with Theorem 34 immediately implies that, as $n \to \infty$, the dimension function on $\mathbb{Y}_d(dn)$ is strictly maximized by the diagram $\Lambda \in \mathbb{Y}_d(dn)$ whose normalized row-lengths are given by the zeros $z_1 > \cdots > z_d$ of the Hermite polynomial. Recall that the Hermite polynomials are given explicitly by

$$H_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k d!}{k!(d-2k)!}(2x)^{d-2k}.$$

The first few Hermite polynomials are:

$$\begin{aligned}
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 = (x + \frac{1}{\sqrt{2}})(x - \frac{1}{\sqrt{2}}) \\
H_3(x) &= 8x^3 - 12x = (x + \sqrt{\frac{3}{2}})x(x - \sqrt{\frac{3}{2}}) \\
&\vdots
\end{aligned}$$

Since $H_d(-x) = (-1)^d H_d(x)$, the Hermite polynomial is an odd function when d is odd and an even function when d is even. In either case, the roots of $H_d(x)$ have the symmetry

$$(z_1, \ldots, z_d) = (-z_d, \ldots, -z_1). \tag{4.21}$$

In particular, $z_1 + \cdots + z_d = 0$, so that W attains its maximum in Ω_d.

It follows that $\Lambda \subset R(d, 2n)$ and $\Lambda^* = \Lambda$, asymptotically, because of the symmetry (4.21). In particular, Λ does not contribute to the error term $E_d(n)$. Moreover diagrams in the neighbourhood of Λ, which are of high dimension, are contained in the rectangle $R(d, 2n)$ and therefore make zero contribution to the error term

$$\sum_{\substack{\nu \in \mathbb{Y}_d(dn) \\ \nu_1 > 2n}} (\dim \nu)^2.$$

Diagrams in the neighbourhood of Λ are also close to being self-complementary, and therefore make small contributions to the error term

$$\frac{1}{2} \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\dim \mu - \dim \mu^*)^2.$$

This provides strong heuristic evidence in favour of the asymptotic negligibility of the error term $E(d, n)$ and is probably convincing enough to pass as a proof in a physics paper.

A key property of the roots of orthogonal polynomials is that they are real, distinct, and interlacing. Stieltjes was aware of these phenomena and sought an electrostatic interpretation of them. In particular, he considered the possibility of describing the ground state of a system of identically charged particles on the line as the zero set of the Hermite polynomial.

The Coulomb gas on $\mathbb{R}$ consists of a system $t_1 > \cdots > t_d$ of d identical point charges on $\mathbb{R}$ in the presence of a potential well located at 0, as in Figure 4.4. The charges are uniformly attracted by the potential well, but also repel one another due to their like charge.

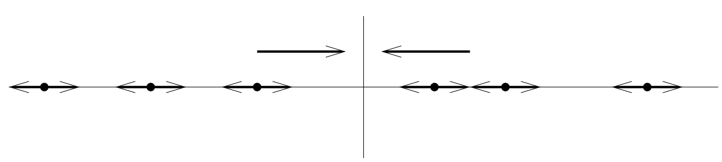


Figure 4.4: Identical point charges in the presence of a potential well.

The physical origin of the model justifies the assumption that the potential well is harmonic, and that the force of repulsion between two particles $t_i > t_j$ is proportional to $-\log(t_i - t_j)$ (these assumptions are consequences of solving a Poisson-type differential equation, see [F] for details). Thus a natural candidate for the Hamiltonian W which describes the potential energy of any particular configuration of the particles is

$$W(t_1, \ldots, t_d) = \frac{1}{2}\sum_{i=1}^{d} t_i^2 - \sum_{1\le i<j\le d} \log|t_i - t_j|. \tag{4.22}$$

Thus Stieltjes result shows that the ground state of this logarithmically interacting gas of particles is achieved uniquely at the zeros of the Hermite polynomials. We will come back to this idea shortly in the context of the eigenvalues of random Hermitian matrices.

Note that the potential W has the following symmetry property:

$$W(t_1, \ldots, t_d) = W(-t_d, \ldots, -t_1). \tag{4.23}$$

This can be checked directly from the definition of W, but it is also physically obvious: reflecting any given configuration of the charges about the potential well leaves preserves the energy. The symmetry is checked directly from the definition of W as follows: we have

$$\sum_{i=1}^{d}(-y_{d-i+1})^2 = \sum_{i=1}^{d} y_{d-i+1}^2 = \sum_{i=1}^{d} y_i^2,$$

which is the stated symmetry for the quadratic component of W, and

$$\sum_{1\le i<j\le d}\log(-y_{d-i+1} + y_{d-j+1}) = \sum_{1\le i<j\le d}\log(y_{d-j+1} - y_{d-i+1}) = \sum_{1\le i<j\le d}\log(y_i - y_j),$$

which is the stated symmetry for the logarithmic component of W.

4.7 Proof of the AKT

In this section we prove the AKT (Theorem 30). It turns out that the symmetry (4.23) is the key to the proof[1]. We work with the following deformations of the partition sum expressions of $u(d, dn)$ and $t(d, 2n)$:

$$u(d, dn; \beta) = \sum_{\lambda \in \mathbb{Y}_d(dn)} (\dim \lambda)^{\beta}$$

$$t(d, 2n; \alpha, \beta) = \sum_{\substack{\mu \in \mathbb{Y}_d(dn) \\ \mu \subseteq R(d,2n)}} (\dim \mu)^{\alpha} (\dim \mu^*)^{\beta - \alpha}.$$

The following is the main result of this Chapter.

Theorem 36 (Generalized Asymptotic Knuth Theorem)**.** *For any fixed integer $d \geq 1$ and real numbers $0 \leq \alpha < \beta$ we have*

$$u(d, dn; \beta) \sim t(d, 2n; \alpha, \beta)$$

as $n \to \infty$.

Proof. Our strategy is quite straightforward: we will show that, with the right scaling, both of these sums converge to two *a priori* different integrals. We then use the symmetry (4.23) to show that these integrals are in fact equal, without actually evaluating them.

Define a sequence of functions $f_n : \overline{\Omega_d} \to \mathbb{R}$ by

$$f_n(y_1, \ldots, y_d) = C_{d,n} \dim(n + y_1\sqrt{n}, \ldots, n + y_d\sqrt{n}),$$

where $C_{d,n}$ is the scaling constant from Theorem 34 and dim is the extended dimension function. Then Theorem 34 asserts pointwise convergence of this sequence on Ω_d :

$$\lim_{n \to \infty} f_n(y_1, \ldots, y_d) = e^{-W(y_1, \ldots, y_d)}$$

[1] I thank Andrei Okounkov for clarifying this point

for all $(y_1, \ldots, y_d) \in \Omega_d$. We will also need to know that the sequence f_n is bounded by an integrable function.

Lemma 37 (Śniady [Sni], Matsumoto [Mat]). *There exists a function $g \in L^1(\Omega_d)$ such that*

$$f_n(y_1, \ldots, y_d) \leq g(y_1, \ldots, y_d)$$

for any $(y_1, \ldots, y_d) \in \Omega_d$ and all $n \geq 1$.

The detailed construction of the dominating function g is given in the papers of Śniady and Matsumoto.

Now recall the definition of $u(d, dn; \beta)$, the sum we want to estimate:

$$\begin{aligned} u(d, dn; \beta) &= \sum_{\lambda \in \mathbb{Y}_d(dn)} (\dim \lambda)^\beta \\ &= \sum_{\substack{dn \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \\ \lambda_1 + \cdots + \lambda_d = dn}} \dim(\lambda_1, \ldots, \lambda_d)^\beta. \end{aligned}$$

Changing to normalized coordinates and scaling, we have

$$\begin{aligned} C_{d,n}^\beta u(d, dn; \beta) &= C_{d,n}^\beta \sum_{\substack{(d-1)\sqrt{n} \geq \tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_d \geq -\sqrt{n} \\ \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_d = 0}} \dim(n + \tilde{\lambda}_1 \sqrt{n}, \ldots, n + \tilde{\lambda}_d \sqrt{n})^\beta \\ &= \sum_{\substack{(d-1)\sqrt{n} \geq \tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_d \geq -\sqrt{n} \\ \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_d = 0}} f_n(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_d)^\beta, \end{aligned}$$

where the sum now goes in steps of $\frac{1}{\sqrt{n}}$, since each $\tilde{\lambda}_i$ is the left-hand endpoint of an interval of the form $[\frac{1}{\sqrt{n}}(k - n), \frac{1}{\sqrt{n}}(k + 1 - n))$ for some $k \in \{0, \ldots, dn\}$.

First, let us underestimate this sum: we have

$$C_{d,n}^\beta \left(\frac{1}{\sqrt{n}}\right)^{d-1} u(d, dn; \beta) \geq \sum_{\substack{(d-1)\sqrt{n} > \tilde{\lambda}_1 > \cdots > \tilde{\lambda}_d > -\sqrt{n} \\ \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_d = 0}} \left(\frac{1}{\sqrt{n}}\right)^{d-1} f_n(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_d \sqrt{n})^\beta.$$

The sum on the right is a Riemann sum, and we thus have

$$\begin{aligned}\lim_{n\to\infty} C_{d,n}^{\beta}\left(\frac{1}{\sqrt{n}}\right)^{d-1} u(d,dn;\beta) &\geq \lim_{n\to\infty}\int_{\Omega_d} f_n(y_1,\ldots,y_d)^{\beta} d\mathbf{y} \\ &= \int_{\Omega_d} \lim_{n\to\infty} f_n(y_1,\ldots,y_d)^{\beta} d\mathbf{y} \\ &= \int_{\Omega_d} e^{-\beta W(y_1,\ldots,y_d)} d\mathbf{y},\end{aligned}$$

where interchanging the limit and the integral is justified by the dominated convergence theorem.

Next, we overestimate: we have

$$C_{d,n}^{\beta}\left(\frac{1}{\sqrt{n}}\right)^{d-1} u(d,dn;\beta) \leq \sum_{\substack{(d-1)\sqrt{n}\geq\tilde{\lambda}_1\geq\cdots\geq\tilde{\lambda}_d\geq-\sqrt{n} \\ \tilde{\lambda}_1+\cdots+\tilde{\lambda}_d=0}} \left(\frac{1}{\sqrt{n}}\right)^{r(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_d)-1} f_n(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_d\sqrt{n})^{\beta},$$

where $r(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_d)$ is the number of distinct elements of $\tilde{\lambda}$. Thus

$$\begin{aligned}\lim_{n\to\infty} C_{d,n}^{\beta}\left(\frac{1}{\sqrt{n}}\right)^{d-1} u(d,dn;\beta) &\leq \lim_{n\to\infty}\int_{\overline{\Omega_d}} f_n(y_1,\ldots,y_d)^{\beta} d\mathbf{y} \\ &= \int_{\overline{\Omega_d}} \lim_{n\to\infty} f_n(y_1,\ldots,y_d)^{\beta} d\mathbf{y} \\ &= \int_{\Omega_d} e^{-\beta W(y_1,\ldots,y_d)} d\mathbf{y},\end{aligned}$$

where interchanging the limit and the integral is again justified by the dominated convergence theorem, and the last line is justified by the fact that $\overline{\Omega_d}/\Omega_d$ has measure zero.

Thus, by the Squeeze Theorem, we have that

$$\lim_{n\to\infty} C_{d,n}^{\beta}\left(\frac{1}{\sqrt{n}}\right)^{d-1} u(d,dn;\beta) = \int_{\Omega_d} e^{-\beta W(y_1,\ldots,y_d)} d\mathbf{y}.$$

By precisely the same argument applied to the sum

$$t(d,2n;\alpha,\beta) = \sum_{\substack{2n\geq\mu_1\geq\cdots\geq\mu_d\geq 0 \\ \mu_1+\cdots+\mu_d=dn}} \dim(\mu_1,\ldots,\mu_d)^{\alpha}\dim(2n-\mu_d,\ldots,2n-\mu_1)^{\beta-\alpha},$$

we arrive at the asymptotic

$$\lim_{n\to\infty} C^{\beta}_{d,n}\left(\frac{1}{\sqrt{n}}\right)^{d-1} t(d,2n;\alpha,\beta) = \int_{\Omega_d} e^{-\alpha W(y_1,\ldots,y_d)} e^{-(\beta-\alpha)W(-y_d,\ldots,-y_1)} d\mathbf{y}.$$

The equality of the two integrals now follows immediately from the symmetry

$$W(t_1,\ldots,t_d) = W(-t_d,\ldots,-t_1).$$

□

Noting that $u(d,dn) = u(d,dn;2)$ and $t(d,2n) = t(d,2n;1,2)$, the Asymptotic Knuth Theorem (Theorem 30) is proved.

4.8 The single scaling limit and random matrix theory

As we saw in Theorem 34, there is a close connection between the asymptotics of $u(d,N)$ and a Coulomb gas of d interacting particles on $\mathbb{R}$. Let us now carry this connection through to its natural conclusion.

Suppose that we wish to model the Coulomb gas probabalistically. The basic formalism of statistical mechanics (see [Bax82]) dictates that each configuration

$$t_1 > \cdots > t_d \tag{4.24}$$

of the charges be assigned the Gibbs measure

$$\frac{e^{-\beta W(t_1,\ldots,t_d)}}{\frac{1}{d!}\Psi_d(\beta)}, \tag{4.25}$$

where the weight $e^{-\beta W}$ is the *Boltzmann factor*, and

$$\frac{1}{d!}\Psi_d(\beta) = \frac{1}{d!}\int_{\mathbb{R}^d} e^{-\beta W(x_1,\ldots,x_d)} d\mathbf{x} = \int_{\mathfrak{W}_d} e^{-\beta W(t_1,\ldots,t_d)} d\mathbf{t} \tag{4.26}$$

is the *partition function*, which makes (4.25) into a probability measure. From the point of view of statistical physics, β is the inverse temperature parameter, i.e. it is defined by

$$\beta := \frac{1}{k_B T}, \tag{4.27}$$

where k_B is Boltzmann's constant and T is the temperature at which the system is in thermal equilibrium. We will see momentarily how to evaluate the partition function $\Psi_d(\beta)$ exactly for arbitrary β.

From our point of view, the significance of the Coulomb gas resides in the fact that we have realized it *combinatorially.* Theorem 34 shows that our "ensemble" $\mathbb{Y}_d(dn)$ of Young diagrams with at most d rows, which we equip with the "energy" functional

$$\lambda \mapsto \dim \lambda, \tag{4.28}$$

becomes the Coulomb gas in the $n \to \infty$ limit, with the normalized row-lengths playing the role of the gaseous particles. The proof of Theorem 36 shows that the asymptotic enumeration of permutations with increasing subsequence length bounded by d corresponds to the evaluation of the partition function of the Coulomb gas at inverse temperature $\beta = 2$, while the asymptotic enumeration of involutions with bounded increasing subsequence length is the partition function at $\beta = 1$ — almost. The glitch is that, due to their origin as the row-lengths of a partition, the normalized coordinates $y_1 > \cdots > y_d$ satisfy a linear constraint and are thus confined to the configuration space Ω_d, which is a linear submanifold of $\mathfrak{W}_d$. Thus we need to express the "partition function"

$$Z_d(\beta) = \int_{\Omega_d} e^{-\beta W(y_1,\dots,y_d)} d\mathbf{y}$$

in terms of $\Psi_d(\beta)$. This was done by Regev:

Lemma 38 (Regev, [Reg81]). *For any $d \geq 1$ and $\beta > 0$,*

$$Z_d(\beta) = \frac{1}{d!}\sqrt{\frac{\beta}{2\pi d}}\Psi_d(\beta).$$

We finally come to the evaluation of $\Psi_d(\beta)$, which is known as *Mehta's integral formula*. In order to explain the context of Mehta's formula, we need to briefly discuss the origins of the field in which it was obtained, namely random matrix theory.

A fundamental principle of quantum mechanics is that the energy levels of a quantum system are encoded in the pure point spectrum of of its Hamiltonian H, which is a Hermitian operator on (typically infinite dimensional) Hilbert space. In the 1950's, Wigner [Wig55, Wig58] suggested that the Hamiltonian could be modelled by a large random Hermitian matrix, the idea being that if the matrix model is chosen correctly one ought to recover salient properties of the Hamiltonian in the limit of large matrix dimension.

A quantum mechanical system may or may not have a certain intrinsic symmetry, called *time reversal symmetry*. The presence or absence of this symmetry must be built into the matrix model of the Hamiltonian. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and set $\beta := \dim_{\mathbb{R}} \mathbb{K}$. Consider a family z_{ij}, $1 \leq i, j \leq d$ of d^2 i.i.d standard $\mathbb{K}$-Gaussian random variables. In the case $\beta = 1$, this means that the z_{ij} are chosen independently from the Gaussian measure on $\mathbb{R}$ with density

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx, \tag{4.29}$$

while in the case $\beta = 2$ the z_{ij} are chosen independently from the Gaussian measure on $\mathbb{C}$ with density

$$\frac{1}{\pi}e^{-|z|^2}dz. \tag{4.30}$$

The (real or complex) *Ginibre matrix* is the $d \times d$ random matrix

$$X_d = (z_{ij})_{1 \leq i,j \leq d}. \tag{4.31}$$

The (real or complex) *Wigner matrix* is the $d \times d$ self-adoint random matrix

$$H_d = \frac{1}{2}(X_d + X_d^*). \tag{4.32}$$

The real Wigner matrix models the Hamiltonian of a quantum system with time reversal symmetry, while the complex Wigner matrix models the Hamiltonian of a quantum system without time reversal symmetry. The reason we begin with the assumption that the entries of X_d are independent Gaussians is twofold: this is both the easiest choice and the best choice. Regarding the first point, the Gaussian distribution is among the most familiar probability measures and thus presents itself as a canonical choice. Regarding the second point, it can be shown (see [Meh04, F]) that the joint distribution of the entries of H_d maximizes the Shannon entropy over a large class of probability measures on Hermitian matrices. Thus Gaussian Wigner matrices are in a sense the "most random" Hermitian matrix models that one can select. It is rather remarkable that these maximally random models are also exactly solvable.

Let $E_1 \geq \cdots \geq E_d$ be the ordered list of eigenvalues of the Wigner matrix H_d. Then $(E_1, \dots, E_d)$ is a random point in the closed Weyl chamber $\overline{\mathfrak{W}_d}$.

Theorem 39 (Wigner [Wig55]). *The distribution of the random point $(E_1, \dots, E_d)$ in $\overline{\mathfrak{W}_d}$ has density*

$$\frac{e^{-\beta W(t_1,\dots,t_d)}}{\frac{1}{d!}\Psi_d(\beta)}. \tag{4.33}$$

Thus we have arrived at the Coulomb gas model from yet another direction: it is statistically identical to the spectrum of the Wigner matrix H_d (for $\beta = 1, 2$).

Since the distribution of H_d is invariant under conjugation by orthogonal matrices when $\beta = 1$ and unitary matrices when $\beta = 2$, the Coulomb gas model is known as the *Guassian Orthogonal Ensemble* (GOE) at inverse temperature $\beta = 1$ and the *Gaussian Unitary Ensemble* (GUE) at inverse temperature $\beta = 2$. From our point of view, the asymptotic enumeration of involutions with bounded increasing subsequence length corresponds to the evaluation of the GOE partition function $\Psi_d(1)$, while the asymptotic enumeration of permutations with bounded increasing subsequence length corresponds to the evaluation of the GUE partition function $\Psi_d(2)$ (modulo Lemma 38). The asymptotics of $u(d, N; \beta)$ for general β correspond to the evaluation of Coulomb gas partition $\Psi_d(\beta)$ for general β.

Dyson and Mehta [DM63] were able to evaluate $\Psi_d(\beta)$ directly for $\beta = 1, 2, 4$. Their result led them to conjecture the general formula

$$\Psi_d(\beta) = (2\pi)^{\frac{d}{2}} \beta^{-\frac{d}{2} - \frac{\beta d(d-1)}{4}} \prod_{i=1}^{d} \frac{\Gamma(1 + i\frac{\beta}{2})}{\Gamma(1 + \frac{\beta}{2})}, \tag{4.34}$$

valid for arbitrary $\beta > 0$. This was eventually proved by Bombieri, who realized that it could be deduced from a more general integral formula due to Selberg. The interesting history of these events is recounted in [FW]. Thus the asymptotics of $u(d, N; \beta)$ can be determined for arbitrary β.

We end this chapter by remarking that, for $\beta = 2$, the argument we have presented avoids the evaluation of any and all integrals. It draws on the intuition that an asymptotic version of Knuth's theorem should hold, a conjecture which is borne out by the fact that the potential function W cannot "see" the difference between a partition μ and its complement μ^*. Given Lemma 38, this argument can be viewed as an independent evaluation of the GUE partition function $\Psi_d(2)$.

Chapter 5

Determinantal Generating Functions

5.1 Introduction

Recall that

$$F_N(x) = \mathsf{P}(\text{lis}_N \leq x)$$

is the cumulative distribution function of lis_N. In order to go beyond the single scaling limit and prove theorems such as Theorem 27 and Theorem 28, one needs an expression for $F_N(x)$ which is well-suited to refined asymptotic analysis. Such an expression can be obtained by considering the parameter N corresponding to the degree of the symmetric group $S(N)$ as a Poisson random variable. That is, we suppose

$$N \sim \text{Poisson}(t), \tag{5.1}$$

where $t > 0$ is the intensity of the Poissonization, i.e. the mean value of N (the symbol "$\sim$" here means "distributed as," not "asymptotic to"). Concretely, this means that N is a discrete random variable with probability mass function

$$\mathsf{P}(N = k) = e^{-k} \frac{t^k}{k!} [k \geq 0]. \tag{5.2}$$

After this *Poissonization*, the distribution function $F_N(x)$ is itself a random variable (for each fixed $x \in \mathbb{R}$). The value of the Poissonization trick resides in the fact that the expected value of $F_N(x)$ takes a very special determinantal form.

Theorem 40 (Gessel [Ges90]). *Let $d \geq 1$ be a fixed positive integer, and let $N \sim$ Poisson(t). Then*

$$\mathbb{E}(F_N(d)) = e^{-t} \det(I_{j-i}(2\sqrt{t}))_{1 \leq i,j \leq d},$$

where $I_k(t)$ denotes the modified Bessel function of the first kind of order k.

The presentation of $\mathbb{E}(F_N(d))$ as a Toeplitz determinant of Bessel functions has several important consequences.

First, every Toeplitz determinant is automatically related to the Circular Unitary Ensemble from random matrix theory via the Heine-Szëgo identity, or more generally the Heine-Szëgo-Bump-Diaconis identity. This relationship, which gives a very tangible connection between increasing subsequences and random matrices valid at the exact (i.e. non-asymtptotic) level, is the subject of Chapter 6. The connection between the Poissonization of the distribution $F_N(x)$ and compact matrix models is, for instance, the root reason behind Ulam's asymptotics: as discovered by Johansson [Joh98], the sharp rise in $F_N(x)$ from close to 0 to close to 1 around the critical point $x \approx 2\sqrt{N}$ is a consequence of the Gross-Witten phase transition [20] in the corresponding unitary matrix model.

Second, the asymptotic analysis of Toeplitz determinants is a venerable field of analysis, replete with powerful techniques which provide a template for the asymptotic analysis of $F_N(x)$. By combining existing classical methods with new analytic machinery inspired by the techniques of random matrix theory, Baik, Deift, and Johansson [BDJ99] deduced Theorem 28 from Theorem 40. Thus Theorem 40 is the foundation on which Theorems 27 and 28 are built.

The goal of this Chapter is to give a new and completely combinatorial proof of Theorem 40. The method here actually produces a rather general determinantal generating function identity, related to *Toeplitz minors*, of which Theorem 40 is a special case. This chapter is completely combinatorial — random matrix consequences are postponed until Chapter 6.

In the latter part of this chapter, we discuss a relationship between Theorem 40 and a certain discrete model from statistical mechanics, namely Fisher's *random-turns* vicious walker model. Random-turns vicious walkers were introduced by Fisher in order to model wetting and melting in certain two-dimensional lattice systems, see [Fis84]. The model consists of d particles ("walkers") initially occupying some given sites on the integer lattice $\mathbb{Z}$. At each instant of discrete time, a single random particle makes a random unit jump left or right, subject to the condition that no two walkers are permitted to occupy the same lattice site simultaneously. The walkers thus model interacting particles, and the adjective "vicious" serves as a reminder of this fact.

The *refined partition function* $Z_d(L^{b_k}R^{a_k}\dots L^{b_1}R^{a_1};\mu,\lambda)$ of the model records the number of ways in which the walkers can move from initial sites μ to terminal sites λ by executing a_1 positive jumps, followed by b_1 negative jumps $\dots$ followed by a_k positive jumps, followed by b_k negative jumps. The connection between the refined partition function of the random-turns model and increasing subsequence theory was first noted by Forrester, who proved the following result.

Theorem 41 (Forrester [For01]). *Let $\rho = \rho_1 > \cdots > \rho_d$ be any d adjacent sites on the integer lattice $\mathbb{Z}$. For any word W_n in L and R with n occurrences of L and n occurrences of R,*

$$Z_d(W_n; \rho, \rho) = u_d(n),$$

the number of permutations in $S(n)$ with no decreasing subsequence of length $d+1$.

In this chapter we will show that Theorems 40 and 41 are equivalent, and can be obtained as a relatively straightforward consequence of the commutativity of the lowering and raising operators on a particular graded graph $\mathbb{W}_d$, which we call the *Weyl graph.*

5.2 Increasing subsequences and the closed Weyl graph

As in Chapter 4, let us denote by

$$u_d(n) = \sum_{\lambda \in \mathbb{Y}_d(n)} (\dim \lambda)^2 \tag{5.3}$$

the number of permutations in $S(n)$ with no decreasing subsequence of length $d+1$. According to the RSK correspondence, $u_d(n)$ is equal to the number of closed walks on the Young graph $\mathbb{Y}$ of the form

$$\emptyset \underbrace{\nearrow \nearrow \cdots \nearrow}_{n} \underbrace{\searrow \searrow \cdots \searrow}_{n} \emptyset \tag{5.4}$$

which only visit diagrams with $\leq d$ rows. This can be formulated in the language of graded graphs. Given a graded graph (G, r), define the *refined partition function* of G by

$$Z_G(W; u, v) = [v]W(u), \tag{5.5}$$

where u, v are vertices of G and $W = L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}$ is a word in the lowering and raising operators on G. Recall from Chapter 3 that the refined partition function has the following combinatorial interpretation: $Z_G(W; u, v)$ is equal to the number of walks $u \to v$ on G of the form

$$u \underbrace{\nearrow\nearrow\cdots\nearrow}_{a_1}\underbrace{\searrow\searrow\cdots\searrow}_{b_1}\cdots\underbrace{\nearrow\nearrow\cdots\nearrow}_{a_k}\underbrace{\searrow\searrow\cdots\searrow}_{b_k} v. \tag{5.6}$$

In Chapter 3, we presented a theorem of Stanley which asserted that

$$Z_{\mathbb{Y}}(L^n R^n; \emptyset, \emptyset) = n!. \tag{5.7}$$

We now wish to determine

$$Z_{\mathbb{Y}_d}(L^n R^n; \emptyset, \emptyset) = u_d(n), \tag{5.8}$$

where $\mathbb{Y}_d$ is the induced subgraph of the Young graph $\mathbb{Y}$ whose vertices are Young diagrams of height at most d. A natural thing to do would be to emulate the proof of (5.7) given in Chapter 3, which used the fact (Theorem 23) that the lowering and raising operators on $\mathbb{Y}$ satisfy the Heisenberg commutation relation $LR - RL = I$. Unfortunately, it turns out that the lowering and raising operators on $\mathbb{Y}_d$ are somewhat badly behaved, and there is no nice characterization of their commutator. However, we can *embed* $\mathbb{Y}_d$ in a larger graded graph whose commutator is easily characterized.

Let $\overline{\mathbb{W}}_d$ be the graph with vertex set

$$\overline{\mathbb{W}}_d = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d : \lambda_1 \geq \cdots \geq \lambda_d\}, \tag{5.9}$$

with two vertices adjacent if and only if they are unit Euclidean distance apart. Since $\overline{\mathbb{W}}_d$ is obtained by intersecting the integer lattice $\mathbb{Z}^d$ with the closed Weyl chamber $\overline{\mathfrak{W}_d}$, we will call $\overline{\mathbb{W}}_d$ the *closed Weyl graph.* There is a natural rank function r on $\overline{\mathbb{W}}_d$ defined by

$$\overline{r}(\lambda) := \sum_{i=1}^{d} \lambda_i. \tag{5.10}$$

Thus $(\overline{\mathbb{W}}_d, \overline{r})$ is a graded graph.

There is an obvious embedding $\mathbb{Y}_d \hookrightarrow \overline{\mathbb{W}}_d$ of the height-restricted Young graph into the closed Weyl graph: simply map each diagram of $\mathbb{Y}_d$ to the vector of its row-lengths, as in Figure 5.1.

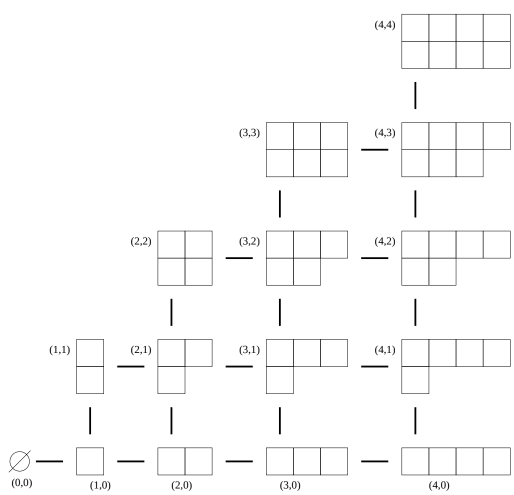


Figure 5.1: Embedding $\mathbb{Y}_2$ in $\overline{\mathbb{W}}_2$.

This is an embedding of graded graphs, by which we mean that it is rank preserving: the rank $|\lambda|$ of a diagram $\lambda \in \mathbb{Y}_d$ is equal to its rank $\overline{r}(\lambda)$ viewed as a lattice point. Thus

$$Z_{\mathbb{Y}_d}(L^n R^n; \emptyset, \emptyset) = Z_{\overline{\mathbb{W}}_d}(L^n R^n; \mathbf{0}^d, \mathbf{0}^d), \tag{5.11}$$

where on the right hand side L, R are the lowering and raising operators on $\overline{\mathbb{W}}_d$ and $\mathbf{0}^d = (0, \dots, 0)$ is the origin in $\mathbb{R}^d$.

Theorem 42. *The lowering and raising operators on the closed Weyl graph $\overline{\mathbb{W}}_d$ commute.*

Proof. Let $\mu, \lambda \in \mathbb{W}_d$ be arbitrary vertices. We check that $[\lambda]LR(\mu) = [\lambda]RL(\mu)$. Consider separately the cases $\mu \neq \lambda$ and $\mu = \lambda$.

Case $\mu \neq \lambda$. In this case, the coefficient $[\lambda]LR(\mu)$ is either 0 or 1. Indeed, if there exist i, j such that

$$\mu \nearrow \mu + \mathbf{e}_i \searrow \mu + \mathbf{e}_i - \mathbf{e}_j = \lambda, \tag{5.12}$$

where $\mathbf{e}_i, \mathbf{e}_j$ are standard basis vectors, then there is only one choice of i, j and $i \neq j$. But then

$$\mu \searrow \mu - \mathbf{e}_j \nearrow \mu - \mathbf{e}_j + \mathbf{e}_i = \lambda, \tag{5.13}$$

is the unique RL-walk from μ to λ, hence $[\lambda]LR(\mu) = [\lambda]RL(\mu) = 1$. On the other hand, if no LR-walk from μ to λ exists then no RL-walk from μ to λ exists, and $[\lambda]LR(\mu) = [\lambda]RL(\mu) = 0$.

Case $\mu = \lambda$. In this case, let r be the number of distinct entries of $\mu = \lambda$, i.e. λ is of the form

$$\lambda = (\underbrace{\lambda_{i_1}, \dots, \lambda_{i_1}}_{a_1}, \underbrace{\lambda_{i_2}, \dots, \lambda_{i_2}}_{a_2}, \dots, \underbrace{\lambda_{i_r}, \dots, \lambda_{i_r}}_{a_r}) \tag{5.14}$$

with $\lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_r}$ and $a_1 + a_2 + \cdots + a_r = d$. Then $[\lambda]LR(\lambda) = r$, since we can obtain λ from λ in r ways by first raising and then lowering the leftmost occurrence of λ_{i_j}, $1 \leq j \leq r$. Similarly, $[\lambda]RL(\lambda) = r$, since we can obtain λ from λ in r ways by first lowering and then raising the rightmost occurrence of λ_{i_j}, $1 \leq j \leq r$.

□

For any graph G, graded or not, we can define its *unrefined partition function* by

$$Z_G(N; u, v) = \#\{\text{walks } u \to v \text{ of length } N \text{ on } G\}. \tag{5.15}$$

In the case of the Weyl graph $\overline{W}_d$, the commutativity of the lowering and raising operators allows us to solve for the refined partition function in terms of the unrefined one.

Corollary 43. *Let $\mu, \lambda \in \overline{\mathbb{W}}_d$ be any two vertices of the closed Weyl graph verifying $\overline{r}(\mu) \leq \overline{r}(\lambda)$. Then*

$$Z_{\overline{\mathbb{W}}_d}(N; \mu, \lambda) = \binom{2n + \overline{r}(\lambda) - \overline{r}(\mu)}{n} Z_{\overline{\mathbb{W}}_d}(W_n; \mu, \lambda)[N = 2n + \overline{r}(\lambda) - \overline{r}(\mu)],$$

where $W_n \in \{L, R\}^*$ is any *operator verifying*

$$\deg_L(W_n) = n, \ \deg_R(W_n) = n + \overline{r}(\lambda) - \overline{r}(\mu).$$

Proof. Clearly, any walk $\mu \to \lambda$ on $\overline{\mathbb{W}}_d$ which takes n $\searrow$-steps must take $n+\overline{r}(\lambda)-\overline{r}(\mu)$ $\nearrow$-steps. Thus a walk $\mu \to \lambda$ of length N exists if and only if N is of the form $N = 2n + \overline{r}(\lambda) - \overline{r}(\mu)$.

Now,

$$Z_{\overline{\mathbb{W}}_d}(N; \mu, \lambda) = \sum_{W_n} Z_{\overline{\mathbb{W}}_d}(W_n; \mu, \lambda),$$

where the sum runs over all operators W_n verifying

$$\deg_L(W_n) = n, \ \deg_R(W_n) = n + \overline{r}(\lambda) - \overline{r}(\mu).$$

By Theorem 42, each term in this sum has the same value. Hence the sum is equal to the number of terms times the value of any individual term. Since there are $\binom{2n+\overline{r}(\lambda)-\overline{r}(\mu)}{n}$ words W_n over the alphabet $\{L, R\}^*$ meeting the condition

$$\deg_L(W_n) = n,\ \deg_R(W_n) = n + \overline{r}(\lambda) - \overline{r}(\mu),$$

the assertion is proved. □

As a particular case of Corollary 43, we have that

$$\binom{2n}{n} u_d(n) = \binom{2n}{n} Z_{\overline{\mathbb{W}}_d}(L^n R^n; \mathbf{0}^d, \mathbf{0}^d) = Z_{\overline{\mathbb{W}}_d}(2n; \mathbf{0}^d, \mathbf{0}^d). \tag{5.16}$$

This reduces the computation of $u_d(n)$ to the enumeration *all* closed walks of length $2n$ on $\overline{\mathbb{W}}_d$ based at the origin. This latter enumeration problem can be solved using the geometry of the Weyl chamber.

5.3 Reflection principle and the open Weyl graph

Let $\mathbb{W}_d$ be the induced subgraph of $\overline{\mathbb{W}}_d$ with vertex set

$$\mathbb{W}_d = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d : \lambda_1 > \cdots > \lambda_d\}. \tag{5.17}$$

Since $\mathbb{W}_d$ is obtained by intersecting the open Weyl chamber $\mathfrak{W}_d$ with the integer lattice $\mathbb{Z}^d$, we will call $\mathbb{W}_d$ the *open Weyl graph*. Evidently, $\overline{\mathbb{W}}_d$ and $\mathbb{W}_d$ are isomorphic graphs: the isomorphism $\overline{\mathbb{W}}_d \to \mathbb{W}_d$ is simply translation along the vector $\rho = (d, d-1, \dots, 2, 1)$, see Figure 5.2.

If we make $\mathbb{W}_d$ into a graded graph by equipping it with the rank function

$$r(\lambda) = \sum_{i=1}^{d} \lambda_i - \frac{1}{2} d(d+1), \tag{5.18}$$

then the translation $\lambda \mapsto \lambda + \rho$ is an isomorphism of graded graphs. The unrefined

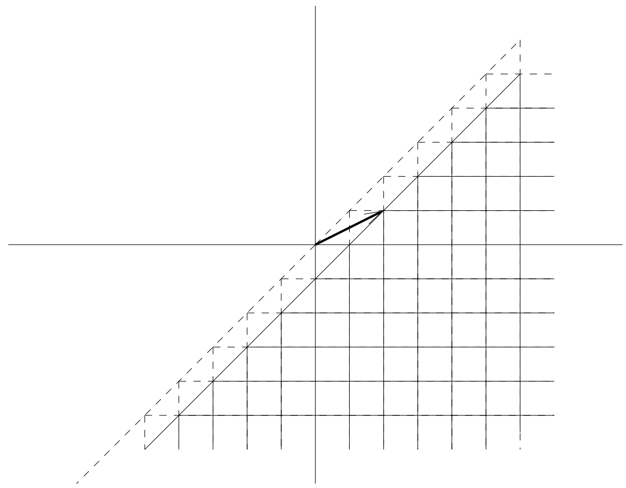


Figure 5.2: Graph isomorphism $\overline{\mathbb{W}}_2 \to \mathbb{W}_2$.

partition function of the open Weyl graph $\mathbb{W}_d$ can be computed using the Anrdré-Gessel-Zeilberger *reflection principle*.

Let us return for a moment to the setting of a general graph G. For each pair of vertices $u, v \in G$, consider the exponential generating function

$$\mathcal{Z}_G(t; u, v) := \sum_{N \geq 0} Z_G(N; u, v) \frac{t^N}{N!}. \tag{5.19}$$

Clearly, $e^{-t}\mathcal{Z}_G(t; u, v)$ is the expected number of walks $u \to v$ of length N on G under the assumption $N \sim \text{Poisson}(t)$. We will compute the Poissonization (5.19) for $G = \mathbb{Z}^d$, and then use this result together with the reflection principle to do the same for $G = \mathbb{W}_d$.

Let us start with the simplest case where $d = 1$. For any integer $k \in \mathbb{Z}$, we compute the number $Z_{\mathbb{Z}}(N; k, k+q)$ of walks on $\mathbb{Z}$ from k to $k+q$ in N steps. Evidently the problem is translation invariant, and thus depends only on N and q and not on k. Moreover, we can assume that $q \geq 0$, since clearly $Z_{\mathbb{Z}}(N; k, k+q) = Z_{\mathbb{Z}}(N; k, k-q)$

by reflection about k. We then have

$$Z_{\mathbb{Z}}(N; k, k+q) = \binom{2n+q}{n}[N = 2n+q], \tag{5.20}$$

where n represents the number of negative steps taken. Thus

$$\mathcal{Z}_{\mathbb{Z}}(t; k, k+q) = \sum_{N\geq 0} Z_{\mathbb{Z}}(N; k, k+q)\frac{t^N}{N!} = \sum_{n\geq 0} \frac{t^{2n+q}}{n!(n+q)!}. \tag{5.21}$$

The function $I_q(t)$ defined by

$$I_q(2t) := \mathcal{Z}_{\mathbb{Z}}(t; k, k+q) \tag{5.22}$$

is a classically studied special function known as the *modified Bessel function* of the first kind of order q. For $q \in \mathbb{Z}$, the combinatorial definition of I_q shows that

$$I_q(2t) = [z^q]e^{t(z+z^{-1})}. \tag{5.23}$$

To see this, consider $e^{t(z+z^{-1})}$ as a generating function for walks on $\mathbb{Z}$ with z marking a positive unit step and z^{-1} marking a negative unit step. Formula (5.23) shows that that the modified Bessel functions have an interpretation as the Fourier coefficients of an L^1-function on the unit circle $\mathbb{T}$, namely

$$e^{t(z+z^{-1})} = \sum_{q=-\infty}^{\infty} I_q(2t)z^q. \tag{5.24}$$

We will use this fact in Chapter 6.

To recapitulate, we have just shown that the partition function counting walks on $\mathbb{Z}$ in unit steps has the Poissonization

$$\mathcal{Z}_{\mathbb{Z}}(t; u, v) = I_{v-u}(2t). \tag{5.25}$$

Now, any walk $\mu \to \lambda$ on $\mathbb{Z}^d$ is simply a shuffle of independent walks on $\mu_i \to \lambda_i$ on $\mathbb{Z}$. Thus, by a general property of exponential generating functions (namely the product lemma for labelled structures, see e.g. [GJ83], Chapter 3), we have that

$$\mathcal{Z}_{\mathbb{Z}^d}(t;\mu,\lambda) = \prod_{i=1}^{d} I_{\lambda_i-\mu_i}(2t), \tag{5.26}$$

Finally, we come to the case of the open Weyl graph $G = \mathbb{W}_d$.

Theorem 44. *For any two vertices $\mu, \lambda \in \mathbb{W}_d$, we have*

$$\mathcal{Z}_{\mathbb{W}_d}(t;\mu,\lambda) = \det(I_{\lambda_i-\mu_j}(2t))_{1\le i,j\le d}.$$

Proof. The proof relies on the André-Gessel-Zeilberger reflection principle [And87, GZ92] for the enumeration of lattice walks confined to a Weyl chamber, in the simplest case where the chamber is of type A and the lattice is $\mathbb{Z}^d$. In this case the reflection principle asserts that

$$Z_{\mathbb{W}_d}(N;\mu,\lambda) = \sum_{\sigma\in S(d)} (-1)^\sigma Z_{\mathbb{Z}^d}(N;\sigma(\mu),\lambda), \tag{5.27}$$

where the symmetric group $S(d)$ acts on $\mathbb{Z}^d$ by permuting coordinates.

Applying the reflection principle together with the definition of the determinant, we have

$$\begin{aligned}
\mathcal{Z}_{\mathbb{W}_d}(t;\mu,\lambda) &= \sum_{N\ge 0} Z_{\mathbb{W}_d}(N;\mu,\lambda)\frac{t^N}{N!} \\
&= \sum_{N\ge 0}\sum_{\sigma\in S(d)} (-1)^\sigma Z_{\mathbb{Z}^d}(N;\sigma(\mu),\lambda)\frac{t^N}{N!} \\
&= \sum_{\sigma\in S(d)} (-1)^\sigma \sum_{N\ge 0} Z_{\mathbb{Z}^d}(N;\sigma(\mu),\lambda)\frac{t^N}{N!} \\
&= \sum_{\sigma\in S(d)} (-1)^\sigma \prod_{i=1}^{d} I_{\lambda_i-\mu_{\sigma(i)}}(2t) \\
&= \det(I_{\lambda_i-\mu_j}(2t))_{1\le i,j\le d}.
\end{aligned}$$

□

Corollary 45. *For any two vertices $\mu, \lambda \in \overline{\mathbb{W}}_d$, we have*

$$\mathcal{Z}_{\overline{\mathbb{W}}_d}(t;\mu,\lambda) = \det(I_{\lambda_i-\mu_j+j-i}(2t))_{1\leq i,j\leq d}.$$

Proof. Repeat the proof of Theorem 44, replacing μ with $\mu+\rho$ and λ with $\lambda+\rho$. ☐

Example 46. The number of closed walks $(0,0) \to (0,0)$ of length 4 on $\mathbb{Z}^2$ is

$$\left[\frac{t^4}{4!}\right] I_0(2t)^2 = 36.$$

The number of closed walks $(0,0) \to (0,0)$ of length 4 on $\overline{\mathbb{W}}_2$ is

$$\left[\frac{t^4}{4!}\right] \begin{vmatrix} I_0(2t) & I_1(2t) \\ I_{-1}(2t) & I_1(2t) \end{vmatrix} = \left[\frac{t^4}{4!}\right](I_0(2t)^2 - I_1(2t)^2) = 12.$$

5.4 A general determinantal identity

We can now state and prove a rather general generating function identity for the *refined* partition function of the Weyl graph.

Theorem 47. *Let $\mu, \lambda \in \overline{\mathbb{W}}_d$ be vertices of the closed Weyl graph verifying $\overline{r}(\mu) \leq \overline{r}(\lambda)$. Then*

$$\sum_{n\geq 0} Z_{\overline{\mathbb{W}}_d}(W_n;\mu,\lambda)\frac{t^{2n+\overline{r}(\lambda)-\overline{r}(\mu)}}{n!(n+\overline{r}(\lambda)-\overline{r}(\mu))!} = \det(I_{\lambda_i-\mu_j+j-i}(2t))_{1\leq i,j\leq d},$$

where (W_n) is any *sequence of words in the raising and lowering operators verifying*

$$\deg_L W_n = n, \ \deg_R W_n = n+\overline{r}(\lambda)-\overline{r}(\mu).$$

Example 48. The number of walks $(0,0) \to (0,0)$ of length 4 on $\overline{\mathbb{W}}_2$ which take two positive steps and two negative steps in any fixed order (i.e. one of $LLRR$, $LRLR$, $LRRL$, $RLRLR$, R
is

$$\left[\frac{t^4}{2!2!}\right] \begin{vmatrix} I_0(2t) & I_1(2t) \\ I_{-1}(2t) & I_1(2t) \end{vmatrix} = \left[\frac{t^4}{2!2!}\right](I_0(2t)^2 - I_1(2t)^2) = 2.$$

Gessel's identity (Theorem 40) is the special case of Theorem 47 obtained by setting $\mu = \lambda = \mathbf{0}^d$ and $W_n = L^n R^n$. The original proof of Gessel's identity used fairly heavy machinery from symmetric function theory, see [Ges90]. A combinatorial proof relying on the notion of "Toeplitz points" was found later found by Gessel, Weinstein, and Wilf [GWW98], but it only applies in the case $\mu = \lambda = \mathbf{0}^d$ and $W_n = L^n R^n$. Theorem 47 was recently obtained by Xin [Xin09] in the case where $\mu, \lambda \in \overline{\mathbb{W}}_d$ may be arbitrary, but W_n is still restricted to be of the "unimodal" form form $L^n R^{n+\overline{r}(\lambda)-\overline{r}(\mu)}$. Xin's method uses the so-called "Stanton-Stembridge trick."

It should be emphasized that the reflection principle is a well-known tool in enumerative combinatorics. The simple approach to Gessel's identity given here is a result of combining the reflection principle with the commutativity of the lowering and raising operators on the Weyl graph.

5.5 Random-turns vicious walkers and Forrester's theorem

The connection between Fisher's random-turns vicious walker model and the refined partition function of the Weyl graph $\mathbb{W}_d$, is through the notion of *configuration space*. The configuration space of a physical system is the set of all possible positions of its constituents. If the system is continuous, the configuration space is a manifold and one studies properties of the system by studying the geometry of the configuration manifold.

The random-turns model is discrete, and therefore one views its configuration space as a graph. More precisely, the configuration graph of d random-turns walkers

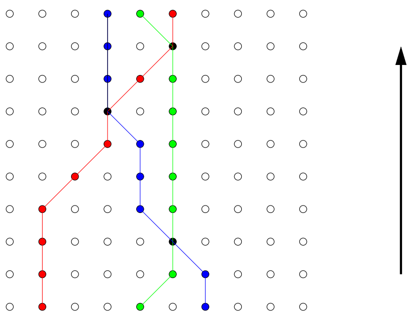


Figure 5.3: Time evolution of friendly random-turns walkers

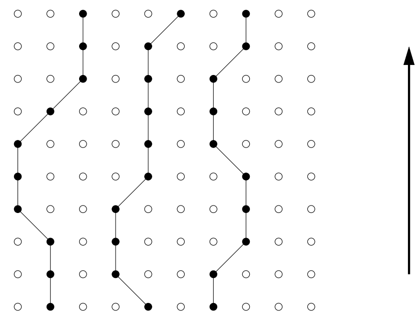


Figure 5.4: Time evolution of vicious random-turns walkers

on $\mathbb{Z}$ is precisely the open Weyl graph $\mathbb{W}_d$. The number of ways in which the walkers can move from state μ to state λ in a sequence of left and right jumps compatible with a given word $W = L^{b_k}R^{a_k}\dots L^{b_1}R^{a_1}$ is precisely $Z_{\mathbb{W}_d}(W;\mu,\lambda)$. Thus Forrester's observation (Theorem 41) that the number of walks which have the particles initially and terminally positioned on adjacent sites ρ depends only on the number of L's and R's in W and not on their order is a special case of the commutativity of L and R viewed as raising and lowering operators on $\mathbb{W}_d$. The fact that this number moreover corresponds to the enumeration of permutations with increasing subsequence length bounded by d is just the fact that the height restricted Young graph $\mathbb{Y}_d$ embeds in $\overline{\mathbb{W}}_d$, which is in turn isomorphic to $\mathbb{W}_d$.

5.6 A brief description of the Tracy-Widom distribution

In this section we will briefly collect the ingredients which go into the definition of the Tracy-Widom distribution function $F(t)$ which appeared in Theorem 28. The construction of $F(t)$ is somewhat involved, but is presented concisely in [BDJ99].

First, we need to extend our definition of the modified Bessel function $I_q(t)$, which so far is only valid for $q \in \mathbb{Z}$. This is easily done by setting

$$I_\alpha(2t) = \sum_{n\geq 0} \frac{t^{2n+\alpha}}{\Gamma(n+1)\Gamma(n+\alpha+1)}, \tag{5.28}$$

a definition which is good for arbitrary $t \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\{-1,-2,-3,\dots\}$. However there isn't really any issue with poles of the Gamma function since we know from our combinatorial approach that $I_q(t) = I_{-q}(t)$ for $q \in \mathbb{Z}$, and this relation patches the problem at the negative integers. Note that $I_\alpha(t) \neq I_{-\alpha}(t)$ for non-integer α.

The modified Bessel function is typically introduced as an analytic object: it is one of two linearly independent solutions to the modified Bessel differential equation

$$x^2y'' + xy' - (x^2+\alpha^2)y = 0. \tag{5.29}$$

The other solution to (5.29) is called the *MacDonald function*. The MacDonald function is related to I_α by

$$K_\alpha(t) = \frac{\pi}{2}\frac{I_{-\alpha}(t) - I_\alpha(t)}{\sin(\alpha\pi)}, \tag{5.30}$$

which we will take as the definition of K_α. If $\alpha \in \mathbb{Z}$, then K_α is defined as the limit $K_\alpha(t) = \lim_{z\to\alpha} K_z(t)$, which exists. Further analytic properties of Bessel functions and their relatives can be found for instance in [BSMM03].

The *Airy function* $Ai(t)$ is defined in terms of the MacDonald function (and hence in terms of the modified Bessel function) by

$$Ai(t) = \frac{1}{\pi}\sqrt{\frac{t}{3}}K_{1/3}(\frac{2}{3}t^{\frac{3}{2}}). \tag{5.31}$$

Consider now the differential equation

$$u_{xx} = 2u^3 + xu. \tag{5.32}$$

This is called the *Painlevé II* equation, and it is known that it has a unique solution u with the asymptotics $u \sim -Ai(x)$ as $x \to \infty$. This unique solution is called the *Hastings-Macleod* solution. The Tracy-Widom distribution function is defined by

$$F(t) := e^{-\int_t^\infty (x-t)u^2(x)dx}, \tag{5.33}$$

where $u(x)$ is the Hastings-Macleod solution to Painlevé II.

The eponymous distribution function $F(t)$ was first encountered by Tracy and Widom [45] in their investigations of limit law of the largest eigenvalue of the Hermitian Wigner matrix H_N, which we constructed in Chapter 4. They proved the following result.

Theorem 49 (Tracy-Widom, [45])**.** *Let $E_{1,N}$ denote the largest eigenvalue of the random Hermitan matrix H_N. Then for any $t \in \mathbb{R}$*

$$\lim_{N\to\infty} \mathsf{P}(E_{1,N} \leq 2\sqrt{N} + \frac{t}{N^{1/6}}) = F(t).$$

Combining Theorems 28 and 49, one arrives at the statement that the limiting distributions of the normalized random variables

$$\frac{\mathrm{lis}_N - 2\sqrt{N}}{N^{1/6}} \text{ and } N^{1/6}(E_{1,N} - 2\sqrt{N}) \tag{5.34}$$

exist and coincide. Thus, after appropriate centering and scaling, the length of the longest increasing subsequence in a random permutation is asymptotically distributed as the largest eigenvalue of a random Hermitian matrix with Gaussian entries.

Chapter 6

Compact Matrix Models

6.1 Introduction

In this chapter, we will discuss the connection between the determinantal generating functions of the previous chapter and the Circular Unitary Ensemble of matrices sampled randomly from the Haar probability measure on the unitary group.

The connection between Toeplitz determinants and random unitary matrices is important and well-known, see e.g. [Joh97] for many applications. Indeed the fact that, via Gessel's identity, the Poissonization of the cumulative distribution function $F_N(x)$ for increasing subsequences can be expressed as a unitary matrix integral is a key ingredient in Johansson's proof [Joh98] of Ulam's asymptotics for the expectation of lis_N; this is the avenue by which contact is made with earlier work of Gross and Witten [20] on unitary matrix models.

In Chapter 5, we saw that walks on the Weyl graph, or equivalently random-turns vicious walkers, are a natural combinatorial generalization of increasing subsequences in permutations. The generalization of Gessel's identity that holds for random-turns walkers results in a determinant which is not a Toeplitz determinant, but rather a *Toeplitz minor*. Bump and Diaconis [BD02] extended aspects of the theory of Toeplitz determinants to the setting of Toeplitz minors. In particular, they obtained an extension of the Heine-Szegö identity, which converts Toeplitz determinants into averages over the unitary group, and an extension of the strong Szegö limit theorem, which gives the asymptotics of certain Toeplitz determinants in the limit where the size of the matrix goes to infinity. These results therefore apply to the Toeplitz minors encountered in Chapter 5, and here we develop the connection.

Finally, we go on to consider random matrices constructed by deleting rows and columns from Haar-distributed random unitary matrices. These matrices are called *truncated* Haar unitaries. Truncated unitary matrices were introduced by Sommers and Zyczkowski [SZ00] in the context of scattering theory, and since then several authors have studied probabalistic aspects of their eigenvalues, which are random points in the unit disc. Examples include the determinantal structure of the correlation functions [PR04, Kri09], and large deviation laws for the empirical eigenvalue distribution [PR05]. We will instead consider averages over the ensembles of truncated random unitary matrices, and investigate combinatorial interpretations of such averages. Since vicious walkers generalize increasing subsequences, and since truncated Haar unitaries generalize Haar unitaries, one hopes that the two generalizations are consonant and that truncated Haar unitaries are related to vicious walkers. We show that this is in fact the case.

6.2 Toeplitz determinants and the Heine-Szegö identity

Let $f \in L^1(\mathbb{T})$ be a complex-valued function on the unit circle with Fourier expansion

$$f(z) = \sum_{q=-\infty}^{\infty} a_q z^q. \tag{6.1}$$

Recall that any matrix whose entries are constant along diagonals is called a *Toeplitz matrix*. The $d \times d$ *Toeplitz matrix with symbol* f is by definition the Toeplitz matrix built from the Fourier coefficients of f by setting

$$T_d(f) = (a_{j-i})_{1 \le i,j \le d} = \begin{bmatrix} a_0 & a_1 & \dots & a_{d-1} \\ a_{-1} & a_0 & \dots & a_{d-2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1-d} & a_{2-d} & \dots & a_0 \end{bmatrix}. \tag{6.2}$$

The determinant $\det T_d(f)$ is called the Toeplitz determinant with symbol f and denoted $D_d(f)$. Note that we have encountered Toeplitz determinants already in Gessel's identity (Theorem 40).

A classical result of Heine and Szegö expresses the Toeplitz determinant $D_d(f)$ as an integral over the group of $d \times d$ unitary matrices $\mathcal{U}(d)$ against the normalized Haar measure dU. Define a function

$$\Phi_f : \mathcal{U}(d) \to \mathbb{C} \tag{6.3}$$

on unitary matrices by

$$\Phi_f(U) = \prod_{i=1}^{d} f(E_i), \tag{6.4}$$

where $E_1, \dots, E_d$ are the eigenvalues of the unitary matrix U. The following result relating $D_d(f)$ to the average value of $\Phi_f(U)$ is known as the *Heine-Szegö identity*, see [BD02, Joh97] for a proof and further references.

Theorem 50 (Heine-Szegö identity)**.** *For any* $f \in L^1(\mathbb{T})$ *with* $D_d(f)$ *and* Φ_f *defined as above, we have*

$$D_d(f) = \int_{\mathcal{U}(d)} \Phi_f(U)dU.$$

Example 51. Let $t > 0$ be a positive real number, and put

$$f(z) = e^{t(z+z^{-1})} \in L^1(\mathbb{T}). \tag{6.5}$$

The Fourier expansion of f is

$$f(z) = \sum_{q=-\infty}^{\infty} I_q(2t)z^q, \tag{6.6}$$

and the function on unitary matrices associated to f is

$$\Phi_f(U) = \prod_{E_i} e^{t(E_i+E_i^{-1})} = e^{\sum_{E_i} t(E_i+E_i^{-1})} = e^{t\,\mathrm{Tr}(U+U^*)}. \tag{6.7}$$

Thus

$$D_d(f) = \det(I_{j-i}(2t))_{1\le i,j\le d} = \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)}dU, \tag{6.8}$$

by the Heine-Szegö identity.

On the other hand, by Gessel's identity (Theorem 40), the Poissonized distribution function of lis_N has expected value

$$\mathbb{E}(F_N(d)) = e^{-t}\det(I_{j-i}(2\sqrt{t})_{1\le i,j\le d}. \tag{6.9}$$

One thus arrives at the following remarkable identity between expectations:

$$\mathbb{E}(F_N(d)) = e^{-t}\mathbb{E}(e^{\sqrt{t}\,\mathrm{Tr}(U_d+U_d^*)}), \tag{6.10}$$

where N is Poisson of mean t and U_d is a random matrix drawn from Haar measure on the unitary group $\mathcal{U}(d)$.

In more combinatorial terms, the above can be stated as follows. According to

Gessel's identity, the number $u_d(n)$ of permutations in $S(n)$ with no decreasing subsequence of length $d+1$ is given by

$$u_d(n) = \left[\frac{t^{2n}}{n!n!}\right] D_d(f). \tag{6.11}$$

Consequently

$$u_d(n) = \left[\frac{t^{2n}}{n!n!}\right] \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} dU \tag{6.12}$$

$$= \left[\frac{t^{2n}}{n!n!}\right] \sum_{a,b\geq 0} \int_{\mathcal{U}(d)} (\mathrm{Tr}\, U)^a (\mathrm{Tr}\, U^*)^b \frac{t^{a+b}}{a!b!} \tag{6.13}$$

$$= \left[\frac{t^{2n}}{n!n!}\right] \sum_{a\geq 0} \int_{\mathcal{U}(d)} |\,\mathrm{Tr}\, U|^{2a} \frac{t^{2a}}{a!a!} \tag{6.14}$$

$$= \int_{\mathcal{U}(d)} |\,\mathrm{Tr}\, U|^{2n} dU. \tag{6.15}$$

Thus we obtain the remarkable integral representation

$$u_d(n) = \int_{\mathcal{U}(d)} |\,\mathrm{Tr}\, U|^{2n} dU, \tag{6.16}$$

an identity due to Rains [Rai98].

Rains' identity is interesting in both directions. For example, combining it with Knuth's theorem one has the pretty formula

$$\int_{\mathcal{U}(2)} |\,\mathrm{Tr}\, U|^{2n} dU = \mathrm{Cat}_n, \tag{6.17}$$

and combining it with the asymptotic version of Knuth's theorem proved in Chapter 4 gives the asymptotic growth rate of the moments of $|\,\mathrm{Tr}\, U_d|^2$, where U_d is a $d \times d$ Haar-distributed random unitary matrix.

The integral representation

$$\mathbb{E}(F_N(d)) = e^{-t} \int_{\mathcal{U}(d)} e^{\sqrt{t}\,\mathrm{Tr}(U+U^*)} dU \tag{6.18}$$

is the starting point of Johansson's theorem (Theorem 27), which we recall is the double scaling limit

$$\lim_{N\to\infty} F_N(s\sqrt{N}) = F(0)[s=2] + [s>2], \tag{6.19}$$

where $F(0) \approx 0.9694$ is the Tracy-Widom distribution function at $t = 0$. Johansson [Joh98] provides a beautiful argument showing how this step function behaviour, and hence Ulam's asymptotics

$$\mathbb{E}(\mathrm{lis}_N) \sim 2\sqrt{N}, \tag{6.20}$$

can be deduced from the formula

$$\lim_{d\to\infty} \frac{1}{d^2} \log \int_{\mathcal{U}(d)} e^{\frac{ds}{2}\operatorname{Tr}(U+U^*)} dU = \frac{s^2}{4}[0 \leq s \leq 1] + (s - \frac{\log s}{2} - \frac{3}{4})[s > 1], \tag{6.21}$$

which is due to Gross and Witten [20]. The limit (6.21) is associated with a phase transition in the "free energy" of lattice gauge theory in the limit $d \to \infty$. Indeed, physicists have long studied integrals of the form

$$\int_{\mathcal{U}(d)} e^{d\operatorname{Tr}(AU+BU^*)} dU, \tag{6.22}$$

where $A, B \in \mathcal{M}_d(\mathbb{C})$ are constant matrices, under the name *external source integrals*; see [20, BG80] for the physicists' perspective.

The usefulness of the interplay between Toeplitz determinants and unitary matrix integrals is greatly enhanced by the availaibility of powerful techniques for computing Toeplitz determinant asymptotics. The archetypal result of this sort is due to Szegö, a proof of which can be found in [BD02].

Theorem 52 (Strong Szegö Limit Theorem)**.** *Suppose that $f \in L^1(\mathbb{T})$ is expressed in terms of an auxiliary Laurent series*

$$\sum_{k=-\infty}^{\infty} c_k z^k$$

as the exponential of that series:

$$f(z) = e^{\sum_{k=-\infty}^{\infty} c_k z^k}.$$

Then, under mild conditions on the coefficients c_k*,*

$$D_d(f) \sim e^{dc_0 + \sum_{k\geq 1} kc_k c_{-k}}$$

as $d \to \infty$*.*

Example 53. Let us take our favourite Toeplitz determinant

$$\det(I_{j-i}(2t))_{1\leq i,j\leq d}, \tag{6.23}$$

which has symbol

$$f(z) = e^{tz^{-1}+tz}. \tag{6.24}$$

This symbol is exactly of the required form to use Szegö's theorem, with

$$c_k = t[k=-1] + t[k=1]. \tag{6.25}$$

Thus

$$\lim_{d\to\infty} \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} dU = \lim_{d\to\infty} \det(I_{j-i}(2t))_{1\leq i,j\leq d} = e^{t^2}, \tag{6.26}$$

and we have d-independent asymptotics for this unitary matrix integral. Of course we knew this already, since we arrived at the determinant/integral from the series

$$\sum_{n\geq 0} u_d(n) \frac{t^{2n}}{n!n!}, \tag{6.27}$$

and obviously $\lim_{d\to\infty} u_d(n) = n!$.

6.3 Toeplitz minors and the Bump-Diaconis extension

If M is a matrix, a *minor* of M is the determinant of a submatrix of M. Probably the first place one encounters minors is in linear algebra class, where it is usually proved that the k^{th} coefficient in the characteristic polynomial of M is the sum of all $k \times k$ principal minors of M.

Let $f \in L^1(\mathbb{T})$ have Fourier coefficients $(a_q)_{q\in\mathbb{Z}}$, and let $\mu, \lambda \in \mathbb{Y}_d$ be partitions with at most d parts. Form the $d \times d$ matrix

$$T_d(f;\mu,\lambda) := (a_{\lambda_i-\mu_j+j-i})_{1\le i,j\le d}. \tag{6.28}$$

The determinant of $T_d(f;\mu,\lambda)$ will be denoted $D_d(f;\mu,\lambda)$. When $\mu = \lambda = \emptyset$, $T_d(f;\mu,\lambda)$ is exactly the Toeplitz matrix $T_d(f)$. When μ,λ are non-trivial, $T_d(f;\mu,\lambda)$ is no longer a Toeplitz matrix, but it can be identified with a submatrix of a larger Toeplitz.

Example 54. Consider the simplest non-trivial example where $\mu = \emptyset, \lambda = (1)$. Then

$$T_4(f;\emptyset,\lambda) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_{-1} & a_0 & a_1 & a_1 \\ a_{-2} & a_{-1} & a_0 & a_1 \\ a_{-3} & a_{-2} & a_{-1} & a_0 \end{bmatrix}. \tag{6.29}$$

This is exactly the submatrix of the Toeplitz matrix $T_5(f)$ obtained by striking out the second row and first column.

In general, it is true that $T_d(f;\mu,\lambda)$ can be obtained from a larger Toeplitz matrix by striking out appropriate rows and columns. Thus the determinant $D_d(f;\mu,\lambda)$ is called a *Toeplitz minor*. Bump and Diaconis [BD02] extended both the Heine-Szegö identity and the Szegö Strong Limit Theorem to Toeplitz minors. In order to describe their result, we need to introduce the characters of the unitary group $\mathcal{U}(d)$.

A group homorphism

$$\varphi : \mathcal{U}(d) \to \mathcal{U}(d') \tag{6.30}$$

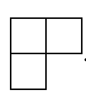
is called a *polynomial representation* if each entry of the output matrix $\varphi(U)$ is a polynomial in the entries of the input matrix U. The irreducible polynomial representations of $\mathcal{U}(d)$ are parameterized by the set $\mathbb{Y}_d$ of Young diagrams with at most d rows. If $\lambda \in \mathbb{Y}_d$, then the associated irreducible character $\operatorname{Tr} \varphi_\lambda(U)$ is a symmetric polynomial in the eigenvalues $E_1, \dots, E_d$ of U. It is called the *Schur polynomial* and denoted s_λ.

Schur polynomials admit a beautiful combinatorial description. A *semi-standard Young tableau* of shape λ is a filling of the cells of λ with positive integers which weakly increase along rows and strictly increase along columns. For a positive integer d, let $\mathrm{STab}_d(\lambda)$ denote the set of semi-standard Young tableaux of shape λ with entries chosen from the set $[d] = \{1, \dots, d\}$. Clearly, $\mathrm{STab}_d(\lambda)$ is empty if $\ell(\lambda) > d$. The Schur polynomial s_λ in d variables is the following generating function for tableaux $T \in \mathrm{STab}_d(\lambda)$:

$$s_\lambda(x_1, \dots, x_d) = \sum_{T \in \mathrm{STab}_d(\lambda)} x_1^{\#1(T)} x_2^{\#2(T)} \dots x_d^{\#d(T)}, \tag{6.31}$$

where $\#k(T)$ denotes the number of cells in T labelled k.

Example 55. Let

$\lambda =$  . (6.32)

Then

$$\mathrm{STab}_3(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \right\} \tag{6.33}$$

so the Schur polynomial of shape λ in three variables is

$$s_\lambda(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + 2x_1x_2x_3 + x_2^2x_3 + x_3x_2^2. \tag{6.34}$$

Since s_λ is the character of the representation of $\mathcal{U}(d)$ labelled by λ, the dimension of this representation is

$$s_\lambda(I_d) = |\,\mathrm{STab}_d(\lambda)|, \tag{6.35}$$

where I_d is the $d \times d$ identity matrix. For this number there is the following variant of the hook-length formula, called the *hook-content formula*:

$$s_\lambda(I_d) = \prod_{\square \in \lambda} \frac{d + c(\square)}{h(\square)}. \tag{6.36}$$

Here $h(\square)$ is the hook-length, and $c(\square)$ is a new combinatorial parameter called *content*, which is by definition the column index of $\square$ minus the row index of $\square$. These facts are all standard elements of the representation theory of the classical groups, see [Bum04, 17].

Having defined the Schur polynomials, we can now state Bump and Diaconis' extension of the Heine-Szegö identity.

Theorem 56 (Bump-Diaconis, [BD02])**.** *For any $f \in L^1(\mathbb{T})$ and any partitions $\lambda, \mu \in \mathbb{Y}_d$,*

$$D_d(f; \mu, \lambda) = \int_{\mathcal{U}(d)} \Phi_f(U) s_\mu(U) s_\lambda(U^*) dU,$$

where $s_\mu(U)$ is the Schur polynomial evaluated at the eigenvalues of U, and similarly for $s_\lambda(U^*)$.

Bump and Diaconis [BD02] also obtain a generalization of the Strong Szegö Limit

Theorem which is valid for Toeplitz minors. The application of this extension presented below only requires the degenerate case $\mu = \emptyset$ of their Theorem, see [BD02] for the case where $\mu \neq \emptyset$.

As in the original Szegö theorem, the Bump-Diaconis limit theorem applies to Toeplitz minors whose symbol f is presented as the exponential of an auxiliary Laurent series, viz.

$$f(z) = e^{\sum_{k=-\infty}^{\infty} c_k z^k}. \tag{6.37}$$

Let $\lambda \in \mathbb{Y}_d$ be a partition, and put $m = |\lambda|$. If $\pi \in S(m)$ is a permutation, define

$$\Delta_f(\pi) = \prod_{k \geq 1} (kc_k)^{\gamma_k(\pi)}, \tag{6.38}$$

where $\gamma_k(\pi)$ denotes the number of k-cycles in π. Note that Δ_f is evidently a central function.

Theorem 57 (Bump-Diaconis, [BD02])**.** *Let $f \in L^1(\mathbb{T})$ be represented as the exponential of the auxiliary Laurent series*

$$\sum_{k=-\infty}^{\infty} c_k z^k.$$

Let $\lambda \in \mathbb{Y}_d(m)$ be a partition. Then, under mild conditions on the coefficients c_k, we have

$$D_d(f; \emptyset, \lambda) \sim \left(\frac{1}{m!} \sum_{\pi \in S(m)} \chi^\lambda(\pi) \Delta_f(\pi) \right) e^{dc_0 + \sum_{k=1}^{\infty} kc_k c_{-k}}$$

as $d \to \infty$.

Note that the second factor in this asymptotic formula is precisely the asymptotic from the original Szegö limit theorem. The Bump-Diaconis extension says that in order to account for the partition λ we must scale by the factor

$$\langle \chi^\lambda, \Delta_f \rangle_{S(m)}. \tag{6.39}$$

In Chapter 5, we found that generalizations of Gessel's identity which deal with the enumeration of walks on the Weyl graph, or equivalently the enumeration of sequences of configurations of vicious random-turns walkers, involve Toeplitz minors (Theorem 47). Thus the generating function identities of Chapter 5 fit nicely with the Bump-Diaconis theory of Toeplitz minors, and this provides a very natural and direct connection between vicious walkers, which generalize increasing subsequences in permutations, and random unitary matrices.

Theorem 58 ([Nov09(1)]). *Let $\mu, \lambda \in \mathbb{Y}_d$ be partitions viewed as vertices of the Weyl graph $\overline{\mathbb{W}}_d$. Assume that $|\mu| \leq |\lambda|$. The number of walks $\mu \to \lambda$ on $\overline{\mathbb{W}}_d$ which take $n \geq 0$ negative steps is*

$$Z_{\overline{\mathbb{W}}_d}(2n+|\lambda|-|\mu|;\mu,\lambda) = \left[\frac{t^{2n+|\lambda|-|\mu|}}{(2n+|\lambda|-|\mu|)!}\right] \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} s_\mu(U) s_\lambda(U^*) dU.$$

Moreover, for any word W_n in the raising and lowering operators on $\overline{\mathbb{W}}_d$ which has n L's and $n+|\lambda|-|\mu|$ R's, the number of walks $\mu \to \lambda$ on $\overline{\mathbb{W}}_d$ compatible with this pattern is

$$Z_{\overline{\mathbb{W}}_d}(W_n;\mu,\lambda) = \left[\frac{t^{2n+|\lambda|-|\mu|}}{n!(n+|\lambda|-|\mu|)!}\right] \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} s_\mu(U) s_\lambda(U^*) dU.$$

Because of the isomorphism $\mathbb{W}_d \to \overline{\mathbb{W}}_d$, these statements are equivalent, respectively, to matrix integral representations of the unrefined and refined partition function of the random-turns model of vicious walkers. In the vicious walker language, Theorem 58 was proved by Adler and Van Moerbeke [AV05] using different methods.

6.4 Truncation

We will now apply the Bump-Diaconis theorems on Toeplitz minors together with the results of Chapter 5 in order to derive some combinatorial properties of truncated random unitary matrices. Truncated random unitary matrices were introduced by Sommers and Zyczkowski [SZ00] in the context of scattering theory, and have since then received further attention from physicists [FyoKho07, ForKri09] and probabalists [PR04, PR05, Kri09]. These studies have dealt primarily with properties of the spectrum, and to the author's knowledge combinatorial aspects of the ensemble averages have not previously been considered.

Consider the linear map

$$T : \mathcal{M}_{d+1}(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C}) \tag{6.40}$$

which acts by deleting the last row and column of a matrix. T is called the *truncation* operator.

Example 59. T maps a 3×3 matrix onto its 2×2 principal submatrix:

$$T \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}. \tag{6.41}$$

More generally, for any non-negative integer $q \geq 0$, the q-fold composition

$$T^{(q)} : \mathcal{M}_{d+q}(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C}) \tag{6.42}$$

maps a $(d+q) \times (d+q)$ matrix onto its $d \times d$ principal submatrix.

Observe that T is a contractive mapping, where the norm on $\mathcal{M}_d(\mathbb{C})$ is the operator norm $\| \cdot \|$, and hence $T^{(q)}$ is a contractive mapping for any $q \geq 0$. Thus the restriction of $T^{(q)}$ to the unitary group $\mathcal{U}(d+q)$ is a map

$$T^{(q)} : \mathcal{U}(d+q) \to \mathcal{B}(d), \tag{6.43}$$

where

$$\mathcal{B}(d) = \{P \in \mathcal{M}_d(\mathbb{C}) : \|P\| \leq 1\} \tag{6.44}$$

is the closed unit ball in the normed space $\mathcal{M}_d(\mathbb{C})$. Note that $\mathcal{B}(d)$ is itself the semigroup of linear contractions of Euclidean space $\mathbb{C}^d$.

Let $\tau_d^{(q)}$ denote the pushforward of Haar measure dU on $\mathcal{U}(d+q)$ under the q-fold truncation map. This is a probability measure on the contraction semigroup $\mathcal{B}(d)$. When $q = 0$, it is simply the Haar measure supported on $\mathcal{U}(d) \subset \mathcal{B}(d)$. Hence $\tau_d^{(q)}$ is a deformation of Haar measure on $\mathcal{U}(d)$ depending on the parameter $q \geq 0$. A random contraction matrix $P_d^{(q)}$ whose distribution in $\mathcal{B}(d)$ is $\tau_d^{(q)}$ is said to come from the ensemble of *truncated random unitary matrices*. Note that this is actually a family of ensembles depending on the discrete parameter q.

Theorem 60 (Neretin, [Ner02]). *For $1 \leq q < d$ the probability measure $\tau_d^{(q)}$ is singular and supported on a subset of the boundary $\partial\mathcal{B}(d)$. For $q \geq d$, the probability measure $\tau_d^{(q)}$ is absolutely continous and has density*

$$\frac{H_{R(d,q)}}{\pi^{d^2} H_{R(d,q-d)}} \det(I - P^* P)^{q-d} dP$$

with respect to Lebesgue measure on $\mathcal{B}(d)$.

Here the scalar $H_{R(d,q)}$ is the hook-product of the $d \times q$ rectangular Young diagram, i.e.

$$H_{R(d,q)} = \prod_{i=0}^{d-1} \frac{(q+i)!}{i!}. \tag{6.45}$$

Consider a random matrix U_d drawn from the Haar measure on $\mathcal{U}(d)$. The eigenvalues $E_1, \ldots, E_d$ are a random point process on the unit circle $\mathbb{T}$. An effective way to describe a random point process is to give its correlation functions. The *k-point correlation function*

$$\rho_k(z_1, \ldots, z_k) \tag{6.46}$$

is by definition the probability density that an infinitesimal neighbourhood of each of the points $z_1, \ldots, z_k$ contains a point of the spectrum of U_d. If we can solve exactly for all correlation functions of the spectrum of a random matrix, then we have an *exactly solvable* random matrix model. The Haar-distributed random unitary matrix is exactly solvable. The form of its correlation functions is a fundamental result in random matrix theory, see [Meh04].

Theorem 61. *The k-point correlation function of U_d is given by*

$$\rho_k(z_1, \ldots, z_k) = \det(K_d(z_i, z_j))_{1 \leq i,j \leq k},$$

where

$$K_d(z, w) = \frac{1}{2\pi} \sum_{j=0}^{d-1} z^j \overline{w}^j.$$

A random point process whose correlation functions are specified in this way by the determinant of a bivariate function K, called the *correlation kernel*, is called a *determinantal point process*. Determinantal point processes have many remarkable properties, see [Sos00, HKPV06]. It is a well-known phenomenon in random matrix theory that the spectra of "nice" random matrix models tend to be determinantal processes. As discovered by Sommers and Zyczkowski, the deformation $U_d \to P_d^{(q)}$ preserves the determinantal structure of the correlation functions.

Theorem 62 (Sommers-Zyczkowski, [SZ00])**.** *Let $d, q \geq 1$, and let $P_d^{(q)}$ be a random* *contraction matrix whose distribution in $\mathcal{B}(d)$ is $\tau_d^{(q)}$. Then the spectrum of $P_d^{(q)}$ is a determinantal point process in the unit disc $\mathbb{D}$ with kernel*

$$K_d^{(q)}(z,w) = \frac{q}{\pi}(1-|z|^2)^{\frac{q-1}{2}}(1-|w|^2)^{\frac{q-1}{2}} \sum_{j=0}^{d-1} \binom{q+j}{j} z^j \overline{w}^j.$$

We can use the Sommers-Zyczkowski theorem to prove the following result, which is well-known in the literature but usually proved by different arguments, see e.g. [PR04, Kri09].

Theorem 63. *For any fixed $d \geq 1$, the scaled contraction matrix $q^{1/2}P_d^{(q)}$ converges to the Ginibre matrix X_d of i.i.d standard complex Gaussians in the limit $q \to \infty$.*

Proof. In the statement of the theorem, by "converges" we mean that the correlation functions of the spectrum of $q^{1/2}P_d^{(q)}$ converge pointwise to the correlation functions of X_d as $q \to \infty$. Since the determinant is continuous, we only have to check this for the kernel.

Scaling the matrix $P_d^{(q)}$ by the factor $q^{1/2}$ corresponds to the following scaling of the kernel:

$$q^{-1}K_d^{(q)}(q^{-1/2}z, q^{-1/2}w) = \frac{1}{\pi}(1-\frac{|z|^2}{q})^{\frac{q-1}{2}}(1-\frac{|w|^2}{q})^{\frac{q-1}{2}} \sum_{j=0}^{d-1} \binom{q+j}{z} q^{-j} z^j \overline{w}^j.$$

Now since

$$\binom{q+j}{j} \sim \frac{q^j}{j!}$$

as $q \to \infty$, we have the pointwise limit

$$\lim_{q\to\infty} q^{-1}K_d^{(q)}(q^{-1/2}z, q^{-1/2}w) = \frac{1}{\pi} e^{\frac{1}{2}(|z|^2+|w|^2)} \sum_{j=0}^{d-1} \frac{1}{j!} z^j \overline{w}^j,$$

which is well-known [Meh04] to be the correlation kernel of the (determinantal) eigenvalue process of X_d. □

This theorem shows how naturally $P_d^{(q)}$ interpolates between the Circular Unitary Ensemble (case $q = 0$) and Ginibre's Unitary Ensemble (case $q = \infty$).

Given that the truncation operation preserves so much of the probabalistic structure of Haar unitaries, it is natural to wonder whether it also preserves their combinatorial structure. Our next theorem shows that this is in fact the case.

Theorem 64 ([Nov09(1)]). *Let $d \geq 1$ and $q \geq 0$, and consider the integral*

$$\mathcal{Z}_d(t;q) := \frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP)$$

over the contraction semigroup against the deformed Haar measure. Then

$$\mathcal{Z}_d(t;q) = \sum_{N \geq 0} Z_d(N;q) \frac{t^N}{N!},$$

where $Z_d(N;q)$ is the number of ways in which d random-turns vicious walkers can move from adjacent initial sites $(d, d-1, \dots, 1)$ to shifted sites $(q+d, q+d-1, \dots, q+1)$ in N jumps.

Proof. In order to prove this, we will make use of the so-called *Bosonic Colour-Flavour Transformation* due to Wei and Wettig [WW05], which arose in the context of lattice gauge theory. This is the matrix integral identity

$$\int_{\mathcal{U}(d+q)} e^{\mathrm{Tr}(Y^*UX+X^*U^*Y)} dU = H_{R(d,q)} \int_{\mathcal{U}(d)} e^{\mathrm{Tr}(UY^*Y+X^*XU^*)} \det(UY^*X)^{-q} dU, \quad (6.47)$$

which holds for any two rectangular matrices $X, Y \in \mathcal{M}_{(d+q)\times d}(\mathbb{C})$ such that the product Y^*X is non-singular. Despite the daunting name, the proof of this identity is a fairly straightforward application of character expansion: expand each integrand as a power series, convert powers of the trace into linear combinations of Schur functions, and use Schur orthogonality.

Let $X = Y$ be rectangular matrices with all entries on the main diagonal equal to t, and all other entries equal to zero. Then the Colour-Flavour Transformation specializes to the identity

$$\frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP) = \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} \det(U^*)^q dU. \tag{6.48}$$

It is easy to check from the combinatorial definition of Schur polynomials that a power of the determinant is a rectangular Schur function, i.e.

$$\det(U^*)^q = s_{R(d,q)}(U^*). \tag{6.49}$$

Thus, by the Bump-Diaconis identity relating unitary matrix integrals to Toeplitz minors, we have

$$\frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP) = \det(I_{q+j-i}(2t))_{1\le i,j\le d}. \tag{6.50}$$

From our results on determinantal generating functions in Chapter 5, we know that this Toeplitz minor is the claimed generating function for random-turns vicious walkers. □

Some remarks regarding this Theorem are in order:

1. The unitary integral

$$\int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} \det(U^*)^q dU \tag{6.51}$$

which appeared in the proof of the above theorem is known in the physics literature as the *Leutwyler-Smilga integral*, see [LS92].

2. Note that the Toeplitz minor $\det(I_{q+j-i}(2t))_{1\le i,j\le d}$ which arises in the fundamental identity

$$\frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP) = \det(I_{q+j-i}(2t))_{1\le i,j\le d} \tag{6.52}$$

is canonical; it is the determinant of the $d \times d$ upper right corner of the Toeplitz matrix $T_{d+q}(f)$, with f the symbol

$$f(z) = e^{tz^{-1}+tz}. \tag{6.53}$$

3. From the point of view of Fisher's model, the fact that we consider configurations of vicious walkers that start and end on adjacent sites is canonical. Indeed, if we consider the the walkers as representing mutually attracting particles, then the ground states of the system correspond to walkers positioned on adjacent sites. See Fisher's article [Fis84] for the physical significance of "reunions" of vicious walkers.

4. From the point of view of walks on the Weyl graph, or "lattice walks in a Weyl chamber" (this is the more usual phrasing in combinatorics), we are considering walks on $\mathbb{Z}^d$ which remain in the type A chamber and begin and end on the line formed by the mutual intersection of the walls which define the chamber. These walks are precisely the ones for which the constraint of confinement to the chamber is most stringent, i.e. a particle in random motion deep in the chamber is unlikely to encounter a wall, whereas one near the boundary line likely will.

5. In terms of random walkers, the previous remark says that, on a short time scale, there is no apparent difference between a vicious walker model and a friendly walker model if the walkers are initially positioned at very distant lattice sites. The characteristics of a vicious walker model are most pertinent when the initially occupied sites are close together.

6. Remark that in Neretin's theorem (Theorem 60), the distribution of $P_d^{(q)}$ in $\mathcal{B}(d)$

is singular for $q < d$ and absolutely continuous for $q \geq d$. This "phase transition" has significance in the vicious walker interpretation: $q \geq d$ corresponds to a *proper shift* of the walkers, i.e. they experience a reunion at previously unoccupied sites.

7. Thus the most interesting configurations of vicious walkers/walks in a Weyl chamber are coupled to a *parametric* matrix model, by which we mean that the dependence on the endpoints of the walk is encoded in the measure, not in the integrand.

Theorem 64 is not merely a restatement of a Toeplitz determinant identity — it enables us to perform new calculations regarding the partition function $Z_d(N;q)$ of mutually-attracting random-turns particles.

Theorem 65 (Walk-to-infinity with finite back-tracking, [Nov09(1)])**.** *Let $d \geq 1$ and $n \geq 0$ be fixed. Then*

$$Z_d(2n+dq;q) \sim (2\pi)^{\frac{1-d}{2}} \left(\prod_{i=0}^{d-1} i!\right) \frac{1}{n!} d^{3n+dq+\frac{1}{2}} q^{n+\frac{1-d^2}{2}}$$

as $q \to \infty$.

Proof. Extracting the coefficient of $t^{2n+dq}/(2n+dq)!$ in the series expansion of the matrix integral $\mathcal{Z}_d(t;q)$, we obtain the formula

$$Z_d(N;q) = [N = 2n+dq] \frac{(2n+dq)!}{H_{R(d,q)} n!n!} \int_{\mathcal{B}_d} |\operatorname{Tr} P|^{2n} \tau_d^{(q)}(dP) \tag{6.54}$$

(this is a q-dependent generalization of Rains' result (6.16)). The $q \to \infty$ asymptotics of the scaling factor outside the integral are easily computed using Stirling's formula. As for the integral itself, we known that $q^{1/2} P_d^{(q)}$ converges to the Ginibre matrix X_d :

$$\lim_{q\to\infty} q^n \int_{\mathcal{B}(d)} |\operatorname{Tr} P|^{2n} \tau_d^{(q)}(dP) = \mathbb{E}(|\operatorname{Tr} X_d|^{2n}). \tag{6.55}$$

Now, the entries of X_d are i.i.d copies of the standard complex Gaussians z. The moments of z are given by

$$\mathbb{E}(z^m \overline{z}^n) = [m = n]n!. \tag{6.56}$$

By independence, $\operatorname{Tr} X_d$ is just a complex Gaussian of variance d. Hence

$$\int_{\mathcal{B}(d)} |\operatorname{Tr} P|^{2n} \tau_d^{(q)}(dP) \sim \frac{d^n n!}{q^n} \tag{6.57}$$

as $q \to \infty$, and putting this together with the required Stirling formula computation produces the asymptotic formula in the statement of the theorem. □

Before moving on let us remark on one nice feature of the above asymptotic formula. The formula deals with the situation when the particles in Fisher's model move to very distant sites by executing random-turns, with only a fixed number n of negative turns allowed. We note that Theorem 65 shows that when $n < \frac{d^2-1}{2}$ the asymptotic growth in question is of the form

$$\frac{\text{exponential in } q}{\text{polynomial in } q}, \tag{6.58}$$

(damped exponential), whereas when $n > \frac{d^2-1}{2}$ we have the growth rate

$$(\text{exponential in } q)(\text{polynomial in } q) \tag{6.59}$$

(faster than exponential). Thus there is a critical point in the model at $n = \frac{d^2-1}{2}$, and this could be physically meaningful.

So far we have used the Gaussian limit of truncated random unitary matrices to make asymptotic statements about the vicious walker model. Now let us go in the other direction and use combinatorics to say something about truncated random unitary matrices.

As already mentioned, $P_d^{(q)}$ interpolates between a Haar unitary U_d and a Ginibre matrix X_d, with $q = 0$ corresponding to the unitary case and $q = \infty$ corresponding to the Gaussian case. There is significant interest in the case where q is positive and finite, since this represents a matrix model which is neither unitary nor Gaussian but rather an intermediate state between the two. The limit where q is fixed, positive, and finite and $d \to \infty$ was called the *weakly non-unitary limit* in [SZ00]. We will now obtain an extension of the asymptotic formula

$$\lim_{d\to\infty} \int_{\mathcal{U}(d)} e^{t\,\mathrm{Tr}(U+U^*)} dU = e^{t^2} \tag{6.60}$$

in the weakly non-unitary case, using the Bump-Diaconis extension of Szegö's theorem[1].

Theorem 66. *For fixed $t > 0$ and $q \geq 0$,*

$$\lim_{d\to\infty} \frac{t^{dq}}{H_{R(d,q)}} \int_{\mathcal{B}(d)} e^{t\,\mathrm{Tr}(P+P^*)} \tau_d^{(q)}(dP) = \frac{e^{t^2}}{q!} \sum_{k=0}^{q} h(q,k) t^k,$$

where $h(q,k)$ is the number of permutations in $S(q)$ with exactly k fixed points.

Proof. The integral in question equals the Toeplitz minor

$$D_d(f; \emptyset, (q)) = \det(I_{q+j-i}(2t))_{1\leq i,j\leq d}, \tag{6.61}$$

and the asymptotics of this minor can be worked out using the Bump-Diaconis extension to Szegö's limit theorem.

According to this extension, we have the asymptotics

$$D_d(f; \emptyset, (q)) \sim \frac{e^{t^2}}{q!} \sum_{\pi \in S(q)} \chi^{(q)}(\pi) \Delta_f(\pi) \tag{6.62}$$

as $d \to \infty$. Recall that f is the symbol

$$f(z) = e^{tz^{-1}+tz}, \tag{6.63}$$

[1] I thank Boris Khoruzhenko for suggesting to me the problem of computing the asymptotics of this integral in the weakly non-unitary limit

and this explains the factor e^{t^2} which comes from the part of the Bump-Diaconis formula corresponding to the original Szegö theorem. Since there is only one term of positive degree in the auxiliary Laurent series expressing f, we have

$$\Delta_f(\pi) = \prod_{k\geq 1}(kc_k)^{\gamma_k(\pi)} = t^{\gamma_1(\pi)}, \tag{6.64}$$

where $\gamma_1(\pi)$ is the number of one-cycles (a.k.a fixed points) in π. Since our partition $\lambda = (q)$ is a single row, $\chi^{(q)}$ is the trivial character of $S(q)$. Hence we have the asymptotics

$$D_d(f;\emptyset,(q)) \sim \frac{e^{t^2}}{q!}\sum_{\pi\in S(q)} t^{\gamma_1(\pi)} = \frac{e^{t^2}}{q!}\sum_{k=0}^{q} h(q,k)t^k, \tag{6.65}$$

which completes the proof. □

Chapter 7

Hermitian Matrix Models and Genus Expansion: Brief Review

7.1 Introduction

Historically, the development of Wigner's Gaussian Unitary Ensemble and Dyson's Circular Unitary Ensemble has proceeded in tandem. This can be seen in many of the disciplines which employ random matrices. The canonical example is of course mathematical physics, where the use of Hermitan matrix models in quantum gravity and string theory predates the use of unitary matrix models. However, unitary matrix models have gained prominence in their own right, due largely to their role in lattice gauge theory and quantum chromodynamics [20, BG80, DZ83]. In number theory, the Hilbert-Polya-Montgomery-Dyson ansatz relating zeroes of the Riemann zeta function and eigenvalues of GUE matrices has to some extent been superseded by the analogy between moments of the zeta function and moments of CUE characteristic polynomials [CD03]. In probability theory and statistics, interest in Gaussian random matrices is classical [Wis28] whereas the study of Haar-distributed random matrices is more recent [DS94]. Finally, in combinatorics, the use of Hermitian matrix integrals as generating functions for maps on surfaces [23] predates interest in unitary matrix integrals as generating functions for tableaux-theoretic objects [Rai98].

Our goal in the next Chapter will be to identify and analyze the CUE analogue of the well-known *genus expansion* for the moments of the spectral measure of a GUE random matrix. In order to set the stage, this expository Chapter contains a brief review of the genus expansion of Hermitian matrix models.

7.2 Spectral measure and Wigner's semicircle law

Starting with a Ginibre matrix X_N, whose entries are i.i.d complex Gaussians of mean zero and variance $1/N$, one can construct a GUE Hermitian matrix H_N. This construction is straightforward: put

$$H_N := \frac{1}{2}(X_N + X_N^*),$$

the average of X_N and its adjoint. An object of principal interest in the analysis of H_N is the *spectral measure*, also known as the empirical eigenvalue distribution. It is well-known that in the limit $N \to \infty$ the spectral measure converges weakly to a deterministic limit distribution; this is Wigner's semicircle law, which we now briefly describe.

Using the fact that the entries of H_N are independent (modulo the self-adjointness condition) and have a Gaussian law, one can easily show that the distribution of H_N in the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices has density

$$\frac{1}{Z_N} e^{-\frac{N}{2}\operatorname{Tr}(H^2)} dH, \tag{7.1}$$

where

$$Z_N = \int_{\mathcal{H}_N} e^{-\frac{N}{2}\operatorname{Tr}(H^2)} dH \tag{7.2}$$

is the partition function. This density on $\mathcal{H}_N$ is called the *GUE measure*.

The *spectral measure* μ_N of H_N is by definition the random measure on $\mathbb{R}$ which places mass $\frac{1}{N}$ on each of the eigenvalues of $E_1 \geq \cdots \geq E_N$ of H_N, with multiplicity. That is,

$$\mu_N(\{x\}) = \frac{1}{N}\sum_{i=1}^{N}[x = E_i] \tag{7.3}$$

for each $x \in \mathbb{R}$. The moments of the spectral measure are

$$\rho_N(n) := \int_{\mathbb{R}} x^n \mu_N(dx) = \frac{1}{NZ_N}\int_{\mathcal{H}_N} \operatorname{Tr}(H^n) e^{-\frac{N}{2}\operatorname{Tr}(H^2)} dx. \tag{7.4}$$

The linearity of the trace together with the fact that the GUE measure gives the same weight to H and $-H$ implies that the odd moments of the spectral measure vanish.

The sequence of spectral measures μ_N converges weakly to a deterministic limit measure μ. This limit measure has density

$$\mu(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}[-2 \leq x \leq 2]dx, \tag{7.5}$$

is called the *semicircle measure*. This fact was first proved by Wigner, and is accordingly known as *Wigner's semicircle law*. No thesis which deals with random matrices is complete without a picture of the semicircle density, see Figure 7.1. Note that the name "semicircle" is erroneous; the curve is in fact a semi-ellipse with major axis 4 and minor axis $\frac{2}{\pi} \approx 0.64$.

Wigner's law has a beautiful combinatorial proof, which has long been known to physicists and was independently rediscovered by Harer and Zagier [23] in 1986. Let $\varepsilon_g(n)$ denote the number of ways in which the sides of a $2n$-gon can be glued together in pairs in order to produce a compact orientable surface of genus g. For example, the

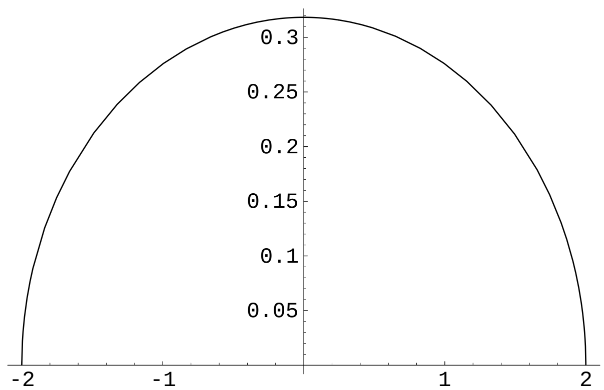


Figure 7.1: The semicircle density

glueings of a square or hexagon result in either the sphere or the torus, as pictured in Figure 7.2.

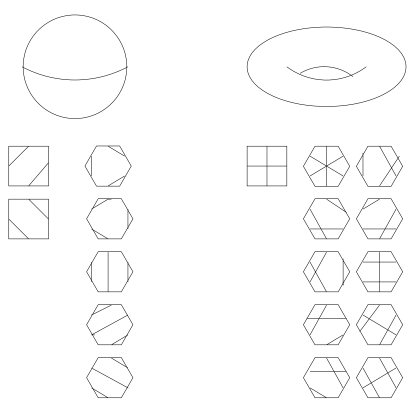


Figure 7.2: Glueing polygons to make surfaces

Theorem 67 (Harer-Zagier, [23]). *The even moments* $\rho_N(2n)$ *of the spectral measure* μ_N *have the* $\frac{1}{N}$*-expansion*

$$\rho_N(2n) = \varepsilon_0(n) + \frac{\varepsilon_1(n)}{N^2} + \frac{\varepsilon_2(n)}{N^4} + \dots. \tag{7.6}$$

The proof of Theorem 67 uses the Wick formula for Gaussian random variables, see Zvokine [Zvo97] or Nica and Speicher [NicSpei06].

Example 68. From Figure 7.2, we have

$$\rho_N(4) = 2 + \frac{1}{N_2} \tag{7.7}$$

$$\rho_N(6) = 5 + \frac{10}{N^2}. \tag{7.8}$$

The next two moments are

$$\rho_N(8) = 14 + \frac{70}{N^2} + \frac{21}{N^4} \tag{7.9}$$

$$\rho_N(10) = 42 + \frac{420}{N^2} + \frac{483}{N^4}. \tag{7.10}$$

The Harer-Zagier theorem is a bridge between random matrix theory and the combinatorics of *maps on surfaces*, and it is valuable from both points of view. From the perspective of random matrix theory, the Harer-Zagier theorem provides a direct combinatorial proof of Wigner's semicircle law as follows. First, it follows immediately from Theorem 67 that

$$\rho_N(2n) = \varepsilon_0(n) + O(\frac{1}{N^2}), \tag{7.11}$$

and in particular

$$\lim_{N\to\infty} \rho_N(2n) = \varepsilon_0(n). \tag{7.12}$$

One can show that, as Figure 7.2 suggests, a glueing of the $2n$-gon results in the sphere if and only if it corresponds to a non-crossing pairing of the polygon sides. Since the number of non-crossing pairings of a $2n$-element linearly ordered set is the Catalan number, we have

$$\varepsilon_0(n) = \mathrm{Cat}_n\,. \tag{7.13}$$

Thus the moment generating function of μ_N satisfies

$$\lim_{N\to\infty} M_{\mu_N}(t) = \sum_{n\geq 0} \mathrm{Cat}_n \frac{t^{2n}}{(2n)!} = \frac{I_1(2t)}{t}. \tag{7.14}$$

It follows that the characteristic function of μ_N satisfies

$$\lim_{N\to\infty} M_{\mu_N}(it) = \frac{J_1(-2t)}{t}, \tag{7.15}$$

where $J_\alpha(t)$ is the non-modified Bessel function, which is related to $I_\alpha(t)$ by $I_\alpha(t) = i^{-\alpha} J_\alpha(it)$. Using the integral representation

$$J_1(t) = \frac{1}{\pi}\int_0^\pi \cos(\theta - t\sin\theta)d\theta \tag{7.16}$$

we can compute the Fourier transform of $\frac{J_1(-2t)}{t}$ and thereby recover the semicircle density.

Chapter 8

Unitary Matrix Integrals and Jucys-Murphy Elements

8.1 Preamble

Let $\mathcal{U}(N)$ denote the compact group of $N \times N$ complex unitary matrices equipped with normalized Haar measure dU. A random Haar-distributed matrix U_N from $\mathcal{U}(N)$ is said to be from the *Circular Unitary Ensemble* (CUE). Our main goal in this Chapter is to address the question: what is the analogue of genus expansion for the CUE?

Let us begin by arguing that the analogue we seek does not in fact involve the spectral measure of U_N. Let $(E_1 = e^{i\theta_1}, \dots E_N = e^{i\theta_N})$ denote the ordered list of eigenvalues of U_N, listed clockwise such that $2\pi \geq \theta_1 \geq \cdots \geq \theta_N \geq 0$. One then has the following classical result due to Weyl.

Theorem 69 (Weyl integration formula)**.** *The probability density of the random point*

$(E_1, \ldots, E_N) \in \mathbb{T}^N$ *is given by*

$$\frac{1}{Z_N} e^{-2U(\theta_1,\ldots,\theta_N)} d\theta,$$

where

$$U(\theta_1, \ldots, \theta_N) = -\sum_{1 \leq i < j \leq N} \log(\theta_i - \theta_j)$$

and Z_N *is a normalization constant.*

Note that U is a potential on the circle analogous to the potential W on $\mathbb{R}$ studied in Chapter 4, but without the harmonic attractor term. Thus the eigenvalues of U_N behave as mutually repulsive point charges on the unit circle $\mathbb{T}$, and by compactness one expects that with high probability they will be uniformly distributed. It is a result of Diaconis and Shashahani [DS94] that this is indeed the case.

Theorem 70 (Diaconis-Shashahani [DS94])**.** *The spectral measure of* U_N *converges to the uniform measure on* $\mathbb{T}$ *as* $N \to \infty$*.*

Thus, the eigenvalues of U_N form a point process whose bulk behaviour is less interesting than that of GUE matrices H_N. However, the situation is precisely the opposite if we look at the *entries* of U_N. Indeed, the entries of H_N are independent up to the symmetry constraints imposed by selfadjointness. Like H_N, U_N may be constructed starting with the Ginibre matrix X_N, this time by applying the Gram-Schmidt orthonormalization procedure to the columns of X_N. This process introduces complicated dependencies between the entries of U_N, and of principal interest are the *correlation functions*

$$\left\langle \prod_{k=1}^{n} u_{i(k)j(k)} \overline{u_{i'(k)j'(k)}} \right\rangle_N = \int_{\mathcal{U}(N)} \prod_{k=1}^{n} u_{i(k)j(k)} \overline{u_{i'(k)j'(k)}} dU \qquad (8.1)$$

Here $n \geq 1$ is a positive integer, $i, j, i', j' \in [N]^{[n]}$ are indices, and the u_{ij}'s are matrix entries. It is an observation of De Wit and t'Hooft [12] that, like the moments of the spectral measure of a GUE matrix H_N, the correlation functions of a CUE matrix U_N expand perturbatively in powers of $1/N$. The goal of this Chapter is to give a combinatorial interpretation of this perturbative series that parallels the genus expansion in the GUE setting. Quite strikingly, the underlying combinatorics turns out to be that of decomposing symmetric functions of *Jucys-Murphy elements* into linear sums of conjugacy classes. Following our tendency to "put combinatorics first," we begin with a discussion of this problem, and then follow with its connection to the correlations of CUE random matrices. This Chapter consists of joint work with Sho Matsumoto from Nagoya University.

8.2 Introduction

8.2.1 Jucys-Murphy elements

The *Jucys-Murphy elements* are certain commuting elements in the group algebra of the symmetric group $S(n)$. They are defined by

$$J_k = \sum \text{transpositions in } S(k) - \sum \text{transpositions in } S(k-1),\ 1 \leq k \leq n. \quad (8.2)$$

Thus $J_k = (1,k) + \cdots + (k-1,k)$ for $2 \leq k \leq n$ and $J_1 = 0$. The JM elements were introduced by Jucys in [26], and independently rediscovered some years later by Murphy [32]. Although their definition may appear somewhat arbitrary at first glance, these elements have many remarkable properties. Most notably, they play a crucial role in the representation theory of the symmetric groups, a fact which was fully realized relatively recently in the work of Okounkov and Vershik [38, 39].

Jucys' original interest in the elements (8.2) stemmed from their remarkable symmetry properties: although the JM elements do not lie in the center $Z(n)$ of $\mathbb{C}[S(n)]$, symmetric functions of them do. More precisely, let Λ denote the $\mathbb{C}$-algebra of symmetric functions in commuting indeterminates $x_1, x_2, \dots$ and let Ξ_n be the multiset $\{\{J_1, \dots, J_n, 0, 0, \dots\}\}$. Jucys [26] proved that

$$f(\Xi_n) = f(J_1, \dots, J_n, 0, 0, \dots) \in Z(n) \tag{8.3}$$

for every $f \in \Lambda$.

8.2.2 The class expansion problem

Given Jucys' result, it is natural to ask for the multiplicity of a given conjugacy class in $f(\Xi_n)$. More formally, let $\mathfrak{c}_\mu(n)$ denote the (sum of) all permutations in $S(n)$ of reduced cycle-type μ. For example, $\mathfrak{c}_{(0)}(n)$ is the class of the identity in $S(n)$ and $\mathfrak{c}_{(1)}(n)$ is the class of transpositions. Then

$$\{\mathfrak{c}_\mu(n) : \mathrm{wt}(\mu) \leq n\} \tag{8.4}$$

is a linear basis of $Z(n)$, which will be called the *reduced class basis*. Here the weight[1] of a partition is defined by $\mathrm{wt}(\mu) := |\mu| + \ell(\mu)$, where $|\mu|$ is the sum of the parts of μ and $\ell(\mu)$ is the number of parts. We ask for the value of the coefficients $G_\mu(f, n)$ in the linear expansion

$$f(\Xi_n) = \sum_{\mathrm{wt}(\mu) \leq n} G_\mu(f, n) \mathfrak{c}_\mu(n). \tag{8.5}$$

This will be referred to as the *class expansion problem.* Our motives for studying this problem are twofold: on one hand, the class expansion problem is of intrinsic interest,

[1]this non-standard definition of the weight of a partition is very convenient for our purposes. An algebro-geometric interpretation of this weight function can be found in [14, 15].

and on the other it is a useful combinatorial paradigm in the study of unitary matrix integrals. This second point will be explained after we present our combinatorial results.

The computation of $G_\mu(f,n)$ has been considered by various authors for various choices of f. Jucys [26] originally considered the case of the elementary symmetric functions $f = e_k$ and proved that

$$e_k(\Xi_n) = \sum_{|\mu|=k} \mathfrak{c}_\mu(n), \tag{8.6}$$

which says that the permutations appearing in the expansion of $e_k(\Xi_n)$ are precisely those which have $n-k$ cycles. This result was also obtained by Diaconis and Greene in [13]. More recently, Lascoux and Thibon [29], and also Lassalle [27], have considered the class expansion of the power-sums

$$p_k(\Xi_n) = J_1^k + \cdots + J_n^k, \tag{8.7}$$

and obtained information on the coefficients $G_\mu(p_k,n)$. Goulden and Jackson [19] have treated a related problem in the context of star factorizations, thereby generalizing previous work of Pak [40] and Irving and Rattan [25]. Murray [33] has considered the class expansion of complete symmetric functions.

In this article we consider the class expansion problem relative to the linear basis of monomial symmetric functions. However, we begin by studying the class expansion problem from a general perspective, as this highlights several important properties which hold for all symmetric functions. In Section 2 we obtain the following result.

Theorem 71. *Let μ be a fixed partition. Then:*

1. *The map $f \mapsto G_\mu(f,n)$ is a linear function on Λ taking values in the polynomials $\mathbb{C}[n]$.*

2. *$G_\mu(f,n)$ vanishes when* $\deg f < |\mu|$.

3. *When* $\deg f = |\mu|$, *the coefficient* $G_\mu(f) = G_\mu(f,n)$ *is independent of* n.

Part 1 of Theorem 71 asserts polynomial dependence of the coefficient $G_\mu(f,n)$ on the size of the alphabet Ξ_n. We show that this implies and extends a recent result of Stanley [42] on the polynomiality of certain averages over Plancherel-distributed random partitions (see also Olshanski [37]). Parts 2 and 3 of Theorem 71 give vanishing and n-independence criteria for class coefficients. In particular, it is very natural to look for a concrete formula for n-independent coefficients, and in Section 8.4 we solve this problem for the monomial symmetric functions.

8.2.3 Class expansion of monomial symmetric functions

A natural way to extend Jucys' solution of the class expansion problem for elementary symmetric functions is to consider the monomial symmetric functions. Let m_λ be the monomial symmetric function of type λ, and define coefficients $L_\mu^\lambda(n)$ by

$$m_\lambda(\Xi_n) = \sum_{\mathrm{wt}(\mu)\leq n} L_\mu^\lambda(n)\mathfrak{c}_\mu(n). \tag{8.8}$$

That is, $L_\mu^\lambda(n) = G_\mu(m_\lambda, n)$. In Section 8.4 we prove the following theorem, which substantially generalizes Jucys' result for $e_k = m_{(1^k)}$.

Theorem 72. *With $L_\mu^\lambda(n)$ as above, we have the following:*

1. *$L_\mu^\lambda(n)$ is a polynomial function of n.*

2. *$L_\mu^\lambda(n)$ vanishes unless* $|\mu| \leq |\lambda|$.

3. *$L_\mu^\lambda(n)$ vanishes unless* $|\mu| \equiv |\lambda| \mod 2$.

4. *When $|\mu| = |\lambda|$, the n-independent coefficients $L^\lambda_\mu = L^\lambda_\mu(n)$ are given explicitly by*

$$L^\lambda_\mu = \sum_{(\lambda^{(1)},\ldots,\lambda^{(\ell(\mu))})\in\mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} \mathrm{RC}(\lambda^{(i)}), \tag{8.9}$$

where the sum runs over refinement sequences of partitions

$$\mathfrak{R}(\lambda,\mu) = \{(\lambda^{(1)},\ldots,\lambda^{(\ell(\mu))}) : \lambda^{(i)} \vdash \mu_i \text{ and } \lambda = \lambda^{(1)} \cup \cdots \cup \lambda^{(\ell(\mu))}\} \tag{8.10}$$

and $\mathrm{RC}(\lambda)$ denotes the refined Catalan number of type λ.

The refined Catalan numbers appearing in this result will be introduced in Section 8.4 below. These numbers are known explicitly, and have previously appeared in work of Haiman [22] and Stanley [44].

Example 73. For the monomial symmetric function $m_{(3,1)}$ we have

$$m_{(3,1)}(\Xi_n) = 4\mathfrak{c}_{(4)}(n)+\mathfrak{c}_{(3,1)}(n)+2(3n-7)\mathfrak{c}_{(2)}(n)+2(2n-3)\mathfrak{c}_{(1^2)}(n)+\frac{1}{3}n(n-1)(n-2)\mathfrak{c}_{(0)}(n).$$

8.2.4 Class expansion of complete symmetric functions

The complete homogeneous symmetric functions h_k are defined by $h_k = \sum_{\lambda\vdash k} m_\lambda$ for $k \geq 0$, with $h_0 := 1$. Thus h_k is the sum of all monomials of degree k in the variables x_i. Set $F^k_\mu(n) := G_\mu(h_k, n)$. Then $F^k_\mu(n)$ is given by $F^k_\mu(n) = \sum_{\lambda\vdash k} L^\lambda_\mu(n)$, and by Theorem 72 non-zero coefficients must be of the form $F^{|\mu|+2g}_\mu(n)$ for $g \geq 0$. In Section 8.5, we determine the coefficients $F^{|\mu|+2g}_\mu(n)$ exactly when $g = 0$ and μ is arbitrary, and when $\mu = (n-1)$ and g is arbitrary. That is, we determine the n-independent coefficients as well as the coefficient of the class $\mathfrak{c}_{n-1}(n)$ of full cycles. It turns out that in these cases class coefficients are expressed in terms of two classical families of numbers from enumerative combinatorics, namely the ubiquitous *Catalan numbers*

[43, Exercise 6.19]

$$\mathrm{Cat}_m = \frac{1}{m+1}\binom{2m}{m} \tag{8.11}$$

and the more exotic *central factorial numbers* [43, Exercise 5.8]

$$T(m,n) = 2\sum_{k=1}^{n} \frac{k^{2m}(-1)^{n-k}}{(n-k)!(n+k)!}. \tag{8.12}$$

Theorem 74. *With $F_\mu^{|\mu|+2g}(n)$ as above, we have*

1. $F_\mu^{|\mu|} = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i}$, *a product of Catalan numbers.*
2. $F_{(n-1)}^{n-1+2g}(n) = \mathrm{Cat}_{n-1}\, T(n-1+g, n-1)$, *the product of a Catalan number and a central factorial number.*

Part 1 of Theorem 74 was first proved by Murray [33] in the framework of the Farahat-Higman algebra and independently rediscovered by the second author [34] via Collins' work [7] on first-order asymptotics of unitary matrix integrals; here this fact will be deduced from Part 4 of Theorem 72, which is a more general result with a completely combinatorial proof. We believe that Part 2 of Theorem 74, in which the central factorial numbers studied by Riordan and Carlitz appear, is new; the proof of this fact is representation-theoretic and relies on spectral properties of JM-elements.

8.2.5 Unitary matrix integrals and the class expansion problem

In Section 8.6, we apply various properties of the JM-elements to study matrix integrals of the form

$$\left\langle \prod_{k=1}^{n} u_{i(k)j(k)}\overline{u_{i'(k)j'(k)}} \right\rangle_N = \int_{\mathcal{U}(N)} \prod_{k=1}^{n} u_{i(k)j(k)}\overline{u_{i'(k)j'(k)}} dU. \tag{8.13}$$

Here $\mathcal{U}(N)$ is the compact group of $N \times N$ complex unitary matrices equipped with normalized Haar measure dU, $i, j, i', j' \in [N]^{[n]}$ is a quadruple of indices[2], and the u_{ij}'s are matrix elements. Integrals of this type have a long history in mathematical physics, where they are known as *n-point correlation functions*; see e.g. the classic work of De Wit and t'Hooft [12] and Samuel [41] as well as the recent survey by Morozov [31]. In particular, it was observed by De Wit and t'Hooft that for $1 \leq n \leq N$ the n-point functions can be expanded as power series in $1/N$ (called a "perturbative series" in physics), and the question of computing the coefficients in these series was raised. We show that the perturbative series in question are generating functions for class multiplicities in the expansion of complete symmetric functions of JM-elements.

Theorem 75. *Let $1 \leq n \leq N$, and let $\pi \in S(n)$ be a permutation of reduced cycle-type μ. Then*

$$(-1)^{|\mu|} N^{n+|\mu|} \left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \sum_{g \geq 0} \frac{F_\mu^{|\mu|+2g}(n)}{N^{2g}}.$$

Our proof of Theorem 75 goes through the *Weingarten calculus.* The Weingarten calculus gives a rigorous approach to the evaluation of matrix integrals of the type (8.13) based on the invariant theory of the unitary groups. It was developed by Collins and collaborators [7, 11, 2, 3, 10], building on earlier work of Xu [47], in response to the needs of free probability theory, where one needs a good understanding of the large N asymptotics of the n-point functions in order to establish asymptotic freeness results. We show that, much as Wick calculus leads to the map enumeration problem in the Hermitian setting, Weingarten calculus naturally leads to the class expansion problem, the end result being Theorem 75.

[2]We write $[s] = \{1, \ldots, s\}$, and $[s]^{[r]}$ is the set of all functions $i : [r] \to [s]$

The involvement of JM-element theory in questions relating to the n-point functions clarifies many issues. For example, Xu's results on the factorization of first-order asymptotics of the n-point functions [47] and Collins' explicit computation of those asymptotics [7] are natural consequences of the above expansion and Theorem 74. As another example, the mysterious *De Wit-t'Hooft anomalies* from the physics literature [12, 41, 31] obtain a natural spectral interpretation as the eigenvalues of JM-elements in the regular representation of the symmetric group algebra. Most importantly, the class expansion problem provides a framework for computing higher-order corrections in the asymptotic expansion of correlation functions.

Example 76. The expansion of the diagonal n-point function to fourth order is

$$N^n\left\langle \prod_{k=1}^{n} u_{kk}\overline{u_{kk}} \right\rangle_N = 1 + \frac{\frac{1}{2}n(n-1)}{N^2} + \frac{\frac{1}{24}n(n-1)(3n^2+17n-34)}{N^4} + O(\frac{1}{N^6}).$$

8.3 Class expansions in general

8.3.1 Notations: partitions and reduced cycle-type for permutations

We will index conjugacy classes of permutations by reduced cycle-type. This technique is convenient when dealing with conjugacy classes in the infinite symmetric group, see e.g. [30, I-7, Example 24].

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of non-negative integers such that $\ell(\lambda) := \#\{i \geq 1 \mid \lambda_i > 0\}$, the *length* of λ, is finite. Denote by $|\lambda|$ the sum $\sum_{i\geq 1} \lambda_i$ and call it the *size* of λ. If $|\lambda| = n$, we say that λ is a partition of n and often write $\lambda \vdash n$. Define the *weight* of λ to be $\mathrm{wt}(\lambda) := |\lambda| + \ell(\lambda)$. If

$\ell(\lambda) = l$, we define the partition $\tilde{\lambda}$ by $\tilde{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_l - 1)$. For each $n \geq 1$, the map $\lambda \mapsto \tilde{\lambda} = \mu$ gives a bijection between the set of partitions λ of size n and the set of partitions μ of weight at most n. Indeed, the inverse map is given by $\mu \mapsto (\mu_1 + 1, \dots, \mu_{\ell(\mu)} + 1, 1^{n-\mathrm{wt}(\mu)}) = \lambda$. In particular, $|\mu| = |\lambda| - \ell(\lambda)$. If μ, λ are partitions related by $\mu = \tilde{\lambda}$, we say that μ is the *reduction* of λ and, dually, λ is the *inflation* of μ.

We say that a permutation $\sigma \in S(n)$ has *reduced cycle-type* μ if $\mu = \tilde{\lambda}$ and the original cycle-type of σ is $\lambda \vdash n$. Let $C_\mu(n)$ be the conjugacy class in $S(n)$ of permutations of reduced cycle-type μ. The set $C_\mu(n)$ is non-empty if and only if $\mathrm{wt}(\mu) \leq n$. For example, $C_{(0)}(n)$ consists of the identity permutation and $C_{(1)}(n)$ is the class of transpositions in $S(n)$. Note that if μ is the reduced cycle-type of σ, then $|\mu|$ is equal to the minimal length of a factorization of σ into transpositions.

Define $\mathfrak{c}_\mu(n)$ to be the sum $\sum_{\sigma \in C_\mu(n)} \sigma$, which is an element of the center $Z(n)$ of the group algebra $\mathbb{C}[S(n)]$. We have $\mathfrak{c}_\mu(n) = 0$ unless $\mathrm{wt}(\mu) \leq n$. The set $\{\mathfrak{c}_\mu(n) \mid \mathrm{wt}(\mu) \leq n\}$ is the *reduced class basis* of $Z(n)$.

8.3.2 Jucys' result

For completeness, we give a proof of Jucys' theorem that any symmetric function in the JM elements is central. Following Jucys [26], the proof is based on the explicit calculation of elementary symmetric functions in JM elements. Recall that by definition we have

$$e_k(\Xi_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} J_{i_1} J_{i_2} \cdots J_{i_k}.$$

Proposition 77 ([26])**.** *For any $0 \leq k \leq n$, $e_k(\Xi_n)$ is the indicator function of* *permutations with exactly $n-k$ cycles. That is,*

$$e_k(\Xi_n) = \sum_{\mu \vdash k} \mathfrak{c}_\mu(n) = \sum_{\substack{\sigma \in S(n) \\ \nu_n(\sigma)=n-k}} \sigma, \tag{8.14}$$

where $\nu_n(\sigma)$ denotes the number of cycles in σ.

Proof. We proceed by induction on n. If $n=2$, then $e_1(\Xi_2) = J_2 = (1,2)$ and the claim is trivial.

Let $n > 2$ and suppose that (8.14) holds true for $e_k(\Xi_{n-1})$ with any k. We define the projection P_n from $S(n)$ to $S(n-1)$ by

$$P_n(\sigma)(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \neq n \\ \sigma(n) & \text{if } \sigma(i) = n, \end{cases}$$

for $\sigma \in S(n)$ and $1 \leq i \leq n-1$. In other words, $P_n(\sigma)$ is defined to be the permutation whose cycle decomposition is obtained by erasing the letter n in the cycle decomposition of σ. For each $\tau \in S(n-1)$, we have $P_n^{-1}(\tau) = \{\tau(s,n) \mid 1 \leq s \leq n-1\} \cup \{\tau\}$. Here τ is regarded as an element of $S(n)$, which fixes the letter n. Observe $\nu_n(\tau(s,n)) = \nu_{n-1}(\tau)$ and $\nu_n(\tau) = \nu_{n-1}(\tau)+1$. Thus, the right hand side on (8.14) equals

$$\sum_{\tau \in S(n-1)} \sum_{\substack{\sigma \in P_n^{-1}(\tau) \\ \nu_n(\sigma)=n-k}} \sigma = \sum_{\substack{\tau \in S(n-1) \\ \nu_{n-1}(\tau)=n-1-k}} \tau + \sum_{\substack{\tau \in S(n-1) \\ \nu_{n-1}(\tau)=n-k}} \sum_{s=1}^{n-1} \tau(s,n).$$

By the induction hypothesis, the first sum on the right hand side equals $e_k(J_1, \dots, J_{n-1})$ Since $e_k(x_1, \dots, x_n) = e_k(x_1, \dots, x_{n-1}) + e_{k-1}(x_1, \dots, x_{n-1})x_n$, we obtain the equality (8.14) for n. □

Let Λ denote the algebra of symmetric functions over $\mathbb{C}$. Proposition 77 has the following important corollary.

Corollary 78. *For any* $f \in \Lambda$, $f(\Xi_n) = f(J_1, J_2, \dots, J_n, 0, 0, \dots) \in Z(n)$.

Proof. The well-known "fundamental theorem" of symmetric function theory states that $\Lambda = \mathbb{C}[e_k : k = 1, 2, \dots]$. This fact and Proposition 77 together imply the corollary. □

Corollary 79. *Let* z *be an indeterminate. Then*

$$(z + J_1) \dots (z + J_n) = \sum_{\mathrm{wt}(\mu) \le n} z^{n-|\mu|} \mathfrak{c}_\mu(n).$$

Remark 80. Farahat and Higman [17] have shown that the class sums

$$a_k = \sum_{|\mu|=k} \mathfrak{c}_\mu(n), \qquad 0 \le k \le n,$$

generate $Z(n)$: $Z(n) = \mathbb{C}[a_0, \dots, a_n]$. But Jucys' result (Proposition 77) says that $a_k = e_k(\Xi_n)$. Thus we have

$$Z(n) = \{f(\Xi_n) \mid f \in \Lambda\}.$$

8.3.3 Character expansions

For each partition λ of n, let χ^λ be the irreducible character of $S(n)$ associated with λ and $\dim \lambda$ the value of χ^λ at the identity id_n: $\dim \lambda = \chi^\lambda(\mathrm{id}_n)$. The χ^λ form a basis of $Z(n)$.

Let A_λ be the multi-set of contents of λ: $A_\lambda = \{\{j-i \mid 1 \le i \le \ell(\lambda),\ 1 \le j \le \lambda_i\}\}$. For example, $A_{(2,2)} = \{\{1, 0, 0, -1\}\}$.

Let f be any symmetric function. From Corollary 78, the element $f(\Xi_n) = f(J_1, J_2, \dots, J_n, 0, 0, \dots)$ can be expanded as a linear combination of the characters χ^λ. The following proposition is well known.

Proposition 81. *Let f be any symmetric function. Then*

$$f(\Xi_n) = \sum_{\lambda \vdash n} \frac{\dim \lambda}{n!} f(A_\lambda)\chi^\lambda. \tag{8.15}$$

Proof. This formula follows immediately from the following well-known identities

$$f(\Xi_n)\chi^\lambda = f(A_\lambda)\chi^\lambda, \qquad \mathrm{id}_n = \sum_{\lambda \vdash n} \frac{\dim \lambda}{n!}\chi^\lambda,$$

the first of which is due to Jucys [26] while the second is a consequence of the second orthogonality relation for irreducible group characters. □

8.3.4 Class coefficients: general properties

Let $f \in \Lambda$. Consider the class coefficients $G_\mu(f,n)$ defined by

$$f(\Xi_n) = \sum_{\mathrm{wt}(\mu) \leq n} G_\mu(f,n)\mathfrak{c}_\mu(n). \tag{8.16}$$

Theorem 82. *For fixed μ, f the coefficient $G_\mu(f,n)$ has the following properties:*

1. *$G_\mu(f,n)$ is a polynomial function of n.*
2. *If $|\mu| > \deg f$, then $G_\mu(f,n)$ is identically zero.*
3. *If $|\mu| = \deg f$, then $G_\mu(f,n)$ is independent of n.*

The second and third statements of this theorem will be proved in the next section.

Proof of the first statement of Theorem 82. The product $\mathfrak{c}_\lambda(n)\mathfrak{c}_\mu(n)$ in $Z(n)$ is a linear combination of the $\mathfrak{c}_\nu(n)$, say $\mathfrak{c}_\lambda(n)\mathfrak{c}_\mu(n) = \sum_\nu a^\nu_{\lambda\mu}(n)\mathfrak{c}_\nu(n)$ with non-negative integers $a^\nu_{\lambda\mu}(n)$. A classical result of Farahat and Higman [17] asserts that the coefficients $a^\nu_{\lambda\mu}(n)$ are polynomials in n. Thus, for each partition $\rho = (\rho_1, \rho_2, \ldots)$, the coefficients of

$$e_\rho(\Xi_n) = \prod_{i \geq 1} e_{\rho_i}(\Xi_n) = \prod_{i \geq 1}(\sum_{\mu \vdash \rho_i} \mathfrak{c}_\mu(n))$$

in $\mathfrak{c}_\nu(n)$ are also polynomials in n. Since any symmetric function is a finite sum of the e_ρ, the claim follows. □

Proposition 81 and the first statement of Theorem 82 imply the following claim.

Corollary 83. *Let f be a symmetric function and μ a partition such that $\mathrm{wt}(\mu) \leq n$. Then*

$$\sum_{\lambda \vdash n} f(A_\lambda)\chi^\lambda_{\mu\cup(1^{n-|\mu|})}\frac{\dim\lambda}{n!}$$

is a polynomial function of n. Here $\mu \cup (1^{n-|\mu|})$ is the partition of n obtained from μ by adding $n - |\mu|$ parts equal to 1.

Stanley [42] proved Corollary 83 in the case $\mu = (0)$ by a combinatorial argument. The significance of this result is in its interaction with the Plancherel measure on partitions. Recall that the *Plancherel probability measure* on the set of Young diagrams $\mathbb{Y}(n)$ is defined by

$$\mathfrak{P}_n(\lambda) = \frac{(\dim\lambda)^2}{n!},\ \lambda \in \mathbb{Y}(n). \tag{8.17}$$

Thus, when $\mu = (0)$, the sum in Corollary 83 degenerates to the Plancherel expectation

$$\langle f(A_\lambda)\rangle_{\mathfrak{P}_n} = \sum_{\lambda\in\mathbb{Y}(n)} f(A_\lambda)\frac{(\dim\lambda)^2}{n!}, \tag{8.18}$$

and Corollary 83 becomes the statement that this expectation is a polynomial function of n. Olshanski [37] recently gave an alternative proof of this polynomiality property using the theory of shifted symmetric functions, and here we see that it also follows from properties of the JM elements.

8.4 Class expansion of monomial symmetric functions

8.4.1 Monomial symmetric functions

Given a partition λ of length $\ell(\lambda) \leq n$, we denote by m_λ the monomial symmetric polynomial:

$$m_\lambda(x_1, x_2, \ldots, x_n) = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

summed over all distinct permutations $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The power-sum symmetric polynomial is $p_k = m_{(k)}$, the elementary symmetric polynomial is $e_k = m_{(1^k)}$, and the complete symmetric polynomial is $h_k = \sum_{|\lambda|=k} m_\lambda$.

We wish to generalize Jucys' result on the class expansion of elementary symmetric functions $e_k(\Xi_n)$ to arbitrary monomial symmetric polynomials. Let $L^\lambda_\mu(n)$ be the coefficient of $\mathfrak{c}_\mu(n)$ in $m_\lambda(J_1, \ldots, J_n)$:

$$m_\lambda(\Xi_n) = \sum_{\mathrm{wt}(\mu)\leq n} L^\lambda_\mu(n)\mathfrak{c}_\mu(n) \tag{8.19}$$

According to part 1 of Theorem 82, $L^\lambda_\mu(n)$ is a polynomial function of n. Our ultimate goal is to evaluate all $L^\lambda_\mu(n)$. However, unfortunately, for general λ, μ the $L^\lambda_\mu(n)$ appear to be quite complicated (see the examples in the Appendix) and we have not been able to solve this problem in full generality. The result we obtain in this paper is that, if $|\lambda| = |\mu|$, $L^\lambda_\mu(n)$ is independent of n and given explicitly by Theorem 90 below.

8.4.2 Refinements of Catalan numbers

Let $\mathrm{Cat}_r = \frac{1}{r+1}\binom{2r}{r}$ be the rth Catalan number:

r	0	1	2	3	4	5	6	7	8
Cat_r	1	1	2	5	14	42	132	429	1430

It is well known that Catalan numbers satisfy the recurrence

$$\mathrm{Cat}_r = \sum_{q=0}^{r-1} \mathrm{Cat}_q \mathrm{Cat}_{r-1-q}. \tag{8.20}$$

Catalan numbers have a large number of combinatorial interpretations, see e.g. [43, Exercises 6.19].

We will use the following interpretation of the Catalan numbers. For a positive integer k, let $\mathfrak{C}(k)$ be the set of all weakly increasing sequences $(i_1, \dots, i_k)$ of k positive integers satisfying $i_p \geq p$ for $1 \leq p \leq k-1$ and $i_k = k$. For example,

$$\mathfrak{C}(3) = \{(123), (133), (223), (233), (333)\}.$$

As proved in Section 8.4.5 (see also [43, Exercises 6.19 (s)]), the cardinality of $\mathfrak{C}(k)$ equals Cat_k.

Let $(i_1, \dots, i_k)$ be a weakly increasing sequence of k positive integers. We say that $(i_1, \dots, i_k)$ is of type $\lambda \vdash k$ if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a permutation of $(b_1, b_2, \dots)$, where, for each $p \geq 1$, b_p is the multiplicity of p in $(i_1, \dots, i_k)$.

Example 84. The sequences (1233), (1334), and (1134) are of type $(2, 1, 1)$, while the sequences (444477799), (555669999) are of type $(4, 3, 2)$.

Definition. *Given a partition $\lambda \vdash k$, let the* refined Catalan number $\mathrm{RC}(\lambda)$ *be the number of sequences $(i_1, \dots, i_k)$ in $\mathfrak{C}(k)$ of type λ. For convenience, set $\mathrm{RC}(\lambda) = 1$ if λ is the empty partition.*

Example 85. The four sequences (1444), (2444), (3444), (3334) in $\mathfrak{C}(4)$ are all of type $(3, 1)$, and indeed $\mathrm{RC}(3, 1) = 4$. We have $\mathrm{RC}(k) = \mathrm{RC}(1^k) = 1$.

Proposition 86. *The sum of* RC(λ) *over* $\lambda \vdash k$ *equals* Cat_k*:*

$$\sum_{\lambda \vdash k} \mathrm{RC}(\lambda) = \mathrm{Cat}_k.$$

Proof. This is a direct consequence of the fact that $|\mathfrak{E}(k)| = \mathrm{Cat}_k$. □

Example 87. We give some examples of RC(λ) for small $|\lambda|$. "SUM" stands for the sum $\sum_{\lambda \vdash k} \mathrm{RC}(\lambda) = \mathrm{Cat}_k$.

λ	1
RC(λ)	1

λ	2	1^2	SUM
RC(λ)	1	1	2

λ	3	21	1^3	SUM
RC(λ)	1	3	1	5

λ	4	31	2^2	21^2	1^4	SUM
RC(λ)	1	4	2	6	1	14

λ	5	41	32	31^2	2^21	21^3	1^5	SUM
RC(λ)	1	5	5	10	10	10	1	42

The explicit expression of RC(λ) is known and given as follows. See [44] and also [22, Section 2.6 and Section 4.1].

Proposition 88 ([44])**.** *Given a parition* λ*,*

$$\mathrm{RC}(\lambda) = \frac{|\lambda|!}{(|\lambda| - \ell(\lambda) + 1)! \prod_{i \ge 1} m_i(\lambda)!} = \frac{1}{|\lambda|+1} m_\lambda(1^{|\lambda|+1}).$$

Here $m_i(\lambda)$ *is the multiplicity of* i *in* $\lambda = (\lambda_1, \lambda_2, \dots)$*.* □

Note that the number $\mathrm{RC}(a^m) = \frac{1}{(a-1)m+1}\binom{am}{m}$ is often called a higher Catalan number. In particular, $\mathrm{RC}(2^m) = \mathrm{Cat}_m$.

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If $\mathfrak{R}(\lambda, \mu) \neq \emptyset$, then we say that λ is a refinement of μ (see [30, I-6]).

8.4.3 Formula for constant coefficients

Let $k = |\lambda|$. We will prove that $L^\lambda_\mu(n)$ is zero unless $|\mu| \leq k$ and $|\mu| \equiv k \pmod 2$, so that

$$m_\lambda(\Xi_n) = \sum_{r=0}^{[\frac{k}{2}]} \sum_{\substack{\mu \vdash k-2r \\ \ell(\mu) \leq n-k+2r}} L^\lambda_\mu(n) \mathfrak{c}_\mu(n). \tag{8.21}$$

We evaluate coefficients $L^\lambda_\mu(n)$ in the expansion of $m_\lambda(\Xi_n)$ meeting the condition $|\mu| = k$. The following theorem is our main result on constant coefficients in class expansions.

Theorem 90. *Let $\lambda, \mu \vdash k$ and $n \geq k + \ell(\mu)$. Then*

$$L^\lambda_\mu(n) = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell(\mu))}) \in \mathfrak{R}(\lambda,\mu)} \mathrm{RC}(\lambda^{(1)}) \mathrm{RC}(\lambda^{(2)}) \cdots \mathrm{RC}(\lambda^{(\ell(\mu))}). \tag{8.22}$$

In particular, $L^\lambda_\mu = L^\lambda_\mu(n)$ is independent of n, and L^λ_μ is zero unless λ is a refinement of μ.

This theorem implies part 3 of Theorem 82 because monomial symmetric functions form a basis of the algebra of symmetric functions.

Observe that for $\lambda, \mu \vdash k$,

$$L^\lambda_{(k)} = \mathrm{RC}(\lambda), \qquad L^\lambda_{(1^k)} = \delta_{\lambda,(1^k)}, \qquad L^{(k)}_\mu = \delta_{\mu,(k)}, \qquad L^{(1^k)}_\mu = 1, \qquad L^\lambda_\lambda = 1.$$

The equality $L^{(1^k)}_\mu = 1$ is compatible with Proposition 77.

As we saw in the previous subsection, unless $\lambda \leq \mu$, $\mathfrak{R}(\lambda, \mu) = \emptyset$ (see [30, I-6

(6.10)]), and hence $L^\lambda_\mu = 0$. The matrix $(L^\lambda_\mu)_{\lambda,\mu\vdash k}$ is therefore strictly lower unitriangular in the sense of [30, I-6].

We give the proof of Theorem 90 below. The numbers L^λ_μ for $\lambda, \mu \vdash k$, for $k \leq 7$, will be tabulated in the Appendix.

Example 91. By Example 87, Example **??**, and Theorem 90, we have

$$L^{(3,2,2,1)}_{(5,3)} = \mathrm{RC}(3,2)\mathrm{RC}(2,1) + \mathrm{RC}(2,2,1)\mathrm{RC}(3) = 5 \times 3 + 10 \times 1 = 25.$$

8.4.4 Technical lemmas

For each $n \geq 1$, a permutation $\sigma \in S(n)$ can be regarded as a permutation in $S(n+1)$ that fixes the letter $n+1$. Thus, we obtain an embedding $S(n) \subset S(n+1)$, and can define the inductive limit $S(\infty) = \bigcup_{n=1}^{\infty} S(n)$. Under the inclusion $S(n) \subset S(n+1)$, the reduced cycle-type of $\sigma \in S(n)$ is invariant. Define the support of $\sigma \in S(\infty)$ by

$$\mathrm{supp}(\sigma) = \{i \mid \sigma(i) \neq i\}.$$

If the reduced cycle-type of σ is μ, then $|\mathrm{supp}(\sigma)| = \mathrm{wt}(\mu)$.

Lemma 92. *Given a permutation π and a transposition (s,t), let $\Pi = \pi(s,t)$. Suppose that $\Lambda = (\Lambda_1, \Lambda_2, \dots)$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ are reduced cycle-types of Π and π, respectively. Then we have $|\Lambda| = |\lambda| \pm 1$. Furthermore, if $|\Lambda| = |\lambda| + 1$, then $\mathrm{supp}(\Pi) = \mathrm{supp}(\pi) \cup \{s,t\}$, and s, t belong to the same cycle of Π.*

Proof. Given a permutation π and a transposition (s,t), the following four cases occur: (i) $|\mathrm{supp}(\pi) \cap \{s,t\}| = 0$; (ii) $|\mathrm{supp}(\pi) \cap \{s,t\}| = 1$; (iii) $s, t \in \mathrm{supp}(\pi)$, and s, t belong to different cycles of π; (iv) $s, t \in \mathrm{supp}(\pi)$, and s, t belong to the same cycle of π.

For the case (i), we obtain $\Lambda = \lambda \cup (1)$ immediately. In the case (ii), we may suppose $\mathrm{supp}(\pi) \cap \{s, t\} = \{s\}$. Then π has a cycle $(\dots, s, \pi(s), \dots)$, and Π has the cycle $(\dots, s, t, \pi(s), \dots)$. Therefore Λ has a part equal to $\lambda_j + 1$. In the case (iii), π has two cycles of the forms $(\dots, \pi^{-1}(s), s, \pi(s), \dots)$ and $(\dots, \pi^{-1}(t), t, \pi(t), \dots)$. Therefore Π has the combined cycle $(\dots, \pi^{-1}(s), s, \pi(t), \dots, \pi^{-1}(t), t, \pi(s), \dots)$. Thus, a certain part Λ_k of Λ equals $\lambda_i + \lambda_j + 1$ for some $1 \le i < j \le \ell(\lambda)$. In the case (iv), π has a cycle of the form

$$(\dots, \pi^{-1}(s), s, \pi(s), \dots, \pi^{-1}(t), t, \pi(t), \dots),$$

and so Π has divided cycles $(\dots, \pi^{-1}(s), s, \pi(t), \dots)$ and $(\pi(s), \dots, \pi^{-1}(t), t)$. Thus, there are Λ_j and Λ_k equal to $r - 1$ and $\lambda_i - r$ for some λ_i and $r \ge 1$.

For the case (iv), Λ and λ satisfy the identity $|\Lambda| = |\lambda| - 1$. For other cases (i),(ii), and (iii), we have $|\Lambda| = |\lambda| + 1$. The rest claims are seen in the above. □

Corollary 93. *Let σ be a permutation of reduced cycle-type λ. Suppose that σ is expressed as $(s_1, t_1) \cdots (s_p, t_p)$, where $s_i < t_i$ $(1 \le i \le p)$. Then $|\lambda| \le p$ and $|\lambda| \equiv p \pmod 2$.*

This corollary implies part 3 of Theorem 82.

If σ is a permutation of reduced cycle-type $\lambda \vdash r$, and if σ can be expressed by r transpositions

$$\sigma = (s_1, t_1) \cdots (s_r, t_r), \tag{8.23}$$

then we say that (8.23) is a minimal factorization of σ.

Lemma 94. *Let $\lambda \vdash r$ and let σ be a permutation of reduced cycle-type λ. Suppose that σ is expressed as $(s_1, t_1)(s_2, t_2) \cdots (s_r, t_r)$, where $s_i < t_i$ $(1 \le i \le r)$ and $2 \le t_1 \le \cdots \le t_r$. Then $\mathrm{supp}(\sigma) = \{s_1, t_1, s_2, t_2, \dots, s_r, t_r\}$. Furthermore, for each i, the* *letters s_i, t_i belong to the same cycle of σ.*

Proof. For each $1 \leq i \leq r$, define $\sigma_i = (s_1, t_1) \cdots (s_i, t_i)$. It follows by Lemma 92 that the size of the reduced cycle-type of σ_i must be i, and that $\text{supp}(\sigma_i) = \text{supp}(\sigma_{i-1}) \cup \{s_i, t_i\}$. In addition, s_i, t_i belong to the same cycle of σ_i, and therefore to the one of σ. □

Lemma 95. *Let $\tau^{(1)}$ and $\tau^{(2)}$ be permutations such that $i < j$ for all $i \in \text{supp}(\tau^{(1)})$ and $j \in \text{supp}(\tau^{(2)})$. Suppose that reduced cycle-types of $\tau^{(1)}$ and $\tau^{(2)}$ have the weight r_1 and r_2, respectively. Also, suppose that $\sigma := \tau^{(1)}\tau^{(2)}$ has an expression $\sigma = (s_1, t_1) \cdots (s_r, t_r)$, where $r = r_1 + r_2$, $s_i < t_i$ $(1 \leq i \leq r)$, and $2 \leq t_1 \leq \cdots \leq t_r$. Then,*

$$\tau^{(1)} = (s_1, t_1) \cdots (s_{r_1}, t_{r_1}), \qquad \tau^{(2)} = (s_{r_1+1}, t_{r_1+1}) \cdots (s_r, t_r).$$

Proof. By Lemma 94, we see $\text{supp}(\tau^{(1)}) \sqcup \text{supp}(\tau^{(2)}) = \text{supp}(\sigma) = \{s_1, t_1, \ldots, s_r, t_r\}$. Since t_i are not decreasing, there exists an integer p such that $t_1, \ldots, t_p \in \text{supp}(\tau^{(1)})$ and $t_{p+1}, \ldots, t_r \in \text{supp}(\tau^{(2)})$. Furthermore, applying Lemma 94 again, we see that s_i, t_i belong to the same cycle of σ, and so that $\text{supp}(\tau^{(1)}) = \{s_1, t_1, \ldots, s_p, t_p\}$ and $\text{supp}(\tau^{(2)}) = \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}$. In particular, for any $i \in \{s_1, t_1, \ldots, s_p, t_p\}$ and $j \in \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}$, we have $\tau^{(1)}(i) = \sigma(i)$ and $\tau^{(2)}(j) = \sigma(j)$.

Let $\rho^{(1)} = (s_1, t_1) \cdots (s_p, t_p)$ and $\rho^{(2)} = (s_{p+1}, t_{p+1}) \cdots (s_r, t_r)$. Since $\sigma = \rho^{(1)}\rho^{(2)}$ we have $\{s_1, t_1, \ldots, s_p, t_p\} = \text{supp}(\rho^{(1)})$ and $\{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\} = \text{supp}(\rho^{(2)})$. Therefore for any $i \in \{s_1, t_1, \ldots, s_p, t_p\}$ and $j \in \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}$, we have $\rho^{(1)}(i) = \sigma(i)$ and $\rho^{(2)}(j) = \sigma(j)$. This means $\tau^{(1)} = \rho^{(1)}$ and $\tau^{(2)} = \rho^{(2)}$. In particular, the sizes of the reduced cycle-type of $\rho^{(1)}$ and $\rho^{(2)}$ are r_1 and r_2, respectively. By definition of $\rho^{(i)}$ and Corollary 93, we have $r_1 \leq p$ and $r_2 \leq r - p$. But $r = r_1 + r_2$ so that $p = r_1$. Therefore $\tau^{(1)} = \rho^{(1)} = (s_1, t_1) \cdots (s_{r_1}, t_{r_1})$. The desired expression for $\tau^{(2)}$ also follows. □

8.4.5 Expressions for cycles

Let a, r be non-negative integers. Define the set $\mathfrak{E}(a;r)$ by

$$\mathfrak{E}(a;r) = \{(i_1,\dots,i_r) \in \mathbb{Z}^r \mid i_1 \le \dots \le i_r, \quad i_p \ge a+p \ (1 \le p \le r-1), \quad i_r = a+r\}$$

for $r \ge 1$ and let $\mathfrak{E}(a;0) = \emptyset$. The set $\mathfrak{E}(r)$ defined in Section 8.4.2 coincides with $\mathfrak{E}(0;r)$, and the mapping $(i_1,\dots,i_r) \mapsto (a+i_1,\dots,a+i_r)$ gives a bijection from $\mathfrak{E}(r)$ to $\mathfrak{E}(a;r)$. Put

$$\begin{aligned}
\mathfrak{E}_0(a;r) =& \{(i_1,\dots,i_r) \in \mathfrak{E}(a;r) \mid i_p > a+p \ (1 \le p \le r-1)\},\\
\mathfrak{E}_1(a;r) =& \{(i_1,\dots,i_r) \in \mathfrak{E}(a;r) \mid i_1 = a+1, \ i_p > a+p \ (2 \le p \le r-1)\},\\
&\vdots\\
\mathfrak{E}_q(a;r) =& \{(i_1,\dots,i_r) \in \mathfrak{E}(a;r) \mid i_q = a+q, \ i_p > a+p \ (q+1 \le p \le r-1)\},\\
&\vdots\\
\mathfrak{E}_{r-1}(a;r) =& \{(i_1,\dots,i_r) \in \mathfrak{E}(a;r) \mid i_{r-1} = a+r-1\}.
\end{aligned}$$

Then we obtain the decomposition $\mathfrak{E}(a;r) = \bigsqcup_{q=0}^{r-1} \mathfrak{E}_q(a;r)$. For each $(i_1,\dots,i_r) \in \mathfrak{E}_q(a;r)$ with $0 \le q \le r-2$, we have $i_{r-1} = a+r$. Therefore, for each $0 \le q \le r-1$, the mapping

$$(i_1,\dots,i_q,i_{q+1},\dots,i_r) \mapsto ((i_1,\dots,i_q),(i_{q+1},\dots,i_{r-1}))$$

gives a bijection from $\mathfrak{E}_q(a;r)$ to $\mathfrak{E}(a;q) \times \mathfrak{E}(a+q+1;r-1-q)$. Here when either $q=0$ or $q=r-1$, we regard the set $\mathfrak{E}(a;q) \times \mathfrak{E}(a+q+1;r-1-q)$ as $\mathfrak{E}(a+1;r-1)$ or $\mathfrak{E}(a;r-1)$, respectively. Thus, we obtain a natural identification

$$\begin{aligned}
\mathfrak{E}(a;r) &= \mathfrak{E}_0(a;r) \sqcup (\textstyle\bigsqcup_{q=1}^{r-2} \mathfrak{E}_q(a;r)) \sqcup \mathfrak{E}_{r-1}(a;r)\\
&\cong \mathfrak{E}(a+1;r-1) \sqcup (\textstyle\bigsqcup_{q=1}^{r-2} (\mathfrak{E}(a;q) \times \mathfrak{E}(a+q+1;r-1-q))) \sqcup \mathfrak{E}(a;r-1).
\end{aligned} \tag{8.24}$$

In particular, $|\mathfrak{C}(r)| = |\mathfrak{C}(r-1)| + \sum_{q=1}^{r-2} |\mathfrak{C}(q)||\mathfrak{C}(r-1-q)| + |\mathfrak{C}(r-1)|$ for $r \geq 2$. Comparing this equation with (8.20), we have $|\mathfrak{C}(a;r)| = |\mathfrak{C}(r)| = \mathrm{Cat}_r$ for all $r \geq 1$.

For two positive integers a, r, we define the cycle $\xi(a;r)$ of length $r+1$ by

$$\xi(a;r) = (a, a+1, \ldots, a+r).$$

For convenience, we let $\xi(a;0)$ to be the identity permutation. The following proposition is the key to our proof of Theorem 90.

Proposition 96. *Let $t_1, \ldots, t_r$ be positive integers satisfying $2 \leq t_1 \leq \cdots \leq t_r$. The cycle $\xi(a;r)$ may be expressed as a product of r transpositions*

$$\xi(a;r) = (s_1, t_1)(s_2, t_2) \cdots (s_r, t_r), \qquad s_i < t_i \ (1 \leq i \leq r) \tag{8.25}$$

if and only if

$$(t_1, \ldots, t_r) \in \mathfrak{C}(a;r). \tag{8.26}$$

Furthermore, for each $(t_1, \ldots, t_r) \in \mathfrak{C}(a;r)$, the expression (8.25) of $\xi(a;r)$ is unique.

Example 97. Consider the cycle $\xi(1;9) = (1, 2, \ldots, 10)$ and three sequences

$$(3,5,5,5,8,8,8,9,10), \quad (3,4,4,7,7,9,9,10,10), \quad (9,9,9,9,10,10,10,10,10)$$

in $\mathfrak{C}(1;9)$. The corresponding expressions of $\xi(1;9)$ are given as follows:

$$(2,3)(4,5)(3,5)(1,5)(7,8)(6,8)(5,8)(8,9)(9,10),$$

$$(2,3)(3,4)(1,4)(6,7)(5,7)(8,9)(7,9)(9,10)(4,10).$$

$$(8,9)(7,9)(6,9)(5,9)(9,10)(4,10)(3,10)(2,10)(1,10).$$

□

Proof of Proposition 96. We proceed by induction on r. When $r = 1$, since $\xi(a;1) = (a, a+1)$, and since $\mathfrak{C}(a;1)$ consists of a sequence $(a+1)$ of length 1, our claims are trivial. Let $r > 1$ and suppose that for cycles of length $< r+1$, all claims in the theorem hold true.

(i) First, we suppose that the cycle $\xi(a;r)$ is given by the form (8.25). Then we have $t_r = a+r$ because t_r is the maximum among $\mathrm{supp}(\xi(a;r))$, where $\mathrm{supp}(\xi(a;r)) = \{s_1, t_1, \dots, s_r, t_r\}$ by Lemma 94. If we write as $s_r = a+q$ with $0 \le q \le r-1$, we have

$$(s_1, t_1) \cdots (s_{r-1}, t_{r-1}) = (a, a+1, \dots, a+q)(a+q+1, a+q+2, \dots, a+r).$$

By Lemma 95, we see that

$$\begin{aligned} (s_1, t_1) \cdots (s_q, t_q) =& (a, a+1, \dots, a+q), \\ (s_{q+1}, t_{q+1}) \cdots (s_{r-1}, t_{r-1}) =& (a+q+1, a+q+2, \dots, a+r). \end{aligned} \tag{8.27}$$

By the induction hypothesis for cycles of length $q+1$ and of length $r-q$, we have $(t_1, \dots, t_q) \in \mathfrak{E}(a;q)$ and $(t_{q+1}, \dots, t_{r-1}) \in \mathfrak{E}(a+q+1; r-1-q)$. This fact and Equation (8.24) imply $(t_1, \dots, t_q, t_{q+1}, \dots, t_{r-1}, t_r) \in \mathfrak{E}_q(a;r) \subset \mathfrak{E}(a;r)$.

(ii) Next, we suppose $(t_1, \dots, t_r) \in \mathfrak{E}(a;r)$. According to the decomposition $\mathfrak{E}(a;r) = \bigsqcup_{q=0}^{r-1} \mathfrak{E}_q(a;r)$, there exists a unique number q such that $0 \le q \le r-1$ and $(t_1, \dots, t_r) \in \mathfrak{E}_q(a;r)$, and then $(t_1, \dots, t_q) \in \mathfrak{E}(a;q)$ and $(t_{q+1}, \dots, t_{r-1}) \in \mathfrak{E}(a+q+1; r-1-q)$. By the induction assumption, there exist sequences $(s_1, s_2, \dots, s_q)$ and $(s_{q+1}, \dots, s_{r-1})$ satisfying (8.27). Therefore we obtain the expression

$$\xi(a;r) = (s_1, t_1) \cdots (s_q, t_q)(s_{q+1}, t_{q+1}) \cdots (s_{r-1}, t_{r-1})(a+q, a+r),$$

as required.

(iii) It remains to prove the uniqueness of the expression (8.25). Assume that the cycle $\xi(a;r)$ has two expressions

$$(s_1, t_1)(s_2, t_2) \cdots (s_r, t_r) \qquad \text{and} \qquad (s'_1, t_1)(s'_2, t_2) \cdots (s'_r, t_r),$$

where $s_i, s'_i < t_i$ $(1 \le i \le r)$. Write as $s_r = a+q$ and $s'_r = a+q'$. As we saw in the part (i), the sequence $(t_1, \dots, t_r)$ belongs to $\mathfrak{E}_q(a;r) \cap \mathfrak{E}_{q'}(a;r)$. But, since $\mathfrak{E}_q(a;r) \cap \mathfrak{E}_{q'}(a;r) = \emptyset$ if $q \neq q'$, we have $q = q'$ so that $s_r = s'_r$. Now, as like (8.27), we have $(t_1, \dots, t_q) \in \mathfrak{E}(a;q)$ and $(t_{q+1}, \dots, t_{r-1}) \in \mathfrak{E}(a+q+1; r-1-q)$, and

$$(s_1, t_1) \cdots (s_q, t_q) = (s'_1, t_1) \cdots (s'_q, t_q) =(a, a+1, \dots, a+q),$$

$$(s_{q+1}, t_{q+1}) \cdots (s_{r-1}, t_{r-1}) = (s'_{q+1}, t_{q+1}) \cdots (s'_{r-1}, t_{r-1}) =(a+q+1, a+q+2, \dots, a+r).$$

By the induction assumption, we obtain $s_1 = s'_1, \dots, s_q = s'_q, s_{q+1} = s'_{q+1}, \dots, s_{r-1} = s'_{r-1}$. □

8.4.6 Proof of Theorem 72

Recall the definition of the Jucys-Murphy elements: $J_t = \sum_{1 \le s < t} (s,t)$. For a permutation $\sigma \in S(n)$ and a polynomial f in n variables, denote by $[\sigma] f(\Xi_n)$ the multiplicity of σ in $f(J_1, \dots, J_n)$:

$$f(\Xi_n) = \sum_{\sigma \in S(n)} ([\sigma]) f(\Xi_n) \sigma \in \mathbb{C}[S_n].$$

For a partition μ with size k and length l, we define the permutation σ_μ of reduced cycle-type μ by

$$\begin{aligned} \sigma_\mu =&(1, 2, \dots, \mu_1 + 1)(\mu_1 + 2, \dots, \mu_1 + \mu_2 + 2) \cdots (\mu_1 + \cdots + \mu_{l-1} + l, \dots, k + l) \\ =&\xi(1; \mu_1) \xi(\mu_1 + 2; \mu_2) \cdots \xi(\mu_1 + \cdots + \mu_{l-1} + l; \mu_l). \end{aligned}$$

Proposition 98. *Let μ be a partition of k and let $(t_1, \dots, t_k)$ be a sequence of positive integers such that $2 \le t_1 \le \cdots \le t_k$. Then $[\sigma_\mu] J_{t_1} \cdots J_{t_k} = 1$ if $(t_1, \dots, t_k)$ satisfies*

$$(t_{\mu_1 + \cdots + \mu_{i-1} + 1}, \dots, t_{\mu_1 + \cdots + \mu_{i-1} + \mu_i}) \in \mathfrak{E}(\mu_1 + \cdots + \mu_{i-1} + i; \mu_i) \tag{8.28}$$

for all $1 \le i \le \ell(\mu)$, and $[\sigma_\mu] J_{t_1} \cdots J_{t_k} = 0$ otherwise.

Proof. The value $[\sigma_\mu] J_{t_1} \cdots J_{t_k}$ is the number of sequences $(s_1, \dots, s_k)$ satisfying

$$\sigma_\mu = \prod_{i=1}^{\ell(\mu)} \xi(\mu_1 + \cdots + \mu_{i-1} + i; \mu_i) = (s_1, t_1) \cdots (s_k, t_k).$$

By Lemma 95, it equals the number of sequences $(s_1, \dots, s_k)$ satisfying

$$\xi(\mu_1+\cdots+\mu_{i-1}+i; \mu_i) = (s_{\mu_1+\cdots+\mu_{i-1}+1}, t_{\mu_1+\cdots+\mu_{i-1}+1}) \cdots (s_{\mu_1+\cdots+\mu_{i-1}+\mu_i}, t_{\mu_1+\cdots+\mu_{i-1}+\mu_i})$$

for all $1 \le i \le \ell(\mu)$. It follows by Proposition 96 that $[\sigma_\mu] J_{t_1} \cdots J_{t_k}$ equals to 1 if (8.28) holds true for all i, and to 0 otherwise. □

Example 99. Let $2 \le t_1 \le \cdots \le t_6$ and consider $\sigma_{(3,2,1)} = (1,2,3,4)(5,6,7)(8,9)$. Suppose $[\sigma_{(3,2,1)}] J_{t_1} \cdots J_{t_6} = 1$. Then, Proposition 98 claims

$$(t_1, t_2, t_3) \in \mathfrak{E}(1; 3), \qquad (t_4, t_5) \in \mathfrak{E}(5; 2), \qquad (t_6) \in \mathfrak{E}(8; 1).$$

Therefore, $(t_1, t_2) \in \{(2,3), (2,4), (3,3), (3,4), (4,4)\}$, $t_3 = 4$, $t_4 \in \{6, 7\}$, $t_5 = 7$, and $t_6 = 9$. □

As defined in Section 8.4.2, a weakly increasing sequence $(t_1, \dots, t_r)$ is of type $\lambda \vdash r$ with $\ell(\lambda) = l$ if there exists a permutation $(\alpha_1, \dots, \alpha_l)$ of $(\lambda_1, \dots, \lambda_l)$ such that

$$t_1 = t_2 = \cdots = t_{\alpha_1} < t_{\alpha_1+1} = t_{\alpha_1+2} = \cdots = t_{\alpha_1+\alpha_2} < t_{\alpha_1+\alpha_2+1} = \cdots .$$

The monomial symmetric polynomial $m_\lambda(\Xi_n)$, $\lambda \vdash k$, is written as

$$m_\lambda(\Xi_n) = \sum_{\substack{2 \le t_1 \le \cdots \le t_k \le n \\ (t_1, \dots, t_k): \text{type } \lambda}} J_{t_1} J_{t_2} \cdots J_{t_k} = \sum_{\substack{2 \le t_1 \le \cdots \le t_k \le n \\ (t_1, \dots, t_k): \text{type } \lambda}} \sum_{s_1=1}^{t_1-1} \cdots \sum_{s_k=1}^{t_k-1} (s_1, t_1) \cdots (s_k, t_k).$$

By Corollary 93, the element $m_\lambda(\Xi_n)$ is given by the form (8.21).

Let μ be a partition of k. We now evaluate the coefficient $L_\mu^\lambda(n)$ of $\mathfrak{c}_\mu(n)$ in $m_\lambda(J_1, \dots, J_n)$, which equals $L_\mu^\lambda(n) = [\sigma_\mu] m_\lambda(J_1, \dots, J_n)$. By the assumption $n \ge k + \ell(\mu)$, the permutation σ_μ lives in $S(n)$. It follows by Proposition 98 that $L_\mu^\lambda(n)$ is the number of weakly increasing sequences $(t_1, \dots, t_k)$ of type λ, satisfying (8.28) for all $1 \le i \le \ell(\mu)$. If $(t_1, \dots, t_k)$ is such a sequence and if we let $\lambda^{(i)} \vdash \mu_i$ being the type of $(t_{\mu_1+\cdots+\mu_{i-1}+1}, \dots, t_{\mu_1+\cdots+\mu_{i-1}+\mu_i})$, then λ must agree with $\lambda^{(1)} \cup \lambda^{(2)} \cup \cdots$ so that $(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)$. Thus, $L^{\lambda}_{\mu}(n)$ coincides with

$$\sum_{(\lambda^{(1)},\lambda^{(2)},\dots)\in\mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} (\text{the number of sequences in } \mathfrak{E}(\mu_1+\cdots+\mu_{i-1}+i;\mu_i) \text{ of type } \lambda^{(i)})$$

$$= \sum_{(\lambda^{(1)},\lambda^{(2)},\dots)\in\mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} \mathrm{RC}(\lambda^{(i)}).$$

This completely proves Theorem 90.

Let μ be a partition of k. We now evaluate the coefficient $L^{\lambda}_{\mu}(n)$ of $\mathfrak{c}_{\mu}(n)$ in $m_{\lambda}(J_1, \dots, J_n)$, which equals $L^{\lambda}_{\mu}(n) = [\sigma_{\mu}] m_{\lambda}(J_1, \dots, J_n)$. By the assumption $n \ge k + \ell(\mu)$, the permutation σ_{μ} lives in $S(n)$. It follows by Proposition 98 that $L^{\lambda}_{\mu}(n)$ is the number of weakly increasing sequences $(t_1, \dots, t_k)$ of type λ, satisfying (8.28) for all $1 \le i \le \ell(\mu)$. If $(t_1, \dots, t_k)$ is such a sequence and if we let $\lambda^{(i)} \vdash \mu_i$ being the type of $(t_{\mu_1+\cdots+\mu_{i-1}+1}, \dots, t_{\mu_1+\cdots+\mu_{i-1}+\mu_i})$, then λ must agree with $\lambda^{(1)} \cup \lambda^{(2)} \cup \cdots$ so that $(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)$. Thus, $L^{\lambda}_{\mu}(n)$ coincides with

$$\sum_{(\lambda^{(1)},\lambda^{(2)},\dots)\in\mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} (\text{the number of sequences in } \mathfrak{E}(\mu_1+\cdots+\mu_{i-1}+i;\mu_i) \text{ of type } \lambda^{(i)})$$

$$= \sum_{(\lambda^{(1)},\lambda^{(2)},\dots)\in\mathfrak{R}(\lambda,\mu)} \prod_{i=1}^{\ell(\mu)} \mathrm{RC}(\lambda^{(i)}).$$

This completely proves Theorem 90.

8.4.7 Remarks regarding $L^{\lambda}_{\mu}(n)$

Recall the definition (8.19) of $L^{\lambda}_{\mu}(n) = G_{\mu}(m_{\lambda}, n)$. First we deal with $L^{\lambda}_{\mu}(n)$ for general λ, μ. Since J_k is a sum of $k-1$ transpositions, the left hand side of (8.19) is the sum of $m_{\lambda}(0, 1, \dots, n-1)$ permutations. For each partition μ satisfying $\mathrm{wt}(\mu) \le n$, let

$C_\mu(n)$ be the conjugacy class in S_n with reduced cycle-type μ. Then (8.19) implies the identity

$$\sum_{\mu:|\mu|\le|\lambda|,\ |\mu|+\ell(\mu)\le n} L_\mu^\lambda(n)|C_\mu(n)| = m_\lambda(0,1,\ldots,n-1).$$

Here the cardinality of the conjugacy class $C_\mu(n)$ is given by

$$|C_\mu(n)| = \frac{n!}{(n-|\mu|-\ell(\mu))!(\prod_{i\ge1}(\mu_i+1))(\prod_{i\ge1} m_i(\mu)!)}.$$

For example, by the class expansion of $m_{(2,1)}(\Xi_n)$ given in the Appendix, we observe

$$\begin{aligned}&3|C_{(3)}(n)| + |C_{(2,1)}(n)| + \frac{1}{2}(n-2)(n+1)|C_{(1)}(n)|\\ =&\frac{3}{4}n(n-1)(n-2)(n-3) + \frac{1}{6}n(n-1)(n-2)(n-3)(n-4) + \frac{1}{2}(n-2)(n+1)\cdot\frac{1}{2}n(n-1)\\ =&\frac{1}{6}(n-2)(n-1)^2n^2 = m_{(2,1)}(0,1,\ldots,n-1).\end{aligned}$$

We have seen that $L_\mu^\lambda(n)$ is zero unless $|\lambda|\ge|\mu|$. It seems quite difficult to obtain an explicit expression for all $L_\mu^\lambda(n)$. In [18], the coefficient of the identity permutation in $m_{(k)}(\Xi_n)$ is calculated. To obtain the following identity, they employed Lascoux and Thibon's result [29].

$$L_{(0)}^{(2r)}(n) = \sum_{j=1}^{r} T(r,j)\frac{(2j)!}{((j+1)!)^2}n(n-1)\cdots(n-j),$$

where $T(r,j)$ is a central factorial number given by

$$T(r,j) = 2\sum_{i=1}^{j}\frac{(-1)^{j-i}i^{2r}}{(j-i)!\,(j+i)!}. \tag{8.29}$$

In particular, $L_{(0)}^{(2r)}(n)$ is a polynomial of degree $r+1$ in n. For example, $L_{(0)}^{(2)}(n) = \frac{1}{2}n(n-1)$ and $L_{(0)}^{(4)}(n) = \frac{1}{6}n(n-1)(4n-5)$. These coincide with the coefficients given in Appendix.

Next we consider the n-independent coefficients L_μ^λ with $|\lambda|=|\mu|$. A necessary and sufficient condition for the positivity of L_μ^λ is given by using a transition matrix among symmetric functions. In fact, for partitions λ, μ satisfying $|\lambda| = |\mu|$, define integers $B_{\lambda\mu}$ by

$$p_\lambda = \sum_\mu B_{\lambda\mu} m_\mu,$$

where $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ is the power-sum symmetric polynomial. For instance, the list for $B_{\lambda\mu}$ with $\lambda, \mu \vdash 4$ is given as follows:

$\lambda \setminus \mu$	4	31	2^2	21^2	1^4
4	1				
31	1	1			
2^2	1	0	2		
21^2	1	2	2	2	
1^4	1	4	6	12	24

By [30, I, (6.9)], the integer $B_{\lambda\mu}$ is nonzero if and only if λ is a refinement of μ. On the other hand, Theorem 90 implies that L^λ_μ is nonzero if and only if λ is a refinement of μ. Therefore L^λ_μ is zero if and only if $B_{\lambda\mu}$ is zero.

The number $B_{\lambda\mu}$ has a combinatorial interpretation. In fact, $B_{\lambda\mu}$ is the number of *domino tableaux* of shape μ and of weight λ. Here a domino tableau of shape μ and of weight λ is a numbering of the boxes of the Young diagram of μ with positive integers, increasing along each row, such that for each $i \geq 1$ the boxes numbered i form a connected horizontal strip of length λ_i. (See [30, I-6, Example 7].) For example,

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 2 & 3 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 1 & 1 & 4 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 4 \\ \hline 1 & 1 & 3 \\ \hline\end{array}$$

is the list of domino tableaux of shape $(3,3)$ and weight $(2,2,1,1)$, so $B^{(2,2,1,1)}_{(3,3)} = 4$. Is there a similar combinatorial interpretation for nonzero L^λ_μ ?

As a corollary of Theorem 90, the following recurrence formula follows immediately.

Proposition 100. *Let $\lambda, \mu \vdash k$. Let q be the smallest part of μ; $q = \mu_{\ell(\mu)}$. Then*

$$L^\lambda_\mu = \sum_{\nu \vdash q} \mathrm{RC}(\nu) L^{\lambda \setminus \nu}_{(\mu_1,\dots,\mu_{\ell(\mu)-1})},$$

where

$$L^{\lambda\setminus\nu}_{(\mu_1,\dots,\mu_{\ell(\mu)-1})} = \begin{cases} L^\eta_{(\mu_1,\dots,\mu_{\ell(\mu)-1})}, & \text{if there exists a partition } \eta \text{ such that } \lambda = \eta \cup \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Here we may take η as the empty partition (0) and we set $L^{(0)}_{(0)} = 1$.

Observe that

$$\begin{aligned} L^\lambda_{\eta\cup(1^p)} &= L^{\lambda\setminus(1^p)}_\eta && \text{if } |\lambda| = |\eta| + p; \\ L^\lambda_{\eta\cup(2)} &= L^{\lambda\setminus(2)}_\eta + L^{\lambda\setminus(1^2)}_\eta && \text{if } |\lambda| = |\eta| + 2 \text{ and } \eta \text{ has no parts equal to 1}; \\ L^\lambda_{\eta\cup(3)} &= L^{\lambda\setminus(3)}_\eta + 3L^{\lambda\setminus(2,1)}_\eta + L^{\lambda\setminus(1^3)}_\eta && \text{if } |\lambda| = |\eta| + 3 \text{ and } \eta \text{ has no parts less than 3.} \end{aligned}$$

8.5 Class expansion of complete symmetric functions

8.5.1 Constant coefficients

Let $F^k_\mu(n)$ be the coefficient of $\mathfrak{c}_\mu(n)$ in $h_k(J_1,\dots,J_n)$:

$$h_k(\Xi_n) = \sum_{g=0}^{[\frac{k}{2}]} \sum_{\substack{\mu \vdash k-2g \\ \ell(\mu) \le n-k+2g}} F^k_\mu(n) \mathfrak{c}_\mu(n). \tag{8.30}$$

Since $h_k = \sum_{|\lambda|=k} m_\lambda$, we have

$$F_\mu^k(n) = \sum_{|\lambda|=k} L_\mu^\lambda(n).$$

As a corollary of Theorem 90, we obtain the following result.

Theorem 101. *Let* μ *be a partition with* $\mathrm{wt}(\mu) \le n$. *Then*

$$F_\mu^{|\mu|}(n) = \prod_{i\ge 1} \mathrm{Cat}_{\mu_i}. \tag{8.31}$$

In particular, $F_\mu^{|\mu|} = F_\mu^{|\mu|}(n)$ *is independent of* n.

Proof. We have

$$F_\mu^k(n) = \sum_{\lambda \vdash k} L_\mu^\lambda(n) = \sum_{\lambda\vdash k} \sum_{(\lambda^{(1)},\lambda^{(2)},\dots)\in\mathfrak{R}(\lambda,\mu)} \mathrm{RC}(\lambda^{(1)})\mathrm{RC}(\lambda^{(2)})\cdots$$

by Theorem 90. By the definition of $\mathfrak{R}(\lambda,\mu)$, we see that

$$\bigsqcup_{\lambda\vdash k} \mathfrak{R}(\lambda,\mu) = \{(\lambda^{(1)},\lambda^{(2)},\dots) \mid \lambda^{(i)} \vdash \mu_i \ (i\ge 1)\},$$

so that, by Proposition 86,

$$F_\mu^k(n) = \prod_{i\ge1}\Big(\sum_{\lambda^{(i)}\vdash\mu_i} \mathrm{RC}(\lambda^{(i)})\Big) = \prod_{i\ge1}\mathrm{Cat}_{\mu_i}.$$

□

Note that Theorem 101 was first obtained by Murray [33, Corollary 6.4] in the framework of the Farahat-Higman algebra.

Remark 102. For the double covering $\widetilde{S}_n$ of the symmetric group, a result similar to Theorem 101 was recently obtained by Tysse and Wang [45]. They deal with $e_k(M_1^2,\dots,M_n^2)$, where the M_i are elements of the spin group algebra of $\widetilde{S}_n$ called *odd Jucys-Murphy elements.* □

8.5.2 Coefficient of the class of full cycles

The generating function

$$\sum_{k\geq 0} F_\mu^k(n) z^k$$

can be expressed in terms of the irreducible characters χ^λ of the symmetric group $S(n)$ by applying Proposition 81.

Proposition 103. *Let μ be a partition with $\mathrm{wt}(\mu) \leq n$, and $\nu \vdash n$ is the inflation of μ. Then we have the generating function*

$$\sum_{k\geq 0} F_\mu^k(n) z^k = \sum_{\lambda \vdash n} \frac{\chi^\lambda(\nu)}{H_\lambda \prod_{\Box \in \lambda}(1 - c(\Box) z)},$$

where H_λ is the product of hook-lengths over all cells of λ, and $c(\Box)$ is the content of an individual cell.

Proof. Consider the generating function

$$\sum_{k\geq 0} h_k(\Xi_n) z^k$$

as a formal power series over the commutative algebra $Z(n)$. We then have

$$\begin{aligned}\sum_{k\geq 0} h_k(\Xi_n) z^k &= \sum_{k\geq 0}\sum_{\lambda\vdash n} \frac{h_k(A_\lambda)}{H_\lambda}\chi^\lambda z^k, \text{ by Proposition 81 and the hook-length formula} \\ &= \sum_{\lambda\vdash n} \frac{\chi^\lambda}{H_\lambda} \sum_{k\geq 0} h_k(A_\lambda) z^k \\ &= \sum_{\lambda\vdash n} \frac{\chi^\lambda}{H_\lambda \prod_{\Box\in\lambda}(1 - c(\Box) z)},\end{aligned}$$

where H_λ is the hook-product over the cells of λ and $c(\Box)$ denotes the content of a cell. Thus for any particular partition $\nu \vdash n$ we have the generating function

$$\sum_{k\geq 0} F_\mu^k(n) z^k = \sum_{\lambda \vdash n} \frac{\chi^\lambda(\nu)}{H_\lambda \prod_{\Box \in \lambda}(1 - c(\Box) z)}$$

as a formal power series over $\mathbb{C}$, where $\mu = \tilde{\nu}$ is the reduction of ν. ∎

Note that this proposition implies that the generating function $\sum_{k\geq 0} F_\mu^k(n)z^k$ is rational. However, explicitly determining this generating function requires full knowledge of the character table of $S(n)$. In the case where $\mu = (n-1)$, this generating function can be explicitly determined — it turns out to be a scaling of the generating function for central factorial numbers.

The *central factorial number* $T(m,n)$ counts the number of partitions of a set

$$\{1, 1', \dots, m, m'\}$$

of m unmarked and m marked points into n blocks such that, for each block B, if i is the least integer such that either $i \in B$ or $i' \in B$, then $\{i, i'\} \subseteq B$.

Example 104. $T(3,2) = 5$, corresponding to the partitions

$$\begin{aligned} \{1,1',2,2',3,3'\} &= \{1,1',2,2'\} \sqcup \{3,3'\} \\ &= \{1,1',3,3'\} \sqcup \{2,2'\} \\ &= \{2,2',3,3'\} \sqcup \{1,1'\} \\ &= \{1,1',3\} \sqcup \{2,2',3'\} \\ &= \{1,1',3'\} \sqcup \{2,2',3\}. \end{aligned}$$

Clearly, in partitions meeting the above constraint, each block B must be of size at least 2, and thus non-zero central factorial numbers are of the form $T(n+g,n)$ for $g \geq 0$. It is also clear that $T(n,n) = 1$, corresponding to the partition

$$\{1,1'\} \sqcup \dots \sqcup \{m,m'\}.$$

The ordinary generating function for the central factorial numbers is known [43, Exercise 5.8] to be

$$\sum_{m\geq 0} T(m,n)z^m = \frac{z^n}{(1-1^2z)\dots(1-n^2z)}. \tag{8.32}$$

Theorem 105. *For any $n \geq 1$, we have $F^{n-1+2g}_{(n-1)}(n) = \mathrm{Cat}_{n-1} T(n-1+g, n-1)$.*

Proof. We consider the special case of Proposition 103 where $\nu = (n)$, the partition with a single part of size n. In this case it is well-known that the character $\chi^\lambda(\nu)$ vanishes unless λ is a "hook," i.e. a partition of the form $(r, 1^{n-r})$ for some $1 \leq r \leq n$. Moreover, $\chi^{(r,1^{n-r})}(\nu) = (-1)^{n-r}$. This is a special case of the more general Murnaghan-Nakayama rule. Since the content alphabet of a hook is $A_{(r,1^{n-r})} = \{-(n-r-1), \dots, 0, 1, \dots, r-1\}$ we have

$$\sum_{\lambda \vdash n} \frac{\chi^\lambda(\nu)}{H_\lambda \prod_{\Box \in \lambda}(1 - c(\Box)z)} = \sum_{r=1}^{n} \frac{(-1)^{n-r}}{H_{(r,1^{n-r})}(1-z)\dots(1-(r-1)z)(1+z)\dots(1+(n-r-1)z)},$$

which as an irreducible rational function has the form

$$\frac{P_n(z)}{\prod_{r=1}^{n-1}(1-r^2z^2)}$$

for some polynomial $P_n(z)$ of degree at most $n-1$. On the other hand, by Theorem 90, we have

$$\sum_{k \geq 0} F^k_{(n-1)}(n) z^k = \sum_{g \geq 0} F^{n-1+2g}_{(n-1)}(n) z^{n-1+2g} = \mathrm{Cat}_{n-1} z^{n-1} + F^{n+1}_{(n-1)}(n) z^{n+1} + \dots.$$

Consequently we must have $P_n(z) = \mathrm{Cat}_{n-1} z^{n-1}$, and the assertion follows from the generating function (8.32) for the central factorial numbers. □

8.6 Unitary matrix integrals

8.6.1 Weingarten calculus

In this Section, we present the basic aspects of the Weingarten calculus as developed by Collins and collaborators in the series of papers [7, 11, 2, 3, 10], and show how this development leads to a natural and direct connection between unitary matrix integrals and Jucys-Murphy elements.

The Weingarten calculus is a systematic machinery whose purpose is to compute the integral of polynomial functions on compact matrix groups. It centers around three scalar matrices: a projection matrix P, the Gram matrix G associated to this projection, and the *Weingarten matrix* $\mathbb{W} = \mathsf{G}^{-1}$.

Let U_N be a Haar-distributed random unitary matrix from $\mathcal{U}(N)$, and consider the random matrix $U_N^{\otimes n} \otimes \overline{U_N}^{\otimes n}$. Taking the expected value $\langle \cdot \rangle_N$ of each entry of this random matrix produces an $N^{2n} \times N^{2n}$ scalar matrix P. By construction, the entries of P are precisely the n-point functions (8.13), and P is therefore referred to as the *correlation matrix*.

The correlation matrix has a natural representation-theoretic interpretation. Let $V = \mathbb{C}^N$ be the defining representation of $\mathcal{U}(N)$ and V^* the dual representation. Consider the representation $W = V^{\otimes n} \otimes (V^*)^{\otimes n}$, with action denoted ρ. Then, the correlation matrix P is the matrix of the averaging operator

$$\mathrm{av}_{\mathcal{U}(N)}(w) = \int_{\mathcal{U}(N)} \rho(U)(w)dU$$

with respect to the basis $\{\bigotimes_{k=1}^n e_{i(k)} \otimes \bigotimes_{k=1}^n e_{j(k)} : i,j \in [N]^{[n]}\}$, where $\{e_1, \dots, e_N\}$ is the standard orthonormal basis of V. On the other hand, $\mathrm{av}_{\mathcal{U}(N)} \in \mathrm{End}\, W$ is the orthogonal projection of W onto the invariant subspace

$$W^{\mathcal{U}(N)} = \{w \in W : \rho(U)w = w \ \ \forall\, U \in \mathcal{U}(N)\}.$$

Thus, P is the matrix of the orthogonal projection $W \to W^{\mathcal{U}(N)}$ with respect to the standard tensor basis.

Theorem 106 (Baik-Rains [1])**.** *The tensors*

$$T_\pi = \sum_{i \in [N]^{[n]}} [i = i\pi] e_{\pi(1)} \otimes \cdots \otimes e_{\pi(n)}$$

form a basis of $V^{\otimes n}$ as π ranges over the set $S_N(n)$ of permutations in $S(n)$ with no *decreasing subsequence of length* $N+1$.

In general, the coordinates of the orthogonal projection of a finite-dimensional Hilbert space onto a given subspace with respect to a given basis can be computed by considering the Gram matrix associated to that basis. In our setting, the Gram matrix is

$$\mathsf{G} = (\langle T_\sigma | T_\tau \rangle)_{\sigma,\tau \in S_N(n)},$$

where $\langle \cdot | \cdot \rangle$ denotes the inner product on V inherited from the standard one on $\mathbb{C}^N$.

Theorem 107 (Collins-Śniady [11]). *We have*

$$\mathsf{G} = (N^{\#(\sigma\tau^{-1})})_{\sigma,\tau \in S_N(n)},$$

where $\#(\pi)$ *denotes the number of cycles in the permutation* π.

Theorem 108 (Weingarten convolution formula). *For any* $i, j, i', j' \in [N]^{[n]}$ *we have*

$$\begin{aligned}\left\langle \prod_{k=1}^{n} u_{i(k)j(k)} \overline{u_{i'(k)j'(k)}} \right\rangle_N &= \sum_{\sigma,\tau \in S_N(n)} [i = i'\sigma][j = j'\tau] \mathsf{G}^{-1}(\tau, \sigma^{-1}). \\ &= \sum_{\pi \in S_N(n)} \left(\sum_{\tau\sigma^{-1} = \pi} [i = i'\sigma][j = j'\tau] \right) \mathsf{G}^{-1}(\mathrm{id}_n, \pi).\end{aligned}$$

The inverse Gram matrix G^{-1} is called the *Weingarten matrix* and denoted $\mathbb{W}$. The matrices G and $\mathsf{G}^{-1} = \mathbb{W}$ are invertible matrices of order $|S_N(n)| = u(N,n)$. In particular, the entries of G are evidently monomials in N, and hence the entries of $\mathbb{W}$ are rational functions of N. In the *stable range* $N \geq n$, we have $u(N,n) = n!$. Thus the dimensions of $\mathsf{G}, \mathbb{W}$ remain stable at $n!$ once $N \geq n$. Note that in the stable range, we have

$$\left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \mathbb{W}(\mathrm{id}_n, \pi)$$

for each $\pi \in S(n)$, which is a consequence of choosing $i = i' = j = \mathrm{id}_n$ and $j' = \pi$ in the convolution formula. Thus the elements in the first row of the Weingarten matrix define the *Weingarten function* $\mathrm{Wg}_N \in \mathbb{C}[S(n)]$ by

$$\mathrm{Wg}_N(\pi) = \mathbb{W}(\mathrm{id}_n, \pi).$$

8.6.2 Gram matrix in the stable range

Assume that we are in the stable range $N \geq n$. Let z be a complex variable, and define the matrix

$$\mathsf{G}_z = (z^{\#(\sigma\tau^{-1})})_{\sigma,\tau \in S(n)}.$$

Thus the Gram matrix introduced above is $\mathsf{G} = \mathsf{G}_N$.

Proposition 109. *G_z is the matrix of $(z+J_1)\dots(z+J_n)$ in the regular representation of $\mathbb{C}[S(n)]$.*

Proof. By Jucys' theorem, we have

$$(z + J_1)\dots(z + J_n) = \sum_{\sigma \in S(n)} N^{\#(\sigma)}\sigma.$$

Thus, for any $\tau \in S(n)$, we have

$$(z + J_1)\dots(z + J_n)\tau = \sum_{\sigma \in S(n)} N^{\#(\sigma)}(\sigma\tau) = \sum_{\sigma \in S(n)} N^{\#(\sigma\tau^{-1})}\sigma.$$

□

Let us now compute the determinant of G_z.

Theorem 110. *We have*

$$\det \mathsf{G}_z = \prod_{k=1}^{n-1} (z^2 - k^2)^{a_k}$$

for some positive integer exponents a_k summing to $2n!$.

Proof. We compute the determinant of G_z by considering the representation of $(z + J_1)\dots(z+J_n)$ in each irreducible representation V^λ of $S(n)$. We know the eigenvalues of the representation of J_k in V^λ exactly by the Okounkov-Vershik theory: the spectrum of J_k is

$$\{c_T(k) : T \in SYT(\lambda)\},$$

where $c_T(k)$ denotes the content of the cell labelled k in the standard Young tableaux T. Clearly, the contents of any Young diagram with n cells belong to the set

$$\{-(n-1), \dots, 0, \dots, (n-1)\}.$$

The result thus follows from the isotypic decomposition

$$\mathbb{C}[S(n)] \simeq \bigoplus_{\lambda \vdash n} (\dim \lambda) V^\lambda$$

of the group algebra. □

Remark 111. This result shows that in the stable range the entries of the generalized Weingarten matrix $\mathbb{W}_z$ are rational functions of z, whose singularities are confined to the set $\{-(n-1), \dots, 0, \dots, (n-1)\}$ of eigenvalues of the JM-elements in the regular representation. Therefore the entries of $\mathbb{W}_N$ accurately represent the n-point functions in the stable range $N \geq n$, but in the stable range a different Weingarten matrix (corresponding to the basis indexed by permutations from $S_N(n)$) is required. This phenomenon is known as the "De Wit-t'Hooft anomaly" in the physics literature. In particular, Samuel's empirical classification of the anomalies [41] is justified by knowledge of the joint spectra of the JM-elements in the regular representation.

Now, note that

$$(z+J_1)^{-1}\dots(z+J_n)^{-1} = \frac{1}{z^n}\sum_{k\geq 0}(-1)^k h_k(\Xi_n) z^k, \tag{8.33}$$

a generating function for the complete homogeneous symmetric functions in JM-elements. Thus the entries of the generalized Weingarten matrix $\mathbb{W}_z$ are rational functions admitting the expansion

$$(-1)^{|\mu|} z^{n+|\mu|} \mathbb{W}_z(\mathrm{id}_n, \pi) = \sum_{g \geq 0} \frac{F_\mu^{|\mu|+2g}(n)}{z^{2g}}, \tag{8.34}$$

where π is a permutation of reduced cycle-type μ, and taking $z = N$, which is valid in the stable range, gives the proof of the expansion stated in Theorem 75. As corollaries of this argument and Theorems 74, we obtain the following results of Collins.

Corollary 112 (Collins [7])**.** *For $1 \leq n \leq N$ and $\pi \in S(n)$ a permutation of reduced cycle-type μ, we have*

$$(-1)^{|\mu|} N^{n+|\mu|} \left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\pi(k)}} \right\rangle_N = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i} + O(\frac{1}{N^2}).$$

Corollary 113 (Collins [7])**.** *Let $1 \leq n \leq N$ and let $\gamma = (1, 2, \ldots, n) \in S(n)$ be the full cycle. Then*

$$\left\langle \prod_{k=1}^{n} u_{kk} \overline{u_{k\gamma(k)}} \right\rangle_N = \frac{(-1)^{n-1} \mathrm{Cat}_{n-1}}{N(N^2 - 1^2) \ldots (N^2 - (n-1)^2)}.$$

Chapter 9

Conclusion

9.1 A final calculation

In this second-to-last informally written Section, I want to come full circle by applying the theory of the Weingarten function (as developed by Collins and Śniady in [11]) to prove the fundamental identity

$$\boxed{u_d(n) = \int_{\mathcal{U}(d)} |\operatorname{Tr} U|^{2n} dU,} \tag{9.1}$$

which is due to Rains [Rai98]. This beautiful result is perhaps the most succinct and complete incarnation of the link between increasing subsequences and random matrices — it incorporates all the theory we've discussed in an astonishingly compact form. To my mind, the final unifying calculation performed in this chapter serves as a much more satisfying conclusion than a recapitulatory discussion would.

Rains' result is not hard to prove once it's pointed out to you, provided you know the RSK correspondence and the right techniques from representation theory. In Chapter 6 we saw a proof using the Heine-Szegö identity and Gessel's identity (this proof actually shows that Rains' identity is equivalent to Gessel's). Here I'll present a proof of this identity using the Weingarten function. This proof is considerably more involved and probably less elegant than the original proof, even though it has the benefit of producing a slightly more general result at no extra cost. The point is that if the Weingarten function is worth its weight, it should be able to reproduce this important result. I am also very fond of the following argument, which appeared in [Nov07], since it's the first original calculation I performed in graduate school.

Let $d, n \geq 1$ be fixed positive integers, and let $1 \leq k \leq d$ be a third integer. A *weak k-part composition* α of n is a vector

$$\alpha = (a_1, \ldots, a_k) \tag{9.2}$$

of non-negative integers whose coordinates sum up to n. By $\binom{n}{\alpha}$ we mean the multinomial coefficient

$$\binom{n}{\alpha} = \binom{n}{a_1, \ldots, a_k}. \tag{9.3}$$

If α is a weak k-part composition of n, the corresponding *Young subgroup* $S(\alpha)$ is by definition the subgroup of $S(n)$ consisting of permutations which permute $1, \ldots, a_1$ among themselves, $a_1 + 1, \ldots, a_1 + a_2$ among themselves, etc.

Lemma 114. *We have*

$$\int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n dU = n! \sum_{\alpha} \binom{n}{\alpha} \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma),$$

where the outer sum runs over all weak k-part compositions α of n, and the inner sum runs over all permutations in the corresponding Young subgroup.

Proof. By the multinomial theorem, we have

$$(u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n = \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} u^{\alpha} \overline{u}^{\beta}, \tag{9.4}$$

summed over all pairs α, β of weak k-part compositions of n. Here we're using multi-index notation

$$u^\alpha = u_{11}^{a_1} \dots u_{kk}^{b_k}. \tag{9.5}$$

We will apply the Weingarten convolution formula to the integral

$$\int_{\mathcal{U}(d)} u^\alpha \overline{u}^\beta dU \tag{9.6}$$

for a fixed pair of compositions α, β. Implicitly define coordinate functions $i_\alpha, j_\alpha, i_\beta, j_\beta : [n] \to [d]$ by setting

$$u_{i_\alpha(1)j_\alpha(1)} \dots u_{i_\alpha(n)j_\alpha(n)} \overline{u}_{i_\beta(1)j_\beta(1)} \dots \overline{u}_{i_\beta(n)j_\beta(n)} := u^\alpha \overline{u}^\beta \tag{9.7}$$

Applying the Weingarten convolution formula, we have

$$\int_{\mathcal{U}(d)} u^\alpha \overline{u}^\beta dU = \sum_{\sigma,\tau \in S(n)} [i_\alpha = i_\beta \sigma][j_\alpha = j_\beta \tau] \operatorname{Wg}_d(\tau\sigma^{-1}). \tag{9.8}$$

Since we are only taking entries from the diagonal, we have $i_\alpha = j_\alpha$ and $i_\beta = j_\beta$. Moreover, the fibres of these functions are easy to read off:

$$\begin{aligned} i_\alpha^{-1}(1) &= [1, a_1] \\ i_\alpha^{-1}(2) &= [a_1 + 1, a_1 + a_2] \\ &\vdots \\ i_\alpha^{-1}(k) &= [n - a_k + 1, n], \end{aligned}$$

and similarly for β, with a_i's replaced by b_i's. Hence

$$\begin{gathered} [i_\alpha = i_\beta \sigma \text{ on the interval } [1, a_1]] = [1 = i_\beta \sigma \text{ on the interval } [1, a_1]] \\ [i_\alpha = i_\beta \sigma \text{ on the interval } [a_1 + 1, a_1 + a_2]] = [2 = i_\beta \sigma \text{ on the interval } [a_1 + 1, a_1 + a_2]] \\ \vdots \\ [i_\alpha = i_\beta \sigma \text{ on the interval } [n - a_k + 1, n]] = [k = i_\beta \sigma \text{ on the interval } [n - a_k + 1, n]]. \end{gathered}$$

Thus in order for

$$[i_\alpha = i_\beta \sigma] \tag{9.9}$$

to be non-zero, σ must bijectively map $[1, a_1]$ onto $[1, b_1]$, and bijectively map $[a_1 + 1, a_1 + a_2]$ onto $[b_1 + 1, b_1 + b_2]$, etc. Similarly, in order for

$$[j_\alpha = j_\beta \tau] \tag{9.10}$$

to be non-zero, τ must do the same. Thus we see that the expectation $\int u^\alpha \overline{u}^\beta$ is zero unless the following conditions hold:

- $\alpha = \beta$, i.e. they are the same k-part composition of n.
- σ, τ both belong to the Young subgroup $S(\alpha)$.

Thus we see that

$$\begin{aligned}
\int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n dU &= \sum_\alpha \sum_\beta \binom{n}{\alpha} \binom{n}{\beta} \int_{\mathcal{U}(d)} u^\alpha \overline{u}^\beta dU \\
&= \sum_\alpha \binom{n}{\alpha}^2 \sum_{\sigma, \tau \in S(\alpha)} \mathrm{Wg}_d(\tau \sigma^{-1}) \\
&= \sum_\alpha \binom{n}{\alpha}^2 \alpha! \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma) \\
&= n! \sum_\alpha \binom{n}{\alpha} \alpha! \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma).
\end{aligned}$$

□

Note that so far this argument is completely combinatorial, and relies only on the Weingarten convolution formula, without actually accessing the definition of the Weingarten function.

Now we plug in the character expansion

$$\mathrm{Wg}_d(\sigma) = \frac{1}{n!^2} \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)}{s_\lambda(1^d)} \chi^\lambda(\sigma), \tag{9.11}$$

which is written in a slightly different form from that which we derived in the body of the thesis, where we applied the hook-length formula to expand $\dim \lambda$ and the hook-content formula to expand $s_\lambda(1^d)$ as products over the cells of λ. Note also that we sum only over representations of $S(n)$ with at most d rows. This is an extension of the character expansion of Wg_d, due to Collins and Śniady in [11], which makes it valid even when $d \leq n$.

Plugging in the character expansion of Wg_d to the right side of the Lemma and changing the order of summation, we get the daunting triple sum

$$\sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} \sum_\alpha \frac{1}{\alpha!} \sum_{\sigma \in S(\alpha)} \chi^\lambda(\sigma). \tag{9.12}$$

This looks hopeless, but it's not; the innermost sum is just the scalar product of χ^λ, viewed as a character of $S(\alpha)$, with the trivial character of $S(\alpha)$. Thus we can write our triple sum as a double sum:

$$\sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \downarrow^{S(n)}_{S(\alpha)} \rangle_{S(\alpha)}. \tag{9.13}$$

We can now appeal to Frobenius reciprocity:

$$\langle 1, \chi^\lambda \downarrow^{S(n)}_{S(\alpha)} \rangle_{S(\alpha)} = \langle 1 \uparrow^{S(n)}_{S(\alpha)}, \chi^\lambda \rangle_{S(n)}, \tag{9.14}$$

where we've dropped the restriction on χ^λ and induced the trivial character of $S(\alpha)$ to a character of $S(n)$. The last trick we'll pull is to use the characteristic map

$$\mathrm{ch}_n : Z(n) \to \mathbb{C}_{\mathrm{sym}}[x_1, \dots, x_n] \tag{9.15}$$

to transport the whole mess from the center of the symmetric group algebra to the ring of symmetric polynomials. The characteristic map is an isometric isomorphism which takes the inner product on characters to the Hall scalar product on symmetric polynomials (a.k.a integration on the unitary group). It also has the special property that it maps the induction of the trivial character of any Young subgroup to the corresponding complete homogeneous symmetric polynomial:

$$\mathrm{ch}_n(1 \uparrow_{S(\alpha)}^{S(n)}) = h_\alpha, \tag{9.16}$$

and moreover sends an irreducible character to the corresponding Schur polynomial:

$$\mathrm{ch}_n(\chi^\lambda) = s_\lambda. \tag{9.17}$$

The frightening inner product involving restricted and induced characters is now just the following Hall product of symmetric polynomials:

$$\langle h_\alpha, s_\lambda \rangle = K_{\lambda\alpha}, \tag{9.18}$$

where $K_{\lambda\alpha}$ is a Kostka number. The Kostka number counts semi-standard Young tableaux of shape λ and content vector α, and hence

$$\sum_\alpha K_{\lambda\alpha} = s_\lambda(1^k), \tag{9.19}$$

the total number of semi-standard tableaux of shape λ with entries from $\{1, \ldots, k\}$.

Finally, going back to our double-sum to be evaluated, we have

$$\sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \downarrow_{S(\alpha)}^{S(n)} \rangle_{S(\alpha)} = \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} s_\lambda(1^k). \tag{9.20}$$

We've now proved the identity

$$\boxed{\int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n dU = \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} s_\lambda(1^k),} \tag{9.21}$$

which one could also show using the theory of zonal polynomials, but that arguably requires even more background. When $k = d$, this general identity reduces to

$$\int_{\mathcal{U}(d)} |\operatorname{Tr} U|^{2n} dU = \sum_{\lambda \in \mathbb{Y}_d(n)} (\dim \lambda)^2 = u_d(n), \tag{9.22}$$

where the last line uses the RSK correspondence. Thus, we've fully reverse-engineered Rains' identity using the Weingarten function as our principal tool.

9.2 Outlook

In this thesis, we have seen various manifestations of the synergistic relationship between combinatorics and random matrix theory. Taking the longest increasing subsequence problem as our guiding motive, we were led into an arena of interesting problems that exist at the interface of these two seemingly disparate but in fact closely related disciplines.

Of course, there are many things that the author would like to do that have not been accomplished in this article. Let us end by saying some words on a few of these:

1. Asymptotic Knuth Theorem: Could the Asymptotic Knuth Theorem be used to say anything about the double scaling limit? This would involve going back to the fundamental decomposition $u(d, dn) = t(d, 2n) + E(d, n)$, and investigating how the error term can be controlled in the limit where $d, n \to \infty$ simultaneously, with scaling as specified in the Baik-Deift-Johansson Theorem.

2. Truncations of random orthogonal matrices: We have seen that the connection between increasing subsequences and random unitary matrices extends to a connection between random-turns vicious walkers and truncated random unitary matrices. Is it true that connections between fixed-point free involutions and random orthogonal matrices found by Rains [**?**] extend to the setting of truncated orthogonal matrices? If so, what is the combinatorial structure being enumerated?

3. Other classical groups and JM-elements: Very recently, the connection between Jucys-Murphy elements and unitary matrix integrals that was the subject of the last Chapter has been extended to orthogonal matrix integrals by Zinn-Justin [48]. Does this connection also extend to the symplectic group? Or perhaps even to the "easy" compact quantum groups axiomatized recently by Banica and Speicher [6]? In particular, on the quantum group side, research into this question could lead to interesting quantum analogues of the JM-elements.

END

Chapter 10

Conclusion

10.1 A final calculation

In this second-to-last informally written Section, I want to come full circle by applying the theory of the Weingarten function (as developed by Collins and Śniady in [11]) to prove the fundamental identity

$$\boxed{u_d(n) = \int_{\mathcal{U}(d)} |\operatorname{Tr} U|^{2n} dU,} \tag{10.1}$$

which is due to Rains [Rai98]. This beautiful result is perhaps the most succinct and complete incarnation of the link between increasing subsequences and random matrices — it incorporates all the theory we've discussed in an astonishingly compact form. To my mind, the final unifying calculation performed in this chapter serves as a much more satisfying conclusion than a recapitulatory discussion would. Rains' result is not hard to prove once it's pointed out to you, provided you know the RSK correspondence and the right techniques from representation theory. In Chapter 6 we saw a proof using the Heine-Szegö identity and Gessel's identity (this proof actually shows that Rains' identity is equivalent to Gessel's). Here I'll present a proof of this identity using the Weingarten function. This proof is considerably more involved and probably less elegant than the original proof, even though it has the benefit of producing a slightly more general result at no extra cost. The point is that if the Weingarten function is worth its weight, it should be able to reproduce this important result. I am also very fond of the following argument, which appeared in [Nov07], since it's the first original calculation I performed in graduate school.

Let $d, n \geq 1$ be fixed positive integers, and let $1 \leq k \leq d$ be a third integer. A *weak k-part composition* α of n is a vector

$$\alpha = (a_1, \dots, a_k) \tag{10.2}$$

of non-negative integers whose coordinates sum up to n. By $\binom{n}{\alpha}$ we mean the multinomial coefficient

$$\binom{n}{\alpha} = \binom{n}{a_1, \dots, a_k}. \tag{10.3}$$

If α is a weak k-part composition of n, the corresponding *Young subgroup* $S(\alpha)$ is by definition the subgroup of $S(n)$ consisting of permutations which permute $1, \dots, a_1$ among themselves, $a_1 + 1, \dots, a_1 + a_2$ among themselves, etc.

Lemma 115. *We have*

$$\int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n dU = n! \sum_{\alpha} \binom{n}{\alpha} \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma),$$

where the outer sum runs over all weak k-part compositions α of n, and the inner sum runs over all permutations in the corresponding Young subgroup.

Proof. By the multinomial theorem, we have

$$(u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n = \sum_{\alpha} \sum_{\beta} \binom{n}{\alpha} \binom{n}{\beta} u^{\alpha} \overline{u}^{\beta}, \tag{10.4}$$

summed over all pairs α, β of weak k-part compositions of n. Here we're using multi-index notation

$$u^{\alpha} = u_{11}^{a_1} \dots u_{kk}^{b_k}. \tag{10.5}$$

We will apply the Weingarten convolution formula to the integral

$$\int_{\mathcal{U}(d)} u^{\alpha}\overline{u}^{\beta} dU \tag{10.6}$$

for a fixed pair of compositions α, β. Implicitly define coordinate functions $i_\alpha, j_\alpha, i_\beta, j_\beta : [n] \to [d]$ by setting

$$u_{i_\alpha(1)j_\alpha(1)} \dots u_{i_\alpha(n)j_\alpha(n)}\overline{u}_{i_\beta(1)j_\beta(1)} \dots \overline{u}_{i_\beta(n)j_\beta(n)}$$

:= u$^{\alpha}\overline{u}^{\beta}$(10.7)Applying the Weingarten convolution formula, we have

$$\int_{\mathcal{U}(d)} u^{\alpha}\overline{u}^{\beta} dU = \sum_{\sigma,\tau \in S(n)} [i_\alpha = i_\beta \sigma][j_\alpha = j_\beta \tau] \operatorname{Wg}_d(\tau\sigma^{-1}). \tag{10.8}$$

Since we are only taking entries from the diagonal, we have $i_\alpha = j_\alpha$ and $i_\beta = j_\beta$. Moreover, the fibres of these functions are easy to read off:

$$\begin{aligned} i_\alpha^{-1}(1) &= [1, a_1] \\ i_\alpha^{-1}(2) &= [a_1 + 1, a_1 + a_2] \\ &\vdots \\ i_\alpha^{-1}(k) &= [n - a_k + 1, n], \end{aligned}$$

and similarly for β, with a_i's replaced by b_i's. Hence

$$[i_\alpha = i_\beta\sigma \text{ on the interval } [1, a_1]] = [1 = i_\beta\sigma \text{ on the interval } [1, a_1]]$$

$$[i_\alpha = i_\beta\sigma \text{ on the interval } [a_1 + 1, a_1 + a_2]] = [2 = i_\beta\sigma \text{ on the interval } [a_1 + 1, a_1 + a_2]]$$

$$\vdots$$

$$[i_\alpha = i_\beta\sigma \text{ on the interval } [n - a_k + 1, n]] = [k = i_\beta\sigma \text{ on the interval } [n - a_k + 1, n]].$$

Thus in order for

$$[i_\alpha = i_\beta \sigma] \tag{10.9}$$

to be non-zero, σ must bijectively map $[1, a_1]$ onto $[1, b_1]$, and bijectively map $[a_1 + 1, a_1 + a_2]$ onto $[b_1 + 1, b_1 + b_2]$, etc. Similarly, in order for

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- $\alpha = \beta$, i.e. they are the same k-part composition of n.
- σ, τ both belong to the Young subgroup $S(\alpha)$.

Thus we see that

$$\begin{aligned} \int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n \overline{(u_{11} + \cdots + u_{kk})}^n dU &= \sum_\alpha \sum_\beta \binom{n}{\alpha} \binom{n}{\beta} \int_{\mathcal{U}(d)} u^\alpha \overline{u}^\beta dU \\ &= \sum_\alpha \binom{n}{\alpha}^2 \sum_{\sigma, \tau \in S(\alpha)} \mathrm{Wg}_d(\tau \sigma^{-1}) \\ &= \sum_\alpha \binom{n}{\alpha}^2 \alpha! \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma) \\ &= n! \sum_\alpha \binom{n}{\alpha} \alpha! \sum_{\sigma \in S(\alpha)} \mathrm{Wg}_d(\sigma). \end{aligned}$$

□

Note that so far this argument is completely combinatorial, and relies only on the Weingarten convolution formula, without actually accessing the definition of the Weingarten function.

Now we plug in the character expansion

$$\mathrm{Wg}_d(\sigma) = \frac{1}{n!^2} \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)}{s_\lambda(1^d)} \chi^\lambda(\sigma), \tag{10.11}$$

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We can now appeal to Frobenius reciprocity:

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$$\sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} \sum_\alpha \langle 1, \chi^\lambda \downarrow_{S(\alpha)}^{S(n)} \rangle_{S(\alpha)} = \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} s_\lambda(1^k). \tag{10.20}$$

We've now proved the identity

$$\boxed{\int_{\mathcal{U}(d)} (u_{11} + \cdots + u_{kk})^n (\overline{u_{11} + \cdots + u_{kk}})^n dU = \sum_{\lambda \in \mathbb{Y}_d(n)} \frac{(\dim \lambda)^2}{s_\lambda(1^d)} s_\lambda(1^k),} \tag{10.21}$$

which one could also show using the theory of zonal polynomials, but that arguably requires even more background. When $k = d$, this general identity reduces to

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END

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