On Malliavin Calculus and Concentration Inequalities

by

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Abstract

We prove new abstract results concerning concentration inequalities and density estimates for Malliavin differentiable random variables. The efficacy of these results are demonstrated by practical computations, such as the calculation of novel concentration inequalities for $Z = \max_{1\leq i\leq n} N_i - E\left[\max_{1\leq i\leq n} N_i\right]$ where the $\{N_i\}_{i=1,\ldots,n}$ are Normal random variables, and $\int_0^1 B_s^4 ds - \frac{3}{4H+1}$ where $\{B_s, s \in [0,1]\}$ is a fractional Brownian motion with Hurst parameter H, as well as the derivation of non-asymptotic confidence intervals for the Hurst parameter of a fractional Brownian motion.

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Table of Contents

Abstract i
Acknowledgments ii
Table of Contents iii
Chapter 1 Introduction 1
Chapter 2 Malliavin Calculus 6
2.1 Wiener Chaos and Stochastic Integrals 6
2.1.1 The Wiener Chaos Expansion 6
2.1.2 Multiple Wiener-Itô Integrals 7
2.2 The Malliavin Derivative 9
2.3 The Adjoint of the Malliavin Derivative 14
2.3.1 Definition and Basic Results 14
2.3.2 Properties of the Skorohod Integral 17
2.3.3 The Skorohod Integral as an Extension of the Itô Integral 20
2.3.4 Stochastic Integral Representation of Wiener Functionals 22
2.4 The Ornstein-Uhlenbeck Semigroup 23
Chapter 3 Density Estimates and Concentration Inequalities 28
3.1 Density Estimates 28
3.2 Concentration Inequalities – Upper Bounds 34
3.3 Concentration Inequalities – Lower Bounds 48
3.4 Concentration Inequalities - Upper and Lower Bounds 51
Chapter 4 Conclusion 53
Bibliography 54

Chapter 1

Introduction

This chapter will document some of the basic elements of stochastic calculus, from the martingale upcrossing theorem to stochastic differential equations. It is not intended as didactic, but rather as broadly expository, hitting some key points in the theory. Many basic definitions will be assumed. These results will not necessarily be used later, either in the exposition of the Malliavin calculus (in Chapter 2), or in the original work on density estimates and concentration inequalities. The theorem statements which follow are based heavily on [5]. Every result in this chapter assumes that the filtration $\{\mathcal{F}_t\}$ satisfies the 'usual conditions' - in particular, $\{\mathcal{F}_t\}$ is right-continuous, and $\mathcal{F}_0$ contains all P-null events in $\mathcal{F}$, where $(\Omega, \mathcal{F}, P)$ is the ambient probability space.

The upcrossing and downcrossing inequalities. Let $\{X_t, 0 \leq t < \infty\}$ be a real-valued stochastic process. Let F be a finite subset of $[0, \infty)$. Let $\alpha < \beta$. Define the random quantities $\tau_j(\omega)$ and $\sigma_j(\omega)$, for $j \geq 1$, as follows:

$$\tau_1(\omega) = \min\{t \in F : X_t(\omega) \leq \alpha\}. \tag{1.1}$$

We then define recursively:

$$\begin{aligned} \sigma_j(\omega) &= \min\{t \in F : t \geq \tau_j, X_t(\omega) \geq \beta\},\ j \geq 1 \\ \tau_j(\omega) &= \min\{t \in F : t \geq \sigma_{j-1}, X_t(\omega) \leq \alpha\},\ j \geq 2. \end{aligned} \tag{1.2}$$

The convention adopted here is that the minimum of the empty set is ∞. Define then the 'number of upcrossings' by the quantity $U_F(\alpha, \beta, X(\omega))$, which is the largest j such that $\sigma_j(\omega)$ is finite. We extend this definition for the case where F is not a finite set by:

$$U_I(\alpha, \beta, X(\omega)) = \sup_{F \subset I, F \text{ finite}} U_F(\alpha, \beta, X(\omega)), \tag{1.3}$$

where I is any subset of $\mathbb{R}$. We define the 'number of downcrossings', denoted by $D_I(\alpha, \beta, X(\omega))$, in an analogous manner. The following theorem characterizes upper bounds on the expected number of upcrossing and downcrossings of a submartingale.

Theorem 1.1. (from [5])
Let $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a right-continuous sample paths. Let $[\sigma, \tau]$ be a subinterval of $[0, \infty)$, and let $\alpha < \beta$. Then we have:

$$\begin{aligned} E[U_{[\sigma,\tau]}(\alpha, \beta, X(\omega))] &\leq \frac{E[X_\tau^+] + |\alpha|}{\beta - \alpha} \\ E[D_{[\sigma,\tau]}(\alpha, \beta, X(\omega))] &\leq \frac{E[(X_\tau - \alpha)^+]}{\beta - \alpha}. \end{aligned} \tag{1.4}$$

Submartingale convergence. The following theorem gives a sufficient condition for a submartingale to admit an integrable last element.

Theorem 1.2. (from [5])
Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a submartingale with right-continuous sample paths. Assume that $\sup_{t \geq 0} E[X_t^+] < \infty$. Then there exists a random variable X_∞, such that $\lim_{t \to \infty} X_t = X_\infty$ almost surely and $E[|X_\infty|] < \infty$.

Optional sampling. The optional sampling theorem in a certain sense generalizes the submartingale property of a stochastic process. More than simply saying that for every $s < t$, we have $E[X_t | \mathcal{F}_s] \geq X_s$, under certain conditions a similar property holds when we consider X stopped at random times S and T. The following theorem makes this precise.

Theorem 1.3. (from [5])
Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a submartingale with right-continuous sample paths. Assume that X admits a last element X_∞. Let S, T be optional times of the filtration, such that $S \leq T$ almost surely. Then almost surely:

$$E[X_T | F_{S^+}] \geq X_S. \tag{1.5}$$

The Itô rule. This rule is the stochastic calculus analog of the 'chain rule'. It states that a sufficiently regular function of a semi-martingale is still a semi-martingale, and gives its decomposition.

Theorem 1.4. (from [5])
Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous semi-martingale with decomposition:

$$X_t = X_0 + M_t + B_t, \ t \geq 0, \tag{1.6}$$

where M is a continuous local martingale and B is a continuous process of bounded variation. Let $f \in C^2(\mathbb{R})$. Then we have for every $t \geq 0$:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dB_s + \frac{1}{2} \int_0^t f''(X_s) d\langle M \rangle_s, \tag{1.7}$$

where the first integral on the right hand side of (1.7) is interpreted as an Itô integral and the others are standard Lebesgue-Stieltjes integrals.

We can write the Itô rule formally using differential notation; more specifically, under the same assumptions as the previous theorem, we can write the Itô rule as:

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t \\ &= f'(X_t)dM_t + f'(X_t)dB_t + f''(X_t)d\langle M\rangle_t. \end{aligned} \tag{1.8}$$

The Burkholder-Davis-Gundy inequalities. The next inequality, often referred to as the BDG-inequality, characterizes the even moments of the supremum of the magnitude of a continuous local martingale in terms of its quadratic variation. We define for $t \geq 0 : M_t^* = \sup_{0\leq s\leq t} |M_s|$.

Theorem 1.5. (from [5])
Let M be a continuous local martingale, T be a stopping time, and $m \geq 0$. Then there exist universal constants k_m and K_m, which depend on m but not on M or T, such that:

$$k_m E[\langle M\rangle_T^m] \leq E[(M_T^*)^{2m}] \leq K_m E[\langle M\rangle_T^m]. \tag{1.9}$$

Representing continuous martingales in terms of Brownian motion. The next theorem states that continuous local martingales are stochastic integrals of some kernel processes with respect to Brownian motion, on an extended probability space.

Theorem 1.6. (from [5])
Let $M = \{M_t = (M_t^{(1)}, ..., M_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$, where $M^{(i)}$ is a continuous local martingale, for every $1 \leq i \leq d$. Assume further that for every $1 \leq i, j \leq d$, the cross-variation process $\langle M^{(i)}, M^{(j)}\rangle_t$ is an absolutely continuous function of t, P-almost surely. Then there exists an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and d-dimensional Brownian motion $B = \{B_t = (B_t^{(1)}, ..., B_t^{(d)}), \tilde{\mathcal{F}}_t; 0 \leq t < \infty\}$ and a matrix-valued process $X = \{X_t = (X_t^{(i,j)})_{i,j=1,...,n}, \tilde{\mathcal{F}}_t; 0 \leq t < \infty\}$ satisfying:

$$\tilde{P}\left(\int_0^t (X_s^{(i,j)})^2 ds < \infty\right) = 1,\ 0 \leq t < \infty,\ 1 \leq i, j \leq d. \tag{1.10}$$

Morever, we can choose B and X such that $\tilde{P}$-almost surely:

$$\begin{aligned} M_t^{(i)} &= \sum_{j=1}^{d} \int_0^t X_s^{(i,j)} dW_s^{(j)},\ 1 \leq i \leq d, \\ \langle M^{(i)}, M^{(j)}\rangle_t &= \sum_{k=1}^{d} \int_0^t X_s^{(i,k)} X_s^{(j,k)} ds. \end{aligned} \tag{1.11}$$

The Girsanov theorem. This theorem states that given a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, a 'translated' version of that process is still a Brownian motion with respect to a different underlying probability measure, $\tilde{P}$.

Theorem 1.7. (from [5])
Let $B = \{B_t = (B_t^{(1)}, ..., B_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. Let $X = \{X_t = (X_t^{(1)}, ..., X_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ be a process satisfying:

$$P\left(\int_0^t (X_s^{(i)})^2 ds < \infty\right) = 1, \ 0 \leq t < \infty, \ 1 \leq i \leq d. \tag{1.12}$$

Define $Z_t(X) = \exp\left(\sum_{i=1}^d \int_0^t X_s^{(i)} dW_s^{(i)} - \frac{1}{2}\int_0^t ||X_s||^2 ds\right)$. Assume that $Z(X) = \{Z_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale. Define the process $\tilde{B} = \{\tilde{B}_t = (\tilde{B}_t^{(1)}, ..., \tilde{B}_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$, where

$$\tilde{B}_t^{(i)} = B_t^{(i)} - \int_0^t X_s^{(i)} ds. \tag{1.13}$$

Then for each fixed $T > 0$, the process $(\tilde{B}_t, \mathcal{F}_t; 0 \leq t \leq T)$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}_T, \tilde{P}_T)$, where $\tilde{P}_T(A) = E[\mathbf{1}_A Z_T(X)]$ for every $A \in \mathcal{F}_T$.

Stochastic differential equations. We state here existence and uniqueness results for weak and strong solutions to stochastic differential equations - i.e., equations of the form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{1.14}$$

where $b_i(t,x), \sigma_{i,j}(t,x); 1 \leq i \leq d, 1 \leq j \leq r$ are functions from $[0,\infty) \times \mathbb{R}^d$ into $\mathbb{R}$, and $W = \{W_t; 0 \leq t < \infty\}$ is an r-dimensional Brownian motion.

Our first result is a sufficient condition for 'strong existence' of solutions to (1.14). For a formal definition of 'strong existence', see [5]. Broadly speaking, a strong solution to (1.14) is a process defined with respect to a given probability space $(\Omega, \mathcal{F}, P)$, Brownian motion W, and initial condition ξ, adapted with respect to the filtration generated by W (augmented by the P-null sets and $\sigma(\xi)$), and satisfying (1.14).

Theorem 1.8. (from [5])
Let $b(t,x)$ and $\sigma(t,x)$ satisfy both global Lipschitz and linear growth conditions in the space variable - i.e., there exists $K > 0$ such that for every $x, y \in \mathbb{R}^d$, for every $t \geq 0$:

$$\begin{aligned} &||b(t,x) - b(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K||x - y|| \\ &||b(t,x)||^2 + ||\sigma(t,x)||^2 \leq K^2(1 + ||x||^2). \end{aligned} \tag{1.15}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $W = (W_t, \mathcal{F}_t^W; 0 \leq t < \infty)$ be an r-dimensional Brownian motion on this space, and let ξ be an $\mathbb{R}^d$-valued random variable independent of W such that $E[||\xi||^2] < \infty$. Let $\{\mathcal{F}_t\}$ be the filtration defined by:

$$\mathcal{F}_t = \sigma(\sigma(\mathcal{F}_t^W \cup \sigma(\xi)) \cup \mathcal{N}), \tag{1.16}$$

where $\mathcal{N} = \{N \in \Omega : \exists A \in \sigma(\sigma(\mathcal{F}_t^W \cup \sigma(\xi)), t \geq 0), P(A) = 0, N \subset A\}$. Then there exists a strong solution to (1.14), with respect to the probability space $(\Omega, \mathcal{F}, P)$, the Brownian motion W, the initial condition ξ, and the filtration $\{\mathcal{F}_t\}$.

Our next result is a sufficient condition for 'strong uniqueness' for (1.14). For a formal definition of the term 'strong uniqueness', we refer the reader to [5]. Broadly speaking however, strong uniqueness is said to hold for (1.14) if, having fixed a probability space, initial condition, and Brownian motion, any two strong solutions X and $\tilde{X}$ must be versions of one another.

Theorem 1.9. (from [5])
Assume that the $b(t,x)$ and $\sigma(t,x)$ are locally Lipschitz in the space variable - i.e., for every $n \geq 1$ there exists $K_n > 0$ such that for every $t \geq 0$, $x, y \in \mathbb{R}^d$ such that $||x||, ||y|| \leq n$, we have:

$$||b(t,x) - b(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K_n ||x - y||. \tag{1.17}$$

Then strong uniqueness holds for (1.14).

The next result is a sufficient condition for existence of a 'weak solution' to the time-homogeneous version of (1.14):

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \tag{1.18}$$

Again, for a formal definition of the 'weak solution', see [5], but broadly speaking, a weak solution to (1.18) is a triple (X, W), $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ where the process X satisfies (1.18) and both X and W are adapted to $\{\mathcal{F}_t\}$.

Theorem 1.10. (from [5])
If the coefficients $b, \sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}$ in (1.18) are bounded and continuous, and if the distribution μ satisfies:

$$\int_{\mathbb{R}^d} ||x||^{2m} \mu(dx) < \infty \text{ for some } m > 1, \tag{1.19}$$

then (1.18) admits a weak solution with initial distribution μ.

The next theorem gives a sufficient condition for a kind of uniqueness in probability law for solutions to the stochastic differential equation:

$$dX_t = \underline{(}t, X_t)dt + dW_t, \ 0 \leq t \leq T. \tag{1.20}$$

Theorem 1.11. (from [5])
If $(X^{(i)}, W^{(i)})$, $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)})$, $\{\mathcal{F}_t^{(i)}\}$, $i = 1, 2$, are two weak solutions to (1.20) which have the same initial distribution, and if, moreover,

$$P^{(i)}\left(\int_0^T ||b(t, X_t^{(i)})||^2 dt < \infty\right) = 1, \tag{1.21}$$

then the processes $X^{(1)}$ and $X^{(2)}$ are identical in law.

Chapter 2

Malliavin Calculus

This chapter introduces the fundamental elements of the Malliavin calculus: the Wiener chaos expansion, the Malliavin derivative, the Skorohod integral, and the generator of the Ornstein-Uhlenbeck semigroup. The treatment is based on [9]. Some proofs are included, in particular for those results where the proof illuminates some aspect of the result.

2.1 Wiener Chaos and Stochastic Integrals

2.1.1 The Wiener Chaos Expansion

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let H be a separable Hilbert space, equipped with an inner product denoted by $\langle \cdot, \cdot \rangle_H$. Then an *isonormal Gaussian process* is defined as a Gaussian family $W = \{W(h), h \in H\}$ where each $W(h)$ is a centered Gaussian random variable, and such that for all $h, g \in H$, we have: $E[W(h)W(g)] = \langle h, g \rangle_H$.

Remark 2.2. By a simple calculation, an easy consequence of the above definition that the mapping $h \mapsto W(h)$ is linear. This means we can simplify the above definition by removing the assumption that the family $\{W(h), h \in H\}$ is a Gaussian family, since this is implied by the linearity of the mapping.

Remark 2.3. Kolmogorov's Extension theorem ([5]) guarantees that given any Hilbert space H, there exists a corresponding probability space $(\Omega, \mathcal{F}, P)$ and a family $W = \{W(h), h \in H\}$ of random variables defined on that space, such that together they form an isonormal Gaussian process.

Definition 2.4. We define the *Hermite polynomials* as the family $\{H_n\}_{n=0}^{\infty}$ where $H_0(x) = 1$ for all $x \in \mathbb{R}$ and

$$H_n(x) = \frac{(-1)^n}{n!} \exp(\frac{x^2}{2}) \frac{d^n}{dx^n}(\exp(\frac{-x^2}{2})), \ x \in \mathbb{R}. \tag{2.1}$$

Hermite polynomials and Gaussian random variables satisfy an orthogonality property, precisely stated in the following lemma. The proof of the lemma is simply calculus.

Lemma 2.5. (from [9])
Let X and Y be two random variables with a joint Normal distribution, each centered and with unit variance. Then for all $n, m \geq 0$, we have:

$$E[H_n(X)H_m(Y)] = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{n!}(E[XY])^n & \text{if } n = m \end{cases} \tag{2.2}$$

Lemma 2.6. (from [9])
Let $F(x,t) = \exp(tx - \frac{t^2}{2})$. Then we have

$$F(x,t) = \sum_{n=0}^{\infty} t^n H_n(x).$$

In the sequel, we will assume the notation that H is a Hilbert space, $(\Omega, \mathcal{F}, P)$ is a probability space, and $W = \{W(h), h \in H\}$ is a corresponding isonormal Gaussian process. We will also denote: $\mathcal{G} = \sigma(\{W(h) : h \in H\})$.

Definition 2.7. For every $n \geq 1$, we define $\mathcal{H}_n$ as the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(W(h)), h \in H, ||h||_H = 1\}$, and we define $\mathcal{H}_0$ as the constants. We call $\mathcal{H}_n$ the *Wiener chaos of order n.*

The following theorem establishes the 'Wiener chaos decomposition'.

Theorem 2.8. (from [9])
The space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:

$$L^2(\Omega, \mathcal{G}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{2.3}$$

2.1.2 Multiple Wiener-Itô Integrals

In the sequel, we assume that $(T, \mathcal{B}, \mu)$ is a measure space where μ is a non-atomic measure, and that the separable Hilbert space H is given by: $H = L^2(T, \mathcal{B}, \mu)$. Note that in this case, the isonormal Gaussian process $W = \{W(h) : h \in H\}$ is determined by the family $\{W(1_A) : A \in \mathcal{B}, \mu(A) < \infty\}$. Moreover, we can consider W as a $L^2(\Omega, \mathcal{F}, P)$-valued measure on the space $(T, \mathcal{B})$, where $W(A) := W(1_A)$ for every $A \in \mathcal{B}$. We say that W is an $L^2(\Omega)$-valued Gaussian measure on $(T, \mathcal{B})$, which we shall call the *white noise based on μ.*

Remark 2.9. The white noise based on μ takes on independent values for disjoint measurable subsets of T and $W(A)$ has law given by $N(0, \mu(A))$.

We now construct the multiple stochastic integral - concretely, we wish to define the multiple stochastic integral $I_m(f)$ of a function $f \in L^2(T^m, \mathcal{B}^m, \mu^m)$. Fix $m \geq 1$, and let $\mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$. We define first the stochastic integral for integrands f in the space $\mathcal{E}_m$ of 'elementary functions' - i.e., functions of the form:

$$f(t_1, ..., t_m) = \sum_{i_1,...,i_m=1}^{n} a_{i_1,...,i_m} 1_{A_{i_1} \times ... \times A_{i_m}}(t_1, ..., t_m), \tag{2.4}$$

where $A_1, ..., A_n$ are pairwise disjoint elements of $\mathcal{B}_0$ and the sum is taken over only those tuples $(i_1, ..., i_m)$ which contain no repeated entries. We define:

$$I_m(f) = \sum_{i_1,...,i_m=1}^{n} a_{i_1,...,i_m} W(A_{i_1}) \cdots W(A_{i_m}). \tag{2.5}$$

Remark 2.10. The following are properties of the multiple stochastic integral thus defined:

1. I_m is linear.
2. $I_m(f) = I_m(\tilde{f})$ where $\tilde{f}$ is the symmetrization of f, given by:
$$\tilde{f}(t_1, ..., t_m) = \frac{1}{m!} \sum_{\sigma} f(t_{\sigma(1)}, ...t_{\sigma(m)}), \tag{2.6}$$
where σ runs over all permutations of $\{1, ..., m\}$.
3. The multiple stochastic integral satisfies an orthogonality property - precisely:
$$E[I_m(f)I_q(g)] = \begin{cases} 0 & \text{if } m \neq q \\ m!\langle \tilde{f}, \tilde{g} \rangle_{L^2(T^m)} & \text{if } m = q \end{cases} \tag{2.7}$$

We now extend this definition of the multiple stochastic integral from the space of elementary functions, $\mathcal{E}_m$, to $L^2(T^m)$. This is done by showing that $\mathcal{E}_m$ is dense in $L^2(T^m)$ - more specifically, by showing that a function of the form $1_{A_1 \times ... \times A_m},\ A_i \in \mathcal{B},\ \mu(A_i) < \infty,\ i = 1, ..., m$, can be approximated arbitrarily well by elements of $\mathcal{E}_m$. The assumption that μ is atomless is crucial to the proof of this claim. This can be done without difficulty. The properties described in Remark 2.10 also hold for the extended multiple stochastic integral.

We can characterize the Wiener chaos expansion in an alternate way using the relationship between the Hermite polynomials and multiple stochastic integrals:

Theorem 2.11. (from [9])
Let H_m be the m^{th} Hermite polynomial and let $h \in H = L^2(T)$ be such that $||h||_H = 1$. Then we have:

$$m! H_m(W(h)) = \int_{T^m} h(t_1) \cdots h(t_m) W(dt_1) \cdots W(dt_m). \tag{2.8}$$

As a result, the Wiener chaos expansion of $F \in L^2(\Omega, \mathcal{G}, P)$ can be written as:

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{2.9}$$

where $f_0 = E[F]$, and I_0 is the identity mapping, and $f_m \in L^2(T^m)$ are symmetric and uniquely determined by F.

2.2 The Malliavin Derivative

As before, we will assume that the Hilbert space H has the form $H = L^2(T, \mathcal{B}, \mu)$, where μ is a σ-finite atomless measure, and $W = \{W(h), h \in H\}$ is an isonormal Gaussian process associated with this Hilbert space, defined on a probability space $(\Omega, \mathcal{F}, P)$. We make the additional assumption that $\mathcal{F} = \sigma(W)$.

Definition 2.12. We define $\mathcal{S}$ as the class of *smooth random variables:*

$$\mathcal{S} = \{f(W(h_1), ..., W(h_n)) : f \in C_p^\infty(\mathbb{R}^n), h_1, ..., h_n \in H, n \geq 1\}, \tag{2.10}$$

where $C_p^\infty(\mathbb{R}^n)$ is the set of all $f \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that f and all its partial derivatives have polynomial growth.

We now define the *Malliavin derivative* operator for elements of $\mathcal{S}$.

Definition 2.13. Let $F \in \mathcal{S}$. Then we define the Malliavin derivative DF as the stochastic process $\{D_t F, t \in T\}$, given by:

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t). \tag{2.11}$$

Note that we can consider DF as an element of $L^2(T \times \Omega)$, or as an element of $L^2(\Omega; H)$.

The following lemma is an 'integration by parts' formula.

Lemma 2.14. (from [9])
Let $F \in L^2(\Omega, \mathcal{G}, P)$ be a smooth random variable and let $h \in H = L^2(T)$. Then:

$$E[\langle DF, h\rangle_H] = E[FW(h)]. \tag{2.12}$$

Proof. Note first that we can assume without loss of generality that $||h|| = 1$. We can also assume without loss of generality that:

$$F = f(W(e_1), ..., W(e_n)), \tag{2.13}$$

where $e_1, ..., e_n \in H$ are orthonormal, and $h = e_1$. In this case, if we let $\phi(x)$ denote the standard normal density on $\mathbb{R}^n$: $\phi(x) = (2\pi)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)$, we have:

$$\begin{aligned}
E[\langle DF, h\rangle_H] &= E\left[\left\langle \sum_{j=1}^n \frac{\partial}{\partial x_j} f(W(e_1), ..., W(e_n))e_j, e_1 \right\rangle\right] \\
&= E\left[\frac{\partial}{\partial x_1} f(W(e_1), ..., W(e_n))\right] \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_1} f(x)\phi(x)dx \\
&= \int_{\mathbb{R}^n} f(x)x_1\phi(x)dx \\
&= E[FW(e_1)] \\
&= E[FW(h)].
\end{aligned} \tag{2.14}$$

This completes the proof. □

Lemma 2.15. (from [9])
Let F and G be smooth elements of $L^2(\Omega, \mathcal{G}, P)$, and let $h \in H$. Then:

$$E[G\langle DF, h\rangle_H] = E[-F\langle DG, h\rangle_H + FGW(h)]. \tag{2.15}$$

Proof. Simply apply the previous lemma to the product FG. □

We can consider taking iterated derivatives of certain random variables. Notationally, if F is a smooth random variable and k is an integer, we define: $D^k_{t_1,...,t_k}F = D_{t_1}D_{t_2}\cdots D_{t_k}F$. Note that D^kF is a measurable function on the product space $T^k \times \Omega$, defined almost everywhere with respect to the measure $\mu^k \times P$.

Definition 2.16. For every $p \geq 1$ and any $k \in \mathbb{N}$, we define a norm on $\mathcal{S}$ given by:

$$||F||_{k,p} = \left[E[|F|^p] + \sum_{j=1}^{k} E[||D^jF||^p_{L^2(T^j)}]\right]^{\frac{1}{p}}. \tag{2.16}$$

Definition 2.17. We denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $||\cdot||_{k,p}$.

Then we can extend the domain of definition of the operator D in $L^p(\Omega)$ to $\mathbb{D}^{1,p}$.

The chain rule for the Malliavin derivative can be extended to a more general context by the previous definition, by approximating the random variable F using smooth random variables.

Proposition 2.18. (from [9])
Let $F = (F^1, ..., F^m)$ be a random vector, such that $F^i \in \mathbb{D}^{1,p}$ for $i = 1, ..., m$, and some $p \geq 1$. Let $\phi \in C^1(\mathbb{R}^m)$, such that all the partial derivatives of ϕ are bounded. Then $\phi(F) \in \mathbb{D}^{1,p}$, and

$$D(\phi(F)) = \sum_{i=1}^{m} \frac{\partial \phi}{\partial x_i}(F)DF^i. \tag{2.17}$$

We state the following technical lemma.

Lemma 2.19. (from [9])
Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{D}^{1,2}$ which converge in $L^2(\Omega)$ to F. Assume that

$$\sup_n E[||DF_n||^2] < \infty. \tag{2.18}$$

Then $F \in \mathbb{D}^{1,2}$, and moreover the sequence $\{DF_n\}_{n=1}^{\infty}$ converges weakly in $L^2(T \times \Omega)$ to DF.

Another extension of the Malliavin chain rule concerns Lipschitz functions, and is considered in the following proposition.

Proposition 2.20. (from [9])
Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be globally Lipschitz with Lipschitz constant K - i.e., for every $x, y \in \mathbb{R}^m$:

$$|\phi(x) - \phi(y)| \leq K||x - y||. \tag{2.19}$$

Let $F = (F_1, ..., F_m)$ be a random vector with $F_i \in \mathbb{D}^{1,2}$, for $i = 1, ..., m$. Then $\phi(F) \in \mathbb{D}^{1,2}$, and there exists a random vector $(G_1, ..., G_m)$ bounded by K such that

$$D\phi(F) = \sum_{i=1}^{m} G_i DF_i. \tag{2.20}$$

Proof. Note that if ϕ is in $C^1(\mathbb{R}^m)$, then this result reduces to Proposition 2.18. The proof here proceeds by a regularization argument. We introduce the sequence of regularization kernels $\{\alpha_n\}_{n=1}^{\infty}$, where $\alpha \in C_0^{\infty}(\mathbb{R}^m)$ is a non-negative function which has support given by the unit ball in $\mathbb{R}^m$ such that $\int_{\mathbb{R}^m} \alpha(x)dx = 1$, and $\alpha_n(x) = n^m\alpha(nx)$. We define a sequence of regularized approximations of ϕ, denoted by $\{\phi_n\}_{n=1}^{\infty}$, where $\phi_n := \alpha_n * \phi$. Each ϕ_n is infinitely differentiable, and $\phi_n(x)$ converges uniformly to $\phi(x)$. Moreover, each ϕ_n has bounded derivatives, and thus we can apply Proposition 2.18 to deduce that $\phi_n(F) \in \mathbb{D}^{1,2}$ for every n, and

$$D(\phi_n(F)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi_n(F)DF_i. \tag{2.21}$$

Note now that the uniform convergence of the ϕ_n to ϕ guarantees that $\phi_n(F)$ converges to $\phi(F)$ in $L^2(\Omega)$. Moreover, since the sequence $\{||D(\phi_n(F))||\}_{n=1}^{\infty}$ is uniformly bounded, we can apply Lemma 2.19 to conclude that $\phi(F) \in \mathbb{D}^{1,2}$, and the sequence $\{D(\phi_n(F))\}_{n=1}^{\infty}$ converges weakly in $L^2(T \times \Omega)$ to $D(\phi(F))$. Now note that the sequence $\{\nabla\phi_n(F)\}_{n=1}^{\infty}$ is uniformly bounded by K, and hence has a weakly convergent subsequence $\{\nabla\phi_{n(k)}(F)\}_{k=1}^{\infty}$, with limit denoted by $G = (G_1, ..., G_m)$, which is itself bounded by K. Hence taking the limit as $n \to \infty$ in 2.21, we obtain the desired result. □

Remark 2.21. If the random vector F admits a density with respect to the Lebesgue measure on $\mathbb{R}^m$, then $G_i = \frac{\partial\phi}{\partial x_i}$ in (2.20).

Suppose now that F is a square-integrable random variable which admits a Wiener chaos expansion of the form:

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{2.22}$$

where the f_m are symmetric elements of $L^2(T^m)$. The following proposition characterizes the conditions on the kernels f_m which will guarantee that F is in the domain of the derivative operator, as well giving the Wiener chaos expansion of the derivative of F in terms of these kernels.

Proposition 2.22. (from [9])
Let $F \in L^2(\Omega, \mathcal{F}, P)$, with Wiener chaos expansion of the form (2.22). Then $F \in \mathbb{D}^{1,2}$ if and only if:

$$\sum_{m=1}^{\infty} mm! ||f_m||^2_{L^2(T^m)} < \infty, \tag{2.23}$$

in which case:

$$D_t F = \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t)), \tag{2.24}$$

and $E[\int_0^T (D_t F)^2 \mu(dt)] = \sum_{m=1}^{\infty} mm! ||f_m||^2_{L^2(T^m)}$.

Proof. The proof consists of three steps.

1. First, assume that $F = H_m(W(h))$, where $h \in H$ with $||h|| = 1$. Then by Theorem 2.11, we have that $F = \frac{1}{m!} I_m(h^{\otimes m})$. That is, F has a chaos decomposition with kernel functions identically zero except for the case $f_m = \frac{1}{m!} h^{\otimes m}$. We have that $F \in \mathcal{S}$, and that:

   $$D_t F = H'_m(W(h))h(t) = H_{m-1}(W(h))h(t).$$

   Thus: $D_t F = m I_{m-1}(f_m(\cdot, t))$.

2. Let now $F \in L^2(\Omega, \mathcal{F}, P)$ with chaos decomposition of the form (2.22). Suppose also that (2.23) holds. Let

   $$F^N = \sum_{m=0}^{N} I_m(f_m). \tag{2.25}$$

   We have that $F_N \to F$ in $L^2(\Omega)$. From Step 1, we have that each $F^N \in \mathbb{D}^{1,2}$, and that $D_t F^N = \sum_{m=1}^{N} m I_{m-1}(f_m(\cdot, t))$. Then (2.23) guarantees that $DF^N \to \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t))$ in $L^2(T \times \Omega)$ as $N \to \infty$. Thus $F \in \mathbb{D}^{1,2}$ and (2.24) holds, by the fact that the operator D is closed.

3. Assume now that $F \in \mathbb{D}^{1,2}$. Let g be an elementary symmetric function and let $G = I_n(g)$. Note that we can apply the integration by parts formula in Lemma 2.15 for $F \in \mathbb{D}^{1,2}$ and G defined as before. Thus for every $h \in H$, we have:

   $$\begin{aligned} \lim_{N\to\infty} E[\langle DF^N, h\rangle G] &= \lim_{N\to\infty} E[-F^N \langle DG, h\rangle + F^N G W(h)] \\ &= E[-F\langle DG, h\rangle + FGW(h)] \\ &= E[\langle DF, h\rangle G]. \end{aligned} \tag{2.26}$$

We will show that (2.23) holds under our assumption that $F \in \mathbb{D}^{1,2}$ by writing

$$E\left[\sum_{i=1}^{\infty}\sum_{m=0}^{\infty}\left(mI_{m-1}\left(\int_T f_m(\cdot,t)e_i(t)\mu(dt)\right)\right)^2\right] \tag{2.27}$$

in two different ways, where $\{e_i, i \geq 1\}$ is an orthonormal basis of H. First note that by (2.7), we can write (2.27) equivalently as (2.23). Now note that for $N > n$, the expectation $E[\langle DF^N, h\rangle G]$ is equal to:

$$E\left[(n+1)I_n\left(\int_T f_{n+1}(\cdot,t)h(t)\mu(dt)\right)G\right], \tag{2.28}$$

where this again follows from the orthogonality property of the multiple stochastic integral (cf. (2.7)). Therefore, the projection of $\langle DF, h\rangle$ on the n-th Wiener chaos is equal to $(n+1)I_n\left(\int_T f_{n+1}(\cdot,t)h(t)\mu(dt)\right)$. Consequently, we can also write (2.27) as:

$$\sum_{i=1}^{\infty} E[\langle DF, e_i\rangle^2] = ||DF||^2_{L^2(T\times\Omega)} < \infty. \tag{2.29}$$

This completes the proof. □

Proposition 2.22 shows that the Malliavin derivative of an m-th order stochastic integral is obtained by removing one of the integrals, leaving the last argument of the kernel function as a free parameter, and multiplying by m.

Remark 2.23. From Proposition 2.22 we see that if $F \in \mathbb{D}^{1,2}$ such that $DF = 0$, then $F = E[F]$.

Remark 2.24. Applying the chain rule (in particular, Proposition 2.18) to a function $\phi \in C_0^\infty(\mathbb{R})$ satisfying that $\phi(x) = x^2$ for $x \in [0,1]$, we see that $A \in \mathcal{F}$ satisfies that $1_A \in \mathbb{D}^{1,2}$ if and only if $P(A)$ is zero or one. More specifically, note that $D\mathbf{1}_A = D\mathbf{1}_A^2 = 2\mathbf{1}_A D\mathbf{1}_A$. Thus $D\mathbf{1}_A = 0$, and by the first part of the remark, we have that $\mathbf{1}_A = P(A)$.

The following lemma is a technical result. Dealing with notation first, let $A \in \mathcal{B}$. Denote by $\mathcal{F}_A$ the σ-field (completed with respect to the measure P) generated by the random variables $\{W(B), B \subset A, B \in \mathcal{B}, \mu(B) < \infty\}$.

Lemma 2.25. (from [9])
Suppose that F is a square-integrable random variable with chaos expansion $F = \sum_{m=0}^{\infty} I_m(f_m)$. Let $A \in \mathcal{B}$. Then

$$E[F|\mathcal{F}_A] = \sum_{m=0}^{\infty} I_m(f_m \mathbf{1}_A^{\otimes m}). \tag{2.30}$$

Proof. It suffices to assume that $F = I_m(f_m)$, where f_m is an elementary function - i.e., $f_m \in \mathcal{E}_m$. Also, by linearity we can assume that the kernel f_m is of the form $\mathbf{1}_{B_1 \times ... \times B_m}$, where $B_1, ..., B_m$ are mutually disjoint sets of finite measure. In this case we have

$$\begin{aligned} E[F|\mathcal{F}_A] &= E[I_m(\mathbf{1}_{B_1 \times ... \times B_m})] \\ &= E[W(B_1) \cdots W(B_m)|\mathcal{F}_A] \\ &= E[\prod_{i=1}^{m}(W(B_i \cap A) + W(B_i \cap A^c))|\mathcal{F}_A] \\ &= E[W(B_1 \cap A)W(B_2 \cap A) \cdots W(B_m \cap A)] \\ &= E[I_m(\mathbf{1}_{B_1,...,B_m}\mathbf{1}_A^{\otimes m})] \end{aligned}$$

□

Proposition 2.26. (from [9])
Suppose that F belongs to $\mathbb{D}^{1,2}$, and let $A \in \mathcal{B}$. Then the conditional expectation $E[F|\mathcal{F}_A]$ also belongs to the space $\mathbb{D}^{1,2}$, and we have

$$D_t(E[F|\mathcal{F}_A]) = E[D_tF|\mathcal{F}_A]\mathbf{1}_A(t)$$

almost everywhere in $T \times \Omega$.

Proof. By Lemma 2.25 and Proposition 2.22 we have

$$\begin{aligned} D_t(E[F|\mathcal{F}_A]) &= \sum_{m=1}^{\infty} mI_{m-1}(f_m(\cdot, t)\mathbf{1}_A^{\otimes(m)})\mathbf{1}_A(t) \\ &= E[D_tF|\mathcal{F}_A]\mathbf{1}_A(t). \end{aligned}$$

□

2.3 The Adjoint of the Malliavin Derivative

2.3.1 Definition and Basic Results

In this section we will define and discuss the adjoint of the derivative operator. Since $D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(T \times \Omega)$, the adjoint is a mapping $\delta : Dom\delta \subset L^2(T \times \Omega) \to L^2(\Omega)$ satisfying, for every $Z \in \mathbb{D}^{1,2}, u \in Dom\delta$:

$$E[Z\delta(u)] = E[\langle DZ, u\rangle_H].$$

We now make this more precise.

Definition 2.27. We define the *adjoint* of the Malliavin derivative operator D as the mapping $\delta : Dom\delta \subset L^2(T \times \Omega) \to L^2(\Omega)$, with domain $Dom\delta$ given by:

$$\{u \in L^2(T \times \Omega) : |E[\langle DF, u\rangle]| \leq c||F||_2 \text{ for every } F \in \mathbb{D}^{1,2}, \text{ where } c \text{ is a constant depending only on } u\}. \tag{2.31}$$

For u in $Dom\delta$, $\delta(u)$ is defined to be the unique element of $L^2(\Omega)$ satisfying for every $F \in \mathbb{D}^{1,2}$:

$$E[F\delta(u)] = E[\langle DF, u\rangle], \tag{2.32}$$

where by definition we have the right hand side of (2.32) is equal to $E\left[\int_T D_t F u_t \mu(dt)\right]$. Note that δ is sometimes referred to as the *divergence operator*.

Remark 2.28. The adjoint is a closed operator.

Remark 2.29. We will refer to $\delta(u)$ as the "Skorohod stochastic integral" of the process u, and denote it by $\delta(u) = \int_T u_t dW_t$.

The following technical lemma is a representation result in $L^2(T \times \Omega)$.

Lemma 2.30. (from [9])
Let $u \in L^2(T\times\Omega)$. Then there exists a sequence of measurable kernels $\{f_m(t_1, ..., t_m, t)\}_{m=0}^{\infty} \subset L^2(T^{m+1})$, each f_m being symmetric in its first m arguments, such that:

$$u(t) = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t)). \tag{2.33}$$

Furthermore:

$$E\left[\int_T u^2(t)\mu(dt)\right] = \sum_{m=0}^{\infty} m!||f_m||^2_{L^2(T^{m+1})}. \tag{2.34}$$

The proof of Lemma 2.30 is omitted, but we note that for each fixed t, $u(t) \in L^2(\Omega)$ and thus admits a Wiener chaos decomposition with kernels $\{f_m(\cdot, t)\}_{m=0}^{\infty}$ depending on the parameter t. It remains to show that these kernels are measurable in all $(m + 1)$ arguments and that the series (2.33) converges. This is done via an approximation argument.

The following proposition characterizes the divergence operator and its domain in terms of the Wiener chaos expansion.

Proposition 2.31. (from [9])
Let $u \in L^2(T \times \Omega)$, such that $u(t) = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t))$. Then $u \in Dom\delta$ if and only if

$$\sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m) \tag{2.35}$$

converges in $L^2(\Omega)$, and in this case $\delta(u)$ is given by the sum in (2.35).

Proof. Assume that $G = I_n(g)$ is a multiple stochastic integral of order n, where $g \in L^2(T^n)$ is a symmetric function. Then we have:

$$\begin{aligned} E\left[\int_T u_t D_t G \mu(dt)\right] &= \sum_{m=0}^{\infty} \int_T E[I_m(f_m(\cdot,t)) n I_{n-1}(g(\cdot,t))]\mu(dt) \\ &= \int_T E[I_{n-1}(f_{n-1}(\cdot,t)) n I_{n-1}(g(\cdot,t))]\mu(dt) \\ &= n(n-1)! \int_T \langle f_{n-1}(\cdot,t), g(\cdot,t)\rangle_{L^2(T^{n-1})}\mu(dt) \\ &= n! \langle f_{n-1}, g\rangle_{L^2(T^n)} \\ &= n! \langle \tilde{f}_{n-1}, g\rangle_{L^2(T^n)} \\ &= E\left[I_n(\tilde{f}_{n-1}) I_n(g)\right] \\ &= E\left[I_n(\tilde{f}_{n-1}) G\right]. \end{aligned}$$

Assume first now that $u \in Dom\delta$. Then the previous calculation, together with (2.32), implies that $I_n(\tilde{f}_{n-1})$ is the projection of $\delta(u)$ onto the Wiener chaos of order n. Therefore $\delta(u)$ is indeed given by the sum in (2.35), which is converges in $L^2(\Omega)$.

Now assume to the contrary that the series in (2.35) holds, and denote its sum by V. Then from the previous calculation we obtain:

$$E\left[\int_T u_t D_t \left(\sum_{n=0}^{N} I_n(g_n)\right)\mu(dt)\right] = E\left[V \sum_{n=0}^{N} I_n(g_n)\right] \leq ||V||_2 || \sum_{n=0}^{N} I_n(g_n)||_2$$

for all $N \geq 0$, where the last inequality is an application of Cauchy-Schwarz. Therefore for any $F \in \mathbb{D}^{1,2}$ which admits a finite Wiener chaos expansion, we have:

$$E\left[\int_T u_t D_t F \mu(dt)\right] \leq ||V||_2 ||F||_2$$

This relation can be extended to any $F \in \mathbb{D}^{1,2}$, and thus referring to (2.31), we see that $u \in Dom\delta$.

□

Remark 2.32. Since $I_m(f_m) = I_m(\tilde{f}_m)$, we could have written (2.35) in terms of $I_m(f_m)$. However, it will be useful to express the norm of $\delta(u)$ in terms of the symmetrized kernels. Also note that since each kernel $f_m(t_1, ..., t_m, t)$ is already symmetric in its first m arguments, we have:

$$\tilde{f}_m(t_1, ..., t_m, t) = \frac{1}{m+1}\left(f_m(t_1, ..., t_m, t) + \sum_{i=1}^{m} f_m(t_1, ..., t_{i-1}, t, t_{i+1}, ..., t_i)\right). \tag{2.36}$$

Remark 2.33. As an immediate consequence of Proposition 2.31, we can characterize $Dom\delta$ as those processes $u \in L^2(T \times \Omega)$ which satisfy:

$$\sum_{m=0}^{\infty}(m+1)!||\tilde{f}_m||^2_{L^2(T^{m+1})} < \infty. \tag{2.37}$$

2.3.2 Properties of the Skorohod Integral

First note that it is immediate from the definition of the Skorohod integral that it is linear, and also that for every $u \in Dom\delta$, we have $E[\delta(u)] = 0$.

Denote by $\mathcal{S}_H$ the set of smooth, elementary processes - i.e., processes of the form:

$$u(t) = \sum_{j=1}^{n} F_j h_j(t), \tag{2.38}$$

where the F_j are smooth random variables and each $h_j \in L^2(T)$. Note that such u belong to $Dom\delta$, and by Lemma 2.15, for every $G \in \mathbb{D}^{1,2}$:

$$\begin{aligned} E\left[G\left(\sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n}\langle DF, h_j\rangle_H\right)\right] &= \sum_{j=1}^{n} E[F_j\langle DG, h_j\rangle_H] \\ &= E\left[\langle DG, \sum_{j=1}^{n} F_j h_j\rangle\right] \\ &= E[\langle DG, u\rangle], \end{aligned} \tag{2.39}$$

which implies that $\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n}\langle DF, h_j\rangle_H$. Notice that a consequence of Proposition 2.26 is that if F is measurable $\mathcal{F}_A$, where $A \in \mathcal{B}$, then the process $D_t F$ is zero almost everywhere in $\Omega \times A^c$. Therefore if each h_j is of the form 1_{A_j} for some $A_j \in \mathcal{B}$, and each F_j is measurable $\mathcal{F}_{A^c}$, then we have that $\delta(u) = \sum_{j=1}^{n} F_j W(h_j)$.

Definition 2.34. The subset of $L^2(T \times \Omega)$ given by processes u satisfying that $u(t) \in \mathbb{D}^{1,2}$ for almost all t, and admitting a measurable version of the two parameter process $D_s u_t$ satisfying $E\left[\int_T \int_T (D_s u_t)^2 \mu(ds)\mu(dt)\right] < \infty$, is denoted by $\mathbb{L}^{1,2}$. If the process u has the expansion given in (2.33), then $u \in \mathbb{L}^{1,2}$ if and only if the series

$$\int_T \int_T E\left[\left|\sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, s, t))\right|^2\right]\mu(ds)\mu(dt) = \sum_{m=1}^{\infty} m m! ||f_m||^2_{L^2(T^{m+1})}$$

is convergent. $\mathbb{L}^{1,2}$ is a Hilbert space with the norm

$$||u||^2_{1,2} = ||u||^2_{L^2(T\times\Omega)} + ||Du||^2_{L^2(T^2\times\Omega)}.$$

Remark 2.35. Recalling the inequality

$$||\tilde{f}_m||_{L^2(T^{m+1})} \leq ||f_m||_{L^2(T^{m+1})}$$

and (2.37), it follows that $\mathbb{L}^{1,2} \subset Dom\delta$.

The following proposition characterizes the commutativity relationship between the Malliavin derivative and the Skorohod integral.

Proposition 2.36. (from [9])
Assume that $u \in \mathbb{L}^{1,2}$ and that for almost every $t \in T$ the process $\{D_t u_s, s \in T\}$ is Skorohod integrable. Assume also that there is a version of the process $\{\int_T D_t u_s dW_s, t \in T\}$ which is in $L^2(T \times \Omega)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and we have

$$D_t(\delta(u)) = u_t + \int_T (D_t u_s) dW_s. \tag{2.40}$$

Proof. Let $u_t = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t))$ be the Wiener chaos expansion of u, as in Lemma 2.33. Then we have

$$\begin{aligned} D_t(\delta(u)) &= D_t\left(\sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m)\right) \\ &= \sum_{m=0}^{\infty} (m+1) I_m(\tilde{f}_m(\cdot, t)) \\ &= \sum_{m=0}^{\infty} I_m(f_m(\cdot, t)) + \sum_{m=0}^{\infty} I_m\left(\sum_{i=1}^{m} f_m(t_1, ..., \hat{t}_i, ..., t_m, t, t_i)\right) \\ &= u_t + \sum_{m=0}^{\infty} m I_m(\text{symm} f_m(\cdot, t, \cdot)), \end{aligned} \tag{2.41}$$

where $\text{symm} f_m(\cdot, t, \cdot)$ denotes the symmetrization of the function

$$(t_1, ..., t_m) \to f_m(t_1, ..., t_{m-1}, t, t_m).$$

Note that we used (2.36) to establish the third equality in (2.41). But we also have that:

$$\int_T (D_t u_s) dW_s = \int_T \left(\sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t, s))\right) dW_s,$$

which is the same as $\sum_{m=0}^{\infty} m I_m(\text{symm} f_m(\cdot, t, \cdot))$ due to (2.31).

□

The following proposition characterizes the expectation of the product of two Skorohod integrals.

Proposition 2.37. (from [9])
Assume that $u, v \in \mathbb{L}^{1,2}$. Then we have

$$E[\delta(u)\delta(v)] = \int_T E[u_t v_t]\mu(dt) + \int_T \int_T E[D_s u_t D_t v_s]\mu(ds)\mu(dt). \tag{2.42}$$

Proof. We assume without loss of generality that u has a Wiener chaos expansion, in the sense of Lemma 2.30, and thus that $\delta(u) \in \mathbb{D}^{1,2}$. No generality is lost because by a density argument, a general process u can be approximated by such processes with finite Wiener chaos expansions. Therefore, we obtain

$$\begin{aligned}
E[\delta(u)\delta(v)] &= E\left[\int_T v_t D_t(\delta(u))\mu(dt)\right] \text{ by (2.32)} \\
&= E\left[\int_T v_t \left(u_t + \int_T (D_t u_s) dW_s\right)\mu(dt)\right] \text{ by Proposition 2.36} \\
&= E\left[\int_T u_t v_t \mu(dt)\right] + \int_T E\left[v_t \left(\int_T (D_t u_s) dW_s\right)\right]\mu(dt) \\
&= E\left[\int_T u_t v_t \mu(dt)\right] + \int_T E\left[\int_T D_t u_s D_s v_t \mu(ds)\right]\mu(dt) \text{ by (2.32)}
\end{aligned}$$

which gives the desired result, by a last application of Fubini's theorem. □

Remark 2.38. Note that we require that u and v belong to $\mathbb{L}^{1,2}$ in order to give the expression in (2.42) meaning.

The following proposition characterizes the Skorohod integral of a Skorohod integrable process multiplied by a random variable.

Proposition 2.39. (from [9])
Suppose that $u \in Dom\delta$ and $F \in \mathbb{D}^{1,2}$. Define

$$A = F \int_T u_t dW_t - \int_T (D_t F) u_t \mu(dt). \tag{2.43}$$

Then if $A \in L^2(\Omega)$, we have that $Fu \in Dom\delta$ and $\delta(Fu) = A$.

Proof. Suppose that $G = g(W(h_1), ..., W(h_n))$, where g is a continuously differentiable function with bounded partial derivatives. Then we have

$$\begin{aligned}
E\left[\int_T (D_t G) F u_t \mu(dt)\right] &= \int_T E[u_t(D_t(FG) - G D_t F)]\mu(dt) \text{ by Proposition 2.18} \\
&= E\left[G(F\delta(u) - \int_T u_t D_t F \mu(dt))\right] \text{ by (2.32)}
\end{aligned}$$

and the result follows. □

The following proposition characterizes which processes u in $L^2(T \times \Omega)$ can be written as DF for some $F \in \mathbb{D}^{1,2}$.

Proposition 2.40. (from [9])
Suppose that $u \in L^2(T \times \Omega)$, with Wiener chaos expansion as in (2.33). Then there exists a random variable $F \in \mathbb{D}^{1,2}$ such that $u = DF$ if and only if the kernels f_m (as in (2.33)) are symmetric in all $(m+1)$-arguments.

Proof. If $F \in \mathbb{D}^{1,2}$ then it admits a Wiener chaos expansion of the form (2.9), where the f_m are symmetric in all arguments, and Proposition 2.22 shows that DF has the desired form. Conversely, given $u \in L^2(T \times \Omega)$ with chaos expansion as in (2.33) with symmetric kernels f_m, define

$$F = \sum_{m=0}^{\infty} \frac{1}{m+1} I_{m+1}(f_m).$$

Then this series converges in $\mathbb{D}^{1,2}$ and $DF = u$. □

Proposition 2.41. (from [9])
Let $u \in L^2(T \times \Omega)$. Then there exists $F \in \mathbb{D}^{1,2}$ and a process $u^0 \in L^2(T \times \Omega)$ such that $E[\langle DG, u^0 \rangle] = 0$ for every $G \in \mathbb{D}^{1,2}$, $u^0 \in Dom\delta$, $\delta(u^0) = 0$, and

$$u = DF + u^0.$$

Proof. By Proposition 2.40, the subspace of $L^2(T \times \Omega)$ given by $\{DF, F \in \mathbb{D}^{1,2}\}$ is closed. Therefore any $u \in L^2(T \times \Omega)$ can be orthogonally decomposed, by the projection theorem for Hilbert spaces, into a sum of the desired form. The Skorohod integrability of u^0 and the form of its Skorohod integral (as stipulated in the statement of the proposition) follow directly from Definition 2.27. □

2.3.3 The Skorohod Integral as an Extension of the Itô Integral

The following lemma shows that $\mathbb{L}^{1,2}$ is a strict subset of $Dom\delta$ - i.e., there exist processes outside $\mathbb{L}^{1,2}$ which are Skorohod integrable.

Lemma 2.42. (from [9])
Let $A \in \mathcal{B}_0$ and let $F \in L^2(\Omega)$ such that F is measurable with respect to the σ-field $\mathcal{F}_{A^c}$. Then the process $F\mathbf{1}_A \in Dom\delta$ and

$$\delta(F\mathbf{1}_A) = FW(A).$$

Proof. Note first that by the remark following (2.39) we have that the Skorohod integral of $\mathbf{1}_A(t)$ is give by $W(A)$. If we assume first that $F \in \mathbb{D}^{1,2}$, then we can apply Proposition 2.39 to get

$$\delta(F\mathbf{1}_A) = FW(A) - \int_T D_t F \mathbf{1}_A(t) \mu(dt) = FW(A),$$

where the second equality follows from Proposition 2.26 (using the assumption that $F \in \mathcal{F}_{A^c}$). The Proposition now follows immediately by noting that an arbitrary $F \in L^2(\Omega)$ such that

F is measurable $\mathcal{F}_{A^c}$ can be approximated by a sequence $\{F_n\}_{n=1}^{\infty} \subset \mathbb{D}^{1,2}$, each measurable $\mathcal{F}_{A^c}$. Then the result follows by the fact that the Skorohod integal is a closed operator.

□

We now remark that Brownian motion can be realized as an isonormal Gaussian process.

Remark 2.43. Suppose that $T = [0,1] \times \{1, ..., d\}$ and the atomless measure μ on T is given by the product of the Lebesgue measure and the uniform measure assigning a mass of one to each element of $\{1, ..., d\}$. As usual, $H = L^2(T)$. By part 2 of Remark 2.2, there exists a corresponding probability space $(\Omega, \mathcal{F}, P)$ and an isonormal Gaussian process $\{W(h), h \in H\}$ defined on this space. Consider the family defined by

$$W^i(t) = W(\mathbf{1}_{[0,t]\times\{i\}}),\ 0 \leq t \leq 1, 1 \leq i \leq d.$$

Then $W = \{(W^1, ..., W^d\}$ is a d-dimensional Brownian motion - in particular, it is an independent, zero-mean Gaussian process with covariance function $E[W^i(s)W^i(t)] = s \wedge t$.

Within the context the previous remark, define now L_a^2 as the closed subspace of $L^2(T \times \Omega)$ constituted by the processes adapted to the filtration $\{\mathcal{F}_t\}$ where $\mathcal{F}_t = \mathcal{F}_{[0,t]}$. Then we have the following proposition, characterizing the Itô integral as a special case of the Skorohod integral.

Proposition 2.44. (from [9])
L_a^2 is contained in Domδ, and by restricting the Skorohod integral to L_a^2, we recover the Itô integral, in the sense that

$$\delta(u) = \sum_{i=1}^{d} \int_0^1 u_t^i dW_t^i.$$

Proof. We first show the result holds for u which are elementary, adapted processes - i.e., of the form

$$u_t = \sum_{j=1}^{n} F_j \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where $F_j \in L^2(\Omega, \mathcal{F}_{t_j}, P; \mathbb{R}^d)$, and $0 \leq t_1 < \cdots < t_{n+1} \leq 1$. Then we have that $\mu(\mathbf{1}_{(t_j,t_{j+1}]}) < \infty$ and F is measurable with respect to $\mathcal{F}_{(t_j,t_{j+1}]^c}$, and so we apply Lemma 2.42 to obtain that $u \in Dom\delta$ and

$$\delta(u) = \sum_{i=1}^{d} \sum_{j=1}^{n} F_j^i (W^i(t_{j+1}) - W^i(t_j)). \tag{2.44}$$

Note that any element of L_a^2 can be approximated in $L^2(T \times \Omega)$ by a sequence $\{u^n\}$ of elementary adapted processes. Moreover, (2.44) implies that $\delta(u^n)$ is given by the Itô integral of u^n. Therefore, since the Skorohod integral is a closed operator, we have that $L_a^2 \subset Dom\delta$ and $\delta(u)$ is given by the Itô integral of u for every $u \in L_a^2$. □

2.3.4 Stochastic Integral Representation of Wiener Functionals

Let $W = \{W_t, t \in [0,1]\}$ be a one-dimensional Brownian motion. Recall from Theorem 1.6 that if $F \in L^2(\Omega)$ is measurable with respect to $\sigma(W)$, then there exists an adapted, square-integrable process $\phi \in L^2_a$ such that

$$F = E[F] + \int_0^1 \phi(t) dW_t.$$

In the special case that $F \in \mathbb{D}^{1,2}$, we can precisely characterize the kernel ϕ in terms of the Malliavin derivative and conditional probability, in the sense of the following proposition, known as the Clark-Ocone representation formula.

Proposition 2.45. (from [9])
Let $F \in \mathbb{D}^{1,2}$, and let W be a one-dimensional Brownian motion. Then:

$$F = E[F] + \int_0^1 E[D_t F | \mathcal{F}_t] dW_t. \tag{2.45}$$

Proof. Suppose $F = \sum_{m=0}^{\infty} I_m(f_m)$. Hence we have

$$\begin{aligned} E[D_t F|\mathcal{F}_t] &= \sum_{m=1}^{\infty} m E[I_{m-1}(f_m(\cdot, t))|\mathcal{F}_t] \\ &= \sum_{m=1}^{\infty} m I_{m-1}(f_m(t_1, ..., t_{m-1}, t)\mathbf{1}_{[t_1 \vee ... \vee t_{m-1} \leq t]}), \end{aligned}$$

where the first equality follows from Proposition 2.22 and the second equality follows from Lemma 2.25. Let now

$$g_m(t_1, ..., t_{m-1}, t) = m f_m(t_1, ..., t_{m-1}, t)\mathbf{1}_{[t_1 \vee ... \vee t_{m-1} \leq t]}.$$

Then it follows from (2.36) that $\tilde{g}_m$, the symmetrization of g, is given by

$$\tilde{g}_m(t_1, ..., t_{m-1}, t) = f_m(t_1, ..., t_{m-1}, t).$$

Thus, using Proposition 2.31, we compute

$$\delta(E[D_t F|\mathcal{F}_t]) = \sum_{m=1}^{\infty} I_m(f_m) = F - E[F].$$

This completes the proof since the adaptedness of the process $\{E[D_t F|\mathcal{F}_t]\}$ gives that the Skorohod integral coincides with the Itô integral, by Proposition 2.44.

□

2.4 The Ornstein-Uhlenbeck Semigroup

For a random variable $F \in L^2(\Omega)$, we will denote by $J_n(F)$ the orthogonal projection of F onto the n-th Wiener chaos - i.e., if $F = \sum_{m=0}^{\infty} I_m(f_m)$, then $J_n(F) = I_n(f_n)$.

Definition 2.46. The *Ornstein-Uhlenbeck semigroup* is the parametrized family of contraction operators $\{T_t, t \geq 0\}$ on $L^2(\Omega)$ defined by the relation

$$T_t(F) = \sum_{n=0}^{\infty} e^{-nt} J_n(F), \tag{2.46}$$

for any $F \in L^2(\Omega)$.

Remark 2.47. Given an isonormal Gaussian process W defined on the probability space $(\Omega, \mathcal{F}, P)$, we will refer in the sequel to an *independent copy* of W, denoted as W', where both W and W' can be considered as defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. By this we mean, that given W, we define two isonormal processes X and X' on $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$ by

$$(X(h))(\omega, \omega') = (W(h))(\omega), \text{ for every } (\omega, \omega') \in \Omega \times \Omega'$$
$$(X'(h))(\omega, \omega') = (W(h))(\omega') \text{ for every } (\omega, \omega') \in \Omega \times \Omega'$$

where $\Omega = \Omega'$, $P = P'$, $\mathcal{F} = \sigma\{X(h), h \in H\}$, $\mathcal{F}' = \{X'(h), h \in H\}$. With some abuse of notation, we redefine X as W and X' as W'.

We now describe an alternate procedure for defining the Ornstein-Uhlenbeck semigroup, and then show it is equivalent to the above definition. Let then $W' = \{W'(h), h \in H\}$ be an independent copy of $\{W(h), h \in H\}$. We define a new isonormal Gaussian process $Z = \{Z(h), h \in H\}$, for any fixed $t > 0$, by

$$Z(h) = e^{-t} W(h) + \sqrt{1 - e^{-2t}} W'(h), \quad h \in H.$$

It is immediate and Z is zero-mean, Gaussian, and that

$$E[Z(h_1)Z(h_2)] = e^{-2t}\langle h_1, h_2 \rangle_H + (1 - e^{-2t})\langle h_1, h_2 \rangle_H = \langle h_1, h_2 \rangle_H.$$

With another abuse of notation, define W also as the canonical mapping $W : \Omega \to \mathbb{R}^H$, where $(W(\omega))(h) = (W(h))(\omega)$. Define W' in the analogous way as the canonical mapping $W' : \Omega' \to \mathbb{R}^H$. Note that if $F \in L^2(\Omega, \mathcal{F}, P)$ then there exists a measurable, P-a.e. defined mapping $\psi_F : \mathbb{R}^H \to \mathbb{R}$ such that $F = \psi_F \circ W$ ([8]). Therefore, the random variable $\psi_F(Z(\omega, \omega')) = \psi_F(e^{-t}W(\omega) + \sqrt{1 - e^{-2t}}W'(\omega'))$ is well defined almost surely $P \times P'$. Proposition 2.49 below establishes 'Mehler's formula'. It's proof however, requires the following lemma.

Lemma 2.48. (from [9])
The family $\{e^{W(h)} : h \in H\}$ *is complete in* $L^2(\Omega, \mathcal{F}, P)$.

Proposition 2.49. (from [9])
In the context of the notation just established, we have for any $F \in L^2(\Omega)$, $t > 0$*:*

$$T_t(F) = E'[\psi_F(e^{-t}W + \sqrt{1 - e^{-2t}}W')], \tag{2.47}$$

where E' *denotes the expectation with respect to* P'.

Proof. We first note that the mapping $F \mapsto E'[\psi_F(e^{-t}W + \sqrt{1 - e^{-2t}}W')]$ is a contraction on $L^p(\Omega, \mathcal{F}, P)$ for every $p \geq 1$ since

$$\begin{aligned} E[|E'[\psi_F(e^{-t}W + \sqrt{1 - e^{-2t}}W')]|^p] &\leq E[E'[|\psi_F(e^{-t}W + \sqrt{1 - e^{-2t}}W')|^p]] \\ &= E[|F|^p]. \end{aligned}$$

We must now show that (2.46) and (2.47) coincide for every $F \in L^2(\Omega, \mathcal{F}, P)$. By Lemma 2.48, it is sufficient to show this for $\{F = \exp\left(W(h) - \frac{1}{2}||h||_H^2\right), h \in H\}$. We have

$$\begin{aligned} &E'\left[\exp\left(e^{-t}W(h) + \sqrt{1 - e^{-2t}}W'(h) - \frac{1}{2}||h||_H^2\right)\right] \\ &= \exp\left(e^{-t}W(h) - \frac{1}{2}e^{-2t}||h||_H^2\right) = \sum_{n=0}^{\infty} e^{-nt}||h||_H^n H_n\left(\frac{W(h)}{||h||_H}\right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} I_n(h^{\otimes n}). \end{aligned}$$

where the first equality is a direct calculation (recalling that $W(h) \sim \mathcal{N}(0, ||h||_H^2)$, the second equality follows from Lemma 2.6, and the third equality follows from Theorem 2.11. On the other hand,

$$\begin{aligned} T_t(F) &= T_t\left(\sum_{n=0}^{\infty} \frac{1}{n!} I_n(h^{\otimes n})\right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} I_n(h^{\otimes n}), \end{aligned}$$

where the first equality follows from Lemma 2.6, and second equality follows from (2.46). This completes the proof. □

Remark 2.50. We can characterize Mehler's formula in a slightly different, yet equivalent manner, as follows. Given a random variable $Z \in \mathbb{D}^{1,2}$, we can write $DZ = \Phi_Z(X)$, where Φ_Z is a measurable mapping from $\mathbb{R}^H \to H$, determined $P \circ W^{-1}$ almost surely. Then for any $u \geq 0$, we have

$$T_u(DZ) = E'[\Phi_Z(e^{-u}W + \sqrt{1 - e^{-2u}}W')], \tag{2.48}$$

Remark 2.51. The Ornstein-Uhlenbeck operators $\{T_t, t \geq 0\}$ satisfy for all t:

1. T_t is non-negative - i.e., $F \geq 0$ implies $T_t(F) \geq 0$.
2. T_t is symmetric - i.e., for all $F, G \in L^2(\Omega, \mathcal{F}, P)$:

$$E[GT_t(F)] = E[FT_t(G)] = \sum_{n=0}^{\infty} e^{-nt} E[J_n(F)J_n(G)].$$

We will now define and characterize the infinitesimal generator of the Ornstein-Uhlenbeck semi-group.

Definition 2.52. We define the operator $L : DomL \subset L^2(\Omega) \to L^2(\Omega)$ by the relation

$$LF = \sum_{n=0}^{\infty} -nJ_n(F), \tag{2.49}$$

for every $F \in DomL$, where

$$DomL = \{F \in L^2(\Omega), F = \sum_{n=0}^{\infty} I_n(f_n) : \sum_{n=1}^{\infty} n^2 n! ||f_n||^2_{L^2(T^n)} < \infty\}. \tag{2.50}$$

Note that $DomL \subset \mathbb{D}^{1,2}$, L is a symmetric operator (in the sense that $E[FLG] = E[GLF]$ for every $F, G \in DomL$), and L is an unbounded operator.

Definition 2.53. A *strongly continuous semi-group* is a family of operators $\{R_t, t \geq 0\}$, operating on some Banach space X, satisfying

- T_0 is the identity mapping on X
- $T_{t+s} = T_t T_s$ for every $t, s \geq 0$
- $||T_t x - x|| \to 0$ as $t \to 0$, for every $x \in X$

Note that the Ornstein-Uhlenbeck operators are a strongly continuous semi-group.

Definition 2.54. The *infinitesimal generator* of a strongly continuous semi-group $\{R_t, t \geq 0\}$ defined on a Banach space X is an operator on X, call it A, defined by

$$Ax = \lim_{t \to 0} \frac{1}{t}(R_t - R_0)(x) \tag{2.51}$$

where $DomA$ is the set of all points in X where this limit exists. It can be shown that $DomA$ is dense in X.

The next proposition says that the operator L is the infinitesimal generator of the Ornstein-Uhlenbeck semi-group.

Proposition 2.55. (from [9])
The infinitesimal generator of the Ornstein-Uhlenbeck semi-group $\{T_t, t \geq 0\}$ is given by the operator L.

Proof. Assuming that $F \in DomL$, we have that

$$\left|\left|\frac{1}{t}(T_tF - F) - LF\right|\right|^2_{L^2(\Omega)} = \left|\left|\sum_{n=0}^{\infty}\left(\frac{1}{t}(e^{-nt}-1)+n\right)J_nF\right|\right|^2_{L^2(\Omega)}$$
$$= \sum_{n=0}^{\infty}\left[\frac{1}{t}(e^{-nt}-1)+n\right]^2 E[|J_nF|^2] \to_{t\to\infty} 0.$$

Suppose now to the converse that $\lim_{t\to 0}\frac{1}{t}(T_tF - F) = G$ in $L^2(\Omega)$. Then we have that

$$J_nG = \lim t \to 0 \frac{1}{t}(T_tJ_nF - J_nF) = -nJ_nF.$$

Therefore $F \in DomL$ and $LF = G$. □

The next proposition characterizes a relation between the Malliavin derivative, the Skorohod integral, and the infinitesimal generator of the Ornstein-Uhlenbeck semi-group.

Proposition 2.56. (from [9])
For every $F \in L^2(\Omega)$, we have that $F \in DomL$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in Dom\delta$. Also, for every $F \in DomL$, we have

$$\delta DF = -LF. \tag{2.52}$$

Proof. Assume first that $F(=\sum_{n=0}^{\infty} I_n(f_n)) \in \mathbb{D}^{1,2}$ and $DF \in Dom\delta$. Let $G = I_n(g)$ be an element of the n-th Wiener chaos. Then we have

$$\begin{aligned} E[G\delta DF] &= E[\langle DG, DF\rangle_H] \text{ (by the duality property of } \delta) \\ &= n^2(n-1)!\langle g, f_n\rangle_{L^2(T^n)} \\ &= nE[GJ_nF] \end{aligned}$$

Therefore $J_n\delta DF = nJ_nF$, and hence $F \in DomL$ and $\delta DF = -LF$. Conversely, assume that $F \in DomL$; then since $DomL \subset \mathbb{D}^{1,2}$, we have that $F \in \mathbb{D}^{1,2}$. Let $G \in \mathbb{D}^{1,2}$ be arbitrary, with Wiener chaos expansion $G = \sum_{n=0}^{\infty} I_n(g_n)$. Then we have

$$\begin{aligned} E[\langle DG, DF\rangle_H] &= \sum_{n=1}^{\infty} n^2(n-1)!\langle g_n, f_n\rangle_{L^2(T^n)} \\ &= \sum_{n=1}^{\infty} nE[J_nGJ_nF] = -E[GLF]. \end{aligned}$$

Hence $DF \in Dom\delta$ and $\delta DF = -LF$. □

The following two propositions characterize the behavior of L as that of a second-order differential operator.

Proposition 2.57. (from [9])
The space $\mathcal{S}$ of smooth random variables is contained in $DomL$, and the action of L on $F \in \mathcal{S}$ of the form $F = f(W(h_1), ..., W(h_n))$ (with $f \in C_p^\infty(\mathbb{R}^n)$) is given by

$$LF = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(W(h_1), ..., W(h_n))\langle h_i, h_j \rangle_H - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))W(h_i).$$

Proof. We have (by definition) that $F \in \mathbb{D}^{1,2}$ and that

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t).$$

Hence $DF \in \mathcal{S}_H \subset Dom\delta$ (c.f. (2.38)) and the line before it) and thus (c.f. the line following (2.39)),

$$\delta DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))W(h_i) - \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(W(h_1), ..., W(h_n))\langle h_i, h_j \rangle_H.$$

Now applying Proposition 2.56, the proof is complete. □

We state the following more general result without proof.

Proposition 2.58. (from [9])
Let $F = (F_1, ..., F_m)$ be a vector of elements in $\mathbb{D}^{2,4}$, and let $\phi : \mathbb{R}^m \to \mathbb{R}$ be twice continuously differentiable with bounded partial derivatives up the second order. Then $\phi(F) \in DomL$ and

$$L(\phi(F)) = \sum_{i,j=1}^{m} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(F)\langle DF^i, DF^j \rangle_H + \sum_{i=1}^{m} \frac{\partial \phi}{\partial x_i}(F)LF^i.$$

Chapter 3

Density Estimates and Concentration Inequalities

In this chapter we will present the results obtained in [7], and demonstrate some extensions and generalizations of these results, as well as computing some new concentration inequalities.

One fact which is implicitly used often throughout this chapter, is that for $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$, the support of ρ is a potentially unbounded interval containing zero - i.e., $supp(\rho) = [\alpha, \beta]$, where $-\infty \leq \alpha < 0 < \beta \leq +\infty$ ([9]).

3.1 Density Estimates

The next result gives an expression for the density of a Malliavin differentiable random variable in terms of a function f. It's utility lies in the fact that f, defined below, is readily estimated in a range of practical examples. Using this result, it is possible to obtain global bounds on the density functions of certain random variables.

Definition 3.1. Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Consider the random variable $E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Since this random variable is measurable with respect to $\sigma(Z)$, there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ so that:

$$f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]. \tag{3.1}$$

Notationally, we write

$$f(z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z = z]. \tag{3.2}$$

Remark 3.2. By [6], we have that $f(z) \geq 0$ on the support of Z.

Theorem 3.3. (from [7])
Let $Z \in \mathbb{D}^{1,2}$ with zero mean, and let $f(z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z = z]$ (as above). Then Z has a density ρ if and only if $f(Z) > 0$ almost surely. In this case, the density ρ is supported on an interval $[\alpha, \beta]$, and for almost every $z \in supp(\rho)$, we have

$$\rho(z) = \frac{E[|Z|]}{2f(z)} \exp\left(-\int_0^z \frac{x}{f(x)} dx\right). \tag{3.3}$$

The following is an interesting criterion for normality of Z, in terms of the function f.

Corollary 3.4. (from [7])
Let $Z \in \mathbb{D}^{1,2}$, and assume Z admits a density ρ; let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Then Z is a Gaussian if and only if $Var(f(Z)) = 0$.

Proof. We can assume without loss of generality that $E[Z] = 0$. Note that we can compute $Var(Z)$ using the following integration by parts formula, picking $e(z) = z$, although in general $e : \mathbb{R} \to \mathbb{R}$ can be any C^1 function with bounded derivative:

$$\begin{aligned} E[Ze(Z)] &= E[(LL^{-1}Z)e(Z)] \text{ (where } LL^{-1}Z = Z \text{ since } E[Z] = 0) \\ &= E[(\delta D(-L^{-1}e(Z)))Z] \text{ (from Proposition 2.56)} \\ &= E[\langle De(Z), -DL^{-1}Z\rangle_H] \text{ (from (2.32))} \\ &= E[e'(Z)\langle DZ, -DL^{-1}Z\rangle_H] \text{ (by Proposition 2.18)} \end{aligned} \tag{3.4}$$

Therefore, the condition $\mathrm{Var}(f(Z)) = 0$ is equivalent to

$$f(Z) = \mathrm{Var}(Z) \text{ almost surely.}$$

Now note that for $g : \mathbb{R} \to \mathbb{R}$ a continuous function with compact support, and let G denote any antiderivative of g. Note that G is bounded. We have

$$\begin{aligned} E[g(Z)\langle DZ, -DL^{-1}Z\rangle_H] &= E[G(Z)Z] \text{ by (3.4)} \\ &= \int_{\mathbb{R}} G(z)z\rho(z)dz \\ &= \int_{\mathbb{R}} g(z)\left(\int_z^\infty y\rho(y)dy\right)dz \text{ (obtained via integration by parts)} \\ &= E\left[g(Z)\frac{\int_Z^\infty y\rho(y)dy}{\rho(Z)}\right]. \end{aligned}$$

Therefore we have shown

$$f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z] = \frac{\int_Z^\infty y\rho(y)dy}{\rho(Z)} \text{ almost surely.} \tag{3.5}$$

Let $Z \sim \mathcal{N}(0, \sigma^2)$. Using (3.5), we immediately check that $f(Z) = \sigma^2$ almost surely. Conversely, if $f(Z) = \sigma^2$ almost surely, then Theorem 3.3 implies that Z has a density ρ given by $\rho(z) = \frac{E[|Z|]}{2\sigma^2}\exp\left(-\frac{-z^2}{2\sigma^2}\right)$ for almost all $z \in \mathbb{R}$, from which we immediately deduce that $Z \sim \mathcal{N}(0, \sigma^2)$. □

If f can be uniformly upper and lower bounded, then we have the corresponding upper and lower bounds on the density function of Z expressed below.

Corollary 3.5. (from [7])
Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$ and $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. If there exists $\sigma_{min}, \sigma_{max} > 0$ such that

$$\sigma_{min}^2 \leq f(Z) \leq \sigma_{max}^2 \text{ almost surely,}$$

then Z has a density ρ satisfying, for almost all $z \in \mathbb{R}$

$$\frac{E[|Z|]}{2\sigma_{max}^2} \exp\left(-\frac{z^2}{2\sigma_{min}^2}\right) \leq \rho(z) \leq \frac{E[|Z|]}{2\sigma_{min}^2} \exp\left(-\frac{z^2}{2\sigma_{max}^2}\right)$$

Proof. This follows immediately from Theorem 3.3. □

The following proposition provides a computationally tractable method for computing $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$.

Proposition 3.6. (from [7])
Write $DZ = \Phi_Z(X)$ with a measurable function $\Phi_Z : \mathbb{R}^H \to \mathbb{R}$. We have

$$f(Z) = \int_0^\infty e^{-u} \mathbb{E}[\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_H | Z] du, \tag{3.6}$$

where X' stands for an independent copy of X, and is such that X and X' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$ - (see Remark 2.47). Here $\mathbb{E}$ denotes the mathematical expectation with respect to $P \times P'$.

We have stated Theorem 3.3 and Proposition 3.6 without proof because we will now prove more general versions of these results. Our proofs follow closely those from [7], with some modifications.

Theorem 3.7. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by the relation (cf. Definition 3.1)*

$$g(Z) = E[\langle DZ, -DL^{-1}(h(Z))\rangle_H | Z] \tag{3.7}$$

for some Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ satisfying that $E[h(Z)] = 0$, $h(z) > 0$ for $z > 0$, and $h(z) < 0$ for $z < 0$. We have that Z admits a density ρ if and only if $\langle DZ, -DL^{-1}Z\rangle > 0$. In this case, ρ is supported on an interval $[\alpha, \beta]$, and for almost every $z \in supp(\rho)$:

$$\rho(z) = \frac{E[|h(Z)|]}{2g(z)} \exp\left(-\int_0^z \frac{h(x)}{g(x)} dx\right). \tag{3.8}$$

Proof. We split the proof into several steps.

Step 1: An integration by parts formula. For any $f : \mathbb{R} \to \mathbb{R}$ of class C^1 with bounded derivative, we have

$$
\begin{aligned}
E[h(Z)f(Z)] &= E[LL^{-1}h(Z)f(Z)] \text{ (where } LL^{-1}h(Z) = Z \text{ since } E[h(Z)] = 0) \\
&= E[(-\delta DL^{-1}h(Z))f(Z)] \text{ (by Proposition 2.56)} \\
&= E[\langle Df(Z), -DL^{-1}h(Z)\rangle] \text{ (from (2.32))} \\
&= E[f'(Z)\langle DZ, -DL^{-1}h(Z)\rangle] \text{ (by Proposition 2.18).}
\end{aligned} \tag{3.9}
$$

Step 2: A key formula. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support, and let F denote any antiderivative of f. Note that F is bounded. Assuming that Z admits a density ρ, we have

$$
\begin{aligned}
E[f(Z)\langle DZ, -DL^{-1}(h(Z))\rangle_H] &= E[F(Z)h(Z)] \text{ by (3.9)} \\
&= \int_{\mathbb{R}} F(z)h(z)\rho(z)dz \\
&= \int_{\mathbb{R}} f(z)\left(\int_z^\infty h(y)\rho(y)dy\right)dz \text{ (obtained via integration by parts)} \\
&= E\left[f(Z)\frac{\int_Z^\infty h(y)\rho(y)dy}{\rho(Z)}\right].
\end{aligned}
$$

where the integration by parts follows from the observation that

$$
\int_z^\infty h(y)\rho(y)dy \to 0 \text{ as } |z| \to \infty,
$$

where this holds for $z \to +\infty$ since $h(Z) \in \mathbb{D}^{1,2} \subset L^2(\Omega) \subset L^1(\Omega)$ (see Proposition 2.20), and for $z \to -\infty$ since $E[h(Z)] = 0$. Therefore we have shown

$$
g(Z) = E[\langle DZ, -DL^{-1}(h(Z))\rangle_H | Z] = \frac{\int_Z^\infty h(y)\rho(y)dy}{\rho(Z)} \text{ almost surely.} \tag{3.10}
$$

Step 3: Existence of the density. Taking h to be the identity function in (3.10), we can conclude that if Z admits a density, then $\langle DZ, -DL^{-1}Z\rangle > 0$ almost surely. To prove the converse, assume now that $\langle DZ, -DL^{-1}Z\rangle > 0$ almost surely. Then applying the integration by parts formula (3.9) with h equal to the identity, together with an approximation argument, yields

$$
E[\mathbf{1}_B(Z)\langle DZ, -DL^{-1}Z\rangle] = E[Z\int_{-\infty}^Z \mathbf{1}_B(y)dy], \tag{3.11}
$$

where B is any Borel set in $\mathbb{R}$. This relation shows that if the Lebesgue measure of the set B is zero, then $E[\mathbf{1}_B(Z)\langle DZ, -DL^{-1}Z\rangle = 0$. Together with the assumption on the positivity of $\langle DZ, -DL^{-1}Z\rangle$, this relation yields that $P(Z \in B) = 0$, hence the measure induced by Z is absolutely continuous with respect to the Lebesgue measure, and we conclude via the

Radon-Nikodym theorem that Z admits a density.

Step 4: Proof of (3.8). Let $\phi : \mathbb{R} \to \mathbb{R}$ be still defined by $\phi(z) = \int_z^\infty h(y)\rho(y)dy$. On one hand, we have $\phi'(z) = -h(z)\rho(z)$ for almost all $z \in supp(\rho)$. On the other hand, by (3.10), we have, for almost all $z \in supp(\rho)$,

$$\phi(z) = \rho(z)g(z). \tag{3.12}$$

By putting these two facts together, we get the following ordinary differential equation satisfied by ϕ:

$$\frac{\phi'(z)}{\phi(z)} = -\frac{h(z)}{g(z)} \text{ for almost all } z \in supp(\rho). \tag{3.13}$$

Note that dividing by $\phi(z)$ is well defined since $\phi(z)$ does not vanish for $z \in supp(\rho)$ by definition; similarly, also note that $g(z)$ does not vanish for $z \in supp(\rho)$ by (3.10). Integrating this relation over the interval $[0, z]$ yields

$$\log(\phi(z)) = \log(\phi(0)) - \int_0^z \frac{h(x)}{g(x)}dx.$$

Taking the exponential and using the fact that $\phi(0) = \frac{1}{2}E[|h(Z)|]$, we get

$$\phi(z) = \frac{1}{2}E[|h(Z)|] \exp\left(-\int_0^z \frac{h(x)}{g(x)}dx\right).$$

Finally, the desired conclusion follows from (3.12).

□

Proposition 3.8. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz with $E[h(Z)] = 0$. Write $DZ = \Phi_Z(X)$ with a measurable function $\Phi_Z : \mathbb{R}^H \to \mathbb{R}$ and $D(h(Z)) = \Phi_{h(Z)}(X)$ with a measurable function $\Phi_{h(Z)} : \mathbb{R}^H \to \mathbb{R}$. We have*

$$g(Z) = \int_0^\infty e^{-u}\mathbb{E}[\langle \Phi_Z(X), \Phi_{h(Z)}(e^{-u}X + \sqrt{1 - e^{-2u}}X')\rangle_H | Z]du \tag{3.14}$$

where X' stands for an independent copy of X, and is such that X and X' are defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$. Here, $\mathbb{E}$ denotes the mathematical expectation with respect to $P \times P'$.

Proof. Without loss of generality, we can assume that $H = L^2(T, \mathcal{B}, \mu)$, where $(T, \mathcal{B})$ is a measurable space and μ is a σ-finite measure without atoms, since H is a separable Hilbert space. Let us consider the chaos expansion of $h(Z)$, given by $h(Z) = \sum_{m=1}^\infty I_m(f_m)$, with $f_m \in L^2_s(T^m)$. Therefore $-L^{-1}(h(Z)) = \sum_{m=1}^\infty \frac{1}{m} I_m(f_m)$ and

$$-D_t L^{-1}(h(Z)) = \sum_{m=1}^{\infty} I_{m-1}(f_m(\cdot,t)),\ t \in T.$$

On the other hand, we have $D_t(h(Z)) = \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot,t))$. Thus

$$\int_0^\infty e^{-u} T_u(D_t h(Z)) du = \int_0^\infty e^{-u} \left(\sum_{m=1}^{\infty} m e^{-(m-1)u} I_{m-1}(f_m(\cdot,t)) \right) du \ = \sum_{m=1}^{\infty} I_{m-1}(f_m(\cdot,t)).$$

Consequently,

$$-DL^{-1}h(Z) = \int_0^\infty e^{-u} T_u(D(h(Z)))du.$$

By Mehler's formula (see (2.48)), and since $D(h(Z)) = \Phi_{h(Z)}(X)$ by assumption, we deduce that

$$-DL^{-1}(h(Z)) = \int_0^\infty e^{-u} E'[\Phi_{h(Z)}(e^{-u}X + \sqrt{1-e^{-2u}}X')]du.$$

Using $E[E'[...]|Z] = \mathbb{E}[...|Z]$, the desired conclusion follows. □

Let's now compute an example to demonstrate these results. We follow [7].

Example 3.9. (from [7]). Let $N \sim \mathcal{N}_n(0,K)$ with $K \in \mathcal{S}_n^+(\mathbb{R})$. We assume that each N_i has the form $X(h_i)$, for a certain centered isonormal process X (over some Hilbert space H) and certain functions $h_i \in H$. Let $Z = \max N_i - E[\max N_i]$, and set

$$I_u = \text{argmax}_{1\le i\le n}(e^{-u}X(h_i) + \sqrt{1-e^{-2u}}X'(h_i)) \text{ for } u \ge 0.$$

Lemma 3.10. *For any $u \ge 0$, I_u is a well-defined random element of $\{1,...,n\}$. Moreover, $Z \in \mathbb{D}^{1,2}$ and we have $DZ = \Phi_Z(N) = h_{I_0}$.*

Proof. Fix $u \ge 0$. For any $i \ne j$, we have

$$P(e^{-u}X(h_i) + \sqrt{1-e^{-2u}}X'(h_i) = e^{-u}X(h_j) + \sqrt{1-e^{-2u}}X'(h_j))$$
$$= P(e^{-u}X(h_i) - e^{-u}X(h_j) = \sqrt{1-e^{-2u}}X'(h_j) - \sqrt{1-e^{-2u}}X'(h_i)) = 0$$

where the last equality follows from the independence of X and X'. Thus the random variable I_u is a well-defined element of $\{1,...,n\}$. Now if Δ_i denotes the set $\{x \in \mathbb{R}^n : x_j \le x_i \text{ for all j}\}$, observe that $\frac{\partial}{\partial x_i} max = \mathbf{1}_{\Delta_i}$ almost everywhere. The desired conclusion follows from the Lipschitz version of the chain rule (see Proposition 2.20), and the following Lipschitz property of the max function, which is easily proved by induction on $n \ge 1$:

$$|\max(y_1,...,y_n) - \max(x_1,...,x_n)| \le \sum_{i=1}^{n} |y_i - x_i| \text{ for any } x,y \in \mathbb{R}^n.$$

□

In particular, we deduce from Lemma 3.10 that

$$\langle \Phi_Z(X), \Phi_Z(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_H = K_{I_0,I_u}.$$

Thus by Theorem 3.3 and Proposition 3.6, if there exists $\sigma_{min}, \sigma_{max} > 0$ such that $\sigma_{min}^2 \leq K_{ij} \leq \sigma_{max}^2$ for any $i, j \in \{1, ..., n\}$, then Z has a density ρ satisfying

$$\frac{E[|Z|]}{2\sigma_{max}^2} \exp\left(-\frac{z^2}{2\sigma_{min}^2}\right) \leq \rho(z) \leq \frac{E[|Z|]}{2\sigma_{min}^2} \exp\left(-\frac{z^2}{2\sigma_{max}^2}\right). \tag{3.15}$$

for almost all $z \in \mathbb{R}$. This completes the example.

3.2 Concentration Inequalities – Upper Bounds

Theorem 3.11. (from [7])
Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$, $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$, and fix $\alpha \geq 0$ and $\beta > 0$. Assume that

1. *$f(Z) \leq \alpha Z + \beta$ almost surely;*
2. *Z has a density ρ.*

Then, for all $z > 0$, we have

$$P(Z \geq z) \leq \exp\left(-\frac{z^2}{2\alpha z + 2\beta}\right). \tag{3.16}$$

We now state and prove a more general version of Theorem 3.11, in terms of the function g.

Theorem 3.12. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Let $g(Z) = E[\langle DZ, -DL^{-1}(h(Z))|Z]$, where $h : \mathbb{R} \to \mathbb{R}$. Fix $\alpha \geq 0$ and $\beta > 0$. Assume that*

1. *$E[h(Z)] = 0$;*
2. *$h(z)$ is non-decreasing for $z \geq 0$;*
3. *h is Lipschitz;*
4. *$g(Z)h'(Z) \leq \alpha h(Z) + \beta$ almost surely;*
5. *Z has a density ρ.*

Then for all $z > 0$ we have

$$P(Z \geq z) \leq \exp\left(-\frac{h(z)^2}{2\alpha h(z) + 2\beta}\right). \tag{3.17}$$

Proof. For any $A > 0$, define $m_A(\theta) = E[e^{\theta h(Z)} 1_{[Z \leq A]}]$. Then $m'_A(\theta) = E[h(Z) e^{\theta h(Z)} 1_{[Z \leq A]}]$. We have:

$$\begin{aligned} m'_A(\theta) &= \int_{-\infty}^{A} h(z)\rho(z)e^{\theta h(z)}dz \\ &= -e^{\theta h(A)} \int_{A}^{\infty} h(y)\rho(y)dy + \int_{-\infty}^{A} \left(\int_{z}^{\infty} h(y)\rho(y)dy \right) e^{\theta h(z)} \theta h'(z) dz \\ &\leq \int_{-\infty}^{A} \theta h'(z) e^{\theta h(z)} g(z)\rho(z)dz \\ &= E[g(Z) 1_{[Z \leq A]} \theta h'(Z) e^{\theta h(Z)}], \end{aligned} \tag{3.18}$$

where the inequality follows from dropping a negative term and invoking relation (3.10). Thus $m'_A(\theta) \leq E[g(Z) 1_{[Z \leq A]} \theta h'(Z) e^{\theta h(Z)}]$. Now applying assumption 4, we get:

$$m'_A(\theta) \leq \theta\alpha m'_A(\theta) + \theta\beta m_A(\theta).$$

That is, for any $\theta \in (0, 1/\alpha)$:

$$\frac{m'_A(\theta)}{m_A(\theta)} \leq \frac{\theta\beta}{1 - \theta\alpha}.$$

By integration and since $m_A(0) = P(Z \leq A) \leq 1$, this gives, for any $\theta \in (0, 1/\alpha)$:

$$m_A(\theta) \leq \exp\left(\int_0^{\theta} \frac{\beta u}{1 - \alpha u} du \right) \leq \exp\left(\frac{\beta\theta^2}{2(1 - \theta\alpha)} \right).$$

Using Fatou's lemma (as $A \to \infty$) in the previous relation implies:

$$E[e^{\theta h(Z)}] \leq \exp\left(\frac{\beta\theta^2}{2(1 - \theta\alpha)} \right)$$

for all $\theta \in (0, 1/\alpha)$. Therefore, for all $\theta \in (0, 1/\alpha)$, we have

$$P(Z \geq z) \leq P(e^{\theta h(Z)} \geq e^{\theta h(z)}) \leq e^{-\theta h(z)} E[e^{\theta h(Z)}] \leq \exp\left(\frac{\beta\theta^2}{2(1 - \theta\alpha)} - \theta h(z) \right).$$

Choosing $\theta = \frac{h(z)}{\alpha h(z) + \beta} \in (0, 1/\alpha)$ gives the desired result.

□

We will now show that by informed choice of the function h, Theorem 3.12 can yield different (and, depending on the context, potentially sharper) concentration inequalities than those yielded by Theorem 3.11, by Theorem 3.3, or the Borel-Sudakov inequalities ([7]).

Example 3.13. Just as in Example 3.9, let $N \sim \mathcal{N}_n(0, K)$ with $K \in \mathcal{S}_n^+(\mathbb{R})$. We assume that each N_i has the form $X(h_i)$, for a certain centered isonormal process X (over some Hilbert space H) and certain functions $h_i \in H$. Let $Z = \max N_i - E[\max N_i]$, and set

$$I_u = \text{argmax}_{1 \le i \le n}(e^{-u}X(h_i) + \sqrt{1 - e^{-2u}}X'(h_i)) \text{ for } u \ge 0,$$

where X' is an independent copy of X. Consider the function

$$h(z) = \begin{cases} -\gamma & : z < -1 \\ \gamma z & : z \in [-1, 0) \\ z + z^2 & : z \in [0, B) \\ B + B^2 & : z \ge B \end{cases} \tag{3.19}$$

where $\gamma > 0$ is a constant to be chosen later, and $B > 0$ is arbitrary. Note that h is Lipschitz and that h' is given almost everywhere by

$$h'(z) = \begin{cases} 0 & : z < -1 \\ \gamma & : z \in [-1, 0) \\ 2z + 1 & : z \in [0, B) \\ 0 & : z \ge B \end{cases} \tag{3.20}$$

In order to apply Theorem 3.12, we must have that $E[h(Z)] = 0$. It is known that Z admits a density ρ, so we have

$$\begin{aligned} E[h(Z)] &= \int_{\mathbb{R}} h(x)\rho(x)dx \\ &= -\gamma \int_{-\infty}^{-1} \rho(x)dx + \gamma \int_{-1}^{0} x\rho(x)dx + \int_0^B (x + x^2)\rho(x)dx + (B + B^2) \int_B^{\infty} \rho(x)dx. \end{aligned}$$

Therefore picking

$$\gamma = \frac{\int_0^B (x + x^2)\rho(x)dx + (B + B^2)P(Z \ge B)}{P(Z \le -1) - \int_{-1}^0 x\rho(x)dx}, \tag{3.21}$$

we get $E[h(Z)] = 0$. By Lemma 3.10 we have again that $DZ = \Phi_Z(X) = h_{I_0}$. Also, $\Phi_{h(Z)}(X) = D(h(Z))$; we can apply Proposition 2.20 since h is a Lipschitz function. Then we get $D(h(Z)) = h'(Z)DZ$. Therefore

$$\begin{aligned} &\Phi_{h(Z)}(e^{-u}X + \sqrt{1 - e^{-2u}}X') \\ &= h' \left(\left(\max_{1 \le i \le n} e^{-u}X(h_i) + \sqrt{1 - e^{-2u}}X'(h_i) \right) - E\left[\max_{1 \le i \le n} e^{-u}X(h_i) + \sqrt{1 - e^{-2u}}X'(h_i) \right] \right) h_{I_u} \end{aligned}$$

Hence we have

$$\langle \Phi_Z(X), \Phi_{h(Z)}(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_H = K_{I_0,I_u} h'(A - E[A]),$$

where $A = \left(\max_{1\le i\le n} e^{-u}X(h_i) + \sqrt{1-e^{-2u}}X'(h_i)\right)$. Hence defining $\sigma^2_{\max} = \max_{1\le i,j\le n} K_{i,j}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\langle \Phi_Z(X), \Phi_{h(Z)}(e^{-u}X + \sqrt{1-e^{-2u}}X')\rangle_H | Z\right] \le \sigma^2_{\max}\left(1+\gamma+2\mathbb{E}[(A-E[A])\mathbf{1}_{([A-E[A])\in[0,B]]}|Z]\right)\\
&\le \sigma^2_{\max}\left((1+\gamma+2\left|E[A]\right|) + 2\mathbb{E}\left[\left(\max_{1\le i\le n} e^{-u}X(h_i) + \max_{1\le i\le n}\sqrt{1-e^{-2u}}X'(h_i)\right)\mathbf{1}_{[(A-E[A])\in[0,B]]}\middle| Z\right]\right)\\
&\le \sigma^2_{\max}\left((1+\gamma+2|E[A]|) + 2e^{-u}|Z| + 2e^{-u}\left|E\left[\max_{1\le i\le n} X(h_i)\right]\right| + 2\sqrt{1-e^{-2u}}\left|\left|\left(\max_{1\le i\le n} X(h_i)\right)\right|\right|_{L^2(\Omega)}\right),
\end{aligned}$$

where the second inequality follows from splitting the expectation, applying the Cauchy-Schwarz inequality to each piece, and using the independence of X' and X. Applying now Proposition 3.8 we get

$$g(Z) \le \sigma^2_{max}|Z| + k$$

where

$$k = \sigma^2_{\max}\left(1+\gamma+2|E[A]| + \left|E\left[\max_{1\le i\le n} X(h_i)\right]\right| + 2\int_0^\infty e^{-u}\sqrt{1-e^{-2u}}du\left|\left|\left(\max_{1\le i\le n} X(h_i)\right)\right|\right|_{L^2(\Omega)}\right).$$

Hence we have

$$g(z)h'(z) \le \begin{cases} 0 & : z < -1\\ (\gamma\sigma^2_{\max})|z| + \gamma k & : z\in[-1,0)\\ (2\sigma^2_{\max})z^2 + (\sigma^2_{\max}+2k)z + k & : z\in[0,B)\\ 0 & : z\ge B\end{cases}$$

and

$$\alpha h(z) + \beta = \begin{cases} -\alpha\gamma + \beta & : z < -1\\ \alpha\gamma z + \beta & : z\in[-1,0)\\ \alpha z^2 + \alpha z + \beta & : z\in[0,B)\\ \alpha(B+B^2)+\beta & : z\ge B\end{cases}$$

We can pick $\alpha > 0$ and $\beta > 0$ to satisfy assumption 4 of Theorem 3.12; for example, $\alpha = \max\{2\sigma^2_{\max}, 2k+\sigma^2_{\max}\}$ and $\beta = \max\{k, \gamma(k+\sigma^2_{\max}+\alpha), \alpha\gamma\}$. Thus for some particular choices of α and β, all the conditions of Theorem 3.12 are satisfied, and in particular we have that for $z\in[0,B]$:

$$P(Z \geq z) \leq \exp\left(-\frac{(z+z^2)^2}{2\alpha(z+z^2)+2\beta}\right).$$

We would like this bound to hold asymptotically - in other words, we wish it to hold for $z \in [0,\infty)$, rather than $z \in [0,B)$. This is still achievable however, by noting first the Borel-Sudakov inequality, which concerns the distribution of the same random variable Z:

$$P(Z \geq z) \leq \exp\left(-\frac{z^2}{2\sigma_{\max}^2}\right) \text{ for } z > 0.$$

Referring to (3.21), and integrating by parts, we see that as B increases, γ correspondingly increases as well, but the Borel-Sudakov inequality guarantees that $\lim_{B\to\infty}\gamma < \infty$. Call this limit γ_∞. Picking α_∞ and β_∞ in terms of γ_∞, we see that

$$P(Z \geq z) \leq \exp\left(-\frac{(z+z^2)^2}{2\alpha_\infty(z+z^2)+2\beta_\infty}\right) \text{ for } z \in [0,\infty). \tag{3.22}$$

We can compare this bound to the Borel-Sudakov inequality, or the even sharper result we get from [7]:

$$P(Z \geq z) \leq \frac{E[|Z|]\sigma_{\max}^2}{2\sigma_{\min}^2}\frac{1}{z}\exp\left(-\frac{z^2}{2\sigma_{\max}^2}\right). \tag{3.23}$$

We obtain this result by the classical inequality $\int_z^\infty e^{-\frac{y^2}{2}}dy \leq \frac{1}{z}e^{-\frac{z^2}{2}}$ applied to the density estimate (3.15). Certainly (3.23) asymptotically improves the Borel-Sudakov inequality, but the relative sharpness of (3.23) and (3.22) is not clear, due to the dependence of (3.22) on α_∞ and β_∞. However, this calculation shows that Theorem 3.12 does in fact yield different and interesting concentration inequalities from those currently known.

Lemma 3.14 is adapted from the proof of Theorem 3.7.

Lemma 3.14. *Assume that $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$, and assume that Z admits a density ρ. Let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Let $\phi(Z) = \int_Z^\infty y\rho(y)dy$. Then we have: $\phi(z) = \frac{E[|Z|]}{2}\exp\left(-\int_0^z \frac{y}{f(y)}dy\right)$, for every $z \in supp(\rho)$.*

Proof. Since $Z \in \mathbb{D}^{1,2}$, it is known that $\text{supp}(\rho) = [\alpha,\beta]$, where $-\infty \leq \alpha < \beta \leq +\infty$. Therefore from (3.5), we have

$$f(z)\rho(z) = \phi(z) \text{ almost surely on } (\alpha,\beta). \tag{3.24}$$

Note that $\phi'(z) = -z\rho(z)$. Thus multiplying (3.24) on both sides by $-z$ and rearranging, we get

$$\frac{\phi'(z)}{\phi(z)} = -\frac{z}{f(z)} \text{ almost surely on } (\alpha, \beta).$$

Dividing by $f(z)$ is well-defined since $f(z) > 0$ almost surely for $z \in (\alpha, \beta)$ (c.f. (3.5)). Note that $0 = E[Z] = E[Z_+] - E[Z_-]$ so that $E[|Z|] = E[Z_+] + E[Z_-] = 2E[Z_+] = 2\phi(0)$, so integrating the above relation we obtain

$$\phi(z) = \frac{E[|Z|]}{2} \exp\left(-\int_0^z \frac{y}{f(y)} dy\right) \text{ almost surely on } (\alpha, \beta). \tag{3.25}$$

□

Note that this proof followed the proof of Step 4 of Theorem 3.7, with h equal to the identity.

Theorem 3.15. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Assume Z admits a density ρ. Let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Then we have for $z \in (0, \infty) \cap supp(\rho)$:*

$$P(Z \geq z) \leq \frac{E[|Z|]}{2} \frac{1}{z} \exp\left(-\int_0^z \frac{y}{f(y)} dy\right).$$

Proof. Let $\phi(z) = \int_z^\infty y\rho(y)dy$. By Lemma 3.14 we have for every $z \in supp(\rho)$:

$$\phi(z) = \frac{E[|Z|]}{2} \exp\left(-\int_0^z \frac{y}{f(y)} dy\right). \tag{3.26}$$

Also, integration by parts yields

$$\phi(z) = zP(Z \geq z) + \int_z^\infty P(Z \geq y)dy. \tag{3.27}$$

Hence we have $P(Z \geq z) \leq \frac{\phi(z)}{z}$, and plugging in the expression for $\phi(z)$ from (3.25), we obtain the desired result. □

Remark 3.16. Note that the previous theorem holds only for the case where $z \in (0, \infty) \cap supp(\rho)$. However, it can still be applied to estimate the left hand tail distribution of Z. Consider for example the case that there exists a function $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\langle DZ, -DL^{-1}Z\rangle \leq h(Z). \tag{3.28}$$

If we define now $Y := -Z$, then the linearity of the inner product and the Malliavin operators L and D imply that:

$$\begin{aligned} \langle DY, -DL^{-1}Y\rangle &= \langle D(-Z), -DL^{-1}(-Z)\rangle \\ &= \langle DZ, -DL^{-1}Z\rangle \\ &\leq h(Z) = h(-Y). \end{aligned} \tag{3.29}$$

Therefore we have for $z \in (0,\infty) \cap -supp(\rho)$:

$$P(Z \leq -z) = P(Y \geq z) \leq \frac{E[|Z|]}{2} \frac{1}{z} \exp\left(-\int_0^z \frac{x}{h(-x)} dx\right). \tag{3.30}$$

As an application of the previous theorem, consider the following corollary, where we assume that f admits an almost sure affine upper bound.

Corollary 3.17. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Assume Z admits a density ρ. Let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$ satisfy*

$$f(Z) \leq \alpha Z + \beta \text{ almost surely,}$$

for some $\alpha, \beta > 0$. Then we have for $z > 0$:

$$P(Z \geq z) \leq \frac{E[|Z|]}{2} \exp\left(-\frac{\beta \log(\beta)}{\alpha^2}\right) \frac{(\alpha z + \beta)^{\beta/\alpha^2}}{z} \exp\left(-\frac{z}{\alpha}\right).$$

Proof. Given our present assumption of a linear upper bound on f, we have for every $z \in supp(\rho)$:

$$\begin{aligned} \exp\left(-\int_0^z \frac{y}{f(y)} dy\right) &\leq \exp\left(-\int_0^z \frac{y}{\alpha y + \beta} dy\right) \\ &= \exp\left(\frac{-\alpha z + \beta \log(\alpha z + \beta)}{\alpha^2} - \frac{\beta \log(\beta)}{\alpha^2}\right) \end{aligned} \tag{3.31}$$

Applying Theorem 3.15, we obtain the result for $z \in (0,\infty) \cap supp(\rho)$. In fact, it holds for every $z > 0$, for the trivial reason that for $z \in (0,\infty) \cap supp(\rho)^c$, we have $P(Z \geq z) = 0$. $\square$

Comparing Corollary 3.17 with Theorem 3.11, we see that Corollary 3.17 yields an asymptotically sharper bound. We now show some examples to demonstrate the utility of these results.

Example 3.18. Let $(B_t, t \geq 0)$ denote a fractional Brownian motion process with Hurst index $H \in (0,1)$. Note that such a process can be realized as an isonormal Gaussian process. In particular, we can consider the Hilbert space $\mathcal{H}$ defined as the closure of the space of step functions on the set $\mathbb{R}_{\geq 0}$ with respect to the inner product given by:

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle = E[B_t B_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{3.32}$$

In this case we have that $B_t := B(\mathbf{1}_{[0,t]})$. We will consider obtaining concentration inequalities on the random variable $Z = Z_T := \int_0^T B_s^4 ds - E[\int_0^T B_s^4 ds]$. We can without loss of generality assume that $T = 1$ by the scaling property of fractional Brownian motion. Hence we will consider the random variable $Z = \int_0^1 B_s^4 ds - \frac{3}{4H+1}$. We wish to upper bound the expression $\langle DZ, -DL^{-1}Z\rangle$ in order to apply Theorem 3.15. Since we must compute $L^{-1}Z$, we must first

compute the Wiener chaos expansion of the random variable Z. To this end, for every $n \geq 1$, let $\Pi_n = \{t_i\}_{i=0}^n$ where $0 = t_0 < t_1 < ... < t_n = 1$. Define $||\Pi_n|| = \max\{t_i - t_{i-1}, i = 1, ..., n\}$. Assume that $\lim_{n\to\infty} ||\Pi_n|| = 0$. Then we have:

$$
\begin{aligned}
Z &= \int_0^1 B_s^4 ds - \frac{3}{4H+1} \\
&= \lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) B_{t_i}^4 - \frac{3}{4H+1} \\
&= \lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^4 - \frac{4}{4H+1} \\
&= \lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 \\
&\left(B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^4 - 6B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^2 + 3 + 6B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^2 - 3 - 6 + 6\right) - \frac{3}{4H+1} \\
&= \underbrace{\lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 H_4\left(B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)\right)}_{\in \mathcal{H}_4} + \\
&\underbrace{\lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 6H_2\left(B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)\right)}_{\in \mathcal{H}_2} + \underbrace{\lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 3 - \frac{3}{4H+1}}_{=0}
\end{aligned}
\tag{3.33}
$$

We have decomposed Z into a sum of two components, belonging to the fourth and second Wiener chaos spaces respectively. Having expressed the integral defining Z in an equivalent limit form, in order to unravel the Wiener chaos expansion, we now repackage the resulting sum of limits back into integral form - for example, the $\mathcal{H}_4$ component of Z can be expressed as follows:

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 H_4\left(B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^n (t_i - t_{i-1}) ||\mathbf{1}_{[0,t_i]}||^4 \left(B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^4 - 6B\left(\frac{\mathbf{1}_{[0,t_i]}}{||\mathbf{1}_{[0,t_i]}||}\right)^2 + 3\right) \\
&= \int_0^1 t^{4H}\left(\frac{B_t^4}{t^{4H}} - 6\frac{B_t^2}{t^{2H}} + 3\right) dt \\
&= \int_0^1 \left(B_t^4 - 6t^{2H} B_t^2 + 3t^{4H}\right) dt.
\end{aligned}
\tag{3.34}
$$

Repeating this repackaging process with the $\mathcal{H}_2$ part of Z, we can write Z as:

$$Z = 6\int_0^1 \left(t^{2H}B_t^2 - t^{4H}\right)dt + \int_0^1 \left(B_t^4 - 6t^{2H}B_t^2 + 3t^{4H}\right)dt, \tag{3.35}$$

where the first and second components of the sum belong to $\mathcal{H}_2$ and $\mathcal{H}_4$ respectively. We now compute $L^{-1}Z$:

$$L^{-1}Z = -\frac{1}{4}\int_0^1 \left(B_t^4 - 6t^{2H}B_t^2 + 3t^{4H}dt\right) - \frac{1}{2}6\int_0^1 \left(t^{2H}B_t^2 - t^{4H}\right)dt. \tag{3.36}$$

We compute also the Malliavin derivative of Z:

$$DZ = D\left(\int_0^1 B_t^4 dt - \frac{3}{4H+1}\right) = \int_0^1 4B_t^3 \mathbf{1}_{[0,t]}dt. \tag{3.37}$$

We compute $DL^{-1}Z$ in an analogous manner, based on (3.36), and finally we see that:

$$\begin{aligned}\langle DZ, -DL^{-1}Z\rangle &= \langle \int_0^1 4B_t^3\mathbf{1}_{[0,t]}dt, \int_0^1 (B_s^3 + 3s^{2H}B_s)\mathbf{1}_{[0,s]}ds\rangle \\ &= \int\int_{[0,1]^2} (4B_t^3)(B_s^3 + 3s^{2H}B_s)E[B_tB_s]dtds.\end{aligned} \tag{3.38}$$

We now upper bound this expression in terms of Z:

$$\begin{aligned}\langle DZ, -DL^{-1}Z\rangle &\leq \| \int\int_{[0,1]^2} (4B_t^3)(B_s^3 + 3s^{2H}B_s)E[B_tB_s]dtds\| \\ &\leq \int 4\int_{[0,1]^2} |B_t|^3|B_s^3 + 3s^{2H}B_s|\frac{3}{2}dtds \\ &= 6\int_0^1 |B_t|^3 dt \int_0^1 |B_s^3 + 3s^{2H}B_s|ds \\ &\leq 6\int_0^1 |B_t|^3 dt \left(\int_0^1 |B_s|^3 ds + 3\int_0^1 |B_s|ds\right) \\ &= 6\left[\left(\int_0^1 |B_t|^3 dt\right)^2 + 3\int_0^1 |B_t|^3 dt \int_0^1 |B_t|dt\right] \\ &= 6\left[\left(\int_0^1 (|B_t|^4)^{\frac{3}{4}}dt\right)^2 + 3\int_0^1 (|B_t|^4)^{\frac{3}{4}}dt \int_0^1 (|B_t|^4)^{\frac{1}{4}}dt\right] \\ &\leq 6\left[(Z + \frac{3}{4H+1})^{\frac{3}{2}} + 3(Z + \frac{3}{4H+1})^{\frac{3}{4}}(Z + \frac{3}{4H+1})^{\frac{1}{4}}\right] \\ &= 6\left[(Z + \frac{3}{4H+1})^{\frac{3}{2}} + 3(Z + \frac{3}{4H+1})\right]\end{aligned} \tag{3.39}$$

Once we show that Z admits a density, we will be in a position to apply Theorem 3.15. We could apply the result from [10], which states that an element of $\mathbb{D}^{1,2}$ admitting a finite Wiener chaos expansion has a density. Since we showed that $Z \in \mathcal{H}_2 \oplus \mathcal{H}_4$, it has a density. We could alternately take a more hand-on approach and apply the Bouleau-Hirsch criterion from [9], which says that for any $Z \in \mathbb{D}^{1,2}$, the almost sure positivity of $||DZ||$ is a sufficient condition for Z to admit a density. We compute:

$$\begin{aligned} ||DZ||^2 &= \langle DZ, DZ \rangle \\ &= \langle 4\int_0^1 B_s^3 ds, 4\int_0^1 B_t^3 dt \rangle \\ &= 16 \int\int_{[0,1]^2} B_s^3 B_t^3 E[B_s B_t] ds dt \\ &= 16 \int\int_{[0,1]^2} B_s^3 B_t^3 \tilde{E}[\tilde{B}_s \tilde{B}_t] ds dt \\ &= 16\tilde{E}\left[\left(\int_0^1 B_t^3 \tilde{B}_t dt\right)^2\right] \end{aligned} \tag{3.40}$$

where $\tilde{B}$ is a fractional Brownian motion with Hurst parameter H which is independent of B. The above calculation shows that $||DZ|| = 0$ if and only if $\int_0^1 B_t^3 \tilde{B}_t dt = 0$ for almost every path of $\tilde{B}$. This is equivalent to the condition that the path $(B_t^3, t \in [0,1])$ be identically zero, which of course has probability zero. Hence Z does admit a density, and we can apply Theorem 3.15 to say that for $z > 0$:

$$P(Z \geq z) \leq \frac{E[|Z|]}{2} \frac{1}{z} \exp\left(-\int_0^z \frac{y}{6\left[(y + \frac{3}{4H+1})^{\frac{3}{2}} + 3(y + \frac{3}{4H+1})\right]} dy\right). \tag{3.41}$$

Upper bounding $E[|Z|]$ by $\frac{6}{4H+1}$, and evaluating the integral inside the exponential, we can simplify this expression to conclude that for $z > 0$:

$$P(Z \geq z) \leq K \frac{\left(\sqrt{c+z} + 3\right)^{1-\frac{1}{9}c} \left(c+z\right)^{\frac{1}{18}c}\right)}{z} e^{-\frac{1}{3}\sqrt{c+z}}, \tag{3.42}$$

where $K = \frac{c}{2}(\sqrt{c}+3)^{-1+\frac{c}{9}} c^{-\frac{c}{18}} e^{\frac{\sqrt{c}}{3}}$, and $c = \frac{3}{4H+1}$. Note that we can also use Malliavin techniques to get a lower bound on $P(Z \geq z)$:

$$
\begin{aligned}
P(Z \geq z) &= P(\int_0^1 B_s^4 ds - \frac{3}{4H+1} \geq z) \\
&= P(\int_0^1 B_s^4 ds \geq z + \frac{3}{4H+1}) \\
&\geq P(|\int_0^1 B_s ds|^4 \geq z + \frac{3}{4H+1}) \\
&= P(|\int_0^1 B_s ds| \geq (z + \frac{3}{4H+1})^{\frac{1}{4}}).
\end{aligned}
\tag{3.43}
$$

Note now that $N := \int_0^1 B_s ds$ is a Normally distributed random variable - this follows from Corollary 3.4, since computing $\langle DN, -DL^{-1}N \rangle$, we see that it is a constant, hence guaranteeing that N is normally distributed. By Fubini we see that N has zero mean, and we compute it's variance as follows:

$$
\begin{aligned}
E[N^2] &= \int\int_{[0,1]^2} E[B_s B_t] ds dt \\
&= \int_0^1 \left(\int_0^t \left(s^{2H} + t^{2H} + (t-s)^{2H} \right) ds \right) dt \\
&= \frac{1}{2H+2}.
\end{aligned}
\tag{3.44}
$$

We can now use the classical inequality $\int_z^\infty e^{-y^2/2} dy \geq \frac{z}{1+z^2} e^{-z^2/2}$ for $z > 0$, to deduce that for $z > 0$:

$$
P(Z \geq z) \geq \sqrt{\frac{2}{\pi}} \left(\frac{t(z)}{1 + t(z)^2} \right) e^{-t(z)^2/2},
\tag{3.45}
$$

where $t(z) = \sqrt{2H+2} \left(z + \frac{3}{4H+1} \right)^{\frac{1}{4}}$. Hence we have characterized by both upper and lower bounds the rate of decay of the tail distribution of Z. The upper bound we derived is, as far as we know, the first such bound in the literature.

Example 3.19. This example concerns estimating the Hurst parameter of a fractional Brownian motion process which has been observed at a finite number of times. Let now $B = \{B_t, t \in [0,1]\}$ denote a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Assume that we observe B at the times $\{k/n, k = 0, ..., n-1\}$. We will establish non-asymptotic confidence intervals for H on the basis of these observations. The technique considered here is based on [1], although we consider a different statistic and establish different confidence intervals. The result here could be substantially sharpened without great difficulty, but that is beyond the scope of this work.

The estimation of self-similarity indices, such as the Hurst parameter, is useful in applications ranging from physics to time-series analysis to mathematical finance, and has generated

a substantial literature - see for example [11], [4], [3]. The result below is remarkable inasmuch as it provides an explicit, non-asymptotic confidence interval; thus if a fBm process is observed at some finite number of points in a practical application, we can deduce explicit bounds on the likelihood of the parameter H being in some interval.

Consider the family $\{X_k\}_{k=0,\dots,n-1}$ where $X_k := n^H \left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}} \right)$. Note that this family has covariance function ρ given by, for every $k \in \mathbb{Z}$:

$$\rho(k) = \frac{1}{2} \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H} \right). \tag{3.46}$$

Assume in the sequel that the $H \leq H^* \in (0, \frac{1}{2})$. This guarantees that $\sum_t |\rho_H(t)| \leq \sum_t |\rho_{H^*}(t)| < \infty$ ([6]), which we will need later. Define now

$$Z_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X_k^4 - 3\sqrt{n}. \tag{3.47}$$

Note that $E[Z] = 0$, since $E[(B_t^H - B_s^H)^{2k}] = \frac{(2k)!}{k! 2^k} |t-s|^{2Hk}$ for every $k \geq 1$ ([8]). Although it is not relevant for the coming calculations, it is interesting to note that since $x^4 = H_4(x) + 6H_2(x) + 3$, we have:

$$\begin{aligned} \sum_{k=0}^{n-1} X_k^4 &= \sum_{k=0}^{n-1} (H_4(X_k) + 6H_2(X_k) + 3) \\ &= \sqrt{n}\sigma_1 c_n \sum_{k=0}^{n-1} H_4(X_k) + \sqrt{n}\sigma_2 6 k_n \sum_{k=0}^{n-1} H_2(X_k) + 3n, \end{aligned} \tag{3.48}$$

where σ_1, σ_2 are constants and c_n, k_n are c functions of n. We rearrange to get:

$$Z_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X_k^4 - 3\sqrt{n} = \sigma_1 \underbrace{\left(c_n \sum_{k=0}^{n-1} H_4(X_k) \right)}_{\to N(0,1) \text{ in law}} + 6\sigma_2 \underbrace{\left(k_n \sum_{k=0}^{n-1} H_2(X_k) \right)}_{\to N(0,1) \text{ in law}}. \tag{3.49}$$

The statistic Z_n can be written as a sum of two n-dependent random variables, each of which converge in law to a standard Normal distribution ([6]). This in some sense 'motivates' the choice of Z_n as the statistic under consideration.

We wish now to compute concentration inequalities involving Z_n based on Theorem 3.15, hence we will compute $\langle DZ_n, DL^{-1}Z_n \rangle$. To do so, we must compute the action of the operator L on Z_n, and for this purpose we must decompose Z_n into its Wiener chaos expansion:

$$Z_n = \underbrace{\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_4(X_k)}_{\in \mathcal{H}_4} + \underbrace{\frac{6}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(X_k)}_{\in \mathcal{H}_2}. \tag{3.50}$$

Therefore:

$$-L^{-1} Z_n = \frac{1}{4} \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_4(X_k) + \frac{1}{2} \frac{6}{\sqrt{n}} \sum_{k=0}^{n-1} H_2(X_k). \tag{3.51}$$

We will omit the mechanical details of some of the calculations that follow, since similar calculations appear previously. To slightly lighten notation we now define $c = \frac{1}{\sqrt{n}}$. We have now:

$$\begin{aligned}
\langle DZ_n, -DL^{-1} Z_n \rangle &= \left\langle c \sum_{k=0}^{n-1} 4X_k^3 h_k, c \sum_{j=0}^{n-1} (X_j^3 + 3X_j) h_j \right\rangle \\
&= c^2 \sum_{k,j=0}^{n-1} 4X_k^3 (X_j^3 + 3X_j) \langle h_k, h_j \rangle \\
&= c^2 \sum_{k,j=0}^{n-1} 4X_k^3 (X_j^3 + 3X_j) \rho_H(k-j) \rangle \\
&\leq 4c^2 \sum_{t \in \mathbb{Z}} |\rho_H(t)| \left(\sum_{j=0}^{n-1} |X_j^3 + 3X_j| \right) \left(\sum_{j=0}^{n-1} |X_j|^3 \right) \\
&\leq 4c^2 \sum_{t \in \mathbb{Z}} |\rho_H(t)| \left(\sum_{j=0}^{n-1} |X_j^3| + 3 \sum_{j=0}^{n-1} |X_j| \right) \left(\sum_{j=0}^{n-1} |X_j|^3 \right) \\
&= 4c^2 \sum_{t \in \mathbb{Z}} |\rho_H(t)| \left[\left(\sum_{j=0}^{n-1} |X_j^3| \right)^2 + 3 \left(\sum_{j=0}^{n-1} |X_j| \right) \left(\sum_{j=0}^{n-1} |X_j^3| \right) \right] \\
&= 4c^2 \sum_{t \in \mathbb{Z}} |\rho_H(t)| \left[\left(\sum_{j=0}^{n-1} (|X_j|^4)^{3/4} \right)^2 + 3 \left(\sum_{j=0}^{n-1} (|X_j|^4)^{1/4} \right) \left(\sum_{j=0}^{n-1} (|X_j|^4)^{3/4} \right) \right] \\
&= 4c^2 \sum_{t \in \mathbb{Z}} |\rho_H(t)| \left[\left(\sum_{j=0}^{n-1} |X_j|^4 \right)^{\frac{3}{2}} + 3 \left(\sum_{j=0}^{n-1} |X_j|^4 \right)^{\frac{1}{4}} \left(\sum_{j=0}^{n-1} |X_j|^4 \right)^{\frac{3}{4}} \right] \\
&= \frac{4}{\sqrt{n}} \sum_{t \in \mathbb{Z}} |\rho_{H^*}(t)| \left[n^{1/4} (Z_n + 3\sqrt{n})^{3/2} + 3\sqrt{n} (Z_n + 3\sqrt{n}) \right]
\end{aligned} \tag{3.52}$$

More concisely, we can summarize the previous calculation by:

$$\langle DZ_n, -DL^{-1}Z_n\rangle \leq h(Z_n), \tag{3.53}$$

where $h(x) = \frac{4}{\sqrt{n}}\sum_{t\in\mathbb{Z}}|\rho_{H^*}(t)|\left[n^{1/4}(x+3\sqrt{n})^{3/2} + 3\sqrt{n}(x+3\sqrt{n})\right]$. Moreover, referring to Remark 3.16, we know also that for $z \in (0, 3\sqrt{n})$:

$$P(Z \leq -z) \leq \frac{E[|Z|]}{2}\frac{1}{z}\exp\left(-\int_0^z \frac{x}{h(-x)}dx\right). \tag{3.54}$$

Note that the function h is strictly increasing, and hence for $x \in (0, 3\sqrt{n})$, we have $h(-x) \leq h(x)$. Putting this into (3.54) we see that $P(Z \leq -z) = P(Z \geq z)$ for $z \in (0, 3\sqrt{n})$. Since for $z \in [3\sqrt{n}, \infty)$, we have $P(Z \leq -z) = 0$, we see that for $z > 0 : P(Z \leq -z) \leq P(Z \geq z)$. Therefore, by Theorem 3.15, for $z > 0$:

$$\begin{aligned} P(|Z| \geq z) &= P(Z \geq z) + P(Z \leq -z) \\ &\leq 2P(Z \geq z) \\ &\leq E[|Z|]\frac{1}{z}\exp\left(-\int_0^z \frac{x}{h(x)}dx\right), \end{aligned} \tag{3.55}$$

where, as before, $h(x) = \frac{4}{\sqrt{n}}\sum_{t\in\mathbb{Z}}|\rho(t)|\left[n^{1/4}(x+3\sqrt{n})^{3/2} + 3\sqrt{n}(x+3\sqrt{n})\right]$. We can evaluate the integral in the above expression and obtain a closed form for our upper bound for $P(|Z| \geq z)$; however, in the present context, we will simply denote the final expression in (3.55) by $w(z)$, and note that in summary, for every $z > 0$, $P(|Z| \geq z) \leq w(z)$. Note that we can equivalently write this as, for $a > 0$, $P(|Z| \leq a) \geq 1 - w(a)$.

Note now that for $0 < a < 3\sqrt{n}$:

$$\begin{aligned} \{|Z| \leq a\} &= \left\{\left| n^{4H\ 1/2}\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)^4 - 3\sqrt{n}\right| \leq a\right\} \\ &= \left\{\frac{1}{4}\left(\frac{\log(3\sqrt{n}-a) - \log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)^4)}{\log(n)} + \frac{1}{2}\right) \leq H\right. \\ &\left.\leq \frac{1}{4}\left(\frac{\log(3\sqrt{n}+a) - \log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\right)^4)}{\log(n)} + \frac{1}{2}\right)\right\}. \end{aligned} \tag{3.56}$$

Note the result from [2] which says that for $p > 0$,

$$n^{pH-1}\sum_{j=0}^{n-1}|B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}|^p \to E[|B^H_1|^p] \text{ in } L^1 \text{ as } {}^{n\to\infty}. \tag{3.57}$$

Hence, if we denote:

$$
\begin{aligned}
I_l(n) &:= \frac{1}{4}\left(\frac{\log(3\sqrt{n}-a)-\log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}}-B_{\frac{k}{n}}\right)^4)}{\log(n)}+\frac{1}{2}\right)\\
I_r(n) &:= \frac{1}{4}\left(\frac{\log(3\sqrt{n}+a)-\log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}}-B_{\frac{k}{n}}\right)^4)}{\log(n)}+\frac{1}{2}\right),
\end{aligned}
\tag{3.58}
$$

we see that $\lim_{n\to\infty} I_l(n) = \lim_{n\to\infty} I_r(n) = H$ almost surely - i.e., the confidence interval collapses to H as $n \to \infty$. Also note that:

$$
|I_r(n) - I_l(n)| = \frac{1}{4}\frac{1}{\log(n)}\log\left(\frac{3\sqrt{n}+a}{3\sqrt{n}-a}\right). \tag{3.59}
$$

so that the length of the confidence interval decreases to zero as n increases at rate $O\left(\frac{1}{\sqrt{n}\log(n)}\right)$, for every fixed a.

We summarize our results in the following proposition, based on [1].

Proposition 3.20. *Let $B = \{B_t, t \in [0,1]\}$ be a fractional Brownian motion, with Hurst parameter $H \le H^* \in (0, 1/2)$. Fix $n \ge 2$, and let $a \in (0, 3\sqrt{n})$. Then we have:*

$$
P(I_l(n) \le H \le I_r(n)) \ge 1 - w(a), \tag{3.60}
$$

where:

$$
\begin{aligned}
I_l(n) &:= \frac{1}{4}\left(\frac{\log(3\sqrt{n}-a)-\log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}}-B_{\frac{k}{n}}\right)^4)}{\log(n)}+\frac{1}{2}\right),\\
I_r(n) &:= \frac{1}{4}\left(\frac{\log(3\sqrt{n}+a)-\log(\sum_{k=0}^{n-1}\left(B_{\frac{k+1}{n}}-B_{\frac{k}{n}}\right)^4)}{\log(n)}+\frac{1}{2}\right),\\
w(a) &= 6\sqrt{n}\frac{1}{a}\exp\left(-\int_0^a \frac{x}{h(x)}dx\right),
\end{aligned}
\tag{3.61}
$$

where $h(x) = \frac{4}{\sqrt{n}}\sum_{t\in\mathbb{Z}}|\rho_{H^*}(t)|\left[n^{1/4}(x+3\sqrt{n})^{3/2} + 3\sqrt{n}(x+3\sqrt{n})\right]$.

3.3 Concentration Inequalities – Lower Bounds

Theorem 3.21. (from [7])
Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$, and fix $\sigma_{\min}, \alpha > 0$ and $\beta > 1$. Assume that $f(Z) \ge \sigma_{\min}^2$ almost surely. This assumption guarantees the existence of a density ρ for Z, by Theorem

3.3. Assume also that the function $e(x) = x^{1+\beta}\rho(x)$ is decreasing on $[\alpha, +\infty)$. Then for all $z \geq \alpha$, we have

$$P(Z \geq z) \geq \frac{1}{2}\left(1 - \frac{1}{\beta}\right) E[|Z|]\frac{1}{z}\exp\left(-\frac{z^2}{2\sigma_{\min}^2}\right).$$

We state this result without proof because we will now give and prove a more general version of it - namely, Theorem 3.22.

Theorem 3.22. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Fix $\alpha > 0, \beta > 1$. Assume that Z admits a density ρ, and the function $h(x) = x^{1+\beta}\rho(x)$ is decreasing on $[\alpha, +\infty)$. Then for all $z \in [\alpha, \infty) \cap supp(\rho)$, we have*

$$P(Z \geq z) \geq \left(1 - \frac{1}{\beta}\right)\frac{E[|Z|]}{2}\frac{1}{z}\exp\left(-\int_0^z \frac{y}{f(y)}dy\right). \tag{3.62}$$

Proof. First, we apply integration by parts to get, for every $z \in \mathbb{R}$:

$$\phi(z) = \int_z^\infty y\rho(y)dy = zP(Z \geq z) + \int_z^\infty P(Z \geq y)dy. \tag{3.63}$$

By assumption 3, for any $y > z \geq \alpha$, such that $z \in supp(\rho)$, we have

$$\frac{y\rho(y)}{z\rho(z)} \leq \left(\frac{z}{y}\right)^\beta. \tag{3.64}$$

Then we have for any $z \in [\alpha, \infty) \cap supp(\rho)$:

$$P(Z \geq z) = z\rho(z)\int_z^\infty \frac{1}{y}\frac{y\rho(y)}{z\rho(z)}dy \leq \frac{z\rho(z)}{\beta}. \tag{3.65}$$

In fact, the last relation holds for every $z \geq \alpha$, since for $z \geq \alpha \cap supp(\rho)^c$, we have $P(Z \geq z) = 0$ and $\rho(z) = 0$, hence the inequality holds trivially. Putting this relation into (3.63), we obtain for every $z \geq \alpha$:

$$\phi(z) \leq zP(Z \geq z) + \frac{1}{\beta}\int_z^\infty y\rho(y)dy = zP(Z \geq z) + \frac{1}{\beta}\phi(z). \tag{3.66}$$

Rearranging the previous equation we get for every $z \geq \alpha$:

$$P(Z \geq z) \geq \left(1 - \frac{1}{\beta}\right)\frac{\phi(z)}{z}.$$

Plugging in the expression for $\phi(z)$ from Lemma 3.14, which holds for every $z \in supp(\rho)$, we have the desired result. □

Note that if f admits a uniform lower bound, namely $\sigma_{\min}$, then Theorem 3.22 reduces to Theorem 3.21. However, Theorem 3.22 is in addition applicable to random variables Z whose corresponding f decays to zero, but does not admit a uniform lower bound. The following theorem repackages a result from [7].

Theorem 3.23. (from [7])
Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Assume that there exists $\sigma_{\min} > 0$ such that $f(Z) \geq \sigma_{\min}^2$ almost surely. Then for every $z > 0$ we have

$$P(Z \geq z) \geq \frac{E[|Z|]}{2} \frac{\sigma_{\min}^4 z^2}{(\sigma_{\min}^2 + z^2)^2} \frac{\exp\left(-\frac{z^2}{\sigma_{\min}^2}\right)}{\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) f(y) dy}. \tag{3.67}$$

Proof. By Theorem 3.3 we have that Z admits a density ρ, and moreover

$$\rho(z) \geq \frac{E[|Z|]}{2f(z)} \exp\left(-\frac{z^2}{2\sigma_{\min}^2}\right).$$

Let $\Psi(z)$ denote the unnormalized Gaussian tail $\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) dy$. We can write, using the Schwarz inequality,

$$\begin{aligned}
\Psi^2(z) &= \left(\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) \sqrt{f(y)} \frac{1}{\sqrt{g(y)}} dy \right)^2 \\
&\leq \int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) f(y) dy \times \int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) \frac{1}{f(y)} dy
\end{aligned}$$

so that

$$\begin{aligned}
P(Z \geq z) &= \int_z^\infty \rho(y) dy \\
&\geq \frac{E[|Z|]}{2} \int_z^\infty e^{-\frac{y^2}{(2\sigma_{\min}^2)}} \frac{1}{f(y)} dy \\
&\geq \frac{E[|Z|]}{2} \frac{\Psi(z)^2}{\int_z^\infty e^{-\frac{y^2}{(2\sigma_{\min}^2)}} f(y) dy}.
\end{aligned}$$

Using the classical inequality $\int_z^\infty e^{-\frac{y^2}{2}} dy \geq \frac{z}{1+z^2} e^{-z^2/2}$, we get

$$P(Z \geq z) \geq \frac{E[|Z|]}{2} \frac{\sigma_{\min}^4 z^2}{(\sigma_{\min}^2 + z^2)^2} \frac{\exp\left(-\frac{z^2}{\sigma_{\min}^2}\right)}{\int_z^\infty \exp\left(-\frac{y^2}{2\sigma_{\min}^2}\right) f(y) dy}. \tag{3.68}$$

□

The next theorem is a lower bound analog of Theorem 3.15.

Theorem 3.24. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$, and assume Z admits a density ρ. Let $g(Z) = E[\langle DZ, -DL^{-1}(h(Z))\rangle | Z]$, where $h : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, non-decreasing on $[0, \infty)$, $h(0) \geq 0$, and satisfies $E[h(Z)] = 0$. Then we have for $z \in (0, \infty) \cap supp(\rho)$:*

$$P(Z \geq z) \geq \frac{E[|h(Z)|]}{2} \frac{1}{h(\infty)} \exp\left(- \int_0^z \frac{h(y)}{g(y)} dy \right) \tag{3.69}$$

where the lower bound term in (3.69) is understood to be zero if $h(\infty) = \infty$.

Proof. Let $\phi(z) = \int_z^\infty h(y)\rho(y)dy$. From (3.25) we have for every $z \in supp(\rho)$:

$$\phi(z) = \frac{E[|h(Z)|]}{2} \exp\left(- \int_0^z \frac{h(y)}{g(y)} dy \right). \tag{3.70}$$

We can integrate the expression $\int_z^\infty h(y)\rho(y)dy$ by parts to obtain

$$\begin{aligned} \phi(z) &= \int_z^\infty h(y)\rho(y)dy \\ &= -\left[h(y) \int_y^\infty \rho(x)dx \right]_{y=z}^\infty + \int_z^\infty h'(y) \left(\int_y^\infty \rho(x)dx \right) dy \\ &= h(z)P(Z \geq z) + \int_z^\infty h'(y)P(Z \geq y)dy. \end{aligned} \tag{3.71}$$

Hence we have $P(Z \geq z) \leq \frac{\phi(z)}{h(z)}$, and plugging in the expression for $\phi(z)$ from (3.70), we obtain the upper bound on $P(Z \geq z)$ in (3.69). Note now that, from (3.71), we have:

$$\phi(z) = h(z)P(Z \geq z) + \int_z^\infty h'(y)P(Z \geq y)dy \quad \leq h(z)P(Z \geq z) + \int_z^\infty h'(y)dyP(Z \geq z),$$

from which we obtain $\phi(z) \leq P(Z \geq z)(h(\infty))$ by the fundamental theorem of calculus; rearranging, we obtain the lower bound in (3.69). □

3.4 Concentration Inequalities - Upper and Lower Bounds

The following proposition characterizes a case where simultaneous lower and upper bounds on the function f jointly yield simultaneous upper and lower bounds on the tail distribution of Z, neither of which would be attainable with simply the upper or lower bound on f individually.

Proposition 3.25. *Let $Z \in \mathbb{D}^{1,2}$ with $E[Z] = 0$. Let $f(Z) = E[\langle DZ, -DL^{-1}Z\rangle_H | Z]$. Assume that there exists $\sigma_{\min}, \alpha, \beta > 0$ such that*

$$\sigma^2_{\min} \leq f(Z) \leq \alpha Z + \beta.$$

Then we have

$$\frac{E[|Z|]}{2} \frac{\sigma^4_{\min} z^2}{(\sigma^2_{\min} + z^2)^2} \frac{\exp\left(-\frac{z^2}{\sigma^2_{\min}}\right)}{\alpha\sigma^2_{\min} \exp\left(-\frac{z^2}{2\sigma^2_{\min}}\right) + \beta\sqrt{\frac{\pi}{2}}\sigma_{\min} erfc\left(\frac{z}{\sqrt{2}\sigma_{\min}}\right)}$$
$$\leq P(Z \geq z) \leq \frac{E[|Z|]}{2} \exp\left(-\frac{\beta \log(\beta)}{\alpha^2}\right) \frac{(\alpha z + \beta)^{\beta/\alpha^2}}{z} \exp\left(-\frac{z}{\alpha}\right),$$

where erfc is the complementary error function - specifically

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Proof. This is a direct calculation based on Theorem 3.23 and Corollary 3.17, noting that the assumption in Corollary 3.17 that Z admits a density is satisfied on the basis of the assumption that $f(Z) \geq \sigma^2_{\min}$ combined with Theorem 3.3. □

Chapter 4

Conclusion

We have presented new density formulas and concentration inequalities derived using the Malliavin calculus, and demonstrated their originality and applicability by computing bounds on the tail-distributions of several random variables. These bounds are largely unattainable using existing techniques. A crucial aspect of these formulas (e.g. Theorem 3.7 and Theorem 3.12) is their dependence on the function h. The arbitrariness in the choice of h presents the opportunity to pick an h tailor-made to the random variable under consideration, informed by the desire to obtain the tightest possible bounds on the tail-distribution, or potentially to obtain bounds of a particular form.

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